

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

**ON EQUIMATCHABLE GRAPHS AND
WELL-INDUMATCHED GRAPHS**

YASEMİN BÜYÜKÇOLAK
A THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
DEPARTMENT OF MATHEMATICS

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T.C.
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İYİ-İNDÜKLENMİŞ ÇİZGELER
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DOKTORA TEZİ
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GEBZE
2022

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SUMMARY

This thesis investigates problems in graph theory, mostly matchings of a graph. A graph is called equimatchable if every maximal matching in the graph is a maximum matching. Many researchers provided some structural and algorithmic results on equimatchable graphs; particularly, some authors provided some partial results on equimatchable graphs with no triangle, in other words, equimatchable triangle-free graphs. In this thesis, we first extend these particular results by providing a full characterization of equimatchable triangle-free graphs by identifying the structure of such graph families. This characterization yields a linear time algorithm that recognizes whether a given non-bipartite graph is equimatchable and triangle-free.

Furthermore, some authors characterized equimatchable bipartite graphs, which does not provide a structural insight, whereas some authors provided only a partial characterization for equimatchable bipartite graphs. An equimatchable graph is called edge-critical if every graph obtained by the removal of one of its edges is not equimatchable. We first point out that it is sufficient to focus on the structure of edge-critical equimatchable bipartite graphs instead of the general equimatchable bipartite graphs. Hence, in this thesis, we provide a structural characterization for edge-critical equimatchable bipartite graphs.

A graph is called well-indumatched if every maximal induced matching in the graph is a maximum induced matching. In the literature, some authors provided a characterization of well-indumatched acyclic graphs and some partial results on well-indumatched unicyclic graphs. A pseudoforest is a graph in which every connected component contains at most one cycle, that is, each connected component is a unicyclic graph or a tree. In this thesis, we finally extend the result in the literature about well-indumatched unicyclic graphs by providing a complete characterization of well-indumatched pseudoforests by identifying the structure of such graph families.

Keywords: Matching, Equimatchable, Well-Indumatched, Triangle-Free Graphs, Bipartite Graphs, Unicyclic Graphs, Pseudoforests.

ÖZET

Bu tez, çizge teorisindeki sorunları, çoğunlukla bir grafiğin eşleşmesini araştırır. Grafikteki her maksimum eşleşme bir maksimum eşleşme ise, bir grafiğe eşiteşleştirilebilir denir. Birçok araştırmacı eşiteşleştirilebilir çizgeler üzerinde bazı yapısal ve algoritmik sonuçlar sağladılar. Özellikle, üçgensiz eşiteşleştirilebilir çizgelerde bazı kısmi sonuçlar elde ettiler. Bu tezde, ilk önce üçgensiz eşiteşleştirilebilir çizgelerdeki sonuçları, bu tür çizge ailelerinin yapısını tanımlayarak, eşiteşleştirilebilir üçgensiz çizgelerin tam karakterizasyonunu sağlayacak şekilde genişletiyoruz. Bu karakterizasyon, verilen iki parçalı olmayan bir çizgenin eşiteşleştirilebilir ve üçgensiz olup olmadığını tanıyan doğrusal bir zaman algoritması verir.

Ayrıca, literatürde eşiteşleştirilebilir iki parçalı çizgeler için yapısal olarak çok fazla bilgi sağlamayan ya da sadece bazı kısmi sonuçlar sağlayan karakterizasyonlar vardır. Kenarlarından birinin çıkarılmasıyla elde edilen her çizgenin eşiteşleştirilemez olması durumunda, eşiteşleştirilebilir bir çizge kenar-kritik olarak adlandırılır. Bu tezde, öncelikle, genel eşiteşleştirilebilir iki parçalı çizgeler yerine, kenar-kritik eşiteşleştirilebilir iki parçalı çizgelerin yapısına odaklanmanın yeterli olduğunu gösteriyoruz. Daha sonra, kenar-kritik eşiteşleştirilebilir iki parçalı çizgeler için yapısal bir karakterizasyon sağlıyoruz.

Çizgedeki her maksimum indüklenmiş eşleşme bir maksimum indüklenmiş eşleşme ise, bu çizgeye iyi-indüklenmiş denir. Literatürde iyi-indüklenmiş asiklik çizgelerin bir karakterizasyonunu ve iyi-indüklenmiş tek döngülü çizgeler üzerinde de bazı kısmi sonuçlar vardır. Sözde orman, her bağlı bileşenin en fazla bir döngü içerdiği, yani her bağlı bileşenin tek döngülü bir çizge veya bir ağaç olduğu bir çizgedir. Bu tezde, son olarak tek döngülü iyi-indüklenmiş çizgelerle ilgili sonuçları, çizge ailelerinin yapısının belirleyerek, iyi-indüklenmiş sahte ormanların tam bir karakterizasyonunu sağlayarak genişletiyoruz.

Anahtar Kelimeler: Eşleşme, Eşiteşlenebilir, İyi-İndüklenmiş, Üçgensiz Çizgeler, İki Parçalı Çizgeler, Tek Döngülü Çizgeler, Sözde Ormanlar.

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LIST of ABBREVIATIONS and ACRONYMS

<u>Abbreviations</u> <u>and Acronyms</u>	<u>Explanations</u>
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<i>EFC</i> -graphs	: Equimatchable factor-critical graphs.
<i>EB</i> -graphs	: Equimatchable bipartite graphs.
MIM	: Maximum independent set problem.
MVC	: Minimum vertex cover problem.
MMM	: Minimum maximal matching problem.
MMIM	: Minimum maximal induced matching problem.
P	: Polynomial-time
NP	: Nondeterministic polynomial-time.
NP-hard	: Nondeterministic polynomial-time hardness.
NP-complete	: Nondeterministic polynomial-time complete.

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1. INTRODUCTION

This thesis investigates a number of problems in graph theory from the structural point of view. A graph is an abstract mathematical structure representing a set of objects and their interrelations, where objects refer to vertices, and the relations between pairs of objects correspond to edges in the graph. Graphs can be used to model many types of relations between objects in physical, chemical, biological, computer and social systems. Hence, many practical and theoretical problems can be represented by graphs. For instance, in chemistry, graphs are used in modelling of a molecule, where vertices and edges represent atoms and bonds, respectively. This representation is useful in computer processing of molecular structures, ranging from chemical editors to database searching. In computational neuroscience, graphs can be used to represent functional connections between brain areas that interact to give rise to various cognitive processes, where the vertices represent different areas of the brain and the edges represent the connections between those areas. In computer science, graphs are used to represent networks of communication, data organization, computational devices, the flow of computation, etc. Besides, the development of algorithms to handle graphs is one of the major interests in computer science. In sociology, graph theory is also used as a way of analyzing acquaintanceship, friendship, influence, or collaboration in social community.

In graph theory, a number of problems are related with the vertices of graphs. An independent vertex set (or shortly independent set) is a set of mutually non-adjacent vertices in a graph. That is, it is a set of vertices such that there is no edge connecting any two vertices in it. In a graph, an independent set is called maximal independent set if it cannot be extended to a larger independent set, and called maximum independent set if it is an independent set of largest size. In 1968, Plummer [10] introduced well-covered graphs; a graph is called well-covered if every maximal independent set in the graph is a maximum independent set, that is, every maximal independent set in the graph has the same cardinality. The structure of well-covered graphs is a widely studied topic in the graph theory literature (see [10–17]).

An independent edge set in a graph is a set of mutually non-incident edges, which is also called a matching. That is, it is a set of edges with no common end vertices. In a graph, a matching is called maximal matching if it cannot be extended to a larger

matching, and called maximum matching if it is a matching of maximum size. As a matching analogue of well-covered property, a graph is defined to be equimatchable if every maximal matching in the graph has the same cardinality, in other words, every maximal matching is also a maximum matching. The concept of equimatchability was introduced in 1974 independently by Grünbaum [1], Lewin [18] and Meng [19].

In the literature, the structure of equimatchable graphs have been extensively studied by several authors (see [2, 3, 5, 7, 8, 20–22]). Since the works [7] and [8] provided only a partial characterization of equimatchable graphs with no triangle, a.k.a, equimatchable triangle-free graphs, we first investigate the general structure of equimatchable triangle-free graphs. In this thesis, we extend the results in [7] and [8] by providing a complete structural characterization of equimatchable triangle-free graphs via identifying graph families. Our characterization yields a linear time algorithm that recognizes whether a given non-bipartite graph is equimatchable and triangle-free. On the other hand, although the work [2] gave a characterization of equimatchable bipartite graphs, it does not provide much insight about the structure of equimatchable bipartite graphs. Besides, the authors in [7] provided only a partial characterization for equimatchable bipartite graphs. Motivated by the lack of a structural characterization for the general case of equimatchable bipartite graphs, we subsequently study equimatchable bipartite graphs. For an equimatchable graph, an edge is said to be a critical-edge if the graph obtained by the removal of this edge is not equimatchable. An equimatchable graph is called edge-critical if every edge is critical. In this thesis, by showing that it is sufficient to focus on the structure of edge-critical equimatchable bipartite graphs instead of the general equimatchable bipartite graphs, we also provide a structural characterization for edge-critical equimatchable bipartite graphs.

An induced matching in a graph is a matching such that no two endpoints of the edges in the matching are joined by an edge of the graph. In a graph, an induced matching is called maximal induced matching if there is no other induced matching containing it, and called maximum induced matching if it is an induced matching of maximum size. The restriction of equimatchability to induced matchings is introduced by Baptiste et al. [23] in 2017 as follows: A graph is well-indumatched if every maximal induced matching in the graph is also a maximum induced matching, that is, every maximal induced matching has the same cardinality. The authors in [9] provided a complete

structural characterization of well-indumatched trees, and also showed that there are infinitely many well-indumatched unicyclic graphs of girth k , where $k \in \{3, 5, 7\}$ or k is an even integer greater than 2 by identifying some well-indumatched graph families. However, this result is not a complete characterization for well-indumatched unicyclic graphs. Hence, in this thesis, we finally focus on well-indumatched unicyclic graphs. A pseudoforest is a graph in which every connected component contains at most one cycle, that is, each connected component is either a unicyclic graph or a tree. In particular, we extend the result in [9] about well-indumatched unicyclic graphs by providing a complete structural characterization of well-indumatched pseudoforests by identifying graph families.

Before formally introducing the main themes and problems considered in this thesis, we first present the terminologies used in graph theory and complexity theory in the following section. Then, in the subsequent section, we mention the points that motivate us to research the topics discussed in this thesis. In Chapter 2, we provide a comprehensive literature review on these topics, namely, equimatchable graphs and well-indumatched graphs. In Chapter 3, we structurally characterize the class of triangle-free equimatchable graphs by identifying such graph families. Besides, we obtain a linear time recognition algorithm for non-bipartite graphs in this graph class. In Chapter 4, we give a complete structural characterization for edge-critical equimatchable bipartite graphs. In Chapter 5, we investigate well-indumatched graphs. More specifically, we provide a complete structural characterization for well-indumatched unicyclic graphs by identifying such graph families.

1.1. Definitions and Notations

The notation used in this thesis is mostly standard and can be found in [24] and [25]. In this section, we first give graph-theoretical definitions and notations that will be frequently used in the forthcoming sections and then mention some fundamental concepts about computational complexity. Further definitions which are not included here will be defined when they are needed.

Formally, a graph G is an ordered pair $(V(G), E(G))$, where $V(G)$ is the set of vertices (also called nodes or points) and $E(G)$ is the set of edges (also called links or lines), that are two-element subsets of $V(G)$. For the sake of brevity, the notation

$G = (V, E)$ is used if G is clear from the context. In general, a graph is a representation of a set of objects and their binary relations, where objects refers to vertices and the relations defined on some pairs of vertices correspond to edges. An undirected graph is a graph in which edges have no orientation, that is, the edges are unordered pairs of vertices. A simple graph is an undirected graph having neither loops, that is, an edge with identical end vertices, nor multiple edges, that is, edges having the same pair of end vertices. In this thesis, we only consider simple graphs.

A subgraph of a graph $G = (V(G), E(G))$ is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G , then G is called a supergraph of H . A subgraph H of G is called spanning subgraph of G if $V(H) = V(G)$. For a graph G and $U \subseteq V(G)$, the subgraph induced by U , denoted by $G[U]$ and called induced subgraph of G , is the graph whose vertex set is U and whose edge set consists of all of the edges in $E(G)$ that have both end vertices in U . The difference $G \setminus H$ of two graphs G and H is defined as the subgraph induced by the difference of their vertex sets, that is, $G \setminus H = G[V(G) \setminus V(H)]$. For a graph G and a vertex v of G , the subgraph induced by $V(G) - \{v\}$ is denoted by $G - v$ for the sake of brevity. Besides, for an edge e of G , the graph $G(V, E \setminus \{e\})$ is denoted by $G - e$ for the sake of brevity. Furthermore, the graph obtained by joining the vertices of u and v of a graph G is denoted by $G + uv$.

The cardinality of the vertex set of a graph G , that is $|V(G)|$, is called the order of G . Particularly, G is called odd (resp. even) if $|V(G)|$ is odd (resp. even). For the sake of brevity, the edge with end vertices u and v , that is, two-element subset $\{u, v\}$, will be denoted by uv . For an edge uv of G , the vertices u and v are said to be joined by the edge uv . Furthermore, the vertices u and v are said to be incident to the edge uv , and vice versa. Two edges which are incident to the same vertex are also said to be incident (or adjacent) to each other. The vertices which are incident to the same edge are said to be adjacent. For a vertex v of G , any vertex which is adjacent to v is called a neighbor of v ; and the set of all vertices that are adjacent to v is called the neighborhood of a vertex v and is denoted by $N_G(v)$. The closed neighborhood $N_G[v]$ of vertex v is used to denote the set $N_G(v) \cup \{v\}$. Moreover, the degree of a vertex v , denoted by $d_G(v)$, is the cardinality of the neighborhood of v ; that is, $d_G(v) = |N_G(v)|$. We omit the index of the neighborhoods $N_G(v)$ and $N_G[v]$ or the degree $d_G(v)$ of a vertex v if G is clear from the context. A vertex of degree zero in G is referred to as an isolated vertex

of G , while a vertex of degree one in G is a pendant vertex (or a leaf) of G . Besides, a vertex which is the neighbor of a leaf vertex is called a stem. An edge of a graph is said to be pendant edge if one of its end vertices is a pendant vertex. A pendant edge which is incident to a vertex of degree 2 is called a good pendant edge. If all vertices of G have the same degree, say d , then G is said to be a regular graph of degree d or d -regular. In particular, a regular graph of degree 3 is called a cubic graph.

A complete graph is a graph in which each pair of vertices is joined by an edge, that is, all vertices are adjacent to each other. Note that a complete graph of order n , denoted by K_n , is a regular graph of degree $n - 1$. In particular, the complete graph K_3 is called triangle. The complement $\bar{G}$ of a graph G is the graph with vertex set $V(G)$ where two vertices are adjacent in $\bar{G}$ if and only if these vertices are not adjacent in G . It is easy to see that the graph $\bar{K}_n$ has n vertices and no edges; and it is referred to as the empty graph of order n . A bipartite graph is a graph whose vertices can be partitioned into two disjoint sets V_1 and V_2 , which are called partite sets, such that every edge connects a vertex of V_1 to a vertex of V_2 . If $|V_1| = |V_2|$, then we say that G is balanced. A complete bipartite graph, denoted by $K_{n,m}$, is a bipartite graph with partite sets $|V_1| = n$ and $|V_2| = m$ such that every vertex of V_1 is adjacent to every vertex of V_2 and vice versa. In particular, the complete bipartite graph $K_{1,3}$ is called claw.

A graph G is said to be H -free if it does not contain H as an induced subgraph. For instance, a claw-free graph and a triangle-free graph are graphs in which no induced subgraph is a claw and a triangle, respectively. The line graph $L(G)$ of a graph G is the graph having vertex set $E(G)$ such that two vertices in $L(G)$ are adjacent if and only if their corresponding edges in G are incident to each other. Furthermore, a planar graph is a graph that can be drawn on the plane without edges intersecting each other.

A walk consists of an alternating sequence of vertices and edges that begin and end with the same vertex and whose consecutive elements are incident. A trail is a walk without repeated edges, whereas a *path* is a walk without repeated vertices. Alternatively, a path P is defined as a sequence of distinct vertices with the property that each vertex in the sequence is adjacent to the vertex next to it. It is usually denoted by the sequence $(v_0, v_1, \dots, v_n)$ of consecutive vertices, whereas the vertices v_0 and v_n are called the end vertices of P . An arbitrary path on n vertices is denoted by P_n . The length of P_n is defined as the number of edges in P_n , that is, $n - 1$. The distance $d_G(u, v)$ between

two vertices u and v in G is the length of a shortest path between u and v , whereas the distance $d_G(e_1, e_2)$ between two edges e_1 and e_2 in G is the length of a shortest path between e_1 and e_2 , that is, $d_G(e_1, e_2) = \min\{d_G(u, v) | u \in e_1, v \in e_2\}$. The k^{th} power of a graph G , denoted by G^k , is a graph with the same set of vertices in which two vertices are adjacent when their distance in G is at most k . Particularly, G^2 is called the square of G and G^3 is called the cube of G .

A graph G is said to be connected if for each pair of vertices u and v in G there exists a path with end vertices u and v in G ; otherwise, G is said to be disconnected. A maximal connected subgraph of G , that is, a subgraph without any other connected subgraph of G containing it, is called a component of G . Notice here that each vertex of G belongs to exactly one component of G , and so does each edge of G . For a graph G , $c(G)$ denotes the number of components in G . A set S of vertices of a graph G such that $c(G \setminus S) > c(G)$ is called a cut-set. If $S \subseteq V(G)$ is a cut-set of size k , S is also called a k -cut. A vertex v is called a cut-vertex if $\{v\}$ is a cut-set. A graph is said to be k -connected (or k -vertex-connected) if its cut-sets have at least k vertices, where $k \geq 1$. The connectivity (or vertex-connectivity) of a graph is defined as the largest k for which the graph is k -connected.

A connected regular graph of degree 2 is called a cycle. Thus, a cycle C is a path with the same end vertices, that is, $C = \{v_0 v_1 v_2 \dots v_{n-1} v_n v_0\}$. Similar to a path, the length of a cycle C is defined as the number of edges in C . Note that the length of a cycle on n vertices is n . Particularly, a cycle of length 3 is identical to a complete graph K_3 , which is also called a triangle. An arbitrary cycle on n vertices is denoted by C_n . A cycle is an even cycle if its length is even, whereas it is an odd cycle if its length is odd. The girth of a graph G is the length of a shortest cycle in G and is denoted by $g(G)$. A paw is a C_3 with one pendant vertex and a diamond is a C_4 with an extra edge joining two nonconsecutive edges. If a graph does not contain any cycles, then it is said to be acyclic (or forest). Each connected component of an acyclic graph is called a tree. Thus, a tree is a connected graph in which any two vertices are joined by exactly one path. A spanning tree of a graph G is a spanning subgraph of G which is a tree. If a graph contains exactly one cycle, then it is said to be unicyclic. A pseudoforest is a graph in which every connected component contains at most one cycle. A connected component of a pseudoforest is called pseudotree, that is, a connected unicyclic graph or a tree.

An independent set (also known as stable set) in a graph G is defined to be a set of mutually non-adjacent vertices in G . In a graph, a given independent set is called maximal independent set if it cannot be extended to a larger independent set by adding some of its vertices, and called maximum independent set if it is an independent set of maximum size. The size of maximum independent set in G is called the independence number of G and denoted by $\alpha(G)$. A vertex cover of a graph G is a subset $Q \subseteq V(G)$ that contains at least one end vertex of every edge. Thus, each edge in $E(G)$ is incident to at least one vertex of Q . In a graph, a given vertex cover is called a minimal vertex cover if it cannot be reduced to a smaller vertex cover by removing some of its vertices, and called a minimum vertex cover if it is a vertex cover of minimum size. The vertex cover number of graph G , denoted by $\tau(G)$, is the size of a minimum vertex cover in G . It is easy to see that for any graph of G , the complement of a vertex cover of G is an independent set in G , and vice versa. It follows that the complement of a minimum vertex cover is a maximum independent set and vice-versa, that is, $\alpha(G) + \tau(G) = |V(G)|$.

A matching in a graph G is defined to be a set of non-incident edges in $E(G)$. The vertices incident to the edges of a matching M are said to be saturated by M ; the others are said to be unsaturated or exposed by M . A matching M is called a maximal matching in G if there is no other matching of G that contains M , and called a maximum matching of G if it is a matching of maximum size. The matching number of graph G , denoted $\nu(G)$, is the size of a maximum matching in G . An induced matching in a graph is a matching such that no two of its edges are joined by an edge of the graph. In a graph, an induced matching is called a maximal matching if there is no other induced matching containing it, and called a maximum matching if it is an induced matching of maximum size. The size of a maximum induced matching of G is called induced matching number and denoted by $\nu_i(G)$. It is easy to see that the induced matching number of G is at most the matching number of G ; that is, $\nu_i(G) \leq \nu(G)$. A matching M in G is a perfect matching if M saturates all vertices in G , that is, $V(M) = V(G)$. A graph G is factor-critical if $G - v$ has a perfect matching for every vertex v of G . For a vertex v , a matching M is called a matching isolating v if $\{v\}$ is a component of $G \setminus V(M)$. Given a matching M in a graph G , an alternating path with respect to M (or shortly M -alternating path) is a path in which edges alternate between those in M and those not in M . An M -augmenting path is an M -alternating path with both end

vertices unsaturated by M .

In computational theory, a decision problem is a problem that can be posed as a yes–no question, whereas an optimization problem is a problem of finding the best solution from all possible solutions. Thus, unlike decision problems, for which there is only one correct answer for each input, optimization problems are concerned with finding the best answer to a particular input. Since an algorithm’s running time varies among different inputs of the same size, the worst-case time complexity, which is the maximum amount of time required for inputs of a given size, is commonly considered. An algorithm is said to have polynomial time complexity if its running time is upper bounded by a polynomial expression in the size of the input for the algorithm. The complexity classes of computational problems are defined in terms of the computational difficulty of solving these problems. P (polynomial time) is a fundamental complexity class that contains all decision problems that can be solved in polynomial time. NP (nondeterministic polynomial time) is a complexity class that contains all decision problems for which the instances where the answer is “yes” have proofs that can be verified in polynomial time. NP-complete (nondeterministic polynomial-time complete) is a complexity class that contains the decision problems in NP for which it is possible to reduce any other NP problem to them in polynomial time. Thus, if some NP-complete problem has a polynomial time algorithm finding a solution, all problems in NP also do. However, although a solution to an NP-complete problem can be verified in polynomial-time, there is no known way to find a solution in polynomial-time. It implies that the NP-complete problems represent the hardest problems in NP. NP-hard (nondeterministic polynomial-time hardness) is a complexity class that contains problems that are at least as hard as the NP-complete problems. Notice that NP-hard problems do not have to be in NP, and they do not have to be decision problems. Thus, it also includes optimization problems.

The P versus NP problem is one of the most important open problems in computer science. It asks whether every problem whose solution can be verified in polynomial time can also be solved in polynomial time. If $P = NP$, then all of the NP problems can be solved deterministically in polynomial time. This implies that all NP problems are essentially the same problem stated in different terms. If $P \neq NP$, which is widely believed, then there are problems in NP that are harder to compute than to verify. That

is, in such a case, the problems in NP could not be solved in polynomial time although a solution could be verified in polynomial time.

1.2. Motivation

In this section, we discuss the motivation that lead us to investigate the topics considered in this thesis; namely: triangle-free equimatchable graphs, equimatchable bipartite graphs and well-indumatched pseudoforests.

The problem of finding a maximum independent set in a general graph is one of six classes of NP-hard problems stated in [26]; that is, there is no polynomial-time algorithm to find a maximum independent set in a general graph unless $P \neq NP$. On the other hand, a maximal independent set in a graph can be easily constructed by a greedy algorithm. Hence, the problem of finding a maximum independent set in a graph is obviously simple when every maximal independent set in the graph is a maximum independent set, that is, every maximal independent set in the graph have the same cardinality. Such a graph is called well-covered, and this property was introduced by Plummer [10] in 1968 in terms of vertex covers. For comprehensive results in well-covered graphs, we refer the reader to [10–17].

A maximum matching of a given graph can be found in polynomial-time by the well-known Edmond's Augmenting Path Algorithm [27]. However, the problem of finding a maximal matching of minimum size is NP-hard for general graphs [25]; that is, it cannot be solved by an algorithm in polynomial-time for a general graph unless $P \neq NP$. It is easy to see that the problem of finding a maximal matching of minimum size is polynomially solvable in a graph by greedily constructing any maximal matching when every maximal matching in the graph has the same size. By this motivation, in 1974, Grünbaum [1], Lewin [18] and Meng [19] independently introduced the matching analogue of well-coverness property: a graph is defined to be equimatchable if every maximal matching in the graph has the same cardinality, in other words, every maximal matching is also a maximum matching. In 1984, Lesk et al. [2] formally introduced equimatchable graphs and provided a structural characterization of equimatchable graphs via Gallai-Edmonds decomposition, yielding a polynomial-time recognition algorithm.

The structure of equimatchable graphs has been extensively studied by several authors. As one of the first studies, in 1979, Sumner [20] characterized the equimatchable

graphs with a perfect matching, which are called randomly matchable graphs. In 1986, Favaron [3] investigated equimatchable factor-critical graphs with vertex connectivity one and two. The work in [2] gave also a characterization of equimatchable bipartite graphs in terms of subsets of neighborhoods of vertices in the smaller partite set. Although it is a complete characterization of equimatchable bipartite graphs, it does not provide much insight about the structure of equimatchable bipartite graphs. In 2010, Frendrup et al. [7] provided a structural characterization of equimatchable graphs with girth at least five. Particularly, they showed that an equimatchable graph with girth at least five is either one of C_5 and C_7 or a member of a graph family which consists of K_2 and all bipartite graphs with partite sets V_1 and V_2 such that all vertices in V_1 are stems and no vertex from V_2 is a stem. Note that it is only a partial characterization of equimatchable bipartite graphs, namely, a characterization of equimatchable bipartite graphs with girth at least six. Besides, the work in [21] reformulated the characterization of equimatchable bipartite graphs given in [2] as follows: A connected bipartite graph is equimatchable if and only if each of its maximal matchings saturates all vertices in the smaller partite set. Furthermore, Akbari et al. [8] partially characterized equimatchable regular graphs by showing that for an odd positive integer r , if G is a connected equimatchable r -regular graph, then $G \in \{K_{r+1}, K_{r,r}\}$. A particular result in [8] is that the only connected triangle-free equimatchable r -regular graphs are C_5 , C_7 and $K_{r,r}$ where r is a positive integer.

In this thesis, since the works [7] and [8] provided only a partial characterization of equimatchable graphs with girth at least 4, that is, equimatchable graphs with no triangle, we first investigate equimatchable triangle-free graphs. In Section 3, we first point out that non-factor-critical equimatchable triangle-free graphs correspond to equimatchable bipartite graphs, which have already been characterized in [2]. We then discuss the structure of factor-critical equimatchable triangle-free graphs. Therefore, we extend both the result by Frendrup et al. [7] and the result by Akbari et al. [8] on triangle-free equimatchable graphs by providing a complete structural characterization of triangle-free equimatchable graphs by identifying graph families. Our characterization yields a linear time algorithm that recognizes whether a given non-bipartite graph is equimatchable and triangle-free. This complexity improves the individual complexities of the individual recognition algorithms for equimatchable and triangle-free graphs, which are already

known in the literature.

Moreover, despite the results given in [2], [7] and [21] about equimatchable bipartite graphs, there exists a lack of a structural characterization for the general case of equimatchable bipartite graphs in the literature. Motivated by this fact, we also study equimatchable bipartite graphs in this thesis. We show that the reformulation given in [21] for the characterization of equimatchable bipartite graphs, any bipartite supergraph of an equimatchable bipartite graph obtained by joining some pair of non-adjacent vertices from different partite sets is also an equimatchable bipartite graph with the same vertex partition. Hence, we consider equimatchable bipartite subgraphs of an equimatchable bipartite graph with the same vertex partition. For an equimatchable graph, an edge is said to be a critical-edge if the graph obtained by the removal of this edge is not equimatchable. An equimatchable graph is called edge-critical, denoted by *ECE*, if every edge is critical. In Section 4, we first show that it is sufficient to focus on the structure of edge-critical equimatchable bipartite graphs instead of the structure of the general case of equimatchable bipartite graphs. Then, we extend the partial characterization of Frendrup et al. [7] to all equimatchable bipartite graphs by showing at least one of the following is true: each vertex in the smaller partite set of an equimatchable bipartite graph is a stem or a vertex of an induced $K_{2,2}$. Afterwards, we provide a complete structural characterization for the general case of bipartite ECE-graphs.

The restriction of equimatchability to induced matchings was introduced by Baptiste et al. [23] in 2017. They defined a graph to be *well-indumatched* if every maximal induced matching in the graph is also a maximum induced matching, that is, every maximal induced matching has the same cardinality. The authors in [23] proved that recognizing a well-indumatched graph is a co-NP-complete problem, in other words, recognizing a non-well-indumatched graph is an NP-complete problem, even for $(2P_5, K_{1,5})$ -free graphs. In 2019, Akbari et al. [9] investigated the structure of well-indumatched acyclic and unicyclic graphs. As a main result, they provided a complete structural characterization of well-indumatched acyclic graphs, that is, well-indumatched trees. This characterization yielded a linear time recognition algorithm. In [9], the authors also showed that there are infinitely many well-indumatched unicyclic graphs of girth k , where $k \in \{3, 5, 7\}$ or k is an even integer greater than 2 by identifying some

well-indumatched graph families. In this thesis, since the result in [9] does not give a complete characterization for well-indumatched unicyclic graphs, we finally focus on well-indumatched pseudoforests, that is, well-indumatched graphs whose each connected component contains at most one cycle. In Section 5, we first give some results on the general structure of well-indumatched graphs. Noting that well-indumatched pseudoforests are the disjoint union of well-indumatched trees and well-indumatched unicyclic graphs, we then study the structure of well-indumatched pseudoforests. First, we examine the class of well-indumatched trees in the light of the characterization given in [9]. We then provide a complete structural characterization of well-indumatched unicyclic graphs by using our results about well-indumatched trees. In particular, we identify all well-indumatched unicyclic graph families.

2. LITERATURE REVIEW

In this chapter, we give a survey of the results on equimatchable graphs and well-covered graphs.

2.1. Equimatchable graphs

The problem of finding a maximum independent set of a graph, which is called Maximum Independent Set (MIS), is a classical combinatorial optimization problem. MIS is known to be NP-hard in general [25]. That is, it is unlikely that there exists a polynomial-time algorithm to find a maximum independent set in a general graph unless $P = NP$. However, the following greedy algorithm shows that a maximal independent set of a graph G can be obtained in polynomial-time: Choose any vertex v_1 in G , place it in a set I , delete $N[v_1]$ from G and obtain a graph G_{v_1} . Then choose any vertex v_2 in G_{v_1} , put it also into set I , delete $N[v_2]$ from G_{v_1} and obtain a graph $(G_{v_1})_{v_2}$, so on. If you continue this process until obtaining an empty graph, the set I obtained in polynomial-time is obviously a maximal independent set in G . The problem of finding a minimum vertex cover in a graph, which is called Minimum Vertex Cover (MVC), is another well-known optimization problem, and is known to be NP-hard [25]. Thus, it does not admit a polynomial-time algorithm if $P \neq NP$. Note here that for any graph of G , the complement of a vertex cover of G is an independent set in G , and vice versa. It follows that the complement of a minimum vertex cover is a maximum independent set and vice-versa, that is, $\alpha(G) + \tau(G) = |V(G)|$. Thus, a minimal vertex cover in a graph can be obtained in polynomial-time by constructing a maximal independent set via the greedy algorithm. Therefore, MVC can be solved in polynomial-time in a graph when every minimal vertex cover in the graph is a minimum vertex cover, in other words, every minimal vertex cover in the graph has the same size.

A graph G is called well-covered if every minimal vertex cover of G is a minimum vertex cover, that is, every minimal vertex cover in G has the same cardinality. Alternatively, a well-covered graph can be defined as a graph of which every maximal independent set of G is a maximum independent set, that is, all of whose maximal independent sets have the same size. Although the concept of well-coveredness was introduced in 1970 by Plummer [10] in terms of vertex coverings, in subsequent re-

search, well-covered graphs were generally studied from independent sets point of view. In works [17] and [12], it is independently proved that the recognition of a well-covered graph is a co-NP-complete problem, that is, the property of being not well-covered is NP-complete. Thus, it is unlikely that there exists a characterization of well-covered graphs which yields a polynomial-time recognition algorithm unless $P = NP$. However, there are some subclasses where well-covered graphs can be recognized in polynomial-time such as bipartite well-covered graphs [11], cubic well-covered graphs [16], well-covered graphs of girth 5 or greater [15] and well-covered graphs that contain neither 4-cycles nor 5-cycles [14]. Particularly, Tankus and Tarsi [13] provided a polynomial-time algorithm which decides whether a claw-free graph is well covered.

Matching can be seen as the edge analogue of an independent set in a graph G , that is, a set of non-incident edges in $E(G)$. Hence, similar to a maximum independent set, a maximal matching in a graph G can be found in polynomial-time via a simple greedy algorithm as follows: Choose any edge $e_1 = u_1v_1$ in G , place it in a set M , delete u_1 and v_1 from G and obtain a graph, say G_{e_1} . Then choose any edge $e_2 = u_2v_2$ in G_{e_1} , place it also into set M , delete u_2 and v_2 from G_{e_2} and obtain a graph $(G_{e_1})_{e_2}$, so on. Continue this process until obtaining an empty graph. It is clear that the set M obtained in polynomial-time is a maximal matching in G . Besides, a maximum matching is also a maximal matching and the well-known Edmond's Augmenting Path Algorithm given in [27] can find a maximum matching of a given graph in polynomial time. However, the problem of finding a maximal matching of minimum size, which is called Minimum Maximal Matching (MMM), is NP-hard for general graphs [25]. Since the early 80s, MMM has been extensively studied due to its wide range of applications such as in a telephone switching network built to route phone calls from incoming lines to outgoing trunks and in finding a stable matching between institutions and their applicants. In 1980, Yannakakis and Gavril [28] showed that MMM is essentially equal to the problem of finding an edge dominating set of minimum size, where edge dominating set in a graph G is defined as a subset D of $E(G)$ such that every edge not in D has a common end vertex with an edge in D . MMM remains an NP-hard problem even on several graph classes such as k -regular bipartite graphs for any fixed $k \geq 3$ [29], planar bipartite and planar cubic graphs [30]. Therefore, it is clear that MMM, which is NP-hard in general, is polynomially solvable in a graph when every maximal matching in the graph

has the same size, in other words, every maximal matching is a maximum matching.

A graph G is called equimatchable if every maximal matching of G is also a maximum matching of G , that is, every maximal matching of G has the same cardinality. Thus, the equimatchability property is accepted as the edge analogue of the well-coveredness property. The problem of characterizing equimatchable graphs, with a different name matching perfect, was first posed by Grünbaum [1] in 1974. In the same year, Meng [19] and Lewin [18] independently gave two different characterizations of equimatchable graphs. Nevertheless, none of the characterizations given in [19] and [18] was a “good” characterization of equimatchability in terms of providing a polynomial-time algorithm for the recognition of equimatchable graphs.

In 1984, Lesk et al. [2] formally introduced the notion of equimatchability and gave a structural characterization of equimatchable graphs via Gallai-Edmonds decomposition given in [31]. This characterization yields a polynomial-time algorithm, namely in time $O(n^{4.5})$, for deciding whether a given graph of order n is equimatchable. Note that a set of edges in a graph is a matching if and only if the corresponding set of vertices in its line graph is an independent set. This implies that a graph is equimatchable if and only if its line graph is well-covered. It is easy to see that every line graph is claw-free. Therefore, the polynomial-time recognition algorithm for well-covered claw-free graphs given in [13] determines also the equimatchability of a given graph with m edges in polynomial-time, namely in time $O(m^6)$. In 2014, Demange and Ekim [4] gave an alternative characterization of equimatchable graphs using alternating chain-based arguments in some properly constructed auxiliary graphs. This leads to an algorithm for deciding whether a general graph of order n with m edges is equimatchable with better time complexity, namely in time $O(n^2m)$.

The structure of equimatchable graphs has been extensively studied by several researchers. Since a graph is equimatchable if and only if each of its components is equimatchable, the researchers investigated connected equimatchable graphs. One of the first studies is the characterization of equimatchable graphs with a perfect matching, which are called randomly matchable graphs. In other words, every maximal matching in a randomly matchable graph is a perfect matching. In 1979, the characterization of randomly matchable graphs was provided by Sumner [20] as follows:

Theorem 2.1: ([20]) A connected graph G is randomly matchable if and only if G is the

complete graph K_{2n} or complete bipartite graph $K_{n,n}$ for $n \geq 1$.

In 1984, Lesk et al. [2] investigated the general structure of all equimatchable graphs, that is, not only randomly matchable graphs but also equimatchable graphs without a perfect matching. A significant consequence of this work is that any 2-connected equimatchable graph which is not randomly matchable is either bipartite or factor-critical. In work [2] and later studies, the researchers focused on the structure of these graph subclasses in general. The authors in [2] provided characterizations for equimatchable bipartite graphs and equimatchable factor-critical graphs in the following ways:

Theorem 2.2: ([2]) A connected bipartite graph $G = (U \cup V, E)$ with $|U| \leq |V|$ is equimatchable if and only if for all $u \in U$, there exists a non-empty $X \subseteq N(u)$ such that $|N(X)| \leq |X|$.

Theorem 2.3: ([2]) Let G be a factor-critical graph. Then the following are equivalent:

- i) G is equimatchable*
- ii) For every pair u, v of independent vertices and for every matching M of $G \setminus \{u, v\}$, there is a third vertex w adjacent to u or to v which is not covered by M .*
- iii) for every pair u, v of independent points there exist disjoint sets $S_1, S_2, \dots, S_k \subseteq N(u) \cup N(v)$ such that:*
 - $|S_i|$ is odd for $1 \leq i \leq k$;*
 - no line joins S_i to S_j for $i \neq j$;*
 - $|N(S_1, S_2, \dots, S_k)| < k$, where*

$$N(S_1, S_2, \dots, S_k) = N(\cup_{i=1}^k S_i) - \cup_{i=1}^k S_i - \{u, v\}.$$

In 1986, Favaron [3] investigated the structure of equimatchable factor-critical graphs. A characterization of graphs having a cut-set of size one was provided by Favaron [3] as follows:

Theorem 2.4: ([3]) A graph with vertex connectivity 1, that is, a graph with a cut-vertex, is an equimatchable factor-critical graph if and only if:

- i) *There exists exactly one cut-vertex v ,*
- ii) *Every component of $G - v$ is either K_{2n} or $K_{n,n}$,*
- iii) *v is adjacent to at least two adjacent vertices of every component of $G - v$.*

The following results given in [3] provide a characterization of graphs having a cut-set of size two:

Lemma 2.5: ([3]) Let G be an equimatchable factor-critical graph of order at least 4 and connectivity 2, and let $S = s_1, s_2$ be a 2-cut of G . Then, $V(G) - S$ has precisely two components one of which has odd number of vertices and the other has even number of vertices.

Theorem 2.6: ([3]) Let G be an equimatchable factor-critical graph satisfying the conditions given in Lemma 2.5. Let A and B denote the even and odd components of $V(G) - S$, respectively. Let $|A| = 2q$, q be a positive integer, $|B| = 2p + 1$, and p be a nonnegative integer. Let e_1, e_2 be two distinct vertices of A , neighboring in G with s_1 and s_2 , respectively. For $p > 1$, let o_1, o_2 be two distinct vertices of B , neighboring in G with s_1, s_2 respectively. Then G has the following structure:

- i) *B is one of the four graphs K_{2p+1} , $K_{2p+1} - o_1o_2$, $K_{p,p+1}$ or $K_{p,p+1} + o_1o_2$ and in the two last cases o_1 and o_2 belong to the $(p + 1)$ -stable set of $K_{p,p+1}$. Thus, there exists a Hamiltonian path in $(S \cup B)$ with endpoints s_1 and s_2 .*
- ii) *$(A - e_1, e_2)$ is either K_{2q-2} or $K_{q-l, q-l}$ and, if $|B| > 1$, then A is either K_{2q} , or $K_{q,q}$. Furthermore, there exists a Hamiltonian path in $(S \cup A)$ with endpoints s_1 and s_2 .*

Furthermore, in [3], the author proved that every 2-connected equimatchable factor-critical graph is Hamiltonian, that is, it possesses a cycle passing through each vertex of the graph exactly once.

In 2003, Kawarabayashi et al. [32] determined all graphs in two important subclasses of equimatchable graphs, namely equimatchable 3-connected planar graphs and equimatchable cubic graphs, using Gallai–Edmonds decomposition. As an immediate consequence, they first showed the following result:

Lemma 2.7: ([32]) Let G be a factor-critical equimatchable 3-connected planar graph, v be a vertex of G , and M_v be a minimal matching isolating v . Then every component of $G \setminus (V(M) \cup \{v\})$ is randomly matchable; particularly, either K_2 , K_4 or C_4 .

Hence, the authors in [32] proved that there are exactly 23 equimatchable 3-connected planar graphs and none of them is bipartite. In the following result, the authors in [32] also showed that there are only two cubic equimatchable graphs:

Theorem 2.8: ([32]) If G is a connected cubic equimatchable graph, then G is either isomorphic to K_4 or to $K_{3,3}$.

In 2009, Kawarabayashi and Plummer [33] showed that in a surface of any fixed genus, equimatchable graphs have bounded size; that is, the order of G is bounded from above. Then, they extended the results in work [32] by proving that there are only a finite number of equimatchable 3-connected graphs with fixed genus.

In 2010, Frendrup et al. [7] studied the structure of equimatchable graphs with a focus on girth conditions. They provided a structural characterization of equimatchable graphs with girth at least five as follows:

Theorem 2.9: ([7]) Let G be a connected equimatchable graph with girth at least 5. Then $G \in \mathcal{F} \cup \{C_5, C_7\}$, where $\mathcal{F}$ is the family of graphs containing K_2 and all connected bipartite graphs with partite sets V_1 and V_2 such that all vertices in V_1 are stems and no vertex from V_2 is a stem.

An important conclusion of the work in [7] is that all equimatchable graphs with an even girth at least six are bipartite. Thus, there is no equimatchable graph with an odd girth at least nine. Furthermore, the authors in [7] proved that the only equimatchable graphs with girth five and seven are the cycle of length five and seven, respectively.

In 2015, Eiben and Kotrbčik [6] extended the result in Lemma 2.7 to 2-connected equimatchable factor-critical graphs using Theorem 2.6 as follows:

Theorem 2.10: ([6]) Let G be a 2-connected equimatchable factor-critical graph. Let v be a vertex of G and M be a minimal matching isolating v . Then $G \setminus (V(M) \cup \{v\})$ is isomorphic to K_{2n} or $K_{n,n}$ for some nonnegative integer n .

They used this result to improve the upper bounds on the maximum number of vertices of 2-connected equimatchable factor-critical graphs.

In 2015, Eiben and Kotrbčik [5] described the structure of k -connected equimatchable factor-critical graphs with respect to their minimum vertex cuts by extending Theorem 2.4 and 2.6 to graphs with higher connectivity in the following way:

Theorem 2.11: ([5]) Let G be a k -connected equimatchable factor-critical graph with at least $2k + 3$ vertices and a k -cut S such that $G - S$ has two components with at least 3 vertices, where $k \geq 3$. Then $G - S$ has exactly two components and both are complete graphs. In particular, if $k \geq 4$, then the independence number of G is 2.

Furthermore, the authors in [5] showed that a 2-connected odd graph G with at least 4 vertices has independence number at most 2 if and only if G is equimatchable and factor-critical and $G + e$ is equimatchable for every edge $e \in E(\overline{G})$.

In 2015, Dibek et al. [34] investigated forbidden induced subgraphs of equimatchable graphs. Since equimatchability is not a hereditary property, that is, since it is not necessarily preserved by induced subgraphs, the authors first showed that equimatchable graphs do not have a forbidden subgraph characterization. They introduced the notion of hereditary equimatchable graphs as equimatchable graphs whose all induced subgraphs are also equimatchable. They then characterized these graphs in the following way:

Lemma 2.12: ([34]) The following are equivalent:

- i) G is a connected hereditary equimatchable graph.
- ii) G is $(P_4, \text{diamond}, \text{paw})$ -free.
- iii) G is a complete graph or a complete bipartite graph.

The authors also provided a structural result for equimatchable graphs as follows:

Lemma 2.13: ([34]) If G is an equimatchable graph with an induced subgraph C isomorphic to a cycle C_{2k+1} for some $k \geq 2$, then G is factor-critical.

As a main consequence, the authors in [34] provided the following structural result based on Gallai-Edmonds decomposition and the structure of factor-critical equimatchable graphs in [6]:

Theorem 2.14: ([34]) Equimatchable graphs are C_{2k+1} -free for any $k \geq 4$.

Thus, equimatchable graphs do not contain an odd cycle of length at least nine as an induced subgraph.

In 2017, Deniz et al. [35] defined and studied two extensions of equimatchability by introducing two new graph parameters that measure how far a graph is from being equimatchable. The first parameter, called the matching gap and denoted by $\mu(G)$,

measures the difference between the sizes of a maximum matching and a minimum maximal matching. It is clear that G is equimatchable if and only if $\mu(G) = 0$. Given a maximal matching M in a graph G , if G has an M -augmenting path isomorphic to a P_4 , then M is with an augmenting P_4 . The authors in [35] provided a characterization of graphs with $\mu(G) = 1$, which is called almost equimatchable, in the following way:

Theorem 2.15: ([35]) A graph G is almost equimatchable if and only if the following two conditions hold:

- i) G contains a maximal matching with an augmenting P_4 .*
- ii) Every two maximal matchings with an augmenting P_4 have the same size.*

The second parameter is related to the notion of equimatchable sets; a set of vertices in a graph G is said to be an equimatchable set if all maximal matchings of G saturating the set are of the same size. The second parameter, called the equimatchability defect and denoted by $\eta(G)$, measures the smallest size of an equimatchable set of a graph. It is clear that $\eta(G) = 0$ if and only if G is equimatchable. The authors in [35] developed several inapproximability and parameterized complexity results and algorithms concerning the computation of these two parameters, a characterization of almost equimatchable graphs, exact values of the equimatchability defect of cycles, and sharp bounds for both parameters.

In 2018, Akbari et al. [22] focused on a particular subclass of equimatchable graphs called equimatchable claw-free graphs. They provided a characterization of these graphs by identifying the equimatchable claw-free graph families. This characterization leads to a more efficient recognition algorithm than the composition of best known algorithms that recognize equimatchability [4] and claw-freeness [36] separately.

In 2018, Akbari et al. [8] investigated the structure of equimatchable regular graphs. They extended Theorem 2.8 by proving the following result which is a partial characterization of equimatchable regular graphs:

Theorem 2.16: ([8]) Let r be an odd positive integer. Then every connected equimatchable r -regular graph is isomorphic to either K_{r+1} or $K_{r,r}$.

Motivated by Theorem 2.9 in [7], which classifies the connected equimatchable graphs not containing C_3 and C_4 , the authors in [8] determined all connected equimatch-

able regular graphs not containing C_3 , that is, connected triangle-free equimatchable regular graphs, as follows:

Theorem 2.17: ([8]) Let G be a connected triangle-free equimatchable r -regular graph, where r is a positive integer. Then G is isomorphic to C_5 , C_7 , or $K_{r,r}$.

In 2019, Deniz and Ekim [21] focused on equimatchable graphs such that the removal of an edge from the graph preserves the equimatchability. They called an equimatchable graph G as edge-stable if the graph obtained by the removal of edge e from G , that is, $G - e$, is equimatchable for each $e \in E(G)$. The authors first pointed out that an edge-stable equimatchable graph is either 2-connected factor-critical or bipartite. They then provided a complete characterization of the class of edge-stable equimatchable graphs. This characterization yields a recognition algorithm, which is better than iteratively checking the equimatchability of $G - e$ for every $e \in E(G)$. Lastly, the authors in [21] introduced the notion of criticality which is the opposite of stability; that is, the notion of equimatchable graphs such that the removal of any edge harms the equimatchability. More formally, for an equimatchable graph G , $e \in E(G)$ is defined to be a critical-edge if $G - e$ is not equimatchable. A graph G is called edge-critical equimatchable, denoted by ECE, if G is equimatchable and every $e \in E(G)$ is critical. In [21], Deniz and Ekim provided a complete characterization of factor-critical ECE-graphs with connectivity 2.

2.2. Well-Indumatched Graphs

In a graph G , a set $M \subseteq E(G)$ is called a distance- k matching if all pairwise distances between distinct edges in M are at least k . The concept of distance- k matchings gives a generalization of matchings in a graph. For $k = 1$, it is the usual notion of matching in graphs and for $k = 2$, it corresponds to the notion of induced matching in graphs. Thus, matchings and induced matchings are also known as distance-1 matchings and distance-2 matchings. The special case $k = 3$ is also studied in [37]. Not only for $k = 3$ but also for any $k \geq 1$, the authors in [37] proved that for a graph G , the edge set M is a distance- k matching in G if and only if M is an independent vertex set in $L(G)^k$, that is, the k^{th} power of the line graph of G . In general, almost all of the studies on distance- k matchings deal with the special case $k = 2$, that is, induced

matchings, and do not investigate the cases $k \geq 3$. The concept of induced matching, with a different name 2-separated matchings, was introduced in 1982 by Stockmeyer and Vazirani [38], whereas the name of induced matching was firstly used in 1989 by Cameron [39]. Induced matchings also appeared under the name strong matchings in [40] and [41].

Induced matchings are of theoretical and practical importance as they are used in various kinds of applications. For this reason, many researchers have extensively studied induced matchings. In [38], induced matchings were used to interpret the “risk-free marriage problem” as an extension of the matching problem, which is known also as the marriage problem. Golumbic and Lewenstein [42] used induced matchings to model secure communication channels, VLSI and network flow problems. Induced matchings are also used to model concurrent transmissions in wireless ad hoc networks [43], network gathering problems [44], network testing problems [45], and network scheduling problems [46]. Moreover, Golumbic and Laskar [41] showed that there is an immediate relation between the size of an induced matching and the irredundancy number of a graph. Hocquard et al. [47] also showed that induced matchings are closely related to the strong edge-colorings where every path of three edges needs three colours.

The problem of finding a maximum induced matching, which is called Maximum Induced Matching (MIM), is significant due to its applications as well as its connections to other graph-theoretical problems. In 1982, MIM was originally introduced in [38] as “risk-free marriage problem”. The authors in [38] proved that MIM is NP-hard in general. The same result was independently proven also by Cameron [39]. Besides, MIM remains NP-hard even for very restricted graph classes such as bipartite graphs of vertex degree at most 3 [48], line graphs [49], planar graphs of vertex degree at most 4 [50] and even for cubic planar graphs [51]. On the other hand, MIM is polynomially solvable under substantial restrictions, for instance: for trees and interval graphs [42], circular arc graphs [41], chordal graphs [39], weakly chordal graphs [52] and some other graph classes [53–55]. Furthermore, in 2003, Kobler and Rotics [56] proved that the graphs where the sizes of a maximum matching and a maximum induced matching coincide, can be recognized in polynomial-time. By extending this result, in 2005, Cameron and Walker [57] provided a complete structural characterization of these graphs. We refer the reader to [42, 58] for a more comprehensive literature review on

MIM from a computational complexity point of view.

The problem of finding a maximal induced matching of minimum size, which is called Minimum Maximal Induced Matching (MMIM), is an understudied problem related to the induced matchings. In 1989, Cameron [39] provided that induced matchings in a graph G correspond to independent sets in $L^2(G)$, the square of the line graph of G ; it implies that maximal induced matchings in G correspond to maximal independent sets in $L^2(G)$. In 1973, Berge [59] proved that any maximal independent set is an independent dominating set, and vice versa, where independent dominating set in a graph G is defined as an independent subset S of $V(G)$ such that every vertex not in S is adjacent to a vertex in S . Hence, it is easy to observe that MMIM in a graph G corresponds to the problem of finding an independent dominating set of minimum size in $L^2(G)$. In works [60–62], it is proved that the problem of finding an independent dominating set of minimum size is solvable in polynomial-time for chordal and weakly chordal graphs, circular-arc graphs, and AT-free graphs. Thus, it can be easily seen that MMIM can be solved in polynomial time for these graph classes. In 2004, Orlovich and Zverovich [63] showed that MMIM is an NP-hard problem in graphs having all of their maximal induced matchings of size either k or $k + 1$ for some integer $k \geq 1$. In 2008, Orlovich et al. [64] showed that MMIM is NP-hard even in bipartite graphs of maximum degree 4.

As the induced matching version of equimatchability, a graph G is called well-indumatched if every maximal induced matching of G is also a maximum induced matching of G , that is, every maximal induced matching of G has the same cardinality. The notion of well-indumatched graphs was introduced by Baptiste et al. [23] in 2017. In this work, the authors posed the problem of recognizing well-indumatched graphs. By showing the difficulty of finding two maximal induced matchings of distinct sizes, they first proved that recognizing a non-well-indumatched graph is an NP-complete problem. That is, recognizing a well-indumatched graph is a co-NP-complete problem. Then they extended this result by showing that recognizing a graph that has maximal induced matchings of at most t distinct sizes is a co-NP-complete problem for any $t \geq 2$. Furthermore, in [23], Baptiste et al. showed that recognizing a well-indumatched graph is a co-NP-complete problem even for $(2P_5, K_{1,5})$ -free graphs, and for the same graphs, the problem of recognizing a graph that has maximal induced matchings of at most t

distinct sizes is co-NP-complete for any given $t \geq 1$.

The decision versions of the well-known problems of finding maximum independent set, minimum dominating set, and maximum independent dominating set are known to be NP-complete for general graphs (see [25]). The authors in [23] examined the computational complexity status of these problems in the class of well-indumatched graphs. They proved that all these problems remain NP-complete for well-indumatched graphs.

In graph theory literature, there are not yet many studies on the structure of well-indumatched graphs (see [9, 23]). Since a graph is well-indumatched if and only if each of its components is well-indumatched, the researchers investigated connected equimatchable graphs. First, it is easy to see that the class of well-indumatched graphs is not hereditary. In other words, there is no a forbidden induced subgraph characterization for well-indumatched graphs; for instance, the path P_7 is a well-indumatched graph, whereas the path P_5 , which is an induced subgraph of P_7 , is not a well-indumatched graph. In [23], the authors introduced a new hereditary notion for well-indumatched graphs: A subgraph F of a graph G is called co-indumatching if F can be obtained by removal of some induced matching M (possibly $M = \emptyset$) of G . The set of all co-indumatching subgraphs in G is denoted by $CIMSub(G)$.

Proposition 2.18: ([23]) Let G be a graph and H be a co-indumatching subgraph of G . Then the relation $CIMSub(H) \subseteq CIMSub(G)$ holds.

A class $\mathcal{M}$ of graphs is called *co-indumatching hereditary* if $CIMSub(G) \subseteq \mathcal{M}$ for any $G \in \mathcal{M}$. In [23], authors consider the heredity property of well-indumatched graphs in this point of view and proved the following result:

Proposition 2.19: ([23]) The class of well-indumatched graphs is a co-indumatching hereditary class.

In [23], the authors posed also the decision problem of recognizing co-indumatching subgraphs, which is co-indumatching subgraph problem. The problem aims to determine whether a given induced subgraph of a graph is a co-indumatching subgraph. In the following result, they established the hardness of the recognition problem:

Theorem 2.20: ([23]) co-indumatching subgraph problem is an NP-complete problem.

Moreover, authors in [23] defined a hereditary subclass of the class of well-indumatched graphs as follows: A graph G is called perfectly well-indumatched if every induced subgraph of G is well-indumatched. In the next result, they structurally characterized perfectly well-indumatched graphs in terms of forbidden induced subgraphs as follows:

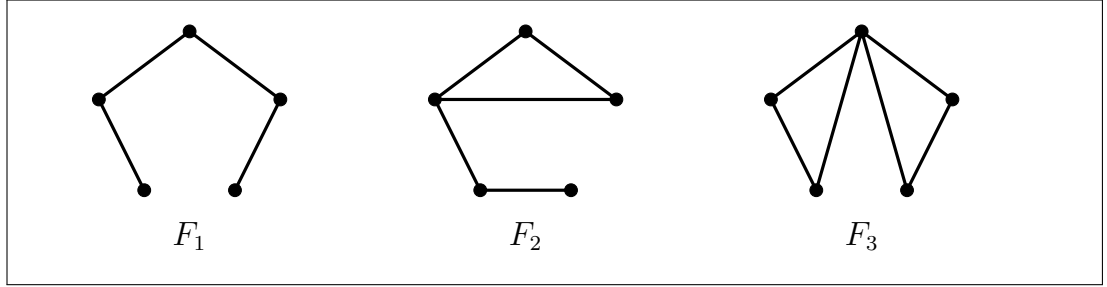


Figure 2.1: The minimal forbidden induced subgraphs for perfectly well-indumatched graphs.

Theorem 2.21: ([23]) For a graph G , the following statements are equivalent:

- i) G is a perfectly well-indumatched graph.
- ii) G is a (F_1, F_2, F_3) -free graph.
- iii) $(L(G))^2$ is a (K_4-e) -free graph.

where F_1 , F_2 and F_3 are depicted in Figure 2.1.

In the literature, the graphs F_2 and F_3 are known as *kite* and *butterfly*, respectively, whereas the graph F_1 is a path P_5 .

Corollary 2.22: ([23]) The problem of finding a dominating set of minimum size is an NP-complete problem for perfectly well-indumatched graphs.

Theorem 2.23: ([23]) The problem of finding a independent set and of maximum size and the problem of finding an independent dominating set of minimum size can be solved in polynomial time for perfectly well-indumatched graphs, even in their weighted versions.

A graph G is called reduced if no two vertices of G have the same set of neighbors. For each set of vertices with the same neighbors in a graph G , the resulting graph obtained by the removal of all of them except one is a uniquely defined reduced graph,

which is called the reduction of G and denoted by $R(G)$. In the next result, in 2019, Akbari et al. [9] showed that it is sufficient to investigate reduced well-indumatched graphs instead of general well-indumatched graphs:

Lemma 2.24: ([9]) A graph G is well-indumatched if and only if the reduction $R(G)$ is well-indumatched.

The following is a reformulation of Proposition 2 in [23]:

Lemma 2.25: ([9]) Let G be a well-indumatched graph and $F_0 \subseteq E(G)$ be an induced matching. If F is the set of all edges covered by F_0 , then the graph obtained by the removal of edges in F is well-indumatched.

Lemma 2.26: ([9]) Let T be a tree with a longest path $P = v_1 \dots v_k$ such that the degree of each vertex of P is at most 3, and those of degree 3 are incident to a pendant edge. Then, T is well-indumatched if and only if $k \in \{1, 2, 3, 4\}$ or $k = 7$ and $d(v_4) = 2$.

The authors in [9] also provided the complete structural characterization of well-indumatched paths and cycles as follows:

Lemma 2.27: ([9]) The following statements hold:

- i) The path P_n is well-indumatched if and only if $n \in \{1, 2, 3, 4, 7\}$.*
- ii) The cycle C_n is well-indumatched if and only if $n \in \{3, 4, \dots, 8, 11\}$.*

The next results about the induced paths are valid not only for well-indumatched trees but also for general well-indumatched graphs:

Lemma 2.28: ([9]) Let G be a graph. Then the following statements hold:

- i) If $v_1 v_2 v_3 v_4 v_5$ is a path in G such that $d(v_1) = d(v_5) = 1$ and $d(v_2) = 2$, then G is not well-indumatched.*
- ii) If $v_1 v_2 v_3 v_4 v_5 v_6$ is a path in G such that $d(v_1) = d(v_6) = 1$ and $d(v_2) = d(v_5) = 2$, then G is not well-indumatched.*

Lemma 2.29: ([9]) Let G be a connected well-indumatched graph and $e = uv$ be a cut-edge of G . If u is an end vertex of a path of length at least two which does not contain v , then the component of $G - e$ containing v is well-indumatched.

In the work [9], the authors provided a simple characterization of well-indumatched trees yielding a linear time recognition algorithm by using good pendant edges as follows:

Theorem 2.30: ([9]) Let T be a reduced tree of order at least 5, and M be the set of good pendant edges of T . T is well-indumatched if and only if each edge of T is covered by exactly one edge of M .

Corollary 2.31: ([9]) Given a tree T , it can be decided in linear time whether T is well-indumatched.

A graph G is said to be minimal well-indumatched with property $\mathcal{P}$ if G is a well-indumatched graph with property $\mathcal{P}$ and has no proper well-indumatched subgraph with property $\mathcal{P}$. In work [9], the authors characterize all minimal well-indumatched graphs of girth at least 9 as follows:

Theorem 2.32: ([9]) The graph G is a minimal well-indumatched graph of girth at least 9 if and only if either $G = C_{11}$ or G is a reduced unicyclic graph of even girth $g \geq 10$ with the unique cycle $C = v_1v_2\dots v_gv_1$, such that for each i , T_{v_i} is alternately of Type (i) and Type (ii).

3. TRIANGLE-FREE EQUIMATCHABLE GRAPHS

Triangle-free equimatchable graphs, that is, equimatchable graphs with girth at least 4, form a significant and large subclass of equimatchable graphs. The authors in [7] provided a characterization for equimatchable graphs with girth at least five. In particular, Theorem 2.9 in Chapter 2 states that an equimatchable graph with girth at least five is either one of C_5 and C_7 or a member of a graph family which contains K_2 and all bipartite graphs with partite sets V_1 and V_2 such that all vertices in V_1 are stems and no vertex from V_2 is a stem. Furthermore, in work [8], authors partially characterized equimatchable regular graphs by showing that for an odd positive integer r , if G is a connected equimatchable r -regular graph, then $G \in \{K_{r+1}, K_{r,r}\}$, see in Theorem 2.16 in Chapter 2. A particular result of [8] stated in Theorem 2.17 is that the only connected triangle-free equimatchable r -regular graphs are C_5 , C_7 and $K_{r,r}$, where r is a positive integer. In this chapter, we extend these results by providing a complete structural characterization of triangle-free equimatchable graphs, that is, equimatchable graphs with girth at least 4. More specifically, we identify all equimatchable triangle-free graph families. Our characterization yields a linear time algorithm that recognizes whether a given non-bipartite graph is equimatchable and triangle-free.

We start by providing some results on equimatchable graphs, which are frequently used in our arguments throughout this chapter. For the sake of brevity, we denote equimatchable factor-critical graphs by EFC-graphs.

Lemma 3.1: A factor-critical graph G is equimatchable if and only if there is no independent set I of size 3 such that $G \setminus I$ has a perfect matching.

Proof. We first remark that a factor-critical graph G has a maximum matching of size $[|V(G)| - 1]/2$.

$(\Rightarrow)$ Let G be an EFC-graph. Then all maximal matchings of G have size $[|V(G)| - 1]/2$. Assume to the contrary that there is an independent set I of G with 3 vertices such that $G \setminus I$ has a perfect matching M . It implies that M is a maximal matching of G of size $[|V(G)| - 3]/2$, contradiction. Therefore, we conclude that for every independent set I of size 3, $G \setminus I$ has no perfect matching.

$(\Leftarrow)$ Suppose, conversely, that for all independent sets of size 3, $G \setminus I$ has no perfect matching. Assume that G is not equimatchable. That is, there exists a maximal

matching M of size strictly less than $\lceil |V(G)| - 1 \rceil / 2$ in G . Since M is not a maximum matching by factor-criticality of G , by repetitively using augmenting paths, a maximal matching M^* of size $(\lceil |V(G)| - 1 \rceil / 2) - 1 = \lfloor |V(G)| - 3 \rfloor / 2$ in G can be obtained. This implies that $I = G \setminus V(M^*)$ is an independent set of size 3 and M^* is a perfect matching in $G \setminus I$, contradiction. We conclude that G is equimatchable. $\square$

Lemma 3.2: Let G be a connected EFC-graph and M be a matching of G . Then $G \setminus V(M)$ is equimatchable and factor-critical, and the followings hold:

- i) $G \setminus V(M)$ has exactly one odd component and this component is equimatchable and factor-critical.*
- ii) Even components of $G \setminus V(M)$ are randomly matchable.*

Proof. Let G be a connected EFC-graph and M be a matching of G . That is, every maximal matching of G leaves exactly one vertex exposed.

- i) Notice that G is odd by factor-criticality and $V(M)$ has an even number of vertices. It follows that $G \setminus V(M)$ has an odd number of vertices. Hence, it is easy to see that $G \setminus V(M)$ contains at least one odd component. Indeed, $G \setminus V(M)$ contains exactly one odd component. Otherwise, if $G \setminus V(M)$ has k odd components where $k \geq 2$, then every maximal matching extending M leaves at least k exposed vertices, contradicting with the fact that G is an EFC-graph. Let G^* be the unique odd component of $G \setminus V(M)$. Assume to the contrary that there exists a maximal matching M^* of G^* leaving at least three exposed vertices. It follows that every maximal matching of G extending $M \cup M^*$ leaves at least three exposed vertices, contradicting with the fact that G is an EFC-graph. Hence, every maximal matching of G^* leaves exactly one vertex exposed; that is, G^* is equimatchable and factor-critical.*
- ii) For an even component H of $G \setminus V(M)$, assume to the contrary that there exists a maximal matching M_H of H leaving at least two exposed vertices. It follows that any maximal matching of G extending $M \cup M_H$ leaves at least two exposed vertices, contradicting with the fact that G is an EFC-graph. Hence, every maximal matching of H leaves no vertex exposed, that is, every maximal matching of H is a perfect matching. Hence, H is randomly matchable.*

By combining cases i) and ii), one can easily observe that $G \setminus V(M)$ is equimatchable and factor-critical. $\square$

Lemma 3.3: Let G be a triangle-free graph and C be an induced odd cycle in G . For any two vertices x and y in $G \setminus C$, there exists a vertex $v \in C$ which is adjacent to neither x nor y .

Proof. Let G be a triangle-free graph with an induced cycle C of length $2n + 1$. Let x and y be any two vertices in $G \setminus C$. By triangle-freeness of G , the vertex x (and equivalently y) can be adjacent to at most n vertices in C . Hence, the size of the neighborhood of x and y in C can be at most $2n$, implying that there exists a vertex $v \in C$ which is adjacent to neither x nor y . $\square$

3.1. Structure of Triangle-Free Equimatchable Graphs

This section is devoted to describe the structure of triangle-free equimatchable graphs, that is, equimatchable graphs with girth at least 4. Since a graph is equimatchable if and only if each of its components is equimatchable, it suffices to characterize connected triangle-free equimatchable graphs. To characterize triangle-free equimatchable graphs, we only need to investigate equimatchable graphs with girth exactly 4 due to the characterization of the equimatchable graphs with girth at least five provided in Theorem 2.9 in Chapter 2.

The class of triangle-free equimatchable graphs can be separated into two complementary subclasses, namely non-factor-critical and factor-critical graphs. First, we consider non-factor-critical triangle-free equimatchable graphs. By Lemma 2.13 in Chapter 2 which provides a structural result for equimatchable graphs, not necessarily triangle-free, an equimatchable non-factor-critical graph cannot have an induced cycle C_{2k+1} for $k \geq 2$. Since a characterization for equimatchable bipartite graphs was given in Theorem 2.2, it suffices to concentrate only on triangle-free factor-critical equimatchable graphs, shortly triangle-free EFC-graphs, in the rest of this chapter. More precisely, we focus on triangle-free equimatchable graphs with girth exactly 4 and containing an odd cycle. Note that a factor-critical graph can not be bipartite, since if we choose a vertex from the small partite set (or from any partite set if their cardinalities are equal) there can not be a perfect matching in the rest of the graph. In the next lemma, we show

that a triangle-free EFC-graph cannot have a cut vertex. That is, all graphs we deal with in this chapter are indeed 2-connected.

Lemma 3.4: Let G be a connected triangle-free EFC-graph. Then G is 2-connected, that is, G has no cut-vertex.

Proof. Let G be a connected triangle-free EFC-graph. Assume that G has a cut-vertex v . By Theorem 2.4, v must be adjacent to at least two adjacent vertices in each component of $G - v$, which contradicts with the assumption that G is triangle-free. $\square$

By Theorem 2.9, it is clear that connected triangle-free EFC-graphs correspond to disjoint union of C_5 , C_7 , and equimatchable graphs with girth exactly 4 and containing an odd cycle. On the other hand, by Theorem 2.14, it is easy to observe the following result without proof:

Corollary 3.5: Let G be a connected triangle-free EFC-graph. Then G contains an induced odd cycle C of length five or seven. Particularly, G does not contain an induced odd cycle of any other length.

Therefore, it only remains to analyze the structure of connected triangle-free EFC-graphs with an induced odd cycle of length five or seven.

Throughout the rest of the paper, G denotes a connected triangle-free EFC-graph and C denotes an induced odd cycle in G . By Corollary 3.5, C is either C_5 or C_7 . Since both G and C are odd, $G \setminus C$ has an even number of vertices. Note here that $G \setminus C$ is allowed to be empty, that is, G can be C_5 or C_7 itself. In the next theorem, we prove that the removal of an odd cycle from G results in a randomly matchable triangle-free graph, that is, disjoint union of $K_{n,n}$ for some nonnegative integer n .

Theorem 3.6: Let G be a connected triangle-free EFC-graph. For an induced odd cycle C in G , every component of $G \setminus C$ is isomorphic to $K_{n,n}$ for some nonnegative integer n .

Proof. Let G be a connected triangle-free EFC-graph. By Corollary 3.5, there exists an induced odd cycle C of length 5 or 7 in G . If G is C_5 or C_7 , we are done. If not, let M be a minimal matching isolating a vertex v in C such that M contains exactly two edges of C ; that is, the neighbors of v in C are saturated by the edges from C . By Lemma 3.4, G is 2-connected. Hence, by Theorem 2.10 and by triangle-freeness of

G , $G \setminus (V(M) \cup \{v\})$ is isomorphic to $K_{n,n}$ for some positive integer n . Notice that if C is the cycle of length seven, then $G \setminus (V(M) \cup \{v\})$ contains an edge of C . We define a matching $M' \subset M$ containing all edges of M except the edges of C , that is, $M' = M \setminus E(C)$. Therefore, in order to describe the structure of $G \setminus C$, it is sufficient to show how edges of M' are connected to the edges of $G \setminus C$.

Claim 3.1: Any two edges of M' induce either a $2K_2$ or a C_4 in G .

Proof of Claim 3.1 Let xy and $x'y'$ be two edges of M' such that x and x' are adjacent to a vertex v of C . If there exists no edge joining an endpoint of xy and an endpoint of $x'y'$ in G , then the edges xy and $x'y'$ form a $2K_2$. Suppose that there exists an edge joining an endpoint of xy and an endpoint of $x'y'$ in G . Note that the vertices x and x' cannot be adjacent by triangle-freeness of G . Let v^* be a neighbor of v in C . Note that the vertices v^* , x and x' form an independent set of size three. If y and y' are adjacent, then $G \setminus \{v^*, x, x'\}$ has a perfect matching which contains the edge yy' , edges of $M' \setminus \{xy, x'y'\}$, a perfect matching of $C \setminus v^*$ and a perfect matching of randomly matchable graph $G \setminus (V(M) \cup \{v\})$. This contradicts with Lemma 3.1. Hence, we deduce that y and y' are nonadjacent in G . Now, assume that there exists an edge xy' but no edge $x'y$ in G . By Lemma 3.3, there exists a vertex v' in C which is adjacent to neither x' nor y . Then the vertices v' , x' and y form an independent set of size three and $G \setminus \{v', x', y\}$ has a perfect matching containing the edge xy' , edges of $M' \setminus \{xy, x'y'\}$, a perfect matching of $C \setminus v'$ and a perfect matching of randomly matchable graph $G \setminus (V(M) \cup \{v\})$, contradicting with Lemma 3.1. Hence, $x'y$ is also an edge so that the vertices x , y , x' , and y' induce a C_4 . $\square$

Claim 3.2: An edge of M' and each edge of the subgraph $G \setminus V(M)$ induce either a $2K_2$ or a C_4 in $G \setminus C$.

Proof of Claim 3.2 By Theorem 2.10, the subgraph $G \setminus (V(M) \cup \{v\})$ is isomorphic to $K_{n,n}$ for some positive integer n . Since $v \in C$, $G \setminus (V(M) \cup \{v\}) = G \setminus V(M)$ in $G \setminus C$. For simplicity, let us denote $G \setminus (V(M) \cup \{v\})$ by $K_{n,n}$ and $G \setminus V(M)$ in $G \setminus C$ by $K_{n,n} \setminus C$. Note that $G \setminus (V(M) \cup \{v\})$ contains either no edge or one edge of C if C is the cycle of length five or seven, respectively. If neither of the endpoints of an edge of M' is adjacent to a vertex of $K_{n,n}$, then such an edge and an edge of $K_{n,n}$ induce a $2K_2$. Now let xy be an edge of M' such that there exists an edge in G joining an endpoint of

xy to a vertex of $K_{n,n} \setminus C$. Without loss of generality, suppose that the vertices y and w are adjacent in G , where w is a vertex in $K_{n,n} \setminus C$. Then we need to show that the vertex x is adjacent to all neighbors of w in $K_{n,n} \setminus C$. Assume to the contrary that there exists a neighbor z of w in $K_{n,n} \setminus C$ such that x and z are nonadjacent. By Lemma 3.3, there exists a vertex $v' \in C$ which is adjacent to neither x nor z . Then the vertices v' , x and z form an independent set of size three and $G \setminus \{v', x, z\}$ has a perfect matching containing the edge wy , edges of $M' \setminus \{xy\}$, a perfect matching of $C \setminus v'$ and a perfect matching of randomly matchable graph $(K_{n,n} \setminus C) \setminus wz$, contradicting with Lemma 3.1. Hence, we deduce that there exists an edge joining x and z in G . This implies that x is adjacent to all neighbors of w in $K_{n,n} \setminus C$. By symmetry, one can observe that the vertex y is also adjacent to all neighbors of z in $K_{n,n} \setminus C$. By triangle-freeness of G , we conclude that $K_{n,n} \setminus C$ and the edge xy form a randomly matchable graph $K_{n+1,n+1} \setminus C$ in $G \setminus C$; that is, xy and each edge of $K_{n,n} \setminus C$ induce a C_4 in G . $\square$

Claim 3.3: If two edges of M' induce a C_4 with the same edge in $G \setminus C$, then these two edges of M' induce also a C_4 in G .

Proof of Claim 3.3 Let xy and $x'y'$ be two edges of M' , $K_{n,n}$ represent the subgraph $G \setminus (V(M) \cup \{v\})$ and let wz be an edge in $K_{n,n} \setminus C$. Suppose that both xy and $x'y'$ induce a C_4 with the edge wz in $K_{n,n}$. Without loss of generality, assume that the vertices x and x' are adjacent to the vertex w and the vertices y and y' are adjacent to the vertex z in G . Note here that x cannot be adjacent to x' , and y cannot be adjacent to y' by triangle-freeness of G . Assume $xy' \notin E(G)$. By Lemma 3.3, there exists a vertex $v' \in C$ which is adjacent to neither x nor y' . Then the vertices v' , x and y' form an independent set of size three and $G \setminus \{v', x, y'\}$ has a perfect matching containing the edges $x'w$, zy , edges of $M' \setminus \{xy, x'y'\}$, a perfect matching of $C \setminus v'$ and a perfect matching of $(K_{n,n} \setminus C) \setminus wz$. Since it contradicts with Lemma 3.1, we deduce that there exists an edge joining x and y' in G . By symmetry, $xy' \in E(G)$. $\square$

By combining Claim 3.1, 3.2 and 3.3, it is easy to verify that an edge of M' induces either a $2K_2$ or a C_4 with all edges of a component of $G \setminus C$. Hence, we conclude that each component of $G \setminus C$ is isomorphic to a $K_{n,n}$ for some positive integer n . $\square$

Corollary 3.7: Let G be a connected triangle-free EFC-graph. Then, there are no two vertex-disjoint induced odd cycles in G and the length of any induced odd cycle is either five or seven.

In order to show how components of $G \setminus C$ are connected to C , we first focus on how each edge of $G \setminus C$ is connected to C . The next lemma guarantees that each edge of $G \setminus C$ induces a C_4 with at least one edge of C in G .

Lemma 3.8: Let G be a connected triangle-free EFC-graph and C be an induced odd cycle in G . Then each edge of $G \setminus C$ induces a C_4 with at least one edge of C . Furthermore, the neighborhood of the endpoints of an edge of $G \setminus C$ in C induce a path.

Proof. Let G be a connected triangle-free EFC-graph, C be an induced odd cycle in G , and H be a component of $G \setminus C$. By Theorem 3.6, H is isomorphic to $K_{n,n}$ for some nonnegative integer n . Consider an edge joining a vertex of H and a vertex of C , say the edge xv , where $x \in H$ and $v \in C$. Such an edge always exists in G since G is connected. Then we need to show that all neighbors of x in H are adjacent to a neighbor of v in C . Assume to the contrary that there exists a neighbor of x in H , say y , that is adjacent to none of the neighbors v' and v'' of v in C . Then the vertices y, v' and v'' form an independent set of size three and $G \setminus \{y, v', v''\}$ has a perfect matching containing the edges xv , a perfect matching of the path $C \setminus \{v, v', v''\}$ which has an even number of vertices, and a perfect matching of randomly matchable graph $G \setminus (C \cup \{x, y\})$. Since it contradicts with Lemma 3.1, we deduce that there exists an edge joining y and at least one of $\{v', v''\}$. Hence, all neighbors of x in H are adjacent to at least one of the vertices v' and v'' . Without loss of generality, assume that y is adjacent to v' in C . In a similar way, one can observe that all neighbors of y in H are adjacent to at least one neighbor of v' in C . Therefore, we conclude that each edge in H induce a C_4 with at least one edge in C since H is isomorphic to $K_{n,n}$ for some nonnegative integer n . In addition, it follows from the above discussion that the neighborhood of the endpoints of an edge of $G \setminus C$ in C induce a path, as desired. $\square$

The next lemma, which describes induced triangle-free EFC subgraphs of G , is the primary tool for our characterization of triangle-free EFC-graphs.

Lemma 3.9: Let G be a connected triangle-free EFC-graph and C be an induced odd cycle in G . Then every subgraph of G induced by the vertices of C together with a randomly matchable subgraph of $G \setminus C$ is a connected triangle-free EFC-graph.

Proof. Let G be a connected triangle-free EFC-graph, C be an induced odd cycle in G , and H be a randomly matchable subgraph of $G \setminus C$. By Theorem 3.6, every component

of $G \setminus C$ is isomorphic to $K_{n,n}$ for some nonnegative integer n . Then $(G \setminus C) \setminus H$ is also a randomly matchable subgraph of $G \setminus C$, say H' . Hence, a subgraph of G induced by the vertices of C and H can be considered as an induced subgraph $G \setminus V(M)$ of G , where M is a perfect matching of H' . Let G' be such an induced subgraph. By Lemma 3.8, G' is connected. By Lemma 3.2, G' is a triangle-free EFC graph. This completes the proof. $\square$

Therefore, each subgraph of G induced by the vertices of C and an edge of $G \setminus C$ is indeed a connected triangle-free EFC-graph.

Corollary 3.10: The graphs $C_5 \cup K_2$ and $C_7 \cup K_2$ are forbidden subgraphs for connected triangle-free EFC graphs.

The next theorem provides the structural characterization of connected triangle-free EFC-graphs containing only one edge after the removal of an induced odd cycle. Note here that such graphs are the smallest connected triangle-free EFC-graphs except C_5 and C_7 .

Theorem 3.11: Let G be a connected triangle-free EFC-graph and C be an induced odd cycle in G . Then a subgraph of G induced by the vertices of C and an edge in $G \setminus C$ is isomorphic to one of the graphs given in Figure 3.1.

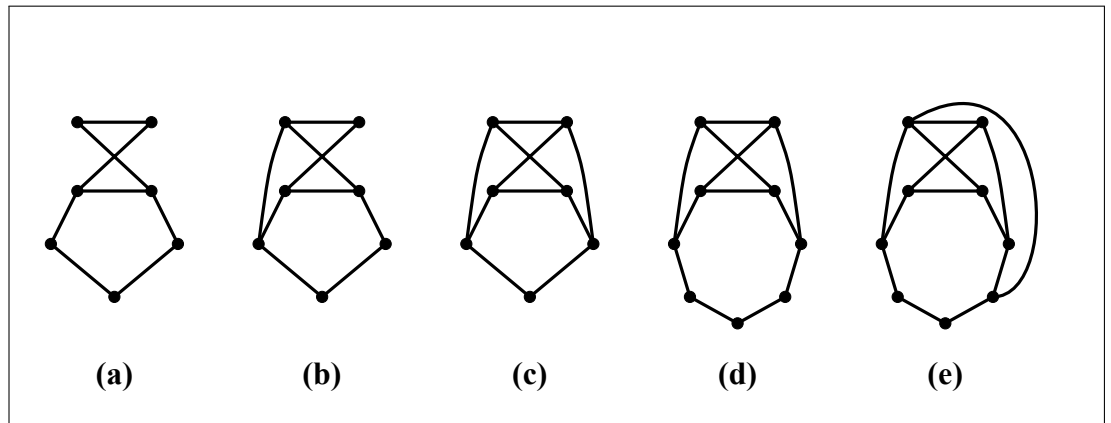


Figure 3.1: Subgraphs of G induced by the vertices of C and an edge of $G \setminus C$.

Proof. Let G be a connected triangle-free EFC-graph and C be an induced odd cycle in G . Recall that C is a cycle of length five or seven. Let G' be a subgraph of G induced by the vertices of C and an edge of $G \setminus C$. By Lemma 3.9, G' is a connected triangle-free

EFC-graph with induced odd cycle C such that $G' \setminus C$ is isomorphic to K_2 . By Lemma 3.8 and Lemma 3.3, it is easy to see that the number of vertices in C that are adjacent to the vertices of $G' \setminus C$ can be at least two and at most four when C is a cycle of length five, and at least two and at most six when C is a cycle of length seven. Besides, all such vertices of C are consecutive in either case of C . For a cycle C of length five, we obtain the graphs (a), (b) and (c) given in Figure 3.1. One can easily verify that all of these graphs are connected triangle-free EFC-graphs. On the other hand, for a cycle C of length seven, one can observe that the only triangle-free EFC-graphs are the graphs d and e given in Figure 3.1, that is, the graphs having four or five consecutive vertices in C that are adjacent to the vertices of $G' \setminus C$. In fact, in all other graphs one can find an independent set of size three containing at least two vertices of C such that the remaining graph after the removal of such an independent set has a perfect matching, contradicting with Lemma 3.1 and hence implying that such graphs are not EFC-graphs. This completes the proof. $\square$

Corollary 3.12: Let G be a connected triangle-free EFC-graph and C be an induced odd cycle in G . The number of vertices in C that are adjacent to the vertices of an edge in $G \setminus C$ is at least two or four, where C is a cycle of length five or seven, respectively. Furthermore, all such vertices of C are consecutive in either case of C .

In the next lemma, we show that if any two edges of $G \setminus C$ induce a C_4 with the same edge in C , then these two edges of $G \setminus C$ belong to the same component of $G \setminus C$.

Lemma 3.13: Let G be a connected triangle-free EFC-graph and C be an induced odd cycle in G . Then two edges from different components of $G \setminus C$ cannot induce a C_4 with the same edge in C .

Proof. Let G be a connected triangle-free EFC-graph and C be an induced odd cycle in G . Let xy and wz be two edges from different components of $G \setminus C$, that is, xy and wz induce a $2K_2$ in $G \setminus C$. Let the consecutive vertices of C be labeled by v_1, v_2, v_3, v_4, v_5 , if necessary v_6 and v_7 , depending on the length of C . Assume to the contrary that both edges xy and wz induce a C_4 with the edge v_2v_3 , where x and w are adjacent to v_2 , and y and z are adjacent to v_3 . If x and v_4 are nonadjacent in G , the vertices x, z and v_4 form an independent set of size 3. Then $G \setminus \{x, z, v_4\}$ has a perfect matching containing the edges wv_2, yv_3 , a perfect matching of the path $C \setminus \{v_2, v_3, v_4\}$ which

has an even number of vertices, and a perfect matching of randomly matchable graph $G \setminus (C \cup \{x, y, w, z\})$. Since it contradicts with Lemma 3.1, we deduce that x and v_4 are adjacent in G . By symmetry, we also deduce that z and v_1 are adjacent in G .

Suppose that C is a cycle of length five in G . If v_5 is adjacent to neither w nor y in G , the vertices w, y and v_5 form an independent set of size 3. Then $G \setminus \{w, y, v_5\}$ has a perfect matching containing the edges xv_4, zv_1, v_2v_3 and a perfect matching of randomly matchable graph $G \setminus (C \cup \{x, y, w, z\})$. By Lemma 3.1, it follows that v_5 is adjacent to at least one of $\{w, y\}$. Without loss of generality, we assume that v_5 is adjacent to w . Then, the vertices w, y and v_4 form an independent set of size 3 and $G \setminus \{w, y, v_4\}$ has a perfect matching containing the edges xv_2, zv_3, v_1v_5 , and a perfect matching of randomly matchable graph $G \setminus (C \cup \{x, y, w, z\})$, contradicting Lemma 3.1. Hence, the edges xy and wz cannot induce a C_4 with the same edge of C , where C is a cycle of length five.

Suppose that C is a cycle of length seven. If v_7 is adjacent to neither w nor y in G , the vertices w, y and v_7 form an independent set of size 3. Then $G \setminus \{w, y, v_7\}$ has a perfect matching containing the edges $zv_1, xv_4, v_2v_3, v_5v_6$, and a perfect matching of randomly matchable graph $G \setminus (C \cup \{x, y, w, z\})$. That is, by Lemma 3.1, v_7 is adjacent to at least one of $\{w, y\}$. In the case where the edge yv_7 exists, the vertices w, y and v_1 form an independent set of size 3 and $G \setminus \{w, y, v_1\}$ has a perfect matching containing the edges $xv_2, zv_3, v_4v_5, v_6v_7$, and a perfect matching of randomly matchable graph $G \setminus (C \cup \{x, y, w, z\})$. Then, Lemma 3.1 implies that the vertex v_7 is adjacent to w but not to y in G . By symmetry, we also deduce that the vertex v_5 is adjacent to y but not to w in G . It follows that the vertices w, y and v_6 form an independent set of size 3 and $G \setminus \{w, y, v_6\}$ has a perfect matching containing the edges $zv_3, xv_2, v_1v_7, v_4v_5$, and a perfect matching of randomly matchable graph $G \setminus (C \cup \{x, y, w, z\})$, a contradiction with Lemma 3.1. Therefore, we conclude that both xy and wz cannot induce a C_4 with the same edge of C , where C is a cycle of length seven. $\square$

By Lemma 3.9, a subgraph of G induced by the vertices of C and two edges from different components of $G \setminus C$ is also a connected triangle-free EFC-graph. From this viewpoint, the next theorem provides the structural characterization of connected triangle-free EFC-graphs containing only a $2K_2$ after the removal of an induced odd cycle.

Theorem 3.14: Let G be a connected triangle-free EFC-graph and C be an induced odd cycle in G . Then a subgraph of G induced by the vertices of C and two edges from different components of $G \setminus C$ is isomorphic to one of the graphs given in Figure 3.2.

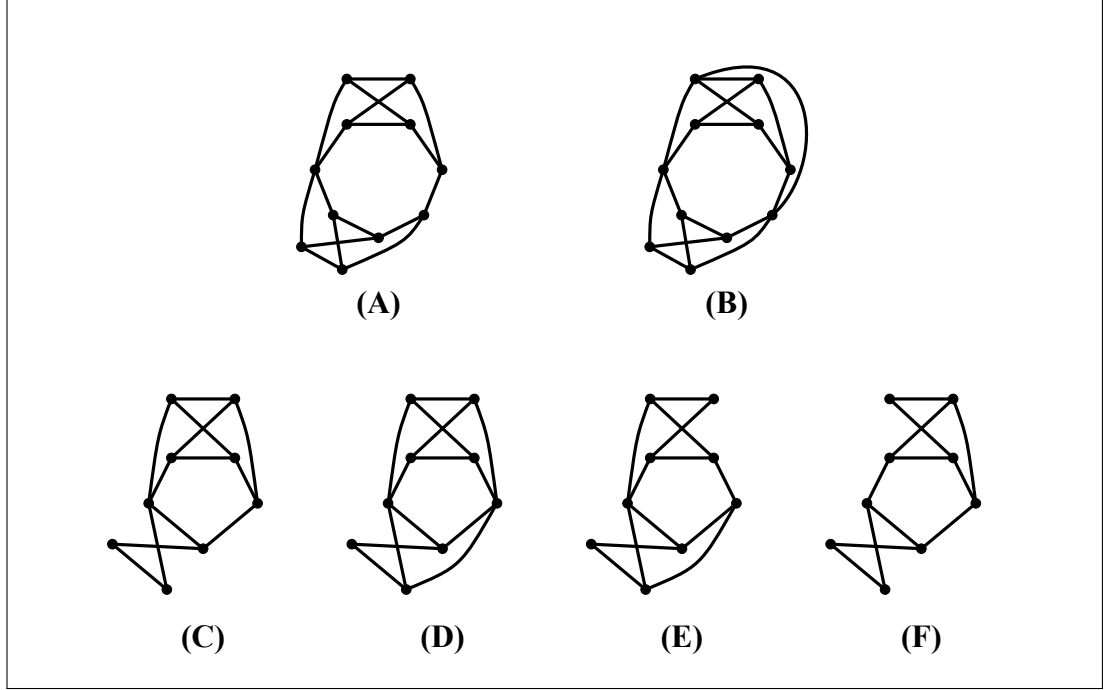


Figure 3.2: Subgraphs of G induced by the vertices of C and two edges from different components of $G \setminus C$.

Proof. Let G be a connected triangle-free EFC-graph and C be an induced odd cycle in G . Recall that C is a cycle of length five or seven. Let G' be a subgraph of G induced by the vertices of C and two edges, say xy and wz , from different components of $G \setminus C$. By Lemma 3.9, G' is a connected triangle-free EFC-graph with induced odd cycle C such that $G' \setminus C$ is isomorphic to $2K_2$. Let S_{xy} and S_{wz} denote the set of vertices of C that are adjacent to the vertices of xy and wz , respectively. Note here that, by Corollary 3.12, the vertices in each of S_{xy} and S_{wz} are consecutive in C . Remark that, by Theorem 3.11, both subgraphs of G' induced by the vertices of C and each of xy and wz are also connected triangle-free EFC-graphs given in Figure 3.1. That is, we only need to figure out how two independent edges xy and wz can form a graph of Figure 3.1 with the same induced odd cycle. Furthermore, by Lemma 3.13, an edge of C cannot induce a C_4 with both edges xy and wz , that is, the vertices of xy and wz have no common consecutive neighbors in C . We then need to examine two disjoint and complementary cases: the vertices of xy and wz have common nonconsecutive neighbors in C and

have no common neighbor in C .

We first consider the case where the vertices of xy and wz have at least one common neighbor in C . For a cycle C of length seven, by Theorem 3.11, each of S_{xy} and S_{wz} contains four or five vertices (see the graphs (d) and (e) in Figure 3.1). By Lemma 3.13 and the cardinality of C , one can observe that at least one of the sets S_{xy} and S_{wz} have size four, whereas the other may have size four or five. Hence, we deduce that G' is isomorphic to one of the graphs given in Figures 3.2(A) and 3.2(B), where C is a cycle of length seven. For a cycle C of length five, by Theorem 3.11, each of S_{xy} and S_{wz} contains at least two and at most four vertices (see the graphs in Figure 3.1(a), Figure 3.1(b) and Figure 3.1(c)). Then, it is easy to verify that the only possible ordered pairs $(|S_{xy}|, |S_{wz}|)$ are $(2, 4)$, $(3, 3)$ and $(3, 4)$. Otherwise, in the other pairs either there exists an independent set I of size three such that $G' \setminus I$ has a perfect matching, contradicting with Lemma 3.1, or there exists an edge of C forming a C_4 with two different edges in $G' \setminus C$, contradicting Lemma 3.13. (Note that this latter case corresponds to the cardinalities $(4, 4)$ for S_{xy} and S_{wz} , respectively.) Therefore, G' is isomorphic to one of the graphs given in Figures 3.2(C), 3.2(D) and 3.2(E), where C is a cycle of length five.

We now consider the case where the vertices of xy and wz have no common neighbor in C . For a cycle C of length seven, by Corollary 3.12, each of S_{xy} and S_{wz} contains at least four vertices. Then, the vertices of xy and wz must be adjacent to at least eight vertices in C , contradicting with the cardinality of C . Therefore, in this case there is no triangle-free EFC-graph where C is a cycle of length seven. On the other hand, for a cycle C of length five, each of S_{xy} and S_{wz} contains at least two vertices by Corollary 3.12 (see the graphs in Figures 3.1(a), 3.1(b) and 3.1(c)). By the cardinality of C , the possible ordered pairs $(|S_{xy}|, |S_{wz}|)$ are $(2, 2)$ and $(2, 3)$. However, in the case with ordered pair $(2, 2)$, one can easily verify that there exists an independent set I of size three such that $G' \setminus I$ has a perfect matching, contradicting with Lemma 3.1. Therefore, the only possible ordered pair $(|S_{xy}|, |S_{wz}|)$ is $(2, 3)$. Hence, G' is isomorphic to the graph given in Figure 3.2(F), where C is a cycle of length five. $\square$

Corollary 3.15: Let G be a connected triangle-free EFC-graph and C be an induced odd cycle in G . The vertices of two edges from different components of $G \setminus C$ dominate the vertices of C . That is, for any two edges inducing a $2K_2$ in $G \setminus C$, each vertex of C

is adjacent to at least one vertex of these edges.

The following result specifies the maximum number of components in $G \setminus C$. One can easily verify this result by combining Theorems 3.11 and 3.14.

Corollary 3.16: Let G be a connected triangle-free EFC-graph and C be an induced odd cycle in G . Then the subgraph $G \setminus C$ of G has at most two components.

3.2. Triangle-Free EFC-Graph Families

The goal of this section is to obtain the families of triangle-free EFC-graphs. Recall that G is a connected triangle-free EFC-graph with an induced odd cycle C of length five or seven. In the case where $G \setminus C$ is empty, G is trivially C_5 or C_7 . Here, we analyze the non-trivial cases; that is, $G \setminus C$ is nonempty. The number of components of $G \setminus C$ is denoted by $c(G \setminus C)$. By Corollary 3.16, $G \setminus C$ contains at most two components; that is, $c(G \setminus C) \leq 2$. Theorem 3.6 states that every component of $G \setminus C$ is a randomly matchable triangle-free graph. By Lemma 3.9, the vertices of C and a component of $G \setminus C$ induce a connected triangle-free EFC-subgraph of G . In Section 3.2.1, we characterize connected triangle-free EFC graphs remaining connected after the removal of an induced odd cycle. In Section 3.2.2, we provide a characterization of connected triangle-free EFC-graphs containing exactly two components after the removal of an induced odd cycle by using the characterization in Section 3.2.1.

Let us now provide the definition of twin-free graphs in order to give explicit descriptions of triangle-free EFC-graph families. Two vertices u and v are called twins if they have the same set of neighbors; that is, $N(u) = N(v)$. In a set of twins, each pair of vertices are twins. A graph is called *twin-free* if it does not have any twins. For a given graph, a twin-free graph can be obtained by contracting each set X of twins in the graph into a single vertex with multiplicity $|X|$. In our graph family descriptions, we use the following notation: Let H be a graph on k vertices $v_1, v_2, \dots, v_k$ and let $m_1, m_2, \dots, m_k$ be nonnegative integers denoting the multiplicities of these vertices, respectively. Then $H(m_1, m_2, \dots, m_k)$ denotes the graph obtained from H by iteratively replacing each vertex v_i with an independent set of m_i vertices, each of which having the same neighborhood as v_i . Notice that all vertices in the independent set replacing v_i are indeed twins. Clearly, $H = H(1, \dots, 1)$ since all multiplicities are 1. To describe the

structure of triangle-free EFC-graph families, we will use the twin-free graph G^* given in Figure 3.3.

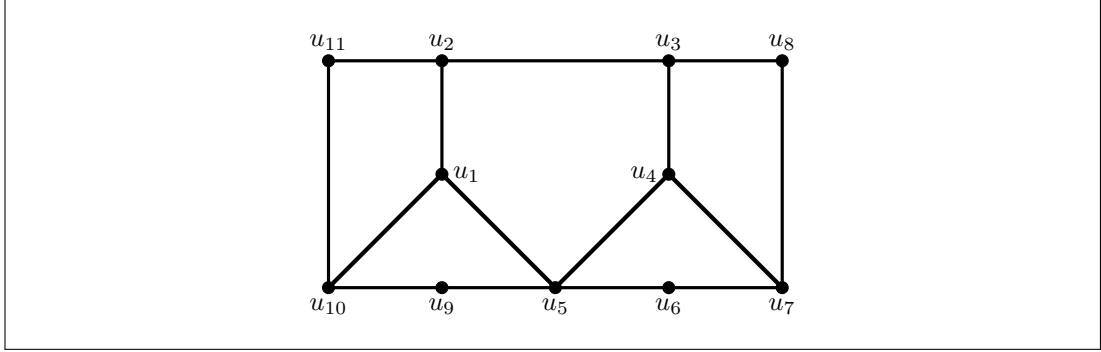


Figure 3.3: Twin-free graph G^* with vertices $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}$.

3.2.1. Triangle-Free EFC Graphs with $c(G \setminus C) = 1$

In this subsection, we study connected triangle-free EFC-graphs remaining connected after the removal of an induced odd cycle. We will show that the class of such EFC-graphs corresponds to the following graph class:

Definition 3.1: The graph class $\mathcal{F}$ is the union of the following graph families:

- $\mathcal{F}_{11} = \{G^*(1, 1, 1, 1, 1, n, n, 0, 0, 0, 0) \mid n \geq 1\}$
- $\mathcal{F}_{12} = \{G^*(1, 1, 1, 0, 1, n+1, n+1, 1, 0, 0, 0) \mid n \geq 1\}$
- $\mathcal{F}_{21} = \{G^*(1, 1, 1, n-r-s+1, 1, r, n, s, 0, 0, 0) \mid n \geq 1, n-1 \geq r \geq 1, n-1 \geq s \geq 1, n \geq r+s\}$
- $\mathcal{F}_{22} = \{G^*(1, 1, 1, n-r-s, 1, r+1, n+1, s+1, 0, 0, 0) \mid n \geq 1, n-1 \geq r \geq 1, n-1 \geq s \geq 1, n \geq r+s\}$
- $\mathcal{F}_3 = \{G^*(1, 1, r+1, s+1, 1, 0, n-s, n-r, 0, 0, 0) \mid n \geq 1, n-1 \geq r \geq 1, n-1 \geq s \geq 1\}$
- $\mathcal{F}_4 = \{G^*(r+1, n+1, s+1, 1, 1, 0, 0, 0, 0, 0, n-r-s) \mid n \geq 1, n-1 \geq r \geq 1, n-1 \geq s \geq 1, n \geq r+s\}$

where G^* is the twin-free graph depicted in Figure 3.3.

The next result proves one direction as follows:

Proposition 3.17: If $G \in \mathcal{F}$ where $\mathcal{F} = \mathcal{F}_{11} \cup \mathcal{F}_{12} \cup \mathcal{F}_{21} \cup \mathcal{F}_{22} \cup \mathcal{F}_3 \cup \mathcal{F}_4$, then G is a connected triangle-free EFC-graph remaining connected after the removal of an induced odd cycle.

Proof. For any $G \in \mathcal{F}$, one can check that there is no independent set I of size three in G such that $G \setminus I$ has a perfect matching. Hence, G is an EFC-graph by Lemma 3.1. All the other properties are easily verifiable. $\square$

To prove the other direction, we suppose that $G \setminus C$ is connected; that is, $c(G \setminus C) = 1$. By Theorem 3.6, $G \setminus C$ is isomorphic to $K_{n,n}$ for some positive integer n . By Corollary 3.12, the number of vertices in C that are adjacent to the vertices of $G \setminus C$ is at least two or four, where C is a cycle of length five or seven, respectively. Besides, all such vertices of C are consecutive in either case of C . Hence, we observe that the number of vertices of C that are nonadjacent to the vertices of $G \setminus C$ is at most three, and all such vertices are consecutive in either case of C . This observation allows us to analyze connected triangle-free EFC-graphs with $c(G \setminus C) = 1$ in the disjoint and complementary cases specified in Propositions 3.18, 3.19, 3.21, 3.22, and Lemma 3.20.

Proposition 3.18: Let G be a connected triangle-free EFC-graph having an induced odd cycle C such that $c(G \setminus C) = 1$. If C has exactly three vertices that are nonadjacent to the vertices of $G \setminus C$, then $G \in \mathcal{F}_{11} \cup \mathcal{F}_{12}$. In particular, $G \in \mathcal{F}_{11}$ if C is a cycle of length five and $G \in \mathcal{F}_{12}$ if C is a cycle of length seven.

Proof. Let G be a connected triangle-free EFC-graph having an induced odd cycle C such that $c(G \setminus C) = 1$. By Corollary 3.5, C is a cycle of length five or seven. By Theorem 3.6, $G \setminus C$ is isomorphic to $K_{n,n}$ for some positive integer n . Let the partite sets of $G \setminus C$ be labeled by A and B . Let the consecutive vertices of C be labeled by v_1, v_2, v_3, v_4, v_5 , if necessary v_6 and v_7 , depending on the length of C . Suppose that three vertices of C are nonadjacent to the vertices of $G \setminus C$. By Corollary 3.12, these vertices of C are consecutive in either case of C . Without loss of generality, let v_1, v_2 , and v_3 be nonadjacent to the vertices of $G \setminus C$.

In the case where C is a cycle of length five, by Theorem 3.11, the vertices v_4, v_5 and the endpoints of each edge in $G \setminus C$ induce the subgraph in Figure 3.1 (a). It follows that an endpoint of each edge in $G \setminus C$ is adjacent to v_4 and the other endpoint is adjacent to v_5 . Since G is triangle-free, it follows that all vertices in a partite set of

$G \setminus C$, say A , are adjacent to v_4 and all vertices in the other partite set B are adjacent to v_5 . Then, the vertices v_4, v_5 , and the vertices in A, B induce a complete bipartite graph $K_{n+1, n+1}$. Notice that any pair of vertices in A are twins and any pair of vertices in B are twins. By representing the vertices v_1, v_2, v_3, v_4 and v_5 in C with the vertices u_1, u_2, u_3, u_4 and u_5 , respectively, and by representing the vertices in A and B with the vertices u_7 and u_6 , respectively, in the graph G^* given in Figure 3.3, it is easy to verify that G belongs to the graph family $\mathcal{F}_{11}$.

Similarly, in the case where C is a cycle of length seven, by Theorem 3.11, the vertices v_4, v_5, v_6, v_7 and the endpoints of each edge in $G \setminus C$ induce the subgraph in Figure 3.1 (d). It follows that an endpoint of each edge in $G \setminus C$ is adjacent to both v_4 and v_6 and the other endpoint is adjacent to both v_5 and v_7 . Since G is triangle-free, it follows that all vertices in a partite set of $G \setminus C$, say A , are adjacent to v_4 and v_6 and all vertices in the other partite set B are adjacent to v_5 and v_7 . Hence, the vertices v_4, v_5, v_6, v_7 , and the vertices in A, B induce a complete bipartite graph with a missing edge, $K_{n+2, n+2} - v_4v_7$. Any pair of the vertices in A or B are twins. Note that the vertex v_5 and any vertex in A are also twins, and the vertex v_6 and any vertex in B are also twins. By representing the vertices $v_1, v_2, v_3, v_4, v_5, v_6$, and v_7 in C with $u_1, u_2, u_3, u_8, u_7, u_6$, and u_5 , respectively, in the graph G^* given in Figure 3.3, we conclude that G belongs to the graph family $\mathcal{F}_{12}$. $\square$

Proposition 3.19: Let G be a connected triangle-free EFC-graph having an induced odd cycle C such that $c(G \setminus C) = 1$. If C has exactly two vertices that are nonadjacent to the vertices of $G \setminus C$, then $G \in \mathcal{F}_{21} \cup \mathcal{F}_{22}$. In particular, $G \in \mathcal{F}_{21}$ if C is a cycle of length five and $G \in \mathcal{F}_{22}$ if C is a cycle of length seven.

Proof. Let G be a connected triangle-free EFC-graph having an induced odd cycle C such that $c(G \setminus C) = 1$. By Corollary 3.5, C is a cycle of length five or seven. By Theorem 3.6, $G \setminus C$ is isomorphic to $K_{n, n}$ for some positive integer n . Let the partite sets of $G \setminus C$ be labeled by A and B . Let the consecutive vertices of C be labeled by v_1, v_2, v_3, v_4, v_5 , if necessary v_6 and v_7 , depending on the length of C . Suppose that two vertices of C are nonadjacent to the vertices of $G \setminus C$. By Corollary 3.12, these vertices of C are consecutive in either case of C . Without loss of generality, let v_1 and v_2 be nonadjacent to the vertices of $G \setminus C$.

Let C be a cycle on five vertices. By Theorem 3.11, the vertices v_3, v_4, v_5 and the

endpoints of each edge in $G \setminus C$ induce the subgraph in Figure 3.1 (a) or (b). It follows that an endpoint of each edge in $G \setminus C$ is adjacent to the vertex v_4 whereas the other endpoint is adjacent to at least one of v_3 and v_5 . Particularly, if an endpoint of an edge in $G \setminus C$ is adjacent to only one of v_3 and v_5 , it is the graph in Figure 3.1 (a); and if an endpoint of an edge in $G \setminus C$ is adjacent to both v_3 and v_5 , it is the graph in Figure 3.1 (b). Hence, the triangle-freeness of G implies that all vertices in a partite set of $G \setminus C$, say A , are adjacent to v_4 and all vertices in the other partite set B are adjacent to at least one of v_3 and v_5 . Note here that the vertex v_4 and the vertices in A and B induce a complete bipartite graph $K_{n+1,n}$; and the vertex v_4 and any vertex in B that is adjacent to both v_3 and v_5 are twins. Therefore, by representing the vertices v_1, v_2, v_3, v_4 , and v_5 with the vertices u_1, u_2, u_3, u_4 , and u_5 , respectively, and the vertices in A with the vertex u_7 and by representing the vertices in B which are adjacent to only v_3 with the vertex u_8 and are adjacent to only v_5 with the vertex u_6 in the graph G^* given in Figure 3.3, one can observe that G belongs to the graph family $\mathcal{F}_{21}$.

Now let C be a cycle on seven vertices. By Theorem 3.11, the vertices v_3, v_4, v_5, v_6, v_7 and the endpoints of each edge in $G \setminus C$ induce the subgraph in Figure 3.1 (d) or (e). It follows that an endpoint of each edge in $G \setminus C$ is adjacent to the vertex v_5 and at least one of the vertices v_3 and v_7 whereas the other endpoint is adjacent to both v_4 and v_6 . Particularly, if an endpoint of an edge in $G \setminus C$ is adjacent to only one of v_3 and v_7 , it is the graph in Figure 3.1 (d); and if an endpoint of an edge in $G \setminus C$ is adjacent to both v_3 and v_7 , it is the graph in Figure 3.1 (e). Since G is triangle-free, all vertices in a partite set of $G \setminus C$, say A , are adjacent to the vertices v_4 and v_6 , and all vertices in the other partite set B are adjacent to the vertex v_5 and are adjacent to at least one of the vertices v_3 and v_7 . It follows that the vertices v_4, v_5, v_6 , and the vertices in A and B induce a complete bipartite graph $K_{n+2,n+1}$. Note here that the vertex v_5 and any vertex in A are twins. Furthermore, the vertex v_4 and any vertex in B being adjacent to v_3 and v_5 but not to v_7 are twins, and the vertex v_6 and any vertex in B being adjacent to v_5 and v_7 but not to v_3 are twins. Hence, by representing the vertices $v_1, v_2, v_3, v_4, v_5, v_6$, and v_7 with the vertices $u_1, u_2, u_3, u_8, u_7, u_6$, and u_5 , respectively, and by representing the vertices in B which are adjacent to both v_3 and v_7 with the vertex u_4 in graph G^* given in Figure 3.3, it is easy to verify that G belongs to the graph family $\mathcal{F}_{22}$. $\square$

We now consider the case where C has at most one vertex that is nonadjacent to

the vertices of $G \setminus C$. In the next result, we show that there is no triangle-free EFC-graph G in such a case where C is a cycle of length seven. In Propositions 3.21 and 3.22, we prove that triangle-free EFC-graph G belongs to $\mathcal{F}_3$ or $\mathcal{F}_4$ in such a case where C is a cycle of length five.

Lemma 3.20: Let G be a connected triangle-free EFC-graph having an induced odd cycle C of length seven such that $c(G \setminus C) = 1$. Then, at least two (consecutive) vertices of C are nonadjacent to the vertices of $G \setminus C$.

Proof. Let G be a connected triangle-free EFC-graph having an induced odd cycle C of length seven such that $c(G \setminus C) = 1$. By Theorem 3.6, $G \setminus C$ is isomorphic to $K_{n,n}$ for some positive integer n . Let the consecutive vertices of C be labeled by $v_1, v_2, v_3, v_4, v_5, v_6$, and v_7 . Let xy be an edge in $G \setminus C$. Note that the vertices x and y belong to the different partite sets of $G \setminus C$. By Theorem 3.11 and Corollary 3.12, $N(x) \cup N(y)$ contains at least four consecutive vertices, say v_1, v_2, v_3 , and v_4 . Without loss of generality, let x be adjacent to v_1 and v_3 , and y be adjacent to v_2 and v_4 .

We first show that the vertex v_6 is nonadjacent to the vertices of $G \setminus C$. Conversely, we suppose that the vertex v_6 is adjacent to a vertex $w \in G \setminus C$. Let w and x be in the same partite set of $G \setminus C$; that is, $wy \in E(G \setminus C)$. It follows that the vertices v_3, v_5 , and v_7 induce an independent set of size 3 in G and $G \setminus \{v_3, v_5, v_7\}$ has a perfect matching containing the edges v_1v_2, v_4y, v_6w , and a perfect matching of randomly matchable graph $(G \setminus C) \setminus \{w, y\}$ which is isomorphic to $K_{n-1, n-1}$. It gives a contradiction with Lemma 3.1. Hence, we conclude that the vertex v_6 is adjacent to none of the vertices of $G \setminus C$.

We now show that at least one of the vertices v_5 and v_7 is nonadjacent to the vertices of $G \setminus C$. Assume to the contrary that there exist two vertices a and b in $G \setminus C$ such that the vertices v_5 and v_7 are adjacent to a and b , respectively. Notice that a and b cannot belong to the different partite sets of $G \setminus C$; that is, $ab \notin E(G \setminus C)$. Otherwise, by contradicting with Lemma 3.1, the vertices v_1, v_4 , and v_6 induce an independent set of size 3 in G , and $G \setminus \{v_1, v_4, v_6\}$ has a perfect matching containing the edges v_2v_3, v_5a, v_7b , and a perfect matching of randomly matchable graph $(G \setminus C) \setminus \{a, b\}$ which is isomorphic to $K_{n-1, n-1}$. Thus, the vertices a and b belong to the same partite set of $G \setminus C$. Note that the vertices x and y belong to the different partite sets of $G \setminus C$. In the case where a, b and x belong to the same partite set of $G \setminus C$, we observe that the

vertices v_1, v_3 , and v_6 induce an independent set of size 3 in G . Besides, $G \setminus \{v_1, v_3, v_6\}$ has a perfect matching containing the edges v_2y, v_4v_5, v_7b , and a perfect matching of randomly matchable graph $(G \setminus C) \setminus \{b, y\}$, by contradicting with Lemma 3.1. Since G is triangle-free, it is clear that b and x cannot refer to the same vertex. The same result holds even if a and x or a and b refer to the same vertex. Similarly, in the case where a, b and y belong to the same partite set of $G \setminus C$, the vertices v_2, v_4 , and v_6 induce an independent set of size 3 in G . Since $G \setminus \{v_2, v_4, v_6\}$ has a perfect matching containing the edges v_1v_7, v_5a, v_3x , and a perfect matching of randomly matchable graph $(G \setminus C) \setminus \{a, x\}$, it contradicts with Lemma 3.1. The vertices a and y cannot refer the same vertex; and the same result holds even if b and y or a and b refer to the same vertex. Thus, we came to the conclusion that there are no such a and b vertices in $G \setminus C$. Therefore, the proof is complete. $\square$

Proposition 3.21: Let G be a connected triangle-free EFC-graph having an induced odd cycle C such that $c(G \setminus C) = 1$. If C has exactly one vertex that is nonadjacent to the vertices of $G \setminus C$, then $G \in \mathcal{F}_3$. In particular, C is a cycle of length five.

Proof. Let G be a connected triangle-free EFC-graph having an induced odd cycle C such that $c(G \setminus C) = 1$. Suppose that one vertex of C is nonadjacent to the vertices of $G \setminus C$. By Lemma 3.20, C is a cycle of length five. By Theorem 3.6, $G \setminus C$ is isomorphic to $K_{n,n}$ for some positive integer n . Let the partite sets of $G \setminus C$ be labeled as A and B , and let the consecutive vertices of C be labeled as v_1, v_2, v_3, v_4 , and v_5 . Without loss of generality, let v_1 be adjacent to none of the vertices of $G \setminus C$. Then, there exist the vertices a and b in $G \setminus C$ such that the vertices v_2 and v_5 are adjacent to a and b , respectively.

By Theorem 3.11, each neighbor of a in $G \setminus C$ is adjacent to the vertex v_3 and each neighbor of b in $G \setminus C$ is adjacent to the vertex v_4 . By the triangle-freeness of G , it is easy to see that a and b has no common neighbor in $G \setminus C$. Since $G \setminus C$ is isomorphic to $K_{n,n}$, we conclude that the vertices a and b belong to the different partite sets of $G \setminus C$, say $a \in A$ and $b \in B$. It follows that each vertex of A in $G \setminus C$ is adjacent to the vertex v_4 and each vertex of B in $G \setminus C$ is adjacent to the vertex v_3 . Hence, the vertices v_3, v_4 and the vertices in A and B induce a complete bipartite graph $K_{n+1, n+1}$. Note here that the vertex v_3 and any vertex in A being adjacent to both v_2 and v_4 are twins, and the vertex v_4 and any vertex in B being adjacent to both v_3 and v_5 are twins. Hence,

by representing vertices v_1, v_2, v_3, v_4 , and v_5 with the vertices u_1, u_2, u_3, u_4 , and u_5 , respectively, and by representing the vertices in A being nonadjacent to v_2 with the vertex u_8 and the vertices in B being nonadjacent to v_5 with the vertex u_{11} in graph G^* given in Figure 3.3, it is easy to verify that G belongs to the graph family $\mathcal{F}_3$. $\square$

Proposition 3.22: Let G be a connected triangle-free EFC-graph having an induced odd cycle C such that $c(G \setminus C) = 1$. If each vertex in C is adjacent to at least one vertex of $G \setminus C$, then $G \in \mathcal{F}_4$. In particular, C is a cycle of length five.

Proof. Let G be a connected triangle-free EFC-graph having an induced odd cycle C such that $c(G \setminus C) = 1$. Suppose that there is no vertex in C being adjacent to none of the vertices of $G \setminus C$; that is, each vertex in C is adjacent to at least one vertex of $G \setminus C$. Then, by Lemma 3.20, C is a cycle of length five. By Theorem 3.6, $G \setminus C$ is isomorphic to $K_{n,n}$ for some positive integer n . Let the partite sets of $G \setminus C$ be labeled by A and B , and the consecutive vertices of C be labeled by v_1, v_2, v_3, v_4 , and v_5 . By the triangle-freeness of G , it is easy to see that all neighbors of each vertex in C belong to the same partite set of $G \setminus C$.

Let xy be an edge in $G \setminus C$, where $x \in A$ and $y \in B$. By Lemma 3.8, xy induces a C_4 with at least one edge of C , say v_1v_5 with edges xv_1 and yv_5 . It follows that $N_{G \setminus C}(v_1) \subseteq A$ and $N_{G \setminus C}(v_5) \subseteq B$. Since $y \in B$, by Theorem 3.11, each vertex in A is adjacent to at least one of v_1 and v_4 . Since $x \in A$, by Theorem 3.11, each vertex in B is adjacent to at least one of v_2 and v_5 . We now consider the neighbors of v_3 in $G \setminus C$. Since all neighbors of v_3 belong to the same partite set of $G \setminus C$, without loss of generality, we suppose that $N_{G \setminus C}(v_3) \subseteq A$. Thus, there exists a vertex $w \in A$ which is adjacent to v_3 . Then, Theorem 3.11 implies that each vertex in B is adjacent to at least one of v_2 and v_4 .

At this point, we have that each vertex in B is adjacent to not only at least one of v_2 and v_5 but also at least one of v_2 and v_4 . Since a vertex in B cannot be adjacent to both v_4 and v_5 by the triangle-freeness of G , it is easy to observe that each vertex in B is adjacent to the vertex v_2 . Particularly, y is adjacent to both v_2 and v_5 . Then, by Theorem 3.11, it follows that each vertex in A is adjacent to at least one of v_1 and v_3 . Remark that each vertex in A is also adjacent to at least one of v_1 and v_4 . Since a vertex in A cannot be adjacent to both v_3 and v_4 by the triangle-freeness of G , each vertex in A is indeed adjacent to the vertex v_1 . Hence, we make a result that the vertices

v_1, v_2 and the vertices in A and B induce a complete bipartite graph $K_{n+1, n+1}$ in G . Remark that $N_{G \setminus C}(v_3) \subseteq A$ and $N_{G \setminus C}(v_5) \subseteq B$.

We now consider the neighbors of v_4 in $G \setminus C$. Without loss of generality, we suppose that $N_{G \setminus C}(v_4) \subseteq B$; that is, there exists a vertex $z \in B$ which is adjacent to v_4 . By Theorem 3.11, each vertex in A is adjacent to at least one of v_3 and v_5 . Since $N_{G \setminus C}(v_5) \subseteq B$, it is easy to see that each vertex in A is adjacent to the vertex v_3 . Hence, we conclude that the subgraph of G induced by the vertices v_1, v_2, v_3 and the vertices in A and B is a complete bipartite graph $K_{n+2, n+1}$. Note here that the vertex v_2 and any vertex in A are twins. Furthermore, the vertex v_3 and any vertex in B being adjacent to both v_2 and v_4 are twins; and the vertex v_1 and any vertex in B being adjacent to v_2 and v_5 are twins. Notice also that $N_B(v_4) \cap N_B(v_5) = \emptyset$ and there may exist some vertices in B which are adjacent to neither v_4 nor v_5 . Therefore, by representing the vertices v_1, v_2, v_3, v_4 , and v_5 with the vertices u_1, u_2, u_3, u_4 , and u_5 , respectively, and by representing the vertices in B being adjacent to none of v_4 and v_5 with the vertex u_{11} , in graph G^* given in Figure 3.3, we conclude that G belongs to the graph family $\mathcal{F}_4$. $\square$

We summarize the results in this subsection as follows:

Theorem 3.23: A graph G is a connected triangle-free EFC-graph having an induced odd cycle C such that $c(G \setminus C) = 1$ if and only if $G \in \mathcal{F}$, where $\mathcal{F} = \mathcal{F}_{11} \cup \mathcal{F}_{12} \cup \mathcal{F}_{21} \cup \mathcal{F}_{22} \cup \mathcal{F}_3 \cup \mathcal{F}_4$.

Proof. One direction easily follows from Proposition 3.17. We proceed as follows to prove the other direction: Let G be a connected triangle-free EFC-graph having an induced odd cycle C such that $c(G \setminus C) = 1$. Let S be the set of vertices in C which are nonadjacent to the vertices of $G \setminus C$. By Corollary 3.12, we observe that $|S| \leq 3$. If $|S| = 3$, then by Proposition 3.18, G is a member of $\mathcal{F}_{11} \cup \mathcal{F}_{12}$. If $|S| = 2$, then by Proposition 3.19, G is a member of $\mathcal{F}_{21} \cup \mathcal{F}_{22}$. If C is a cycle of length seven, then by Lemma 3.20, there is no such graph G in the case $|S| \leq 1$. Finally, if C is a cycle of length five, then by Propositions 3.21 and 3.22, G is a member of $\mathcal{F}_3 \cup \mathcal{F}_4$ in the case $|S| \leq 1$. $\square$

3.2.2. Triangle-Free EFC Graphs with $c(G \setminus C) = 2$

In this subsection, we study connected triangle-free EFC-graphs containing two components after the removal of an induced odd cycle. We will show that the class of such EFC-graphs corresponds to the following graph class:

Definition 3.2: The graph class $\mathcal{G}$ contains the following graph families:

- $\mathcal{G}_{11} = \{G^*(m+1, m+1, 1, 0, 1, 1, n+1, n+1, 0, 0, 0) \mid n \geq 1, m \geq 1\}$
- $\mathcal{G}_{12} = \{G^*(m+1, m+1, 1, n-r-s, 1, r+1, n+1, s+1, 0, 0, 0) \mid n \geq 1, m \geq 1, n-1 \geq r \geq 1, n-1 \geq s \geq 1, n \geq r+s\}$
- $\mathcal{G}_{21} = \{G^*(1, 1, 1, n-r-s+1, 1, r, n, s, 0, m, m) \mid n \geq 1, m \geq 1, n-1 \geq r \geq 1, n-1 \geq s \geq 1, n \geq r+s\}$.
- $\mathcal{G}_{22} = \{G^*(1, 1, r+1, s+1, 1, 0, n-s, n-r, 0, m, m) \mid n \geq 1, m \geq 1, n-1 \geq r \geq 1, n-1 \geq s \geq 1\}$
- $\mathcal{G}_{23} = \{G^*(r+1, n+1, s+1, 1, 1, m, m, 0, 0, 0, n-r-s) \mid n \geq 1, m \geq 1, n-1 \geq r \geq 1, n-1 \geq s \geq 1, n \geq r+s\}$
- $\mathcal{G}_{31} = \{G^*(m-k-l+1, 1, 1, n-r-s+1, 1, r, n, s, l, m, k) \mid n \geq 1, m \geq 1, n-1 \geq r \geq 1, n-1 \geq s \geq 1, n \geq r+s, m-1 \geq l \geq 1, m-1 \geq k \geq 1, m \geq k+l\}$
- $\mathcal{G}_{32} = \{G^*(k+1, l+1, 1, n-r-s+1, 1, r, n, s, 0, m-l, m-k) \mid n \geq 1, m \geq 1, n-1 \geq r \geq 1, n-1 \geq s \geq 1, n \geq r+s, m-1 \geq l \geq 1, m-1 \geq k \geq 1, m \geq k+l\}$

where G^* is the twin-free graph depicted in Figure 3.3.

In the next result, we prove one direction as follows:

Proposition 3.24: If $G \in \mathcal{G}$ where $\mathcal{G} = \mathcal{G}_{11} \cup \mathcal{G}_{12} \cup \mathcal{G}_{21} \cup \mathcal{G}_{22} \cup \mathcal{G}_{23} \cup \mathcal{G}_{31} \cup \mathcal{G}_{32}$, then G is a connected triangle-free EFC-graph containing exactly two components after the removal of an induced odd cycle.

Proof. As in Proposition 3.17, one can verify that there is no independent set I of size three in $G \in \mathcal{G}$ such that $G \setminus I$ has a perfect matching; hence by Lemma 3.1, G is an EFC-graph. All the other properties are easily verifiable. $\square$

To prove the other direction, we suppose that $c(G \setminus C) = 2$; that is, $G \setminus C$ contains two components, say H_1 and H_2 . By Theorem 3.6, H_1 and H_2 are isomorphic to $K_{n,n}$ and $K_{m,m}$, respectively, for some nonnegative integers n and m . By Lemma 3.9, both subgraphs of G induced by the vertices of C and H_1 and by the vertices of C and H_2 , that is, $G \setminus H_2$ and $G \setminus H_1$, respectively, are connected triangle-free EFC-graphs. Besides, $(G \setminus H_2) \setminus C$ and $(G \setminus H_1) \setminus C$ are connected; indeed, $(G \setminus H_2) \setminus C = H_1$ and $(G \setminus H_1) \setminus C = H_2$ which are complete bipartite graphs $K_{n,n}$ and $K_{m,m}$, respectively. Hence, by Theorem 3.23, we deduce that $G \setminus H_2$ and $G \setminus H_1$ are members of the graph class $\mathcal{F}$ defined in Section 3.2.1. On the other hand, by Lemma 3.13, any two edges from H_1 and H_2 cannot induce a C_4 with the same edge in C , and by Corollary 3.15, the vertices of these edges dominate the vertices of C . Furthermore, Corollary 3.7 states that there are no two vertex-disjoint induced odd cycles in G . These observations allow us to reduce the rest of this subsection to merging two graphs from $\mathcal{F}$ with each other. Note here that the merged graphs should have the same induced odd cycle C and the components of $G \setminus C$ have at most two nonconsecutive common neighbors in C .

In the next result, we provide a characterization of connected triangle-free EFC-graphs with $c(G \setminus C) = 2$ where C is a cycle of length seven.

Proposition 3.25: Let G be a connected triangle-free EFC-graph having an induced odd cycle C of length seven such that $c(G \setminus C) = 2$. Then $G \in \mathcal{G}_{11} \cup \mathcal{G}_{12}$.

Proof. Let G be a connected triangle-free EFC-graph having an induced odd cycle C of length seven such that $c(G \setminus C) = 2$. Let H_1 and H_2 be the components of $c(G \setminus C)$. By Lemma 3.9 and by Theorem 3.23, induced subgraphs $G \setminus H_2$ and $G \setminus H_1$ of G are both contained in the graph class $\mathcal{F}$. By Propositions 3.18 and 3.19, each of $G \setminus H_2$ and $G \setminus H_1$ of G is a member of the union of the graph families $\mathcal{F}_{12}$ and $\mathcal{F}_{22}$; that is, $G \setminus H_1, G \setminus H_2 \in \mathcal{F}_{12} \cup \mathcal{F}_{22}$. However, by Lemma 3.13, the subgraphs $G \setminus H_2$ and $G \setminus H_1$ cannot both belong to the graph family $\mathcal{F}_{22}$. It follows that at least one of $G \setminus H_2$ and $G \setminus H_1$ is a member of the graph family $\mathcal{F}_{12}$. If $G \setminus H_2$ and $G \setminus H_1$ both belong to the graph family $\mathcal{F}_{12}$, the endpoints of any two edges from H_1 and H_2 and vertices of C induce the graph in Figure 3.2 (A). Then, by representing the cycle C with $u_1 u_2 u_3 u_8 u_7 u_6 u_5$ in graph G^* given in Figure 3.3, it is easy to see that G belongs to the graph family $\mathcal{G}_{11}$.

Suppose that $G \setminus H_2$ belongs to the graph family $\mathcal{F}_{12}$ and $G \setminus H_1$ belongs to the

graph family $\mathcal{F}_{22}$. Since $G \setminus H_1 \in \mathcal{F}_{22}$, there exists an edge xy in H_2 such that the vertices x, y and the vertices of C induce the graph in Figure 3.1 (e). It follows that the vertices x, y , the vertices of C and the endpoints of any edge of H_1 induce the graph in Figure 3.2 (B). Let x be the vertex being adjacent to exactly three vertices of C . Hence, by representing the cycle C with $u_1u_2u_3u_8u_7u_6u_5$ and by representing x and its twins with the vertex u_4 in graph G^* given in Figure 3.3, we conclude that G belongs to the graph family $\mathcal{G}_{12}$. $\square$

We now proceed with the characterization of connected triangle-free EFC-graphs with $c(G \setminus C) = 2$ where C is a cycle of length five. In the next lemma, we show that at least one of $G \setminus H_2$ and $G \setminus H_1$ of G is a graph contained in the graph family $\mathcal{F}_{11}$ or $\mathcal{F}_{21}$.

Lemma 3.26: Let G be a connected triangle-free EFC-graph having an induced odd cycle C of length five such that $c(G \setminus C) = 2$. If H_1 and H_2 are the components of $c(G \setminus C)$, then at least one of the induced subgraphs $G \setminus H_2$ and $G \setminus H_1$ belongs to the union of the graph families $\mathcal{F}_{11}$ and $\mathcal{F}_{21}$.

Proof. Let G be a connected triangle-free EFC-graph having an induced odd cycle C of length five such that $c(G \setminus C) = 2$. Let H_1 and H_2 be the components of $c(G \setminus C)$. By Lemma 3.9 and Theorem 3.23, induced subgraphs $G \setminus H_2$ and $G \setminus H_1$ of G are both contained in the graph class $\mathcal{F}$. Since C is a cycle of length five, each of $G \setminus H_2$ and $G \setminus H_1$ belongs to one of the graph families $\mathcal{F}_{11}$, $\mathcal{F}_{21}$, $\mathcal{F}_3$, and $\mathcal{F}_4$. Notice that $(G \setminus H_2) \setminus C = H_1$ and $(G \setminus H_1) \setminus C = H_2$. By Lemma 3.13 and Corollary 3.15, the vertices of H_1 and H_2 have at most two nonconsecutive common neighbors in C . By definition of the graph families $\mathcal{F}_3$ and $\mathcal{F}_4$, we observe that $G \setminus H_2$ and $G \setminus H_1$ cannot both belong to $\mathcal{F}_3 \cup \mathcal{F}_4$. Therefore, it follows that at least one of $G \setminus H_2$ and $G \setminus H_1$ belongs to the graph families $\mathcal{F}_{11}$ and $\mathcal{F}_{21}$. $\square$

Proposition 3.27: Let G be a connected triangle-free EFC-graph having an induced odd cycle C of length five such that $c(G \setminus C)$ has the components H_1 and H_2 . Then $G \in \mathcal{G}_{21} \cup \mathcal{G}_{22} \cup \mathcal{G}_{23} \cup \mathcal{G}_{31} \cup \mathcal{G}_{32}$.

Proof. Let G be a connected triangle-free EFC-graph having an induced odd cycle C of length five such that $c(G \setminus C) = 2$. Let H_1 and H_2 be the components of $c(G \setminus C)$. By Lemma 3.9 and Theorem 3.23, induced subgraphs $G \setminus H_2$ and $G \setminus H_1$ of G are both

contained in the graph class $\mathcal{F}$. Since C is a cycle of length five, each of $G \setminus H_2$ and $G \setminus H_1$ belongs to the union of the graph families $\mathcal{F}_{11}$, $\mathcal{F}_{21}$, $\mathcal{F}_3$, and $\mathcal{F}_4$. By Lemma 3.26, at least one of the induced subgraphs $G \setminus H_2$ and $G \setminus H_1$ belongs to the union of the graph families $\mathcal{F}_{11}$ and $\mathcal{F}_{21}$.

We first suppose that one of $G \setminus H_2$ and $G \setminus H_1$ belongs to the graph family $\mathcal{F}_{11}$; say $G \setminus H_2 \in \mathcal{F}_{11}$. Then, by definition of $\mathcal{F}_{11}$, each edge of H_1 induce a C_4 with the same edge of C . Since, by Corollary 3.15, any two edges from H_1 and H_2 dominate the vertices of C , it is easy to see that $G \setminus H_1 \notin \mathcal{F}_{11}$. It follows that $G \setminus H_1$ is contained in one of the graph families $\mathcal{F}_{21}$, $\mathcal{F}_3$ and $\mathcal{F}_4$. Hence, by merging a graph from $\mathcal{F}_{11}$ and a graph from $\mathcal{F}_{21}$ via common cycle C represented with $u_1u_2u_3u_4u_5u_1$ in graph G^* given in Figure 3.3, we conclude that G belongs to the graph family $\mathcal{G}_{21}$. Similarly, by merging a graph from $\mathcal{F}_{11}$ and a graph from $\mathcal{F}_3$ or $\mathcal{F}_4$ via common cycle C represented with $u_1u_2u_3u_4u_5u_1$ in graph G^* given in Figure 3.3, we conclude that G belongs to the graph family $\mathcal{G}_{22}$ or $\mathcal{G}_{23}$, respectively.

We now suppose that one of $G \setminus H_2$ and $G \setminus H_1$ belongs to the graph family $\mathcal{F}_{21}$; say $G \setminus H_2 \in \mathcal{F}_{21}$. In the case where $G \setminus H_1 \in \mathcal{F}_{11}$; we previously obtain the graph family $\mathcal{G}_{21}$. Then, we suppose also that $G \setminus H_1 \notin \mathcal{F}_{11}$. Note that $(G \setminus H_2) \setminus C = H_1$ and $(G \setminus H_1) \setminus C = H_2$. By Lemma 3.13 and Corollary 3.15, the vertices of H_1 and H_2 have at most two nonconsecutive common neighbors in C . By definition of the graph families $\mathcal{F}_4$, it follows that $G \setminus H_1 \notin \mathcal{F}_4$. Therefore, by merging a graph from $\mathcal{F}_{21}$ and a graph from $\mathcal{F}_{21}$ or $\mathcal{F}_3$ via common cycle C represented with $u_1u_2u_3u_4u_5u_1$ in graph G^* given in Figure 3.3, it is easy to observe that G belongs to the graph family $\mathcal{G}_{31}$ or $\mathcal{G}_{32}$. $\square$

We summarize the results in this subsection as follows:

Theorem 3.28: *A graph G is a connected triangle-free EFC-graph having an induced odd cycle C such that $c(G \setminus C) = 2$ if and only if $G \in \mathcal{G}$ where $\mathcal{G} = \mathcal{G}_{11} \cup \mathcal{G}_{12} \cup \mathcal{G}_{21} \cup \mathcal{G}_{22} \cup \mathcal{G}_{23} \cup \mathcal{G}_{31} \cup \mathcal{G}_{32}$.*

Proof. One direction easily follows from Proposition 3.24. For the other direction, we proceed as follows: Let G be a connected triangle-free EFC-graph having an induced odd cycle C such that $c(G \setminus C) = 2$ and let H_1 and H_2 be two components of $c(G \setminus C)$. If C is a cycle of length seven, then $G \in \mathcal{G}_{11} \cup \mathcal{G}_{12}$ by Proposition 3.25. If C is a cycle of length five, then by Proposition 3.27, G is a member of $\mathcal{G}_{21} \cup \mathcal{G}_{22} \cup \mathcal{G}_{23} \cup \mathcal{G}_{31} \cup \mathcal{G}_{32}$. Hence, we are done. $\square$

3.3. Main Theorem and Recognition Algorithm

In this section, we first present the main result of this study and then give an efficient recognition algorithm for nonbipartite triangle-free EFC-graphs.

The following theorem, which provides a complete structural characterization of triangle-free equimatchable graphs, is the main theorem of this study:

Theorem 3.29: A graph G is a connected triangle-free equimatchable graph if and only if one of the following holds:

- i) G is an equimatchable bipartite graph, which is characterized in [2].
- ii) G is a C_5 or a C_7 .
- iii) $G \in \mathcal{F} = \mathcal{F}_{11} \cup \mathcal{F}_{12} \cup \mathcal{F}_{21} \cup \mathcal{F}_{22} \cup \mathcal{F}_3 \cup \mathcal{F}_4$.
- iv) $G \in \mathcal{G} = \mathcal{G}_{11} \cup \mathcal{G}_{12} \cup \mathcal{G}_{21} \cup \mathcal{G}_{22} \cup \mathcal{G}_{23} \cup \mathcal{G}_{31} \cup \mathcal{G}_{32}$.

Proof. One direction follows from Proposition 3.17 and Proposition 3.24. We proceed with the other direction. Let G be a connected triangle-free equimatchable graph. If G is a bipartite equimatchable graph, then the case i) holds. Now consider the case where G has an induced odd cycle C of length at least 5. Hence, by Theorem 2.13, G is an EFC-graph. If G has girth at least 5, then the case ii) holds by [7]. Otherwise, by Corollary 3.16, $G \setminus C$ has at most two components. If $G \setminus C$ is connected, then the case iii) holds by Theorem 3.23. If $G \setminus C$ has two components, then the case iv) holds by Theorem 3.28. $\square$

The recognition problem of triangle-free equimatchable graphs is clearly polynomial since each one of these two properties can be tested in polynomial time. Equimatchable graphs can be recognized in time $\mathcal{O}(m \cdot \overline{m})$ (see [4]), where m (resp. $\overline{m}$) is the number of edges (resp. non-edges) of the graph. Triangle-freeness can be recognized in time $\mathcal{O}(m^{\frac{2\omega}{\omega+1}} = \mathcal{O}(m^{1.407})$ (see [65]), where ω is the exponent of the matrix multiplication complexity (the best known exponent is $\omega \approx 2.37286$ (see [66])). Hence, the currently known overall complexity of the recognition of triangle-free equimatchable graphs is $\mathcal{O}(m \cdot (\overline{m} + m^{0.407}))$.

We will now show that for non-bipartite graphs, our characterization yields a linear time recognition algorithm.

Algorithm 1

Require: A non-bipartite graph G .

- 1: Compute a twin-free graph H and multiplicities $n_1, \dots, n_k$ such that $G = H(n_1, \dots, n_k)$.
 - 2: **if** H is isomorphic to neither G^* nor to a relevant subgraph of it **then**
 - 3: **return false**
 - 4: **else**
 - 5: **return true** if and only if $n_1, \dots, n_k$ matches any of the multiplicity patterns in the definitions of $\mathcal{F}$, $\mathcal{G}$, C_5 or C_7
-

Figure 3.4: The recognition algorithm for non-bipartite triangle-free graphs

Corollary 3.30: Given a non-bipartite graph G , Algorithm 1 given in Figure 3.4 can recognize whether G is equimatchable and triangle-free in linear time.

Proof. The correctness of Algorithm 1 given in Figure 3.4 is a direct consequence of Theorem 3.29. For every graph G , there is a twin-free graph H and a vector $(n_1, \dots, n_k)$ of vertex multiplicities such that $G = H(n_1, \dots, n_k)$. The graph H and the vector $(n_1, \dots, n_k)$ can be computed from G in linear time by first computing the modular decomposition of G (see [67]) and then looking for leaves of the modular decomposition tree that are independent sets. Therefore, step 1 can be performed in linear time.

We now note that some entries of the multiplicity vectors allowed for the graph families can be zero. In this case, H is not isomorphic to G^* but to an induced subgraph of it with those specific vertices removed. We refer to those graphs as relevant subgraphs in the algorithm. Observe that C_5 and C_7 are also one of the relevant subgraphs since $C_5 = G^*(1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0)$ and $C_7 = G^*(1, 1, 1, 0, 1, 1, 1, 1, 0, 0, 0, 0)$.

As for step 2, it takes constant time to decide whether an isomorphism exists: if H has more than 11 vertices, it is isomorphic to neither G^* nor to a subgraph of them; otherwise, H has to be compared to each one of these graphs and their relevant subgraphs, where each comparison takes constant time. Finally, step 5 takes constant time.

We conclude that the running time of Algorithm 1 given in Figure 3.4 is dominated

by the running time of step 1, which can be performed in linear time.

□

4. EQUIMATCHABLE BIPARTITE GRAPHS

The class of bipartite graphs is one of the most studied graph classes in graph theory due to its widespread applications. So, bipartite graphs form one of the first graph classes studied under the equimatchability property. The author in [20] proved that the only connected randomly matchable graphs are the complete graphs K_{2n} and complete bipartite graphs $K_{n,n}$ for $n \geq 1$. Besides, the authors in [2] provided a characterization for general equimatchable graphs, that is, not only randomly matchable but also equimatchable graphs without a perfect matching. Particularly, in Theorem 2.2, the authors gave a characterization of equimatchable bipartite graphs in terms of subsets of neighborhoods of vertices in the smaller partite set. Although it is a complete characterization of equimatchable bipartite graphs, it does not provide much insight about the structure of equimatchable bipartite graphs. Besides, this characterization does not lead to an efficient polynomial-time recognition algorithm, namely $O(n^4)$ time recognition algorithm as in work [4], for equimatchable bipartite graphs. Furthermore, the authors in [7] provided a structural characterization of equimatchable graphs with girth at least five; however, it is only a partial characterization of equimatchable bipartite graphs, namely, a characterization of equimatchable bipartite graphs with girth at least six. Motivated by the lack of a structural characterization for the general case of equimatchable bipartite graphs, we investigate equimatchable bipartite graphs in this dissertation.

The following well-known result, which is called Hall's Theorem, gives a necessary and sufficient condition for the existence of a perfect matching in bipartite graphs as follows:

Theorem 4.1: ([68]) A bipartite graph $G = (A \cup B, E)$ contains a matching saturating all vertices in A if and only if it satisfies $|N(S)| \geq |S|$ for all subset $S \subseteq A$.

The authors in [21] provided the following result, which is a more intuitive reformulation of the characterization of EB-graphs in Theorem 2.2 by using Hall's Theorem:

Theorem 4.2: ([21]) Let $G = (U \cup V, E)$ be a connected bipartite graph with $|U| \leq |V|$. Then G is equimatchable if and only if every maximal matching of G saturates all vertices in U .

By this reformulation, any bipartite supergraph of an equimatchable bipartite graph obtained by joining some pair of non-adjacent vertices from different partite sets is also an equimatchable bipartite graph with the same vertex partition. Hence, we intuitively consider equimatchable bipartite subgraphs of an equimatchable bipartite graph with the same vertex partition. For an equimatchable graph, an edge is said to be a critical-edge if the graph obtained by the removal of this edge is not equimatchable. An equimatchable graph is called edge-critical, denoted by *ECE*, if every edge is critical. Notice here that the smallest equimatchable bipartite subgraph of an equimatchable bipartite graph with the same vertex partition is indeed an edge-critical equimatchable bipartite graph with the same vertex partition. That is, each bipartite ECE-graph can be obtained from some equimatchable bipartite graph having the same vertex partition by recursively removing non-critical edges; moreover, each equimatchable bipartite graph can also be constructed from some bipartite ECE-graph by joining some non-adjacent vertices from different partite sets. Hence, it is sufficient to focus on the structure of edge-critical equimatchable bipartite graphs instead of the structure of the general case of equimatchable bipartite graphs.

4.1. Structure of Equimatchable Bipartite Graphs

This section is devoted to investigate the structure of equimatchable bipartite graphs, more simply EB-graphs. Since a graph is equimatchable if and only if all of its components are equimatchable, it suffices to focus on connected EB-graphs. The class of equimatchable graphs is separated into two complementary subclasses, namely factor-critical and non-factor-critical equimatchable graphs. Since a factor-critical (equimatchable) graph cannot be bipartite, all EB-graphs are non-factor-critical. However, the converse is not necessarily true; that is, not all non-factor-critical equimatchable graphs are bipartite. For instance, the graph consisting of two K_4 with one common vertex is a non-factor-critical equimatchable graph that is not bipartite. Therefore, we concentrate only on non-factor-critical equimatchable graphs in this dissertation.

The following well-known structural result, which is called Gallai-Edmonds decomposition, provides an important characterization for general graphs based on maximum matchings as follows:

Theorem 4.3: ([31]) For any graph G , let us denote by $D(G)$ the set of vertices which

are exposed by at least one maximum matching of G and by $A(G)$ the vertices of $V(G) \setminus D(G)$ which are neighbors of at least one vertex of $D(G)$. Let $C(G) = V(G) \setminus (D(G) \cup A(G))$. Then:

- i) Every component of the graph $G[D(G)]$ is factor-critical,
- ii) $G[C(G)]$ has a perfect matching,
- iii) Every maximum matching of G matches every vertex of $A(G)$ to a vertex of a distinct component of $G[D(G)]$.

By definition, for an equimatchable graph G , if G admits a perfect matching then $C(G) = V(G)$. Remark that by Theorem 2.1, a connected EB-graph G admitting a perfect matching is $K_{n,n}$ where $n \geq 1$. If G is a connected equimatchable graph G without a perfect matching, one can observe the following result:

Lemma 4.4: ([2]) Let G be a connected equimatchable graph with no perfect matching. Then $C(G) = \emptyset$ and $A(G)$ is an independent set in G .

By the Gallai-Edmonds decomposition, if G is a connected equimatchable graph without a perfect matching and $A(G) = \emptyset$, then G is factor-critical. That is, the class of EB-graphs does not contain such a graph G since each EB-graph is non-factor-critical. Therefore, it suffices to focus on connected equimatchable graphs without a perfect matching and $A(G) \neq \emptyset$, or equivalently connected non-factor-critical equimatchable graphs without a perfect matching. The following result is not explicitly given in [2], but it is an immediate consequence of Theorems 3 and 4 in [2]:

Lemma 4.5: ([2]) Let G be a connected equimatchable non-factor-critical graph with no perfect matching. Then G is bipartite if and only if each component of $G[D(G)]$ is a singleton.

Corollary 4.6: Let G be a connected EB-graph with no perfect matching. Then, $C(G) = \emptyset$, and each of $D(G)$ and $A(G)$ is a nonempty independent set.

For the rest of the paper, $G = (U \cup V, E)$ denotes a connected EB-graph with $|U| < |V|$. By Theorems 2.1 and 4.2, the only connected EB-graph with equal partite sets is $K_{n,n}$ where $n \geq 1$. In the next lemma, we show that the parts U and V of G correspond to the sets $A(G)$ and $D(G)$, respectively, where $C(G)$ is empty:

Lemma 4.7: Let $G = (U \cup V, E)$ with $|U| < |V|$ be a connected EB-graph and (D, A, C) be its Gallai-Edmonds decomposition. Then we have $C = \emptyset$, $A = U$ and $D = V$.

Proof. Let $G = (U \cup V, E)$ with $|U| < |V|$ be a connected EB-graph and (D, A, C) be its Gallai-Edmonds decomposition. By Corollary 4.6, we have $C = \emptyset$ and each of D and A is a nonempty independent set since G has no perfect matching. By Theorem 4.2, every maximal matching of G saturates all vertices in U . The Gallai-Edmonds decomposition implies that D does not have any vertex from U and then all vertices of U are in A . By Corollary 4.6, we observe that $A = U$ and $D = V$ since A is a nonempty independent set. $\square$

Corollary 4.8: Let $G = (U \cup V, E)$ with $|U| < |V|$ be a connected EB-graph. Then there exists an isolating matching for each $v \in V$ and there is no isolating matching for any $u \in U$.

The following result, which provides a characterization for EB-graphs by using the Gallai-Edmonds decomposition, can be easily verified by combining Theorem 2.1 and Lemma 4.7:

Corollary 4.9: Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected EB-graph with Gallai-Edmonds decomposition (D, A, C) .

- i) If G admits a perfect matching, that is, $|U| = |V|$, then $C = V(G)$, $D = \emptyset$ and $A = \emptyset$. In particular, G is $K_{n,n}$ where $n \geq 1$.*
- ii) If G admits no perfect matching, that is, $|U| < |V|$, then $C = \emptyset$, $A = U$ and $D = V$.*

In the next result, we extend the characterization of Frendrup et al. [7] stated in Theorem 2.9 to all EB-graphs as follows:

Lemma 4.10: Let $G = (U \cup V, E)$ with $|U| < |V|$ be a connected EB-graph. Then each vertex $u \in U$ satisfies at least one of the followings:

- i) u is a stem in G ,*
- ii) u is included in a subgraph $K_{2,2}$ in G .*

Proof. Let $G = (U \cup V, E)$ with $|U| < |V|$ be a connected EB-graph. Let $u \in U$ and $N(u) = \{v_1, v_2, \dots, v_n\}$ be the set of neighbors of u in V . If one vertex in $N(u)$ is a leaf in G , then we are done. Assume to the contrary that none of the vertices in $N(u)$ is a leaf in G . If u is not included in a subgraph $K_{2,2}$ in G , then there is no pair of vertices in $N(u)$ having a common neighbor except u . It implies that there exists a matching isolating u in G . Since it contradicts with Corollary 4.8, we deduce that there exists at least one pair, say $\{v_1, v_2\}$, of vertices in $N(u)$ having a common neighbor except u , say u^* . It follows that the vertices $\{u, v_1, u^*, v_2\}$ induce a $K_{2,2}$ in G , as desired. $\square$

Corollary 4.11: For a connected EB-graph $G = (U \cup V, E)$ with $|U| < |V|$, there is no leaf in U and there is no stem in V .

Thus, each vertex $u \in U$ has at least two neighbors in V . Since G is connected, unless $|U| = 1$, each vertex $u \in U$ has a neighbor $v \in V$ which is not a leaf. Remark that Theorem 2.9 states that in an EB-graph with girth at least 5, all vertices in either U or V are stems, whereas Lemma 4.10 provides that in such a graph, all vertices in U are indeed stems. Hence, Lemma 4.10 together with Theorem 2.9 lead to the following corollary:

Corollary 4.12: Let G be a connected EB-graph with girth at least 6. Then $G \in \mathcal{F}$, where $\mathcal{F}$ is the family of graphs containing K_2 and all connected bipartite graphs with bipartite sets V_1 and V_2 with $|V_1| \leq |V_2|$ such that all vertices in V_1 are stems and no vertex from V_2 is a stem.

Note here that the EB-graph family $\mathcal{F}$ described in Theorem 2.9 and Corollary 4.12 contains not only all EB-graphs with girth at least six but also some EB-graphs with girth less than six, that is, EB-graphs with girth exactly four. On the other hand, it is obvious that $\mathcal{F}$ contains some but definitely not all EB-graphs with girth four. For instance, complete bipartite graphs which are not included in the graph family $\mathcal{F}$ are trivially EB-graphs with girth four. So, in this thesis we deal with EB-graphs with girth exactly four, whereas some of these graphs are contained in the graph family $\mathcal{F}$.

We conclude this section with some observations providing an insight about the structure of connected EB-graphs.

Lemma 4.13: ([21]) Let $G \neq K_2$ be a connected equimatchable graph. Then $\nu(G) = \nu(G \setminus e)$ for any edge $e \in E(G)$.

Proof. Let $G \neq K_2$ be a connected equimatchable graph and $e = uv \in E(G)$. For a maximal matching M of G containing uv , $M \setminus uv$ is a matching of $G \setminus uv$ with size $\nu(G) - 1$. It follows that $\nu(G \setminus uv) \geq \nu(G) - 1$; that is, $\nu(G \setminus uv)$ equals either $\nu(G) - 1$ or $\nu(G)$. Assume to the contrary that $\nu(G \setminus uv) = \nu(G) - 1$. Since $G \neq K_2$ and is connected, there exists a vertex $w \in N(u) \cup N(v)$. Without loss of generality, we say $w \in N(u)$; that is, $wu \in E(G)$. Hence, there also exists another maximal matching M' in G which can be obtained by extending the edge wu . Since G is equimatchable, $|M'| = |M|$. Note here that M' is also a maximal matching in $G \setminus uv$ with size $\nu(G)$, contradicting with the assumption that $\nu(G \setminus uv) = \nu(G) - 1$. Therefore, $\nu(G \setminus uv) = \nu(G)$ for every edge uv . $\square$

In Lemma 4.15, we extend the following known result about the cut vertices in equimatchable graphs to EB-graphs as in the following way:

Lemma 4.14: ([8]) Let G be a connected equimatchable graph with a cut vertex c . Then each component of $G - c$ is also equimatchable.

Lemma 4.15: Let $G = (U \cup V, E)$ with $|U| < |V|$ be a connected EB-graph with a cut vertex c . Then each component of $G - c$ is also a connected EB-graph $H = (U_H \cup V_H, E_H)$ with $|U_H| < |V_H|$ such that $U_H \subseteq U$ and $V_H \subseteq V$.

Proof. Let $G = (U \cup V, E)$ with $|U| < |V|$ be a connected EB-graph with a cut vertex c . Let $H_1, H_2, \dots, H_k$ ($k \geq 2$) be the components of $G - c$. By Lemma 4.14, each H_i is an EB-graph for $i \in 1, 2, \dots, k$. Assume that a connected EB-graph $H = (U_H \cup V_H, E_H)$ with $|U_H| < |V_H|$ is a component of $G - c$ such that $U_H \subseteq U$ and $V_H \subseteq V$ or $U_H \subseteq V$ and $V_H \subseteq U$. Since G is connected, c is adjacent to some vertices of H . Since G is bipartite, c is adjacent to some vertices of only U_H or V_H . Let x be a neighbor of c in H and $e = cx$ be an edge joining c and x in G . We then examine the following complementary cases:

- In the case $c \in V$ in G , $x \in U$ in G . Assume to the contrary that $x \in V_H$ in H . Then, by Corollary 4.8, there exists a matching M_H isolating x in H . Note here that the only neighbor of x in G which is not contained in H is the vertex c . Since c is a cut vertex of G , that is, $k \geq 2$, there exists another component H' of $G - c$. Let x' be a neighbor of c in H' and $e' = cx'$ be an edge joining c

and x' in G . Hence, it is easy to see that the matching $M_H \cup \{e'\}$ is a matching isolating x in G . By Corollary 4.8, it contradicts with $x \in U$ in G . Therefore, we obtain that $x \in U_H$ in the case $c \in V$.

- In the case $c \in U$ in G , $x \in V$ in G . Assume to the contrary that $x \in U_H$ in H . Then, by Corollary 4.8, there exists a matching M isolating x in G . Note that the only neighbor of x in G which is not contained in H is the vertex c . Besides, the vertices c and x have no common neighbor in G , since G is bipartite. Let $e' \in M$ be such that $e' = cx'$ is an edge joining c and $x' \notin N(x)$ in G . Hence, it is easy to see that the matching $M \setminus \{e'\}$ isolates x in H . By Corollary 4.8, it contradicts with the assumption $x \in U_H$ in G . Therefore, we obtain that $x \in V_H$ in the case $c \in U$.

Therefore, we conclude that $U_H \subseteq U$ and $V_H \subseteq V$ irrespective of whether $c \in U$ or $c \in V$. $\square$

In short, Lemma 4.15 implies that for each cut vertex c of G , the components of $G - c$ are induced EB-subgraphs preserving (U, V) -partitions of G .

4.2. Edge-Critical Equimatchable Bipartite Graphs

Recall that for an equimatchable graph, an edge is a critical-edge if the graph obtained by removal of this edge is not equimatchable, and an equimatchable graph is edge-critical if every edge is critical. For the sake of brevity, we will call edge-critical equimatchable graphs as *ECE*-graphs. The goal of this section is to describe the structure of a generating subclass of EB-graphs, namely edge-critical *EB*-graphs.

Notice that the complete graph K_2 (or equivalently the complete bipartite graph $K_{1,1}$) is equimatchable but not edge-critical. In fact, its edge is not critical since the remaining graph obtained by removal of the edge contains only isolated vertices, which is trivially equimatchable. The following results about *ECE*-graphs, not necessarily bipartite, are frequently used in our arguments:

Lemma 4.16: ([21]) Let $G \neq K_2$ be a connected equimatchable graph. Then $uv \in E(G)$ is critical if and only if there is a matching of G containing uv and saturating $N_{G \setminus uv}(\{u, v\})$.

Proof. Let $G \neq K_2$ be a connected equimatchable graph with $uv \in E(G)$.

($\Rightarrow$) Assume that $uv \in E(G)$ is critical; that is, G is equimatchable but $G \setminus uv$ is not equimatchable. Then $G \setminus uv$ admits two maximal matchings M_1 and M_2 with different sizes, say $|M_1| < |M_2|$. It follows that $|M_1| = \nu(G) - 1$ and $|M_2| = \nu(G)$, since $G \setminus uv$ is obtained by the removal of only uv from G . That is, M_2 is a maximal matching of G whereas M_1 is not maximal in G . It implies that M_1 leaves both u and v exposed in $G \setminus uv$ and $M_1 \cup \{uv\}$ is a maximal matching of G . Hence, it follows that M_1 saturates all vertices in $N_{G \setminus uv}(\{u, v\})$, because otherwise M_1 cannot be maximal in $G \setminus uv$. Therefore, the maximal matching $M_1 \cup \{uv\}$ of G is the desired matching.

($\Leftarrow$) Conversely, assume that there is a matching of G containing uv and saturating $N_{G \setminus uv}(\{u, v\})$. Without loss of generality, we extend this matching to a maximal matching M of G containing uv and saturating $N_{G \setminus uv}(\{u, v\})$. It follows that $M \setminus uv$ is a maximal matching in $G \setminus uv$. There also exist another maximal matching M' in $G \setminus uv$ which can be obtained by extending the edge vw for some $w \in N(v)$ with $w \neq u$ or uz for some $z \in N(u)$ with $z \neq v$. Notice that the matching M' has size $\nu(G)$ since it is also a maximal matching of G , which is equimatchable. Hence, $G \setminus uv$ is not equimatchable since the maximal matchings M and M' of G have different sizes $\nu(G)$ and $\nu(G) - 1$. It implies that the edge uv is critical. $\square$

Corollary 4.17: A connected equimatchable graph $G \neq K_2$ is edge-critical if and only if there is a matching containing uv and saturating $N(\{u, v\})$ for every $uv \in E(G)$.

Corollary 4.18: All randomly matchable connected graphs except K_2 are edge-critical.

Recall that $G = (U \cup V, E)$ is a connected EB-graph with $|U| < |V|$. We consider a bipartite supergraph G' of G obtained by joining a pair of non-adjacent vertices of $u \in U$ and $v \in V$ by the edge uv . By Theorem 4.2, a maximal matching of G cannot be extended to a larger matching in G' since maximal matchings of G saturate all vertices in U . It follows that $\nu(G) = \nu(G') = |U|$. Let us consider a minimum maximal matching M of G' which is obtained by extending the edge uv . Note that all edges of M except uv are indeed the edges of G . Then, by Theorem 4.2, the matching $M - uv$ of G is not maximal since it does not saturate the vertex $u \in U$. We extend $M - uv$ to a maximal matching M' in G . Since G is an EB-graph, by Theorem 4.2, we have $\nu(G) = |M'| = |U|$. Since M is a maximal matching of G'

containing the edge $uv \notin E(G)$, we need to add at least one and at most two edges to $M \setminus uv$ in order to extend it to M' ; namely, the edges saturating u and v . Note here that M' has to saturate the vertex u but may not saturate the vertex v . It follows that $|U| \geq |M| \geq |M'| - 1 = |U| - 1$. Thus, $|M|$ is equal to either $|U|$ or $|U| - 1$; equivalently, it is equal to either $\nu(G)$ or $\nu(G) - 1$.

In the case where $|M| = \nu(G)$, by definition of M , all maximal matchings of G' have the same size, namely $|U|$. Thus, the supergraph G' of G is also a connected EB-graph with the same vertex set $U \cup V$. In such a case, uv is a non-critical edge of G' . In the case where $|M| = \nu(G) - 1$, we observe that maximal matchings of G' have size either $|U|$ or $|U| - 1$. It follows that the supergraph G' of G is a connected almost equimatchable bipartite graph with the vertex set $U \cup V$. Besides, the work in [35] provides a characterization for almost equimatchable graphs. All these leads us to the following observation:

Observation 4.1: For a connected EB-graph $G = (U \cup V, E)$ with $|U| < |V|$ except $K_{n,m}$ for some n and m , and a connected bipartite supergraph G' of G obtained by joining a pair of non-adjacent vertices of U and V by an edge, one of the followings holds:

- i) G' is equimatchable such that all maximal matchings of G' have size $|U|$,*
- ii) G' is almost equimatchable such that a maximal matching of G' has size $|U|$ or $|U| - 1$.*

The next result is a characterization for EB-supergraphs of EB-graphs as follows:

Theorem 4.19: Let $G = (U \cup V, E)$ with $|U| < |V|$ be a connected EB-graph except $K_{n,m}$ for some n and m , and let G' be a supergraph of G obtained by joining a pair of non-adjacent vertices of $u \in U$ and $v \in V$ by the edge uv . G' is a connected EB-graph if and only if every maximal matching of G' obtained by extending the edge uv saturates $N_G(v)$.

Proof. Let $G = (U \cup V, E)$ with $|U| < |V|$ be a connected EB-graph except $K_{n,m}$ for some n and m and G' be a supergraph of G obtained by joining a pair of non-adjacent vertices of $u \in U$ and $v \in V$ by the edge uv . It is clear that G' is connected. By Observation 4.1, G' is either equimatchable or almost equimatchable.

($\Rightarrow$) Suppose that G' is an EB-supergraph of G . Then, by Theorem 4.2, every maximal matching of G' saturates all vertices in U . Hence, every maximal matching of G' obtained by extending the edge uv saturates $N_G(v)$.

($\Leftarrow$) Suppose that every maximal matching of G' obtained by extending the edge uv saturates $N_G(v)$. We will show that G' is an EB-graph. Assume to the contrary that G' is almost equimatchable. Then, by Observation 4.1, G' admits maximal matchings of two different sizes, namely $|U| - 1$ and $|U|$. For maximal matchings of G' with $|M| = |U|$, the theorem holds and we are done. Let M be a maximal matching of G' with $|M| = |U| - 1$. Then, M is a maximal matching obtained by extending the edge uv ; otherwise M is a maximal matching of G , which is equimatchable with $\nu(G) = |U|$. Let $u^* \in U$ be the vertex exposed by M . Since $\nu(G) = |U|$, it is obvious that $M \setminus uv$ is not maximal in G ; particularly, $M \setminus uv$ does not saturate the vertices u and u^* in U . Since G is equimatchable with $\nu(G) = |U|$, we can extend the matching $M \setminus uv$ to a maximal matching M^* of G by adding two edges saturating the vertices u and u^* . Since u^* is saturated by all maximal matchings in G and exposed by M in G' , it follows that there exists an edge u^*v in G ; that is, u^* is adjacent to v . Hence, $u^* \in N(v)$ is exposed by M . Therefore, we conclude that every maximal matching of G' with size $|U| - 1$ exposes a vertex of $N(v)$, a contradiction. $\square$

Observation 4.1 and Theorem 4.19 imply that a supergraph G' of G obtained by joining a pair of non-adjacent vertices of $u \in U$ and $v \in V$ by the edge uv is a connected bipartite almost equimatchable graph if and only if there exists a maximal matching of G' obtained by extending the edge uv such that the matching leaves exposed a vertex of $N_G(v)$.

Note here that the largest such EB-supergraph of G is trivially the complete bipartite graph. We intuitively consider EB-subgraphs of G with the same vertex set $U \cup V$. Note that the smallest such EB-subgraph of G , say H , is indeed an edge-critical EB-graph with the same vertex set $U \cup V$; while the other EB-subgraphs with the same vertex set $U \cup V$ are exactly bipartite supergraphs of H . That is, each edge-critical EB-graph can be obtained from some EB-graph having the same vertex set by recursively removing non-critical edges. Therefore, in order to characterize all EB-graphs it is sufficient to characterize all edge-critical EB-graphs; that is, the class of edge-critical EB-graphs form a generating subclass of EB-graphs. Notice here that an edge-critical

EB-graph H may not be connected. Since a graph is equimatchable if and only if each of its components is equimatchable, it suffices to focus on connected edge-critical EB-graphs.

From here onwards, we focus on edge-critical EB-graphs. In the following lemma, we show that edge-critical EB-graphs cannot have a cut vertex; that is, all edge-critical EB-graphs are 2-connected.

Lemma 4.20: Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected edge-critical EB-graph. Then G is 2-connected; that is, G has no cut vertex.

Proof. Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected edge-critical EB-graph. In the case where $|U| = |V|$, by Theorem 4.2, G has a perfect matching. Then by Theorem 2.1, G is $K_{n,n}$ for some $n \geq 1$ and by Corollary 4.18, we omit $K_2 = K_{1,1}$ and we have $n \geq 2$. That is, G is 2-connected and we are done.

We now suppose that $|U| < |V|$. Assume to the contrary that G has a cut vertex c . Let $H_1, H_2, \dots, H_k$ ($k \geq 2$) be connected components of $G - c$ such that $d_i \in H_i$ where $d_1, d_2, \dots, d_k \in N(c)$ and $i \in [k]$. By Lemma 4.15, each H_i is an EB-subgraph preserving (U, V) -partitions of G where $i \in [k]$. By Corollary 4.17, for each edge $e_i = cd_i$, there exists a matching M^i containing the edge cd_i and saturating all vertices in $N(\{c, d_i\})$ for $i \in [k]$. For $j \in [k]$ with $j \neq i$, we define $M_j^i = M^i \cap E(H_j)$ which is a matching in H_j saturating all vertices in $N_{H_j}(c)$. Note that $M_1^i \cup M_2^i \cup \dots \cup M_{i-1}^i \cup M_{i+1}^i \cup \dots \cup M_k^i$ is a matching saturating all neighbors of c except the neighbors in H_i . Without loss of generality, since $k \geq 2$, let us consider the cases $i = 1$ and $i = 2$. For $i = 1$, $M_2^1 \cup M_3^1 \cup \dots \cup M_k^1$ is a matching saturating all neighbors of c except the neighbors in H_1 ; and for $i = 2$, $M_1^2 \cup M_3^2 \cup \dots \cup M_k^2$ is a matching saturating all neighbors of c except the neighbors in H_2 . It follows that $M = M_1^2 \cup M_2^1 \cup M_3^1 \cup \dots \cup M_k^1$ is a matching saturating all neighbors of c in G . Hence, by Corollary 4.8, we conclude that $c \in V$ and $d_1, d_2, \dots, d_k \in U$. By Lemma 4.15, it follows that $d_i \in U_{H_i}$ for $i \in [k]$.

Remember that for each edge $e_i = cd_i$, M^i is a matching containing the edge cd_i and saturating all vertices in $N(\{c, d_i\})$ for $i \in [k]$. By defining $T_i = M^i \cap E(H_i)$ for $i \in [k]$, for each edge $e_i = cd_i$, there also exists a matching T_i in H_i saturating all vertices in $N_{H_i}(d_i)$ for $i \in [k]$. It is easy to see that each T_i is indeed a matching isolating d_i in H_i for $i \in [k]$. Hence, by Corollary 4.8, we conclude that $d_i \in V_{H_i}$,

which is a contradiction. $\square$

The next theorem provides a characterization for edge-critical EB-graphs as follows:

Theorem 4.21: A connected bipartite graph $G = (U \cup V, E)$ with $|U| \leq |V|$ except K_2 is an edge-critical EB-graph if and only if for every $u \in U$, $|N(S)| \geq |S|$ holds for any subset $S \subseteq N(u)$ and the equality holds only for $S = N(u)$.

Proof. Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected bipartite graph except K_2 .

$(\Rightarrow)$ Suppose that G is an edge-critical EB-graph. We will first show that for every $u \in U$, $|N(S)| > |S|$ holds for any subset $S \subset N(u)$. Assume to the contrary that there exists $u \in U$ such that $|N(S)| \leq |S|$ for some $S \subset N(u)$. Then there exists $w \in N(u) \setminus S$. Since $|N(S)| \leq |S|$, there is no matching of G saturating $N_{G \setminus uw}(\{u, w\})$ and containing uw . By Lemma 4.16, G is not an ECE-graph which is a contradiction. Thus, for all $u \in U$, $|N(S)| > |S|$ holds for any subset $S \subset N(u)$. We will now show that $|N(S)| \geq |S|$ for $S = N(u)$. Since G is an EB-graph, by Theorem 2.2, for all $u \in U$, there exists a non-empty $S \subseteq N(u)$ such that $|N(S)| \leq |S|$. Since for all $u \in U$, $|N(S)| > |S|$ holds for any subset $S \subset N(u)$, it follows that for all $u \in U$, $S = N(u)$ satisfies $|N(S)| \leq |S|$. Assume that there exists $u \in U$, $|N(S)| < |S|$ holds for $S = N(u)$. Then, we observe that there is no matching saturating all vertices in $N(u)$. Hence, for any $v \in N(u)$, there is no matching of G saturating $N_{G \setminus uv}(\{u, v\})$ and containing uv . By Lemma 4.16, the edge uv is not critical, contradicting with G being an ECE-graph. That is, for all $u \in U$, $|N(S)| = |S|$ holds for $S = N(u)$. Therefore, for every $u \in U$, $|N(S)| \geq |S|$ for any subset $S \subseteq N(u)$ and the equality holds only for $S = N(u)$, as desired.

$(\Leftarrow)$ Suppose that for every $u \in U$, $|N(S)| \geq |S|$ holds for any subset $S \subseteq N(u)$ and the equality holds only for $S = N(u)$. Since $S = N(u)$ satisfies $|N(S)| = |S|$, G is an equimatchable graph by Theorem 2.2. Assume to the contrary that there exists a vertex v such that $u \in U$, $v \in N(u)$ and $uv \in E(G)$ is not critical; that is, $G \setminus uv$ is equimatchable. Then, by Theorem 2.2, there exists $X \subseteq N_{G \setminus uv}(u)$ such that $|N_{G \setminus uv}(X)| \leq |X|$. Note also that $X \subseteq N_{G \setminus uv}(u) \subset N(u)$. It follows that $|N(X)| = |N_{G \setminus uv}(X)| \leq |X|$. Since $X \subset N(u)$ and $|N(S)| \geq |S|$ holds for

any subset $S \subseteq N(u)$, we also have $|N(X)| \geq |X|$, implying that $|N(X)| = |X|$. However, $X \neq N(u)$ and this contradicts with the fact that $|N(S)| = |S|$ holds only for $S = N(u)$. Therefore, $uv \in E(G)$ is critical for all $uv \in E(G)$ and hence, G is an edge-critical EB-graph. $\square$

If the equality $|U| = |V|$ holds, we can give an explicit characterization for edge-critical EB-graphs as follows:

Proposition 4.22: A connected bipartite graph $G = (U \cup V, E)$ with $|U| = |V|$ is an edge-critical EB-graph if and only if it is isomorphic to $K_{n,n}$ for some $n \geq 2$.

Proof. Let $G = (U \cup V, E)$ with $|U| = |V|$ be a connected bipartite graph except K_2 . Suppose that G is an edge-critical EB-graph. By Theorem 4.2, G has a perfect matching. Then by Theorem 2.1 and Corollary 4.18, G is $K_{n,n}$ for $n \geq 2$. Conversely, by Theorem 2.1 and Corollary 4.18, $K_{n,n}$ is an edge-critical EB-graph for some $n \geq 2$. $\square$

The next theorem provides another characterization for edge-critical EB-graphs in terms of induced subgraphs as follows:

Theorem 4.23: A connected bipartite graph $G = (U \cup V, E)$ with $|U| \leq |V|$ is an edge-critical EB-graph if and only if for any $u \in U$, the subgraph $H = (U_H \cup V_H, E_H)$ of G induced by the vertices $N(u)$ and $N(N(u))$ is a 2-connected balanced bipartite subgraph of G such that for all $v \in V_H$, there exists a perfect matching containing uv in H .

Proof. Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected bipartite graph.

$(\Rightarrow)$ Suppose that G is an edge-critical EB-graph. Let $u \in U$ and H be the subgraph of G induced by the vertices $N(u)$ and $N(N(u))$. By Theorem 4.21, we have $|N(u)| = |N(N(u))|$, implying that $|U_H| = |V_H|$. That is, H is a balanced bipartite subgraph of G . Then by Corollary 4.17, for any neighbor v of u , there exists a matching containing uv and saturating all other neighbors of u in G . It implies that for all $v \in V_H$, there exists a perfect matching containing uv in H since H is a balanced bipartite graph. Notice that by definition of H , u is adjacent to all vertices in V_H , implying that there is no cut vertex in V_H . Suppose that $u^* \in U_H$ is a cut vertex in H . Then, we have $u^* = u$ since u is adjacent to all vertices in V_H . If u is a stem in H , then it is also a stem in G . However, G is 2-connected by Theorem 4.20, contradiction.

Therefore, u is not a stem in H . Since H is a balanced bipartite graph, there exists a component H^* of $H - u$ such that $|U_{H^*}| < |V_{H^*}|$. Then, for any $v \in V_H \setminus V_{H^*}$, there is no perfect matching containing uv in H since $|U_{H^*}| < |V_{H^*}|$, contradicting with Corollary 4.17. Hence, we conclude that u cannot be a cut vertex, which is a contradiction. Therefore, H is a 2-connected balanced bipartite subgraph of G such that for all $v \in V_H$, there exists a perfect matching containing uv in H .

($\Leftarrow$) Suppose that for any $u \in U$, the subgraph $H = (U_H \cup V_H, E_H)$ of G induced by the vertices $V_H = N(u)$ and $U_H = N(N(u))$ is a 2-connected balanced bipartite subgraph of G such that for all $v \in V_H$, there exists a perfect matching containing uv in H . Since for all $v \in V_H$, there exists a perfect matching containing uv in H , $|N(S)| \geq |S|$ holds for any subset $S \subseteq N(u)$. We assume that there exists a $X \subset N(u)$ such that $|N(X)| = |X|$. By definition of H , it is clear that $u \in N(X)$. Then, there exists a vertex w in $V_H \setminus X$, and there exists a perfect matching M containing uw in H . It implies that the vertices in X is saturated by the vertices in $N(X) - u$ in M . It gives a contradiction since $|N(X) - u| < |N(X)| = |X|$. Hence, the equality $|N(S)| = |S|$ holds only for $S = N(u)$. Therefore, by Theorem 4.21, we conclude that G is an edge-critical EB-graph. $\square$

The next result can be verified by Theorem 4.23. Particularly, the subgraph $H = (U_H \cup V_H, E_H)$ of G induced by the vertices $N(u)$ and $N(N(u))$ is a 2-connected, implying that each vertex in $U_H \setminus \{u\}$ has two neighbors in V_H and u is adjacent to all vertices in V_H .

Corollary 4.24: Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected edge-critical EB-graph. If $H = (U_H \cup V_H, E_H)$ is a subgraph of G induced by the vertices $N(u)$ and $N(N(u))$ for any $u \in U$, then all vertices in $U_H \setminus \{u\}$ form a C_4 with u .

We conclude this section with the following result used frequently in the forthcoming sections:

Lemma 4.25: Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected edge-critical EB-graph and $u_1, u_2 \in U$. Then either $N(u_1) = N(u_2)$ or both $N(u_1) \setminus N(u_2)$ and $N(u_2) \setminus N(u_1)$ are nonempty.

Proof. Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected edge-critical EB-graph and $u_1, u_2 \in U$. Without loss of generality, we assume to the contrary that $N(u_1) \subset N(u_2)$.

Let $X = N(u_1) \subset N(u_2)$. By Theorem 4.23, we have $|N(u_1)| = |N(N(u_1))|$, implying that $|N(X)| = |X|$. However, by Theorem 4.21 since for $u_2 \in U$, $|N(S)| \geq |S|$ holds for any subset $S \subseteq N(u_2)$ and the equality holds only for $S = N(u_2)$, contradiction. $\square$

4.3. Some Subclasses of Bipartite ECE-Graphs

In this section, we characterize some special subclasses of edge-critical EB-graphs, specifically, some subclasses of edge-critical EB-graphs defined by vertex ordering properties.

We begin to examine subclasses of edge-critical EB-graphs with the chain ECE-graphs. Note that chain graphs are known as difference graphs and also $2K_2$ -free bipartite graphs. A bipartite graph $G = (U \cup V, E)$ is called a chain graph if the neighborhoods of vertices in both U and V can be linearly ordered with respect to inclusion, that is, $N(u_1) \subseteq N(u_2) \subseteq \dots \subseteq N(u_n)$ and $N(v_1) \subseteq N(v_2) \subseteq \dots \subseteq N(v_m)$. We denote the class of chain graphs by CHAIN and the class of chain ECE-graphs by ECE-CHAIN.

In the following result, we provide a characterization of chain ECE-graphs by proving that ECE-CHAIN is indeed the class of randomly matchable bipartite graphs except $K_{1,1}$:

Theorem 4.26: Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected bipartite graph. Then G is a chain ECE-graph if and only if it is a randomly matchable graph except $K_{1,1}$, that is, G is $K_{n,n}$ for some $n \geq 2$.

Proof. Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected bipartite graph.

($\Rightarrow$) Let G be a chain ECE-graph such that $|U| = n$ and $|V| = m$, where $n, m \geq 1$. Suppose that $N(u_1) \subseteq N(u_2) \subseteq \dots \subseteq N(u_n)$ and $N(v_1) \subseteq N(v_2) \subseteq \dots \subseteq N(v_m)$ are the linear orderings of U and V , respectively. By Lemma 4.25, it is easy to see that $N(u) = V$ for all $u \in U$. By Lemma 4.20, it follows that G is a complete bipartite graph $K_{n,m}$, where $n, m \geq 2$. By Theorem 4.23, we conclude that G is $K_{n,n}$ for some integer $n \geq 2$, as desired.

($\Leftarrow$) Let G be a randomly matchable graph except $K_{1,1}$, that is, $K_{n,n}$ for some integer $n \geq 2$. It is easy to see that G is a chain graph since it is a $2K_2$ -free bipartite

graph. By Corollary 4.18, we conclude that G is a chain ECE-graph. $\square$

We now provide a characterization of bipartite permutation ECE-graphs, which constitute a bipartite superclass of ECE-CHAIN. A graph $G = (V(G), E(G))$ with $V(G) = \{v_1, v_2, \dots, v_n\}$ is said to be a permutation graph if there is a permutation π over $\{1, 2, \dots, n\}$ such that $v_i v_j \in E$ if and only if $(i - j)(\pi^{-1}(i) - \pi^{-1}(j)) < 0$. A permutation graph which is also bipartite is called a bipartite-permutation graph (or bi-permutation graph shortly). We denote the class of bipartite permutation graphs by BIPERMUTATION. To give equivalent definitions of bi-permutation graphs, we need the following definitions of specialized orderings:

Definition 4.1: Given a bipartite graph $G = (U \cup V, E)$, an ordering $\prec_V$ of the vertices in V has the adjacency property if for every vertex $u \in U$, $N(u)$ consists of vertices that are consecutive in the ordering $\prec_V$.

Definition 4.2: Given a bipartite graph $G = (U \cup V, E)$, an ordering $\prec_V$ of the vertices in V has the enclosure property if for every pair of vertices $u_1, u_2 \in U$ such that $N(u_1) \subset N(u_2)$, vertices in $N(u_2) \setminus N(u_1)$ occur consecutively in $\prec_V$.

Remark that adjacency and enclosure properties can be defined on both partite sets of a bipartite graph G , that is, on both U and V .

Definition 4.3: A strong ordering of the vertices of a bipartite graph $G = (U \cup V, E)$ consists of an ordering $\prec_U$ of U and an ordering $\prec_V$ of V such that for all $uv, u'v' \in E$, where u, u' are in U and v, v' are in V , if $u \prec_U u'$ and $v' \prec_V v$, then $uv', u'v \in E$.

The next theorem provides two characterizations of bi-permutation graphs as follows:

Theorem 4.27: ([69]) The following statements are equivalent for a bipartite graph $G = (U \cup V, E)$:

- i) G is a bipartite permutation graph,
- ii) There is a strong ordering of $U \cup V$,
- iii) There exists an ordering of U (or V) which has the adjacency and enclosure properties.

Note here that if the orderings $\prec_U$ and $\prec_V$ of U and V satisfy the adjacency and enclosure properties, it does not imply that the pair $(\prec_U, \prec_V)$ forms a strong ordering of G . Note here that if the orderings $\prec_U$ and $\prec_V$ of U and V satisfy the adjacency and enclosure properties, $(\prec_U, \prec_V)$ is not necessarily the pair that forms a strong ordering of G ; that is, some pair other than $(\prec_U, \prec_V)$ may be a strong ordering of $U \cup V$. By the proof of Theorem 4.27 given in [69], if the graph G is connected, the statements 2 and 3 in Theorem 4.27 can be combined as follows:

Corollary 4.28: Let $(\prec_U, \prec_V)$ be a strong ordering of a connected bi-permutation graph $G = (U \cup V, E)$. Then both $\prec_U$ and $\prec_V$ have the adjacency property and the enclosure property.

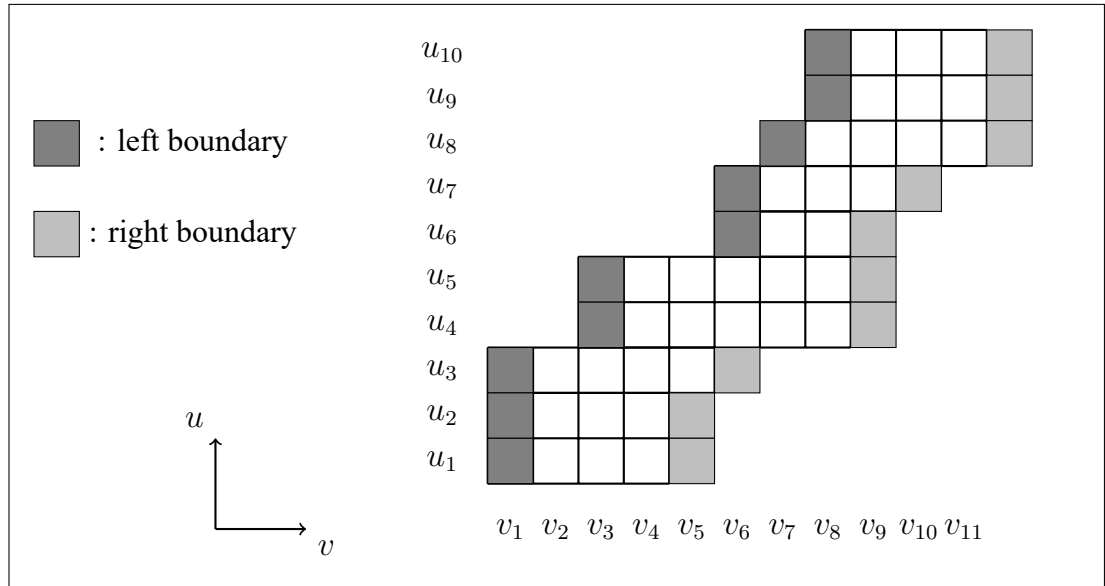


Figure 4.1: Rectilinear polygon representing a bi-permutation graph

We suppose that $G = (U \cup V, E)$ is a connected bi-permutation graph with a strong ordering $(\prec_U, \prec_V)$. The work [70] shows that G can be expressed as a rectilinear polygon: there exists a square at the position (u_i, v_j) for each edge $u_i v_j$ where $u_i \in U$ and $v_j \in V$. Note that each row and column corresponds to the neighborhood of a vertex in U and V , respectively. By Corollary 4.28, both $\prec_U$ and $\prec_V$ have the adjacency property and the enclosure property. In the work [70], authors also proved that by the adjacency property, each row and column of the polygon contains a continuous block of squares. Additionally, they proved that by the adjacency and enclosure properties, the right and left boundary of the polygon are nondecreasing block of squares from bottom

to top; that is, the leftmost and rightmost squares of rows form an ordered set of squares which is nondecreasing with respect to the ordering $\prec_V$ (see Figure 4.1). Notice that by Lemma 4.25, for connected edge-critical EB-graphs, any row of the polygon cannot be contained by another row. Thus, edge-critical EB-graphs trivially satisfy the enclosure property. Therefore, we obtain the following result:

Observation 4.2: ([70]) If G is a connected bi-permutation ECE-graph, then the polygon representation of G has a shape with the following properties (given in Figure 4.2):

- i) Each row (column) of the polygon contains a continuous block of squares,*
- ii) A row of the polygon is not contained by another row,*
- iii) The right (left) boundary of the polygon is nondecreasing block of squares from bottom to top.*

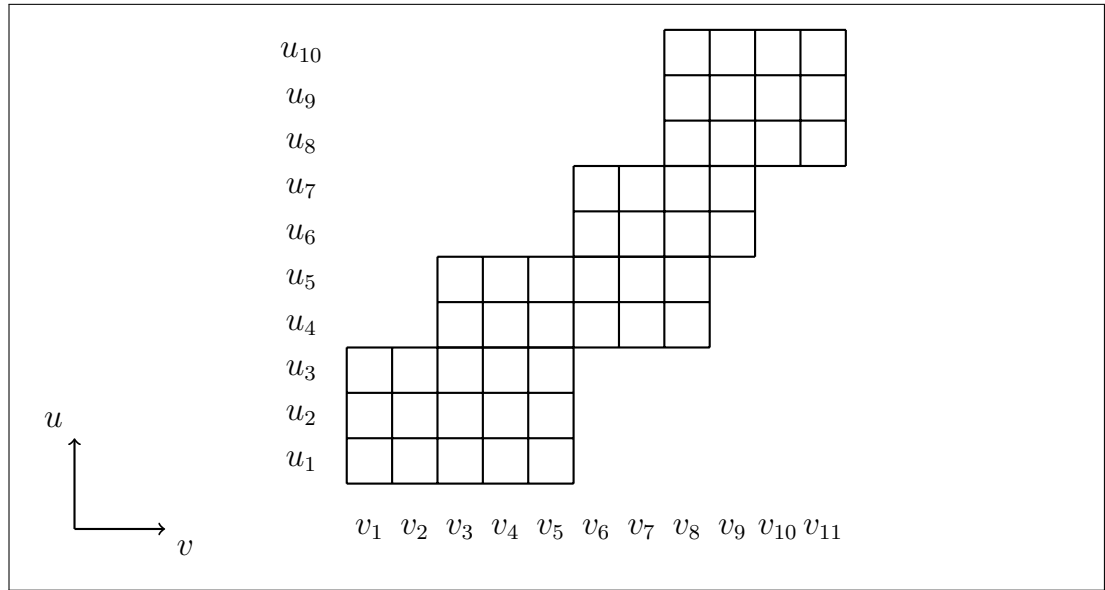


Figure 4.2: Rectilinear polygon representing a bi-permutation ECE-graph

We here introduce covering subgraphs of connected bi-permutation ECE-graphs as follows: Suppose that $G = (U \cup V, E)$ is a connected bi-permutation ECE-graph with a strong ordering $(\prec_U, \prec_V)$. Let $\{C_1, C_2, C_3, \dots, C_k\}$ be a smallest covering set $\mathcal{C}$ of U , that is, $\bigcup_i C_i = U$, such that for each $C_i, i \in \{1, \dots, k\}$, any two vertices in C_i have the same neighborhood in V . It is clear that $C_i \cap C_j = \emptyset$ for all $i \neq j$ whereas $N(C_i) \cap N(C_j)$ may not be empty; that is, the set $\mathcal{C}$ forms indeed a partition of vertices in U . Considering the polygon in Figure 4.2, it can be easily noticed that for each C_i ,

the vertices in C_i are consecutive in the ordering $\prec_U$ and the vertices in $N(C_i)$ are consecutive in the ordering $\prec_V$. We also observe that for each C_i , $C_i \cup N(C_i)$ induces a complete bipartite subgraph, say G_i , in G . The sequence of complete bipartite graphs $G_1, G_2, \dots, G_k$ form a set $\mathcal{G}$ of complete bipartite covering subgraphs of G , which are consecutive with respect to the ordering $(\prec_U, \prec_V)$.

Note that by Theorem 4.21, $|C_i| \leq |N(N(C_i))|$ always holds for each i . By Theorem 4.23, $|N(N(C_i))| = |N(C_i)|$ also holds for each i since all vertices in C_i have the same the neighborhood. Since G is connected, it follows that if $|C_i| = |N(C_i)|$ for some i , then G is trivially a complete bipartite graph $K_{n,n}$ for $n \geq 2$; that is, G is a chain ECE-graph. Hence, we assume that $|C_i| < |N(C_i)|$ for each i ; that is, G is not a chain ECE-graph. For each C_i , let the vertices l_i and r_i be the leftmost and rightmost vertices, respectively, of $N(C_i)$ in the ordering $\prec_V$. That is, the leftmost and rightmost vertices in C_i are the vertices with the smallest and largest indices, respectively, in $N(C_i)$.

For the case where rectilinear polygon has equal dimensions, that is, where $|U| = |V|$, the polygon is indeed a square with size $n = |U|$. Note here that the square polygon has four corner squares at the positions u_1v_1, u_1v_n, u_nv_1 and u_nv_n . A diagonal of a square polygon is defined as a line segment having as its end points two corners u_1v_1 and u_nv_n ; that is, the set of ordered pairs at the positions $u_1v_1, u_2v_2, \dots, u_{n-1}v_{n-1}, u_nv_n$. In the forthcoming part of this section, we will also use line segments being parallel to the diagonal of a square polygon; namely, sets of ordered pairs at the positions $u_iv_1, u_{i+1}v_2, \dots, u_{i+k}v_{k+1}$ for each i where $k = |U| - i$ or $u_1v_j, u_2v_{j+1}, \dots, u_{t+1}v_{j+t}$ for each j where $t = |V| - j$. Note here that the nearest parallel line segments to the diagonal occurs in $i = 2$ and $j = 2$.

The next result gives an insight about the structure of the polygon in Figure 4.2 as follows:

Lemma 4.29: Let G be a connected bi-permutation ECE-graph. Then the polygon corresponding to the subgraph of G induced by the vertices of $N(C_i)$ and $N(N(C_i))$ for each C_i has a shape with the following properties (as in Figure 4.3):

- i) Each row and column of the polygon contains a continuous block of squares,
- ii) The right and left boundary of the polygon are nondecreasing blocks of squares from bottom to top,

- iii) The polygon has equal dimensions, namely $|N(C_i)|$; in fact, $|N(C_i)| = |N(N(C_i))| = |N(l_i) \cup N(r_i)| = |N(l_i)| + |N(r_i)| - |C_i|$,
- iv) The diagonal contains a continuous block of squares; in particular, it contains all unit squares,
- v) Each two consecutive line segments being parallel to the diagonal contain a continuous block of squares; in particular, they are completely full of squares.

Proof. Let G be a connected bi-permutation ECE-graph. For each C_i , each subgraph of G induced by the vertices of $N(C_i)$ and $N(N(C_i))$, say G_i , corresponds a rectilinear polygon, which is indeed a section of the polygon (like given in Figure 4.3). Hence, the properties i) and ii) follow from Observation 4.2. By Theorem 4.23, each G_i is a 2-connected balanced bipartite subgraph of G with a perfect matching. By G_i being balanced bipartite graph, the polygon has equal dimensions, namely $|N(C_i)|$. In particular, by definition and also by property (a), $|N(C_i)| = |N(N(C_i))| = |N(l_i) \cup N(r_i)| = |N(l_i)| + |N(r_i)| - |C_i|$; that is, the property iii) holds. By combining the properties i), ii) and iii) and the existence of a perfect matching in G_i , we conclude that the diagonal contains all unit squares; that is, the diagonal contains a continuous block of squares. Hence, 2-connectedness of G_i also implies that two consecutive line segments being parallel to the diagonal are completely full of squares; that is, these line segments contain a continuous block of squares. Therefore, the properties iv) and v) hold. $\square$

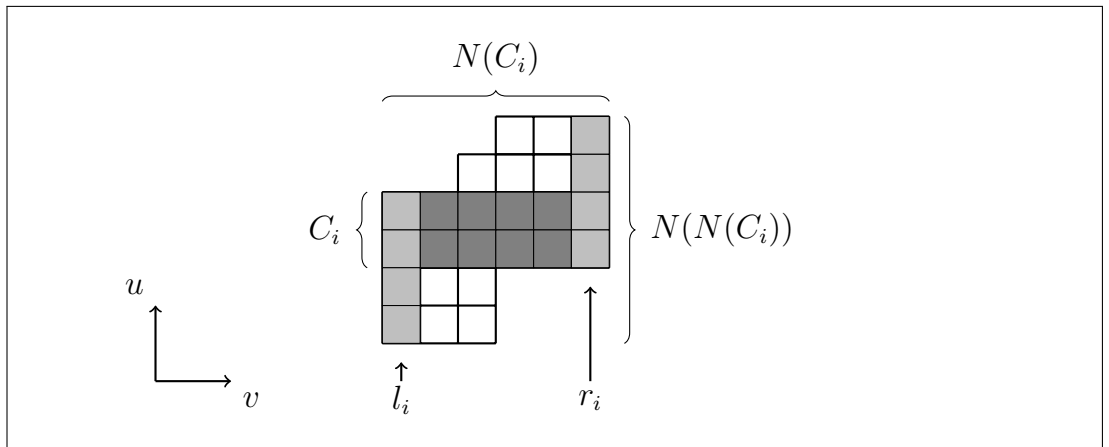


Figure 4.3: A rectilinear polygon constructed from subgraph induced by the vertices $N(C_i)$ and $N(N(C_i))$

The next theorem provides a characterization of bi-permutation ECE-graphs as in the following way:

Theorem 4.30: Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a bi-permutation graph with the strong ordering $(\prec_U, \prec_V)$. Then G is a bi-permutation ECE-graph if and only if G has the polygon representation $\mathcal{P}$ with respect to $(\prec_U, \prec_V)$ such that

- i) Each row and column of the polygon contain a continuous block of squares,*
- ii) The right and left boundary of the polygon are nondecreasing from bottom to top,*
- iii) The polygon has a (minimal) covering with complete bipartite subgraphs $G_1 = (U_1 \cup V_1, E_1)$, $G_2 = (U_2 \cup V_2, E_2)$, ..., $G_k = (U_k \cup V_k, E_k)$ with $N_G(U_i) = V_i$ as follows: for each U_i , each section of the polygon induced by the vertices of $N(U_i)$ and $N(N(U_i))$ has equal dimensions, namely $|N(U_i)| = |N(N(U_i))| = |N(l_i) \cup N(r_i)| = |N(l_i)| + |N(r_i)| - |U_i|$. Moreover, each two consecutive line segments being parallel to the diagonal contain a continuous block of squares; indeed, they are completely full of squares.*

Proof. Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a bi-permutation graph with the strong ordering $(\prec_U, \prec_V)$.

$(\Rightarrow)$ Suppose that G is a bi-permutation ECE-graph. The polygon representation $\mathcal{P}$ with respect to $(\prec_U, \prec_V)$ satisfies the conditions i) and ii) by Observation 4.2, and the conditions iii) by Lemma 4.29.

$(\Leftarrow)$ Suppose that G has the polygon representation $\mathcal{P}$ with respect to $(\prec_U, \prec_V)$ satisfying the conditions i), ii) and iii). Let the complete bipartite subgraphs $G_1 = (U_1 \cup V_1, E_1)$, $G_2 = (U_2 \cup V_2, E_2)$, ..., $G_k = (U_k \cup V_k, E_k)$ of G with $N_G(U_i) = V_i$ be a (minimal) covering of G satisfying the condition iii). We need to show that G is an ECE-graph. Since for each U_i , each section of the polygon induced by the vertices of $N(U_i)$ and $N(N(U_i))$ has equal dimensions, for any $u \in U$, the subgraph $H = (U_H \cup V_H, E_H)$ of G induced by the vertices $N(u)$ and $N(N(u))$ is a balanced bipartite subgraph of G . By the condition iii), for such an induced subgraph H , the diagonal and each two consecutive line segments being parallel to the diagonal in the polygon representation of H are completely full of squares. The existence of the diagonal being completely full of squares implies the existence of a perfect matching in H . The existence of two consecutive line segments being completely full of squares implies that

the removal of two vertices does not disconnect H ; that is, H is 2-connected. Hence, we conclude that H is a 2-connected balanced bipartite graph with a perfect matching. Therefore, by Theorem 4.23, it follows that G is a bi-permutation ECE-graph. $\square$

We now move on to convex and biconvex graphs, which are bipartite superclasses of BIPERMUTATION. A bipartite graph $G = (U \cup V, E)$ is convex on U or V if there is an ordering of U or V , respectively, that fulfills the adjacency property. An ordering having the adjacency property is also called a convex ordering. A bipartite graph G is called a convex graph if it is convex on U or V and is called a biconvex (or doubly convex) graph if it is convex on both U and V . For more information we refer the reader to [71] and [72]. We denote the class of convex graphs and biconvex graphs by CONVEX and BICONVEX. By definition, it is clear that biconvex graphs form a subclass of convex graphs. Therefore, the following proper inclusion relations between the subclasses of bipartite graphs given in [53] can be easily derived:

$$\text{CHAIN} \subset \text{BIPERMUTATION} \subset \text{BICONVEX} \subset \text{CONVEX} \quad (4.1)$$

We denote the class of convex ECE-graphs and biconvex ECE-graphs by ECE-CONVEX and ECE-BICONVEX. The following result provides a characterization of biconvex ECE-graphs by showing that ECE-BICONVEX precisely corresponds to ECE-BIPERMUTATION:

Theorem 4.31: Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected bipartite graph. Then G is a biconvex ECE-graph if and only if it is a bi-permutation ECE-graph.

Proof. Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected bipartite graph.

$(\Rightarrow)$ Suppose that G is a biconvex ECE-graph with orderings $\prec_U$ and $\prec_V$ of U and V , respectively, having the adjacency property. By Lemma 4.25, there is no pair of vertices $u_1, u_2 \in U$ such that $N(u_1) \subset N(u_2)$. Hence, the ordering $\prec_V$ of V trivially satisfies the enclosure property. That is, the ordering $\prec_V$ of V has both adjacency and enclosure properties. It follows that G is a bi-permutation ECE-graph by Theorem 4.27.

$(\Leftarrow)$ Suppose that G is a bi-permutation ECE-graph. Since bi-permutation graphs form a proper subclass of biconvex graphs, it is obvious that G is also a biconvex ECE-graph. $\square$

Corollary 4.32: Each biconvex ECE-graph $G = (U \cup V, E)$ with $|U| \leq |V|$ has a strong ordering on $U \cup V$.

We then focus on the class of convex ECE-graphs, that is, ECE-CONVEX. Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected convex ECE-graph; that is, G is convex on at least one of U and V . Hence, convex ECE-graphs can be separated into two sets, namely convex ECE-graphs on U and convex ECE-graphs on V , and these cases can be examined separately. It is easy to see that the intersection of these sets forms the class of biconvex ECE-graphs. In this study, we characterize convex ECE-graphs on V , while the characterization of convex ECE-graphs on U remains an open question. For the sake of brevity, we denote the class of convex ECE-graphs on V by ECE-CONVEXonV.

We suppose that G is a convex ECE-graph on V with the convex ordering $\prec_V$ of V . For each $u \in U$, let the vertices l_u and r_u be the leftmost and rightmost vertices, respectively, of $N(u)$ in the interval of the ordered vertices of V . That is, the leftmost and rightmost vertices are the vertices with the smallest and largest indices, respectively, in $N(u)$. We define the *clusters* of vertices in U as follows: two vertices u_1 and u_2 of U are said to be in the same cluster if $l_{u_1} = l_{u_2}$; for the sake of brevity, $l_1 = l_2$. By Lemma 4.25, we have $N(u_1) = N(u_2)$ for the vertices u_1 and u_2 in the same cluster. It implies that a vertex $v \in V$ is adjacent to either all or none of vertices in a cluster C of U ; we say that $v \in V$ is adjacent to a cluster C or not. It follows that for each cluster C in U , the vertices in C and $N(C)$ form a complete bipartite subgraph $K_{|C|, |N(C)|}$ of G . By definition, the intersection of two clusters in U is empty, whereas union of all clusters cover U . That is, the collection of all clusters in U forms disjoint sets covering U .

We sort the clusters in U as follows: if $l_i < l_j$ for $u_i \in C_i$ and $u_j \in C_j$ then $C_i < C_j$. Let the set $\mathcal{C} = \{C_1, C_2, C_3, \dots, C_k\}$ be a set of clusters in U and $\prec_U$ be the ordering of vertices on U with respect to $\mathcal{C}$; that is, the vertices in the same cluster are consecutive and the vertices from different clusters have the same ordering of their clusters. By Theorem 4.23, we have $|C| \leq |N(C)|$ for any cluster C in U . Since G is connected, it follows that if there exists a cluster C with $|C| = |N(C)|$, then G trivially has only one cluster, implying that G is $K_{n,n}$ for some $n \geq 1$; particularly, G is a chain ECE-graph.

In the following result, we provide a characterization for convex ECE-graphs on V by showing that the class of convex ECE-graphs on V corresponds to ECE-BICONVEX:

Theorem 4.33: Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected bipartite graph. Then G is a convex ECE-graph on V if and only if it is a biconvex ECE-graph.

Proof. Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected bipartite graph.

$(\Rightarrow)$ Suppose that G is a convex ECE-graph on V with the convex ordering $\prec_V$ of V . Let $\mathcal{C} = \{C_1, C_2, C_3, \dots, C_k\}$ be a set of sorted clusters in U and $\prec_U$ be the ordering of vertices on U with respect to $\mathcal{C}$. Recall that a vertex $v \in V$ is adjacent to either all or none of vertices in a cluster C of U . Besides, by definition of $\prec_U$, a vertex $v \in V$ cannot be adjacent to nonconsecutive clusters. It follows that the ordering $\prec_U$ on U with respect to sorted clusters $\mathcal{C}$ has also the adjacency property. Thus, G is convex on U and hence G is a biconvex ECE-graph.

$(\Leftarrow)$ Suppose that G is a biconvex ECE-graph. By the relation (4.1), it is obvious that G is a convex ECE-graph both on U and V . $\square$

We summarize the results on convex ECE-graphs on V in the following way:

Theorem 4.34: The following statements are equivalent for a connected bipartite graph G :

- i) G is a convex ECE-graph on V ,
- ii) G is a biconvex ECE-graph,
- iii) G is a bi-permutation ECE-graph.

Proof. $(i) \Leftrightarrow ii)$ By Theorem 4.33.

$(ii) \Leftrightarrow iii)$ By Theorem 4.31. $\square$

In a nutshell, we have extend the relations between the subclasses of bipartite graphs given in (4.1) to the subclasses of bipartite ECE-graphs in the following way:

$$\text{ECE-CHAIN} \subset \text{ECE-BIPERMUTATION} \equiv \text{ECE-BICONVEX} \equiv \text{ECE-CONVEXonV} \quad (4.2)$$

5. WELL-INDUMATCHED PSEUDOFORESTS

Since a graph is well-indumatched if and only if each of its components is well-indumatched, it suffices to study connected well-indumatched graphs. Besides, by Lemma 2.24, we only need to investigate reduced well-indumatched graphs. Therefore, we can restrict the study of well-indumatched graphs to connected reduced graphs. Unless otherwise stated, all well-indumatched graphs are assumed to be connected and reduced in the rest of this paper.

The next result proves that the resulting graph obtained by the removal of end-vertices of a cut edge in a well-indumatched graph is well-indumatched.

Lemma 5.1: Let G be a well-indumatched graph and uv be a cut edge of G such that none of u and v is a leaf. Then each component of $G - uv$ is well-indumatched.

Proof. Let G be a well-indumatched graph, $e = u_1u_2$ be a cut edge where none of u_1 and u_2 is a leaf, and let H_1 and H_2 be the components of $G - uv$ that contain u_1 and u_2 , respectively. Since none of u_1 and u_2 is a leaf, there exists an edge $f_i \in E(H_i)$ for $i = 1, 2$ that is incident to u_i . Without loss of generality, we extend the edge f_2 to a maximal induced matching M in G . Since f_2 covers all edges that are incident to u_1 , M does not contain an edge incident to u_1 . Now, we define M_i to be the set of all edges of M in H_i ; that is, $M \cap E(H_1) = M_1$, $M \cap E(H_2) = M_2$ and $M_1 \cup M_2 = M$. Then, since M does not contain any edge incident to u_1 , M_1 is a maximal induced matching of $H_1 - u_1$. Remark that M_1 is an empty set when $H_1 - u_1$ is a null graph. For any two maximal induced matchings M'_1 and M''_1 of $H_1 - u_1$ with different sizes, $M'_1 \cup M_2$ and $M''_1 \cup M_2$ are maximal induced matchings of G with different sizes. Since it contradicts with the well-indumatchability of G , it follows that $H_1 - u_1$ is well-indumatched. By symmetry, $H_2 - u_2$ is also well-indumatched. Therefore, the proof is complete. $\square$

A cut edge is called trivial if it is a pendant edge; otherwise, it is called nontrivial. Considering the resulting graph obtained by the removal of the end-vertices of a nontrivial cut edge, it is easy to observe the following result:

Corollary 5.2: Let G be a well-indumatched graph and c be a cut vertex of G . If there exists at least two nontrivial cut edges of G incident to c , then each component of $G - c$ is a well-indumatched graph.

The converse of Lemma 5.1 and Corollary 5.2 is not necessarily true. For a graph G and cut edge uv of G , G may not be a well-indumatched graph even if each component of $G \setminus \{u, v\}$ is well-indumatched. For instance, the path $P_8 = v_1v_2v_3v_4v_5v_6v_7v_8$ is not well-indumatched, whereas $G \setminus \{v_4, v_5\}$ is the union of two well-indumatched paths, namely $v_1v_2v_3$ and $v_6v_7v_8$, and also $G \setminus \{v_4\}$ is the union of two well-indumatched paths, namely $v_1v_2v_3$ and $v_5v_6v_7v_8$.

The next lemma proves that the graph obtained by the removal of the neighborhood of a cut vertex in a well-indumatched graph is also well-indumatched even if the cut vertex is a stem.

Lemma 5.3: Let G be a well-indumatched graph and c be cut vertex of G . Then each component of $G \setminus N[c]$ is well-indumatched.

Proof. Let G be a well-indumatched graph and c be cut vertex of G . Suppose that $H_1, H_2, \dots, H_k$ are the components of $G - c$, where $k \geq 2$. Let $e_i = v_i c$ with $v_i \in H_i$ being an edge joining the cut vertex c and the component H_i , where $i \in [k]$. For a fixed i , the edge e_i covers all edges incident to the vertices of $N_{H_j}(c)$, where $j \neq i$ and $j \in [k]$. Since the edge e_i cannot cover any other edges in H_j , where $j \neq i$ and $j \in [k]$, by Lemma 2.25, we conclude that each $H_j \setminus N_{H_j}(c)$ is well-indumatched, where $j \neq i$ and $j \in [k]$. Since it works for any fixed $i \in [k]$, we can generalize that each $H_i \setminus N_{H_i}(c)$ is well-indumatched where $i \in [k]$. Noting that each $H_i \setminus N_{H_i}(c)$, where $i \in [k]$, is indeed a component of $G \setminus N[c]$, we complete the proof. $\square$

Notice here that the converse of Lemma 5.3 is not necessarily true. That is, for a graph G and cut vertex c of G , G may not be a well-indumatched graph even if each component of $G \setminus N[c]$ is well-indumatched. For instance, the path $P_9 = v_1v_2v_3v_4v_5v_6v_7v_8v_9$ is not well-indumatched whereas $G \setminus N[v_5]$ is the union of two well-indumatched paths, namely $v_1v_2v_3$ and $v_7v_8v_9$.

In the next result, we prove that there is no well-indumatched graph with a cut edge which is not covered by a pendant edge.

Lemma 5.4: For a well-indumatched graph G , each cut edge of G is covered by a pendant edge of G .

Proof. Let G be a well-indumatched graph and uv be a cut edge in G . We assume to the contrary that uv is covered by none of the pendant edges of G . Suppose that H

and H' are the components of $G - uv$ that contain u and v , respectively. Suppose also that $\mathcal{H} = \{H_1, H_2, \dots, H_k\}$ and $\mathcal{H}' = \{H'_1, H'_2, \dots, H'_{k'}\}$ are the sets of components of $G \setminus \{u, v\}$ that is contained in H and H' , respectively. Notice here that none of $\mathcal{H}$ and $\mathcal{H}'$ is an empty set because otherwise uv is a pendant edge covering itself. For each $i \in [k]$, let $u_i \in N_{H_i}(u)$, and for each $j \in [k']$, let $v_j \in N_{H'_j}(v)$. Note also that none of H_i and H'_j is K_1 because otherwise uv is incident to a pendant edge uu_i or vv_j , respectively. Besides, none of u_i and v_j is a stem in G ; that is, there is no leaf vertex u_i^* and v_j^* being adjacent to u_i and v_j in H_i and H'_j , respectively, because otherwise uv is covered by a pendant edge $u_i u_i^*$ or $v_j v_j^*$, respectively.

Let M_i be a maximal induced matching in H_i saturating u_i for $i \in [k]$, and let M'_j be a maximal induced matching in H'_j saturating v_j for $j \in [k']$. Then, it follows that $M = \bigcup_{i=1}^k M_i \cup \bigcup_{j=1}^{k'} M'_j$ is a maximal induced matching in G . On the other hand, we extend the edge uv to a maximal induced matching M^* in G . It is obvious that M^* does not contain an edge incident to u_i , $i \in [k]$, or v_j , $j \in [k']$. We define M_i^* as the set of all edges of M^* in H_i for $i \in [k]$ and $(M^*)'_j$ as the set of all edges of M^* in H'_j for $j \in [k']$. That is, $M^* \cap E(H_i) = M_i^*$, $M^* \cap E(H'_j) = (M^*)'_j$ and $M = \bigcup_{i=1}^k M_i^* \cup \bigcup_{j=1}^{k'} (M^*)'_j \cup \{uv\}$. Since each u_i and v_j is neither a leaf nor a stem in G , none of M_i^* and $(M^*)'_j$ is an empty set. By Lemma 5.1, each H_i , $i \in [k]$, and H'_j , $j \in [k']$, is well-indumatched. Hence, we have $|M_i| = |M_i^*|$ and $|M'_j| = |(M^*)'_j|$ for $i \in [k]$, $j \in [k']$. It follows that $|M^*| = |M| + 1 > |M|$, that is, there are two maximal induced matchings of G with different sizes, a contradiction. $\square$

5.1. Well-Indumatched Trees

Remark that, by definition, a tree can be reduced to a tree having at most one leaf adjacent to each vertex. Let $\mathcal{T}_1$ be the class of all reduced well-indumatched trees of order less than five. It is easy to observe that $\mathcal{T}_1$ only contains the well-indumatched paths of order at most four, that is, $\mathcal{T}_1 = \{P_1, P_2, P_4\}$. Notice here that P_3 is not a reduced tree. Let $\mathcal{T}_2$ be the class of all reduced well-indumatched trees of order at least 5. All well-indumatched trees in $\mathcal{T}_2$ are characterized in Theorem 2.30. Hence, we obtain the following result:

Lemma 5.5: For each $T_i \in \mathcal{T}_i$, $i \in \{1, 2\}$,

- i) the induced matching number of T_1 is at most 1 and
- ii) the induced matching number of T_2 equals the number of good pendant edges in T_2 .

Proof. Let $T_i \in \mathcal{T}_i$ for $i \in \{1, 2\}$. It is obvious that the induced matching number of P_1 is 0 and the induced matching number of P_2 and P_4 is 1. Then, the induced matching number of T_1 is at most 1. By the characterization in Theorem 2.30, good pendant edges form an induced matching in T_2 . Hence, the result follows. $\square$

We denote the class of all reduced well-indumatched trees by $\mathcal{T}$. It is easy to see that each $T \in \mathcal{T}$ is either a reduce path of order at most four, that is, P_1 , P_2 and P_4 , or a tree characterized in Theorem 2.30. Therefore, $\mathcal{T}$ is disjoint union of $\mathcal{T}_1$ and $\mathcal{T}_2$; that is, $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$.

Let $e = v_1v_2$ be a good pendant edge in a well-indumatched tree $T \in \mathcal{T}_2$, where v_1 is a leaf and v_2 is a stem in T . We define the unit $U_T(e)$ as the subtree of T induced by the endvertices of edges covered by e . By definition of good pendant edge, v_1 has unique vertices, namely v_2 and v_3 , at distances 1 and 2, respectively, from v_1 , while more than one vertex may be at distance 3 from v_1 (see Figure 5.1). Note that the reduction of $U_T(e)$ is P_4 . Besides, a tree is well-indumatched if and only if each of its components are well-indumatched. Since each induced subgraph of T properly containing a unit of T has an induced matching of size at least 2, it follows that units of T are maximal well-indumatched subtrees of T having induced matching number 1.

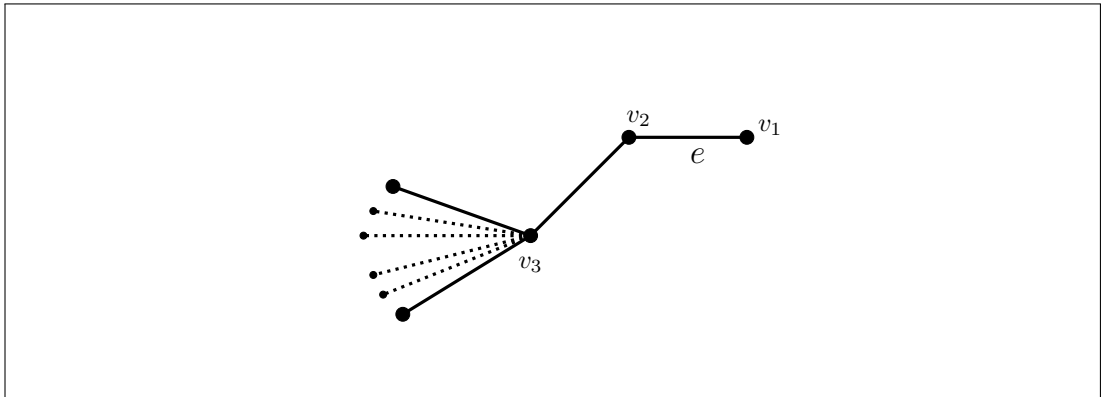


Figure 5.1: Structure of a unit in a well-indumatched tree in $\mathcal{T}_2$

Let $U(T)$ be the set of all units in T . Since each unit in $U(T)$ is associated with a unique good pendant edge in T , there exists a one-to-one correspondence between

$U(T)$ and the set of good pendant edges of T . This implies the following observation:

Observation 5.1: For a well-indumatched tree $T \in \mathcal{T}_2$, the number of units in T equals the induced matching number of T ; that is, $|U(T)| = \nu_i(T)$.

Lemma 5.6: For a well-indumatched tree $T \in \mathcal{T}_2$, every maximal induced matching of T contains exactly one edge from each unit of T .

Proof. Let T be a well-indumatched tree in $\mathcal{T}_2$, e be a good pendant edge in T and $U_T(e)$ be a unit of T . We assume that M is a maximal induced matching containing none of the edges of $U_T(e)$ in T . By the structure of the units, no edge of M covers the edge e . Then, $M \cup \{e\}$ is an induced matching in T , which contradicts with the maximality of M . Furthermore, since units are maximal well-indumatched subtrees with induced matching number 1, any induced matching in T cannot contain two edges of $U_T(e)$. Hence, the proof is complete. $\square$

Specifically, the set of good pendant edges forms a maximal induced matching in T .

Taking into consideration the structure of units in $U(T)$, the vertex set $V(T)$ can be divided into four disjoint subsets $V_1(T)$, $V_2(T)$, $V_3(T)$ and $V_4(T)$ such that $V_1(T)$ is the set of leaf vertices of good pendant edges in T , $V_2(T)$ is the set of stems of good pendant edges in T , $V_3(T)$ is the set of non-leaf neighbors of vertices in $V_2(T)$ and $V_4(T)$ is the set of non-stem neighbors of vertices in $V_3(T)$. For a better understanding, in Figure 5.2, we label the vertices of a well-indumatched tree T in $\mathcal{T}_2$ such as $v_i \in V_i(T)$ where $i \in \{1, 2, 3, 4\}$. By Theorem 2.30, noting that every unit in $U(T)$ contains a unique vertex from each of $V_1(T)$, $V_2(T)$ and $V_3(T)$, we obtain the following observation:

Observation 5.2: For a well-indumatched tree $T \in \mathcal{T}_2$, the sizes of the sets $V_1(T)$, $V_2(T)$ and $V_3(T)$ are equal to the induced matching number of T ; that is, $|V_1(T)| = |V_2(T)| = |V_3(T)| = \nu_i(T)$.

By definition, two units in $U(T)$ may share at most one vertex, namely a vertex of $V_4(T)$. Thus, two units in $U(T)$ are attached to each other at a vertex of $V_4(T)$. Since a unit of T may be attached to a number of units in T , it follows that a unit of T may contain more than one vertex of $V_4(T)$. In other words, it contains at least a vertex of $V_4(T)$. In

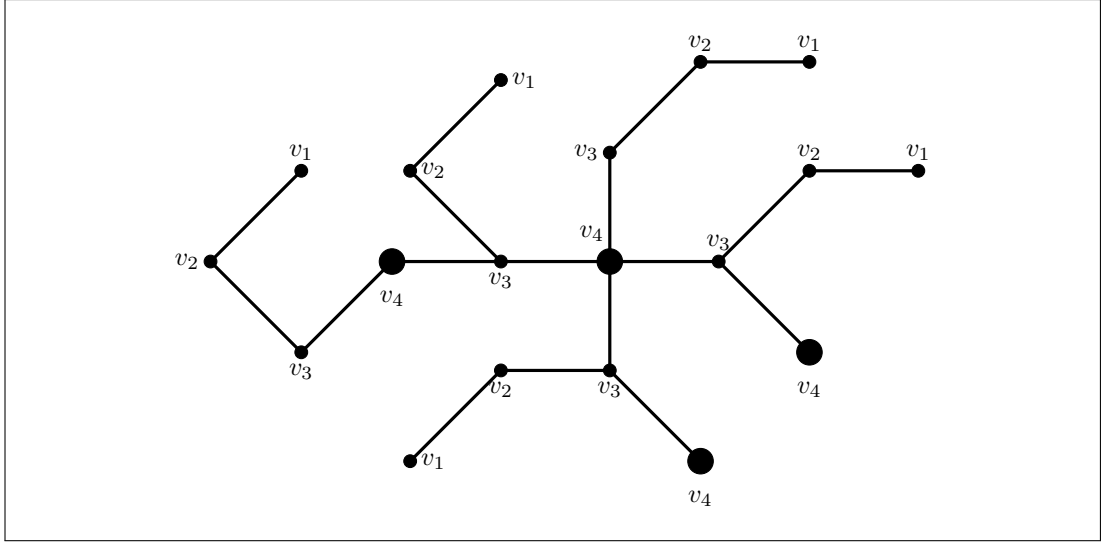


Figure 5.2: Labelling of vertices in a well-indumatched tree in $\mathcal{T}_2$

particular, if a vertex of $V_4(T)$ is uniquely contained in a unit of $U(T)$, then the vertex is a leaf of a non-good pendant edge in T .

By Theorem 2.30, for $T \in \mathcal{T}_2$, any two units in $U(T)$ are either disjoint or attached to each other at a shared vertex called unit-cut vertex; particularly a vertex of $V_4(T)$. Then, it is easy to observe that T decomposes into multiple well-indumatched trees glued at unit-cut vertices of T (see Figure 5.3). Note here that the unit-cut vertex continues to exist in all resulting well-indumatched trees. Then, the number of resulting well-indumatched trees obtained by decomposing T from a fixed unit-cut vertex equals the degree of the unit-cut vertex. Besides, it is easy to see that any well-indumatched tree $T \in \mathcal{T}_2$ with $\nu_i(T) \geq 2$ contains a unit-cut vertex. The following result is consistent with Observation 5.1:

Lemma 5.7: For a well-indumatched tree $T \in \mathcal{T}_2$, the sum of the induced matching numbers of resulting well-indumatched trees obtained by decomposing T from a unit cut vertex equals the induced matching number of T . Particularly, by recursively decomposing T until no unit-cut vertex exists, the set of resulting well-indumatched trees form $U(T)$; that is, each resulting well-indumatched tree is a unit of T .

Proof. Let T be a well-indumatched tree in $\mathcal{T}_2$ and v be a unit-cut vertex of T . Suppose that $d(v) = k$ and $T_1, T_2, \dots, T_k$ are the well-indumatched trees obtained by decomposing T from v . Let v_i be a unit-cut vertex of T_i , where $i \in [k]$. Suppose that $d(v_i) = k_i$ and $T_{i_1}, T_{i_2}, \dots, T_{i_{k_i}}$ are the well-indumatched trees obtained by de-

composing T_i from v_i . We continue this process until no unit-cut vertex exists in any resulting well-indumatched tree. Then, by definition of unit-cut vertex, each resulting well-indumatched tree obtained by decomposing T until no unit-cut vertex exists is a unit of T . Thus, by Observation 5.1, the number of these resulting well-indumatched trees is equal to the induced matching number $\nu_i(T)$ of T .

Without loss of generality, let T decompose from a unit-cut vertex into two well-indumatched trees T^* and T' . By the above argument, the number of resulting well-indumatched trees obtained by decomposing T^* , T' and T until no unit-cut vertex exists is equal to $\nu_i(T^*)$, $\nu_i(T')$ and $\nu_i(T)$, respectively. Note that, by definition of decomposition, T^* and T' are the edge-disjoint well-indumatched subtrees of T . Then, each unit of T^* and T' is also a unit of T , implying that $\nu_i(T^*) + \nu_i(T') \leq \nu_i(T)$. In other respects, by definition again, each unit of T is a unit of either T^* or T' , which implies that $\nu_i(T) \leq \nu_i(T^*) + \nu_i(T')$. Therefore, we conclude that $\nu_i(T^*) + \nu_i(T') = \nu_i(T)$, as desired. $\square$

Therefore, a well-indumatched tree T in $\mathcal{T}_2$ can be decomposed into a well-indumatched tree T^* and a unit U such that the unit-cut vertex continues to exist in both T^* and U , which is called unit reduction. In other words, a well-indumatched tree in $\mathcal{T}_2$ can be reduced to a smaller well-indumatched tree by removing a unit. By Lemma 5.7, unit reduction operation preserves the well-indumatchability; particularly, $\nu_i(T^*) = \nu_i(T) - \nu_i(U) = \nu_i(T) - 1$. More generally, a well-indumatched tree in $\mathcal{T}_2$ decomposes into its units in such a way that every unit-cut vertex of the well-indumatched tree continues to exist in each unit containing it, which is called unit decomposition. Lemma 5.7 guarantees that unit decomposition operation preserves the well-indumatchability and the induced matching number in total.

As a converse operation of decomposition of a well-indumatched tree in $\mathcal{T}_2$, two well-indumatched trees in $\mathcal{T}_2$ can be extended to a larger well-indumatched tree in $\mathcal{T}_2$ as follows: Let T_1 and T_2 be two vertex-disjoint well-indumatched trees in $\mathcal{T}_2$. T_1 and T_2 are joined by reducing two vertices from $V_4(T_1)$ and $V_4(T_2)$ to a single vertex (see Figure 5.3). It can be easily observed that the resulting tree, say T^* , is a well-indumatched tree by transforming all vertices in both $V_i(T_1)$ and $V_i(T_2)$ to $V_i(T^*)$ for $i \in \{1, 2, 3, 4\}$. Note also that the joining vertex of T^* is a unit-cut vertex in T^* . The following result provides the converse of Lemma 5.7:

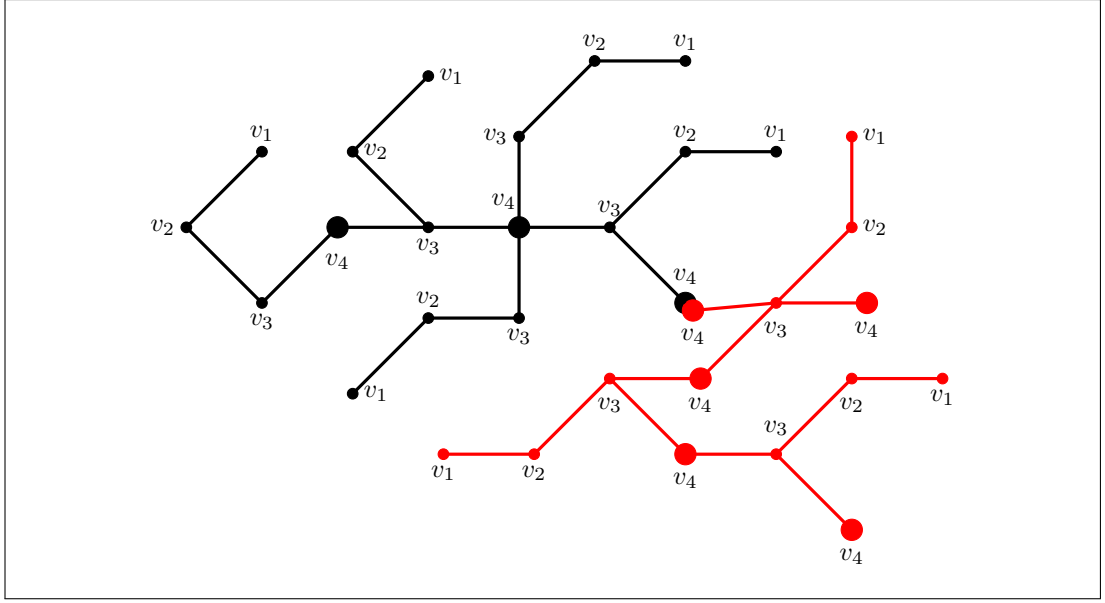


Figure 5.3: Decomposition and extension of a well-indumatched tree in $\mathcal{T}_2$

Lemma 5.8: For any vertex-disjoint well-indumatched trees T_1 and T_2 in $\mathcal{T}_2$, the sum of the induced matching numbers of T_1 and T_2 equals the induced matching number of the resulting well-indumatched tree obtained by extending T_1 and T_2 .

Proof. Let T_1 and T_2 be two vertex-disjoint well-indumatched trees in $\mathcal{T}_2$, and let $v_1 \in V_4(T_1)$ and $v_2 \in V_4(T_2)$. Suppose that T is a well-indumatched tree extended from T_1 and T_2 by reducing two vertices from $V_4(T_1)$ and $V_4(T_2)$ to a single vertex. Then, the well-indumatched tree T decomposes into the well-indumatched trees T_1 and T_2 . By Lemma 5.7, it follows that $\nu_i(T_1) + \nu_i(T_2) = \nu_i(T)$. $\square$

By the same argument in the extension of well-indumatched trees in $\mathcal{T}_2$, for a unit U , a well-indumatched tree in $T \in \mathcal{T}_2$ can be extended to a larger well-indumatched tree T^* by reducing a vertex from $V_4(T)$ and a leaf of U to a single vertex. This operation is called unit extension. Adapting Lemma 5.8, we observe that $\nu_i(T^*) = \nu_i(T) + \nu_i(U) = \nu_i(T) + 1$ and the joining vertex of T^* is a unit-cut vertex in T^* . Hence, it is easy to see that unit decomposition operation preserves the well-indumatchability and the induced matching number in total.

5.2. Well-Indumatched Unicyclic Graphs

A connected graph containing a unique cycle is called 1-tree or unicyclic. As with a tree, a 1-tree is reduced if and only if each of its vertices is adjacent to at most one leaf vertex. Thus, a 1-tree can be reduced by deleting pendant edges being adjacent with the same vertex except for one pendant edge. Throughout this section, we assume that well-indumatched 1-trees are reduced.

Let $\mathcal{G}$ be the class of well-indumatched 1-trees. Considering each vertex in the cycle as a root, a well-indumatched 1-tree in $\mathcal{G}$ can be seen as a union of vertex-disjoint rooted trees such that the roots are joined by edges consecutively. For a graph $G \in \mathcal{G}$, the set of such rooted trees is denoted by $R(G)$, and the maximum size of depths of rooted trees in $R(G)$ is denoted by $m(G)$. Then, the class $\mathcal{G}$ can be divided into two disjoint subclasses, say $\mathcal{G}'$ and $\mathcal{G}^*$, concerning the maximum depth of the rooted trees. For a well-indumatched 1-tree G , $G \in \mathcal{G}'$ if $m(G) \leq 2$ and $G \in \mathcal{G}^*$ if $m(G) > 2$.

Let G be a well-indumatched 1-tree with the unique cycle C and let $v_i u_i$ be the cut edges of G for $i \in [k]$, such that v_i is a vertex of C and the depth of the rooted tree T_{v_i} is greater than two. The graph obtained by removing each edge $v_i u_i$ gives a unique decomposition $D(G)$ of G into trees and a 1-tree. This is a decomposition of G into pseudotrees, i.e., a pseudotree decomposition of G . It is easy to see that for any well-indumatched 1-tree in $\mathcal{G}^*$, its pseudotree decomposition contains at least two components. Noting that the graphs K_1 and K_2 are trivially well-indumatched, the pseudotree decomposition of any well-indumatched 1-tree in $\mathcal{G}'$ does not contain a non-trivial component except the graph itself. Furthermore, trees in $D(G)$ are clustered with respect to their associated vertices in C , which we refer to as *the representation classes* of trees in $D(G)$. That is, all trees in the same representation class are associated with the same vertex of C , and any two trees from different representation classes are associated with different vertices of C .

In the next result, we show that the pseudotree decomposition of any well-indumatched 1-tree in $\mathcal{G}^*$ is the disjoint union of some well-indumatched trees and a well-indumatched 1-tree.

Lemma 5.9: For any well-indumatched 1-tree $G \in \mathcal{G}^*$, the pseudotree decomposition $D(G)$ of G is a decomposition of G into vertex-disjoint well-indumatched pseudotrees,

where only one of which is a well-indumatched 1-tree. Particularly, such a component of $D(G)$ is a well-indumatched 1-tree in $\mathcal{G}'$ and all other components of $D(G)$ are well-indumatched trees in $\mathcal{T}$.

Proof. Let G be a well-indumatched 1-tree in $\mathcal{G}^*$ with the unique cycle C . By definition of $\mathcal{G}^*$, there exist some vertices $v_1, v_2, \dots, v_k$ in C such that the depth of the rooted tree T_{v_i} , where $i \in [k]$, is greater than two. Thus, for a fixed $i \in [k]$, the rooted tree T_{v_i} contains a path of length at least 3 starting with v_i , say the path $v_i u_i x_i y_i$. Then, $v_i u_i$ is a cut edge of G . Let H_i^1 and H_i^2 be the components of $G \setminus v_i u_i$ that contain v_i and u_i , respectively. By definition, u_i is an endvertex of a path of length at least two which does not contain v_i . Then, by Lemma 2.29, the component of $G \setminus v_i u_i$ containing v_i is well-indumatched. Since $v_i \in C$, it is easy to see that v_i is also an endvertex of a path of length at least two which does not contain u_i . By Lemma 2.29, it follows that the component of $G \setminus v_i u_i$ containing u_i is also well-indumatched. Therefore, $G \setminus v_i u_i$ has exactly two components, one of which is a well-indumatched tree and the other is a well-indumatched 1-tree. By definition, the graph obtained by repeatedly removing each edge $v_i u_i$ forms the pseudotree decomposition $D(G)$ of G . By definition of $\mathcal{G}'$, it is clear that all components of $D(G)$ are well-indumatched trees except one which is a well-indumatched 1-tree in $\mathcal{G}'$, which completes the proof. $\square$

Corollary 5.10: Let G be a well-indumatched 1-tree in $\mathcal{G}^*$, and let $H, T_1, T_2, \dots, T_k$ be the components of $D(G)$ such that $H \in \mathcal{G}'$ and $T_1, T_2, \dots, T_k \in \mathcal{T}$. Then each subgraph induced by H and some T_i , where $i \in [k]$, is a well-indumatched 1-tree in $\mathcal{G}^*$.

Corollary 5.11: Let G be a well-indumatched 1-tree in $\mathcal{G}^*$, and let $T_1, T_2, \dots, T_k$ be well-indumatched trees in $D(G)$. Then, there is no edge of G joining a vertex of T_i and a vertex of T_j , and there is a unique edge of G joining a vertex of H and a vertex of T_i , where $i, j \in [k]$ with $i \neq j$.

We first characterize well-indumatched 1-trees in $\mathcal{G}'$, that is, well-indumatched 1-trees whose pseudotree decomposition contains a single component. Let $\mathcal{G}_0, \mathcal{G}_1$ and $\mathcal{G}_2$ be the classes of well-indumatched 1-trees whose maximum depth of rooted trees are zero, one and two, respectively. That is, $\mathcal{G}'$ is disjoint union of $\mathcal{G}_0, \mathcal{G}_1$ and $\mathcal{G}_2$. In the following results, we give explicit descriptions of the well-indumatched 1-trees in the classes $\mathcal{G}_0, \mathcal{G}_1$ and $\mathcal{G}_2$.

Proposition 5.12: Let G be a well-indumatched 1-tree in $\mathcal{G}_0$, in other words, a well-indumatched cycle. Then G is isomorphic to one of the graphs given in Figure 5.4. Particularly, G is C_3 , C_4 , C_5 , C_6 , C_7 , C_8 or C_{11} .

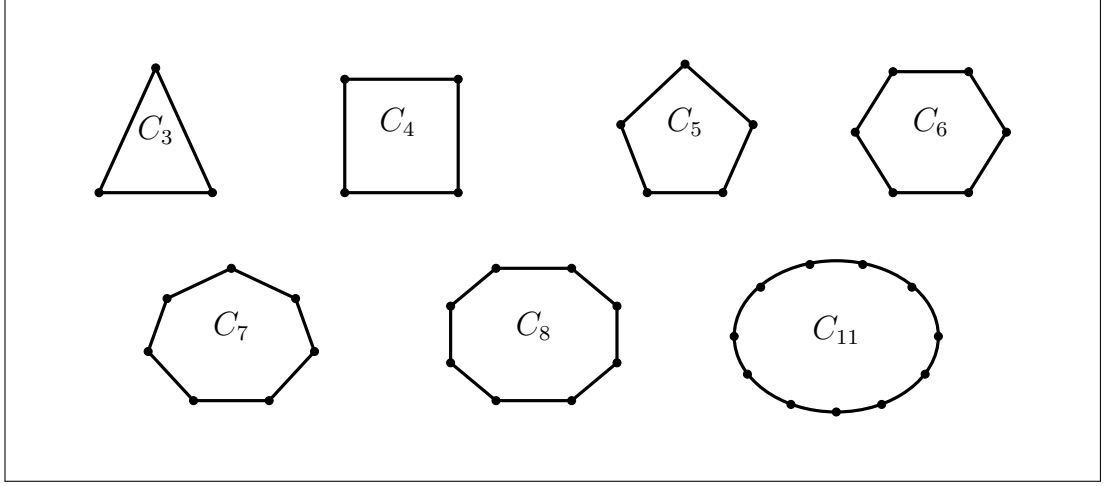


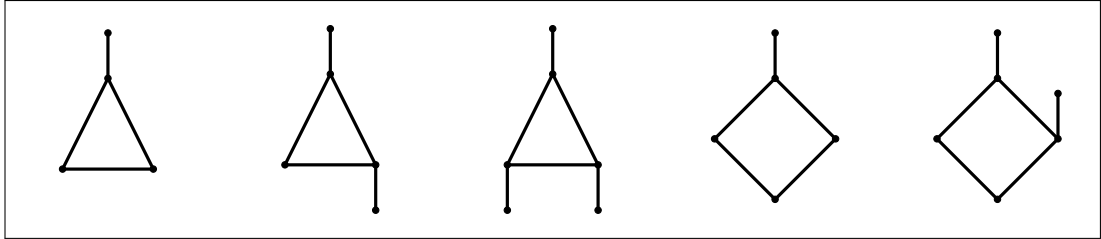
Figure 5.4: Well-indumatched graphs in $\mathcal{G}_0$.

Proof. Let $G \in \mathcal{G}_0$ with the unique cycle C . By definition of $\mathcal{G}_0$, each rooted tree with a root at C is a singleton. It follows that $G \setminus C$ is an empty set; that is, G is the cycle C itself. Hence, by Lemma 2.27, $\mathcal{G}_0 = \{C_3, C_4, C_5, C_6, C_7, C_8, C_{11}\}$ (see Figure 5.4). $\square$

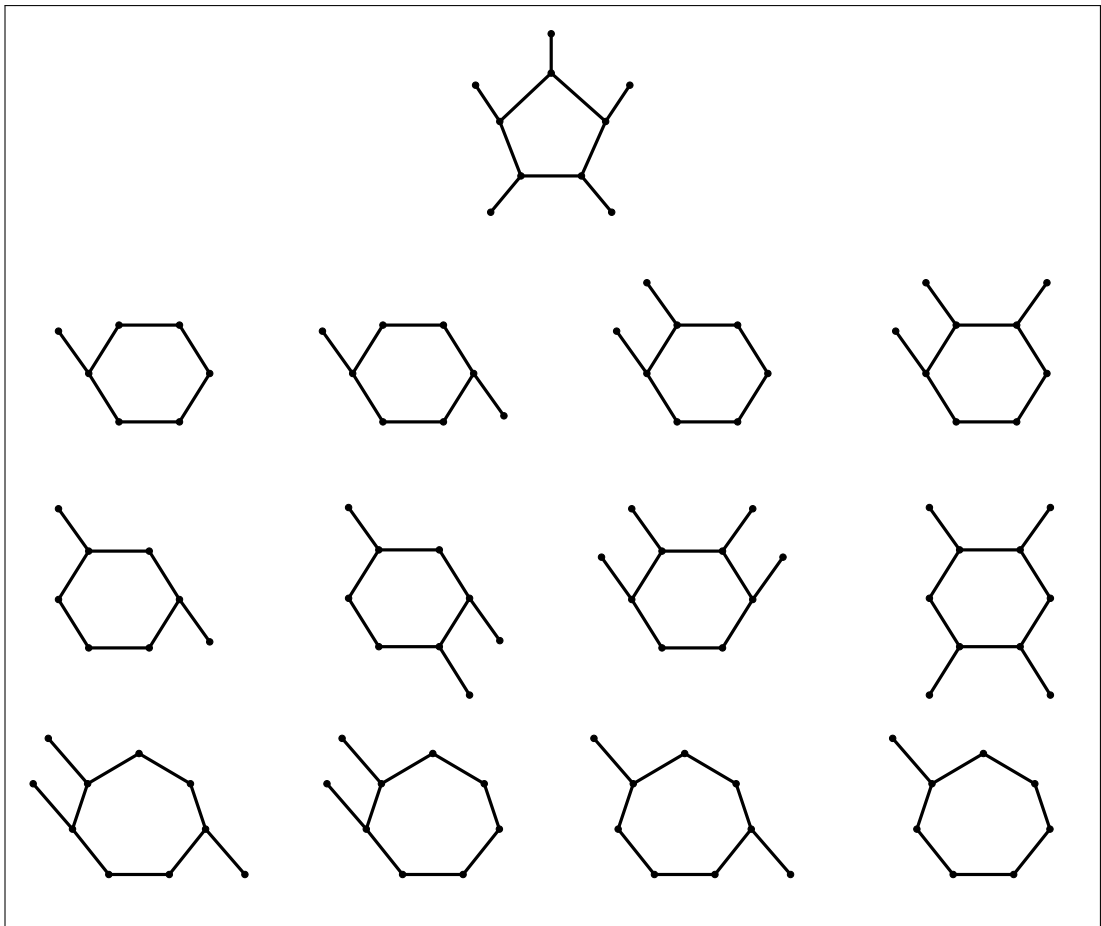
Proposition 5.13: Let G be a well-indumatched 1-tree in $\mathcal{G}_1$. Then G is isomorphic to one of the graphs given in Figure 5.5.

Proof. Let G be a well-indumatched 1-tree in $\mathcal{G}_1$ with the unique cycle C . By definition of $\mathcal{G}_1$, each rooted tree with a root at C is P_1 or P_2 , and there exists at least one rooted tree P_2 . It follows that the degree of each vertex in C is 2 or 3, whereas the degree of each vertex in $G \setminus C$ is 1. Let $v \in C$ with $d(v) = 3$ and $u \in G \setminus C$ be incident to v . Then, uv is a pendant edge of G . Let $F \subset E(G)$ be the set of edges covered by the edge uv and let G^* be the nontrivial connected component of the graph obtained by removing the edges in F .

Suppose that the size of maximal induced matchings in G is 1; that is, $\nu_i(G) = 1$. Then, the edge uv covers all edges in G implying that G is $2K_2$ -free. Hence, it is easy to verify that G is one of the graphs given in Figure 5.5a. Suppose that $\nu_i(G) \geq 2$. Then, C is a cycle of length at least 5. By Lemma 2.25, G^* is a well-indumatched tree



(a) $2K_2$ -free Well-Indumatched Graphs



(b) $3K_2$ -free Well-Indumatched Graphs

Figure 5.5: Well-indumatched unicyclic graphs in $\mathcal{G}_1$.

such that the degree of each of its vertices is at most 3. Hence, by Lemma 2.26, G^* has a longest path $P = v_1 \dots v_k$ where $k \in \{1, 2, 3, 4\}$ or $k = 7$ and $d(v_4) = 2$. In the case where $k \in \{1, 2, 3, 4\}$, we have $\nu_i(G) = 2$ and then G is $3K_2$ -free. Hence, one can verify that G is one of the graphs given in Figure 5.5b. Notice here that these graphs have the unique cycle of length five, six or seven.

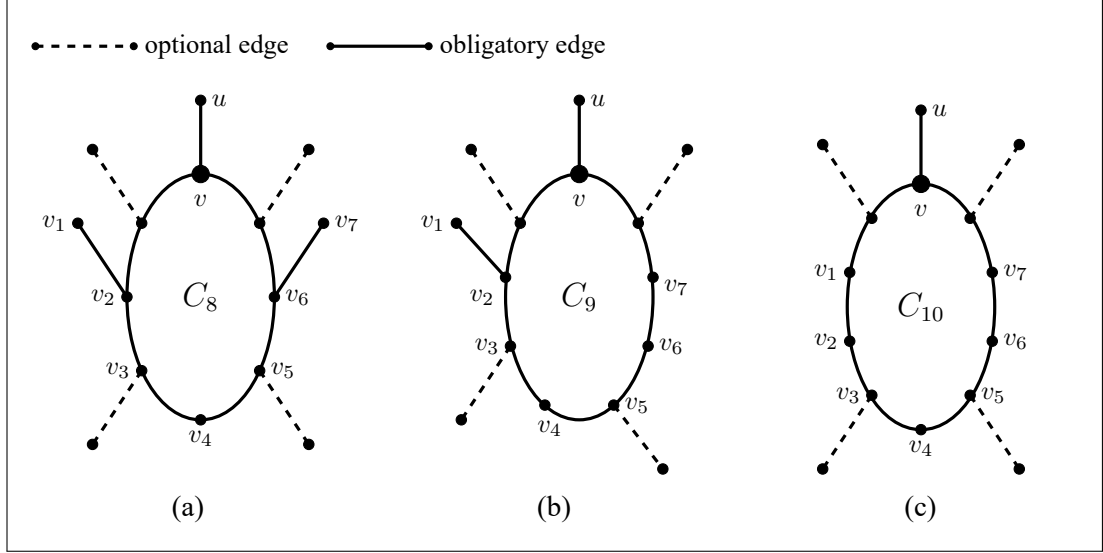


Figure 5.6: Non-well-indumatched graphs with cycles C_8 , C_9 and C_{10} .

We now consider the case where $k = 7$ and $d(v_4) = 2$. Because P_7 is well-indumatched and $\nu_i(P_7) = 2$, G^* is a well-indumatched tree in $\mathcal{T}_2$ such that $\nu_i(G^*) = 2$. By the structure of G^* , the edges v_1v_2 and v_6v_7 are good pendant edges of G^* . Then, we deduce that $V_1(G^*) = \{v_1, v_7\}$, $V_2(G^*) = \{v_2, v_6\}$, $V_3(G^*) = \{v_3, v_5\}$, and $v_4 \in V_4(G^*)$. If both edges v_1v_2 and v_6v_7 are also pendant edges in G , then we obtain the graphs given in Figure 5.6a. If one of the edges v_1v_2 and v_6v_7 is a pendant edge in G , then we obtain the graphs given in Figure 5.6b. If none of the edges v_1v_2 and v_6v_7 is a pendant edge in G , then we obtain the graphs given in Figure 5.6c. Note that these graphs have the unique cycle of length eight, nine, and ten, respectively. However, since all of these graphs have maximal induced matchings of sizes 2 and 3, none of them is well-indumatched. Therefore, the proof is complete. $\square$

Proposition 5.14: Let G be a well-indumatched 1-tree in $\mathcal{G}_2$. Then G is isomorphic to one of the graphs given in Figure 5.7.

Proof. Let G be a well-indumatched 1-tree in $\mathcal{G}_2$ with the unique cycle C . By definition of $\mathcal{G}_2$, each rooted tree with a root at C is P_1 , P_2 , P_3 or P_4 given in Figure 5.8a. Note

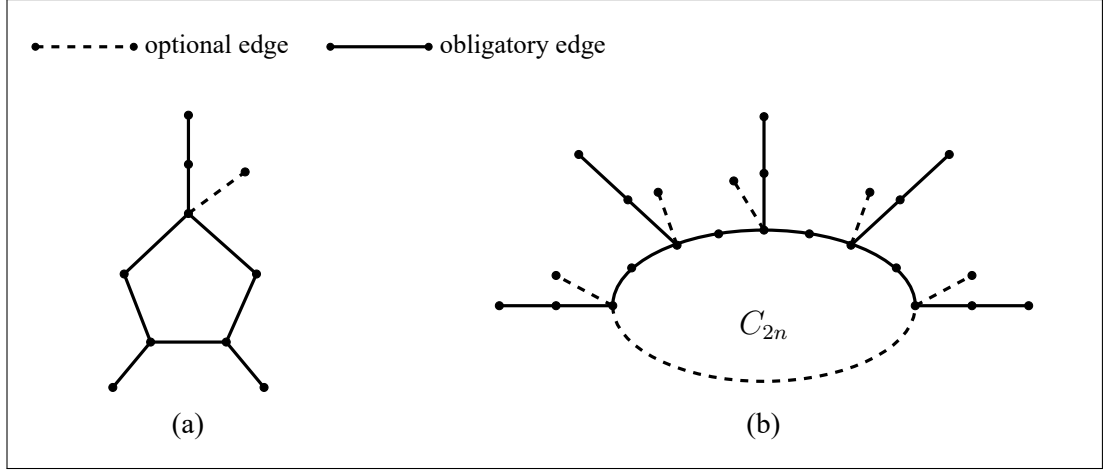


Figure 5.7: Well-indumatched unicyclic graphs in $\mathcal{G}_2$.

here that a rooted tree with a root at C cannot be a P_5 given in Figure 5.8b, because otherwise, by Lemma 2.28, G is not well-indumatched. Let $v \in C$ be such that the depth of the rooted tree T_v is 2. Thus, T_v is P_3 or P_4 . Let u_1 and u_2 be the neighbors of v in C ; that is, $u_1, u_2 \in N_C(v)$. Then, by Lemma 2.28, the rooted trees T_{u_1} and T_{u_2} are P_1 .

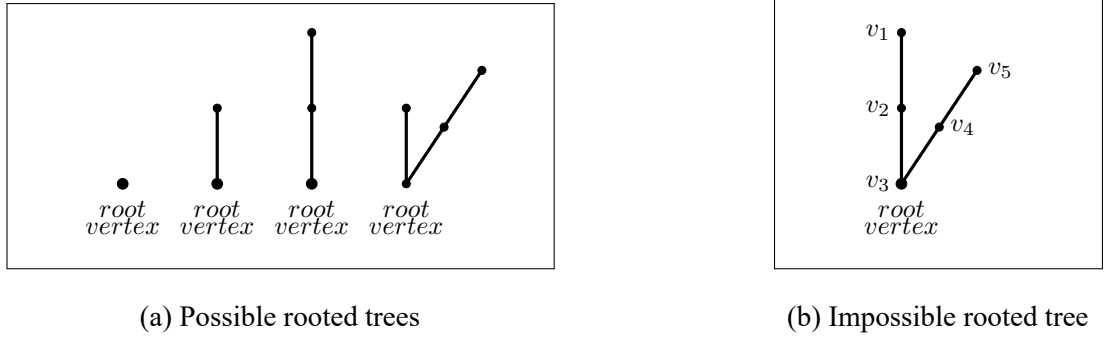


Figure 5.8: The rooted trees in $\mathcal{G}_2$.

The vertex v is a cut vertex of G and $G - v$ has two or three components, where T_v is P_3 or P_4 , respectively. Particularly, $G - v$ has exactly two nontrivial components in both cases, whereas it also has a trivial component in the case T_v is P_4 . Hence, by Corollary 5.2, each component of $G - v$ is a well-indumatched tree. Without loss of generality, let H_1 be the nontrivial component of $G - v$ containing all vertices of $C - v$, H_2 be the other nontrivial component of $G - v$ and H_3 be the trivial component of $G - v$, if exists. Note here that H_2 is a path of length one consisting of an edge e . It follows that the edge e covers all edges incident with the vertex v ; particularly, it covers the edge joining v and H_3 . Since none of the edges in H_1 cover the edge e , it is easy to

observe that we have $\nu_i(G) \geq 2$ and $\nu_i(H_1) = \nu_i(G) - 1$.

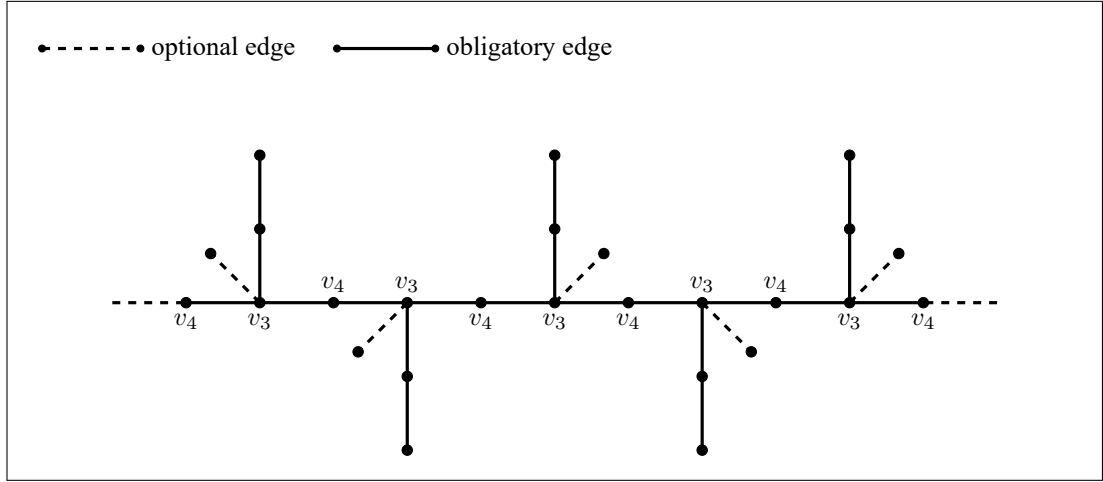


Figure 5.9: Chain of units in C .

If $\nu_i(G) = 2$, then H_1 is a well-indumatched tree with $\nu_i(H_1) = 1$. It follows that H_1 is a well-indumatched tree of order at most 4; that is, there exists a longest path of at most 3. Then, C is a cycle of length three, four, and five. By well-indumatchability of G , it is easy to verify that the length of C is either four or five, but not three. Hence, it is easy to verify that G is either the graph given in Figure 5.7a or the graph given in Figure 5.7b for $n = 2$. Suppose that $\nu_i(G) \geq 3$. Then, H_1 is a well-indumatched tree with $\nu_i(H_1) \geq 2$; that is, H_1 is a well-indumatched tree of order at least 5. By definition, the vertices u_1 and u_2 are leaf vertices of H_1 . Thus, by vertex partition of H_1 , we have $u_1, u_2 \in V_1(H_1) \cup V_4(H_1)$. Assume that u_1 and u_2 are vertices in $V_1(H_1)$. That is, they are leaf vertices of good pendant edges, say e_1 and e_2 , respectively, in H_1 . Then, by Theorem 2.30, the edges covered by e_1 and e_2 cannot be covered by any other good pendant edge of H_1 . On the other hand, since $N_{H_1}(v) = \{u_1, u_2\}$, we observe that $H_1 \setminus \{u_1, u_2\}$ is a component of $G \setminus N[v]$. By Lemma 4.14, it follows that $H_1 \setminus \{u_1, u_2\}$ is also a well-indumatched tree. However, by Theorem 2.30, a graph obtained by the removal of a leaf vertex incident to a good pendant edge of a well-indumatched tree cannot be well-indumatched. Thus, it contradicts with the assumption $u_1, u_2 \in V_1(H_1)$. Hence, we deduce that $u_1, u_2 \in V_4(H_1)$. Let $w_1 \in N_C(u_1)$ and $w_2 \in N_C(u_2)$. Then, by vertex partition of H_1 , $w_1, w_2 \in V_3(H_1)$. It follows that the depth of the rooted trees T_{w_1} and T_{w_2} is 2; that is, each of T_{w_1} and T_{w_2} is P_3 or P_4 . Therefore, by applying the same argument for the vertex v to the vertices w_1 and w_2 and iteratively repeating, we conclude that each vertex of $C - v$ is alternately in either $V_3(H_1)$ or $V_4(H_1)$. Notice

here that the rooted tree with the root at $V_3(H_1)$ is P_3 or P_4 . Hence, H_1 forms a chain of units as given in Figure 5.9. As a result, G is the graph given in Figure 5.7b. The proof is complete. $\square$

As an immediate result of Theorem 5.14, it is easy to obtain the following result given also in [9]:

Corollary 5.15: (Lemma 10 in [9]) For any integer $n \geq 2$, there exists a well-indumatched 1-tree $G \in \mathcal{G}_2$ of girth $g = 2n$, which is of the form given in Figure 5.7b. In particular, if $C = v_1v_2 \dots v_gv_1$ is the unique cycle of G , then the depth of the rooted trees T_{v_i} and $T_{v_{i+1}}$, $i \in [g]$, are alternately zero and two. Furthermore, the size of any maximal induced matching of G is half of g ; that is, $\nu_i(G) = n$.

In the following result, we obtain a complete structural characterization for the graphs in $\mathcal{G}'$, that is, well-indumatched 1-trees whose pseudotree decomposition contains a single component. Recall that if $G \in \mathcal{G}'$, then $m(G) \leq 2$.

Theorem 5.16: A graph G is a well-indumatched 1-tree in $\mathcal{G}'$ if and only if G is isomorphic to one of the graphs in Figures 5.4, 5.5 and 5.7.

Proof. It is easy to verify that all graphs in Figures 5.4, 5.5 and 5.7 are well-indumatched 1-trees of which the maximum depth of rooted trees with a root at the cycle is at most 2. To prove the other direction, we suppose that $G \in \mathcal{G}'$. By definition, the class $\mathcal{G}'$ is divided into three disjoint subclasses, namely $\mathcal{G}_0$, $\mathcal{G}_1$ and $\mathcal{G}_2$. If $G \in \mathcal{G}_0$, $\mathcal{G}_1$, and $\mathcal{G}_2$, then by Propositions 5.12, 5.13 and 5.14, G is one of the graphs given in Figures 5.4, 5.5, and 5.7, respectively. Hence, the proof is complete. $\square$

We are now ready to characterize well-indumatched 1-trees whose pseudotree decomposition contains at least two components; that is, well-indumatched 1-trees in $\mathcal{G}^*$. By Lemma 5.9, the pseudotree decompositions of well-indumatched 1-trees in $\mathcal{G}^*$ is a well-indumatched graph, one of whose components is in $\mathcal{G}'$ and the other components are in $\mathcal{T}$. By Corollary 5.10, the subgraph induced by the vertices of the component in $\mathcal{G}'$ and some components in $\mathcal{T}$ is also a well-indumatched 1-tree in $\mathcal{G}^*$. Recall that $\mathcal{T}$ is the disjoint union of $\mathcal{T}_1$ and $\mathcal{T}_2$, where $\mathcal{T}_1 = \{P_1, P_2, P_4\}$ and the order of each well-indumatched tree in $\mathcal{T}_2$ is at least 5. Furthermore, for a unit U , each well-indumatched tree T in $\mathcal{T}_2$ can be extended to a larger well-indumatched tree with a larger induced

matching number by the unit extension, which reduces two vertices from $V_4(T)$ and a leaf of U to a single vertex. In other side, each well-indumatched tree in $\mathcal{T}_2$ can be reduced to a smaller well-indumatched tree with a smaller induced matching number by the unit reduction in $\mathcal{T}_2$. Note here that the unit extension and the unit reduction increase and decrease induced matching number by one, respectively, and they are inverse operation of each other. Hence, by the minimality, we define the subclass $\mathcal{F}$ of $\mathcal{G}^*$ to be the set of all well-indumatched 1-trees in $\mathcal{G}^*$ having two components in their pseudotree decompositions; particularly, one of them is a well-indumatched 1-tree in $\mathcal{G}'$ and the other is a well-indumatched tree in $\mathcal{T}$ with induced matching number 1. In other words, the well-indumatched 1-trees in $\mathcal{F}$ have only one representative class in their pseudotree decomposition containing only one well-indumatched tree.

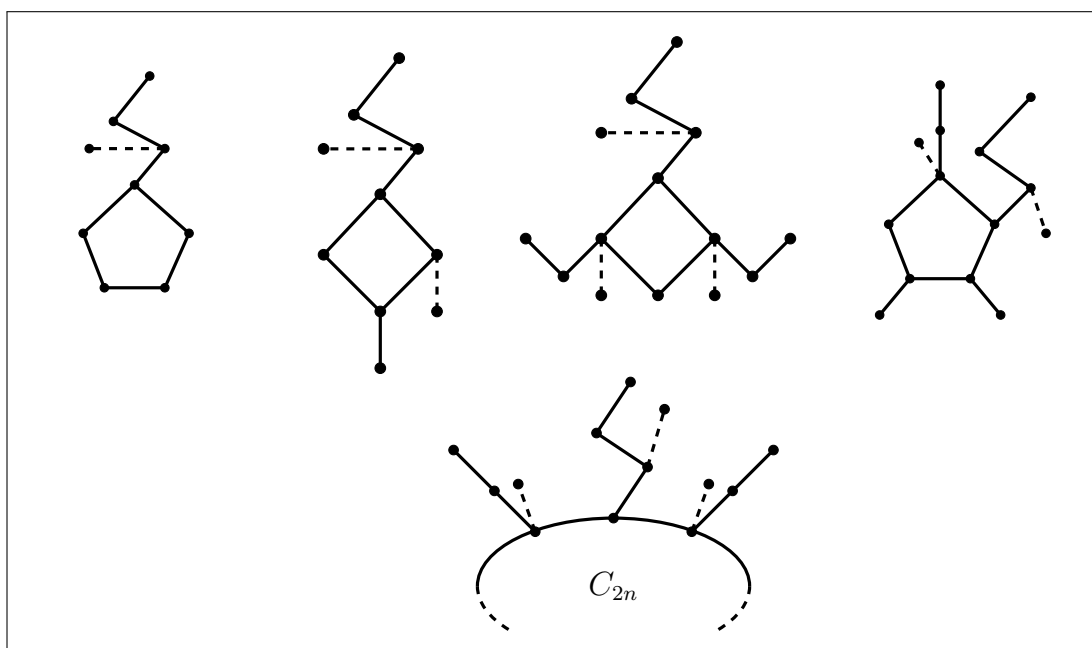
In the next theorem, we provide a complete structural characterization for the graphs in $\mathcal{F}$ as follows:

Theorem 5.17: A graph G is a well-indumatched 1-tree in $\mathcal{F}$ if and only if G is isomorphic to one of the graphs given Figure 5.10.

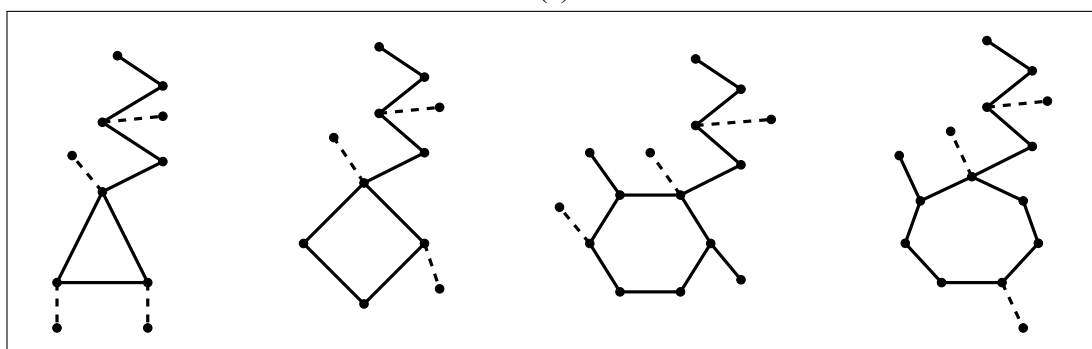
Proof. It can be verified that all graphs in Figure 5.10 are well-indumatched 1-trees in $\mathcal{G}^*$. Besides, none of them has a proper well-indumatched subtree in $\mathcal{G}^*$. To prove the other direction, we suppose that G is a well-indumatched 1-tree in $\mathcal{F}$ with the unique cycle C . Then $D(G)$ has exactly two components such that $H \in \mathcal{G}'$ and $T \in \mathcal{T}$ where $\nu_i(T) = 1$. By definition, it follows that C is a unique cycle in H and there exists a tree T_v rooted at a vertex $v \in C$ such that $T = T_v - v$. By Corollary 5.11, there exists a unique edge $vu \in E(G)$ such that $v \in C$ and $u \in T$. Remark that $\mathcal{T}$ is the disjoint union of $\mathcal{T}_1$ and $\mathcal{T}_2$. If $T \in \mathcal{T}_1$, then by definition, the depth of the rooted tree T_v in G is 3 or 4. By definition, the induced matching number of any well-indumatched tree in $\mathcal{T}_2$ is at least 2. However, by unit decomposition in $\mathcal{T}_2$, T can be supposed to be a unit which is reduced from a well-indumatched tree in $\mathcal{T}_2$. Hence, we may assume that T is a path on four vertices, P_4 . In such a case, the vertices of T are partitioned into four sets $V_1(T)$, $V_2(T)$, $V_3(T)$, and $V_4(T)$; that is, each $V_i(T)$ contains only one vertex, where $i \in \{1, 2, 3, 4\}$. Without loss of generality, let u be a vertex of $V_3(T)$ or $V_4(T)$.

We suppose that either the depth of the rooted tree T_v in G is 3 where $T \in \mathcal{T}_1$ or $u \in V_3(T)$ where T is a unit. In both cases, there exists a path uxy in T such that y is a leaf vertex and x is a stem with degree two in G (see Figure 5.11a). Since $H \in \mathcal{G}'$, the

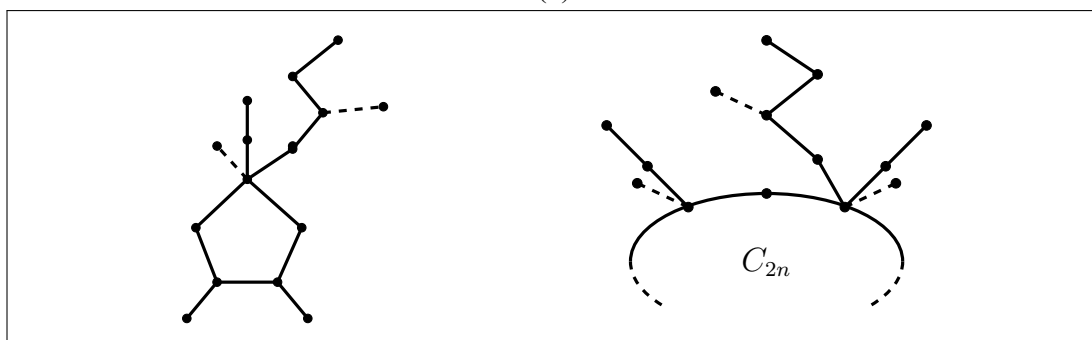
• - - - • optional edge • — • obligatory edge



(a)



(b)



(c)

Figure 5.10: Well-indumatched unicyclic graphs in $\mathcal{F}$

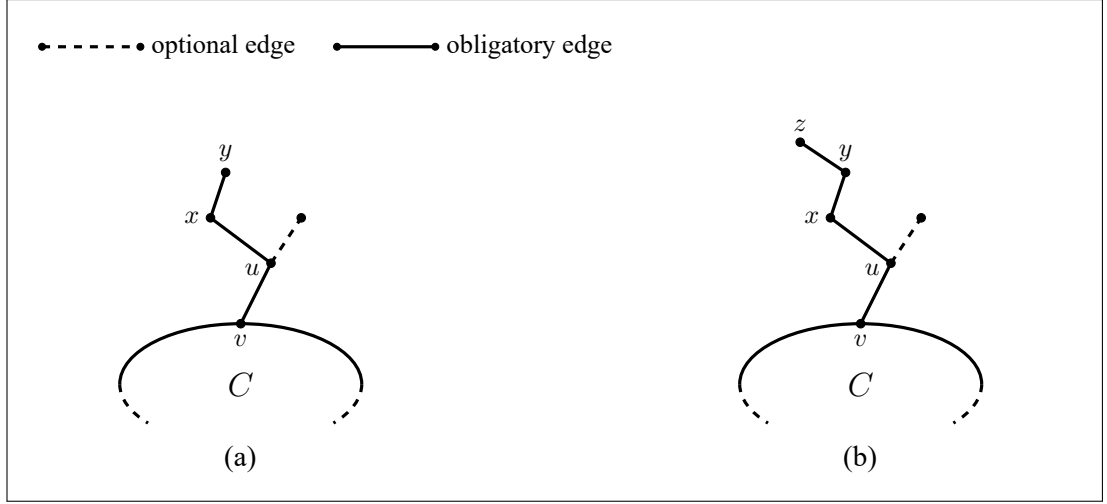


Figure 5.11: The rooted trees in $\mathcal{F}$.

tree rooted at v in H is either P_1 , P_2 , P_3 or P_4 given in Figure 5.8a. Since the degree of x and y in T is one and two, respectively, by Lemma 2.28, the tree rooted at v in H cannot be P_2 , P_3 or P_4 . It follows that $d_H(v) = 2$ and the tree rooted at v in H is P_1 . Furthermore, the edge xy in T cannot be covered by any edge of H , implying that $\nu_i(G) = \nu_i(H) + 1$. By Theorem 5.16, H is isomorphic to one of the graphs in Figures 5.4, 5.5 and 5.7. Notice here that both v and u are cut vertices of G . Then, considering G is a well-indumatched graph with a cut vertex u , by Lemma 4.14, $H - v$ is a well-indumatched tree. Considering G is a well-indumatched graph with a cut vertex v , by Lemma 4.14, $H \setminus N[v]$ is also a well-indumatched tree. Hence, it is easy to see that H is either C_5 from Figure 5.4, C_4 with one or two pendant edges from Figure 5.5a, or any graph from Figure 5.7. As a result, it is easy to verify that G is isomorphic to one of the graphs given in Figure 5.10a.

Now suppose that either the depth of the rooted tree T_v in G is 4 where $T \in \mathcal{T}_1$ or $u \in V_4(T)$ where T is a unit. In both cases, there exists a path $uxyz$ such that z is a leaf vertex, y is a stem with degree two and x is a non-leaf neighbor of y in G (see Figure 5.11b). Then, ux is a cut edge of G such that x is an endvertex of a path of length at least two which does not contain u . By Lemma 2.29, the subgraph H' of G induced by the vertices $V(H) \cup \{u\}$ is also a well-indumatched 1-tree in G' . Besides, we have $\nu_i(G) = \nu_i(H) + 1$ since the edges xy and yz in T cannot be covered by any edge of H . Therefore, by Theorem 5.16, H and H' are independently isomorphic to one of the graphs in Figures 5.4, 5.5 and 5.7. If H' is isomorphic to one of the graphs given

Figure 5.5 and 5.7, then it is easy to verify that G is isomorphic to one of the graphs given in Figure 5.10b and 5.10c, respectively.

In a nutshell, for $G \in \mathcal{F}$, we conclude that in the case where the depth of the rooted tree T_v in G is 3 where $T \in \mathcal{T}_1$ or $u \in V_3(T)$ where T is a unit, G is isomorphic to one of the graphs given in Figure 5.10a, and in the case where the depth of the rooted tree T_v in G is 4 where $T \in \mathcal{T}_1$ or $u \in V_4(T)$ where T is a unit, G is isomorphic to one of the graphs given in Figure 5.10b and 5.10c. This completes the proof. $\square$

As a immediate consequence of Theorems 5.16 and 5.17, it is easy to observe the following extension of the result of Akbari et al. [9] given in Theorem 2.32 by eliminating the girth condition:

Corollary 5.18: The graph G is a minimal well-indumatched graph if and only if G is either $C_3, C_4, C_5, C_6, C_7, C_8, C_{11}$ or isomorphic the graph given in Figure 5.7b. Particularly, for an odd integer $g \geq 9$ and $g \neq 11$, there is no well-indumatched graph of girth g .

Theorems 5.16 and 5.17 also imply the following results, which can be followed by the work [9]:

Corollary 5.19: Let G be a well-indumatched 1-tree in $\mathcal{G}$ with odd girth. Then, the girth of G is 3, 5, 7 or 11. Particularly, there are infinitely many well-indumatched 1-trees with odd girth k , where $k \in \{3, 5, 7\}$, and the cycle C_{11} is the only well-indumatched 1-tree with girth 11.

Corollary 5.20: For every integer $k \geq 2$, there are infinitely many well-indumatched 1-trees with even girth $2k$.

Corollary 5.21: Let G be a well-indumatched 1-tree in $\mathcal{G}$ with girth $g \geq 8$. Then G is bipartite except C_{11} .

By unit extension and unit reduction in $\mathcal{T}_2$, the proof of Theorem 5.17 can be extended to the following results:

Corollary 5.22: For any graph $H \in \mathcal{G}^*$ with two components in $D(H)$, there exists a well-indumatched 1-tree G in $\mathcal{G}^*$ having two components in $D(G)$ such that H is an induced subgraph of G . Particularly, for any graph $H \in \mathcal{F}$, there exists infinitely

many well-indumatched 1-trees in $\mathcal{G}^*$ having two components in their pseudotree decompositions such that H is an induced subgraph of these graphs.

Corollary 5.23: For any graph $G \in \mathcal{G}^*$ such that $G \notin \mathcal{F}$ and G has two components in $D(G)$, there exists a well-indumatched 1-tree H in $\mathcal{G}^*$ with two components in $D(H)$ such that H is an induced subgraph of G . More specifically, $H \in \mathcal{F}$.

Furthermore, by Lemma 5.9 and Corollary 5.10, each well-indumatched 1-tree in $\mathcal{G}^*$ can be extended to a larger well-indumatched 1-tree in $\mathcal{G}^*$ by adding a new well-indumatched tree to its pseudotree decomposition. It is said to be component extension in $\mathcal{G}^*$. Conversely, each well-indumatched 1-tree in $\mathcal{G}^*$ can be reduced to a smaller well-indumatched 1-tree by removal of a well-indumatched tree from its pseudotree decomposition. It is said to be *component reduction* in $\mathcal{G}^*$. Note here that the component extension and the component reduction are inverse operations of each other. Hence, it is easy to observe the following results:

Observation 5.3: For any graph $H \in \mathcal{G}^*$, there exists a well-indumatched 1-tree G in $\mathcal{G}^*$ having more components in $D(G)$ than $D(H)$ such that H is an induced subgraph of G . Particularly, for any graph $H \in \mathcal{F}$, there exists infinitely many well-indumatched 1-trees in $\mathcal{G}^*$ having more than two components in their pseudotree decompositions such that H is an induced subgraph of these graphs.

Observation 5.4: For any graph $G \in \mathcal{G}^*$ with more than two components in $D(G)$, there exists a well-indumatched 1-tree H in $\mathcal{G}^*$ with less components in $D(H)$ than $D(G)$ such that H is an induced subgraph of G . Particularly, for any graph $G \in \mathcal{G}^*$ with more than two components in $D(G)$, there exists a well-indumatched 1-tree H in $\mathcal{F}$ such that H is an induced subgraph of G .

Therefore, under the component extension/reduction operation in $\mathcal{G}^*$ and the unit extension/reduction operation in $\mathcal{T}_2$, each well-indumatched 1-tree in $\mathcal{G}^*$ can be obtained from a graph in $\mathcal{F}$ by some component extensions and/or unit extensions, and each well-indumatched 1-tree in $\mathcal{F}$ can be obtained from a graph in $\mathcal{G}^*$ by some component reductions and/or unit reductions. In the next theorem, we show that the class $\mathcal{F}$ can be seen as a basis set for the class $\mathcal{G}^*$.

Lemma 5.24: The well-indumatched 1-trees in $\mathcal{F}$ form a set of generators for well-indumatched 1-trees in $\mathcal{G}^*$ under the component extension operation in $\mathcal{G}^*$ and the unit extension operation in $\mathcal{T}_2$.

Proof. Let G be a well-indumatched 1-tree in $\mathcal{G}^*$ with the unique cycle C . Let $H \in \mathcal{G}'$ and $T_1, T_2, \dots, T_k \in \mathcal{T}$ be the components of $D(G)$. Without loss of generality, by the component reduction, we remove all well-indumatched tree components except T_1 . By Corollary 5.10, the subgraph H^* of G induced by H and T_1 is a well-indumatched 1-tree in $\mathcal{G}^*$. By Corollary 5.11, there exists a unique edge $vu \in E(G)$ such that $v \in C$ and $u \in T_1$. If $T_1 \in \mathcal{T}_1$, then by definition of $\mathcal{F}$, we trivially have $H^* \in \mathcal{F}$. If $T_1 \in \mathcal{T}_2$, then by unit reduction, we remove all unit in T except the one containing the vertex u . Hence, by Observation 5.23, $H^* \in \mathcal{F}$ and the proof is complete. $\square$

We define the subclass $\mathcal{H}$ of $\mathcal{G}^*$ to be the set of all well-indumatched 1-trees in $\mathcal{G}^*$ having maximum number of representation classes in their pseudotree decompositions. The next theorem provides a complete structural characterization of well-indumatched 1-trees in $\mathcal{H}$ as follows:

Theorem 5.25: A graph G is a well-indumatched 1-tree in $\mathcal{H}$ if and only if G is isomorphic to one of the graphs given Figures 5.12, 5.13 and 5.14.

Proof. It can be verified that all graphs in Figures 5.12, 5.13 and 5.14 are well-indumatched 1-trees. For the other direction, we suppose that G is a well-indumatched 1-tree in $\mathcal{H}$ with the unique cycle C . By definition of $\mathcal{H}$, G has the maximum number of representation classes in $D(G)$. Let $H \in \mathcal{G}'$ and $T_1, T_2, \dots, T_k \in \mathcal{T}$ be the components of $D(G)$. By Corollary 5.10, each subgraph of G induced by H and T_i , where $i \in [k]$, is a well-indumatched 1-tree in $\mathcal{G}^*$. By Lemma 5.24, it follows that each such subgraph is a graph in $\mathcal{F}$ with cycle C . Hence, by Theorem 5.17, each of such induced subgraph is isomorphic to one of the graphs in Figure 5.10. By taking well-indumatched unions of graphs in Figure 5.10a, 5.10b and 5.10c with the common cycle C , it is easy to verify that G is isomorphic to either one of the graphs given Figure 5.12, 5.13 and 5.14, respectively. This completes the proof. $\square$

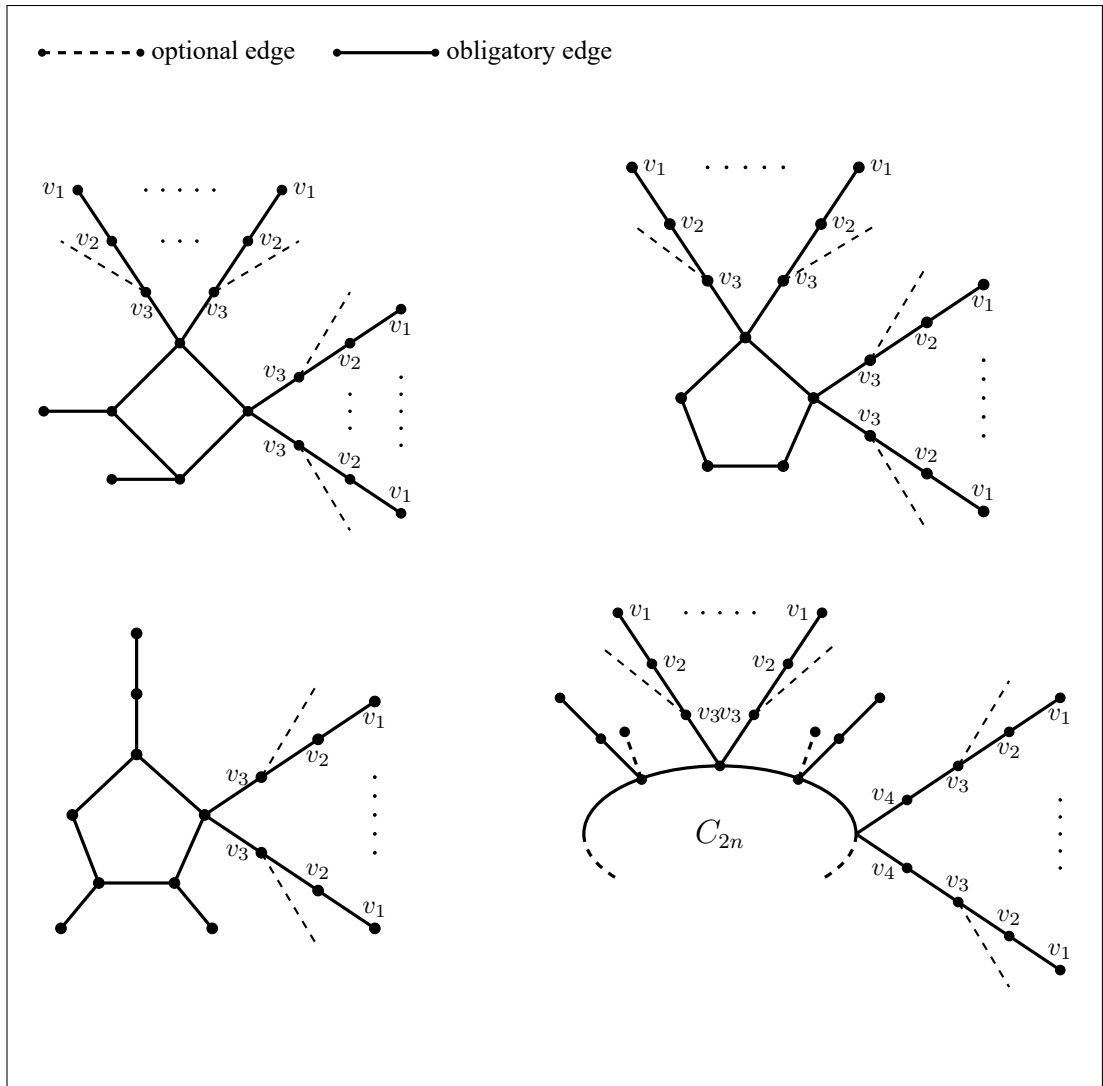


Figure 5.12: Well-indumatched unicyclic graphs in $\mathcal{H}$ obtained by the graphs in Figure 5.10a.

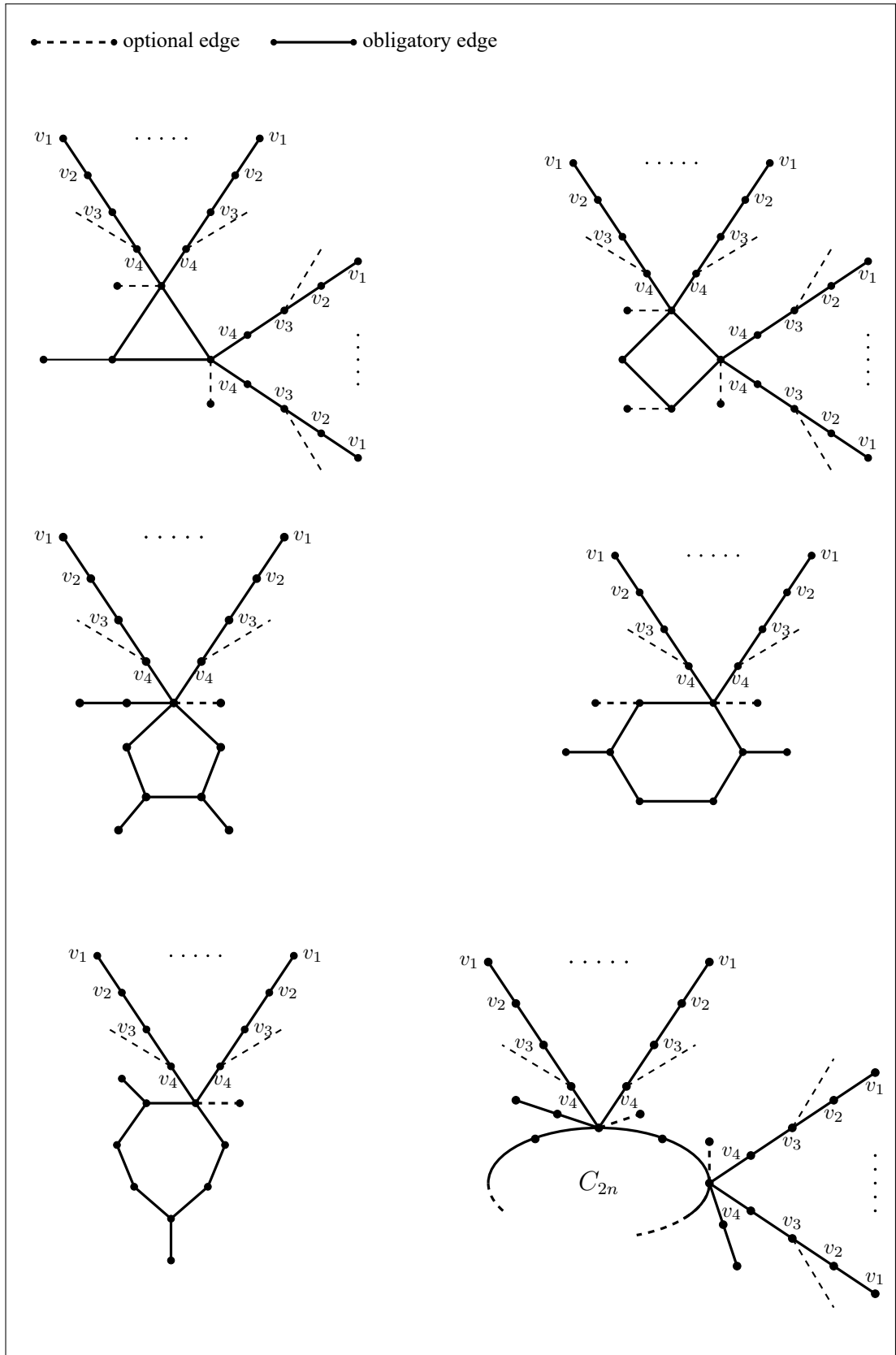


Figure 5.13: Well-indumatched unicyclic graphs in $\mathcal{H}$ obtained by the graphs in Figure 5.10b.

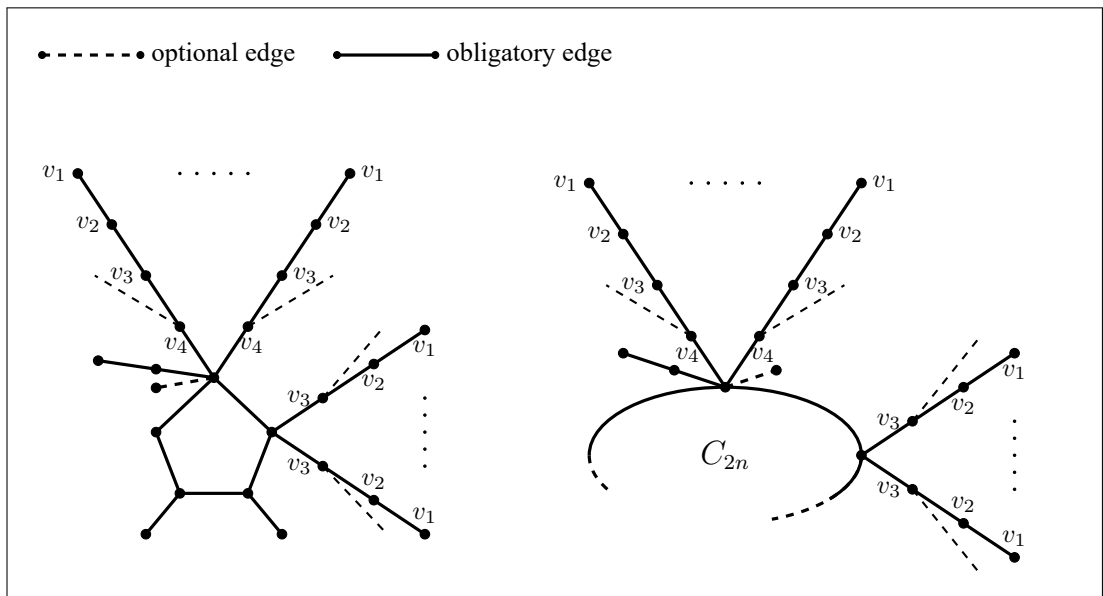


Figure 5.14: Well-indumatched unicyclic graphs in $\mathcal{H}$ obtained by the graphs in Figure 5.10a.

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BIOGRAPHY

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APPENDICES

APPENDIX A: Publications Based on This Thesis

Büyükçolak Y., Gözüpek D., Özkan S., (2022), “Triangle-Free Equimatchable Graphs”, Journal of Graph Theory, Vol. 99-3, 461–484.

Büyükçolak Y., Gözüpek D., Özkan S., (2020), “Equimatchable Bipartite Graphs”, accepted to *Discussiones Mathematicae Graph Theory*.

Büyükçolak Y., Gözüpek D., Özkan S., “Well-Indumatched Pseudoforests”, in preparation.