

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL

**MOTION OF MULTIPLE RIGID BODIES IN CHANNEL FLOW:
IMPACT PHENOMENA**

YASİN SEFA ASLAN

A THESIS OF MASTER OF SCIENCE
DEPARTMENT OF MATHEMATICS

ADVISOR: ASST. PROF. SAMİRE YAZAR

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ABSTRACT

The interaction between fluid and a single body or multiple bodies in a thin straight channel investigated both analytically and numerically. The interaction between free moving bodies (or particles) and the fluid surrounding them is studied. It is motivated by applications in different branches of industry, biomedicine, the environment and science such as flying droplets, ice growth, dust, impacts, food grains, sport, complexity and storms. A two-way nonlinear interaction takes place between the fluid and the body in two dimensional unsteady or quasi-steady flow. The study consists of the following stages; In the first stage, the nonlinear interaction between a small body with a certain thickness and/or camber and the fluid is modelled. The characterisation of this model will be described. Significant dependences on initial conditions and on the comparative body masses and moments of inertia are found for the occurrence of body-body impacts as opposed to wall-body impacts and for the associated impact times. Then, a possible impact between the body and one of the channel walls is discussed in detail and some analytical approach is presented. Finally, the movement of multiple bodies in a quasi-steady flow setting is analysed.

This research is expected to provide improvements in the analysis and simulation studies of some problems in the industrial field and contribute to innovative approaches in the related field. Particle separation systems that provide the separation of foreign particles that can damage aircraft engines and food separation in the field of food technologies are among the main application areas of the model. The study includes numerical results as well as analytical studies.

Keywords: Fluid-Body Interaction, Impact Phenomena, Particle Movement, Body-Body Interaction.

ÖZET

Bu çalışmada ince düz bir kanalda akışkan ile tek bir cisim veya bir kaç cisim arasındaki etkileşim, hem analitik hem de nümerik olarak incelenmiştir. Serbest hareket eden cisimler (veya parçacıklar) ile onları çevreleyen akışkan arasındaki etkileşim incelenmektedir. Bu araştırma, endüstrinin farklı dallarında, biyomedikal alanda, çevre bilimlerinde ve diğer bilimsel alanlardaki uygulamalardan esinlenmiştir. Bu uygulamalar arasında uçan damlacıklar, buz oluşumu, toz çarpışmaları, gıda taneleri, spor, karmaşıklık ve fırtınalar gibi olgular bulunmaktadır. İki boyutlu kararsız veya yarı-kararlı akışta, akışkan ile cisim arasında iki yönlü doğrusal olmayan bir etkileşim meydana gelmektedir. Tez aşağıdaki aşamalardan oluşmaktadır; ilk aşamada, belirli bir kalınlığa ve/veya kamberliğe sahip küçük bir cisim ile akışkan arasındaki doğrusal olmayan etkileşim modellenmiştir. Bu modelin karakterizasyonu açıklanacaktır. Başlangıç koşullarına, gövde kütlelerinin ve atalet momentlerinin karşılaştırılmasına bağlı olarak, gövde-gövde çarpışmalarının duvar-gövde çarpışmalarına göre meydana gelmesi ve buna bağlı çarpışma sürelerinde önemli ilişkiler tespit edilmiştir. Daha sonra, cisim ile kanal duvarlarından biri arasındaki olası bir etki ayrıntılı olarak tartışılacak ve bazı analitik yaklaşımlar sunulacaktır. Son olarak, yarı kararlı akış düzeneğindeki bir kaç tanecik hareketi incelenmiştir.

Bu araştırmanın endüstriyel alandaki bazı problemlerin analiz ve simülasyon çalışmalarında iyileştirmeler sağlaması ve ilgili alanda yenilikçi yaklaşımlara katkıda bulunması beklenmektedir. Uçak motorlarına zarar verebilecek yabancı partiküllerin ayrıştırılmasını sağlayan partikül ayırma sistemleri ve gıda teknolojileri alanında gıda ayrıştırma, modelin başlıca uygulama alanları arasında yer almaktadır. Çalışma, nümerik sonuçların yanı sıra analitik çalışmaları da içermektedir.

Anahtar Kelimeler: Akışkan-Cisim Etkileşimi, Çarpışma Olayı, Parçacık Hareketi, Cisim-Cisim Etkileşimi.

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LIST of ABBREVIATIONS and ACRONYMS

$GTÜ$	:	Gebze Teknik Üniversitesi
NSE	:	Navier Stokes Equation
Re	:	Reynold Number
Fr	:	Froude Number
CoM	:	Center of mass
σ	:	Horizontal position center of mass
h	:	Vertical position center of mass
$\dot{h}$	:	First derivative of h
$\ddot{h}$	:	Second derivative of h
ρ	:	Density
μ	:	Viscosity
u	:	Velocity
p	:	Pressure
g	:	Gravity
∇	:	Gradient operator
$\sin(x)$	:	Sinus function
$O(t)$	:	Big O notation

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1. INTRODUCTION

This study is concerned with the interaction effects associated with single body or two bodies moving freely in fluid flow contained a straight, horizontal thin channel which can bifurcate later on. The motivations here are industrial, biomedical, sports among many others. In industry the principal fields of application include firing bullet-like objects in defense contexts, predicting the impact of a loose piece or ice fragment entering the inlet of an aircraft engine and potentially striking the engine walls causing damage, and studying the movement of wind-blown ice particles along wing surfaces in the context of aerodynamic safety. Additionally, the descent of rice grains through a channel can be demonstrated [1–3].

Biomedical applications are many, including the movement of solids or drugs within the arteries, the transportation of blood clots, and embolization techniques for treating strokes. The study also has potential applications in refining a drug-transport method to detect structural weaknesses in arterial walls or other enclosures. The body's dynamic movement within an artery can lead to shape alterations in a weakened arterial segment, due to its higher elasticity, which can be identified through clinical scans. Moreover, practical interests extend across various sectors including industry, biomedical science, environmental studies, and engineering disciplines, particularly emphasizing the medical implications concerning flow obstruction and the onset of diseases associated with constrictions and branching phenomena [4–6].

Numerous studies have been conducted on fluid-body interaction [7–14]. Our primary focus is to model on inviscid flow in a straight thin channel with their analytical and numerical solutions. The bodies in question are slender and can move freely within the horizontal and straight channel.

The study in [7] focuses on the interactions between a finite number of infinitesimally thin moving bodies or grains and the surrounding fluid within a straight-walled channel. The grains are aligned straight and move freely in a nearly parallel manner in a quasi-inviscid fluid, with the overall motion assumed to be planar. The impact phenomena was investigated when body or grain is flat, i.e. without thickness or camber. Analytical and numerical findings suggest the impact location is at the leading edge of

the grain. The research was carried out by both numerical and analytical investigations as well as comparisons between those two.

Smith and Wilson [8] conducted a study investigating the interaction between a single body with thickness or camber and the flow within a channel with two parallel straight walls in [7]. In [8], the impact phenomena between solids and solids (body and channel wall) is also examined. Here, impact occurs near mid-body station. This research provides a theoretical analysis of fluid-body interaction, highlighting both fluid and body motion and interaction between these two. In this context, clashing describes the solid-solid impacts that occur due to fluid-body interaction, while skimming denotes the oblique impact of a thin body on a fluid surface, leading to subsequent bounces in both regions. The principal novel contribution lies in examining the effects of thickness and camber, which produce a distinctive overall pattern of collisions and rebounds.

Smith and Wilson [15] studied delves into the circumstances determining whether a body, initially or rocking on a solid horizontal plane, is either displaced by hydrodynamic forces or continues to rocking. The nonlinear framework analyzed here encompasses the dynamic interaction between the unsteady motion of an inviscid fluid and a slender mobile body. Different body shapes are studied under various initial conditions. The parameter space controlling these dynamics is found to be complex, with the evolving shape and motion both playing significant roles, influenced by mass scaling and gravitational forces. Notably, the body's lift-off from the surface generally requires fluid flow, but once initiated, it can occur either instantly or over a finite period. This is followed by [11] which can concentrated on a single body motion initially lying on a horizontal surface. The early behaviour of body and lift off criterion as well as large time analysis was carried out. Interesting properties have been found there

The thesis may also contribute to addresses collision dynamics, encompassing bouncing and skimming phenomena, such as shallow-water skipping observed in fluid-body or fluid-fluid impacts, detailed in [16–18]. Additionally, while most research in this field has historically concentrated on two dimensions, a study [18] has extended the analysis to encompass three dimensions.

The execution of this thesis will consist of five main stages. Firstly, we give some preliminaries in order to introduce fluid-body interaction. In the second stage, the

interaction model between the small solid body having a certain thickness and or camber and the nonlinear fluid-body interaction will be described in detail in unsteady flow. This will be followed by numerical solutions which will be obtained for this nonlinear behaviour of the body motion. Then, the impact model that may occur between the body and channel walls will be discussed in detail. In the last stage, multiple bodies interacting each other and with surrounding fluid is examined in a quasi-steady flow setting. The models will be examined with numerical approaches performing in a programming language, e.g. MATLAB.

2. PRELIMINARIES

2.1. Flow Types

In this section, some basic definitions and theorems for various types of fluids will be given and their properties will be introduced.

Definition 2.1: [19] *A steady fluid flow is defined as a state in which the conditions of velocity, pressure, and cross-sectional area remain constant throughout the flow. Conversely, an unsteady fluid flow is characterized by a change in these properties over time. A quasi-steady flow is a flow in which the variables change slowly with time.*

Definition 2.2: [19] *Laminar flow is characterized by smooth fluid motion where velocity, pressure, and cross-sectional area remain constant at a given observation point and time. In this flow type, all streamlines run parallel to each other, typically occurring when the flow velocity is relatively low and the viscosity is high. On the other hand, turbulent flow occurs when the inertial forces of the fluid are dominant, usually resulting in higher velocities and speeds at any given point and time.*

To assess whether fluid flow is laminar or turbulent, the dimensionless Reynolds number is employed.

The Reynolds number, a measure of the ratio of inertial to viscous forces, is expressed as:

$$Re = \frac{\rho \cdot u \cdot l}{\mu}$$

where ρ is the fluid's density μ is the fluid's viscosity and u is the fluid's velocity.

The classification of flow types based on the Reynolds number is as follows:

- If Re is less than 2000, the flow is laminar.
- If Re is greater than 4000, the flow is turbulent.
- If Re falls between 2000 and 4000, the flow is in transition

Definition 2.3: [19] *In uniform fluid flow, the speed of the fluid at any given moment remains constant at any location and has the same value in all directions, with the flow depth remaining unchanged with distance. This type of flow is referred to as uniform fluid flow. Conversely, in non-uniform fluid flow, the velocity of the fluid varies at every*

moment and at any given point in time. Fluid parameters such as pressure, velocity, and speed are in a state of continuous fluctuation.

Definition 2.4: [19] In compressible fluid flow, the fluid's density may vary over time due to the influence of pressure or shear forces acting upon it. Most of the fluids in nature at normal room temperature are incompressible, it is that property of fluid where the density of the fluid does not change upon shear forces acting on it. Gasses can be compressed, whereas liquids cant be generally compressed.

Definition 2.5: [19] A fluid where the flow velocity is directly proportional to the shear stress is known as a Newtonian fluid. The most common examples of Newtonian fluids include water, oils, gasoline, air, whereas blood and liquid plastics are categorized as Non-Newtonian fluids.

Definition 2.6: [19] Irrotational flow is a flow in which no component of the moving fluid rotates in any direction from one instant to the next. Flow with zero viscosity is inviscid flow.

Definition 2.7: [19] One-dimensional fluid flow: When the fluid properties such as velocity, density and pressure vary in one direction.

Two-dimensional fluid flow: When the fluid properties such as velocity, density and pressure vary in two directions.

Definition 2.8: [19] Kutta Condition in fluid mechanics theory states that when a body has a sharp corner or a curved edge, the pressures on the upper and lower surfaces at the trailing edge of the body are equal.

2.2. Nondimensionalization

Definition 2.9: [20] A dimension refers to the measure of a physical quantity, such as length, time, or mass. A unit is the assignment of a numerical value to a dimension, for example, (m) for meters, (sec) for seconds, and (kg) for kilograms. All secondary dimensions can be created by combining the seven fundamental dimensions listed in Table 2.1.

Examples;

$$Velocities = \frac{Lentgh}{Time} = [l/t]$$

Table 2.1: Seven Primary Dimensions.

1. Mass	kilogram (kg)	m
2. Length	meter (m)	l
3. Time	second (sec)	t
4. Temperature	kelvin (K)	T
5. Current	Ampere (A)	I
6. Amount of Light	Candela	C
7. Amount of matter	mole (mol)	N

$$Force = \frac{Mass \cdot Length}{Time^2} = [ml/t^2]$$

Definition 2.10: [20] The law of dimensional homogeneity (DH) states that every additive term in an equation must possess the same dimension.

Example 2.1: Bernoulli Equation;

$$p^* + \frac{1}{2} \rho^* u^{*2} + \rho^* g^* z^* = c^* \quad (2.1)$$

$$\begin{aligned} p^* &= \frac{force}{area} = mass \cdot \frac{length}{time^2} \cdot \frac{1}{length^2} = [m/t^2l] \\ \frac{1}{2} \rho^* u^{*2} &= \frac{mass}{length^3} \cdot \left(\frac{length}{time} \right)^2 = [m/t^2l] \\ \rho^* g^* z^* &= \frac{mass}{length^3} \cdot \frac{length}{time^2} \cdot length = [m/t^2l] \end{aligned}$$

The symbols used in the expression represent the following quantities: u denotes the velocity of fluid flow at a specific point, g stands for the acceleration due to gravity, ρ represents the density of the fluid. p signifies the pressure at any point, z indicates the elevation, and c represents a constant value.

Definition 2.11: [19] By applying the law of dimensional homogeneity, when we divide every term in the equation by a set of variables and constants sharing the same dimensions, the equation becomes dimensionless. During the process of dimensionless conversion, nondimensional parameters frequently emerge, such as the Reynolds number and Froude number.

Example 2.2: To illustrate dimensionlessness, we will consider to nondimensionalize

the Bernoulli equation. The dimensional Bernoulli equation

$$p^* + \frac{1}{2} \rho^* u^{*2} + \rho^* g^* z^* = c^* \quad (2.2)$$

The initial step is to list the dimensions of all the dimensional variables and constants present in the Bernoulli equation.

- Pressure (p^*) : $[m/t^2l]$
- Density (ρ^*) : $[m/l^3]$
- Velocity (u^*) : $[l/t]$
- Acceleration due to gravity (g^*) : $[l/t^2]$
- Elevation (z^*) : $[l]$

We choose scaling parameters for nondimensionalization. In this example, we select length l , a reference velocity u_0 and reference density ρ_0 .

We then apply nondimensionalization to each variable using the chosen scaling parameters:

- Nondimensional Pressure (p):

$$p = \frac{p^*}{\rho_0 u_0^2}$$

- Nondimensional Density (ρ):

$$\rho = \frac{\rho^*}{\rho_0}$$

- Nondimensional Velocity (u):

$$u = \frac{u^*}{u_0}$$

- Nondimensional gravity (g):

$$g = \frac{g^* l}{u_0^2}$$

- Nondimensional Elevation (z):

$$z = \frac{z^*}{l}$$

If the obtained terms p, ρ, u, g, z are substituted in dimensional Bernoulli equation in (2.2), we obtain in (2.3)

$$\rho_0 u_0^2 p + \frac{1}{2} \rho \rho_0 (u_0^2 u^2) + \rho_0 \rho g u_0^2 \frac{1}{l} z l = c. \quad (2.3)$$

The flow is assumed to be incompressible with $\rho = 1$ and each term is divided by $\rho_0 u_0^2$,

$$p + \frac{1}{2}u + gz = \frac{c^*}{\rho_0 u_0^2} = c$$

Since $g = 1/Fr^2$,

$$p + \frac{1}{2}u^2 + \frac{1}{Fr^2}z = c. \quad (2.4)$$

where $Fr = \frac{u_0}{\sqrt{g^* l}}$ the Froude number. This completes the nondimensionalization of Bernoulli equation in (2.2).

Nondimensionalization offers several benefits in an academic context:

Enhanced Understanding of Critical Parameters: It provides a deeper comprehension of pivotal parameters within a problem, thereby enriching the understanding of underlying physical phenomena or system dynamics.

Reduction in Parameter Count: By minimizing the number of parameters, nondimensionalization simplifies the problem, making it more tractable and facilitating clearer analyses.

Facilitates Communication: Expressing equations in a dimensionless form enables smoother and more precise communication among scholars, researchers, and practitioners, especially when discussing intricate concepts or phenomena.

Efficiency in Simulations: With fewer parameters to consider, nondimensionalization leads to a reduction in the number of simulations needed for comprehensive analysis, thus saving time and computational resources.

2.3. Nondimensionalization of Momentum and Continuity Equations

In this section, we outline the fundamental governing equations that are used to define the fluid flow. We define the flow velocity vector and its components as $\vec{u}^* = (u^*, v^*, w^*)$, accompanied by the corresponding Cartesian coordinates (x^*, y^*, z^*) . Additionally, we introduce the pressure p^* , the time parameter t^* , alongside the uniform density ρ^* , dynamic viscosity μ^* and kinematic viscosity $\nu^* (= \mu^*/\rho^*)$ of the fluid. The term f^* denotes external forces acting on the unit volume, such as gravity. We express

the governing equations for fluid motion in dimensional terms,

$$\vec{\nabla}^* \cdot \vec{u}^* = 0, \quad (2.5a)$$

$$\rho \left(\frac{\partial \vec{u}^*}{\partial t^*} + \vec{u}^* \cdot \nabla^* \vec{u}^* \right) = -\nabla^* p^* + \mu \nabla^{*2} \vec{u}^* + \vec{f}^*. \quad (2.5b)$$

These are continuity equation and the Navier–Stokes equation in Cartesian coordinates are, respectively, the two equations in question. The gradient operator $\nabla^* = (\partial_{x^*}, \partial_{y^*}, \partial_{z^*})$.

In order to exploit the advantages of nondimensionalization, it is necessary to nondimensionalize the given dimensional equations. The first step is to indicate the dimensions of all variables. The primary dimensions of all variables and constants used in the equations to make continuity and Navier Stokes equations nondimensional are given in Example 2.2. Subsequently, the scaling parameters must be selected. Let the parameters be the corresponding velocity u_r^* and the corresponding length l_r^* . The dimensional continuity equation for incompressible flow is given by:

$$\nabla^* \cdot \vec{u}^* = 0$$

To nondimensionalize this equation, we define the following dimensionless variables,

$$\vec{x} = \vec{x}^*/l_r, \quad \vec{u} = \vec{u}^*/u_r.$$

Substituting these into the dimensional continuity equation, we get:

Dimensionless velocity:

$$\vec{u}^* = u_r \vec{u}$$

Dimensionless Nabla operator:

$$\nabla^* = \frac{\partial}{\partial \vec{x}^*} = \frac{\partial}{\partial (l_r \vec{x})} = \frac{1}{l_r} \nabla$$

Substituting these into the continuity equation, we have:

$$\nabla^* \cdot \vec{u}^* = \left(\frac{1}{l_r} \nabla \right) \cdot (u_r \vec{u}) = \frac{u_r}{l_r} (\nabla \cdot \vec{u}) = 0$$

Since u_r/l_r is different from zero, the equation can be simplified with u_r/l_r . Hence the non-dimensional continuity equation is obtained,

$$\nabla \cdot \vec{u} = 0. \quad (2.6)$$

The dimensional Navier-Stokes equation for incompressible flow is given by

$$\rho \left(\frac{\partial \vec{u}^*}{\partial t^*} + (\vec{u}^* \cdot \nabla^*) \vec{u}^* \right) = -\nabla^* p^* + \mu \nabla^{*2} \vec{u}^* + \vec{f}^* \quad (2.7)$$

To nondimensionalize these equations, we define the following dimensionless variables as follows

$$\vec{x} = \frac{\vec{x}^*}{l_r}, \quad \vec{u} = \frac{\vec{u}^*}{u_r}, \quad t = \frac{t^* u_r}{l_r}, \quad p = \frac{p^*}{\rho u_r^2}.$$

Substituting these into the dimensional equations, we get:

Dimensionless velocity and time:

$$\vec{u}^* = u_r \vec{u}, \quad t^* = \frac{l_r}{u_r} t \quad (2.8)$$

Time derivative:

$$\frac{\partial \vec{u}^*}{\partial t^*} = \frac{\partial (u_r \vec{u})}{\partial t^*} = u_r \frac{\partial \vec{u}}{\partial t} \cdot \frac{\partial t}{\partial t^*} = u_r \frac{\partial \vec{u}}{\partial t} \cdot \frac{u_r}{l_r} = \frac{u_r^2}{l_r} \frac{\partial \vec{u}}{\partial t} \quad (2.9)$$

Convective term:

$$(\vec{u}^* \cdot \nabla^*) \vec{u}^* = (u_r \vec{u}) \cdot \nabla^* (u_r \vec{u}) = \frac{u_r^2}{l_r} (\vec{u} \cdot \nabla) \vec{u} \quad (2.10)$$

where $\nabla = \frac{\partial}{\partial \vec{x}} = l_r \nabla^*$.

Pressure gradient:

$$\nabla^* p^* = \nabla^* (\rho u_r^2 p) = \rho u_r^2 \nabla p \cdot \frac{1}{l_r} = \frac{\rho u_r^2}{l_r} \nabla p \quad (2.11)$$

Viscous term:

$$\mu \nabla^{*2} \vec{u}^* = \mu \nabla^{*2} (u_r \vec{u}) = \mu u_r \nabla^2 \vec{u} \cdot \frac{1}{l_r^2} = \frac{\mu u_r}{l_r^2} \nabla^2 \vec{u} \quad (2.12)$$

Substituting these terms into the Navier-Stokes equation, we obtain:

$$\rho \left(\frac{u_r^2}{l_r} \frac{\partial \vec{u}}{\partial t} + u_r^2 (\vec{u} \cdot \nabla) \vec{u} \right) = -\frac{\rho u_r^2}{l_r} \nabla p + \frac{\mu u_r}{l_r^2} \nabla^2 \vec{u} + \vec{f}^* \quad (2.13)$$

Simplifying this by dividing with by $\rho u_r^2 / l_r$ gives

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\nabla p + \frac{\mu}{\rho u_r l_r} \nabla^2 \vec{u} + \frac{\vec{f}^* l_r}{\rho u_r^2} \quad (2.14)$$

Noting that $\mu / \rho u_r l_r$ is the inverse of the Reynolds number (Re),

$$\frac{\mu}{\rho u_r l_r} = \frac{1}{Re} \quad (2.15)$$

Finally, defining the dimensionless external force term $\vec{F} = \vec{f}^* l_r / \rho u_r^2$, the non-dimensional Navier-Stokes equation becomes

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\nabla p + \frac{1}{Re} \nabla^2 \vec{u} + \vec{F} \quad (2.16)$$

Assuming that viscous and gravitational effects are negligible, the momentum equation is obtained from the nondimensional Navier-Stokes equation for incompressible flow as follows:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\nabla p \quad (2.17)$$

Next, we see perturbation methods that are used to find on approximate solution to our system later.

2.4. Perturbation Methods

Many problems encountered in mathematics are actually the result of mathematical modeling of physical events that occur in daily life. Exact solutions to these problems, particularly those that include integral and differential equations, are often unattainable. One of the options used in such cases is to approach the solution of the relevant problem or problems with the help of asymptotic expansions. Although they are more demanding, asymptotic expansions have become one of the cornerstones of applied mathematics, engineering sciences, physical mathematics and especially fluid mechanics since the last century [21].

Before defining an asymptotic approximation, it is necessary to introduce order symbols, also known as Landau symbols. These symbols were initially introduced by Bachmann in 1894 and later popularized by Landau in 1909. As a result, they are sometimes referred to as Bachmann–Landau symbols. The rationale behind this is our interest in understanding how functions behave as a parameter, typically denoted as ε , approaches zero. For instance, the function $\phi(\varepsilon) = \varepsilon$ does not approach zero as rapidly as $f(\varepsilon) = \varepsilon^2$ as ε tends to 0, necessitating a notation to represent this observation, [22].

Perturbation techniques are applicable for constructing approximate solutions to differential equations with variable coefficients; however, they cannot yield an exact

solution. These methods rely on identifying a dimensionless parameter that is relatively small in the problem: $\varepsilon \ll 1$.

The fundamental principle and methodology of regular perturbation expansion are as follows:

1. Begin by setting $\varepsilon = 0$ and solving the resulting system (denoted as f_0 for definiteness),
2. Introduce perturbations to the system by allowing ε to be nonzero (though small in magnitude)
3. Express the solution to the modified, perturbed system as a series

$$f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$$

4. Derive an expansion of the governing equations as a series in ε , grouping terms with identical powers of ε ; subsequently solve these equations as needed for the solution [22].

Example 2.3:

$$\frac{dy(x)}{dx} + 3y(x) - \varepsilon y^2(x) = 0, \quad y(0) = 3. \quad (2.18)$$

First, let's start by setting ε to 0 and solve the differential equation:

$$\frac{dy(x)}{dx} + 3y(x) = 0 \quad (2.19)$$

This equation is a simple first-order differential equation. We can solve it using the method of separable variables. Solving the equation:

$$\frac{dy(x)}{dx} = -3y(x) \implies \frac{dy(x)}{y(x)} = -3dx \quad (2.20)$$

Integrating this, we get:

$$\ln |y(x)| = -3x + C_1 \quad (2.21)$$

Since $y(0) = 3$, we have $C_1 = \ln(3)$, and our solution becomes:

$$\ln |y(x)| = -3x + \ln(3) \implies |y(x)| = 3e^{-3x} \quad (2.22)$$

The solution could also be $y(x) = -3e^{-3x}$, however given the initial condition $y(0) = 3$, we take the solution as $y(x) = 3e^{-3x}$. We get the following system.

$$y = 3e^{-3x} + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \varepsilon^3 y_3(x) + \dots \quad (2.23)$$

Given that the initial condition is $y(0) = 3$, it follows that $y_1(0) = y_2(0) = y_3(0) = \dots = 0$. If the resulting system is substituted into the given differential equation in (2.18),

1. Step: Calculate the derivative of $y(x)$.

$$\frac{dy}{dx} = \frac{d}{dx} (3e^{-3x} + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \varepsilon^3 y_3(x) + \dots), \quad (2.24)$$

$$\frac{dy}{dx} = -9e^{-3x} + \varepsilon \frac{dy_1}{dx} + \varepsilon^2 \frac{dy_2}{dx} + \varepsilon^3 \frac{dy_3}{dx} + \dots. \quad (2.25)$$

2. Step: Calculate the square of $y(x)$.

$$y^2(x) = (3e^{-3x} + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \varepsilon^3 y_3(x) + \dots)^2. \quad (2.26)$$

Expanding this expression, we get:

$$y^2(x) = 9e^{-6x} + 6\varepsilon e^{-3x} y_1(x) + 6\varepsilon^2 e^{-3x} y_2(x) + 6\varepsilon^3 e^{-3x} y_3(x) + \varepsilon^2 y_1^2(x) + \dots. \quad (2.27)$$

Now we substitute these expressions back into the original equation and collect terms by powers of ε .

$$-9e^{-3x} + \varepsilon \frac{dy_1}{dx} + \varepsilon^2 \frac{dy_2}{dx} + \varepsilon^3 \frac{dy_3}{dx} + \dots \quad (2.28)$$

$$+ 3 (3e^{-3x} + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \varepsilon^3 y_3(x) + \dots) \quad (2.29)$$

$$- \varepsilon (9e^{-6x} + 6\varepsilon e^{-3x} y_1(x) + \dots) = 0. \quad (2.30)$$

Grouping terms by ε :

$$-9e^{-3x} + 9e^{-3x} + 3\varepsilon y_1(x) + 3\varepsilon^2 y_2(x) + 3\varepsilon^3 y_3(x) + \dots - 9\varepsilon e^{-6x} - 6\varepsilon^2 e^{-3x} y_1(x) - \dots = 0. \quad (2.31)$$

Solving each term individually: 0 th order (ε^0):

$$-9e^{-3x} + 9e^{-3x} = 0 \Rightarrow \text{This is satisfied.} \quad (2.32)$$

At the 1st order (ε^1):

$$\varepsilon \left(\frac{dy_1}{dx} + 3y_1(x) - 9e^{-6x} \right) = 0, \quad (2.33)$$

from which $y_1(x)$ can be solved as:

$$\frac{dy_1}{dx} + 3y_1(x) = 9e^{-6x}, \quad (2.34)$$

To solve this first-order linear differential equation, we utilize the integrating factor method. The integrating factor, $\mu(x)$, is given by: $\mu(x) = e^{\int 3 dx} = e^{3x}$. Multiplying both sides of the differential equation by the integrating factor results in:

$$e^{3x} \frac{dy_1}{dx} + 3e^{3x} y_1(x) = 9e^{-3x}. \quad (2.35)$$

The left-hand side of the equation is the derivative of the product of the integrating factor and $y_1(x)$:

$$\frac{d}{dx} (e^{3x} y_1(x)) = 9e^{-3x}. \quad (2.36)$$

Integrating both sides with respect to x gives

$$e^{3x} y_1(x) = \int 9e^{-3x} dx = -3e^{-3x} + C_1, \quad (2.37)$$

where C_1 is the constant of integration. Here C_1 is found as $C_1 = 3$ due to the initial condition, $y_1(0) = 0$. Substituting back, we obtain,

$$e^{3x} y_1(x) = -3e^{-3x} + 3. \quad (2.38)$$

Finally, solving for $y_1(x)$:

$$y_1(x) = -3e^{-6x} + 3e^{-3x}. \quad (2.39)$$

At the 2nd order (ε^2) we get

$$\varepsilon^2 \left(\frac{dy_2}{dx} + 3y_2(x) - 6e^{-3x} y_1(x) \right) = 0, \quad (2.40)$$

$$\frac{dy_2}{dx} + 3y_2(x) = 6e^{-3x} y_1(x) = 6e^{-3x} (-3e^{-6x} + 3e^{-3x}) = -18e^{-9x} \quad (2.41)$$

$$\frac{dy_2}{dx} + 3y_2(x) = -18e^{-9x} \quad (2.42)$$

Again, this is another linear differential equation. We can use the same integration factor to solve the differential equation. Its solution is found as follows

$$e^{3x} \frac{dy_2}{dx} + 3e^{3x} y_2(x) = -18e^{-6x}. \quad (2.43)$$

The left-hand side of the equation is the derivative of the product of the integrating factor and $y_2(x)$:

$$\frac{d}{dx} (e^{3x} y_2(x)) = -18e^{-6x}. \quad (2.44)$$

$$\frac{d}{dx} (e^{3x} y_2(x)) = -18e^{-6x}. \quad (2.45)$$

Integrating both sides with respect to x gives

$$e^{3x}y_2(x) = \int -18e^{-6x} dx. \quad (2.46)$$

Evaluating the integral on the right-hand side, we obtain

$$e^{3x}y_2(x) = -18 \int e^{-6x} dx = -18 \left(-\frac{1}{6}e^{-6x} \right) = 3e^{-6x}. \quad (2.47)$$

Thus, we have:

$$e^{3x}y_2(x) = 3e^{-6x} + C_2, \quad (2.48)$$

where C_2 is the constant of integration. C_2 is found as $C_2 = -3$ due to the initial condition $y_2(0) = 0$.

$$e^{3x}y_2(x) = 3e^{-6x} - 3. \quad (2.49)$$

Finally, solving for $y_2(x)$:

$$y_2(x) = 3e^{-9x} - 3e^{-3x} \quad (2.50)$$

and substituting these into (2.23) gives

$$y = 3e^{-3x} - 3\varepsilon (e^{-6x} - e^{-3x}) + 3\varepsilon^2 (e^{-9x} - e^{-3x}) + \dots \quad (2.51)$$

Given that each term is of the form $3\varepsilon^n (e^{-3(n+1)x} - e^{-3x})$, we can generalize the solution of the equation as follows

$$y = 3e^{-3x} + \sum_{n=1}^{\infty} 3\varepsilon^n (e^{-3(n+1)x} - e^{-3x}) = 3e^{-3x} + 3e^{-3x} \cdot \frac{\varepsilon e^{-3x}}{1 - \varepsilon e^{-3x}} - 3e^{-3x} \cdot \frac{\varepsilon}{1 - \varepsilon} \quad (2.52)$$

3. THE FLUID BODY INTERACTION

We begin this chapter by introducing fluid-body interaction, investigating this is the main aim of this thesis. First, let us consider a single body between two parallel channel walls as shown in Figure 3.1.

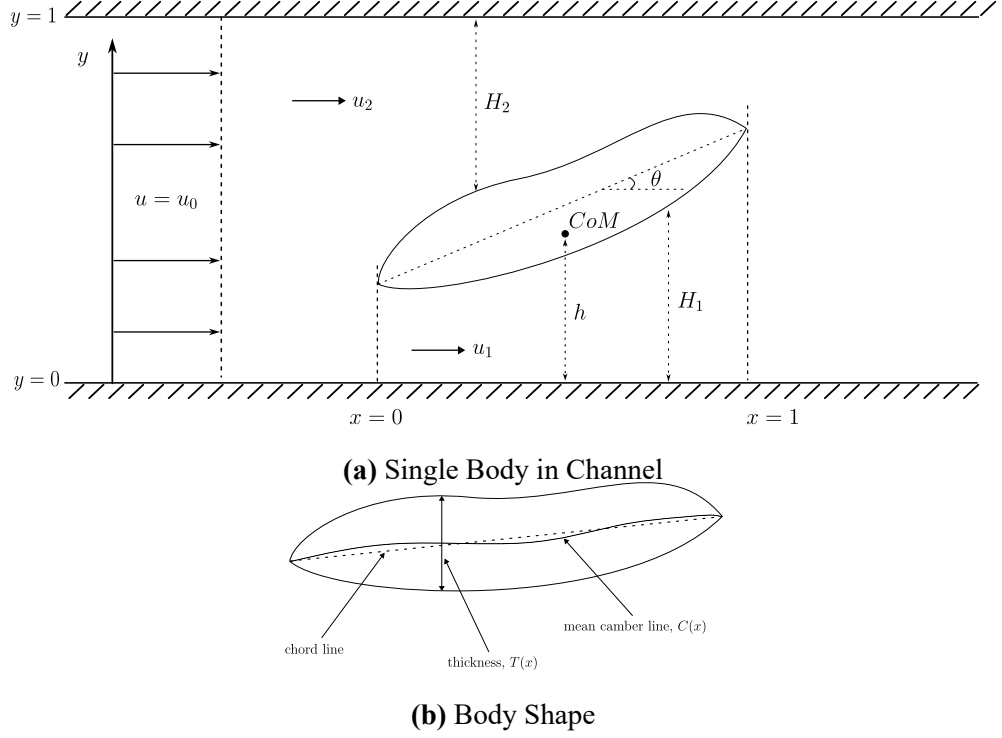


Figure 3.1: The configuration and nomenclature of a single body in the channel.

Here the body or grain with non-zero thickness or camber (or both) is taken to be thin. The initial position of the body may or may not be parallel to the channel walls. The thickness of the body $T(x)$ is defined as the distance between the upper and lower surface of the body, while the camber $C(x)$ is defined to be located in the middle of these surfaces. The chord line, which is the straight line through the leading and trailing edges of the body, makes an angle of θ with the x -axis during the motion. θ is defined as the angle between the chord line and the x -axis, which is inclination of the body. The gravity force acting on the body during movement is ignored. The fluid flow is two-dimensional, unsteady, irrotational in general. The fluid around the body is inviscid, incompressible, Newtonian fluid density ρ . The oncoming flow in the channel is a unsteady uniform flow, and body is free to move.

The interaction occurs in a two-dimensional Cartesian coordinate system, where the coordinates are designated as x and y . The channel walls were represented by the lower wall, $y = 0$, and the upper wall, $y = 1$. The leading and trailing edges of the body are located at $x = 0$ and $x = 1$, i.e. the body presents between $x = 0$ and $x = 1$. The vertical position of the center of mass for the body is defined as h whereas the horizontal position of the center of mass (CoM) for the body is σ .

The fluid-filled gaps between the parallel walls of the channel are designated as H_1 and H_2 between the body and the lower wall and the body and the upper wall, respectively.

The oncoming flow is $u = u_0 = 1$. The horizontal flow velocities u_1 and u_2 are in gap 1 and gap 2, respectively. The velocity vectors are $\vec{u}_1(u_1, v_1) = \vec{u}_1(u_1(x, y), v_1(x, y))$ and $\vec{u}_2(u_2, v_2) = \vec{u}_2(u_2(x, y), v_2(x, y))$ in each gap. From (2.6) we have,

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \quad (3.1a)$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0. \quad (3.1b)$$

We also obtain from (2.17)

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} = -\frac{\partial p_1}{\partial x}, \quad (3.2a)$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + v_2 \frac{\partial u_2}{\partial y} = -\frac{\partial p_2}{\partial x}, \quad (3.2b)$$

$$0 = -\frac{\partial p_1}{\partial y}, \quad (3.2c)$$

$$0 = -\frac{\partial p_2}{\partial y}. \quad (3.2d)$$

The basic case of an incident velocity profile $u_0(y) = 1$ corresponds to uniform or so-called plug flow far upstream. The tangential flow condition on channel walls is

$$v_1 = 0 \quad \text{on} \quad y = 0, \quad (3.3a)$$

$$v_2 = 0 \quad \text{on} \quad y = 1. \quad (3.3b)$$

The oncoming flow velocity and pressure in ahead of the leading edge at position $x = 0$ are respectively, $(u, v, p) = (1, 0, 0)$. Considering equations (3.2c) and (3.2d), the pressures in the two gaps can be taken as functions depending only on x and t .

In order to satisfy the Kutta condition at the trailing edge of the body, it is necessary to have an Euler jump at the leading edge of the body, [23]. An Euler jump or Euler zone is defined as a sudden change in the velocity or pressure of a fluid as it encounters an obstacle [12]. The Kutta condition at the body trailing edge gives equal pressures there, i.e.

$$p_1(1, t) = p_2(1, t). \quad (3.4)$$

where p_1 and p_2 be the fluid pressures in gap 1 and gap 2, respectively. The Euler condition at the leading edge is given as

$$p_1 + \frac{1}{2}u_1^2 = \frac{1}{2} \quad \text{at } x = 0^+, \quad (3.5a)$$

$$p_2 + \frac{1}{2}u_2^2 = \frac{1}{2} \quad \text{at } x = 0^+. \quad (3.5b)$$

We also have,

$$u_t + u u_x = -p_x \quad (3.6)$$

from x -momentum equation. The two-dimensional shallow water equation, expressed as

$$\frac{\partial H}{\partial t} + \frac{\partial(Hu)}{\partial x} + \frac{\partial(Hv)}{\partial y} = 0, \quad (3.7)$$

describes the temporal evolution of the depth of the fluid and its relationship with the flow velocities. Here, H represents the scaled gap widths, u and v are the velocity components in the x and y directions, respectively. The fluid flow equations in each gap now yield the shallow-water relations

$$\frac{\partial H}{\partial t} + \frac{\partial(Hu)}{\partial x} = 0. \quad (3.8)$$

At the (3.8), H denotes the gap width between the channel and the body, that is defined as follows:

$$H_1(x, t) = h(t) + C(x) - \frac{1}{2}T(x) + (x - \sigma)\theta(t), \quad (3.9a)$$

$$H_2(x, t) = 1 - h(t) - C(x) - \frac{1}{2}T(x) - (x - \sigma)\theta(t), \quad (3.9b)$$

where $T(x)$ is thickness of body, $\theta(t)$ is angle of inclination of the body. $C(x)$ is camber, $h(t)$ is y -position of the body centre of mass. The x -position being close to σ .

The body moves in response to fluid dynamic pressure forces acting on it. Therefore, lateral and angular motion balances require body motion equations derived from Newton's second law. We have fluid pressure as the force. $\ddot{h}$ and $\ddot{\theta}$ in the equations are angular velocity and angular acceleration, respectively. Vertical momentum equation is

$$M\ddot{h} = \int (p_1 - p_2)dx, \quad (3.10a)$$

where M is scaled mass of the body. Angular momentum equation is

$$I\ddot{\theta} = \int (x - \sigma) (p_1 - p_2)dx, \quad (3.10b)$$

where I is the scaled moment of inertia of the body.

The total mass balance, then requires;

$$u_1 H_1 + u_2 H_2 = 1 \quad \text{at} \quad x = 1^-, \quad (3.11)$$

by virtue of the incident conditions, $u_0 = 1$, $H_0 = 1$ ahead of the body, where H_0 is the total width of channel when the body is absent. Let us also consider the equation in 3.8.

$$H_t + (uH)_x = 0 \quad (3.12)$$

If both sides of the above equations are regularized, the following equations are obtained:

$$u_1 H_1 = c_1(t) - (x - \sigma) \dot{h}(t) - \frac{1}{2} (x - \sigma)^2 \dot{\theta}(t) \quad (3.13a)$$

$$u_2 H_2 = c_2(t) + (x - \sigma) \dot{h}(t) + \frac{1}{2} (x - \sigma)^2 \dot{\theta}(t) \quad (3.13b)$$

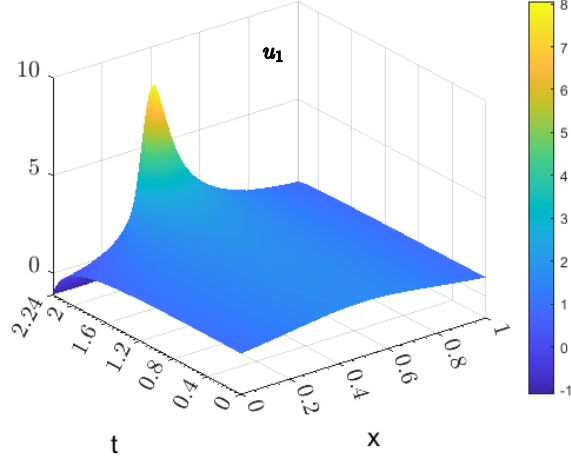
where c_1 and c_2 are constants while σ denotes x -position of center of mass. It is also clear that since $u_1 H_1 + u_2 H_2 = 1$ from total mass balance, the equation

$$c_1(t) + c_2(t) = 1 \quad (3.14)$$

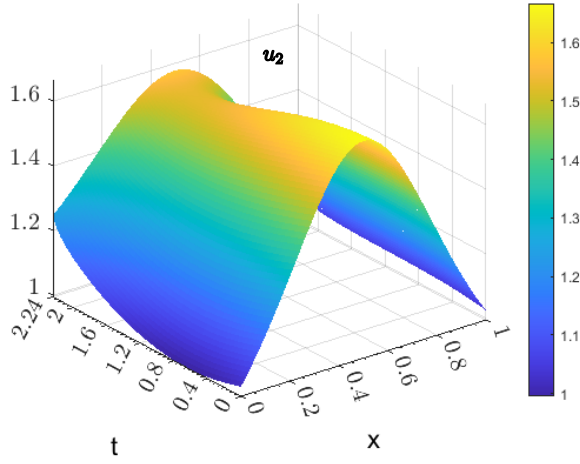
will be obtained.

3.1. Numerical Results

In this section, we present some numerical results for a thick body without any camber, i.e $C(x) = 0$ and $T(x) = 0.4 \sin(\pi x)$.



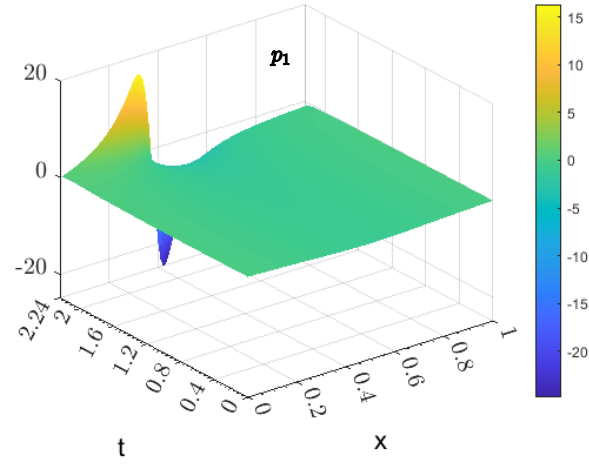
(a) Velocity u_1 .



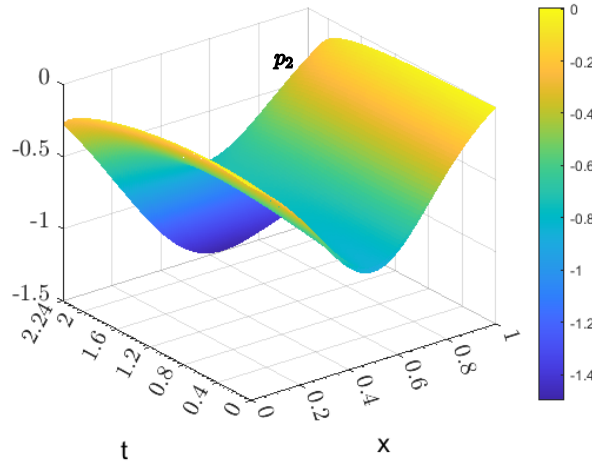
(b) Velocity u_2 .

Figure 3.2: Velocity solutions in each gap.

Figure 3.2 shows the solutions for the velocity of the fluid in each gap. Here, u_1 and u_2 are the velocities in lower and upper gap, respectively. As illustrated in Figure 3.2a, when the value of t is relatively small, the solution u_1 is also relatively small until the impact occurs against the lower wall. In the physical sense, when a body in a fluid impacts one of the walls, the velocity of the fluid is expected to increase at the impact location. This phenomenon can be observed in Figure 3.2a, where u_1 solution increases significantly as the moment of impact $t \sim 2.24$ approaches. Therefore, it can be concluded that the impact occurs at the lower gap. The change in u_2 solution in Figure 3.2b is sufficiently small compared to u_1 , which supports that the impact occurs at the lower wall.



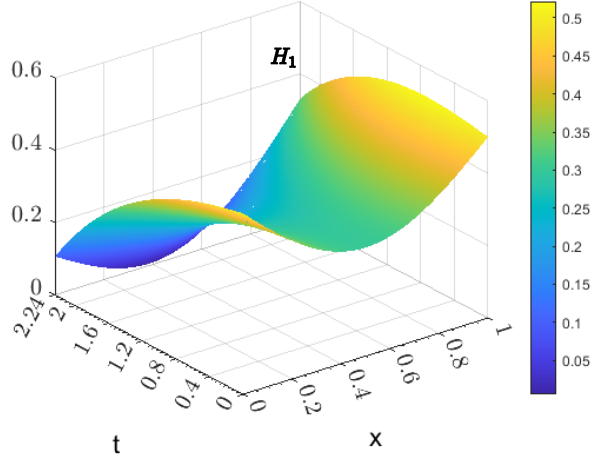
(a) Pressure p_1 .



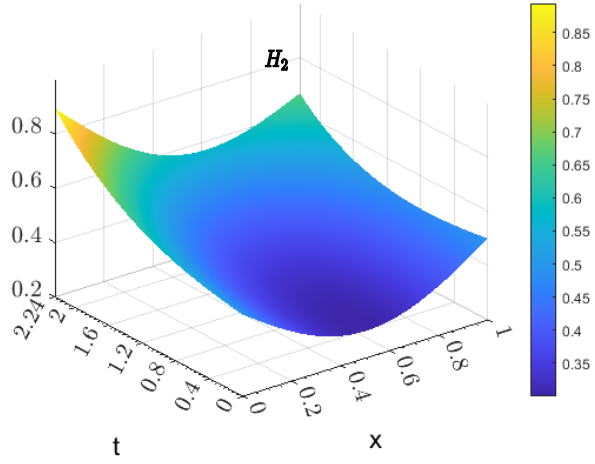
(b) Pressure p_2 .

Figure 3.3: Pressure solutions in each gap.

Similar observations can be made for the pressure solutions presented in Figure 3.3. As illustrated in Figure 3.3a, there is a significant change in the fluid pressure in the lower gap around the moment of impact. Conversely, as seen in Figure 3.3b, there is no notable change in the p_2 solution.



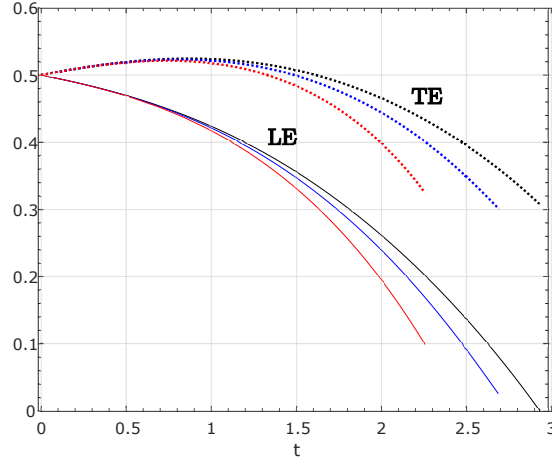
(a) Gap 1 H_1 .



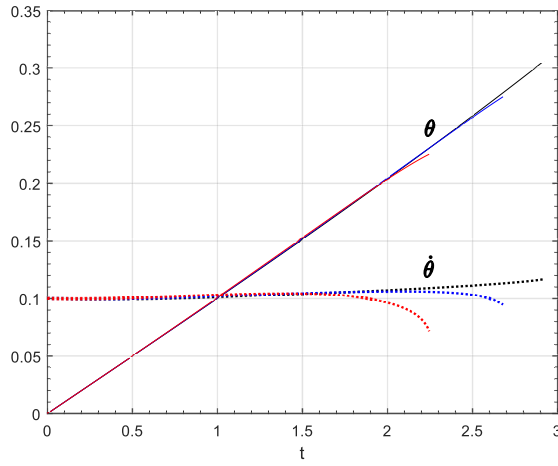
(b) Gap 2 H_2 .

Figure 3.4: Gap 1 and gap 2 solutions.

The gap between the lower surface of the body and the lower wall of the fluid-filled channel H_1 solution is shown in Figure 3.4a, while the gap between the upper surface of the body and the upper wall of the fluid-filled channel H_2 solution is shown in Figure 3.4b. When this solution is analyzed, gap 1 approaches zero around $t = 2.24$ at $x = 0.4$. This indicates that the body is approaching an impact with the lower wall. As expected, gap 2 is increasing, (see 3.4b). In Figure 3.5, the effect of thickness on impact is analyzed.



(a) Body leading-edge (LE) (solid lines) and trailing-edge (TE) (dotted lines) positions.



(b) θ and $\dot{\theta}$.

Figure 3.5: Effect of thickness on impact.

In order to investigate the influence of thickness on the impact, let the thickness be defined as $T(x) = \kappa \sin(\pi x)$ with parameter κ . Here, black lines represent the solutions for $\kappa = 0$, while blue lines are for $\kappa = 0.2$ and red lines for $\kappa = 0.4$. In all cases body has no camber, i.e. $C(x) = 0$. Firstly, when κ is zero, the body is a flat body with no thickness. A flat body is expected to clash against the wall at either the leading edge or the trailing edge of the body. As illustrated in Figure 3.5a, the leading edge, represented by the black solid line, approaches zero around the impact time, $\sim t_c = 2.9$. Consequently, the impact occurs at the leading edge of the body when κ is equal to zero. As the thickness increases, the impact is expected to occur earlier and closer to the center of the body, which makes physical sense. When κ is 0.2 or 0.4, the impact is

observed to occur earlier and closer to the center of the body, as indicated by the blue and red lines in Figure 3.5a, respectively. In Figure 3.5b, it can be observed that both inclination of the body, θ and angular velocity $\dot{\theta}$, increase linearly. In Figure 3.5, the initial conditions are $h(0) = 0.5$, $\dot{h}(0) = 0$; $\theta(0) = 0$, $\dot{\theta}(0) = -0.1$. Also, the scaled mass and moment of inertia values are $M = 1$ and $I = 2$ in turn.

The position of the body at $t = 0$, $t = 1.5$ and $t = 2.23$ are presented in Figure 3.6 with the same conditions.

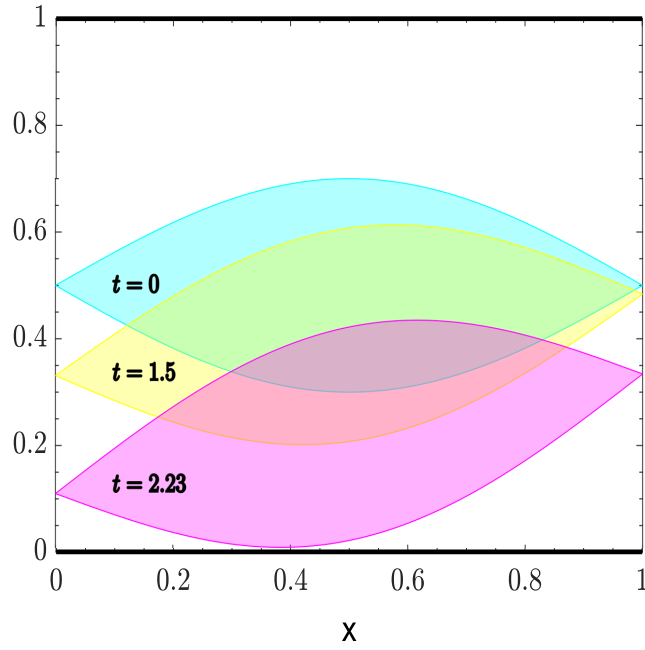


Figure 3.6: The body position.

At time $t = 0$, the body is in its initial position, as illustrated in cyan. As the moment of impact approaches, the body shifts downwards. The body then approaches the lower wall. That is illustrated in yellow and pink.

4. IMPACT PHENOMENA

Following the given results, let us examine the mathematical structure of the fluid-body interaction when an impact occurs for a single body (or particle) in a fluid-filled channel. The aim is to achieve comparable outcomes in both layers by conducting research in the inner and outer layers at the point of impact. This research will employ asymptotic expansions and perturbation methods. We will proceed our analysis with the outer layer.

In [8], the phenomenon is analysed impact for a body with homogeneous density in both analytically and numerically. In this chapter, we investigate the impact of a body with nonhomogeneous density distribution which will be done analytically.

4.1. Outer Layer

Suppose the impact occurs at a point $x = x_0$ between $x = 0$ and $x = 1$. Also, let gap 1 tends to approach zero at time $t = t_0$. In this case, the asymptotic expansions of the time-dependent terms $(t_0 - t)$ can be given as [8]:

$$h(t) = h_{10} + (t_0 - t) h_{11} + (t_0 - t)^{\frac{3}{2}} h_{12} + \dots, \quad (4.1a)$$

$$\theta(t) = \theta_{10} + (t_0 - t) \theta_{11} + (t_0 - t)^{\frac{3}{2}} \theta_{12} + \dots, \quad (4.1b)$$

$$c_1(t) = c_{11} + (t_0 - t)^{\frac{1}{2}} c_{12} + \dots, \quad (4.1c)$$

$$c_2(t) = c_{21} + (t_0 - t)^{\frac{1}{2}} c_{22} + \dots. \quad (4.1d)$$

Here, the coefficients multiplying the powers of $(t_0 - t)$ are of $O(1)$. So for x not equal to x_0 now the gap widths H_1 and H_2 develop according to

$$H_1(x, t) = H_{10}(x) + (t_0 - t) H_{11}(x) + (t_0 - t)^{\frac{3}{2}} H_{12}(x) + \dots \quad (4.2a)$$

$$H_2(x, t) = H_{20}(x) + (t_0 - t) H_{21}(x) + (t_0 - t)^{\frac{3}{2}} H_{22}(x) + \dots \quad (4.2b)$$

Let us substitute the asymptotic expansions given in (4.2a, b) and (4.1a-d) into equations (3.9a, b).

$$\begin{aligned} H_1(x, t) = & \left(h_{10} + (t_0 - t) h_{11} + (t_0 - t)^{\frac{3}{2}} h_{12} + \dots \right) + C(x) - \frac{1}{2} T(x) \\ & + (x - \sigma) \left(\theta_{10} + (t_0 - t) \theta_{11} + (t_0 - t)^{\frac{3}{2}} \theta_{12} + \dots \right) \end{aligned} \quad (4.3a)$$

and

$$H_2(x, t) = 1 - \left(h_{10} + (t_0 - t) h_{11} + (t_0 - t)^{\frac{3}{2}} h_{12} + \dots \right) - C(x) - \frac{1}{2} T(x), \\ - (x - \sigma) \left(\theta_{10} + (t_0 - t) \theta_{11} + (t_0 - t)^{\frac{3}{2}} \theta_{12} + \dots \right). \quad (4.3b)$$

Considering the leading order terms in (4.3a, b), we obtain

$$H_{10}(x) = h_{10} + C(x) - \frac{1}{2} T(x) + (x - \sigma) \theta_{10}, \quad (4.4a)$$

$$H_{20}(x) = 1 - h_{10} - C(x) - \frac{1}{2} T(x) - (x - \sigma) \theta_{10} \quad (4.4b)$$

Note that H_{10} , H_{20} are gap widths at the impact time $t = t_c$. The velocity components are expressed as follows:

$$u_1(x, t) = u_{10}(x) + (t_0 - t)^{\frac{1}{2}} u_{11}(x) + \dots, \quad (4.5a)$$

$$u_2(x, t) = u_{20}(x) + (t_0 - t)^{\frac{1}{2}} u_{21}(x) + \dots. \quad (4.5b)$$

Substituting (4.2a, b) and (4.5a, b) into (3.13a, b) gives

$$\left(H_{10}(x) + (t_0 - t) H_{11}(x) + (t_0 - t)^{\frac{3}{2}} H_{12}(x) + \dots \right) \left(u_{10}(x) + (t_0 - t)^{\frac{1}{2}} u_{11}(x) + \dots \right) \\ = \left(c_{11} + (t_0 - t)^{\frac{1}{2}} c_{12} + \dots \right) - (x - \sigma) \left(h_{11} + \frac{3}{2} (t_0 - t)^{\frac{1}{2}} h_{12} \dots \right) \\ - \frac{1}{2} (x - \sigma)^2 \left(\theta_{11} + \frac{3}{2} (t_0 - t)^{\frac{1}{2}} \theta_{12} + \dots \right), \quad (4.6a)$$

and

$$\left(H_{20}(x) + (t_0 - t) H_{21}(x) + (t_0 - t)^{\frac{3}{2}} H_{22}(x) + \dots \right) \left(u_{20}(x) + (t_0 - t)^{\frac{1}{2}} u_{21}(x) + \dots \right) \\ = \left(c_{21} + (t_0 - t)^{\frac{1}{2}} c_{22} + \dots \right) + (x - \sigma) \left(h_{11} + \frac{3}{2} (t_0 - t)^{\frac{1}{2}} h_{12} \dots \right) \\ + \frac{1}{2} (x - \sigma)^2 \left(\theta_{11} + \frac{3}{2} (t_0 - t)^{\frac{1}{2}} \theta_{12} + \dots \right). \quad (4.6b)$$

Looking at the $O(1)$ terms, we obtain

$$H_{10}(x) u_{10}(x) = c_{11} - (x - \sigma) h_{11} - (x - \sigma)^2 \theta_{11} \quad (4.7a)$$

$$H_{20}(x) u_{20}(x) = c_{21} + (x - \sigma) h_{11} + (x - \sigma)^2 \theta_{11} \quad (4.7b)$$

At $O((t_0 - t)^{\frac{1}{2}})$, we get

$$H_{10}(x) u_{11}(x) = c_{12} - \frac{3}{2} (x - \sigma) h_{12} - \frac{3}{4} (x - \sigma)^2 \theta_{12} \quad (4.8a)$$

and similarly,

$$H_{20}(x)u_{21}(x) = c_{22} + \frac{3}{2}(x - \sigma)h_{12} + \frac{3}{4}(x - \sigma)^2\theta_{12} \quad (4.8b)$$

Now, the leading order terms of the gaps, in (4.4a, b), can be substituted respectively into (4.7a, b), allowing us to determine the leading order terms of the velocity components.

$$u_{10}(x) = \frac{c_{11} - (x - \sigma)h_{11} - (x - \sigma)^2\theta_{11}}{h_{10} + C(x) - \frac{1}{2}T(x) + (x - \sigma)\theta_{10}} \quad (4.9a)$$

$$u_{20}(x) = \frac{c_{21} + (x - \sigma)h_{11} + (x - \sigma)^2\theta_{11}}{1 - h_{10} - C(x) - \frac{1}{2}T(x) - (x - \sigma)\theta_{10}} \quad (4.9b)$$

$$u_{11}(x) = \frac{c_{12} - \frac{3}{2}(x - \sigma)h_{12} - \frac{3}{4}(x - \sigma)^2\theta_{12}}{h_{10} + C(x) - \frac{1}{2}T(x) + (x - \sigma)\theta_{10}} \quad (4.9c)$$

$$u_{21}(x) = \frac{c_{22} + \frac{3}{2}(x - \sigma)h_{12} + \frac{3}{4}(x - \sigma)^2\theta_{12}}{1 - h_{10} - C(x) - \frac{1}{2}T(x) - (x - \sigma)\theta_{10}} \quad (4.9d)$$

Furthermore, substitution of (4.1c, d) into (3.14) shows that

$$\left(c_{11} + (t_0 - t)^{\frac{1}{2}}c_{12} + \dots\right) + \left(c_{21} + (t_0 - t)^{\frac{1}{2}}c_{22} + \dots\right) = 1. \quad (4.10)$$

At $O(1)$ and $O(t_0 - t)^{\frac{1}{2}}$

$$c_{11} + c_{21} = 1 \quad (4.11a)$$

$$c_{12} + c_{22} = 0. \quad (4.11b)$$

must hold. Furthermore, gap pressures are therefore defined as

$$p_1(x, t) = (t_0 - t)^{-\frac{1}{2}}p_{10}(x) + p_{11}(x) + \dots, \quad (4.12a)$$

$$p_2(x, t) = (t_0 - t)^{-\frac{1}{2}}p_{20}(x) + p_{21}(x) + \dots, \quad (4.12b)$$

which we will see later on. Thus, from the momentum equation given in (3.6), we obtain

$$\begin{aligned} & \left(-\frac{1}{2}(t_0 - t)^{-\frac{1}{2}}u_{11}(x) + \dots\right) \\ & + \left(u_{10}(x) + (t_0 - t)^{\frac{1}{2}}u_{11}(x) + \dots\right) \left(u'_{10}(x) + (t_0 - t)^{\frac{1}{2}}u'_{11}(x) + \dots\right) \\ & = - \left[(t_0 - t)^{-\frac{1}{2}}p'_{10}(x) + p'_{11}(x) + \dots\right], \end{aligned} \quad (4.13a)$$

and

$$\begin{aligned}
& \left(-\frac{1}{2}(t_0 - t)^{-\frac{1}{2}}u_{21}(x) + \dots \right) \\
& + \left(u_{20}(x) + (t_0 - t)^{\frac{1}{2}}u_{21}(x) + \dots \right) \left(u'_{20}(x) + (t_0 - t)^{\frac{1}{2}}u'_{21}(x) + \dots \right) \\
& = - \left[(t_0 - t)^{-\frac{1}{2}}p'_{20}(x) + p'_{21}(x) + \dots \right]
\end{aligned} \tag{4.13b}$$

The leading order terms $(t_0 - t)^{-\frac{1}{2}}$ satisfy;

$$-\frac{1}{2}u_{11}(x) = -p'_{10}(x), \tag{4.14a}$$

$$-\frac{1}{2}u_{21}(x) = -p'_{20}(x), \tag{4.14b}$$

where a prime denotes a derivative with respect to x . From the well-known Kutta condition in (3.4), we have

$$(t_0 - t)^{-\frac{1}{2}}p_{10}(1_-) + \dots = (t_0 - t)^{-\frac{1}{2}}p_{20}(1_-) + \dots \tag{4.15a}$$

$$p_{10}(1_-) = p_{20}(1_-) \tag{4.15b}$$

In addition, to Euler condition in (3.5) requires

$$\left((t_0 - t)^{-\frac{1}{2}}p_{10} + \dots \right) + \frac{1}{2} \left(u_{10} + (t_0 - t)^{\frac{1}{2}}u_{11} + \dots \right)^2 = \frac{1}{2} \quad \text{at } x = 0 + .$$

(4.16a)

At the leading order term, we have

$$p_{10}(0_+) = p_{20}(0_+) = 0. \tag{4.16b}$$

Finally, let's consider body-motion relations in (3.10a, b). Using asymptotic expansions for h and θ ,

- $\dot{h} = h_{11} - \frac{3}{2}(t_0 - t)^{\frac{1}{2}}h_{12} + \dots$,
- $\ddot{h} = \frac{3}{4}(t_0 - t)^{-\frac{1}{2}}h_{12} + \dots$,
- $\dot{\theta} = \theta_{11} - \frac{3}{2}(t_0 - t)^{\frac{1}{2}}\theta_{12} + \dots$,
- $\ddot{\theta} = \frac{3}{4}(t_0 - t)^{-\frac{1}{2}}\theta_{12} + \dots$,

and substituting the expansions into (3.10a, b) the solution at the leading order term $(t_0 - t)^{-\frac{1}{2}}$ is found as follows,

$$\frac{3}{4}Mh_{12} = \int (p_{10}(x) - p_{20}(x))dx, \tag{4.17a}$$

$$\frac{3}{4}I\theta_{12} = \int (x - \sigma) (p_{10}(x) - p_{20}(x))dx. \tag{4.17b}$$

Next, we will look at the local analysis. The two predictions for inner and outer layers have to match exactly where overlap occurs.

4.2. Inner Layer

Now we will investigate the inner layer effects operating around the impact location $x = x_0$. In addition to the asymptotic expansions in (4.1)

$$F_1 = F_{10} + (t_0 - t)^{\frac{1}{2}} \xi F_{11} + (t_0 - t) \xi^2 F_{12} + \dots, \quad (4.18a)$$

$$F_2 = F_{20} + (t_0 - t)^{\frac{1}{2}} \xi F_{21} + (t_0 - t) \xi^2 F_{22} + \dots \quad (4.18b)$$

where $F_1(x) = \frac{1}{2}T(x) - C(x)$ and $F_2(x) = \frac{1}{2}T(x) + C(x)$ are functions that depend on x while the terms multiplied with $(t_0 - t)$ and ξ are of order unity, at $O(1)$. Additionally, the appropriate spatial scales are given by

$$x - x_0 = (t_0 - t)^{\frac{1}{2}} \xi, \quad (4.19)$$

where ξ is of order unity. Then, organizing the expansions in (3.9a, b) and (4.5a, b) the new expansions will be as follows.

$$H_1 = (t_0 - t) \hat{H}_1(\xi) + \dots, \quad (4.20a)$$

$$H_2 = (t_0 - t) \hat{H}_2(\xi) + \dots, \quad (4.20b)$$

$$u_1 = (t_0 - t)^{-\frac{1}{2}} \hat{u}_1(\xi) + \dots, \quad (4.20c)$$

$$u_2 = (t_0 - t)^{-\frac{1}{2}} \hat{u}_2(\xi) + \dots. \quad (4.20d)$$

The asymptotic descriptions for H_1 and H_2 are due to parabolicity. Matching with (3.9a) gives,

$$\begin{aligned} (t_0 - t) \hat{H}_1(\xi) + \dots = & \left(h_{10} + (t_0 - t) h_{11} + (t_0 - t)^{\frac{3}{2}} h_{12} + \dots \right) \\ & - \left(F_{10} + (t_0 - t)^{\frac{1}{2}} \xi F_{11} + (t_0 - t) \xi^2 F_{12} + \dots \right) \\ & + \left(x_0 + (t_0 - t)^{\frac{1}{2}} \xi - \sigma \right) \left(\theta_{10} + (t_0 - t) \theta_{11} + (t_0 - t)^{\frac{3}{2}} \theta_{12} + \dots \right). \end{aligned} \quad (4.21)$$

From the definition of impact we require; $h_{10} - F_{10} + (x_0 - \sigma) \theta_{10} = 0$ at $O(1)$. We also note that $F_{11} = \theta_{10}$ as slopes cancel. So, at $O(t_0 - t)$ we get

$$\hat{H}_1(\xi) = h_{11} + (x_0 - \sigma) \theta_{11} - F_{12} \xi^2. \quad (4.22)$$

Similarly, matching with (3.9b) gives

$$\begin{aligned}
(t_0 - t)\hat{H}_2(\xi) + \dots = & 1 - \left(h_{10} + (t_0 - t)h_{11} + (t_0 - t)^{\frac{3}{2}}h_{12} + \dots \right) \\
& - \left(F_{20} + (t_0 - t)^{\frac{1}{2}}\xi F_{21} + (t_0 - t)\xi^2 F_{22} + \dots \right) \\
& - \left(x_0 + (t_0 - t)^{\frac{1}{2}}\xi - \sigma \right) \left(\theta_{10} + (t_0 - t)\theta_{11} + (t_0 - t)^{\frac{3}{2}}\theta_{12} + \dots \right).
\end{aligned} \tag{4.23}$$

This leads to; $h_{10} + F_{20} + (x_0 - \sigma)\theta_{10} = 1$ and $F_{21} + \theta_{10} = 0$. Thus at $O(t_0 - t)^{\frac{1}{2}}$, we similarly get

$$\hat{H}_2(\xi) = -h_{11} - (x_0 - \sigma)\theta_{11} - F_{22}\xi^2. \tag{4.24}$$

The new expansions given for u_1 and u_2 , in (4.20c, d) are substituted into equations (3.13a, b) as follows

$$\begin{aligned}
\left[(t_0 - t)\hat{H}_1(\xi) + \dots \right] \left[(t_0 - t)^{-\frac{1}{2}}\hat{u}_1(\xi) + \dots \right] = & \left(c_{11} + (t_0 - t)^{\frac{1}{2}}c_{12} + \dots \right) \\
& + \left(x_0 + (t_0 - t)^{\frac{1}{2}}\xi - \sigma \right) \left(h_{11} + \frac{3}{2}(t_0 - t)^{\frac{1}{2}}h_{12} + \dots \right) \\
& + \frac{1}{2} \left[\left(x_0 + (t_0 - t)^{\frac{1}{2}}\xi - \sigma \right)^2 \left(\theta_{11} + \frac{3}{2}(t_0 - t)^{\frac{1}{2}}\theta_{12} + \dots \right) \right].
\end{aligned} \tag{4.25}$$

At $O(1)$

$$0 = c_{11} + (x_0 - \sigma)h_{11} + \frac{1}{2}(x_0 - \sigma)^2\theta_{11}, \tag{4.26}$$

and at $O((t_0 - t)^{\frac{1}{2}})$

$$\begin{aligned}
\hat{H}_1(\xi) \cdot \hat{u}_1(\xi) = & c_{12} + \frac{3}{2}(x_0 - \sigma)h_{12} + \frac{3}{4}(x_0 - \sigma)^2\theta_{12} + \xi h_{11} + (x_0 - \sigma)\xi\theta_{11} \\
= & c_{12} + \frac{3}{2}(x_0 - \sigma)h_{12} + \frac{3}{4}(x_0 - \sigma)^2\theta_{12} + (h_{11} + (x_0 - \sigma)\theta_{11})\xi.
\end{aligned} \tag{4.27}$$

Substituting $\hat{H}_1(\xi)$ described in (4.22) into (4.27) gives

$$\hat{u}_1(\xi) = \frac{c_{12} + \frac{3}{2}(x_0 - \sigma)h_{12} + \frac{3}{4}(x_0 - \sigma)^2\theta_{12} + (h_{11} + (x_0 - \sigma)\theta_{11})\xi}{\hat{H}_1(\xi) = h_{11} + (x_0 - \sigma)\theta_{11} - F_{12}\xi^2}. \tag{4.28}$$

Based on equations (3.6) and (4.20c), it follows that the pressure expansion in gap 1 should be $p_1 = (t_0 - t)^{-1}\hat{p}_1(\xi) + \dots$, whereas in gap 2 the pressure $p = p_2$ is considerably smaller. Given the expansions

$$u_1 = (t_0 - t)^{-\frac{1}{2}}\hat{u}_1(\xi) + \dots, \tag{4.29a}$$

$$p_1 = (t_0 - t)^{-1} \hat{p}_1(\xi) + \dots \quad (4.29b)$$

where $\xi = \frac{x-x_0}{(t_0-t)^{1/2}}$, we need to evaluate the equation $u_t + u u_x = -p_x$. Firstly, we compute u_t . Applying the chain rule, we get

$$u_t = \frac{\partial}{\partial t} \left((t_0 - t)^{-\frac{1}{2}} \hat{u}_1(\xi) + \dots \right) = \frac{1}{2} (t_0 - t)^{-\frac{3}{2}} \hat{u}_1(\xi) + (t_0 - t)^{-\frac{1}{2}} \frac{\partial \hat{u}_1(\xi)}{\partial \xi} \frac{\partial \xi}{\partial t} + \dots \quad (4.30a)$$

Considering $\xi = \frac{x-x_0}{(t_0-t)^{1/2}}$, we have

$$\frac{\partial \xi}{\partial t} = \frac{1}{2} (x - x_0) (t_0 - t)^{-\frac{3}{2}}. \quad (4.30b)$$

Thus,

$$u_t = \frac{1}{2} (t_0 - t)^{-\frac{3}{2}} \hat{u}_1(\xi) + \frac{1}{2} (x - x_0) (t_0 - t)^{-2} \frac{\partial \hat{u}_1}{\partial \xi} + \dots \quad (4.30c)$$

We also have

$$u_x = \frac{\partial}{\partial x} \left((t_0 - t)^{-\frac{1}{2}} \hat{u}_1(\xi) + \dots \right) = (t_0 - t)^{-\frac{1}{2}} \frac{\partial \hat{u}_1(\xi)}{\partial \xi} \frac{\partial \xi}{\partial x} + \dots. \quad (4.30d)$$

Given that $\frac{\partial \xi}{\partial x} = (t_0 - t)^{-1/2}$, we have

$$u_x = (t_0 - t)^{-\frac{1}{2}} \frac{\partial \hat{u}_1}{\partial \xi} (t_0 - t)^{-1/2} + \dots = (t_0 - t)^{-1} \frac{\partial \hat{u}_1}{\partial \xi} + \dots \quad (4.30e)$$

We compute $u u_x$ as follows

$$u u_x = \left((t_0 - t)^{-\frac{1}{2}} \hat{u}_1(\xi) + \dots \right) \left((t_0 - t)^{-1} \frac{\partial \hat{u}_1}{\partial \xi} + \dots \right) = (t_0 - t)^{-\frac{3}{2}} \hat{u}_1(\xi) \frac{\partial \hat{u}_1}{\partial \xi} + \dots. \quad (4.30f)$$

Lastly, to find p_x , we apply the chain rule

$$\begin{aligned} p_x &= \frac{\partial}{\partial x} \left((t_0 - t)^{-1} \hat{p}_1(\xi) + \dots \right) = (t_0 - t)^{-1} \frac{\partial \hat{p}_1(\xi)}{\partial \xi} \frac{\partial \xi}{\partial x} + \dots \\ &= (t_0 - t)^{-1} \frac{\partial \hat{p}_1(\xi)}{\partial \xi} (t_0 - t)^{-1/2} + \dots \\ &= (t_0 - t)^{-3/2} \frac{\partial \hat{p}_1}{\partial \xi} + \dots. \end{aligned} \quad (4.30g)$$

Combining these, we get

$$\left(\frac{1}{2} (t_0 - t)^{-\frac{3}{2}} \hat{u}_1(\xi) + \frac{1}{2} (x - x_0) (t_0 - t)^{-2} \frac{\partial \hat{u}_1}{\partial \xi} + \dots \right) + \left((t_0 - t)^{-\frac{3}{2}} \hat{u}_1(\xi) \frac{\partial \hat{u}_1}{\partial \xi} + \dots \right) \quad (4.30h)$$

$$= (t_0 - t)^{-\frac{3}{2}} \frac{\partial \hat{p}_1}{\partial \xi} \dots. \quad (4.30i)$$

Here $(x - x_0) = \xi(t_0 - t)^{1/2}$. Considering the leading order term $(t_0 - t)^{-\frac{3}{2}}$ gives:

$$\frac{1}{2}\hat{u}_1(\xi) + \frac{\xi}{2}\frac{\partial\hat{u}_1}{\partial\xi} + \hat{u}_1(\xi)\frac{\partial\hat{u}_1}{\partial\xi} = -\frac{\partial\hat{p}_1}{\partial\xi}. \quad (4.31)$$

Consequently, integrating both sides with respect to ξ give the pressure term,

$$\hat{p}_1(\xi) = -\frac{\xi}{2}\hat{u}_1(\xi) - \frac{\hat{u}_1^2}{2} - \frac{h_{11} + (x_0 - \sigma)\theta_{11}}{2F_{12}} \quad (4.32)$$

$$= -\frac{1}{2}\left(\xi\hat{u}_1(\xi) + \hat{u}_1^2(\xi) + \frac{h_{11} + (x_0 - \sigma)\theta_{11}}{F_{12}}\right) \quad (4.33)$$

where the integration constant is determined to ensure at large positive or negative ξ , i.e $\xi \rightarrow \pm\infty$. The body movement equations now becomes

$$\frac{3}{4}Mh_{12} = \int \hat{p}_1(\xi)d\xi, \quad (4.34a)$$

$$\frac{3}{4}I\theta_{12} = (x - \sigma) \int \hat{p}_1(\xi)d\xi. \quad (4.34b)$$

In the transition region between the inner and outer regions, the two expansions should give the same result. Thus, the solutions have to match at the leading order.

We now try to write the pressure and velocity solutions in the normalized form.

We introduce $\Omega = \sqrt{\frac{\alpha}{\gamma}}$ where $\alpha = h_{11} + (x_0 - \sigma)\theta_{11}$, $\gamma = -F_{12}$. We also define

$$\beta = c_{12} + \frac{3}{2}(x_0 - \sigma)h_{12} + \frac{3}{4}(x_0 - \sigma)^2\theta_{12} \quad (4.35)$$

Thus $\hat{u}_1(\xi)$ and $\hat{p}_1(\xi)$ solutions in (4.28) and (4.33) can become

$$\hat{u}_1(\xi) = \frac{\beta + \alpha\xi}{\alpha + \gamma\xi^2} \quad (4.36a)$$

and

$$\hat{p}_1(\xi) = -\frac{1}{2}\left(\xi\hat{u}_1(\xi) + \hat{u}_1^2(\xi) - \frac{\alpha}{\gamma}\right) \quad (4.36b)$$

The normalized form is obtained by taking $\beta = \alpha\Omega\beta_n$ and $\xi = \Omega\xi_n$ and substituting these values in (4.36a, b).

$$\hat{u}_1(\xi) = \frac{\alpha\Omega\beta_n + \alpha\Omega\xi_n}{\alpha + \gamma(\Omega\xi_n)^2} = \frac{\alpha\Omega\beta_n + \alpha\Omega\xi_n}{\alpha + \gamma\frac{\alpha}{\gamma}\xi_n^2} = \Omega\frac{\beta_n + \xi_n}{\xi_n^2 + 1}. \quad (4.37a)$$

and for pressure

$$\hat{p}_1(\xi) = -\frac{1}{2}\left[\Omega^2\xi_n\left(\frac{\beta_n + \xi_n}{\xi_n^2 + 1}\right) + \left(\Omega\frac{\beta_n + \xi_n}{\xi_n^2 + 1}\right)^2 - \Omega^2\right] \quad (4.37b)$$

$$= \frac{\Omega^2}{2}\left[1 - \xi_n\left(\frac{\beta_n + \xi_n}{\xi_n^2 + 1}\right) - \left(\frac{\beta_n + \xi_n}{\xi_n^2 + 1}\right)^2\right] \quad (4.37c)$$

$$= \frac{\Omega^2}{2}\left[\frac{1 - \beta_n(\xi_n^3 + 3\xi_n + \beta_n)}{(\xi_n^2 + 1)^2}\right] \quad (4.37d)$$

Now the pressure and velocity solutions can be written as

$$\hat{u}_1(\xi) = \Omega \hat{u}_{1n}, \quad (4.37e)$$

$$\hat{p}_1(\xi) = \frac{\Omega^2}{2} \hat{p}_{1n}, \quad (4.37f)$$

where

$$\hat{u}_{1n} = \frac{\xi_n + \beta_n}{\xi_n^2 + 1} \quad (4.38a)$$

and

$$\hat{p}_{1n} = \frac{1 - \beta_n (\xi_n^3 + 3\xi_n + \beta_n)}{(\xi_n^2 + 1)^2}. \quad (4.38b)$$

Finally the asymptotes for large ξ_n values give

$$\hat{u}_{1n} \sim \xi_n^{-1} + \beta_n \xi_n^{-2} + \dots \quad (4.39a)$$

$$\hat{p}_{1n} \sim -\beta_n \xi_n^{-1} + \dots \quad (4.39b)$$

If we combine these with the velocity and pressure solutions in the region in (4.29a) and (4.29b), we must have

$$u_1 \sim (x - x_0)^{-1} + \beta_n (t_0 - t)^{\frac{1}{2}} (x - x_0)^{-2} + \dots \quad (4.40a)$$

and

$$p_1 \sim -\beta_n (t_0 - t)^{-\frac{1}{2}} (x - x_0)^{-1} + \dots \quad (4.40b)$$

in the outer layer, where x is of order of unity.

We note that the leading edge jump conditions and the trailing edge pressure conditions should be negligible in the inner layer, while their effects must have been seen in the outer layer.

We terminate our description here for the impact process which arises at a finite time.

5. MULTIPLE BODIES INTERACTING IN CHANNEL FLOW

In this chapter, we examine multiple bodies interacting each other and with fluid inside a straight channel.

5.1. Two Bodies

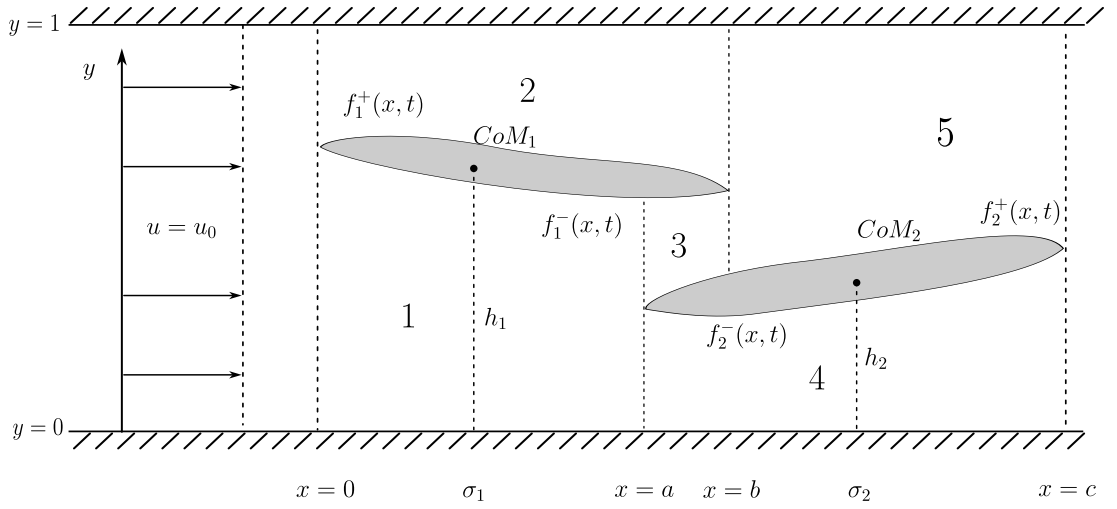


Figure 5.1: The interactive structure for two bodies, showing zones 1-5.

We will consider two thin bodies in channel flow as shown in Figure 5.1. The lower wall of the channel is given by $y = 0$, the upper wall is given by $y = 1$ in nondimensional terms. The first body starts at $x = 0$ and extends to $x = b$, while the second body extends from $x = a$ to $x = c$. Also, the leading edge of the second body is overlapped so that it is between the leading edge and the trailing edge of the body on the first body located on the left.

h_1 and h_2 are the vertical position of the center of mass for first and second body, respectively. Also, the horizontal positions of the center of mass (CoM) are defined as σ_1 and σ_2 for the first and second body in turn. The inclinations of the bodies are $\theta_1(t)$ and $\theta_2(t)$, respectively

There are five gaps ($H_1 - H_5$) between the upper and lower sides of the two bodies and the parallel walls of the fluid-filled channel. Here $f_1^+(x, t)$, $f_2^+(x, t)$ are the upper

surface functions whereas, $f_1^-(x, t)$, $f_2^-(x, t)$ denote the lower surface functions of the first and second bodies respectively. Thus the gap widths can be defined as follows

$$H_1 = f_1^-, H_2 = 1 - f_1^+, H_3 = f_1^- - f_2^+, H_4 = f_2^-, H_5 = 1 - f_2^+.$$

In addition, moving body surfaces can be defined as

$$f_1^-(x, t) = F_1^-(x) + h_1(t) + (x - \sigma_1)\theta_1(t), \quad (5.1a)$$

$$f_1^+(x, t) = F_1^+(x) + h_1(t) + (x - \sigma_1)\theta_1(t), \quad (5.1b)$$

$$f_2^-(x, t) = F_2^-(x) + h_2(t) + (x - \sigma_2)\theta_2(t), \quad (5.1c)$$

$$f_2^+(x, t) = F_2^+(x) + h_2(t) + (x - \sigma_2)\theta_2(t). \quad (5.1d)$$

The first body locates between $x = 0$ and $x = b$ while the second body has its leading edge at $x = a$ and trailing edge at $x = c$. The x -positions of the center of masses are σ_1 and σ_2 . We define the body shape functions as follows: $F_1^- = C_1(x) - T_1(x)/2$, $F_1^+ = C_1(x) + T_1(x)/2$, $F_2^- = C_2(x) - T_2(x)/2$, $F_2^+ = C_2(x) + T_2(x)/2$ where $C_1(x)$, $C_2(x)$ are the camber and $T_1(x)$ and $T_2(x)$ are the thickness of the bodies and are defined as $T_1(x) = \kappa \sin(\frac{\pi x}{b})$ and $T_2(x) = \kappa \sin(\frac{\pi(x-a)}{(c-a)})$.

As previously stated, given that $H_3 = f_1^- - f_2^+$ and $H_4 = f_2^-$, it can be defined as follows:

$$H_3 = [(F_1^-(x) - F_2^+(x)) + (h_1(t) - h_2(t)) + ((x - \sigma_1)\theta_1(t) - (x - \sigma_2)\theta_2(t))], \quad (5.2)$$

$$H_4 = F_2^-(x) + h_2(t) + (x - \sigma_2)\theta_2(t). \quad (5.3)$$

Integrating (3.8) and after some manipulation, we get

$$u_3 H_3 = c_3(t) - [(x - \sigma_1)\dot{h}_1 - (x - \sigma_2)\dot{h}_2] - \frac{1}{2} [(x - \sigma_1)^2 \dot{\theta}_1 - (x - \sigma_2)^2 \dot{\theta}_2], \quad (5.4)$$

and

$$u_4 H_4 = c_4(t) - (x - \sigma_2)\dot{h}_2 - \frac{1}{2}(x - \sigma_2)^2 \dot{\theta}_2. \quad (5.5)$$

Here c_3 and c_4 are constants of integration. The oncoming flow is assumed as $u = u_0 = 1$ and the velocities in the gaps are expressed as u_n ($n = 1 - 5$). The flow is supposed to proceeds from left to right through all region with u_n being positive. Therefore, by expanding the notations and assumptions in the previous chapter, we have the mass balance $H_n u_n = d_n$ ($n = 1 - 5$) where d_n is constant for each gap. This is due to the

assumption that states the density ratio small, i.e., flow can be treated as if quasi-steady. In addition, due to Euler's condition, Bernoulli's principle is also satisfied throughout the region 1-5.

$$p_n + \frac{1}{2}u_n^2 = \frac{1}{2} \quad \text{for } (n = 1 - 5) \quad (5.6)$$

So, we get

$$p_n = \frac{1}{2} - \frac{1}{2} \left(\frac{d_n}{H_n} \right)^2. \quad (5.7)$$

It is anticipated that there will be fluctuations in pressure p and velocity u across the Euler regions in general; however, the quantities in (5.7) will remain uniformly conserved across those gaps. Then, considering the total mass flux, since the incident velocity profile $u_0 = 1$ and the channel width $H_0 = 1$ in the absence of the body gives

$$u_1 H_1 + u_2 H_2 = 1. \quad (5.8)$$

Thus, we find

$$d_1 + d_2 = 1 \quad (5.9)$$

Similarly, we have the following relations

$$u_3 H_3 + u_4 H_4 = u_1 H_1 \quad \rightarrow \quad d_3 + d_4 = d_1 \quad (5.10a)$$

$$u_2 H_2 + u_3 H_3 = u_5 H_5 \quad \rightarrow \quad d_2 + d_3 = d_5 \quad (5.10b)$$

Thus,

$$d_1 + d_2 = d_4 + d_5 = 1 \quad (5.10c)$$

is obtained from the equations. Additionally, from the Bernoulli equation in (5.7) we get the following

$$\frac{d_2^2}{H_2(b, t)^2} = \frac{d_3^2}{H_3(b, t)^2}, \quad (5.10d)$$

and

$$\frac{d_5^2}{H_5(c, t)^2} = \frac{d_4^2}{H_4(c, t)^2}. \quad (5.10e)$$

It can be demonstrated that the equality of the values in (5.10d) with the value $\frac{d_5^2}{H_5(b, t)^2}$ implied by region 5 is guaranteed. Thus we have five equations (5.10d, e) and (5.10a, e) for the five unknowns d_1-d_5 . The balances (5.10d, e) give

$$\frac{d_2}{H_2(b, t)} = \frac{d_3}{H_3(b, t)} = \frac{d_5}{H_5(b, t)} \quad (5.11a)$$

and

$$\frac{d_5}{H_5(c, t)} = \frac{d_4}{H_4(c, t)} \quad (5.11b)$$

since u_n being positive. Therefore, the solutions can be found as

$$d_1(t) = 1 - \frac{H_2(b, t)H_5(c, t)}{H_5(b, t)}, \quad (5.12a)$$

$$d_2(t) = \frac{H_2(b, t)H_5(c, t)}{H_5(b, t)}, \quad (5.12b)$$

$$d_3(t) = \frac{H_3(b, t)H_5(c, t)}{H_5(b, t)}, \quad (5.12c)$$

$$d_4(t) = H_4(c, t), \quad (5.12d)$$

$$d_5(t) = H_5(c, t). \quad (5.12e)$$

In terms of body motion, two bodies must move due to mass acceleration effects.

Therefore,

$$M_1 \ddot{h}_1 = \int_0^a (p_1 - p_2) dx + \int_a^b (p_3 - p_2) dx, \quad (5.13a)$$

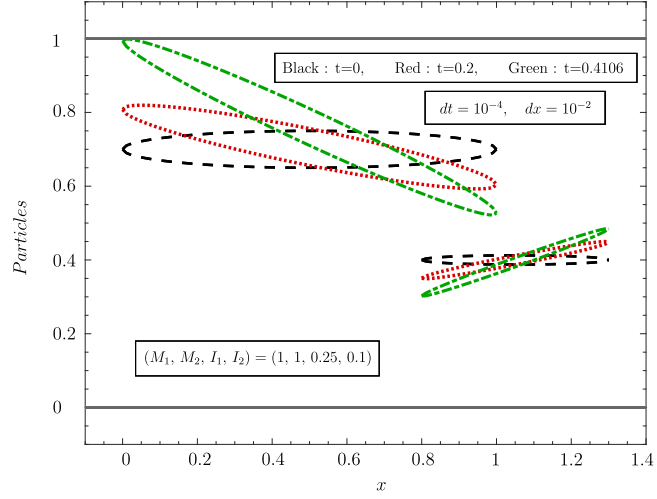
$$I_1 \ddot{\theta}_1 = \int_0^a (x - \sigma_1)(p_1 - p_2) dx + \int_a^b (x - \sigma_1)(p_3 - p_2) dx, \quad (5.13b)$$

for the first body and

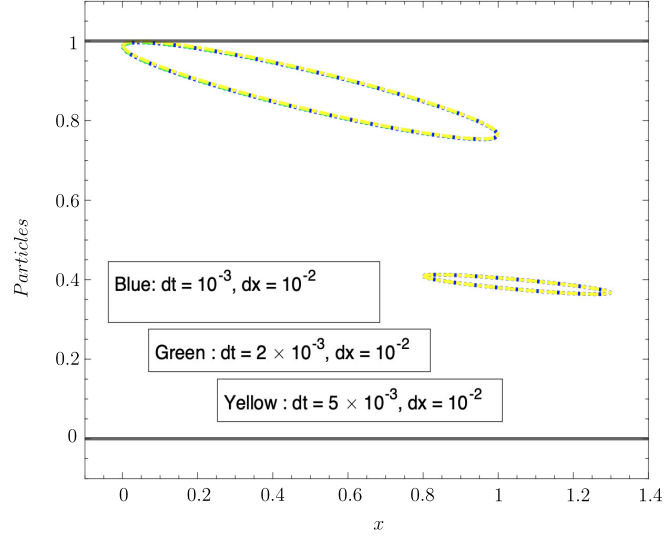
$$M_2 \ddot{h}_2 = \int_b^c (p_4 - p_3) dx + \int_c^d (p_5 - p_4) dx, \quad (5.13c)$$

$$I_2 \ddot{\theta}_2 = \int_b^c (x - \sigma_2)(p_4 - p_3) dx + \int_c^d (x - \sigma_2)(p_4 - p_5) dx, \quad (5.13d)$$

for second body. Here M_1 and M_2 are the masses of the first and second bodies, respectively, whereas I_1 and I_2 are the moments of inertia of the first and second bodies, in turn. In $\ddot{h}_1$, $\ddot{h}_2$, $\ddot{\theta}_1$ and $\ddot{\theta}_2$ represent the vertical acceleration and angular acceleration, respectively. p_n denotes the fluid pressure in the gaps, while u_n represents the fluid velocity in the gaps, respectively (where $n = 1 - 5$).



(a) Two bodies at $t = 0, 0.2, \sim 0.41$.



(b) Grid refinement.

Figure 5.2: Two bodies interacting in channel flow. Grid refinement.

Figure 5.2a illustrates bodies' positions at some specific times, i.e $t = 0, 0.2$ and around 0.41 . The first body clashes against the upper wall of the channel. Grid refinement is made in Figure 5.2b with $(dx, dt) = (10^{-2}, 10^{-3}), (10^{-2}, 2.10^{-3}), (10^{-2}, 5.10^{-3})$.

5.2. Three Bodies

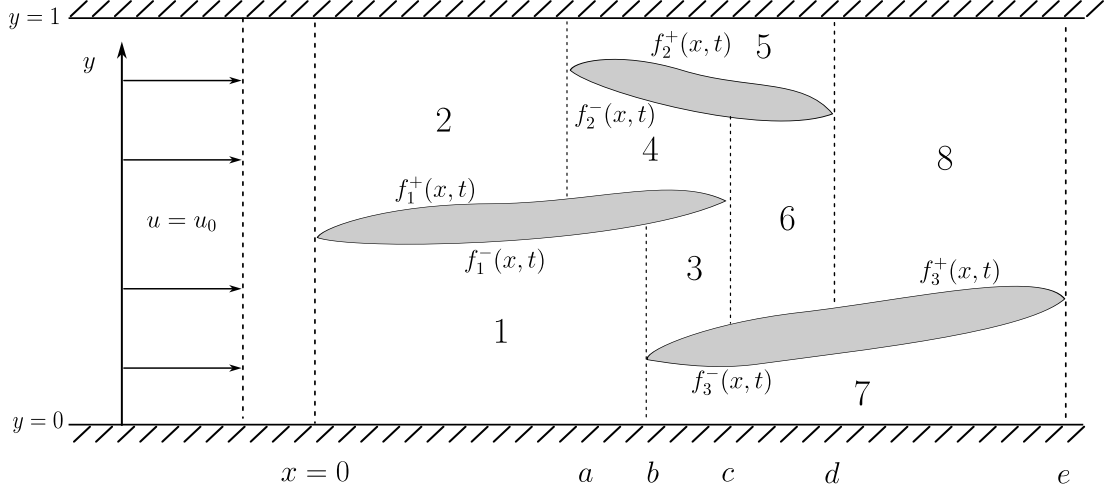


Figure 5.3: Sketch of the three-body problem and its solution structure, showing regions 1-8.

The extension of the model provided for two bodies interacting with a fluid in the previous section to three bodies is shown in Figure 5.3. The lower wall of the channel is given by $y = 0$, the upper wall is $y = 1$ in nondimensional terms. The first body starts at $x = 0$ and extends to $x = c$, the second body extends from $x = a$ to $x = d$, while the third body extends from $x = b$ to $x = e$. The bodies are overlapping as illustrated in Figure 5.3

Here h_1 , h_2 and h_3 are the vertical position of the center of mass for first, second and third bodies, respectively. Similarly, the horizontal positions of the center of mass (CoM) are defined as σ_1 , σ_2 and σ_3 for first, second and third bodies, in turn. The inclinations of the bodies are given $\theta_1(t)$, $\theta_2(t)$ and $\theta_3(t)$, respectively.

Here $f_1^+(x, t)$, $f_2^+(x, t)$, $f_3^+(x, t)$ represent the upper surface functions whereas, $f_1^-(x, t)$, $f_2^-(x, t)$, $f_3^-(x, t)$ denote the lower surface functions of the bodies respectively. There are eight separate long-scale regions, denoted by H_n for $n = 1 - 8$. Similar to the assumptions for the two bodies in the previous section, the flow can be considered as quasi-steady due to the assumption of a small density ratio. Thus, we have the mass balance $H_n(x, t)u_n(x, t) = d_n(t)$ for $n = 1 - 8$. Additionally, due to Euler's condition,

Bernoulli's principle is satisfied throughout regions 1-8,

$$p_n(x, t) + \frac{1}{2}u_n^2(x, t) = \frac{1}{2} \quad \text{for } (n = 1 - 8) \quad (5.14)$$

The governing equations in the eight region are similar to those for two body case,

$$p_n(x, t) = \frac{1}{2} - \frac{1}{2} \left(\frac{d_n(t)}{H_n(x, t)} \right)^2. \quad (5.15)$$

The total mass flux holds,

$$u_1(x, t)H_1(x, t) + u_2(x, t)H_2(x, t) = 1, \quad (5.16)$$

due to the incident velocity profile $u_0 = 1$ and the channel width $H_0 = 1$ in the absence at bodies. Thus, we find

$$d_1(t) + d_2(t) = 1. \quad (5.17)$$

Similarly, we have the following relations,

$$u_3H_3 + u_7H_7 = u_1H_1 \quad \rightarrow \quad d_3(t) + d_7(t) = d_1(t), \quad (5.18a)$$

$$u_4H_4 + u_5H_5 = u_2H_2 \quad \rightarrow \quad d_4(t) + d_5(t) = d_2(t), \quad (5.18b)$$

$$u_3H_3 + u_4H_4 = u_6H_6 \quad \rightarrow \quad d_3(t) + d_4(t) = d_6(t), \quad (5.18c)$$

$$u_5H_5 + u_6H_6 = u_8H_8 \quad \rightarrow \quad d_5(t) + d_6(t) = d_8(t). \quad (5.18d)$$

We therefore obtain,

$$d_7(t) + d_8(t) = d_1(t) + d_2(t) = 1 \quad (5.18e)$$

Due to the well known Kutta condition, the following relations hold,

$$p_3(c_-, t) = p_4(c_-, t) \quad (5.19a)$$

$$p_5(d_-, t) = p_6(d_-, t) \quad (5.19b)$$

$$p_7(e_-, t) = p_8(e_-, t) \quad (5.19c)$$

Additionally, from the Bernoulli principle we get the following,

$$\frac{d_3^2}{H_3(c, t)^2} = \frac{d_4^2}{H_4(c, t)^2} = \frac{d_6^2}{H_6(c, t)^2}, \quad (5.20a)$$

$$\frac{d_5^2}{H_5(d, t)^2} = \frac{d_6^2}{H_6(d, t)^2} = \frac{d_8^2}{H_8(d, t)^2}, \quad (5.20b)$$

$$\frac{d_7^2}{H_7(e, t)^2} = \frac{d_8^2}{H_8(e, t)^2}. \quad (5.20c)$$

The equality of the values in (5.20a) with the value $\frac{d_6^2}{H_6(c,t)^2}$ implied by region 6 is guaranteed by the displayed equations above. Similarly, the equality of the values in (5.20b) with the value $\frac{d_8^2}{H_8(d,t)^2}$ implied by region 8 is also assured by the displayed equations above. Thus we have eight equations (5.20a, c) and (5.18a, e) for the eight unknowns d_1-d_8 . The balances (5.20a, c) give

$$\frac{d_3}{H_3(c,t)} = \frac{d_4}{H_4(c,t)} = \frac{d_6}{H_6(c,t)}, \quad (5.21a)$$

$$\frac{d_5}{H_5(d,t)} = \frac{d_6}{H_6(d,t)} = \frac{d_8}{H_8(d,t)}, \quad (5.21b)$$

$$\frac{d_7}{H_7(e,t)} = \frac{d_8}{H_8(e,t)}. \quad (5.21c)$$

since u_n needs to be positive. Therefore, the solutions can be found as

$$d_1(t) = 1 - \frac{H_4(c,t)H_6(d,t)H_8(e,t)}{H_8(d,t)H_6(c,t)} - \frac{H_5(d,t)H_8(e,t)}{H_8(d,t)}, \quad (5.22a)$$

$$d_2(t) = \frac{H_4(c,t)H_6(d,t)H_8(e,t)}{H_8(d,t)H_6(c,t)} + \frac{H_5(d,t)H_8(e,t)}{H_8(d,t)}, \quad (5.22b)$$

$$d_3(t) = \frac{H_3(c,t)H_6(d,t)H_8(e,t)}{H_8(d,t)H_6(c,t)}, \quad (5.22c)$$

$$d_4(t) = \frac{H_4(c,t)H_6(d,t)H_8(e,t)}{H_8(d,t)H_6(c,t)}, \quad (5.22d)$$

$$d_5(t) = \frac{H_5(d,t)H_8(e,t)}{H_8(d,t)}, \quad (5.22e)$$

$$d_6(t) = \frac{H_6(d,t)H_8(e,t)}{H_8(d,t)}, \quad (5.22f)$$

$$d_7(t) = H_7(e,t), \quad (5.22g)$$

$$d_8(t) = H_8(e,t). \quad (5.22h)$$

In terms of body motion, three bodies must move due to mass acceleration effects.

Therefore, we have the following relations

$$M_1 \ddot{h}_1 = \int_0^a (p_1 - p_2) dx + \int_a^b (p_1 - p_4) dx + \int_b^c (p_3 - p_4) dx, \quad (5.23a)$$

$$I_1 \ddot{\theta}_1 = \int_0^a (x - \sigma_1)(p_1 - p_2) dx + \int_a^b (x - \sigma_1)(p_1 - p_4) dx + \int_b^c (x - \sigma_1)(p_3 - p_4) dx, \quad (5.23b)$$

for the first body,

$$M_2 \ddot{h}_2 = \int_a^c (p_4 - p_5) dx + \int_c^d (p_6 - p_5) dx, \quad (5.23c)$$

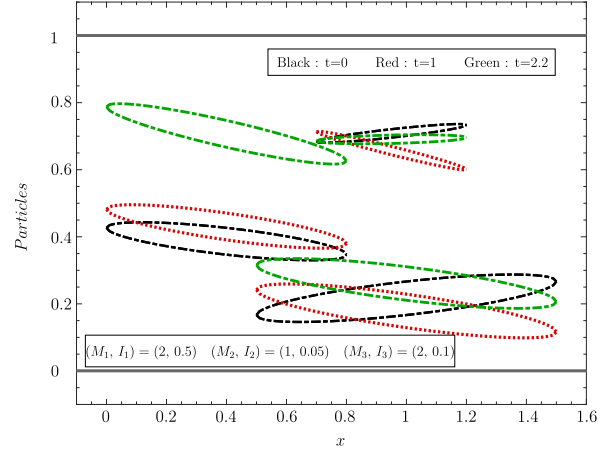
$$I_2 \ddot{\theta}_2 = \int_a^c (x - \sigma_2)(p_4 - p_5) dx + \int_c^d (x - \sigma_2)(p_6 - p_5) dx, \quad (5.23d)$$

for second body, finally,

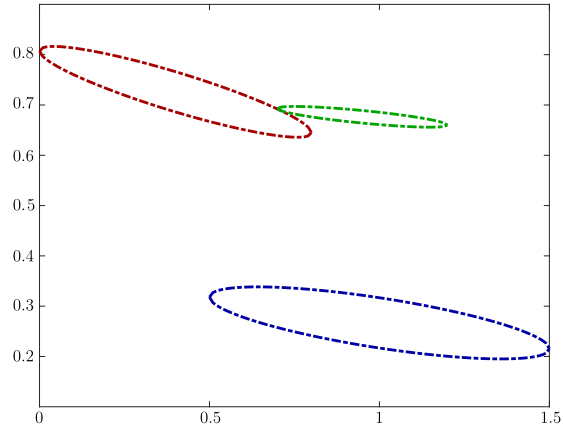
$$M_3 \ddot{h}_3 = \int_b^c (p_7 - p_3) dx + \int_c^d (p_7 - p_6) dx + \int_d^e (p_7 - p_8) dx, \quad (5.23e)$$

$$I_3 \ddot{\theta}_3 = \int_b^c (x - \sigma_3)(p_7 - p_3) dx + \int_c^d (x - \sigma_3)(p_7 - p_6) dx + \int_d^e (x - \sigma_3)(p_7 - p_8) dx, \quad (5.23f)$$

for third body. Here M_1 , M_2 and M_3 are scaled masses of the first, second and third bodies, respectively, whereas I_1 , I_2 and I_3 denote scaled moments of inertia. In (5.23a, f) the values $\ddot{h}_1$, $\ddot{h}_2$, $\ddot{h}_3$ are represent the vertical accelerations while $\ddot{\theta}_1$, $\ddot{\theta}_2$ and $\ddot{\theta}_3$ represent the angular accelerations at each body, respectively. In addition $p_1 - p_8$ denot the fluid pressure in respective gaps, while $u_1 - u_8$ represent the fluid velocity in these eight gaps, respectively.

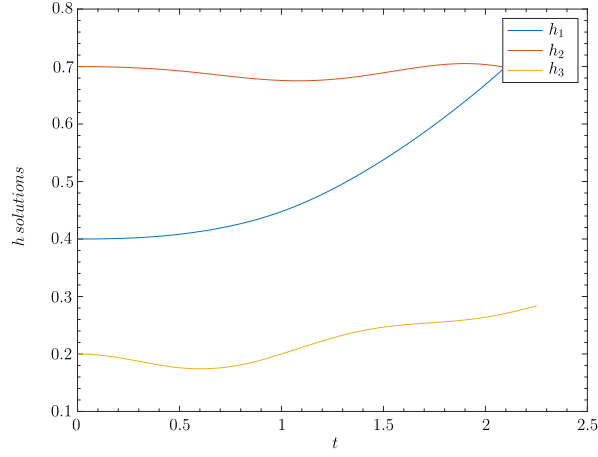


(a) Three bodies at $t = 0, 1, 2.3$.

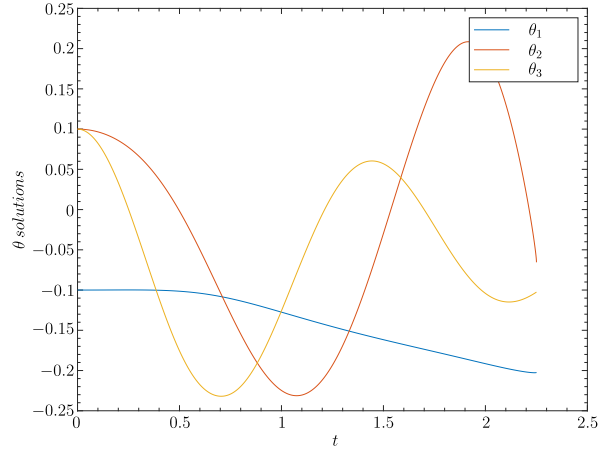


(b) Body positions at time of impact $t_c \sim 2.36$.

Figure 5.4: Three bodies interacting in channel flow.



(a) h_1, h_2, h_3 .



(b) $\theta_1, \theta_2, \theta_3$.

Figure 5.5: Sample behaviour for oscillating three bodies. *CoM* positions of the bodies are $1/3 * l_i$ where l_i is the length of the body i . The length of the bodies $l_1 = 0.8$, $l_2 = 0.5$, $l_3 = 1$.

Sample behaviours are presented in Figure 5.4 and 5.5 for three-body motion. In Figure 5.4b the first body clashes against the second body at around $t = 2.36$. This can be observed in Figure 5.4a for the same parameters and under the same initial conditions. If bodies have nonhomogenous density with $\sigma_i = 1/3l_i$, where l_i is the length of the i th body, the bodies can oscillate as illustrated in Figure 5.5a,b. Further analysis is required to investigate these very interesting findings, see [13].

6. CONCLUSION

The aim of this thesis is to investigate how particles move between parallel walls such as the movement of fuel particles in an exhaust pipe, the movement of blood cells in arteries or the movement of ice particle in sensors on an airfoil. The study focused on a two-way, nonlinear, two-dimensional interaction between the fluid and body or between body and body. This thesis examined the interaction between fluid flow and a single body or two bodies in a straight and thin channel using analytical and numerical methods.

In Chapter 2 some preliminary information is given to understand the concept of fluid-body interaction and fluid dynamics in general. In Chapter 3, the nonlinear interaction model between the thin, small solid body having thickness and/or camber is presented. Based on the previous research, we also established the nondimensional continuity, axial momentum and normal momentum equations of the fluid flow in question, which form a reduced system derived from the Navier-Stokes equations. We then established the boundary conditions the on pressure i.e., the Kutta condition at the trailing edge of the body and Euler jump at the leading edge. Numerical simulations are then presented. A finite difference scheme is employed to solve the nonlinear interaction satisfying Kutta condition at the trailing edge. The effect of thickness on the impact is also seen in the results. In the next chapter, the impact model between a body with nonhomogeneous density distribution and the parallel walls of a fluid-filled channel is discussed in detail. Finally, multiple bodies interacting with each other and with the fluid are studied in a quasi-steady flow setting. We begin our analysis with two bodies and then continue to examine three bodies in flow. Some interesting properties have been arised, which need to be analysed in detail.

The study includes comprehensive analysis and numerical results and provided insights through comparative analyzes between these two approaches. The next research topic could involve the modification of channel wall shapes. For instance, equipment may sustain damage, resulting in rough surfaces. Altering the wall shape function can lead to variations in pressure distributions, significantly influencing particle dynamics. The number of moving bodies can be increased. Further research on four or more

bodies having certain thickness or camber and interacting with surrounding fluid in a quasi-steady or unsteady flow setting can also be interesting.

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BIOGRAPHY

Yasin Sefa ASLAN graduated from Selçuk University, Faculty of Science, Department of Mathematics in 2014. He started his master's degree in 2022 as a student at Gebze Technical University, Faculty of Science, Department of Mathematics. He is married and he has one child.

PUBLICATIONS WITHIN the SCOPE of THESIS STUDY

- [1] Aslan Y. S., Yazar S., (2024), "Free Motion of a Particle Within Fluid Flow: IMPACT PHENOMENA", Workshop On Nonlinear Systems IV, January 02-03, Gebze Technical University, Türkiye.
- [2] Aslan Y. S., Yazar S., (2024), "Interaction Between Grains and Surrounding Fluid: Impact Phenomena", 7th International Hybrid Conference on Mathematical Advances and Applications, May 08-11, Yıldız Technical University, Türkiye.
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- [4] Aslan Y. S., Yazar S., (2024), "Behaviour of a Single Body or Multiple Bodies in Channel Flow", 8th GTU Graduate Research Symposium, May 30-31, Gebze Technical University, Türkiye.