

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

BUCKLING OF ISOTROPIC AND COMPOSITE COLUMNS
WITH UNCERTAIN MATERIAL PROPERTIES AND DIMENSIONS
BY USING CONVEX MODELING

ONUR KAYA
A THESIS SUBMITTED FOR THE DEGREE OF
MASTER OF SCIENCE
DEPARTMENT OF MECHANICAL ENGINEERING

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ASSOC. PROF. DR. AHMET SİNAN ÖKTEM

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T.C.
GEBZE TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

BELİRSİZ MALZEME ÖZELLİKLERİ VE
BOYUTLARA SAHİP İZOTROPİK VE
KOMPOZİT KOLONLARIN KONVEKS
MODEL KULLANILARAK BURKULMASI

ONUR KAYA
YÜKSEK LİSANS TEZİ
MAKİNE MÜHENDİSLİĞİ ANABİLİM DALI

DANIŞMANI
DOÇ. DR. AHMET SİNAN ÖKTEM

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2019

GEBZE TEKNİK ÜNİVERSİTESİ	YÜKSEK LİSANS JÜRİ ONAY FORMU
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SUMMARY

Until now, deterministic approaches have been applied in many studies on buckling of columns. As a matter of fact, since the properties and dimensions of the material which has an important role in the column design, may vary due to the defects in production stages, the deviations in layer thickness for composite materials and many problems such as those, the aforementioned parameters which have a significant effect on the critical buckling load, should be considered as uncertain. Differences in material properties and dimensions may lead to buckling of structures at lower loads and this may cause to human and material losses. Therefore, the issue of uncertainty is a very important fact that allows us to obtain realistic results. As a result of this study, analytical equations in closed form are obtained for engineers and researchers to obtain the lowest linear critical buckling load at different uncertainty levels. Sensitivity analysis reveals that which parameter has the most effect on the lowest buckling load at different uncertainty levels for all cross-section types. In this study Euler-Bernoulli beam theory is used for isotropic columns and first-order shear deformation theory is also used for orthotropic composite column sections. The lowest analytical critical buckling equations at different uncertainty levels in isotropic and composite columns are obtained by using convex model. In order to verify the equations, finite elements modeling of some columns of different cross-sections are performed and the results are compared. Practical formulas are obtained for engineers, especially in the preliminary design stage.

Key Words: Buckling, Convex Modeling, Finite Elements Method, Uncertainty.

ÖZET

Şimdiye kadar kolonlarda burkulmanın incelendiği birçok çalışmada belirli (deterministik) yaklaşımlar uygulanmıştır. Kolon tasarımında önemli role sahip olan malzemenin özellikleri ve boyutları, üretim aşamasında olan hatalardan, kompozit malzemelerde tabaka kalınlıklarında olabilecek sapmalardan ve bunlar gibi birçok problemten dolayı değişkenlik gösterdiğinden, kritik burkulma yüküne önemli derecede etki eden bu parametreler belirsiz olarak tanımlanmalıdır. Malzeme özelliklerinde ve boyutlarında olabilecek farklılıklar yapının daha düşük yüklerde burkulmasına ve bu durum da insan ve malzeme kayıplarına yol açmaktadır. Bu sebeple belirsizlik konusu gerçeğe daha yakın sonuçlar elde etmeye olanak sağlayan önemli bir olgudur. Bu çalışmanın sonucunda mühendisler ve araştırmacılar için farklı belirsizlik seviyelerinde en düşük doğrusal kritik burkulma yükünü elde edebilecekleri kapalı formda analitik denklemler elde edilmiştir. Böylece üretim aşamasına gelinmeden tasarlanacak olan kolonlar için elastik burkulma davranışı tahmin edilmiştir. Hassaslık analiziyle tüm kesit tipleri için hangi parametrenin farklı belirsizlik seviyelerinde en düşük burkulma yüküne en çok etki ettiği ortaya konulmuştur. Belirlenen şartlar altında gerçeğe daha yakın kritik burkulma değerlerini bilmek insan ve malzeme kayıplarının en aza indirilmesine olanak tanır. Dolayısıyla pratik kullanımda en uygun tasarımın elde edilmesi mümkün olur. Bu çalışmada izotropik kolonlar için Euler-Bernoulli kiriş teorisi, ortotropik kompozit kolon kesitleri için ise birinci dereceden kayma deformasyon teorisi kullanılmıştır. İzotropik ve kompozit kolonlarda farklı belirsizlik seviyelerinde olabilecek en düşük analitik kritik burkulma denklemleri konveks model kullanılarak elde edilmiştir. Denklemlerin doğrulanması için farklı kesitteki bazı kolonların sonlu elemanlar modellemesi yapılmış ve sonuçlar karşılaştırılmıştır. Mühendisler için özellikle ön tasarım aşamasında kullanabilecekleri pratik formüller elde edilmiştir.

Anahtar Kelimeler: Belirsizlik, Burkulma, Konveks Model, Sonlu Elemanlar Metodu.

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LIST of ABBREVIATIONS and ACRONYMS

<u>Abbreviations</u>	<u>Explanations</u>
<u>and Acronyms</u>	
D_{ij}	: Bending stiffness matrix expressions
E	: Young's modulus
G_{13}	: Shear modulus in plane 1-3
$P_{cr,r}$	: Critical buckling load for isotropic rectangular column
$P_{cr,c}$	: Critical buckling load for isotropic circular column
$P_{cr,rt}$	: Critical buckling load for isotropic rectangular tube column
$P_{cr,ct}$	: Critical buckling load for isotropic circular tube column
$P_{cr,I}$	: Critical buckling load for isotropic I-section column
$P_{cr,U}$	: Critical buckling load for isotropic U-section column
$P_{cr,L}$	: Critical buckling load for isotropic L-section column
$P_{cr,r}^c$	: Critical buckling load for composite rectangular column
$P_{cr,c}^c$	: Critical buckling load for composite circular column
$P_{cr,rt}^c$	: Critical buckling load for composite rectangular tube column
$P_{cr,ct}^c$	: Critical buckling load for composite circular tube column
$P_{cr,I}^c$	: Critical buckling load for composite I-section column
$P_{cr,U}^c$	: Critical buckling load for composite U-section column
$P_{cr,L}^c$	: Critical buckling load for composite L-section column
$\bar{Q}_{ij}$	: Transformed reduced stiffness matrix expressions

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1. INTRODUCTION

In general, components exposed to axial loads in structures are called columns. Columns are the most common components in the construction used in machinery, aviation and marine industries, which enable the structures to remain robust and stable. Composite structure columns are widely used in industry because they offer high strength-to-weight and high stiffness-to-weight ratios. Knowing the buckling behavior of these structures exposed to axial load contributes to the minimization of life and property losses.

Buckling occurs when a member is axially under compressive load, particularly in thin-walled structures. When the applied load reaches a point that unbalances the column, large lateral deviations occur on the structure and this is expressed as buckling. The critical buckling load is an important limit for the structural stability of the column. Engineers need to be able to predict whether the columns are structurally stable under a certain axial load. Therefore, critical buckling load has an important role in columns.

In many analytical studies that have investigated the buckling of columns, certain (deterministic) approaches have been applied. Barbero and Tomblin (1993), based on the Euler-Bernoulli beam theory, obtained the buckling load equation in the I-section composite columns with the classical lamination theory. However, in the analytical results that were compared experimentally, it was found that the study could be valid only for long columns. For short columns, it was not a study that takes into account the lateral sliding effects that can give accurate results. Akbulut et al. (2010), in the columns of gradually changed cross-section, examined the behavior of buckling based on the first-order shear deformation theory (FSDT). In the region where the column section varies gradually, it was observed that the buckling resistance is increased in case of using curved structure. The results obtained analytically were validated by finite elements method. Schnabl and Planinc (2011), in their statement, presented an effective mathematical model to study the geometrically perfect elastic two-layer composite columns with the sliding behavior of layers. The analytical model was based on the linearized stability theory and could predict the exact critical buckling loads. According to reported information, it has been reported that the negative impact of transverse shear deformation on buckling can be up to 20% for wood composite

columns and up to 40% for composite columns that can exhibit very flexible behavior. As a result of this study, it's seen that how much the use of first-order shear deformation theory in composite columns is important. However, since the results obtained in all reports are found by considering certain (deterministic) approaches, they cannot give information about the real buckling behavior of the columns as a result of fluctuations in material properties and dimensions.

In the literature, except for column structures, there are studies related with buckling analysis that define the material properties, dimensions and applied load of the shell and conical elements as uncertain [Zhang & Ellingwood, 1995; Papadopoulos & Papadrakakis, 2005; Gamez-Montero et al., 2009; Ifayefunmi and Błachut, 2011; Tomás and Tovar, 2012]. These studies, in general, reveal that the issue of uncertainty is very important for structures. Morgan et al. (1965), in the experimental study on elastic stability of cylindrical and conical shell structures, observed that there are differences between the theoretical results and experimental results. They have revealed that this is due to defects in the material. Therefore, the use of uncertainty methods in the analysis of buckling due to axial load in structures is taking crucial role in terms of the results closer to reality. Therefore, the engineering constants and dimensions of the material which have an important role in the column design will differ from the damages occurring during the production stage, from the gaps in the internal structure, from the deviations in the layer thicknesses for the composite materials or from the changes of dimensions that may occur during the production. Therefore, these parameters, which have a significant effect on the critical buckling load, should be defined as uncertain. Differences in material properties and dimensions lead to buckling of the structure at lower loads and this leads to losses of human and material (Figure 1.1) [Web 1, 2019].



Figure 1.1: Damage caused by buckling in the column.

Considering that many creatures shelter, value-added products are stored, various industrial activities are carried out in the structures where column support is required and the earthquake risk of the region where Gebze Technical University is located, knowing the buckling behavior of columns is vital. Convex model, probabilistic and statistical approaches are used to examine the uncertainty for the structures. Probabilistic calculations and statistical approaches require probabilistic density function and large amounts of experimental data. The advantage of the convex model, which can give more accurate results to the users in the relevant field, is that it can be used in uncertainty subjects without the need for experimentation [Ben-Haim ve Elishakoff, 1990; Qui, 2005; Radebe ve Adali, 2014; Oktem ve Adali, 2018]. For this reason, the convex model is used in this study.

Except for the convex model, some uncertainty applications are used based on probabilistic calculations. Likewise, studies on imperfect beams using convex model are also found in the literature [Verhaeghe et al., 2013]. Herein, cosinusoidal polynomials were used to obtain certain (deterministic) buckling loads. Then, probabilistic calculations and convex model were used for uncertainty analysis, a significant similarity was observed between probabilistic calculations and convex

models. Qui et al. (2006) examined the nonlinear buckling of a column. In cases where the deflections in the center of the columns are different from zero, minimum buckling load is determined by using first and second order Taylor series expansions. While the random coefficients of the Fourier series were used in the model based on probabilistic estimation, the upper-lower limit coefficients were used in the convex model. Zhang et al. (2017) examined the geometry of a thin imperfect plate by the method of uncertainty. Trigonometric series were used to determine the defects and Monte Carlo simulation was used to determine the limit value of the largest lateral deflection. However, in all studies reported here, many tests and data need to be collected for the methods based on probabilistic calculations. This is a costly business as well as wasting time. Therefore, with the help of the convex model which does not require any experiment, the results obtained by obtaining the critical buckling load equations of the columns with uncertainty are advantageous compared to the approaches such as probabilistic calculations.

The convex model is a model used to obtain the equation that gives the minimum and maximum critical buckling load at different levels of uncertainty in non-deterministic applications. This model has been applied to exact solutions such as Least-Square Vector Machine (LSSVM) [Zhang et al., 2017], nonlocal plate theory [Radebe and Adali, 2014] and cosinusoidal polynomials [Verhaeghe et al., 2013]. Radebe and Adali reported a study based on uncertainty on a nano-sized plate due to the possible defects that can be made since the material properties at the nano level can vary during production and at the same time it is difficult to measure in dimensions. They obtained a certain critical buckling load equation using nonlocal plate theory and then applied a convex model. Lagrange multipliers were used to calculate the uncertain parameters. Thus, a very useful equation is obtained which provides the minimum and maximum critical buckling loads at different levels of uncertainty. They determined the effect of small-scale changes on the critical buckling load at different levels of uncertainty [Radebe and Adali, 2014]. In another study, Oktem and Adali used the convex model to determine the smallest buckling load at different uncertainty levels in the nanocomposite columns where the material properties were defined as uncertain. The effects of various material parameters on buckling behavior at different levels of uncertainty were investigated. In this study, it has been observed that there are significant losses in critical buckling load as a result of slight changes in the parameters [Oktem and Adali, 2018]. Radebe and Adali

determined the buckling behavior of a composite cylinder in which the material properties were defined as uncertain under externally applied pressure by using a convex model. For hybrid cross-ply cylinders with high rigidity skin and low rigidity core layers, the most cost-effective design problem was examined with material properties that differed around the nominal values, resulting in a very useful buckling equation for engineers in analytically expressed closed form [Radebe and Adali, 2014]. Bi et al. using the convex model and the LSSVM method, they examined whether the cylindrical shells with laminated and functionally graded properties were robust or not. They concluded that even minor changes in the level of uncertainty led to intense fluctuations in critical buckling load. Considering the studies conducted using convex model, it's seen that it is inevitable that small differences in material properties and dimensions which cause changes in critical buckling load should be included in the calculations of buckling [Bi et al., 2013].

When literature is analyzed, it is understood that buckling loads are important in columns with the subject of uncertainty. There is a preliminary study that Kaya and Oktem reported that the material properties and dimensions are considered as uncertain in the isotropic columns in the limited cross-sections of columns and that the critical buckling equations are used analytically by using the convex model [Kaya and Oktem, 2018] and also there is no study in which the critical buckling load is examined with the uncertainty of the material properties and dimensions by using convex model which is investigated in this thesis for the composite structures.

In this study, the fluctuations in critical linear buckling loads were investigated by the uncertainty analysis of the columns with isotropic properties and laminated orthotropic composite structures. Considering the lowest buckling load with the convex model, closed form analytical formulas have been obtained which provide critical buckling load due to uncertain material properties and dimensions of columns. Losses in critical linear buckling loads resulting from the changing of various parameters have been determined at different levels of uncertainty by analytical equations in closed form. Thus, linear elastic buckling behavior can be estimated for the columns to be designed before the production phase is reached. Sensitivity analysis also carried out to understand which parameter has the most effect on the lowest critical buckling load at different uncertainty levels. The analytically obtained equations were confirmed by finite element analysis for some columns. Especially for the engineers, practical formulas were obtained. To know this information in advance,

especially in the preliminary design of the material properties and production methods to determine the most appropriate design can be a reference to the engineers. On the other hand, knowing all the critical linear buckling values closer to reality under all conditions will minimize human and material losses. Analytical equations obtained can be used in construction, marine, aerospace and machinery industries. With the data to be obtained in this study, the number of experiments and analyzes will be reduced; thus, material, energy and time loss will be prevented.

2. METHODOLOGY

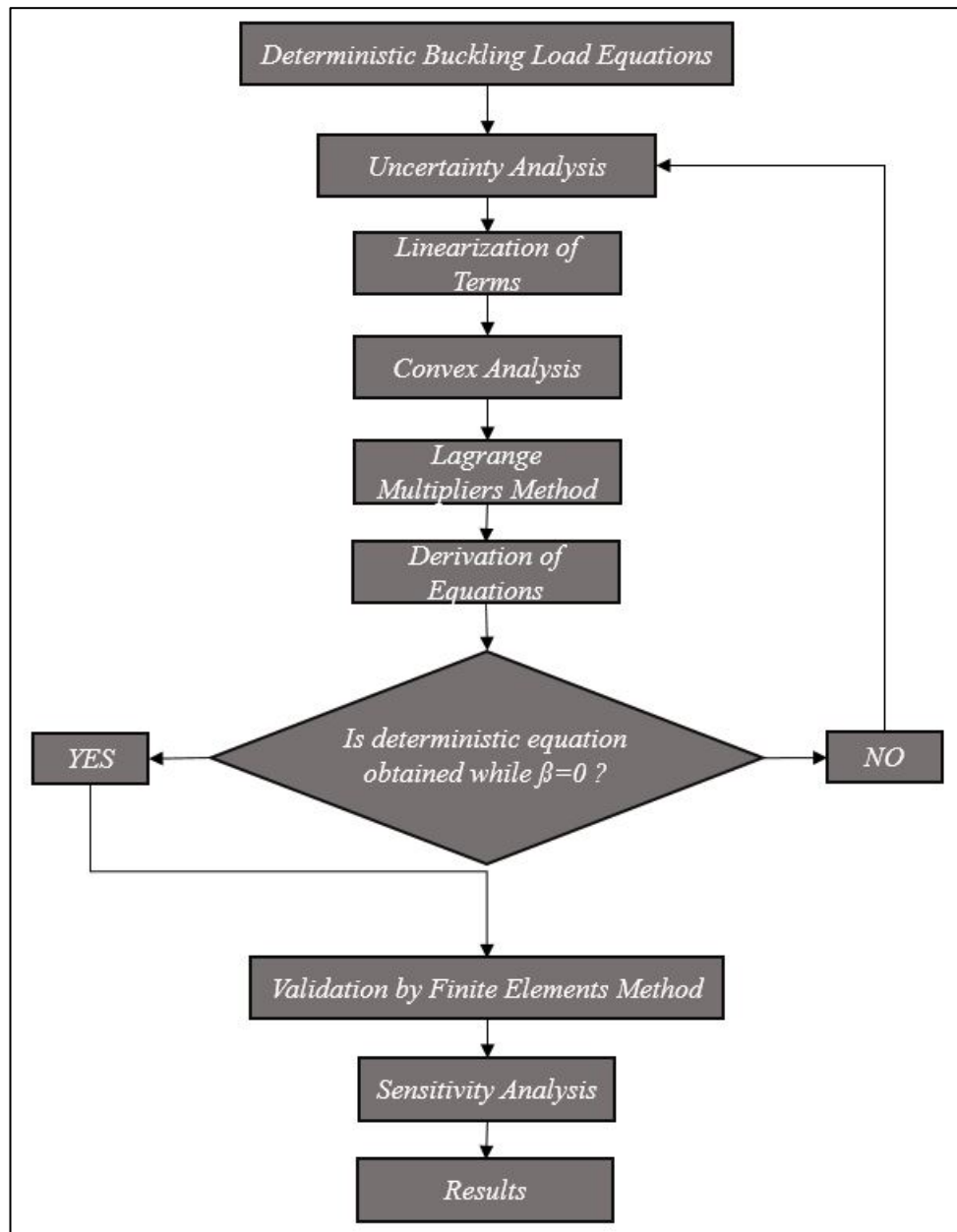


Figure 2.1: Flow chart.

Applied steps to obtain the minimum critical buckling load equations are shown in Figure 2.1 and are examined in detail in the following sections.

2.1. Deterministic Approaches & Problem Definition

The critical buckling equation (2.1) obtained by using Euler - Bernoulli [Euler, 1759] beam theory in isotropic columns and the critical buckling equation (2.10) obtained by first - order shear deformation theory for composite orthotropic materials are used as deterministic approaches to obtain the minimum critical buckling load equations which have uncertain material properties and dimensions. In this study, for the isotropic columns, when the ratio of the column length to the width of its cross-section, in other words, the slenderness ratio is 60 and above [Desai, 2018], critical buckling load values provide appropriate results with compared to experimental data. In this study, Euler-Bernoulli beam theory is used for the values of slenderness ratio which equals 60 or above.

$$P_{cr} = \frac{n^2 \pi^2 E I}{(k L)^2} \quad (2.1)$$

Equation (2.1) shows the critical buckling load based on Euler-Bernoulli beam theory where E is the modulus of elasticity, I is area of moment of inertia, L is column length, n is the buckling mode ($n = 1$ as the fundamental mode in this study) and k is the coefficient that varies for different boundary conditions. All equations are obtained for the simply supported from both ends ($k=1$) and load is applied as a constant force.

For composite columns, instead of using classical lamination theory (CLT), buckling analysis is carried out by taking into account the effects of transverse shear deformations with the aid of first-order shear deformation theory (FSDT), which is especially important in buckling analysis of laminated composites [Berthelot, 1999].

$$hG_{13} \left(\frac{d\varphi_x}{dx} + \frac{d^2\omega_0}{dx^2} \right) - N_0 \frac{d^2\omega_0}{dx^2} = 0 \quad (2.2)$$

$$\left(\frac{D_{11}D_{22} - D_{12}^2}{D_{22}} \right) \frac{d^2\varphi_x}{dx^2} - hG_{13} \left(\varphi_x + \frac{d\omega_0}{dx} \right) = 0 \quad (2.3)$$

where h and N_0 refers to the thickness of the rectangular cross-section and applied load, respectively. Multiplying the expressions (2.2) and (2.3) by width of cross-section (b), equations become as follows:

$$A_{cs}G_{13}\left(\frac{d\varphi_x}{dx} + \frac{d^2\omega_0}{dx^2}\right) - P\frac{d^2\omega_0}{dx^2} = 0 \quad (2.4)$$

$$b\left(\frac{D_{11}D_{22} - D_{12}^2}{D_{22}}\right)\frac{d^2\varphi_x}{dx^2} - A_{cs}G_{13}\left(\varphi_x + \frac{d\omega_0}{dx}\right) = 0 \quad (2.5)$$

The expressions satisfying the boundary conditions for the edges of the column are written as:

$$\varphi_x = A\cos\frac{m\pi x}{L} \quad (2.6)$$

$$\omega_0 = B\sin\frac{m\pi x}{L} \quad (2.7)$$

Substituting expressions (2.6) and (2.7) in equations (2.4) and (2.5), we obtain:

$$A\left(-\frac{m\pi}{L}A_{cs}G_{13}\right) + B\left(\frac{m^2\pi^2}{L^2}(P - A_{cs}G_{13})\right) = 0 \quad (2.8)$$

$$A\left(-\frac{m^2\pi^2}{L^2}bD - A_{cs}G_{13}\right) + B\left(-\frac{m\pi}{L}A_{cs}G_{13}\right) = 0 \quad (2.9)$$

A nonzero solution exists in the case where the determinant of the matrix of coefficients A and B is zero, which leads to:

$$P_{cr} = \frac{m^2\pi^2 A_{cs} G_{13} b D}{m^2\pi^2 b D + A_{cs} G_{13} L^2} \quad (2.10)$$

Deterministic critical buckling load equation for composite columns [Berthelot, 1999] is shown in equation (2.10) where A_{cs} , G_{13} , b , L and m refers to the area of cross-section, shear modulus in plane 1-3, width of section and length of column and buckling mode shape ($m=1$ for fundamental mode), respectively. D is one of the most important parameters that differ in each column type and therefore affects the buckling because it contains terms of the moment of inertia (Eq. 2.11).

$$D = \frac{D_{11} D_{22} - D_{12}^2}{D_{22}} \quad (2.11)$$

The expressions D_{11} , D_{12} and D_{22} are elements of the bending stiffness matrix linking bending moment components to plate curvature. Classical Lamination Theory, based on Kirchoff's hypothesis, gives close results when compared to experimental data in thin columns. However, when the ratio of the width to the thickness is less than 10, it gives results that are not accurate [Turvey, 1995; Berthelot, 1999; Jones, 1999; Reddy, 1999a, 2003b, 2006c; Altenbach and Becker, 2003]. Therefore, in this study, the first-order shear deformation theory is used for all cross-sections where the critical buckling calculation is performed by using uncertainty for composite columns.



Figure 2.2: Column with simply supported from both ends.

The boundary conditions of the orthotropic columns are analyzed for all sections with simply supported from both ends (Figure 2.2). The force acting on the columns is expressed as a constant load.



Figure 2.3: Cross-sections of columns.

Figure 2.3 shows different cross-sections which is analytically calculated minimum critical buckling equations at different levels of uncertainty of the isotropic and composite columns and are the most widely used columns in industry.

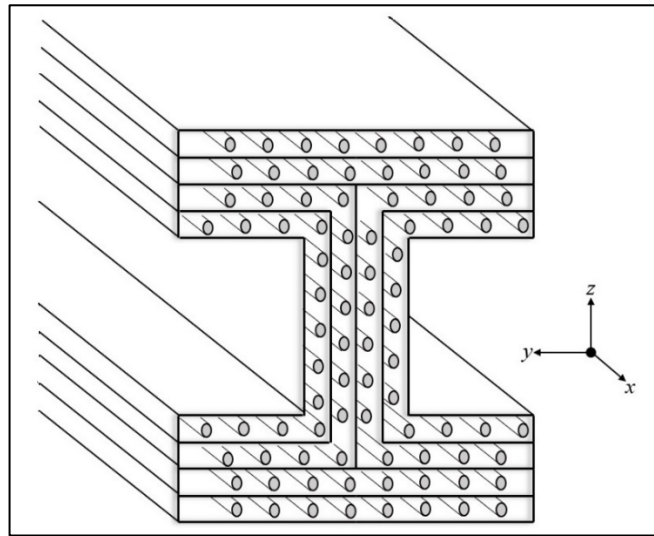


Figure 2.4: Demonstration of laminated composite column with I-section which has symmetrical stacking sequence $[0^\circ/0^\circ]_s$.

Figure 2.4 shows stacking sequence of the I-sectioned composite column. In this thesis, all columns with different cross-sections have the same stacking sequence for composite columns and the derivations of equations are obtained by using same stacking sequence.

2.2. Uncertainty Analysis

In uncertainty analysis, the parameters defined as uncertain by the column types may vary. Since the Euler - Bernoulli beam theory is used for columns with isotropic properties, the modulus of elasticity (E) of the material, the area moment of inertia (I) depending on the cross-sectional geometry and dimensions, and the length of the

column (L) are expressed in equations as uncertain. In the composite columns, the modulus of elasticity acting in the axial direction (E_1), the shear modulus (G_{13}) in the plane (1-3), which is expressed in the equations due to the first order shear deformation theory, the parameters that may arise from the inability to obtain the desired thicknesses in the laminates formed by overlapping and column length (L) are defined as uncertain. Table 2.1 shows the uncertain parameters for each column type.

Table 2.1: Uncertain parameters for each column type.

Cross-Sections	Column Section Code	Uncertain Parameters	
		Isotropic Material	Composite Material
	r	$\bar{E} = E(1 + \delta_1)$ $\bar{b} = b(1 + \delta_2)$ $\bar{f} = f(1 + \delta_3)$ $\bar{L} = L(1 + \delta_4)$	$\bar{E}_1 = E_1(1 + \delta_1)$ $\bar{G}_{13} = G_{13}(1 + \delta_2)$ $\bar{L} = L(1 + \delta_3)$ $\bar{f} = f(1 + \delta_4)$
	c	$\bar{E} = E(1 + \delta_1)$ $\bar{R} = R(1 + \delta_2)$ $\bar{L} = L(1 + \delta_3)$	$\bar{E}_1 = E_1(1 + \delta_1)$ $\bar{G}_{13} = G_{13}(1 + \delta_2)$ $\bar{L} = L(1 + \delta_3)$ $\bar{R} = R(1 + \delta_4)$
	rt	$\bar{E} = E(1 + \delta_1)$ $\bar{b} = b(1 + \delta_2)$ $\bar{f} = f(1 + \delta_3)$ $\bar{h} = h(1 + \delta_4)$ $\bar{L} = L(1 + \delta_5)$	$\bar{E}_1 = E_1(1 + \delta_1)$ $\bar{G}_{13} = G_{13}(1 + \delta_2)$ $\bar{L} = L(1 + \delta_3)$ $\bar{h} = h(1 + \delta_4)$
	ct	$\bar{E} = E(1 + \delta_1)$ $\bar{R} = R(1 + \delta_2)$ $\bar{h} = h(1 + \delta_3)$ $\bar{L} = L(1 + \delta_4)$	$\bar{E}_1 = E_1(1 + \delta_1)$ $\bar{G}_{13} = G_{13}(1 + \delta_2)$ $\bar{L} = L(1 + \delta_3)$ $\bar{h} = h(1 + \delta_4)$
	I	$\bar{E} = E(1 + \delta_1)$ $\bar{b} = b(1 + \delta_2)$ $\bar{h} = h(1 + \delta_3)$ $\bar{L} = L(1 + \delta_4)$	$\bar{E}_1 = E_1(1 + \delta_1)$ $\bar{G}_{13} = G_{13}(1 + \delta_2)$ $\bar{L} = L(1 + \delta_3)$ $\bar{h} = h(1 + \delta_4)$
	U	$\bar{E} = E(1 + \delta_1)$ $\bar{b} = b(1 + \delta_2)$ $\bar{h} = h(1 + \delta_3)$ $\bar{L} = L(1 + \delta_4)$	$\bar{E}_1 = E_1(1 + \delta_1)$ $\bar{G}_{13} = G_{13}(1 + \delta_2)$ $\bar{L} = L(1 + \delta_3)$ $\bar{h} = h(1 + \delta_4)$
	L	$\bar{E} = E(1 + \delta_1)$ $\bar{b} = b(1 + \delta_2)$ $\bar{h} = h(1 + \delta_3)$ $\bar{L} = L(1 + \delta_4)$	$\bar{E}_1 = E_1(1 + \delta_1)$ $\bar{G}_{13} = G_{13}(1 + \delta_2)$ $\bar{L} = L(1 + \delta_3)$ $\bar{h} = h(1 + \delta_4)$

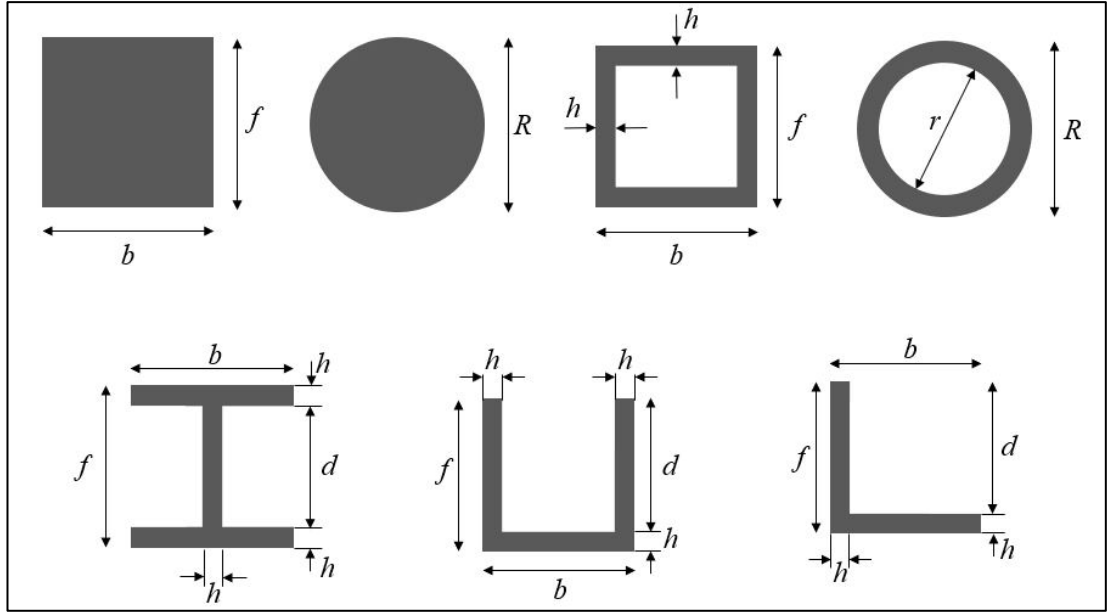


Figure 2.5: Cross-sections.

As the sections shown in Figure 2.5 are the types of columns commonly used in industry, these column types have been studied in this thesis. The critical buckling equations with uncertainty analysis are expressed as follows:

$$\overline{P}_{cr} = \frac{n^2 \pi^2 \overline{EI}}{\overline{L}^2} \quad (2.12)$$

$$\overline{P}_{cr} = \frac{\pi^2 \overline{A}_{cs} \overline{G}_{13} \overline{b} \overline{D}}{\pi^2 \overline{b} \overline{D} + \overline{A}_{cs} \overline{G}_{13} \overline{L}^2} \quad (2.13)$$

Here, over-bar defines the parameters which are taken as uncertain for all equations in this study.

2.3. Convex Analysis

Convex analysis is used to obtain the minimum point of critical buckling load. The critical buckling load equation (P_{cr}), expressed by uncertain parameters, is minimized by the following inequality boundary equation according to the uncertain

parameters (δ_i) for minimizing the critical buckling load by considering the worst-case scenario:

$$\sum_{i=1}^n \delta_i^2 \leq \beta^2 \quad (2.14)$$

As the lowest critical buckling load at different levels of uncertainty is possible when the δ_i parameters are located on the surface of the ellipsoid, the constraint equation is expressed as follows:

$$\sum_{i=1}^n \delta_i^2 - \beta^2 = 0 \quad (2.15)$$

The method of Lagrange multipliers is a very functional method that gives the minimum and maximum values of a multivariable function subject to restriction and is often used in optimization studies in mechanical fields [Stoecker, 1989; Haftka et al., 1990]. In Figure 2.6, the red color restriction function is $g(x, y)$, and the function $f(x, y)$ expressed in blue indicates the function of the maximum and minimum values for the x and y values depending on the restriction function.

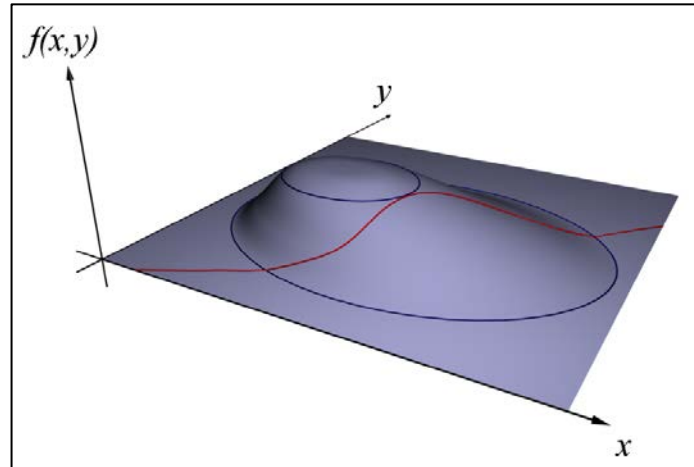


Figure 2.6: Three-dimensional representation of the Lagrange multipliers method.

By using Lagrange multipliers method, the maximum and minimum values of the function $f(x, y)$ are obtained on the $g(x, y)$ equation, which limits the function $f(x, y)$ as shown in Figure 2.6 [Web 2, 2019].

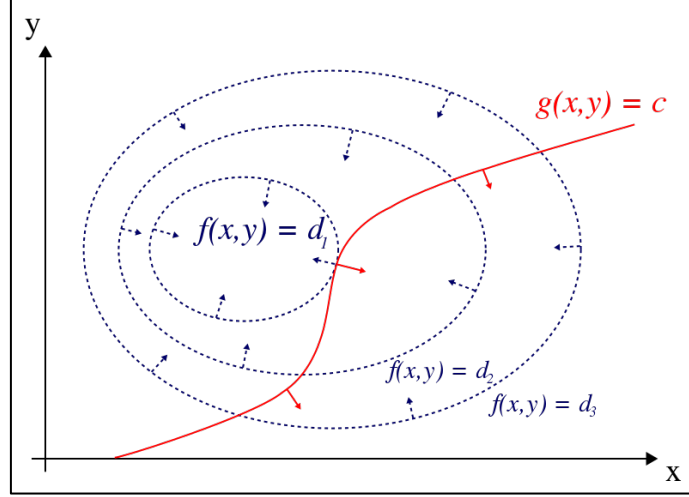


Figure 2.7: Two-dimensional representation of the Lagrange multipliers method.

Figure 2.7 shows the representation contour plot of the Lagrange multipliers method [Web 3, 2019]. Since the level curves of the function $f(x, y)$ are tangent to the boundary $g(x, y)$ curve, the gradient vectors of the functions must be parallel to each other, because the gradients of the functions f and g are perpendicular to these points (Figure 2.7). But there may be differences in the magnitudes of these vectors. Therefore, the presence of a scalar multiplier that equals the gradient vectors is required and mathematically expressed as follows:

$$\nabla f = \lambda \nabla g \quad (2.16)$$

Here, λ is referred to as the Lagrange multiplier. In this study, equation (2.15) is used as constraint equation. The minimized function is the critical buckling load equation. The critical buckling load and the constraint equation are expressed depending on the δ_i variables and the expression that gives the critical buckling load at the point where these points are tangent to the limiting curve is obtained. Thus, the minimum critical buckling load is obtained. In this context, the equation in which the Lagrange method is applied is expressed as follows:

$$L(\delta_i, \lambda) = \bar{P}_{cr} + \lambda \left(\sum_{i=1}^n \delta_i^2 - \beta^2 \right) \quad (2.17)$$

Here, β refers to the level of uncertainty. When β is equal to zero, the equation becomes deterministic. Using the Lagrange multipliers method, the equality of all δ values in terms of β is expressed and thus the equation giving the critical buckling load is determined as a function dependent on the level of uncertainty (β). The obtained analytical equation results in the variation in the minimum critical buckling load for different levels of uncertainty (β).

2.4. Sensitivity Analysis

Sensitivity analysis of uncertain parameters is examined using the following expression:

$$S(\delta_i) = \left| \frac{\partial P_{cr}(\beta)}{\partial \delta_i} \right| \frac{|\delta_i|}{P_{cr}(0)} \quad (2.18)$$

Sensitivity analysis shows which uncertain parameter has the greatest unfavorable effect on critical buckling load. The fact that the unfavorable effects of the parameters in the critical buckling load are known in advance reveal to engineers and researchers that which parameter should be paid more attention especially in the preliminary design stages.

2.5. Finite Element Method

The results obtained by the finite element method are compared to the data of the critical buckling equations based on the level of uncertainty obtained analytically. Critical buckling load values are consistent with each other and examined in detail in the results section. Analysis performed in ANSYS Workbench environment is carried out by defining the composite materials as a shell in ACP (ANSYS Composite PrepPost) module. For the analysis of isotropic columns, simple buckling analyzes are

performed by using 3 – D body. The ‘Eigenvalue Buckling’ module is used for all buckling analyzes.

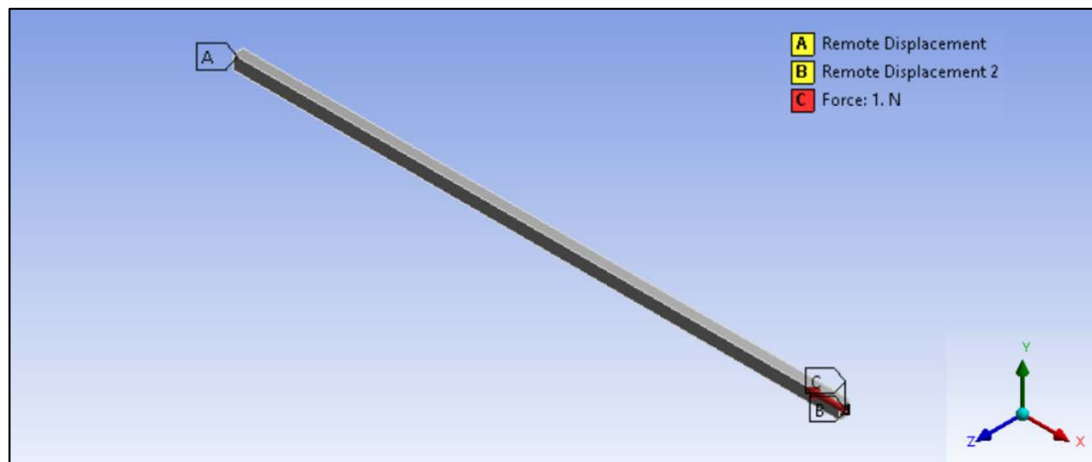


Figure 2.8: Representation of boundary conditions (BCs) for isotropic rectangular column.

Figure 2.8 presents a visual of BCs for isotropic rectangular column. Applied load is taken as 1 N in x direction (Table 2.2). Thus, load multiplier value, which can be read by using Eigenvalue Buckling module of ANSYS Workbench, provides critical buckling load for isotropic rectangular column.

Table 2.2: BCs and mesh information for isotropic rectangular column.

Boundary Conditions		Component (x, y, z)		
Remote Displacement (A)	Displacement	0 mm	0 mm	0 mm
	Rotation	0°	Free	0°
Remote Displacement 2 (B)	Displacement	Free	0 mm	0 mm
	Rotation	0°	Free	0°
Force (C)		-1 N	0 N	0 N
Mesh Properties				
Nodes		85133		
Elements		17280		

Similar steps are carried out for composite I-section column after model is generated in ACP module of ANSYS Workbench (Figure 2.9 and Table 2.3).

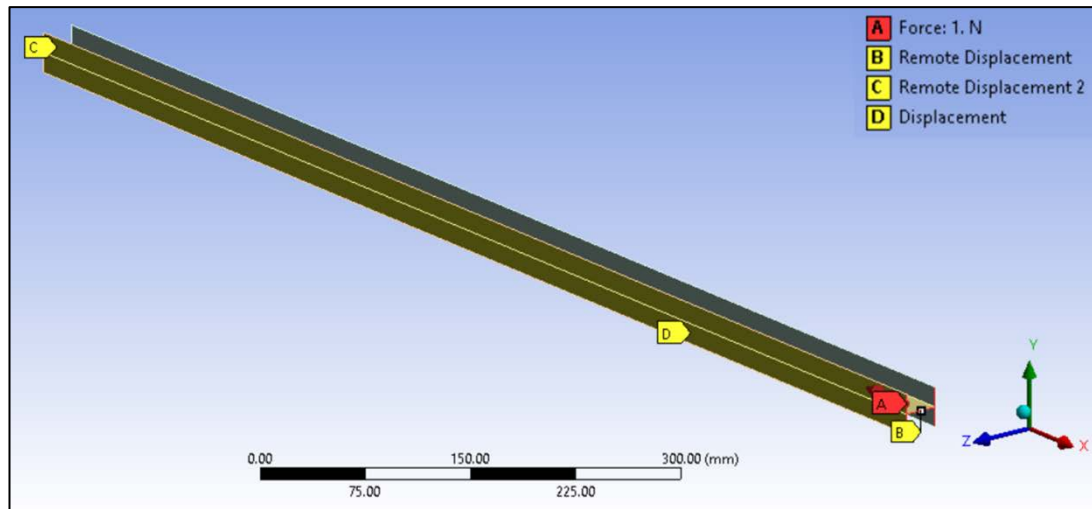


Figure 2.9: Representation of boundary conditions (BCs) for composite I-section column.

Table 2.3: BCs and mesh information for composite I-section column.

Boundary Conditions		Component (x, y, z)		
Remote Displacement (B)	Displacement	Free	0 mm	0 mm
	Rotation	0°	Free	0°
Remote Displacement 2 (C)	Displacement	0 mm	0 mm	0 mm
	Rotation	0°	Free	0°
Displacement (D)	Displacement	Free	0 mm	Free
Force (A)		-1 N	0 N	0 N
Mesh Properties				
Nodes		1494		
Elements		1360		

3. DERIVATIONS OF EQUATIONS

In this section, minimum critical buckling load equations for different cross-sections are derived by using convex modeling with the aid of “Lagrange Multipliers Method” as mentioned in previous section. For all calculations of critical buckling equations, y and z axis are taken as direction 2 and 3 showed in Figure 2.3, respectively.

3.1. Isotropic Rectangular Column

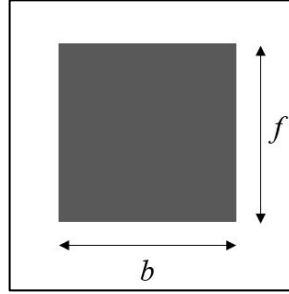


Figure 3.1: Cross-section of isotropic rectangular column.

For isotropic rectangular column (Figure 3.1), the uncertain material properties and dimensions are defined as:

$$\bar{E} = E(1 + \delta_1) \quad (3.1)$$

$$\bar{b} = b(1 + \delta_2) \quad (3.2)$$

$$\bar{f} = f(1 + \delta_3) \quad (3.3)$$

$$\bar{L} = L(1 + \delta_4) \quad (3.4)$$

where the E , b , f and L are the nominal values. The variations (δ_i) are unknown expressions to be determined by convex analysis considering the worst scenario or the least favorable situation. In this study, all equations are derived by taking uncertain

parameters less than 1 ($|\delta_i| \ll 1$). Equation (3.5) and (3.6) demonstrate the changing of cross-sectional area and area of moment of inertia values by uncertain parameters.

$$\bar{A}_{cs} = b(1 + \delta_2) f(1 + \delta_3) \quad (3.5)$$

$$\bar{I}_{yy} = \frac{1}{12} b(1 + \delta_2) f^3(1 + \delta_3)^3 \quad (3.6)$$

$$\frac{1}{(1 + \delta)^n} \cong (1 - n\delta) \quad (3.7)$$

The first-order approximation (3.7) is used in order to linearize the equations for convenience to Lagrange multipliers method. Values of n can be positive or negative. Critical buckling load equation based on Euler-Bernoulli theory is defined as below:

$$\bar{P}_{cr} = \frac{1}{12} \frac{\pi^2 E (1 + \delta_1) b(1 + \delta_2) f^3(1 + \delta_3)^3}{L^2 (1 + \delta_4)^2} \quad (3.8)$$

Here, the equation (3.8) can be linearized as it is shown in equation (3.9).

$$\bar{P}_{cr} \cong \frac{1}{12} \frac{\pi^2 E (1 + \delta_1) b(1 + \delta_2) f^3(1 + \delta_3)^3}{L^2} (1 - 2\delta_4) \quad (3.9)$$

After expanding and neglecting the higher-order terms (due to very small values of higher-order δ_i) of δ_i of equation (3.9), the expression yields as

$$\bar{P}_{cr} \cong \varphi_r (1 + \delta_1 + \delta_2 + 3\delta_3 - 2\delta_4) \quad (3.10)$$

where φ_r is given in the Appendix A. The expression (3.10) subjected to constraint equation (2.15) is minimized by using equations (3.11) - (3.13).

$$L(\delta_i, \lambda) = \bar{P}_{cr} + \lambda \left(\sum_{i=1}^4 \delta_i^2 - \beta^2 \right) \quad (3.11)$$

$$\frac{\partial}{\partial \delta_i} L = 0 \quad (i = 1, 2, 3, 4) \quad (3.12)$$

$$\frac{\partial}{\partial \lambda} L = 0 \quad (3.13)$$

$$\lambda = \pm \frac{1}{2} \frac{\sqrt{15}r}{\beta} \quad (3.14)$$

Uncertain parameters are determined by substituting equation (3.14) to all uncertain expressions (δ_i) to determine in terms of level of uncertainty (β) and we obtain:

$$\delta_1 = -\frac{1}{15} \sqrt{15} \beta \quad (3.15)$$

$$\delta_2 = -\frac{1}{15} \sqrt{15} \beta \quad (3.16)$$

$$\delta_3 = -\frac{1}{5} \sqrt{15} \beta \quad (3.17)$$

$$\delta_4 = \frac{2}{15} \sqrt{15} \beta \quad (3.18)$$

Substituting equations (3.15) - (3.28) into equation (3.8), the minimized critical buckling load equation (3.19) for defined uncertain parameters are determined analytically considering least favorable situation for isotropic rectangular column.

$$\bar{P}_{cr}(\beta) = \frac{1}{1500} \frac{\pi^2 E (\sqrt{15} \beta - 15)^2 b f^3 (5 - \sqrt{15} \beta)^3}{L^2 (2\sqrt{15} \beta + 15)^2} \quad (3.19)$$

For the other cross-sections, similar steps are carried out in order to obtain minimum critical buckling load equations.

3.2. Isotropic Circular Column

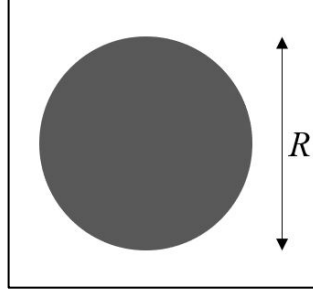


Figure 3.2: Cross-section of isotropic circular column.

For isotropic circular column (Figure 3.2), the uncertain material properties and dimensions are defined as:

$$\bar{E} = E(1 + \delta_1) \quad (3.20)$$

$$\bar{R} = R(1 + \delta_2) \quad (3.21)$$

$$\bar{L} = L(1 + \delta_3) \quad (3.22)$$

where the E , R and L are the nominal values. The steps applied for rectangular column are repeated for circular section. Thus, uncertain values of A_{cs} , I_{yy} and P_{cr} are expressed as below:

$$\bar{A}_{cs} = \frac{1}{4} \pi R^2 (1 + \delta_2)^2 \quad (3.23)$$

$$\bar{I}_{yy} = \frac{1}{64} \pi R^4 (1 + \delta_2)^4 \quad (3.24)$$

The first-order approximation (3.7) is used in order to linearize the equations for convenience to Lagrange multipliers method. For isotropic circular column, critical buckling load equation based on Euler-Bernoulli theory is defined as below:

$$\bar{P}_{cr} = \frac{1}{64} \frac{\pi^3 E (1 + \delta_1) R^4 (1 + \delta_2)^4}{L^2 (1 + \delta_3)^2} \quad (3.25)$$

Here, the equation (3.25) can be linearized as it is shown in equation (3.26).

$$\bar{P}_{cr} \cong \frac{1}{64} \frac{\pi^3 E (1 + \delta_1) R^4 (1 + \delta_2)^4}{L^2} (1 - 2\delta_3) \quad (3.26)$$

After expanding and neglecting the higher-order terms (due to very small values of higher-order δ_i) of δ_i of equation (3.26), the expression yields as

$$\bar{P}_{cr} \cong \varphi_{0,c} (1 + \phi_{1,c} \delta_1 + \phi_{2,c} \delta_2 + \phi_{3,c} \delta_3) \quad (3.27)$$

where

$$\phi_{1,c} = \frac{\varphi_{1,c}}{\varphi_{0,c}} \quad (3.28)$$

$$\phi_{2,c} = \frac{\varphi_{2,c}}{\varphi_{0,c}} \quad (3.29)$$

$$\phi_{3,c} = \frac{\varphi_{3,c}}{\varphi_{0,c}} \quad (3.30)$$

and expressions of φ are given in Appendix A. The expression (3.27) subjected to constraint equation (2.15) is minimized by using equations (3.31) - (3.33).

$$L(\delta_i, \lambda) = \bar{P}_{cr} + \lambda \left(\sum_{i=1}^3 \delta_i^2 - \beta^2 \right) \quad (3.31)$$

$$\frac{\partial}{\partial \delta_i} L = 0 \quad (i=1, 2, 3) \quad (3.32)$$

$$\frac{\partial}{\partial \lambda} L = 0 \quad (3.33)$$

$$\lambda = \pm \frac{1}{2} \frac{\sqrt{\phi_{1,c}^2 + \phi_{2,c}^2 + \phi_{3,c}^2} \phi_{0,c}}{\beta} \quad (3.34)$$

Uncertain parameters are determined by substituting equation (3.34) to all uncertain expressions (δ_i) to determine in terms of level of uncertainty (β) and we obtain:

$$\delta_1 = -\frac{\phi_{1,c}\beta}{\sqrt{\phi_{1,c}^2 + \phi_{2,c}^2 + \phi_{3,c}^2}} \quad (3.35)$$

$$\delta_2 = -\frac{\phi_{2,c}\beta}{\sqrt{\phi_{1,c}^2 + \phi_{2,c}^2 + \phi_{3,c}^2}} \quad (3.36)$$

$$\delta_3 = -\frac{\phi_{3,c}\beta}{\sqrt{\phi_{1,c}^2 + \phi_{2,c}^2 + \phi_{3,c}^2}} \quad (3.37)$$

Substituting equations (3.35) - (3.37) into equation (3.25), the minimized critical buckling load equation (3.38) for defined uncertain parameters are determined analytically considering least favorable situation for isotropic circular column.

$$\bar{P}_{cr}(\beta) = \frac{1}{64} \frac{\pi^3 E R^4 \left(-\phi_{1,c}\beta + \sqrt{\phi_{1,c}^2 + \phi_{2,c}^2 + \phi_{3,c}^2} \right) \left(-\phi_{2,c}\beta + \sqrt{\phi_{1,c}^2 + \phi_{2,c}^2 + \phi_{3,c}^2} \right)^4}{L^2 \left(\phi_{1,c}^2 + \phi_{2,c}^2 + \phi_{3,c}^2 \right)^{3/2} \left(-\phi_{3,c}\beta + \sqrt{\phi_{1,c}^2 + \phi_{2,c}^2 + \phi_{3,c}^2} \right)^2} \quad (3.38)$$

3.3. Isotropic Rectangular Tube Column

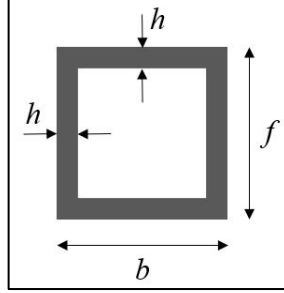


Figure 3.3: Cross-section of isotropic rectangular tube column.

For isotropic rectangular tube column (Figure 3.3), the uncertain material properties and dimensions are defined as:

$$\bar{E} = E(1 + \delta_1) \quad (3.39)$$

$$\bar{b} = b(1 + \delta_2) \quad (3.40)$$

$$\bar{f} = f(1 + \delta_3) \quad (3.41)$$

$$\bar{h} = h(1 + \delta_4) \quad (3.42)$$

$$\bar{L} = L(1 + \delta_5) \quad (3.43)$$

where the E , b , f , h and L are the nominal values.

$$\bar{A}_{cs} = bf(1 + \delta_2)(1 + \delta_3) - [b(1 + \delta_2) - 2h(1 + \delta_4)] \left[\begin{matrix} f(1 + \delta_3) \\ -2h(1 + \delta_4) \end{matrix} \right] \quad (3.44)$$

$$\bar{I}_{yy} = \frac{1}{12} \left[bf^3(1 + \delta_2)(1 + \delta_3)^3 - [b(1 + \delta_2) - 2h(1 + \delta_4)] \right] \left[\begin{matrix} f(1 + \delta_3) \\ -2h(1 + \delta_4) \end{matrix} \right]^3 \quad (3.45)$$

The first-order approximation (3.7) is used in order to linearize the equations for convenience to Lagrange multipliers method. For isotropic rectangular tube column, critical buckling load equation based on Euler-Bernoulli theory is defined as below:

$$\bar{P}_{cr} = \frac{\pi^2 E (1 + \delta_1)}{L^2 (1 + \delta_5)^2} \left(\frac{1}{12} \left[\frac{bf^3 (1 + \delta_2)(1 + \delta_3)^3 - [b(1 + \delta_2) - 2h(1 + \delta_4)]}{[f(1 + \delta_3) - 2h(1 + \delta_4)]^3} \right] \right) \quad (3.46)$$

Here, the equation (3.46) can be linearized as it is shown in equation (3.47).

$$\bar{P}_{cr} \cong \frac{\pi^2 E (1 + \delta_1)}{L^2} \left(\frac{1}{12} \left[\frac{bf^3 (1 + \delta_2)(1 + \delta_3)^3 - [b(1 + \delta_2) - 2h(1 + \delta_4)]}{[f(1 + \delta_3) - 2h(1 + \delta_4)]^3} \right] \right) (1 - 2\delta_5) \quad (3.47)$$

After expanding and neglecting the higher-order terms (due to very small values of higher-order δ_i) of δ_i of equation (3.47), the expression yields as

$$\bar{P}_{cr} \cong \varphi_{0,rt} (1 + \phi_{1,rt} \delta_1 + \phi_{2,rt} \delta_2 + \phi_{3,rt} \delta_3 + \phi_{4,rt} \delta_4 + \phi_{5,rt} \delta_5) \quad (3.48)$$

where

$$\phi_{1,rt} = \frac{\varphi_{1,rt}}{\varphi_{0,rt}} \quad (3.49)$$

$$\phi_{2,rt} = \frac{\varphi_{2,rt}}{\varphi_{0,rt}} \quad (3.50)$$

$$\phi_{3,rt} = \frac{\varphi_{3,rt}}{\varphi_{0,rt}} \quad (3.51)$$

$$\phi_{4,rt} = \frac{\varphi_{4,rt}}{\varphi_{0,rt}} \quad (3.52)$$

$$\phi_{5,rt} = \frac{\varphi_{5,rt}}{\varphi_{0,rt}} \quad (3.53)$$

and expressions of φ are given in Appendix A. The expression (3.46) subjected to constraint equation (2.15) is minimized by using equations (3.54) - (3.56).

$$L(\delta_i, \lambda) = \bar{P}_{cr} + \lambda \left(\sum_{i=1}^5 \delta_i^2 - \beta^2 \right) \quad (3.54)$$

$$\frac{\partial}{\partial \delta_i} L = 0 \quad (i = 1, 2, 3, 4, 5) \quad (3.55)$$

$$\frac{\partial}{\partial \lambda} L = 0 \quad (3.56)$$

$$\lambda = \pm \frac{1}{2} \frac{\sqrt{\phi_{1,rt}^2 + \phi_{2,rt}^2 + \phi_{3,rt}^2 + \phi_{4,rt}^2 + \phi_{5,rt}^2} \varphi_{0,rt}}{\beta} \quad (3.57)$$

Uncertain parameters are determined by substituting equation (3.57) to all uncertain expressions (δ_i) to determine in terms of level of uncertainty (β) and we obtain:

$$\delta_1 = - \frac{\phi_{1,rt} \beta}{\sqrt{\phi_{1,rt}^2 + \phi_{2,rt}^2 + \phi_{3,rt}^2 + \phi_{4,rt}^2 + \phi_{5,rt}^2}} \quad (3.58)$$

$$\delta_2 = - \frac{\phi_{2,rt} \beta}{\sqrt{\phi_{1,rt}^2 + \phi_{2,rt}^2 + \phi_{3,rt}^2 + \phi_{4,rt}^2 + \phi_{5,rt}^2}} \quad (3.59)$$

$$\delta_3 = -\frac{\phi_{3,rt}\beta}{\sqrt{\phi_{1,rt}^2 + \phi_{2,rt}^2 + \phi_{3,rt}^2 + \phi_{4,rt}^2 + \phi_{5,rt}^2}} \quad (3.60)$$

$$\delta_4 = -\frac{\phi_{4,rt}\beta}{\sqrt{\phi_{1,rt}^2 + \phi_{2,rt}^2 + \phi_{3,rt}^2 + \phi_{4,rt}^2 + \phi_{5,rt}^2}} \quad (3.61)$$

$$\delta_5 = -\frac{\phi_{5,rt}\beta}{\sqrt{\phi_{1,rt}^2 + \phi_{2,rt}^2 + \phi_{3,rt}^2 + \phi_{4,rt}^2 + \phi_{5,rt}^2}} \quad (3.62)$$

Substituting equations (3.58) - (3.62) into equation (3.46), the minimized critical buckling load equation for defined uncertain parameters can be determined analytically considering least favorable situation for isotropic rectangular tube column.

3.4. Isotropic Circular Tube Column

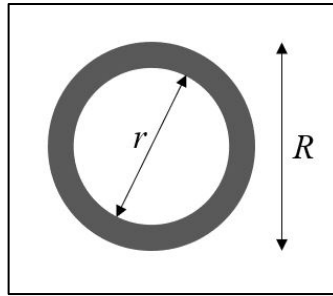


Figure 3.4: Cross-section of isotropic circular tube column.

For isotropic circular tube column (Figure 3.4), the uncertain material properties and dimensions are defined as:

$$\bar{E} = E(1 + \delta_1) \quad (3.63)$$

$$\bar{R} = R(1 + \delta_2) \quad (3.64)$$

$$\bar{h} = h(1 + \delta_3) \quad (3.65)$$

$$\bar{L} = L(1 + \delta_4) \quad (3.66)$$

$$\bar{A}_{cs} = 2\pi h(1 + \delta_3) \left(\frac{1}{2} R(1 + \delta_2) - \frac{1}{2} h(1 + \delta_3) \right) \quad (3.67)$$

$$\bar{I}_{yy} = \frac{1}{64} \pi R^4 (1 + \delta_2)^4 - \frac{1}{64} \pi (R(1 + \delta_2) - 2h(1 + \delta_3))^4 \quad (3.68)$$

The first-order approximation (3.7) is used in order to linearize the equations for convenience to Lagrange multipliers method. For isotropic circular tube column, critical buckling load equation based on Euler-Bernoulli theory is defined as below:

$$\bar{P}_{cr} = \frac{\pi^2 E (1 + \delta_1) \left(\frac{1}{64} \pi \left[R^4 (1 + \delta_2)^4 - [R(1 + \delta_2) - 2h(1 + \delta_3)]^4 \right] \right)}{L^2 (1 + \delta_4)^2} \quad (3.69)$$

Here, the equation (3.69) can be linearized as it is shown in equation (3.70).

$$\bar{P}_{cr} \cong \frac{\pi^2 E (1 + \delta_1)}{L^2} \left(\frac{1}{64} \pi \left[R^4 (1 + \delta_2)^4 - (R + R\delta_2 - 2h - 2h\delta_3)^4 \right] \right) (1 - 2\delta_4) \quad (3.70)$$

After expanding and neglecting the higher-order terms (due to very small values of higher-order δ_i) of δ_i of equation (3.70), the expression yields as

$$\bar{P}_{cr,ct} \cong \varphi_{0,ct} (1 + \phi_{1,ct} \delta_1 + \phi_{2,ct} \delta_2 + \phi_{3,ct} \delta_3 + \phi_{4,ct} \delta_4) \quad (3.71)$$

where

$$\phi_{1,ct} = \frac{\varphi_{1,ct}}{\varphi_{0,ct}}, \quad \phi_{2,ct} = \frac{\varphi_{2,ct}}{\varphi_{0,ct}}, \quad \phi_{3,ct} = \frac{\varphi_{3,ct}}{\varphi_{0,ct}}, \quad \phi_{4,ct} = \frac{\varphi_{4,ct}}{\varphi_{0,ct}} \quad (3.72)$$

and expressions of φ_i are given in Appendix A. The expression (3.71) subjected to constraint equation (2.15) is minimized by using equations (3.73) - (3.75).

$$L(\delta_i, \lambda) = \bar{P}_{cr} + \lambda \left(\sum_{i=1}^4 \delta_i^2 - \beta^2 \right) \quad (3.73)$$

$$\frac{\partial}{\partial \delta_i} L = 0 \quad (i = 1, 2, 3, 4) \quad (3.74)$$

$$\frac{\partial}{\partial \lambda} L = 0 \quad (3.75)$$

$$\lambda = \pm \frac{1}{2} \frac{\sqrt{\phi_{1,ct}^2 + \phi_{2,ct}^2 + \phi_{3,ct}^2 + \phi_{4,ct}^2} \phi_{0,ct}}{\beta} \quad (3.76)$$

Uncertain parameters are determined by substituting equation (3.76) to all uncertain expressions (δ_i) to determine in terms of level of uncertainty (β) and we obtain:

$$\delta_1 = - \frac{\phi_{1,ct} \beta}{\sqrt{\phi_{1,ct}^2 + \phi_{2,ct}^2 + \phi_{3,ct}^2 + \phi_{4,ct}^2}} \quad (3.77)$$

$$\delta_2 = - \frac{\phi_{2,ct} \beta}{\sqrt{\phi_{1,ct}^2 + \phi_{2,ct}^2 + \phi_{3,ct}^2 + \phi_{4,ct}^2}} \quad (3.78)$$

$$\delta_3 = - \frac{\phi_{3,ct} \beta}{\sqrt{\phi_{1,ct}^2 + \phi_{2,ct}^2 + \phi_{3,ct}^2 + \phi_{4,ct}^2}} \quad (3.79)$$

$$\delta_4 = - \frac{\phi_{4,ct} \beta}{\sqrt{\phi_{1,ct}^2 + \phi_{2,ct}^2 + \phi_{3,ct}^2 + \phi_{4,ct}^2}} \quad (3.80)$$

Substituting equations (3.77) - (3.80) into equation (3.69), the minimized critical buckling load equation for defined uncertain parameters are determined analytically considering least favorable situation for isotropic circular tube column.

3.5. Isotropic I-Section Column

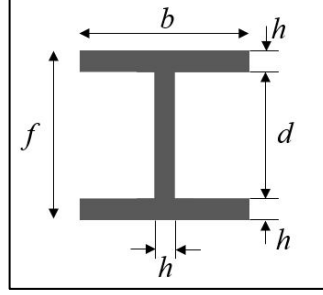


Figure 3.5: Isotropic I-section column.

For isotropic I-section column (Figure 3.5), the uncertain material properties and dimensions are defined as:

$$\bar{E} = E(1 + \delta_1) \quad (3.81)$$

$$\bar{b} = b(1 + \delta_2) \quad (3.82)$$

$$\bar{h} = h(1 + \delta_3) \quad (3.83)$$

$$\bar{L} = L(1 + \delta_4) \quad (3.84)$$

$$\bar{d} = f - 2h(1 + \delta_3) \quad (3.85)$$

$$\bar{A}_{cs} = h(1 + \delta_3) [2b(1 + \delta_2) [f - 2h(1 + \delta_3)]] \quad (3.86)$$

The centroid expressions of I-section are defined as:

$$\bar{y}_c = \frac{1}{2}b(1 + \delta_2) \quad (3.87)$$

$$z_c = \frac{1}{2}f \quad (3.88)$$

Equation (3.89) represent the area of moment of inertia of I-section column using parallel-axis theorem.

$$\bar{I}_{yy} = 2 \left(\frac{\bar{b}\bar{h}^3}{12} + \bar{b}\bar{h} \left(z_c - \frac{\bar{h}}{2} \right)^2 \right) + \frac{\bar{h}\bar{d}^3}{12} \quad (3.89)$$

By linearize the critical buckling load equation based on Euler-Bernoulli beam theory with uncertain material properties and dimensions, equation (3.90) can be obtained as

$$\bar{P}_{cr,I} \cong \varphi_{0,I} \left(1 + \phi_{1,I}\delta_1 + \phi_{2,I}\delta_2 + \phi_{3,I}\delta_3 + \phi_{4,I}\delta_4 \right) \quad (3.90)$$

where

$$\phi_{1,I} = \frac{\varphi_{1,I}}{\varphi_{0,I}}, \quad \phi_{2,I} = \frac{\varphi_{2,I}}{\varphi_{0,I}}, \quad \phi_{3,I} = \frac{\varphi_{3,I}}{\varphi_{0,I}}, \quad \phi_{4,I} = \frac{\varphi_{4,I}}{\varphi_{0,I}} \quad (3.91)$$

and expressions of φ_i are given in Appendix A. The expression (3.90) subjected to constraint equation (2.15) is minimized by using equations (3.92) - (3.94).

$$L(\delta_i, \lambda) = \bar{P}_{cr,I} + \lambda \left(\sum_{i=1}^4 \delta_i^2 - \beta^2 \right) \quad (3.92)$$

$$\frac{\partial}{\partial \delta_i} L = 0 \quad (i = 1, 2, 3, 4) \quad (3.93)$$

$$\frac{\partial}{\partial \lambda} L = 0 \quad (3.94)$$

$$\lambda = \pm \frac{1}{2} \frac{\sqrt{\phi_{1,I}^2 + \phi_{2,I}^2 + \phi_{3,I}^2 + \phi_{4,I}^2} \phi_{0,I}}{\beta} \quad (3.95)$$

Uncertain parameters are determined by substituting equation (3.95) to all uncertain expressions (δ_i) to determine in terms of level of uncertainty (β) and we obtain:

$$\delta_1 = -\frac{\phi_{1,I}\beta}{\sqrt{\phi_{1,I}^2 + \phi_{2,I}^2 + \phi_{3,I}^2 + \phi_{4,I}^2}} \quad (3.96)$$

$$\delta_2 = -\frac{\phi_{2,I}\beta}{\sqrt{\phi_{1,I}^2 + \phi_{2,I}^2 + \phi_{3,I}^2 + \phi_{4,I}^2}} \quad (3.97)$$

$$\delta_3 = -\frac{\phi_{3,I}\beta}{\sqrt{\phi_{1,I}^2 + \phi_{2,I}^2 + \phi_{3,I}^2 + \phi_{4,I}^2}} \quad (3.98)$$

$$\delta_4 = -\frac{\phi_{4,I}\beta}{\sqrt{\phi_{1,I}^2 + \phi_{2,I}^2 + \phi_{3,I}^2 + \phi_{4,I}^2}} \quad (3.99)$$

Substituting equations (3.96) - (3.99) into general uncertain equation, the minimized critical buckling load equation for defined uncertain parameters are determined analytically considering least favorable situation for isotropic I-section column.

3.6. Isotropic U-Section Column

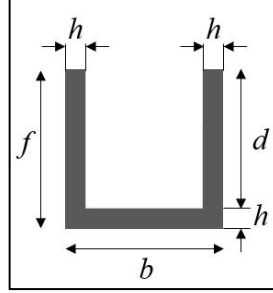


Figure 3.6: Isotropic U-section column.

For isotropic U-section column (Figure 3.6), the uncertain material properties and dimensions are defined as:

$$\bar{E} = E(1 + \delta_1) \quad (3.100)$$

$$\bar{b} = b(1 + \delta_2) \quad (3.101)$$

$$\bar{h} = h(1 + \delta_3) \quad (3.102)$$

$$\bar{L} = L(1 + \delta_4) \quad (3.103)$$

$$\bar{d} = f - h(1 + \delta_3) \quad (3.104)$$

$$\bar{A}_{cs} = h(1 + \delta_3) \left[b(1 + \delta_2) + 2 \left[f - h(1 + \delta_3) \right] \right] \quad (3.105)$$

The centroid expressions of U-section are defined as:

$$\bar{y}_c = \frac{1}{2} b(1 + \delta_2) \quad (3.106)$$

$$\bar{z}_c = \frac{1}{2} \frac{bh\delta_2\delta_3 - 2h^2\delta_3^2 + bh\delta_2 + bh\delta_3 - 4h^2\delta_3 + bh + 2f^2 - 2h^2}{b\delta_2 - 2h\delta_3 + b + 2f - 2h} \quad (3.107)$$

Equation (3.108) represent the area of moment of inertia of U-section column using parallel-axis theorem.

$$\bar{I}_{yy} = \frac{\bar{b}\bar{h}^3}{12} + \bar{b}\bar{h}\left(\bar{z}_c - \frac{\bar{h}}{2}\right)^2 + 2\left(\frac{\bar{h}\bar{d}^3}{12} + \bar{d}\bar{h}\left(\frac{\bar{d}}{2} + \bar{h} - \bar{z}_c\right)^2\right) \quad (3.108)$$

By linearize the critical buckling load equation based on Euler-Bernoulli beam theory with uncertain material properties and dimensions, equation (3.109) can be obtained as:

$$\bar{P}_{cr,U} \cong \varphi_{0,U} \left(1 + \phi_{1,U}\delta_1 + \phi_{2,U}\delta_2 + \phi_{3,U}\delta_3 + \phi_{4,U}\delta_4\right) \quad (3.109)$$

where

$$\phi_{1,U} = \frac{\varphi_{1,U}}{\varphi_{0,U}}, \quad \phi_{2,U} = \frac{\varphi_{2,U}}{\varphi_{0,U}}, \quad \phi_{3,U} = \frac{\varphi_{3,U}}{\varphi_{0,U}}, \quad \phi_{4,U} = \frac{\varphi_{4,U}}{\varphi_{0,U}} \quad (3.110)$$

Expressions of φ_i are given in Appendix A. The expression (3.109) subjected to constraint equation (2.15) is minimized by using equations (3.111) - (3.113).

$$L(\delta_i, \lambda) = \bar{P}_{cr,U} + \lambda \left(\sum_{i=1}^4 \delta_i^2 - \beta^2 \right) \quad (3.111)$$

$$\frac{\partial}{\partial \delta_i} L = 0 \quad (i = 1, 2, 3, 4) \quad (3.112)$$

$$\frac{\partial}{\partial \lambda} L = 0 \quad (3.113)$$

$$\lambda = \pm \frac{1}{2} \frac{\sqrt{\phi_{1,U}^2 + \phi_{2,U}^2 + \phi_{3,U}^2 + \phi_{4,U}^2} \phi_{0,U}}{\beta} \quad (3.114)$$

Uncertain parameters are determined by substituting equation (3.114) to all uncertain expressions (δ_i) to determine in terms of level of uncertainty (β) and we obtain:

$$\delta_1 = -\frac{\phi_{1,U}\beta}{\sqrt{\phi_{1,U}^2 + \phi_{2,U}^2 + \phi_{3,U}^2 + \phi_{4,U}^2}} \quad (3.115)$$

$$\delta_2 = -\frac{\phi_{2,U}\beta}{\sqrt{\phi_{1,U}^2 + \phi_{2,U}^2 + \phi_{3,U}^2 + \phi_{4,U}^2}} \quad (3.116)$$

$$\delta_3 = -\frac{\phi_{3,U}\beta}{\sqrt{\phi_{1,U}^2 + \phi_{2,U}^2 + \phi_{3,U}^2 + \phi_{4,U}^2}} \quad (3.117)$$

$$\delta_4 = -\frac{\phi_{4,U}\beta}{\sqrt{\phi_{1,U}^2 + \phi_{2,U}^2 + \phi_{3,U}^2 + \phi_{4,U}^2}} \quad (3.118)$$

Substituting equations (3.115) - (3.118) into general uncertain equation, the minimized critical buckling load equation for defined uncertain parameters are determined analytically considering least favorable situation for isotropic U-section column.

3.7. Isotropic L-Section Column

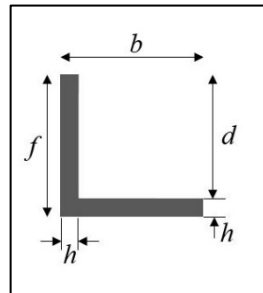


Figure 3.7: Isotropic L-section column.

For isotropic L-section column (Figure 3.7), the uncertain material properties and dimensions are defined as:

$$\bar{E} = E(1 + \delta_1) \quad (3.119)$$

$$\bar{b} = b(1 + \delta_2) \quad (3.120)$$

$$\bar{h} = h(1 + \delta_3) \quad (3.121)$$

$$\bar{L} = L(1 + \delta_4) \quad (3.122)$$

$$\bar{d} = f - h(1 + \delta_3) \quad (3.123)$$

$$\bar{A}_{cs} = h(1 + \delta_3) \left[b(1 + \delta_2) + [f - h(1 + \delta_3)] \right] \quad (3.124)$$

The centroid expressions of L-section are defined as:

$$\bar{y}_c = \frac{1}{2} \frac{b^2 \delta_2^2 - h^2 \delta_3^2 + 2b^2 \delta_2 + fh \delta_3 - 2h^2 \delta_3 + b^2 + fh - h^2}{b \delta_2 - h \delta_3 + b + f - h} \quad (3.125)$$

$$\bar{z}_c = \frac{1}{2} \frac{bh \delta_2 \delta_3 - h^2 \delta_3^2 + bh \delta_2 + bh \delta_3 - 2h^2 \delta_3 + bh + f^2 - h^2}{b \delta_2 - h \delta_3 + b + f - h} \quad (3.126)$$

Equation (3.127) represent the area of moment of inertia of L-section column using parallel-axis theorem.

$$\bar{I}_{yy} = \frac{\bar{b} \bar{h}^3}{12} + \bar{b} \bar{h} \left(\bar{z}_c - \frac{\bar{h}}{2} \right)^2 + \left(\frac{\bar{h} \bar{d}^3}{12} + \bar{d} \bar{h} \left(\frac{\bar{d}}{2} + \bar{h} - \bar{z}_c \right)^2 \right) \quad (3.127)$$

By linearize the critical buckling load equation based on Euler-Bernoulli beam theory with uncertain material properties and dimensions, equation (3.128) can be obtained as:

$$\bar{P}_{cr,L} \cong \varphi_{0,L} (1 + \phi_{1,L} \delta_1 + \phi_{2,L} \delta_2 + \phi_{3,L} \delta_3 + \phi_{4,L} \delta_4) \quad (3.128)$$

where

$$\phi_{1,L} = \frac{\varphi_{1,L}}{\varphi_{0,L}}, \quad \phi_{2,L} = \frac{\varphi_{2,L}}{\varphi_{0,L}}, \quad \phi_{3,L} = \frac{\varphi_{3,L}}{\varphi_{0,L}}, \quad \phi_{4,L} = \frac{\varphi_{4,L}}{\varphi_{0,L}} \quad (3.129)$$

Expressions of φ_i are given in Appendix A. The expression (3.128) subjected to constraint equation (2.15) is minimized by using equations (3.130) - (3.132).

$$L(\delta_i, \lambda) = \bar{P}_{cr,L} + \lambda \left(\sum_{i=1}^4 \delta_i^2 - \beta^2 \right) \quad (3.130)$$

$$\frac{\partial}{\partial \delta_i} L = 0 \quad (i = 1, 2, 3, 4) \quad (3.131)$$

$$\frac{\partial}{\partial \lambda} L = 0 \quad (3.132)$$

$$\lambda = \pm \frac{1}{2} \frac{\sqrt{\phi_{1,L}^2 + \phi_{2,L}^2 + \phi_{3,L}^2 + \phi_{4,L}^2} \varphi_{0,L}}{\beta} \quad (3.133)$$

Uncertain parameters are determined by substituting equation (3.133) to all uncertain expressions (δ_i) to determine in terms of level of uncertainty (β) and we obtain:

$$\delta_1 = - \frac{\phi_{1,L} \beta}{\sqrt{\phi_{1,L}^2 + \phi_{2,L}^2 + \phi_{3,L}^2 + \phi_{4,L}^2}} \quad (3.134)$$

$$\delta_2 = -\frac{\phi_{2,L}\beta}{\sqrt{\phi_{1,L}^2 + \phi_{2,L}^2 + \phi_{3,L}^2 + \phi_{4,L}^2}} \quad (3.135)$$

$$\delta_3 = -\frac{\phi_{3,L}\beta}{\sqrt{\phi_{1,L}^2 + \phi_{2,L}^2 + \phi_{3,L}^2 + \phi_{4,L}^2}} \quad (3.136)$$

$$\delta_4 = -\frac{\phi_{4,L}\beta}{\sqrt{\phi_{1,L}^2 + \phi_{2,L}^2 + \phi_{3,L}^2 + \phi_{4,L}^2}} \quad (3.137)$$

Substituting equations (3.134) - (3.137) into general uncertain equation, the minimized critical buckling load equation for defined uncertain parameters are determined analytically considering least favorable situation for isotropic L-section column.

3.8. Laminated Orthotropic Composite Rectangular Column

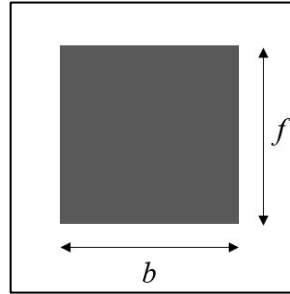


Figure 3.8: Cross-section of composite rectangular column.

For composite rectangular column (Figure 3.8), the uncertain material properties and dimensions are defined as:

$$\bar{E}_1 = E_1(1 + \delta_1) \quad (3.138)$$

$$\bar{G}_{13} = G_{13}(1 + \delta_2) \quad (3.139)$$

$$\bar{L} = L(1 + \delta_3) \quad (3.140)$$

$$\bar{f} = f(1 + \delta_4) \quad (3.141)$$

$$\bar{A}_{cs} = b f(1 + \delta_4) \quad (3.142)$$

$$\bar{v}_{21} = \frac{\nu_{12} E_2}{E_1(1 + \delta_1)} \quad (3.143)$$

Substituting expression (3.143) into transformed reduced stiffness matrix equations for $[0/0]_s$ stacking sequence, expressions (3.144) – (3.146) are obtained as:

$$\bar{Q}_{11,u} = \frac{E_1^2(1 + \delta_1)^2}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.144)$$

$$\bar{Q}_{12,u} = \frac{\nu_{12} E_2 E_1(1 + \delta_1)}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.145)$$

$$\bar{Q}_{22,u} = \frac{E_2 E_1(1 + \delta_1)}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.146)$$

Equation (3.147) represent the area of moment of inertia of composite rectangular column using parallel-axis theorem.

$$\bar{I}_{yy} = \frac{1}{12} b f^3 (1 + \delta_4)^3 \quad (3.147)$$

The bending stiffness matrices are defined as:

$$\begin{aligned} \bar{D}_{11} &= \frac{\bar{I}_{yy}}{b} \bar{Q}_{11,u} \\ &= \frac{1}{12} \frac{f^3 (1 + \delta_4)^3 E_1^2 (1 + \delta_1)^2}{-E_2\delta_{12}^2 + E_1\delta_1 + E_1} \end{aligned} \quad (3.148)$$

$$\begin{aligned}
\bar{D}_{12} &= \frac{\bar{I}_{yy}}{b} \bar{Q}_{12,u} \\
&= \frac{1}{12} \frac{f^3 (1 + \delta_4)^3 v_{12} E_2 E_1 (1 + \delta_1)}{-E_2 v_{12}^2 + E_1 \delta_1 + E_1}
\end{aligned} \tag{3.149}$$

$$\begin{aligned}
\bar{D}_{22} &= \frac{\bar{I}_{yy}}{b} \bar{Q}_{22,u} \\
&= \frac{1}{12} \frac{f^3 (1 + \delta_4)^3 E_2 E_1 (1 + \delta_1)}{-E_2 v_{12}^2 + E_1 \delta_1 + E_1}
\end{aligned} \tag{3.150}$$

$$\begin{aligned}
\bar{D} &= \bar{D}_{11} - \frac{\bar{D}_{12}^2}{\bar{D}_{22}} \\
&= \frac{1}{12} f^3 (1 + \delta_4)^3 E_1 (1 + \delta_1)
\end{aligned} \tag{3.151}$$

By linearize the critical buckling load equation based on FSDT with uncertain material properties and dimensions, equation (3.152) can be obtained as:

$$\bar{P}_{cr,r}^c \cong \frac{\varphi_{0,r}^c (1 + \phi_{1,r}^c \delta_1 + \phi_{2,r}^c \delta_2 + \phi_{4,r}^c \delta_4)}{\zeta_{0,r}^c (1 + \psi_{1,r}^c \delta_1 + \psi_{2,r}^c \delta_2 + \psi_{3,r}^c \delta_3 + \psi_{4,r}^c \delta_4)} \tag{3.152}$$

In order to eliminate the higher order δ_i terms, equation (3.152) is linearized again and defined as:

$$\bar{P}_{cr,r}^c \cong \kappa_r^c \left[1 + (\phi_{1,r}^c - \psi_{1,r}^c) \delta_1 + (\phi_{2,r}^c - \psi_{2,r}^c) \delta_2 - \psi_{3,r}^c \delta_3 + (\phi_{4,r}^c - \psi_{4,r}^c) \delta_4 \right] \tag{3.153}$$

where

$$\kappa_r^c = \frac{\varphi_{0,r}^c}{\zeta_{0,r}^c} \tag{3.154}$$

$$\phi_{1,r}^c = \frac{\varphi_{1,r}^c}{\varphi_{0,r}^c}, \quad \phi_{2,r}^c = \frac{\varphi_{2,r}^c}{\varphi_{0,r}^c}, \quad \phi_{4,r}^c = \frac{\varphi_{4,r}^c}{\varphi_{0,r}^c} \quad (3.155)$$

$$\psi_{1,r}^c = \frac{\zeta_{1,r}^c}{\zeta_{0,r}^c}, \quad \psi_{2,r}^c = \frac{\zeta_{2,r}^c}{\zeta_{0,r}^c}, \quad \psi_{3,r}^c = \frac{\zeta_{3,r}^c}{\zeta_{0,r}^c}, \quad \psi_{4,r}^c = \frac{\zeta_{4,r}^c}{\zeta_{0,r}^c} \quad (3.156)$$

where φ_i and ζ_i are given in Appendix A. The expression (3.153) subjected to constraint equation (2.15) is minimized by using equations (3.157) - (3.159).

$$L(\delta_i, \lambda) = \bar{P}_{cr,r}^c + \lambda \left(\sum_{i=1}^4 \delta_i^2 - \beta^2 \right) \quad (3.157)$$

$$\frac{\partial}{\partial \delta_i} L = 0 \quad (i = 1, 2, 3, 4) \quad (3.158)$$

$$\frac{\partial}{\partial \lambda} L = 0 \quad (3.159)$$

$$\lambda = \pm \frac{1}{2} \frac{\kappa_r^c \omega_r^c}{\beta} \quad (3.160)$$

By substituting the expression (3.160) which leads to minimize or maximize the critical buckling load into equation (3.158), expressions of δ_i are obtained as below:

$$\delta_1 = \frac{(\psi_{1,r}^c - \phi_{1,r}^c) \beta}{\omega_r^c} \quad (3.161)$$

$$\delta_2 = \frac{(\psi_{2,r}^c - \phi_{2,r}^c) \beta}{\omega_r^c} \quad (3.162)$$

$$\delta_3 = \frac{\psi_{3,r}^c \beta}{\omega_r^c} \quad (3.163)$$

$$\delta_4 = \frac{(\psi_{4,r}^c - \phi_{4,r}^c) \beta}{\omega_r^c} \quad (3.164)$$

3.9. Laminated Orthotropic Composite Circular Column

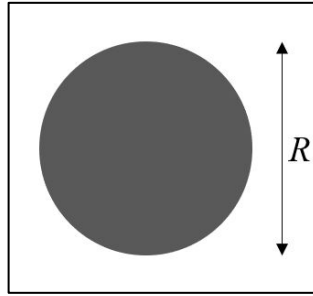


Figure 3.9: Cross-section of composite circular column.

For composite circular column (Figure 3.9), the uncertain material properties and dimensions are defined as:

$$\bar{E}_1 = E_1(1 + \delta_1) \quad (3.165)$$

$$\bar{G}_{13} = G_{13}(1 + \delta_2) \quad (3.166)$$

$$\bar{L} = L(1 + \delta_3) \quad (3.167)$$

$$\bar{R} = R(1 + \delta_4) \quad (3.168)$$

$$\bar{A}_{cs} = \frac{1}{4} \pi \bar{R}^2 (1 + \delta_4)^2 \quad (3.169)$$

$$\bar{v}_{21} = \frac{\nu_{12}E_2}{E_1(1+\delta_1)} \quad (3.170)$$

Substituting expression (3.170) into transformed reduced stiffness matrix equations for $[0/0]_s$ stacking sequence, expressions (3.171) – (3.173) are obtained as:

$$\bar{Q}_{11,u} = \frac{E_1^2(1+\delta_1)^2}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.171)$$

$$\bar{Q}_{12,u} = \frac{\nu_{12}E_2E_1(1+\delta_1)}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.172)$$

$$\bar{Q}_{22,u} = \frac{E_2E_1(1+\delta_1)}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.173)$$

Equation (3.174) represent the area of moment of inertia of composite circular column using parallel-axis theorem.

$$\bar{I}_{yy} = \frac{1}{64}\pi R^4(1+\delta_4)^4 \quad (3.174)$$

The bending stiffness matrices are defined as:

$$\begin{aligned} \bar{D}_{11} &= \frac{\bar{I}_{yy}}{R} \bar{Q}_{11,u} \\ &= \frac{1}{64} \frac{\pi R^3(1+\delta_4)^3 E_1^2(1+\delta_1)^2}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \end{aligned} \quad (3.175)$$

$$\begin{aligned} \bar{D}_{12} &= \frac{\bar{I}_{yy}}{R} \bar{Q}_{12,u} \\ &= \frac{1}{64} \frac{\pi R^3(1+\delta_4)^3 \nu_{12}E_2E_1(1+\delta_1)}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \end{aligned} \quad (3.176)$$

$$\begin{aligned}
\overline{D}_{22} &= \frac{\overline{I}_{yy}}{R} \overline{Q}_{22,u} \\
&= \frac{1}{64} \frac{\pi R^3 (1 + \delta_4)^3 E_2 E_1 (1 + \delta_1)}{-E_2 \nu_{12}^2 + E_1 \delta_1 + E_1}
\end{aligned} \tag{3.177}$$

$$\begin{aligned}
\overline{D} &= \overline{D}_{11} - \frac{\overline{D}_{12}^2}{\overline{D}_{22}} \\
&= \frac{1}{64} \pi R^3 (1 + \delta_4)^3 E_1 (1 + \delta_1)
\end{aligned} \tag{3.178}$$

By linearize the critical buckling load equation based on FSDT with uncertain material properties and dimensions, equation (3.179) can be obtained as:

$$\overline{P}_{cr,c}^c \cong \frac{\varphi_{0,c}^c (1 + \phi_{1,c}^c \delta_1 + \phi_{2,c}^c \delta_2 + \phi_{4,c}^c \delta_4)}{\zeta_{0,c}^c (1 + \psi_{1,c}^c \delta_1 + \psi_{2,c}^c \delta_2 + \psi_{3,c}^c \delta_3 + \psi_{4,c}^c \delta_4)} \tag{3.179}$$

In order to eliminate the higher order δ_i terms, equation (3.179) is linearized again and defined as:

$$\overline{P}_{cr,c}^c \cong \kappa_c^c \left[1 + (\phi_{1,c}^c - \psi_{1,c}^c) \delta_1 + (\phi_{2,c}^c - \psi_{2,c}^c) \delta_2 - \psi_{3,c}^c \delta_3 + (\phi_{4,c}^c - \psi_{4,c}^c) \delta_4 \right] \tag{3.180}$$

where

$$\kappa_c^c = \frac{\varphi_{0,c}^c}{\zeta_{0,c}^c} \tag{3.181}$$

$$\phi_{1,c}^c = \frac{\varphi_{1,c}^c}{\varphi_{0,c}^c}, \quad \phi_{2,c}^c = \frac{\varphi_{2,c}^c}{\varphi_{0,c}^c}, \quad \phi_{4,c}^c = \frac{\varphi_{4,c}^c}{\varphi_{0,c}^c} \tag{3.182}$$

$$\psi_{1,c}^c = \frac{\zeta_{1,c}^c}{\zeta_{0,c}^c}, \quad \psi_{2,c}^c = \frac{\zeta_{2,c}^c}{\zeta_{0,c}^c}, \quad \psi_{3,c}^c = \frac{\zeta_{3,c}^c}{\zeta_{0,c}^c}, \quad \psi_{4,c}^c = \frac{\zeta_{4,c}^c}{\zeta_{0,c}^c} \tag{3.183}$$

where φ_i and ζ_i are given in Appendix A. The expression (3.180) subjected to constraint equation (2.15) is minimized by using equations (3.184) - (3.186).

$$L(\delta_i, \lambda) = \overline{P}_{cr,c}^c + \lambda \left(\sum_{i=1}^4 \delta_i^2 - \beta^2 \right) \quad (3.184)$$

$$\frac{\partial}{\partial \delta_i} L = 0 \quad (i = 1, 2, 3, 4) \quad (3.185)$$

$$\frac{\partial}{\partial \lambda} L = 0 \quad (3.186)$$

$$\lambda = \pm \frac{1}{2} \frac{\kappa_c^c \omega_c^c}{\beta} \quad (3.187)$$

where

$$\begin{aligned} \omega_c^c &= \sqrt{A_c^c + B_c^c - 2\psi_{1,c}^c \phi_{1,c}^c - 2\psi_{2,c}^c \phi_{2,c}^c - 2\psi_{4,c}^c \phi_{4,c}^c} \\ A_c^c &= \left(\psi_{1,c}^c \right)^2 + \left(\psi_{2,c}^c \right)^2 + \left(\psi_{3,c}^c \right)^2 + \left(\psi_{4,c}^c \right)^2 \\ B_c^c &= \left(\phi_{1,c}^c \right)^2 + \left(\phi_{2,c}^c \right)^2 + \left(\phi_{4,c}^c \right)^2 \end{aligned} \quad (3.188)$$

By substituting the expression (3.187) which leads to minimize or maximize the critical buckling load into equation (3.185), expressions of δ_i are obtained as below:

$$\delta_1 = \frac{\left(\psi_{1,c}^c - \phi_{1,c}^c \right) \beta}{\omega_c^c} \quad (3.189)$$

$$\delta_2 = \frac{(\psi_{2,c}^c - \phi_{2,c}^c)\beta}{\omega_c^c} \quad (3.190)$$

$$\delta_3 = \frac{\psi_{3,c}^c \beta}{\omega_c^c} \quad (3.191)$$

$$\delta_4 = \frac{(\psi_{4,c}^c - \phi_{4,c}^c)\beta}{\omega_c^c} \quad (3.192)$$

3.10. Laminated Orthotropic Composite Rectangular Tube Column

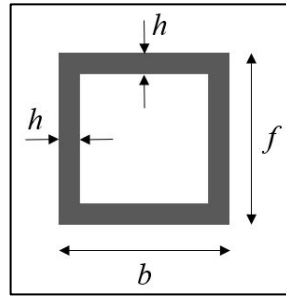


Figure 3.10: Cross-section of composite rectangular tube column.

For composite rectangular tube column (Figure 3.10), the uncertain material properties and dimensions are defined as:

$$\bar{E}_1 = E_1(1 + \delta_1) \quad (3.193)$$

$$\bar{G}_{13} = G_{13}(1 + \delta_2) \quad (3.194)$$

$$\bar{L} = L(1 + \delta_3) \quad (3.195)$$

$$\bar{h} = h(1 + \delta_4) \quad (3.196)$$

$$\bar{d}_1 = f - 2h(1 + \delta_4) \quad (3.197)$$

$$\bar{d}_2 = b - 2h(1 + \delta_4) \quad (3.198)$$

$$\bar{A}_{cs} = bf - [f - 2h(1 + \delta_4)][b - 2h(1 + \delta_4)] \quad (3.199)$$

$$\bar{v}_{21} = \frac{\nu_{12}E_2}{E_1(1 + \delta_1)} \quad (3.200)$$

Substituting expression (3.200) into transformed reduced stiffness matrix equations for $[0/0]_s$ stacking sequence, expressions (3.201) – (3.203) are obtained as:

$$\bar{Q}_{11,u} = \frac{E_1^2(1 + \delta_1)^2}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.201)$$

$$\bar{Q}_{12,u} = \frac{\nu_{12}E_2E_1(1 + \delta_1)}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.202)$$

$$\bar{Q}_{22,u} = \frac{E_2E_1(1 + \delta_1)}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.203)$$

Equation (3.204) represent the area of moment of inertia of composite rectangular tube column using parallel-axis theorem.

$$\bar{I}_{yy} = \frac{bf^3}{12} - \frac{[f - 2h(1 + \delta_4)][b - 2h(1 + \delta_4)]^3}{12} \quad (3.204)$$

The bending stiffness matrices are defined as:

$$\bar{D}_{11} = \frac{\bar{I}_{yy}}{b} \bar{Q}_{11,u} \quad (3.205)$$

$$\bar{D}_{11} \cong -\frac{1}{12} \frac{E_1^2 (1 + \delta_1)^2}{b(-E_2 v_{12}^2 + E_1 \delta_1 + E_1)} \left[\begin{array}{c} -2h\delta_4(b-2h)^2(b+3f-8h) + \\ (-2h+b+f) \left[\begin{array}{c} b(f-2h) \\ (b-f-4h)-8h^3 \end{array} \right] \end{array} \right] \quad (3.206)$$

$$\bar{D}_{12} = \frac{\bar{I}_{yy}}{b} \bar{Q}_{12,u} \quad (3.207)$$

$$\bar{D}_{12} \cong -\frac{1}{12} \frac{v_{12} E_2 E_1 (1 + \delta_1)}{b(-E_2 v_{12}^2 + E_1 \delta_1 + E_1)} \left[\begin{array}{c} -2h\delta_4(b-2h)^2(b+3f-8h) + \\ (-2h+b+f) \left[\begin{array}{c} b(f-2h) \\ (b-f-4h)-8h^3 \end{array} \right] \end{array} \right] \quad (3.208)$$

$$\bar{D}_{22} = \frac{\bar{I}_{yy}}{b} \bar{Q}_{22,u} \quad (3.209)$$

$$\bar{D}_{22} \cong -\frac{1}{12} \frac{E_2 E_1 (1 + \delta_1)}{b(-E_2 v_{12}^2 + E_1 \delta_1 + E_1)} \left[\begin{array}{c} -2h\delta_4(b-2h)^2(b+3f-8h) + \\ (-2h+b+f) \left[\begin{array}{c} b(f-2h) \\ (b-f-4h)-8h^3 \end{array} \right] \end{array} \right] \quad (3.210)$$

$$\bar{D} = \bar{D}_{11} - \frac{\bar{D}_{12}^2}{\bar{D}_{22}} \quad (3.211)$$

$$\bar{D} \cong -\frac{1}{12} \frac{E_1}{b} \left[\begin{array}{c} -2h\delta_4(b-2h)^2(b+3f-8h) + (-2h+b+f) \\ \left[\begin{array}{c} b(f-2h)(b-f-4h)-8h^3 \end{array} \right] \end{array} \right] \quad (3.212)$$

By linearize the critical buckling load equation based on FSDT with uncertain material properties and dimensions, equation (3.213) can be obtained as:

$$\bar{P}_{cr,rt}^c \cong \frac{\varphi_{0,rt}^c \left(1 + \phi_{1,rt}^c \delta_1 + \phi_{2,rt}^c \delta_2 + \phi_{4,rt}^c \delta_4 \right)}{\zeta_{0,rt}^c (1 + \psi_{1,rt}^c \delta_1 + \psi_{2,rt}^c \delta_2 + \psi_{3,rt}^c \delta_3 + \psi_{4,rt}^c \delta_4)} \quad (3.213)$$

In order to eliminate the higher order δ_i terms, equation (3.213) is linearized again and defined as:

$$\bar{P}_{cr,rt}^c \cong \kappa_{rt}^c \left[1 + \left(\phi_{1,rt}^c - \psi_{1,rt}^c \right) \delta_1 + \left(\phi_{2,rt}^c - \psi_{2,rt}^c \right) \delta_2 - \psi_{3,rt}^c \delta_3 + \left(\phi_{4,rt}^c - \psi_{4,rt}^c \right) \delta_4 \right] \quad (3.214)$$

where

$$\kappa_{rt}^c = \frac{\varphi_{0,rt}^c}{\zeta_{0,rt}^c} \quad (3.215)$$

$$\phi_{1,rt}^c = \frac{\varphi_{1,rt}^c}{\varphi_{0,rt}^c}, \quad \phi_{2,rt}^c = \frac{\varphi_{2,rt}^c}{\varphi_{0,rt}^c}, \quad \phi_{4,rt}^c = \frac{\varphi_{4,rt}^c}{\varphi_{0,rt}^c} \quad (3.216)$$

$$\psi_{1,rt}^c = \frac{\zeta_{1,rt}^c}{\zeta_{0,rt}^c}, \quad \psi_{2,rt}^c = \frac{\zeta_{2,rt}^c}{\zeta_{0,rt}^c}, \quad \psi_{3,rt}^c = \frac{\zeta_{3,rt}^c}{\zeta_{0,rt}^c}, \quad \psi_{4,rt}^c = \frac{\zeta_{4,rt}^c}{\zeta_{0,rt}^c} \quad (3.217)$$

where φ_i and ζ_i are given in Appendix A. The expression (3.214) subjected to constraint equation (2.15) is minimized by using equations (3.218) - (3.220).

$$L(\delta_i, \lambda) = \bar{P}_{cr,rt}^c + \lambda \left(\sum_{i=1}^4 \delta_i^2 - \beta^2 \right) \quad (3.218)$$

$$\frac{\partial}{\partial \delta_i} L = 0 \quad (i = 1, 2, 3, 4) \quad (3.219)$$

$$\frac{\partial}{\partial \lambda} L = 0 \quad (3.220)$$

$$\lambda = \pm \frac{1}{2} \frac{\kappa_{rt}^c \omega_{rt}^c}{\beta} \quad (3.221)$$

where

$$\omega_{rt}^c = \sqrt{A_{rt}^c + B_{rt}^c - 2\psi_{1,rt}^c \phi_{1,rt}^c - 2\psi_{2,rt}^c \phi_{2,rt}^c - 2\psi_{4,rt}^c \phi_{4,rt}^c}$$

$$A_{rt}^c = \left(\psi_{1,rt}^c\right)^2 + \left(\psi_{2,rt}^c\right)^2 + \left(\psi_{3,rt}^c\right)^2 + \left(\psi_{4,rt}^c\right)^2 \quad (3.222)$$

$$B_{rt}^c = \left(\phi_{1,rt}^c\right)^2 + \left(\phi_{2,rt}^c\right)^2 + \left(\phi_{4,rt}^c\right)^2$$

By substituting the expression (3.221) which leads to minimize or maximize the critical buckling load into equation (3.219), expressions of δ_i are obtained as below:

$$\delta_1 = \frac{\left(\psi_{1,rt}^c - \phi_{1,rt}^c\right)\beta}{\omega_{rt}^c} \quad (3.223)$$

$$\delta_2 = \frac{\left(\psi_{2,rt}^c - \phi_{2,rt}^c\right)\beta}{\omega_{rt}^c} \quad (3.224)$$

$$\delta_3 = \frac{\psi_{3,rt}^c \beta}{\omega_{rt}^c} \quad (3.225)$$

$$\delta_4 = \frac{\left(\psi_{4,rt}^c - \phi_{4,rt}^c\right)\beta}{\omega_{rt}^c} \quad (3.226)$$

3.11. Laminated Orthotropic Composite Circular Tube Column

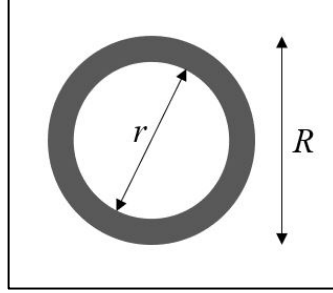


Figure 3.11: Cross-section of composite circular tube column.

For composite circular tube column (Figure 3.11), the uncertain material properties and dimensions are defined as:

$$\bar{E}_1 = E_1(1 + \delta_1) \quad (3.227)$$

$$\bar{G}_{13} = G_{13}(1 + \delta_2) \quad (3.228)$$

$$\bar{L} = L(1 + \delta_3) \quad (3.229)$$

$$\bar{h} = h(1 + \delta_4) \quad (3.230)$$

$$\bar{r} = R - 2h(1 + \delta_4) \quad (3.231)$$

$$\bar{A}_{cs} = \pi \left(\frac{1}{4} R^2 - \frac{1}{4} [R - 2h(1 + \delta_4)]^2 \right) \quad (3.232)$$

$$\bar{\nu}_{21} = \frac{\nu_{12} E_2}{E_1(1 + \delta_1)} \quad (3.233)$$

Substituting expression (3.233) into transformed reduced stiffness matrix equations for $[0/0]_s$ stacking sequence, expressions (3.234) – (3.236) are obtained as:

$$\bar{Q}_{11,u} = \frac{E_1^2(1+\delta_1)^2}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.234)$$

$$\bar{Q}_{12,u} = \frac{\nu_{12}E_2E_1(1+\delta_1)}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.235)$$

$$\bar{Q}_{22,u} = \frac{E_2E_1(1+\delta_1)}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.236)$$

Equation (3.237) represent the area of moment of inertia of composite circular tube column using parallel-axis theorem.

$$\bar{I}_{yy} = \pi \left(\frac{1}{64} R^4 - \frac{1}{64} [R - 2h(1+\delta_4)]^4 \right) \quad (3.237)$$

The bending stiffness matrices are defined as:

$$\bar{D}_{11} = \frac{\bar{I}_{yy}}{R} \bar{Q}_{11,u} \quad (3.238)$$

$$\bar{D}_{11} \cong \frac{1}{8} \frac{\pi h E_1^2 (1+\delta_1)^2}{R(-E_2\nu_{12}^2 + E_1\delta_1 + E_1)} \begin{bmatrix} \delta_4(R-2h)^3 + R(R^2 - 3Rh + 4h^2) \\ -2h^3 \end{bmatrix} \quad (3.239)$$

$$\bar{D}_{12} = \frac{\bar{I}_{yy}}{R} \bar{Q}_{12,u} \quad (3.240)$$

$$\bar{D}_{12} \cong \frac{1}{8} \frac{\pi h \nu_{12} E_2 E_1 (1+\delta_1)}{R(-E_2\nu_{12}^2 + E_1\delta_1 + E_1)} \begin{bmatrix} \delta_4(R-2h)^3 + R(R^2 - 3Rh + 4h^2) \\ -2h^3 \end{bmatrix} \quad (3.241)$$

$$\bar{D}_{22} = \frac{\bar{I}_{yy}}{R} \bar{Q}_{22,u} \quad (3.242)$$

$$\bar{D}_{22} \cong \frac{1}{8} \frac{\pi h E_2 E_1 (1 + \delta_1)}{R(-E_2 v_{12}^2 + E_1 \delta_1 + E_1)} \left[\begin{array}{l} \delta_4 (R - 2h)^3 + R(R^2 - 3Rh + 4h^2) \\ -2h^3 \end{array} \right] \quad (3.243)$$

$$\bar{D} \cong \frac{1}{8} \frac{\pi h E_1}{R} \left[\delta_4 (R - 2h)^3 + R(R^2 - 3Rh + 4h^2) - 2h^3 \right] \quad (3.244)$$

By linearize the critical buckling load equation based on FSDT with uncertain material properties and dimensions, equation (3.245) can be obtained as:

$$\bar{P}_{cr,ct}^c \cong \frac{\varphi_{0,ct}^c \left(1 + \phi_{1,ct}^c \delta_1 + \phi_{2,ct}^c \delta_2 + \phi_{4,ct}^c \delta_4 \right)}{\zeta_{0,ct}^c (1 + \psi_{1,ct}^c \delta_1 + \psi_{2,ct}^c \delta_2 + \psi_{3,ct}^c \delta_3 + \psi_{4,ct}^c \delta_4)} \quad (3.245)$$

In order to eliminate the higher order δ_i terms, equation (3.245) is linearized again and defined as:

$$\bar{P}_{cr,ct}^c \cong \kappa_{ct}^c \left[1 + \left(\phi_{1,ct}^c - \psi_{1,ct}^c \right) \delta_1 + \left(\phi_{2,ct}^c - \psi_{2,ct}^c \right) \delta_2 - \psi_{3,ct}^c \delta_3 + \left(\phi_{4,ct}^c - \psi_{4,ct}^c \right) \delta_4 \right] \quad (3.246)$$

where

$$\kappa_{ct}^c = \frac{\varphi_{0,ct}^c}{\zeta_{0,ct}^c} \quad (3.247)$$

$$\phi_{1,ct}^c = \frac{\varphi_{1,ct}^c}{\varphi_{0,ct}^c}, \quad \phi_{2,ct}^c = \frac{\varphi_{2,ct}^c}{\varphi_{0,ct}^c}, \quad \phi_{4,ct}^c = \frac{\varphi_{4,ct}^c}{\varphi_{0,ct}^c} \quad (3.248)$$

$$\psi_{1,ct}^c = \frac{\zeta_{1,ct}^c}{\zeta_{0,ct}^c}, \quad \psi_{2,ct}^c = \frac{\zeta_{2,ct}^c}{\zeta_{0,ct}^c}, \quad \psi_{3,ct}^c = \frac{\zeta_{3,ct}^c}{\zeta_{0,ct}^c}, \quad \psi_{4,ct}^c = \frac{\zeta_{4,ct}^c}{\zeta_{0,ct}^c} \quad (3.249)$$

where φ_i and ζ_i are given in Appendix A. The expression (3.246) subjected to constraint equation (2.15) is minimized by using equations (3.250) - (3.252).

$$L(\delta_i, \lambda) = \bar{P}_{cr,ct}^c + \lambda \left(\sum_{i=1}^4 \delta_i^2 - \beta^2 \right) \quad (3.250)$$

$$\frac{\partial}{\partial \delta_i} L = 0 \quad (i = 1, 2, 3, 4) \quad (3.251)$$

$$\frac{\partial}{\partial \lambda} L = 0 \quad (3.252)$$

$$\lambda = \pm \frac{1}{2} \frac{\kappa_{ct}^c \omega_{ct}^c}{\beta} \quad (3.253)$$

where

$$\begin{aligned} \omega_{ct}^c &= \sqrt{A_{ct}^c + B_{ct}^c - 2\psi_{1,ct}^c \phi_{1,ct}^c - 2\psi_{2,ct}^c \phi_{2,ct}^c - 2\psi_{4,ct}^c \phi_{4,ct}^c} \\ A_{ct}^c &= \left(\psi_{1,ct}^c \right)^2 + \left(\psi_{2,ct}^c \right)^2 + \left(\psi_{3,ct}^c \right)^2 + \left(\psi_{4,ct}^c \right)^2 \\ B_{ct}^c &= \left(\phi_{1,ct}^c \right)^2 + \left(\phi_{2,ct}^c \right)^2 + \left(\phi_{4,ct}^c \right)^2 \end{aligned} \quad (3.254)$$

By substituting the expression (3.253) which leads to minimize or maximize the critical buckling load into equation (3.251), expressions of δ_i are obtained as below:

$$\delta_1 = \frac{\left(\psi_{1,ct}^c - \phi_{1,ct}^c \right) \beta}{\omega_{ct}^c} \quad (3.255)$$

$$\delta_2 = \frac{\left(\psi_{2,ct}^c - \phi_{2,ct}^c \right) \beta}{\omega_{ct}^c} \quad (3.256)$$

$$\delta_3 = \frac{\psi_{3,ct}^c \beta}{\omega_{ct}^c} \quad (3.257)$$

$$\delta_4 = \frac{(\psi_{4,ct}^c - \phi_{4,ct}^c) \beta}{\omega_{ct}^c} \quad (3.258)$$

3.12. Laminated Orthotropic Composite I-Section Column

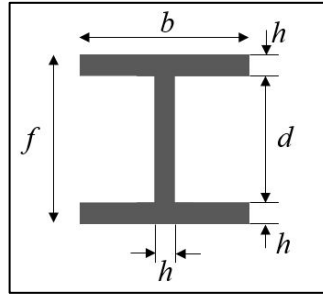


Figure 3.12: Composite I-section column.

For composite I-section column (Figure 3.12), the uncertain material properties and dimensions are defined as:

$$\bar{E}_1 = E_1(1 + \delta_1) \quad (3.259)$$

$$\bar{G}_{13} = G_{13}(1 + \delta_2) \quad (3.260)$$

$$\bar{L} = L(1 + \delta_3) \quad (3.261)$$

$$\bar{h} = h(1 + \delta_4) \quad (3.262)$$

$$\bar{d} = f - 2h(1 + \delta_4) \quad (3.263)$$

$$\bar{A}_{cs} = h(1 + \delta_4) \left[2b + [f - 2h(1 + \delta_4)] \right] \quad (3.264)$$

$$y_c = \frac{1}{2}b \quad (3.265)$$

$$z_c = \frac{1}{2}f \quad (3.266)$$

$$\bar{v}_{21} = \frac{\nu_{12}E_2}{E_1(1+\delta_1)} \quad (3.267)$$

Substituting expression (3.267) into transformed reduced stiffness matrix equations for $[0/0]_s$ stacking sequence, expressions (3.268) – (3.270) are obtained as:

$$\bar{Q}_{11,u} = \frac{E_1^2(1+\delta_1)^2}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.268)$$

$$\bar{Q}_{12,u} = \frac{\nu_{12}E_2E_1(1+\delta_1)}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.269)$$

$$\bar{Q}_{22,u} = \frac{E_2E_1(1+\delta_1)}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.270)$$

Equation (3.271) represent the area of moment of inertia of composite I-section column using parallel-axis theorem.

$$\bar{I}_{yy} \cong \frac{1}{12}h(1+\delta_4) \left[\delta_4 \left(-24h^3 + 8h^2(2b+3f) - 6hf(2b+f) \right) + f^3 \right] + (b-h) \left[6f^2 - 4h(3f-2h) \right] \quad (3.271)$$

The bending stiffness matrices are defined as:

$$\bar{D}_{11} = \frac{\bar{I}_{yy}}{b} \bar{Q}_{11,u} \quad (3.272)$$

$$\bar{D}_{11} \cong \frac{1}{12} \frac{E_1^2 h (1 + \delta_4) (1 + \delta_1)^2}{b (-E_2 v_{12}^2 + E_1 \delta_1 + E_1)} \left[\begin{array}{l} \delta_4 \left(\begin{array}{l} -24h^3 + 8h^2 (2b + 3f) \\ -6hf (2b + f) \end{array} \right) + f^3 \\ + (b - h) [6f^2 - 4h(3f - 2h)] \end{array} \right] \quad (3.273)$$

$$\bar{D}_{12} = \frac{\bar{I}_{yy}}{b} \bar{Q}_{12,u} \quad (3.274)$$

$$\bar{D}_{12} = \frac{1}{12} \frac{E_2 E_1 h v_{12} (1 + \delta_1) (1 + \delta_4)}{b (-E_2 v_{12}^2 + E_1 \delta_1 + E_1)} \left[\begin{array}{l} \delta_4 \left(\begin{array}{l} -24h^3 + 8h^2 (2b + 3f) \\ -6hf (2b + f) \end{array} \right) + f^3 \\ + (b - h) [6f^2 - 4h(3f - 2h)] \end{array} \right] \quad (3.275)$$

$$\bar{D}_{22} = \frac{\bar{I}_{yy}}{b} \bar{Q}_{22,u} \quad (3.276)$$

$$\bar{D}_{22} = \frac{1}{12} \frac{E_2 E_1 h (1 + \delta_1) (1 + \delta_4)}{b (-E_2 v_{12}^2 + E_1 \delta_1 + E_1)} \left[\begin{array}{l} \delta_4 \left(\begin{array}{l} -24h^3 + 8h^2 (2b + 3f) \\ -6hf (2b + f) \end{array} \right) + f^3 \\ + (b - h) [6f^2 - 4h(3f - 2h)] \end{array} \right] \quad (3.277)$$

$$\bar{D} = \bar{D}_{11} - \frac{\bar{D}_{12}^2}{\bar{D}_{22}} \quad (3.278)$$

$$\bar{D} = \frac{1}{12} \frac{E_1 h (1 + \delta_1) (1 + \delta_4)}{b} \left[\begin{array}{l} \delta_4 \left(\begin{array}{l} -24h^3 + 8h^2 (2b + 3f) \\ -6hf (2b + f) \end{array} \right) + f^3 \\ + (b - h) [6f^2 - 4h(3f - 2h)] \end{array} \right] \quad (3.279)$$

By linearize the critical buckling load equation based on FSDT with uncertain material properties and dimensions, equation (3.280) can be obtained as:

$$\bar{P}_{cr,I}^c \cong \frac{\varphi_{0,I}^c \left(1 + \phi_{1,I}^c \delta_1 + \phi_{2,I}^c \delta_2 + \phi_{4,I}^c \delta_4\right)}{\zeta_{0,I}^c (1 + \psi_{1,I}^c \delta_1 + \psi_{2,I}^c \delta_2 + \psi_{3,I}^c \delta_3 + \psi_{4,I}^c \delta_4)} \quad (3.280)$$

In order to eliminate the higher order δ_i terms, equation (3.280) is linearized again and defined as:

$$\bar{P}_{cr,I}^c \cong \kappa_I^c \left[1 + \left(\phi_{1,I}^c - \psi_{1,I}^c \right) \delta_1 + \left(\phi_{2,I}^c - \psi_{2,I}^c \right) \delta_2 - \psi_{3,I}^c \delta_3 + \left(\phi_{4,I}^c - \psi_{4,I}^c \right) \delta_4 \right] \quad (3.281)$$

where

$$\kappa_I^c = \frac{\varphi_{0,I}^c}{\zeta_{0,I}^c} \quad (3.282)$$

$$\phi_{1,I}^c = \frac{\varphi_{1,I}^c}{\varphi_{0,I}^c}, \quad \phi_{2,I}^c = \frac{\varphi_{2,I}^c}{\varphi_{0,I}^c}, \quad \phi_{4,I}^c = \frac{\varphi_{4,I}^c}{\varphi_{0,I}^c} \quad (3.283)$$

$$\psi_{1,I}^c = \frac{\zeta_{1,I}^c}{\zeta_{0,I}^c}, \quad \psi_{2,I}^c = \frac{\zeta_{2,I}^c}{\zeta_{0,I}^c}, \quad \psi_{3,I}^c = \frac{\zeta_{3,I}^c}{\zeta_{0,I}^c}, \quad \psi_{4,I}^c = \frac{\zeta_{4,I}^c}{\zeta_{0,I}^c} \quad (3.284)$$

where φ_i and ζ_i are given in Appendix A. The expression (3.281) subjected to constraint equation (2.15) is minimized by using equations (3.285) - (3.287).

$$L(\delta_i, \lambda) = \bar{P}_{cr,I}^c + \lambda \left(\sum_{i=1}^4 \delta_i^2 - \beta^2 \right) \quad (3.285)$$

$$\frac{\partial}{\partial \delta_i} L = 0 \quad (i = 1, 2, 3, 4) \quad (3.286)$$

$$\frac{\partial}{\partial \lambda} L = 0 \quad (3.287)$$

$$\lambda = \pm \frac{1}{2} \frac{\kappa_I^c \omega_I^c}{\beta} \quad (3.288)$$

where

$$\omega_I^c = \sqrt{A_I^c + B_I^c - 2\psi_{1,I}^c \phi_{1,I}^c - 2\psi_{2,I}^c \phi_{2,I}^c - 2\psi_{4,I}^c \phi_{4,I}^c}$$

$$A_I^c = \left(\psi_{1,I}^c\right)^2 + \left(\psi_{2,I}^c\right)^2 + \left(\psi_{3,I}^c\right)^2 + \left(\psi_{4,I}^c\right)^2 \quad (3.289)$$

$$B_I^c = \left(\phi_{1,I}^c\right)^2 + \left(\phi_{2,I}^c\right)^2 + \left(\phi_{4,I}^c\right)^2$$

By substituting the expression (3.288) which leads to minimize or maximize the critical buckling load into equations (3.286), expressions of δ_i are obtained as below:

$$\delta_1 = \frac{\left(\psi_{1,I}^c - \phi_{1,I}^c\right) \beta}{\omega_I^c} \quad (3.290)$$

$$\delta_2 = \frac{\left(\psi_{2,I}^c - \phi_{2,I}^c\right) \beta}{\omega_I^c} \quad (3.291)$$

$$\delta_3 = \frac{\psi_{3,I}^c \beta}{\omega_I^c} \quad (3.292)$$

$$\delta_4 = \frac{\left(\psi_{4,I}^c - \phi_{4,I}^c\right) \beta}{\omega_I^c} \quad (3.293)$$

3.13. Laminated Orthotropic Composite U-Section Column

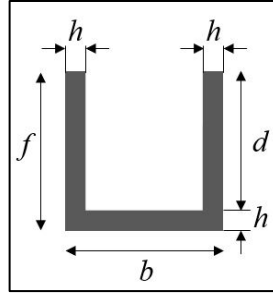


Figure 3.13: Composite U-section column.

For composite U-section column (Figure 3.13), the uncertain material properties and dimensions are defined as:

$$\bar{E}_1 = E_1(1 + \delta_1) \quad (3.294)$$

$$\bar{G}_{13} = G_{13}(1 + \delta_2) \quad (3.295)$$

$$\bar{L} = L(1 + \delta_3) \quad (3.296)$$

$$\bar{h} = h(1 + \delta_4) \quad (3.297)$$

$$\bar{d} = f - h(1 + \delta_4) \quad (3.298)$$

$$\bar{A}_{cs} = h(1 + \delta_4) \left[b + 2 \left[f - h(1 + \delta_4) \right] \right] \quad (3.299)$$

$$y_c = \frac{1}{2}b \quad (3.300)$$

$$\bar{z}_c = \frac{1}{2} \frac{\left(-2h^2\delta_4^2 + bh\delta_4 - 4h^2\delta_4 + bh + 2f^2 - 2h^2 \right)}{\left(-2h\delta_4 + b + 2f - 2h \right)} \quad (3.301)$$

$$\bar{z}_c \cong \frac{1 - 2h^2\delta_4^2 + bh\delta_4 - 4h^2\delta_4 + bh + 2f^2 - 2h^2}{b + 2f - 2h} \left(1 + \frac{2h\delta_4}{b + 2f - 2h} \right) \quad (3.302)$$

$$\bar{v}_{21} = \frac{\nu_{12}E_2}{E_1(1 + \delta_1)} \quad (3.303)$$

Substituting expression (3.303) into transformed reduced stiffness matrix equations for $[0/0]_s$ stacking sequence, expressions (3.304) – (3.306) are obtained as:

$$\bar{Q}_{11,u} = \frac{E_1^2(1 + \delta_1)^2}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.304)$$

$$\bar{Q}_{12,u} = \frac{\nu_{12}E_2E_1(1 + \delta_1)}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.305)$$

$$\bar{Q}_{22,u} = \frac{E_2E_1(1 + \delta_1)}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.306)$$

Equation (3.307) represent the area of moment of inertia of composite U-section column using parallel-axis theorem.

$$\bar{I}_{yy} = \frac{b\bar{h}^3}{12} + b\bar{h}\left(\bar{z}_c - \frac{\bar{h}}{2}\right)^2 + 2\left[\frac{\bar{h}\bar{d}^3}{12} + \bar{d}\bar{h}\left(\frac{\bar{d}}{2} + \bar{h} - \bar{z}_c\right)^2\right] \quad (3.307)$$

The bending stiffness matrices are defined as:

$$\bar{D}_{11} = \frac{\bar{I}_{yy}}{b} \bar{Q}_{11,u} \quad (3.308)$$

$$\bar{D}_{12} = \frac{\bar{I}_{yy}}{b} \bar{Q}_{12,u} \quad (3.309)$$

$$\bar{D}_{22} = \frac{\bar{I}_{yy}}{b} \bar{Q}_{22,u} \quad (3.310)$$

$$\bar{D} = \bar{D}_{11} - \frac{\bar{D}_{12}^2}{\bar{D}_{22}} \quad (3.311)$$

By linearize the critical buckling load equation based on FSDT with uncertain material properties and dimensions, critical buckling load equation is defined as Eq. (3.313). In order to eliminate the higher order δ_i terms, equation (3.313) is linearized again and defined as Eq. (3.314).

$$\bar{D} \cong \frac{1}{12} \frac{E_1 h}{b(b+2f-2h)^2} \left[\begin{array}{c} (1+\delta_1)(b+2f-2h) \left(\begin{array}{c} h^2 \left[\begin{array}{c} 4h(h-1)+b^2 \\ +8bf+24f^2 \end{array} \right] \\ -4f^2 \left[\begin{array}{c} h(3b+4f) \\ -f(2b+f) \end{array} \right] \end{array} \right) \\ -32h^5 + 4h^4(11b+34f) - 4h^3 \left(\begin{array}{c} 5b^2+32bf \\ +56f^2 \end{array} \right) \\ +\delta_4 \left[\begin{array}{c} h^2 \left[\begin{array}{c} 3b^2(b+10f)+16f^2(9b+11f) \\ -8f^2(b+2f) \left[\begin{array}{c} h(3b+4f)-f \left(b+\frac{1}{2}f \right) \end{array} \right] \end{array} \right] \end{array} \right] \end{array} \right] \quad (3.312)$$

$$\bar{P}_{cr,U}^c \cong \frac{\varphi_{0,U}^c (1 + \phi_{1,U}^c \delta_1 + \phi_{2,U}^c \delta_2 + \phi_{4,U}^c \delta_4)}{\zeta_{0,U}^c (1 + \psi_{1,U}^c \delta_1 + \psi_{2,U}^c \delta_2 + \psi_{3,U}^c \delta_3 + \psi_{4,U}^c \delta_4)} \quad (3.313)$$

$$\bar{P}_{cr,U}^c \cong \kappa_U^c \left[1 + \left(\phi_{1,U}^c - \psi_{1,U}^c \right) \delta_1 + \left(\phi_{2,U}^c - \psi_{2,U}^c \right) \delta_2 - \psi_{3,U}^c \delta_3 + \left(\phi_{4,U}^c - \psi_{4,U}^c \right) \delta_4 \right] \quad (3.314)$$

where

$$\kappa_U^c = \frac{\varphi_{0,U}^c}{\zeta_{0,U}^c} \quad (3.315)$$

$$\phi_{1,U}^c = \frac{\varphi_{1,U}^c}{\varphi_{0,U}^c}, \quad \phi_{2,U}^c = \frac{\varphi_{2,U}^c}{\varphi_{0,U}^c}, \quad \phi_{4,U}^c = \frac{\varphi_{4,U}^c}{\varphi_{0,U}^c} \quad (3.316)$$

$$\psi_{1,U}^c = \frac{\zeta_{1,U}^c}{\zeta_{0,U}^c}, \quad \psi_{2,U}^c = \frac{\zeta_{2,U}^c}{\zeta_{0,U}^c}, \quad \psi_{3,U}^c = \frac{\zeta_{3,U}^c}{\zeta_{0,U}^c}, \quad \psi_{4,U}^c = \frac{\zeta_{4,U}^c}{\zeta_{0,U}^c} \quad (3.317)$$

where φ_i and ζ_i are given in Appendix A. The expression (3.314) subjected to constraint equation (2.15) is minimized by using equations (3.318) - (3.320).

$$L(\delta_i, \lambda) = \bar{P}_{cr,U}^c + \lambda \left(\sum_{i=1}^4 \delta_i^2 - \beta^2 \right) \quad (3.318)$$

$$\frac{\partial}{\partial \delta_i} L = 0 \quad (i = 1, 2, 3, 4) \quad (3.319)$$

$$\frac{\partial}{\partial \lambda} L = 0 \quad (3.320)$$

$$\lambda = \pm \frac{1}{2} \frac{\kappa_U^c \omega_U^c}{\beta} \quad (3.321)$$

where

$$\begin{aligned} \omega_U^c &= \sqrt{A_U^c + B_U^c - 2\psi_{1,U}^c \phi_{1,U}^c - 2\psi_{2,U}^c \phi_{2,U}^c - 2\psi_{4,U}^c \phi_{4,U}^c} \\ A_U^c &= \left(\psi_{1,U}^c \right)^2 + \left(\psi_{2,U}^c \right)^2 + \left(\psi_{3,U}^c \right)^2 + \left(\psi_{4,U}^c \right)^2 \\ B_U^c &= \left(\phi_{1,U}^c \right)^2 + \left(\phi_{2,U}^c \right)^2 + \left(\phi_{4,U}^c \right)^2 \end{aligned} \quad (3.322)$$

By substituting the expression (3.321) which leads to minimize or maximize the critical buckling load into equation (3.319), expressions of δ_i are obtained as below:

$$\delta_1 = \frac{(\psi_{1,U}^c - \phi_{1,U}^c)\beta}{\omega_U^c} \quad (3.323)$$

$$\delta_2 = \frac{(\psi_{2,U}^c - \phi_{2,U}^c)\beta}{\omega_U^c} \quad (3.324)$$

$$\delta_3 = \frac{\psi_{3,U}^c \beta}{\omega_U^c} \quad (3.325)$$

$$\delta_4 = \frac{(\psi_{4,U}^c - \phi_{4,U}^c)\beta}{\omega_U^c} \quad (3.326)$$

3.14. Laminated Orthotropic Composite L-Section Column

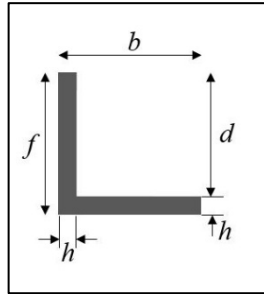


Figure 3.14: Composite L-section column.

For composite L-section column (Figure 3.14), the uncertain material properties and dimensions are defined as:

$$\bar{E}_1 = E_1(1 + \delta_1) \quad (3.327)$$

$$\bar{G}_{13} = G_{13}(1 + \delta_2) \quad (3.328)$$

$$\bar{L} = L(1 + \delta_3) \quad (3.329)$$

$$\bar{h} = h(1 + \delta_4) \quad (3.330)$$

$$\bar{d} = f - h(1 + \delta_4) \quad (3.331)$$

$$\bar{A}_{cs} = h(1 + \delta_4) \left[b + \left[f - h(1 + \delta_4) \right] \right] \quad (3.332)$$

$$\bar{y}_c = \frac{1}{2} \frac{-h^2\delta_4^2 + fh\delta_4 - 2h^2\delta_4 + b^2 + fh - h^2}{-h\delta_4 + b + f - h} \quad (3.333)$$

$$\bar{z}_c = \frac{1}{2} \frac{-h^2\delta_4^2 + bh\delta_4 - 2h^2\delta_4 + bh + f^2 - h^2}{-h\delta_4 + b + f - h} \quad (3.334)$$

$$\bar{y}_c \cong \frac{1}{2} \frac{\left(-h^2\delta_4^2 + fh\delta_4 - 2h^2\delta_4 + b^2 + fh - h^2 \right)}{b + f - h} \left(1 + \frac{h}{b + f - h} \delta_4 \right) \quad (3.335)$$

$$\bar{z}_c \cong \frac{1}{2} \frac{\left(-h^2\delta_4^2 + bh\delta_4 - 2h^2\delta_4 + bh + f^2 - h^2 \right)}{b + f - h} \left(1 + \frac{h}{b + f - h} \delta_4 \right) \quad (3.336)$$

$$\bar{v}_{21} = \frac{\nu_{12}E_2}{E_1(1 + \delta_1)} \quad (3.337)$$

Substituting expression (3.337) into transformed reduced stiffness matrix equations for $[0/0]_s$ stacking sequence, expressions (3.338) – (3.340) are obtained as:

$$\bar{Q}_{11,u} = \frac{E_1^2(1 + \delta_1)^2}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.338)$$

$$\bar{Q}_{12,u} = \frac{\nu_{12}E_2E_1(1 + \delta_1)}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \quad (3.339)$$

$$\bar{Q}_{22,u} = \frac{E_2 E_1 (1 + \delta_1)}{-E_2 \nu_{12}^2 + E_1 \delta_1 + E_1} \quad (3.340)$$

Equation (3.341) represent the area of moment of inertia of composite L-section column using parallel-axis theorem.

$$\bar{I}_{yy} = \left(\frac{b \bar{h}^3}{12} + b \bar{h} \left(\bar{z}_c - \frac{\bar{h}}{2} \right)^2 \right) + \left(\frac{\bar{h} \bar{d}^3}{12} + \bar{d} \bar{h} \left(\frac{\bar{d}}{2} + \bar{h} - \bar{z}_c \right)^2 \right) \quad (3.341)$$

The bending stiffness matrices are defined as:

$$\bar{D}_{11} = \frac{\bar{I}_{yy}}{b} \bar{Q}_{11,u} \quad (3.342)$$

$$\bar{D}_{12} = \frac{\bar{I}_{yy}}{b} \bar{Q}_{12,u} \quad (3.343)$$

$$\bar{D}_{22} = \frac{\bar{I}_{yy}}{b} \bar{Q}_{22,u} \quad (3.344)$$

$$\bar{D} = \bar{D}_{11} - \frac{\bar{D}_{12}^2}{\bar{D}_{22}} \quad (3.345)$$

$$\overline{D} \cong \frac{1}{12} \frac{E_1 h}{b(b+f-h)^2} \left[(1+\delta_1) (b+f-h) \begin{bmatrix} h^4 - 2h^3(b+2f) \\ +h^2(b^2+4bf+6f^2) \\ -2hf^2(3b+2f) \\ +f^3(4b+f) \end{bmatrix} + \delta_4 \begin{bmatrix} -4h^5 + h^4(11b+17f) \\ -2h^3(5b^2+16bf+14f^2) \\ +h^2(3b^3+15b^2f+36bf^2+22f^3) \\ -4f^2(b+f) \left[h(3b+2f) - f \left(b + \frac{1}{4}f \right) \right] \end{bmatrix} \right] \quad (3.346)$$

By linearize the critical buckling load equation based on FSDT with uncertain material properties and dimensions, equation (3.347) can be obtained as:

$$\overline{P}_{cr,L}^c \cong \frac{\varphi_{0,L}^c \left(1 + \phi_{1,L}^c \delta_1 + \phi_{2,L}^c \delta_2 + \phi_{4,L}^c \delta_4 \right)}{\zeta_{0,L}^c (1 + \psi_{1,L}^c \delta_1 + \psi_{2,L}^c \delta_2 + \psi_{3,L}^c \delta_3 + \psi_{4,L}^c \delta_4)} \quad (3.347)$$

In order to eliminate the higher order δ_i terms, equation (3.347) is linearized again and defined as:

$$\overline{P}_{cr,L}^c \cong \kappa_L^c \left[1 + \left(\phi_{1,L}^c - \psi_{1,L}^c \right) \delta_1 + \left(\phi_{2,L}^c - \psi_{2,L}^c \right) \delta_2 - \psi_{3,L}^c \delta_3 + \left(\phi_{4,L}^c - \psi_{4,L}^c \right) \delta_4 \right] \quad (3.348)$$

where

$$\kappa_L^c = \frac{\varphi_{0,L}^c}{\zeta_{0,L}^c} \quad (3.349)$$

$$\phi_{1,L}^c = \frac{\varphi_{1,L}^c}{\varphi_{0,L}^c}, \quad \phi_{2,L}^c = \frac{\varphi_{2,L}^c}{\varphi_{0,L}^c}, \quad \phi_{4,L}^c = \frac{\varphi_{4,L}^c}{\varphi_{0,L}^c} \quad (3.350)$$

$$\psi_{1,L}^c = \frac{\zeta_{1,L}^c}{\zeta_{0,L}^c}, \quad \psi_{2,L}^c = \frac{\zeta_{2,L}^c}{\zeta_{0,L}^c}, \quad \psi_{3,L}^c = \frac{\zeta_{3,L}^c}{\zeta_{0,L}^c}, \quad \psi_{4,L}^c = \frac{\zeta_{4,L}^c}{\zeta_{0,L}^c} \quad (3.351)$$

where φ_i and ζ_i are given in Appendix A. The expression (3.348) subjected to constraint equation (2.15) is minimized by using equations (3.352) - (3.354).

$$L(\delta_i, \lambda) = \bar{P}_{cr,L}^c + \lambda \left(\sum_{i=1}^4 \delta_i^2 - \beta^2 \right) \quad (3.352)$$

$$\frac{\partial}{\partial \delta_i} L = 0 \quad (i = 1, 2, 3, 4) \quad (3.353)$$

$$\frac{\partial}{\partial \lambda} L = 0 \quad (3.354)$$

$$\lambda = \pm \frac{1}{2} \frac{\kappa_L^c \omega_L^c}{\beta} \quad (3.355)$$

where

$$\begin{aligned} \omega_L^c &= \sqrt{A_L^c + B_L^c - 2\psi_{1,L}^c \phi_{1,L}^c - 2\psi_{2,L}^c \phi_{2,L}^c - 2\psi_{4,L}^c \phi_{4,L}^c} \\ A_L^c &= \left(\psi_{1,L}^c \right)^2 + \left(\psi_{2,L}^c \right)^2 + \left(\psi_{3,L}^c \right)^2 + \left(\psi_{4,L}^c \right)^2 \\ B_L^c &= \left(\phi_{1,L}^c \right)^2 + \left(\phi_{2,L}^c \right)^2 + \left(\phi_{4,L}^c \right)^2 \end{aligned} \quad (3.356)$$

By substituting the expression (3.355) which leads to minimize or maximize the critical buckling load into equation (3.353), expressions of δ_i are obtained as below:

$$\delta_1 = \frac{\left(\psi_{1,L}^c - \phi_{1,L}^c \right) \beta}{\omega_L^c} \quad (3.357)$$

$$\delta_2 = \frac{\left(\psi_{2,L}^c - \phi_{2,L}^c\right)\beta}{\omega_L^c} \quad (3.358)$$

$$\delta_3 = \frac{\psi_{3,L}^c \beta}{\omega_L^c} \quad (3.359)$$

$$\delta_4 = \frac{\left(\psi_{4,L}^c - \phi_{4,L}^c\right)\beta}{\omega_L^c} \quad (3.360)$$

4. RESULTS AND DISCUSSIONS

Table 4.1: Nominal dimensions and material properties for isotropic columns.

Column Code	Nominal values of dimensions (mm)					Young's Modulus (GPa)
	L	b	f	h	R	E
r	1800	30	30	-	-	200
c	1800	-	-	-	30	200
rt	1800	30	30	5	-	200
ct	1800	-	-	5	30	200
I	1800	30	30	5	-	200
U	1800	30	30	5	-	200
L	1800	30	30	5	-	200

Table 4.1 and 4.2 represent the values of material properties and its dimensions used in this study for isotropic and composite columns. The mechanical properties of structural steel are used for isotropic columns, while the mechanical properties of carbon / epoxy are used in composite columns.

Table 4.2: Nominal dimensions and materials properties for composite columns.

Column Code	Material Properties (GPa)		Dimensions (mm)				
	E ₁	G ₁₃	b	f	h	R	L
r	121	4.7	30	30	-	-	1000
c	121	4.7	-	-	-	30	1000
rt	121	4.7	30	30	4	-	1000
ct	121	4.7	-	-	4	30	1000
I	121	4.7	30	30	4	-	1000
U	121	4.7	30	30	4	-	1000
L	121	4.7	30	30	4	-	1000

Analytical critical buckling equations obtained for different sections are calculated with the losses in critical buckling load. It is determined which parameter has the most negative effect on the critical buckling load. In the following graph (Figure 4.1), the sections with the losses in normalized critical buckling load according to different levels of uncertainty in isotropic columns are shown.

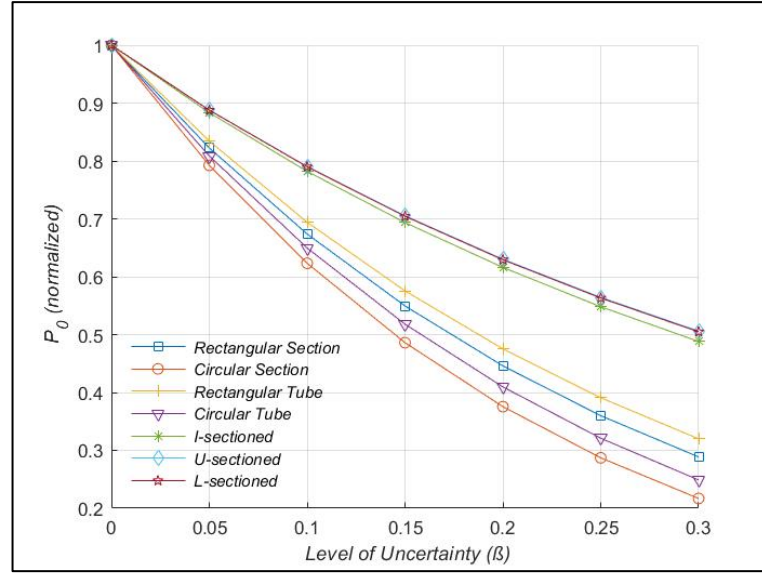


Figure 4.1 Normalized critical buckling load variations for isotropic columns.

Here, the normalized critical buckling load values are calculated by Eq. (4.1).

$$P_0 = \frac{P_{cr}(\beta)}{P_{cr}(0)} \quad (4.1)$$

It should be noted that the rectangular, rectangular tube, circular and circular tube sections of columns are the most affected structures with the increasement of the level of uncertainty. While these sections have loss of critical buckling load about %18, critical buckling loads of the other sections which are I, U and L- sectioned columns, decrease about %10 from the nominal value of critical buckling load when level of uncertainty is equal to 0.05. Rectangular, rectangular tube, circular and circular tube sections should be compared among themselves. Because the calculations are made considering the possibility of error at the outer borders of these columns. Therefore, considering the worst case, the amount of change in the critical buckling load in these columns are higher. In this context, it is observed that the rectangular and rectangular tube columns are more stable at different uncertainty levels than the column with circular and circular tube cross sections. The columns with I, U and L sections also represent similar loss of normalized critical buckling load.

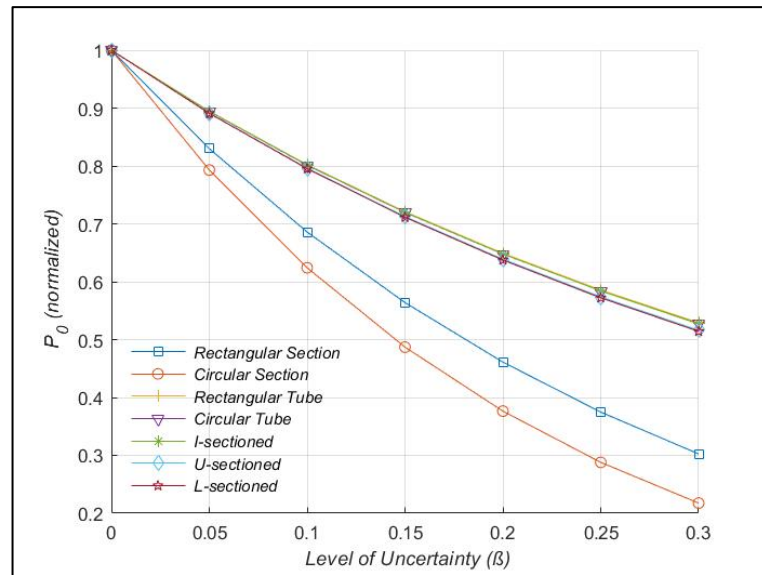


Figure 4.2: Normalized critical buckling load variations for composite columns.

In the graph (Figure 4.2), the sections with the losses in normalized critical buckling load according to different levels of uncertainty in composite columns are represented. The loss of normalized critical buckling loads for rectangular and circular columns are about %17 and %21, respectively. Here, again, the calculations are made considering the possibility of error at the outer borders of these composite columns and should be compared among themselves. Therefore, it can be concluded that the circular composite column is more sensitive than the rectangular composite column at different level of uncertainty.

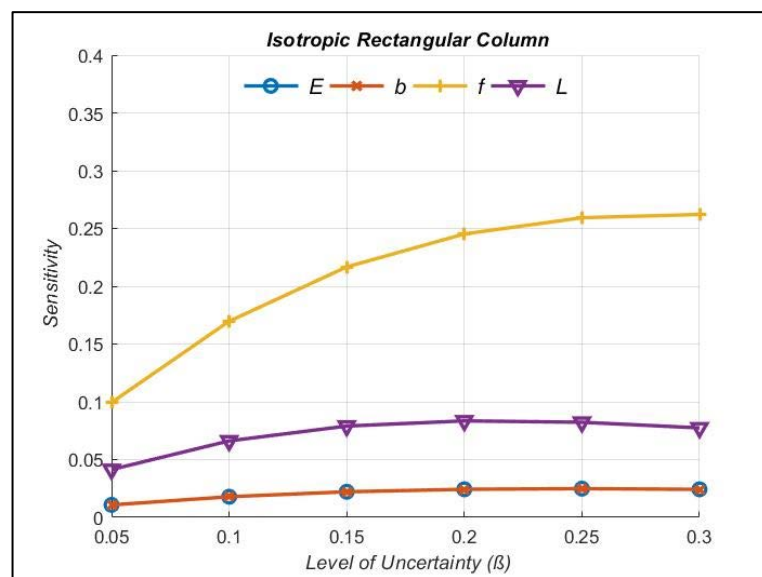


Figure 4.3: Sensitivity analysis of isotropic rectangular column.

For the rest of cross-section types, they all show similar loss of normalized critical buckling load values. Figure 4.3 indicates which one of the parameters have how much effect on the loss of minimum critical buckling load for isotropic rectangular column. It can be deduced that the most important parameter for the defined uncertain parameters in Section 3 is the height of cross-section.

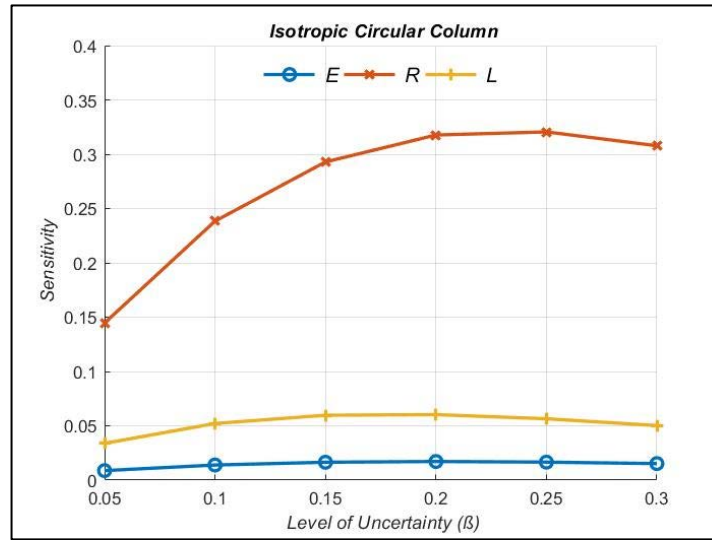


Figure 4.4: Sensitivity analysis of isotropic circular column.

Figure 4.4 represents that the most important parameter is the diameter (R) of cross-section for the isotropic circular column. Sensitivity analysis for all columns reveals which parameter needs to be paid more attention during the production phase.

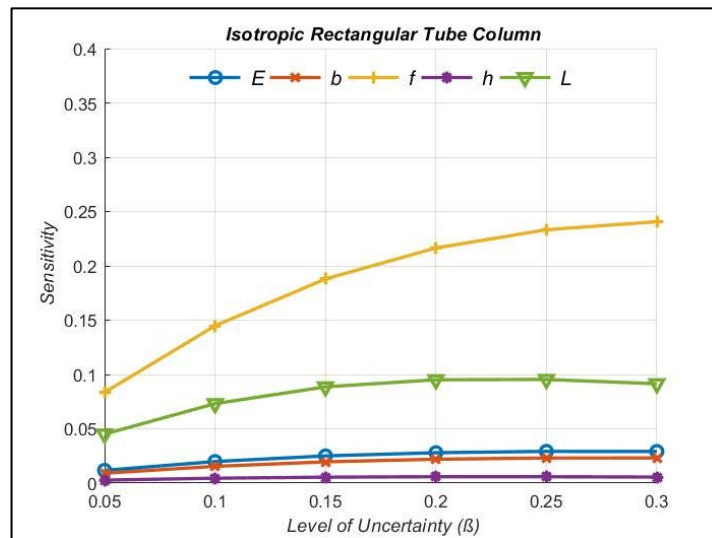


Figure 4.5: Sensitivity analysis of isotropic rectangular tube column.

For the isotropic rectangular tube column, effects of defined uncertain parameters are shown in Figure 4.5. The parameter f has the most effect on the critical buckling load for different level of uncertainties and the length of the column also has the remarkable effects on the critical buckling load. Figure 4.6 also shows the uncertain parameters for isotropic circular tube. Similar to the circular column, the parameter most effects critical buckling load is diameter of the circular tube column.

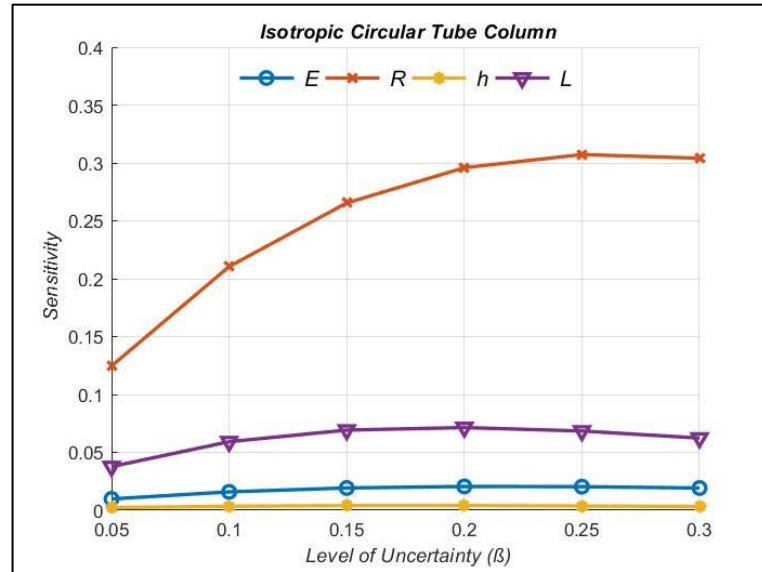


Figure 4.6: Sensitivity analysis of isotropic circular tube column.

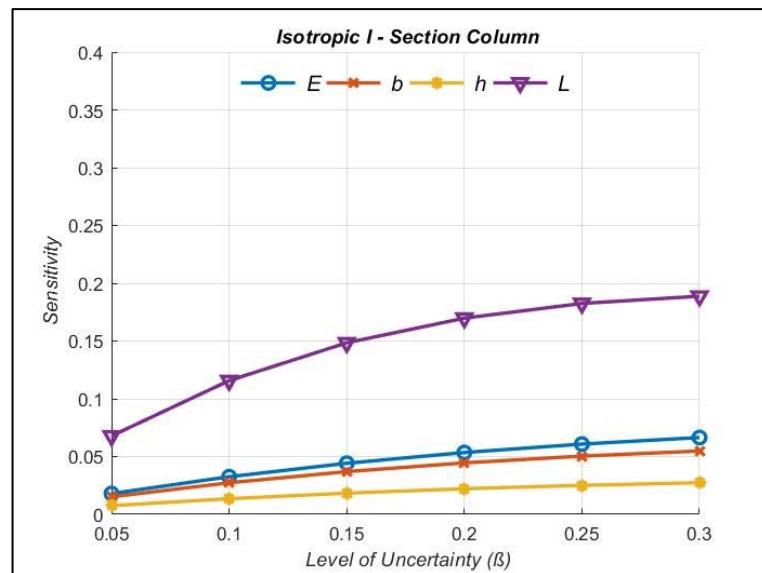


Figure 4.7: Sensitivity analysis of isotropic I-section column.

Sensitivity analysis applied on I, U and L cross-sections that represented in Figure 4.7, Figure 4.8 and Figure 4.9 provide similar characteristic features. The length of the columns for mentioned columns are the most important parameter which has the most effects on critical buckling load adversely.

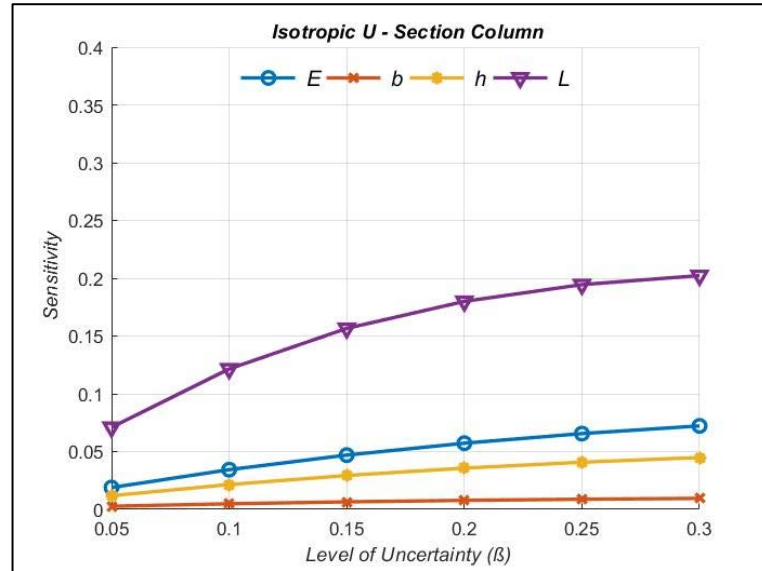


Figure 4.8: Sensitivity analysis of isotropic U-section column.

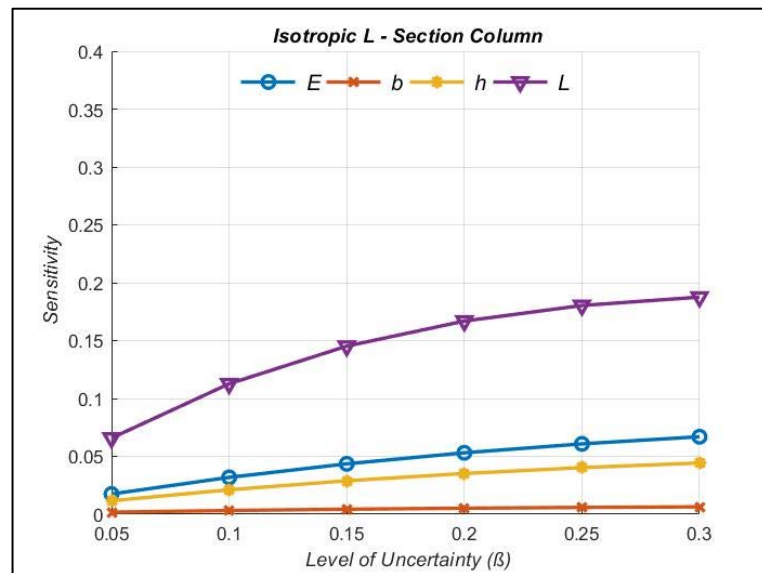


Figure 4.9: Sensitivity analysis of isotropic L-section column.

In general, according to defined uncertain parameters that are represented before, the parameters which has the most adversely effect on the critical buckling load are

vary depend on the geometrical features of cross-sections. Therefore, abovementioned sensitivity results obtained for isotropic columns are taking crucial role for robust designing of columns.

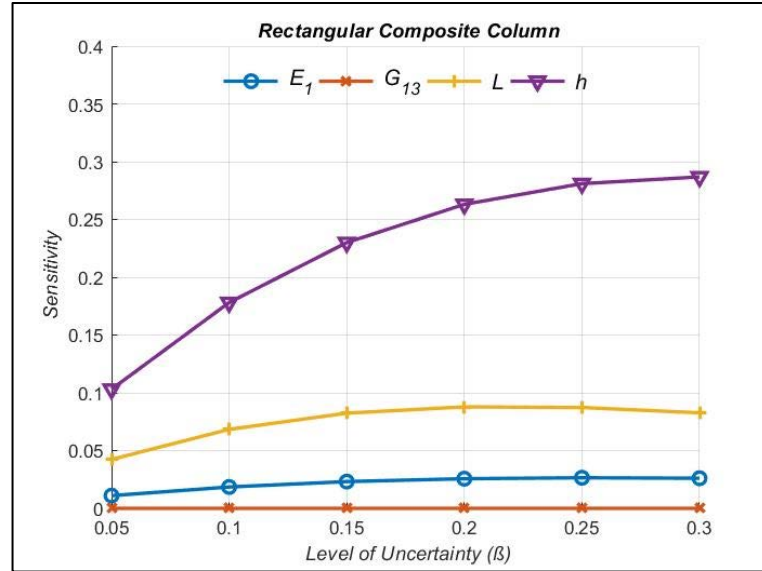


Figure 4.10: Sensitivity analysis of composite rectangular column.

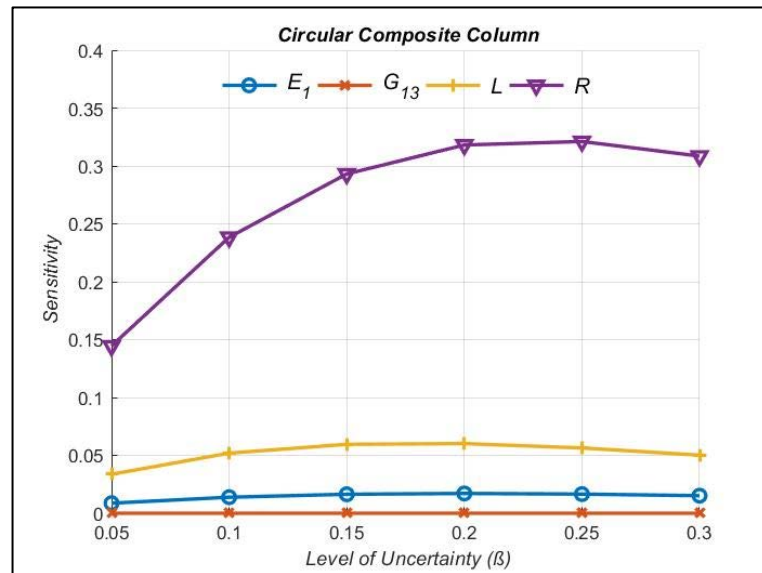


Figure 4.11: Sensitivity analysis of composite circular column.

For composite columns, rectangular and circular sections have similar characteristic features in point of sensitivity analysis. Figure 4.10 and Figure 4.11 show the sensitivity analysis of composite rectangular and circular columns,

respectively. Here, the height of the rectangular cross-section and the diameter of the circular cross-section are the most vital parameters regarding critical buckling loads. For all composite columns, the parameter G_{13} which refers to shear modulus in 1-3 plane has almost no effect on critical buckling load with the increasing level of uncertainty for defined uncertain parameters (Figure 4.10 – Figure 4.16).

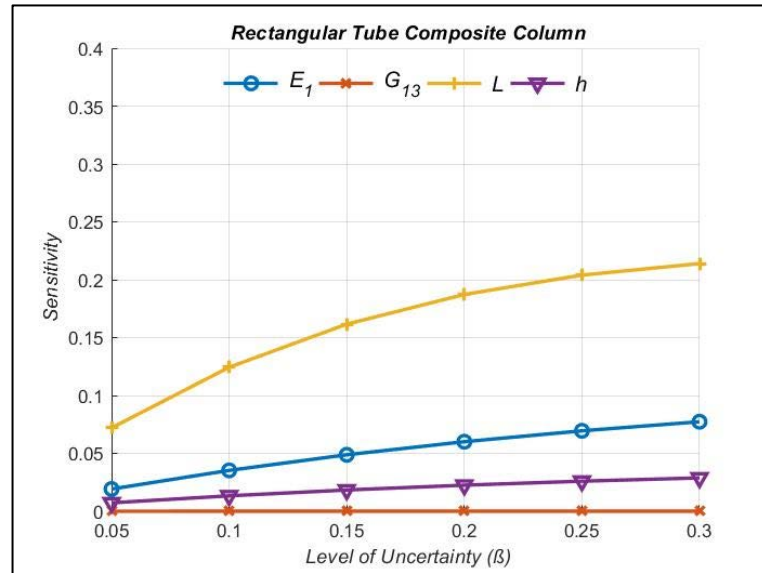


Figure 4.12: Sensitivity analysis of composite rectangular tube column.

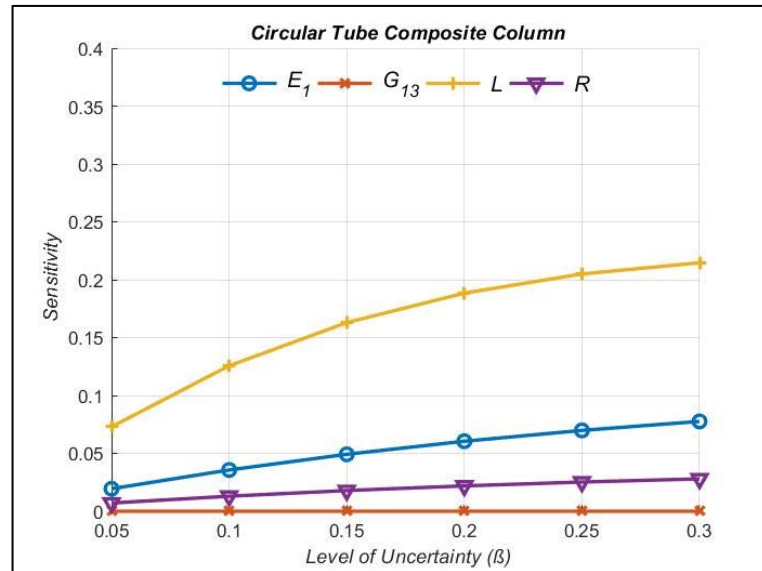


Figure 4.13: Sensitivity analysis of composite circular tube column.

Similar sensitivity analysis results are obtained in composite columns where thicknesses are defined uncertain as well as dimensional column length and material properties.

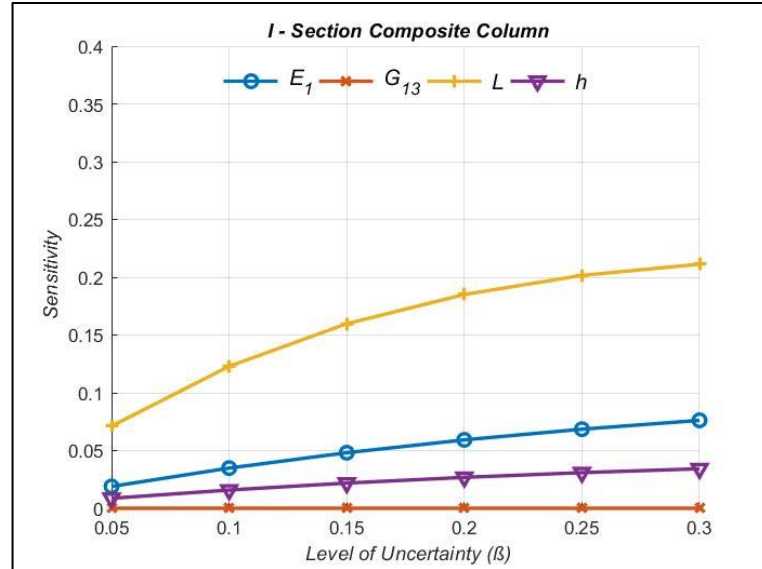


Figure 4.14: Sensitivity analysis of I-section composite column.

These columns showed in figures (Figure 4.12 - Figure 4.16) indicates that the column length, by far, is the most negatively affecting parameter regarding critical buckling load.

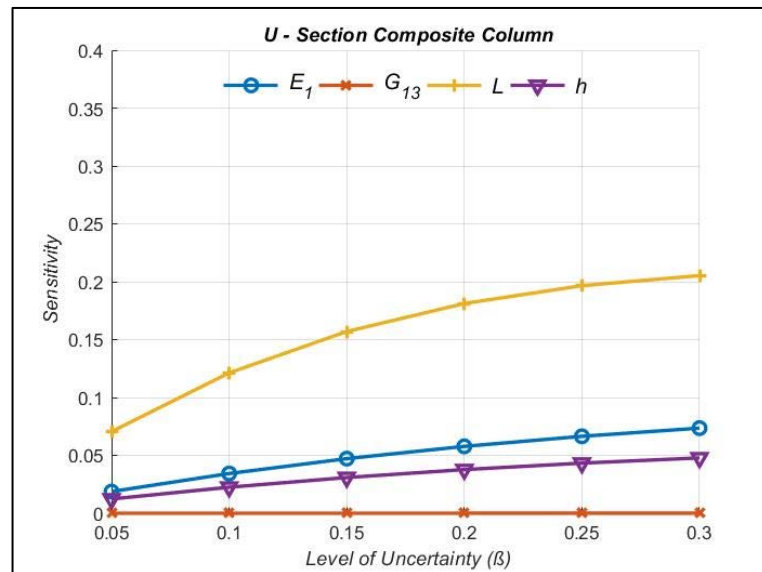


Figure 4.15: Sensitivity analysis of U-section composite column.

The results clearly show that the E_1 and thickness of cross-section also play a significant role considering that there may be variability on the longitudinal Young's modulus and thickness during composite production.

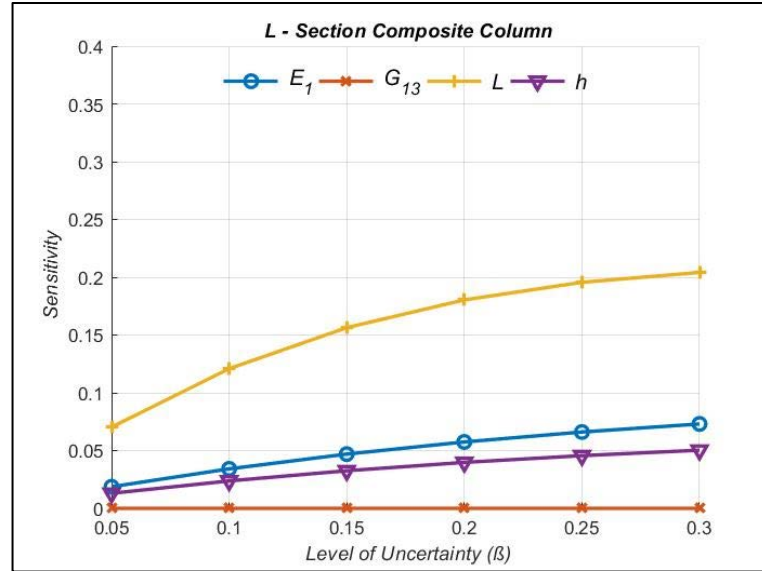


Figure 4.16: Sensitivity analysis of L-section composite column.

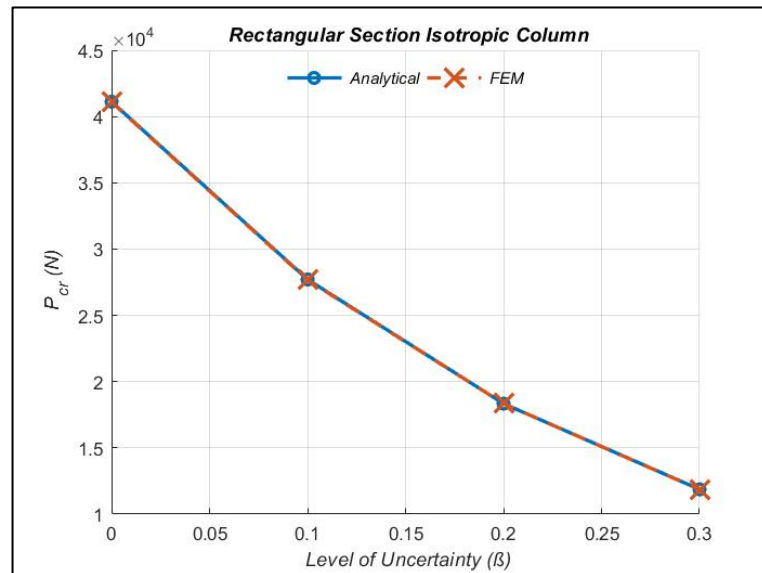


Figure 4.17: Comparison of analytical and FEM results for isotropic rectangular column.

Figure 4.17 and Figure 4.18 indicate the comparison of the results obtained with analytical and finite element method (FEM). For isotropic rectangular and I-section

composite columns, results provide closer values which validate the derived equations in Chapter 3.

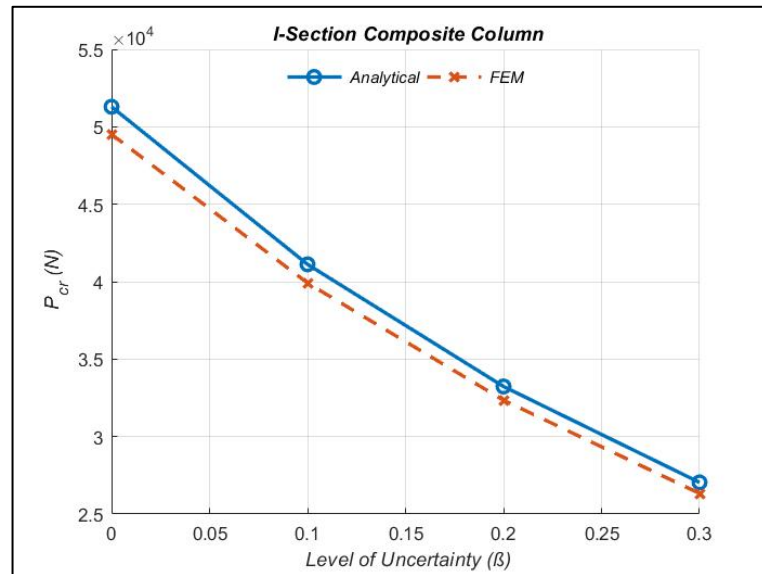


Figure 4.18: Comparison of analytical and FEM results for I-section composite column.

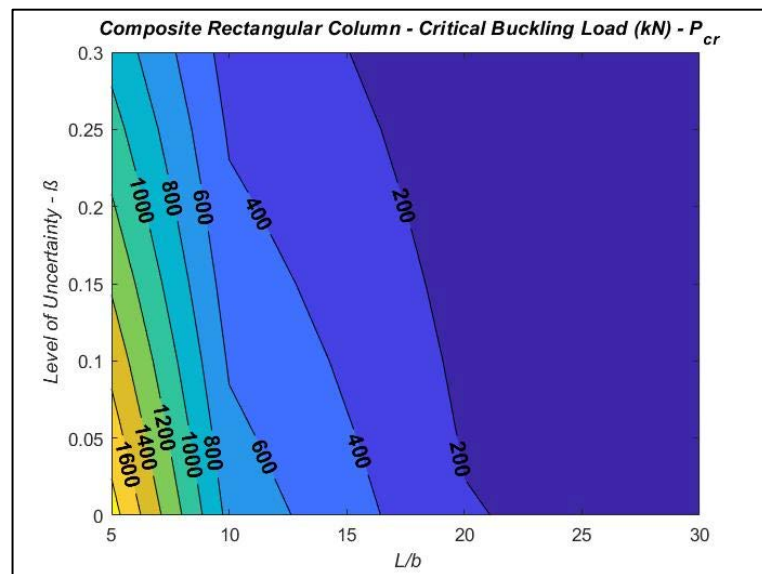


Figure 4.19: Contour plots of composite rectangular column.

Figure 4.19 – 4.25 shows contour plot of critical buckling load at different level of uncertainty and the ratio of L to b or the ratio of L to R. It's seen that even minor variations on the parameters provide considerable loss of critical buckling load values for all composite sections.

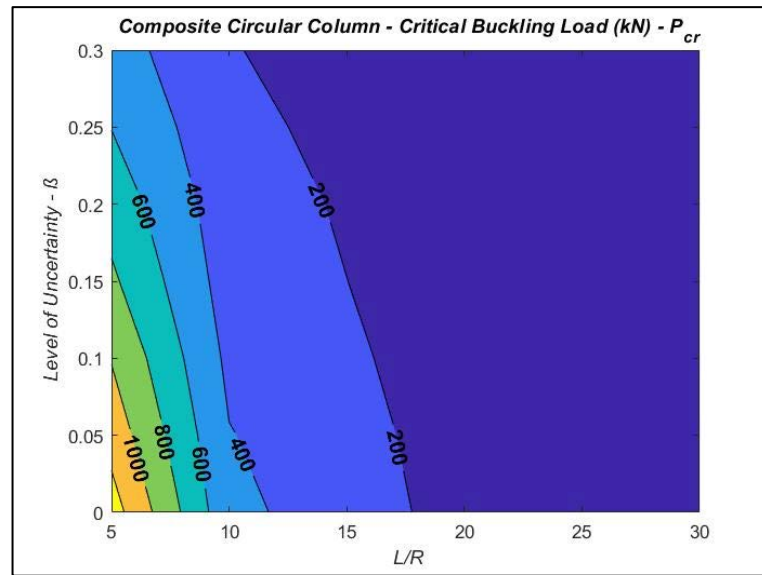


Figure 4.20: Contour plots of composite circular column.

Although the increment of L/b or L/R values are low for range between 5 and 10, the decrement values on critical buckling loads are remarkable for all composite sections as it's represented in contour plots (Figure 4.19 – 4.25).

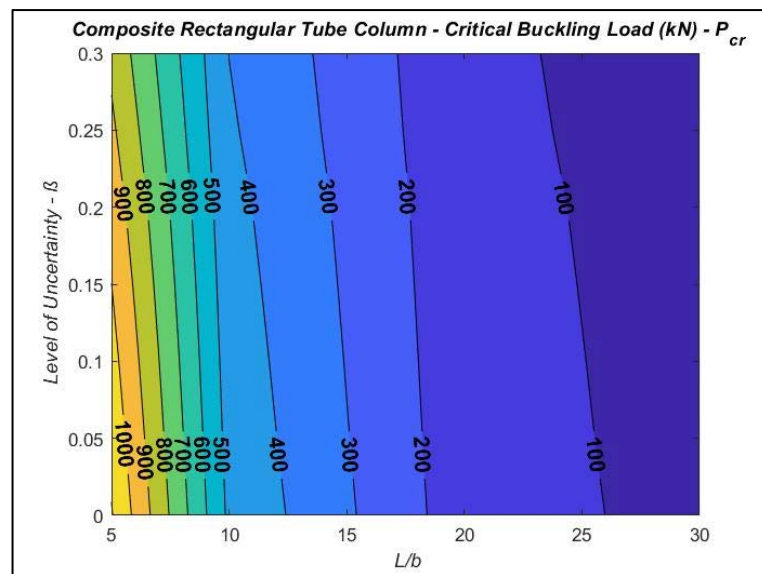


Figure 4.21: Contour plots of composite rectangular tube column.

Also, for the range of L/b or L/R bigger than 10, critical buckling load values are decreasing as L/b or L/R values are increasing but not as much as for the range between 5 and 10.

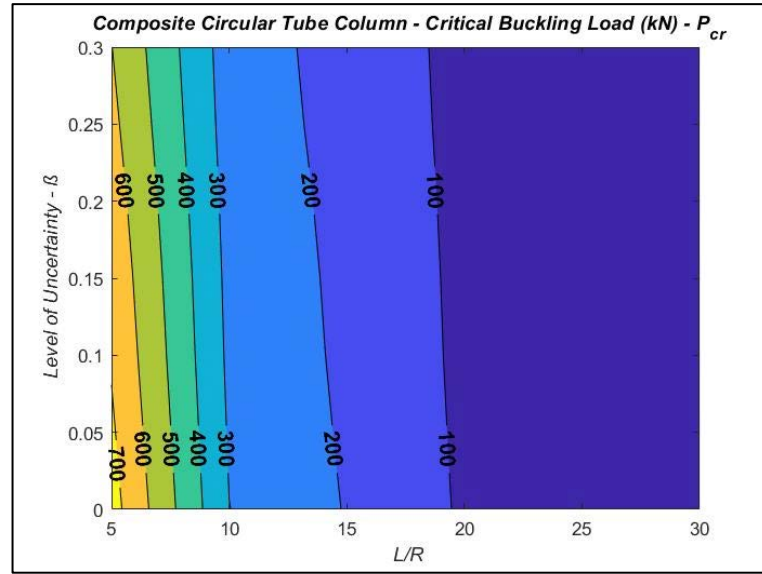


Figure 4.22: Contour plots of composite circular tube column.

Therefore, it can be concluded that the critical L/b or L/R value for composite sections studied in this thesis is 10. Cases for smaller values of L/b or L/R than 10, especially the dimensional uncertainties are should be more taken into account during productions stages of columns (Figure 4.19 – 4.25).

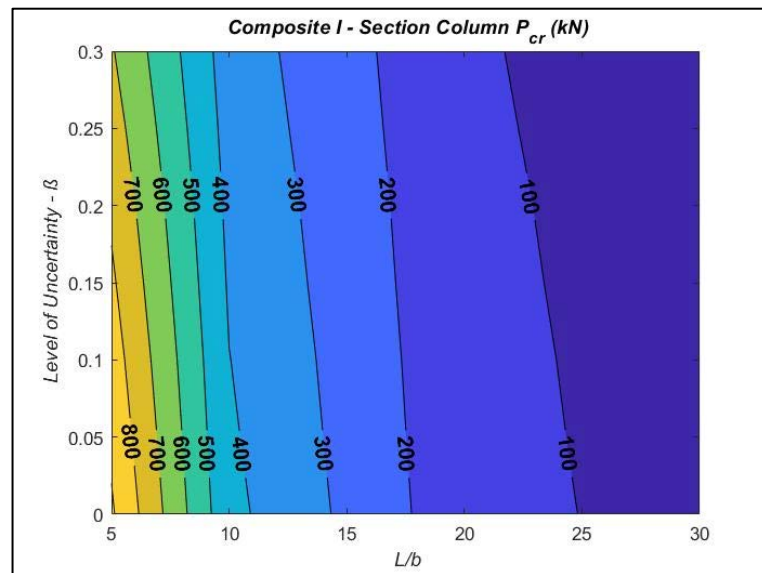


Figure 4.23: Contour plots of composite I-section column.

Also, it's clear that the uncertainty negatively effects the critical buckling load for all L/b or L/R values. Therefore, instead of calculating critical buckling loads using certain values of material properties and its dimensions, the buckling calculations

should be made considering uncertainty which provides to obtain realistic critical buckling load at defined level of uncertainty (Figure 4.19 – 4.25).

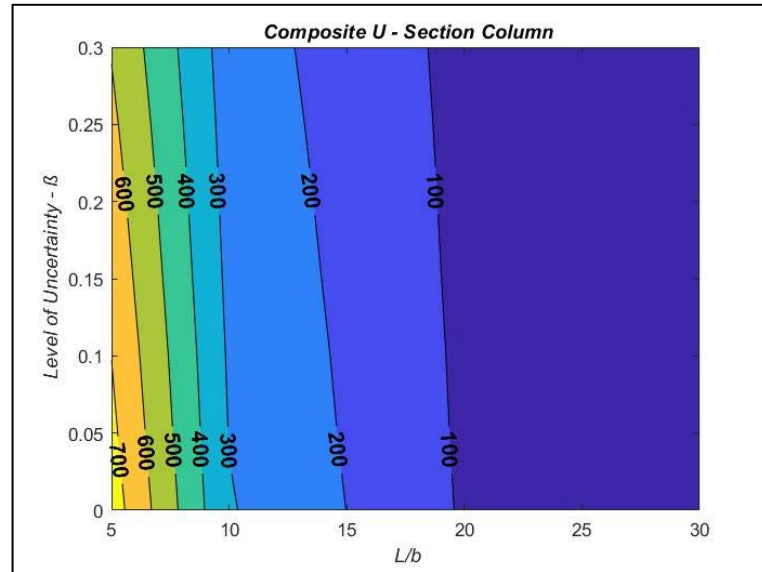


Figure 4.24: Contour plots of composite U-section column.

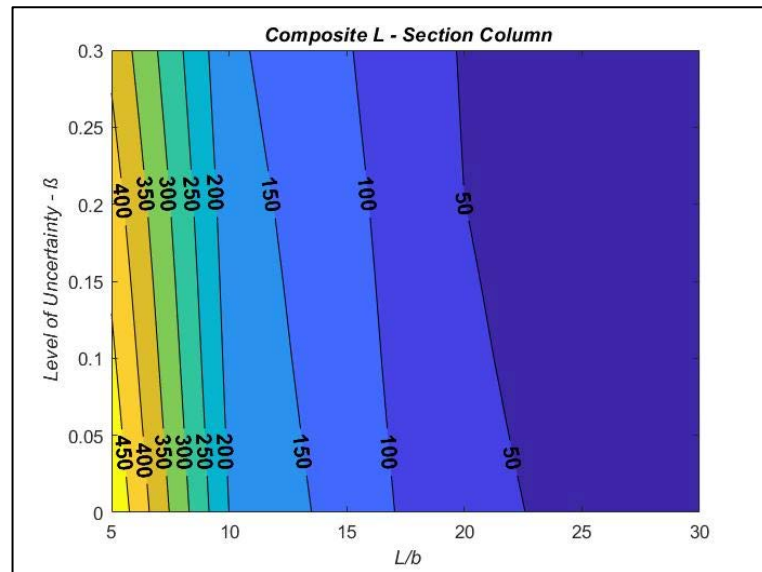


Figure 4.25: Contour plots of composite L-section column.

As a result, critical buckling load equations are obtained analytically for different cross sections of isotropic and composite columns with uncertain material properties and dimensions. In this study, changes in critical buckling load of isotropic and composite columns with uncertain material properties and dimensions are obtained. It

is stated that even small changes in the uncertainty level create significant deviations in critical buckling load. Also, in Appendix B, comparisons of isotropic and composite columns are represented. Normalized critical buckling load values at different level of uncertainties are compared. Decrement on the normalized critical buckling load for composite columns at different level of uncertainties are less than the values of normalized critical buckling load of isotropic columns. As another topic, performing sensitivity analysis, it is calculated which parameter has more effect on the losses in critical buckling load. It's observed that the issue of uncertainty has to be included in the calculations particularly at the preliminary design stages.

In addition, bending and torsion of beams and plates may be studied in the future. Particularly in all areas where the calculations are made at the nano level, many studies on the subject of uncertainty can be done in this area as the deviations from the production at the nano level have significant effects.

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BIOGRAPHY

Onur Kaya was born in İzmir in 1992. He received his Bachelor of Science degree in Mechanical Engineering from Manisa Celal Bayar University, Manisa, Turkey in 2016. He started his graduate studies in Mechanical Engineering at İzmir Kâtip Çelebi University, İzmir, Turkey in 2016 and transferred to Gebze Technical University, Kocaeli, Turkey in 2018.

Since 2018, he has been working at Gebze Technical University as a Research Assistant.

APPENDICES

Appendix A: Expressions for the constants used in Chapter 3.

$$\varphi_r = \frac{1}{12} \frac{\pi^2 E b f^3}{L^2} \quad (\text{A1.1})$$

$$\varphi_{0,c} = \frac{1}{64} \frac{\pi^3 E R^4}{L^2} \quad (\text{A1.2})$$

$$\varphi_{1,c} = \frac{1}{64} \frac{\pi^3 E R^4}{L^2} \quad (\text{A1.3})$$

$$\varphi_{2,c} = \frac{1}{16} \frac{\pi^3 E R^4}{L^2} \quad (\text{A1.4})$$

$$\varphi_{3,c} = -\frac{1}{32} \frac{\pi^3 E R^4}{L^2} \quad (\text{A1.5})$$

$$\varphi_{0,rt} = \frac{1}{6} \frac{\pi^2 E h^2 \left[b \left(3 f^2 - 6 f h + 4 h^2 \right) + (f - 2 h)^3 \right]}{L^2} \quad (\text{A1.6})$$

$$\varphi_{1,rt} = \frac{1}{6} \frac{\pi^2 E h^2 \left[b \left(3 f^2 - 6 f h + 4 h^2 \right) + (f - 2 h)^3 \right]}{L^2} \quad (\text{A1.7})$$

$$\varphi_{2,rt} = \frac{1}{6} \frac{\pi^2 E h b \left(3 f^2 - 6 f h + 4 h^2 \right)}{L^2} \quad (\text{A1.8})$$

$$\varphi_{3,rt} = \frac{\pi^2 E f h \left(\frac{1}{2} f^2 + (b - 2 h)(f - h) \right)}{L^2} \quad (\text{A1.9})$$

$$\varphi_{4,rt} = \frac{1}{6} \frac{\pi^2 E h (3b + f - 8h) (f - 2h)^2}{L^2} \quad (\text{A1.10})$$

$$\varphi_{5,rt} = \frac{1}{3} \frac{\pi^2 E h^2 \left[b(-3f^2 + 6fh - 4h^2) + [(2h - f)^3] \right]}{L^2} \quad (\text{A1.11})$$

$$\varphi_{0,ct} = \frac{1}{8} \frac{\pi^3 E h (R^3 - 3R^2 h + 4Rh^2 - 2h^3)}{L^2} \quad (\text{A1.12})$$

$$\varphi_{1,ct} = \frac{1}{8} \frac{\pi^3 E h (R^3 - 3R^2 h + 4Rh^2 - 2h^3)}{L^2} \quad (\text{A1.13})$$

$$\varphi_{2,ct} = \frac{1}{8} \frac{\pi^3 E h (3R^3 - 6R^2 h + 4Rh^2)}{L^2} \quad (\text{A1.14})$$

$$\varphi_{3,ct} = \frac{1}{8} \frac{\pi^3 E h (R^3 - 6R^2 h + 12Rh^2 - 8h^3)}{L^2} \quad (\text{A1.15})$$

$$\varphi_{4,ct} = \frac{1}{4} \frac{\pi^3 E h (-R^3 + 3R^2 h - 4Rh^2 + 2h^3)}{L^2} \quad (\text{A1.16})$$

$$\varphi_{0,I} = \frac{1}{12} \frac{\pi^2 E h^2 \left[b(6f^2 - 12fh + 8h^2) + [(f - 2h)^3] \right]}{L^2} \quad (\text{A1.17})$$

$$\varphi_{1,I} = \frac{1}{12} \frac{\pi^2 E h^2 \left[b(6f^2 - 12fh + 8h^2) + [(f - 2h)^3] \right]}{L^2} \quad (\text{A1.18})$$

$$\varphi_{2,I} = \frac{1}{6} \frac{\pi^2 E h b (3f^2 - 6fh + 4h^2)}{L^2} \quad (\text{A1.19})$$

$$\varphi_{3,I} = \frac{1}{12} \frac{\pi^2 E h (f - 2h)^2 (6b + f - 8h)}{L^2} \quad (\text{A1.20})$$

$$\varphi_{4,I} = \frac{1}{6} \frac{\pi^2 E h^2 \left[b(-6f^2 + 12fh - 8h^2) + (2h - f)^3 \right]}{L^2} \quad (\text{A1.21})$$

$$\varphi_{0,U} = \frac{1}{12} \frac{\pi^2 E h^2}{L^2 (b + 2f - 2h)} \left[\frac{4h^4 - 4h^3(b + 4f) - 4h(3bf^2 + 4f^3)}{(b^2 + 8bf + 24f^2)h^2 + 8bf^3 + 4f^4} \right] \quad (\text{A1.22})$$

$$\varphi_{1,U} = \frac{1}{12} \frac{\pi^2 E h^2}{L^2 (b + 2f - 2h)} \left[\frac{4h^4 - 4h^3(b + 4f) - 4h(3bf^2 + 4f^3)}{(b^2 + 8bf + 24f^2)h^2 + 8bf^3 + 4f^4} \right] \quad (\text{A1.23})$$

$$\varphi_{2,U} = \frac{1}{12} \frac{\pi^2 E h b}{L^2 (b + 2f - 2h)^2} \left[\frac{4h^4 - 4h^3(b + 2f) - 24f^3h + 12f^4}{h^2(b^2 + 4bf + 16f^2)} \right] \quad (\text{A1.24})$$

$$\varphi_{3,U} = \frac{1}{12} \frac{\pi^2 E h}{L^2 (b + 2f - 2h)^2} \left[\begin{aligned} & -32h^5 + 4h^4(34f + 11b) + \\ & 4h^3(-5b^2 - 32bf - 56f^2) + \\ & h^2(3b^3 + 30b^2f + 144bf^2 + 176f^3) - \\ & 24hf^2(b + 2f)\left(b + \frac{4}{3}f\right) + \\ & 8f^3(b + 2f)\left(b + \frac{1}{2}f\right) \end{aligned} \right] \quad (\text{A1.25})$$

$$\varphi_{4,U} = \frac{1}{6} \frac{\pi^2 E h}{L^2 (b + 2f - 2h)} \begin{bmatrix} -4h^4 + 4h^3(b + 4f) - \\ h^2(b^2 + 8bf + 24f^2) + \\ 4h(3bf^2 + 4f^3) - 8bf^3 - 4f^4 \end{bmatrix} \quad (\text{A1.26})$$

$$\varphi_{0,L} = \frac{1}{12} \frac{\pi^2 E h}{L^2 (b + f - h)} \begin{bmatrix} h^4 - 2h^3(b + 2f) + h^2(b^2 + 4bf + 6f^2) - \\ 2hf^2(3b + 2f) + 4bf^3 + f^4 \end{bmatrix} \quad (\text{A1.27})$$

$$\varphi_{1,L} = \frac{1}{12} \frac{\pi^2 E h}{L^2 (b + f - h)} \begin{bmatrix} h^4 - 2h^3(b + 2f) + h^2(b^2 + 4bf + 6f^2) - \\ 2hf^2(3b + 2f) + 4bf^3 + f^4 \end{bmatrix} \quad (\text{A1.28})$$

$$\varphi_{2,L} = \frac{1}{12} \frac{\pi^2 E h b}{L^2 (b + f - h)^2} \begin{bmatrix} h^4 - 2h^3(b + f) + h^2(b^2 + 2bf + 4f^2) \\ -6f^3h + 3f^4 \end{bmatrix} \quad (\text{A1.29})$$

$$\varphi_{3,L} = \frac{1}{12} \frac{\pi^2 E h}{L^2 (b + f - h)^2} \begin{bmatrix} -4h^5 + h^4(17f + 11b) + 4b^2f^3 - \\ 2h^3(5b^2 + 16bf + 14f^2) + f^5 + \\ h^2(3b^3 + 15b^2f + 36bf^2 + 22f^3) - \\ 4hf^2(3b^2 + 5bf + 2f^2) + 5bf^4 \end{bmatrix} \quad (\text{A1.30})$$

$$\varphi_{4,L} = -\frac{1}{6} \frac{\pi^2 E h}{L^2 (b + f - h)} \begin{bmatrix} h^4 - 2h^3(b + 2f) + h^2(b^2 + 4bf + 6f^2) - \\ 2hf^2(3b + 2f) + 4bf^3 + f^4 \end{bmatrix} \quad (\text{A1.31})$$

$$\varphi_{0,r}^c = \frac{1}{12} \pi^2 b^2 f^4 G_{13} E_1 \quad (\text{A1.32})$$

$$\varphi_{1,r}^c = \frac{1}{12} \pi^2 b^2 f^4 G_{13} E_1 \quad (\text{A1.33})$$

$$\varphi_{2,r}^c = \frac{1}{12} \pi^2 b^2 f^4 G_{13} E_1 \quad (\text{A1.34})$$

$$\varphi_{4,r}^c = \frac{1}{3} \pi^2 b^2 f^4 G_{13} E_1 \quad (\text{A1.35})$$

$$\zeta_{0,r}^c = \frac{1}{12} \pi^2 b f^3 E_1 + b f G_{13} L^2 \quad (\text{A1.36})$$

$$\zeta_{1,r}^c = \frac{1}{12} \pi^2 b f^3 E_1 \quad (\text{A1.37})$$

$$\zeta_{2,r}^c = L^2 b f G_{13} \quad (\text{A1.38})$$

$$\zeta_{3,r}^c = 2 L^2 b f G_{13} \quad (\text{A1.39})$$

$$\zeta_{4,r}^c = \frac{1}{4} \pi^2 b f^3 E_1 + b f G_{13} L^2 \quad (\text{A1.40})$$

$$\varphi_{0,c}^c = \frac{1}{256} \pi^4 R^6 G_{13} E_1 \quad (\text{A1.41})$$

$$\varphi_{1,c}^c = \frac{1}{256} \pi^4 R^6 G_{13} E_1 \quad (\text{A1.42})$$

$$\varphi_{2,c}^c = \frac{1}{256} \pi^4 R^6 G_{13} E_1 \quad (\text{A1.43})$$

$$\varphi_{4,c}^c = \frac{3}{128} \pi^4 R^6 G_{13} E_1 \quad (\text{A1.44})$$

$$\zeta_{0,c}^c = \frac{1}{64} \pi^3 R^4 E_1 + \frac{1}{4} \pi R^2 G_{13} L^2 \quad (\text{A1.45})$$

$$\zeta_{1,c}^c = \frac{1}{64}\pi^3 R^4 E_1 \quad (\text{A1.46})$$

$$\zeta_{2,c}^c = \frac{1}{4}L^2\pi R^2 G_{13} \quad (\text{A1.47})$$

$$\zeta_{3,c}^c = \frac{1}{2}L^2\pi R^2 G_{13} \quad (\text{A1.48})$$

$$\zeta_{4,c}^c = \frac{1}{16}\pi^3 R^4 E_1 + \frac{1}{2}\pi R^2 G_{13} L^2 \quad (\text{A1.49})$$

$$\varphi_{0,rt}^c = -\frac{1}{6}\pi^2 E_1 G_{13} h(b+f-2h)^2 \left[b(f-2h)[b-(f+4h)] - 8h^3 \right] \quad (\text{A1.50})$$

$$\varphi_{1,rt}^c = -\frac{1}{6}\pi^2 E_1 G_{13} h(b+f-2h)^2 \left[b(f-2h)[b-(f+4h)] - 8h^3 \right] \quad (\text{A1.51})$$

$$\varphi_{2,rt}^c = -\frac{1}{6}\pi^2 E_1 G_{13} h(b+f-2h)^2 \left[b(f-2h)[b-(f+4h)] - 8h^3 \right] \quad (\text{A1.52})$$

$$\varphi_{4,rt}^c = -\frac{1}{6}\pi^2 E_1 G_{13} h(b+f-2h) \left[\begin{array}{l} 96h^4 - 16h^3(7b+2f) + \\ 40bh^2(b+f) - \\ 2hb(2b^2+7bf-f^2) + \\ bf(b^2-f^2) \end{array} \right] \quad (\text{A1.53})$$

$$\zeta_{0,rt}^c = (b+f-2h) \left[\begin{array}{l} -\frac{1}{12}\pi^2 E_1 \left[b(f-2h)[b-(f+4h)] - 8h^3 \right] \\ + hL^2 G_{13} \end{array} \right] \quad (\text{A1.54})$$

$$\zeta_{1,rt}^c = -\frac{1}{12}\pi^2 E_1 (b+f-2h) \left[b(f-2h)[b-(f+4h)] - 8h^3 \right] \quad (\text{A1.55})$$

$$\zeta_{2,rt}^c = 2G_{13}L^2h(b + f - 2h) \quad (\text{A1.56})$$

$$\zeta_{3,rt}^c = 4G_{13}L^2h(b + f - 2h) \quad (\text{A1.57})$$

$$\zeta_{4,rt}^c = h \left[\frac{1}{6} \pi^2 E_1 (b - 2h)^2 (b + 3f - 8h) + 2L^2 G_{13} (b + f - 4h) \right] \quad (\text{A1.58})$$

$$\varphi_{0,ct}^c = \frac{1}{8} \pi^4 E_1 G_{13} h^2 (R^2 - 2Rh + 2h^2) (R - h)^2 \quad (\text{A1.59})$$

$$\varphi_{1,ct}^c = \frac{1}{8} \pi^4 E_1 G_{13} h^2 (R^2 - 2Rh + 2h^2) (R - h)^2 \quad (\text{A1.60})$$

$$\varphi_{2,ct}^c = \frac{1}{8} \pi^4 E_1 G_{13} h^2 (R^2 - 2Rh + 2h^2) (R - h)^2 \quad (\text{A1.61})$$

$$\varphi_{4,ct}^c = \frac{1}{4} \pi^4 E_1 G_{13} h^2 (R - h) (R - 2h) (R^2 - 3Rh + 3h^2) \quad (\text{A1.62})$$

$$\zeta_{0,ct}^c = \pi h (R - h) \left[\frac{1}{8} \pi^2 E_1 (R^2 - 2Rh + 2h^2) + L^2 G_{13} \right] \quad (\text{A1.63})$$

$$\zeta_{1,ct}^c = \frac{1}{8} \pi^3 E_1 h (R - h) (R^2 - 2Rh + 2h^2) \quad (\text{A1.64})$$

$$\zeta_{2,ct}^c = \pi G_{13} L^2 h (R - h) \quad (\text{A1.65})$$

$$\zeta_{3,ct}^c = 2\pi G_{13} L^2 h (R - h) \quad (\text{A1.66})$$

$$\zeta_{4,ct}^c = \pi h (R - 2h) \left[\frac{1}{8} \pi^2 E_1 (R - 2h)^2 + L^2 G_{13} \right] \quad (\text{A1.67})$$

$$\varphi_{0,I}^c = \frac{1}{12} \pi^2 E_1 G_{13} h^2 (-2h + 2b + f) \left[\begin{array}{l} 2b(3f^2 - 6fh + 4h^2) \\ + f(f^2 - 6fh + 12h^2) - 8h^3 \end{array} \right] \quad (\text{A1.68})$$

$$\varphi_{1,I}^c = \frac{1}{12} \pi^2 E_1 G_{13} h^2 (-2h + 2b + f) \left[\begin{array}{l} 2b(3f^2 - 6fh + 4h^2) \\ + f(f^2 - 6fh + 12h^2) - 8h^3 \end{array} \right] \quad (\text{A1.69})$$

$$\varphi_{2,I}^c = \frac{1}{12} \pi^2 E_1 G_{13} h^2 (-2h + 2b + f) \left[\begin{array}{l} 2b(3f^2 - 6fh + 4h^2) \\ + f(f^2 - 6fh + 12h^2) - 8h^3 \end{array} \right] \quad (\text{A1.70})$$

$$\varphi_{4,I}^c = 2G_{13}\pi^2 E_1 h^2 \left[\begin{array}{l} 4h^4 - \frac{20}{3}h^3(b+f) + \frac{8}{3}h^2\left(b + \frac{1}{2}f\right)(b+3f) + \\ hf\left(-3b^2 - \frac{9}{2}bf - f^2\right) + f^2\left(b + \frac{1}{2}f\right)\left(b + \frac{1}{6}f\right) \end{array} \right] \quad (\text{A1.71})$$

$$\zeta_{0,I}^c = 2h \left[\begin{array}{l} \frac{1}{4} \pi^2 E_1 \left[\begin{array}{l} -\frac{4}{3}h^3 + h^2\left(\frac{4}{3}b + 2f\right) + f^2\left(b + \frac{1}{6}f\right) \\ + hf(-2b - f) \end{array} \right] \\ + L^2 G_{13} \left(b + \frac{1}{2}f - h\right) \end{array} \right] \quad (\text{A1.72})$$

$$\zeta_{1,I}^c = \frac{1}{12} \pi^2 E_1 h \left[2b(3f^2 - 6fh + 4h^2) + f(f^2 - 6fh + 12h^2) - 8h^3 \right] \quad (\text{A1.73})$$

$$\zeta_{2,I}^c = G_{13} L^2 h (2b + f - 2h) \quad (\text{A1.74})$$

$$\zeta_{3,I}^c = 2G_{13} L^2 h (2b + f - 2h) \quad (\text{A1.75})$$

$$\zeta_{4,I}^c = 2h \left[\frac{1}{4} \pi^2 E_1 (f - 2h)^2 \left(b + \frac{1}{6}f - \frac{4}{3}h\right) + L^2 G_{13} \left(b + \frac{1}{2}f - 2h\right) \right] \quad (\text{A1.76})$$

$$\varphi_{0,U}^c = \frac{1}{12} \pi^2 h^2 E_1 G_{13} \begin{bmatrix} 4h^4 - 4h^3(b+4f) + h^2(b^2 + 8bf + 24f^2) \\ -4hf^2(3b+4f) + 4f^3(2b+f) \end{bmatrix} \quad (\text{A1.77})$$

$$\varphi_{1,U}^c = \frac{1}{12} \pi^2 h^2 E_1 G_{13} \begin{bmatrix} 4h^4 - 4h^3(b+4f) + h^2(b^2 + 8bf + 24f^2) \\ -4hf^2(3b+4f) + 4f^3(2b+f) \end{bmatrix} \quad (\text{A1.78})$$

$$\varphi_{2,U}^c = \frac{1}{12} \pi^2 h^2 E_1 G_{13} \begin{bmatrix} 4h^4 - 4h^3(b+4f) + h^2(b^2 + 8bf + 24f^2) \\ -4hf^2(3b+4f) + 4f^3(2b+f) \end{bmatrix} \quad (\text{A1.79})$$

$$\varphi_{4,U}^c = \frac{1}{3} \pi^2 h^2 G_{13} E_1 \begin{bmatrix} 6h^4 - 5h^3(b+4f) + h^2(b^2 + 8bf + 24f^2) \\ -3hf^2(3b+4f) + 2f^3(2b+f) \end{bmatrix} \quad (\text{A1.80})$$

$$\zeta_{0,U}^c = \frac{1}{12} \frac{h}{k} \left[\pi^2 E_1 \begin{bmatrix} 4h^4 + h^2(b^2 + 8bf + 24f^2) \\ -4h^3(b+4f) + 4f^3(2b+f) \\ -4h(3bf^2 + 4f^3) \end{bmatrix} + 12G_{13}L^2k^2 \right] \quad (\text{A1.81})$$

$$k = 2(f-h) + b \quad (\text{A1.82})$$

$$\zeta_{1,U}^c = \frac{1}{12} \frac{\pi^2 E_1 h}{k} \begin{bmatrix} 4h^3[h - (b+4f)] + h^2(b^2 + 8bf + 24f^2) \\ -4hf^2(3b+4f) + 4f^3(2b+f) \end{bmatrix} \quad (\text{A1.83})$$

$$\zeta_{2,U}^c = G_{13} h L^2 k \quad (\text{A1.84})$$

$$\zeta_{3,U}^c = 2G_{13} h L^2 k \quad (\text{A1.85})$$

$$\zeta_{4,U}^c = \frac{1}{12} \frac{h}{k^2} \left[\pi^2 E_1 \left[\begin{aligned} &-32h^5 + 4h^4(11b + 34f) - \\ &4h^3(5b^2 + 32bf + 56f^2) + \\ &h^2(3b^3 + 30b^2f + 144bf^2 + 176f^3) - \\ &4f^2(b + 2f) \left[6h \left(b + \frac{4}{3}f \right) + 2f \left(b + \frac{1}{2}f \right) \right] \end{aligned} \right] + 12G_{13}L^2[(b + 2f - 4h)k^2] \right] \quad (\text{A1.86})$$

$$\varphi_{0,L}^c = \frac{1}{12} \pi^2 E_1 G_{13} h^2 \left[\begin{aligned} &\left(h^4 - 2h^3(b + 2f) + h^2(b^2 + 4bf + 6f^2) \right) \\ &-2hf^2(3b + 2f) + f^3(4b + f) \end{aligned} \right] \quad (\text{A1.87})$$

$$\varphi_{1,L}^c = \frac{1}{12} \pi^2 E_1 G_{13} h^2 \left[\begin{aligned} &\left(h^4 - 2h^3(b + 2f) + h^2(b^2 + 4bf + 6f^2) \right) \\ &-2hf^2(3b + 2f) + f^3(4b + f) \end{aligned} \right] \quad (\text{A1.88})$$

$$\varphi_{2,L}^c = \frac{1}{12} \pi^2 E_1 G_{13} h^2 \left[\begin{aligned} &\left(h^4 - 2h^3(b + 2f) + h^2(b^2 + 4bf + 6f^2) \right) \\ &-2hf^2(3b + 2f) + f^3(4b + f) \end{aligned} \right] \quad (\text{A1.89})$$

$$\varphi_{4,L}^c = \frac{1}{6} \pi^2 E_1 G_{13} h^2 \left[\begin{aligned} &3h^4 - 5h^3(b + 2f) + 2h^2(b^2 + 4bf + 6f^2) \\ &-3f^2(3b + 2f)h + f^3(4b + f) \end{aligned} \right] \quad (\text{A1.90})$$

$$\zeta_{0,L}^c = \frac{1}{12} \frac{h}{b + f - h} \left[\pi^2 E_1 \left[\begin{aligned} &h^4 - 2h^3(b + 2f) + h^2(b^2 + 4bf + 6f^2) \\ &-2hf^2(+3b + 2f) + f^3(4b + f) \end{aligned} \right] + 12G_{13}L^2[(b + f - h)^2] \right] \quad (\text{A1.91})$$

$$\zeta_{1,L}^c = \frac{1}{12} \frac{\pi^2 E_1 h}{b + f - h} \left[\begin{aligned} &h^4 - 2h^3(b + 2f) + h^2(b^2 + 4bf + 6f^2) \\ &-2hf^2(+3b + 2f) + f^3(4b + f) \end{aligned} \right] \quad (\text{A1.92})$$

$$\zeta_{2,L}^c = G_{13} h L^2 (b + f - h) \quad (\text{A1.93})$$

$$\zeta_{3,L}^c = 2G_{13} h L^2 (b + f - h) \quad (\text{A1.94})$$

$$\zeta_{4,L}^c = \frac{1}{12} \frac{h}{k^2} \left[\pi^2 E_1 \left[\begin{aligned} &-4h^5 + h^4(11b + 17f) - \\ &2h^3(5b^2 + 16bf + 14f^2) + \\ &h^2(3b^3 + 15b^2f + 36bf^2 + 22f^3) - \\ &4f^2(b + f) \left[3h \left(b + \frac{2}{3}f \right) + f \left(b + \frac{1}{4}f \right) \right] \end{aligned} \right] + 12G_{13}L^2 \left[(b + f - 2h)(b + f - h)^2 \right] \right] \quad (\text{A1.95})$$

Appendix B: Comparisons loss of normalized critical buckling load for isotropic and composite columns.

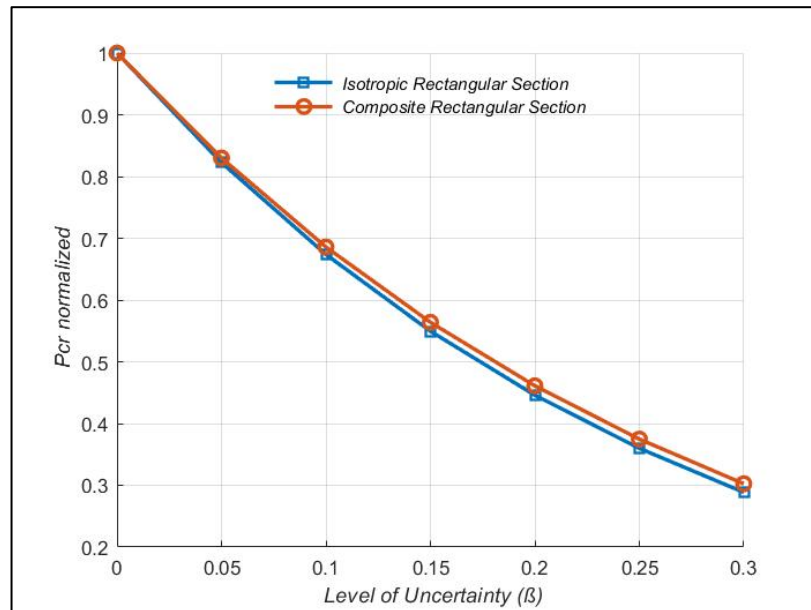


Figure B.1: Loss of normalized critical buckling load for isotropic and composite rectangular section.

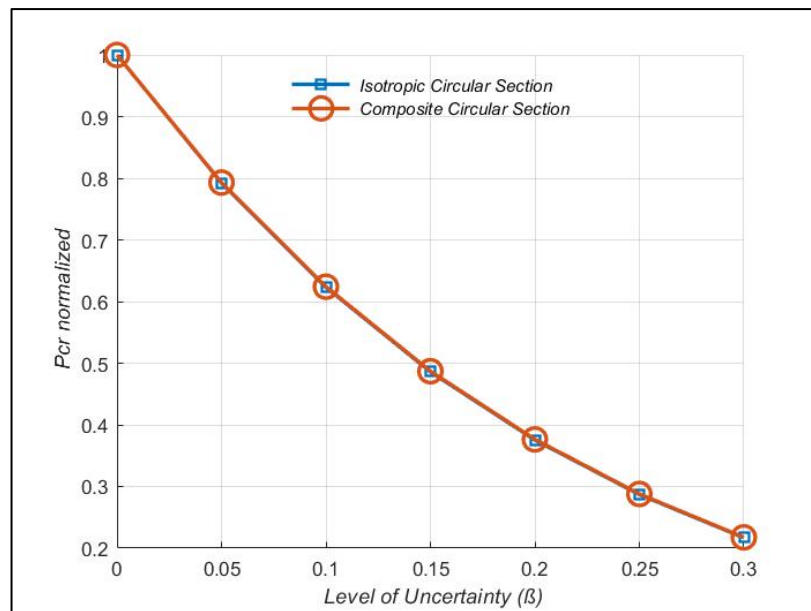


Figure B.2: Loss of normalized critical buckling load for isotropic and composite circular section.

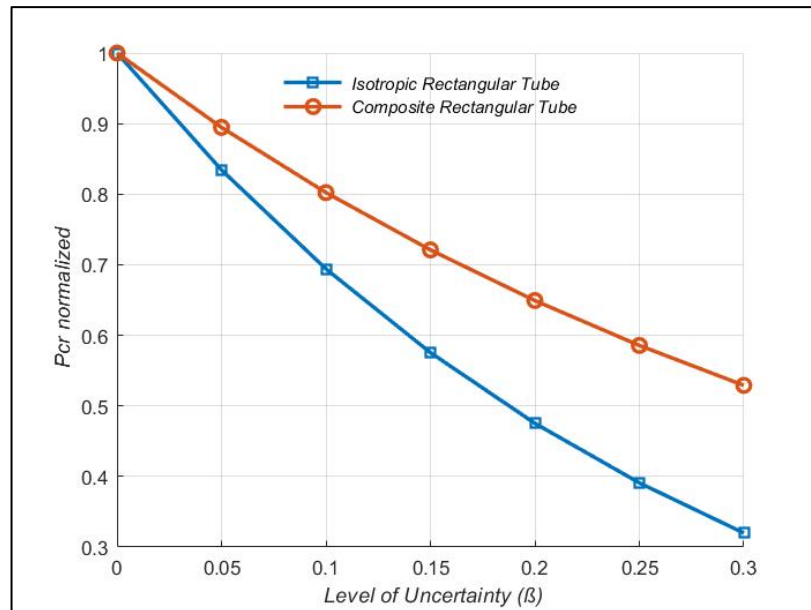


Figure B.3: Loss of normalized critical buckling load for isotropic and composite rectangular tube section.

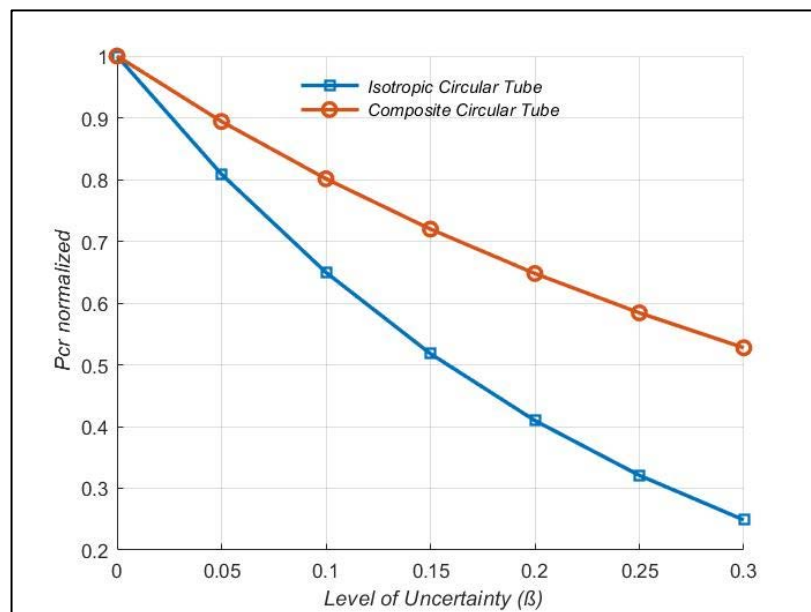


Figure B.4: Loss of normalized critical buckling load for isotropic and composite circular tube section.

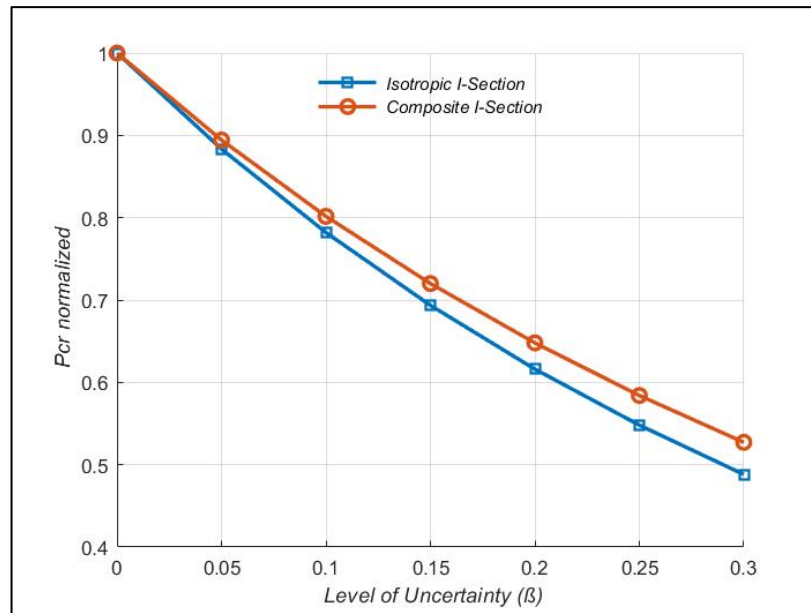


Figure B.5: Loss of normalized critical buckling load for isotropic and composite I – section.

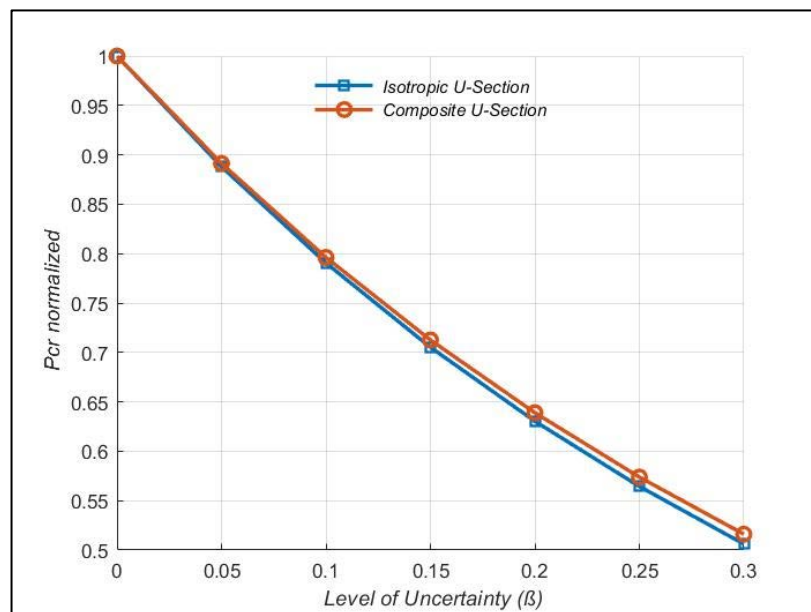


Figure B.6: Loss of normalized critical buckling load for isotropic and composite U – section.

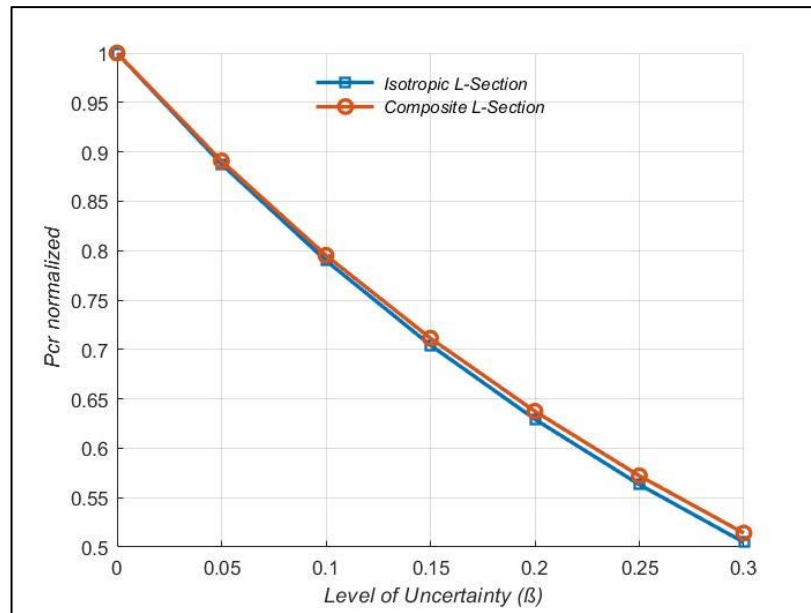


Figure B.7: Loss of normalized critical buckling load for isotropic and composite L – section.