

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL

**STABILITY AND LIPSCHITZ STABILITY CRITERIA OF SET
VALUED DIFFERENTIAL EQUATIONS AND ITS
APPLICATIONS**

TUBA SATILMIŞ

A THESIS OF MASTER OF SCIENCE
DEPARTMENT OF MATHEMATICS

ADVISOR: PROF. DR. COŞKUN YAKAR

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LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

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This thesis is dedicated to the memory of my beloved late father, Asri SATILMIŞ. His unwavering support, wisdom, and guidance continue to inspire me and have left a profound and lasting impact on my life.

ABSTRACT

In this thesis, we examine stability and Lipschitz stability concepts for set-valued differential equations, particularly focusing on systems affected by initial time differences (ITD) and perturbations in the spatial variable. Set-valued systems provide a natural framework for modeling uncertainty, and understanding their long-term behavior requires strong theoretical foundations. Lyapunov's second method has been used as a key analytical tool to study stability without the need for explicit solutions.

Classical stability concepts are extended to the context of set-valued differential equations, and new stability definitions based on ITD are introduced. Lyapunov-like functions are used to derive several results, and a unified stability concept based on two measures is proposed, which integrates existing stability notions. Furthermore, nonlinear variation of parameters theory is extended to account for initial time differences.

Our findings demonstrate that a set-valued differential equation can be stable without necessarily being Lipschitz stable, but if it is Lipschitz stable, it must also be stable. This highlights that Lipschitz stability imposes stricter conditions than classical stability. The results contribute to the development of a more generalized framework for stability analysis, with potential applications in control theory, nonlinear systems, and applied mathematics, where stability plays a central role.

Keywords: Set-Valued Differential Equations, Stability, Lipschitz Stability, Initial Time Difference, Lyapunov Functions, Variation Of Parameters, Nonlinear Systems.

ÖZET

Bu tezde, küme değerli diferansiyel denklemleri için stabilite ve Lipschitz kararlılık incelenmiştir. Özellikle, başlangıç zaman farkları (ITD) ve mekansal değişkenlere bağlı bozulmalarla etkilenen sistemler üzerinde durulmuştur. Küme-değerli sistemler, belirsizlikleri doğal bir şekilde modellemekte olup, uzun vadeli davranışlarının anlaşılması güçlü teorik çerçeveler gerektirir. Lyapunov'un ikinci yöntemi, açık çözümler bulmaya gerek kalmadan kararlılığı incelemek için anahtar bir analitik araç olarak kullanılmaktadır.

Klasik kararlılık kavramları, küme değerli diferansiyel denklemler genişletilmiş ve ITD temelli yeni kararlılık tanımları geliştirilmiştir. Lyapunov benzeri fonksiyonlar kullanılarak çeşitli sonuçlar elde edilmiş ve mevcut yaklaşımları birleştiren iki ölçüye dayalı birleşik bir kararlılık kavramı önerilmiştir. Ayrıca, başlangıç zaman farklarını hesaba katan doğrusal olmayan parametre varyasyonu teorisi detaylandırılmıştır.

Bulgularımız, bir küme değerli diferansiyel denklemin Lipschitz kararlı olup olmadan kararlı olabileceğini, ancak tersinin her zaman geçerli olmadığını göstermektedir; yani, Lipschitz kararlılık, klasik kararlılığı zorunlu kılar. Bu durum, Lipschitz kararlılığın daha kısıtlayıcı bir kriter olduğunu ortaya koymaktadır. Bu göre, kararlılığın merkezi bir rol oynadığı kontrol teorisi, doğrusal olmayan sistemler ve uygulamalı matematik gibi alanlarda potansiyel uygulamalara olanak tanımaktadır.

Anahtar Kelimeler: Küme-Değerli Diferansiyel Denklemler, Kararlılık, Lipschitz Kararlılık, Başlangıç Zaman Farkı, Lyapunov Fonksiyonları, Parametrelerin Değişimi, Doğrusal Olmayan Sistemler.

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LIST OF SYMBOLS VE ABBREVIATIONS

$\mathbb{R}$	: Real Numbers.
$\mathbb{K}(\mathbb{R}^n)$	: Nonempty compact subsets of n dimensional real space.
$\mathbb{C}(\mathbb{R}^n)$	: Nonempty closed subsets of n dimensional real space.
$\mathbb{K}_C(\mathbb{R}^n)$	: Nonempty compact convex subsets of nth dimensional real space.
$\int$	: Integral.
$\lim$	: Limit.
$d_H(B, A)$	: Hausdorff metric of B from A.
$D(A, B)$	: Hausdorff distance between A and B.
$D_H U(t)$	: The Hukuhara derivative.
$U(t)$	: Compact and convex set-valued function that changes over time.
$S_\epsilon(A)$	: $\{x \in \mathbb{R}^n : d(x, A) < \epsilon\}$ (open ball)
$d(x, F(.))$	: $T \rightarrow \mathbb{R}$ is measurable for each $x \in \mathbb{R}^n$.
$\tilde{0}$	: The zero element of the space of $\mathbb{K}_C(\mathbb{R}^n)$, i.e., the singleton $\{0\}$.
iff	: if and only if.
ITD	: Initial Time Difference.
VEq	: Variation Equation.
VPs	: Variation of Parameters.
SDE	: Set Differential Equations.
LS	: Lipschitz Stability.
ITDLS	: Initial Time Difference Lipschitz Stability.
GTÜ	: Gebze Teknik Üniversitesi.

1. INTRODUCTION

Set-valued differential equations (SDEs) offer a powerful framework for modeling uncertainty in dynamic systems where the solution is naturally expressed as a set. These equations have found applications in various fields such as thermal analysis, signal processing, optics, and system control. Theoretical developments by researchers such as Lakshmikantham [22], [1], [23] and C. Yakar [24], [2] have significantly advanced the understanding of existence, uniqueness, and extremal solutions for SDEs in settings such as quasi-linear and Hausdorff spaces [3].

Stability plays a crucial role in the analysis of these equations. Classical stability theory, particularly the use of Lyapunov-like functions, has been extended to set-valued frameworks to evaluate the robustness of systems under perturbations [25], [26], [27]. A special focus has been given to systems with initial time differences (ITD), which require generalized comparison principles and methods such as nonlinear variation of parameters [4], [5].

This thesis investigates both classical and Lipschitz stability properties of SDEs, with a focus on systems influenced by ITD. The concept of Lipschitz stability is studied as a stronger condition that implies stability, although the reverse does not hold in general [6, 5]. By constructing specific set-valued differential systems, we demonstrate this asymmetry, and we develop related criteria using Lyapunov-like functions and ITD-based analysis.

The thesis is structured as follows: we begin with preliminary definitions and theoretical foundations, followed by classical and set-valued notions of stability. Then, we introduce ITD comparison techniques and develop a generalized variation of parameters method. Finally, Lipschitz stability is defined and analyzed within the context of set-valued systems. This study contributes to the literature by clarifying the relationship between classical and Lipschitz stability and by enhancing the theory of stability in set-valued differential systems.

2. BASIC DEFINITIONS

In this chapter, we introduce essential notions from set calculus and set-valued differential equations, which are foundational for the subsequent analysis.

We begin by discussing key concepts and preliminary results needed to explore differential systems involving sets. The Hukuhara difference is defined, and the conditions under which its corresponding derivative exists are analyzed in detail. Additionally, we consider the differentiability of set-valued functions in the context of the Bochner integral, which is central to describing integrability and constructing solutions in this setting.

The metric space $(K_C(\mathbb{R}^n), D)$, where $K_C(\mathbb{R}^n)$ stands for the family of all nonempty compact convex subsets of $\mathbb{R}^n$ endowed with the Hausdorff metric D , is also introduced. This space is known to be complete and serves as a suitable framework for investigating the behavior and stability of set-valued differential inclusions.

These foundational definitions and structures are crucial for the formulation and study of systems whose solutions are not single-valued but rather lie within set spaces. The theoretical background provided in this section builds upon classical references in the literature on differential inclusions and set-valued analysis [20, 28]. For more recent developments regarding the Hukuhara derivative, see for instance [7].

2.1. Preliminaries

Let $A, B \subset \mathbb{R}^n$ be nonempty sets, and let $\mu \in \mathbb{R}$. The Minkowski addition and scalar multiplication of these sets are defined as follows:

$$A + B = \{a + b : a \in A, b \in B\}, \quad \mu A = \{\mu a : a \in A\}.$$

A set $A \subset \mathbb{R}^n$ is said to be convex if for any $a_1, a_2 \in A$ and $\lambda \in [0, 1]$, the combination

$$a = \lambda a_1 + (1 - \lambda) a_2$$

also belongs to A .

We consider the following collections of subsets of $\mathbb{R}^n$:

$$C(\mathbb{R}^n) = \{A \subset \mathbb{R}^n : A \text{ is nonempty and closed}\},$$

$$\mathbb{K}(\mathbb{R}^n) = \{A \subset \mathbb{R}^n : A \text{ is nonempty and compact}\},$$

$$\mathbb{K}_C(\mathbb{R}^n) = \{A \in \mathbb{K}(\mathbb{R}^n) : A \text{ is convex}\}.$$

If $\alpha, \beta \in \mathbb{R}$ and $X, Y, Z \in \mathbb{K}_C(\mathbb{R}^n)$, then

$$X + \tilde{0} = \tilde{0} + X \text{ where } \tilde{0} \text{ is the zero element of } \mathbb{R}^n \text{ treated as singleton}$$

$$(X + Y) + Z = X + (Y + Z),$$

$$X + Y = Y + X,$$

$$X + Z = Y + Z \text{ implies } X = Y,$$

$$\alpha(X + Y) = \alpha X + \alpha Y,$$

$$\alpha(\beta X) = (\alpha\beta)X,$$

$$1X = X.$$

If $\alpha, \beta \geq 0$, then we have the equality $(\alpha + \beta)X = \alpha X + \beta X$.

The set $\mathbb{K}_C(\mathbb{R}^n)$, consisting of all nonempty, compact, convex subsets of $\mathbb{R}^n$, and $A, B, C \in \mathbb{K}_C(\mathbb{R}^n)$; forms a complete metric space when equipped with the Hausdorff metric D , defined as:

$$D[A, B] = \max \left\{ \sup_{x \in B} d(x, A), \sup_{y \in A} d(y, B) \right\},$$

where $d(x, A) = \inf\{\|x - y\| : y \in A\}$.

This metric satisfies the following properties for all $A, B, C \in \mathbb{K}_C(\mathbb{R}^n)$ and $\mu \in \mathbb{R}_+$:

$$D[A, B] = 0 \iff \bar{A} = \bar{B},$$

$$D[A, B] = D[B, A],$$

$$D[A + C, B + C] = D[A, B], \tag{2.1}$$

$$D[\mu A, \mu B] = \mu D[A, B],$$

$$D[A, B] \leq D[A, C] + D[C, B].$$

Given $A, B \in \mathbb{K}_C(\mathbb{R}^n)$, if there exists $C \in \mathbb{K}_C(\mathbb{R}^n)$ such that $A = B + C$, then C is called the Hukuhara difference of A and B , denoted $A - B$.

A mapping $F : \Omega \rightarrow \mathbb{K}_C(\mathbb{R}^n)$ is said to possess a Hukuhara derivative $D_H F(t)$ at $t \in \Omega$ if both limits

$$\lim_{h \rightarrow 0^+} \frac{F(t+h) - F(t)}{h}, \quad \lim_{h \rightarrow 0^+} \frac{F(t) - F(t-h)}{h}$$

exist in the metric space $(\mathbb{K}_C(\mathbb{R}^n), D)$ and are equal. In this case, $D_H F(t) \in \mathbb{K}_C(\mathbb{R}^n)$.

The derivative satisfies linearity properties:

$$D_H(F + G)(t) = D_H F(t) + D_H G(t), \quad D_H(\kappa F)(t) = \kappa D_H F(t), \quad \kappa \in \mathbb{R}.$$

If F is Hukuhara differentiable, then the function $t \mapsto \text{Diam}(F(t))$ is nondecreasing on Ω . Note that the converse is not necessarily true.

By embedding the space $\mathbb{K}_C(\mathbb{R}^n)$ as a cone in an appropriate Banach space, and employing results related to the Bochner integral, one can establish the differentiability of certain set-valued functions. Suppose a function $F : \Omega \rightarrow \mathbb{K}_C(\mathbb{R}^n)$ can be expressed as

$$F(t) = U_0 + \int_0^t \Phi(s) ds,$$

where $U_0 \in \mathbb{K}_C(\mathbb{R}^n)$ and Φ are Bochner integrable. Then F is Hukuhara differentiable almost everywhere and satisfies

$$D_H F(t) = \Phi(t), \quad \text{a.e. on } \Omega.$$

The Hukuhara integral of a set-valued function $F : \Omega \rightarrow \mathbb{K}_C(\mathbb{R}^n)$ over a compact domain $\Omega \subset \mathbb{R}_+$ is defined as

$$\int_{\Omega} F(s) ds = \left\{ \int_{\Omega} f(s) ds : f \text{ is a continuous selector of } F \right\}.$$

If F and G are integrable set-valued functions and $\lambda \in \mathbb{R}$, the following properties hold:

$$\int_{\Omega} (F + G)(s) ds = \int_{\Omega} F(s) ds + \int_{\Omega} G(s) ds,$$

$$\int_{\Omega} (\lambda F)(s) ds = \lambda \int_{\Omega} F(s) ds,$$

$$D[F(\cdot), G(\cdot)] : \Omega \rightarrow \mathbb{R}_+ \text{ is integrable,}$$

$$D \left[\int_{\Omega} F(s) ds, \int_{\Omega} G(s) ds \right] \leq \int_{\Omega} D[F(s), G(s)] ds.$$

Moreover, for any $a \leq c \leq b$, the additivity of the integral holds:

$$\int_a^b F(s) ds = \int_a^c F(s) ds + \int_c^b F(s) ds.$$

Using the Hausdorff metric, for any set $A \in \mathbb{K}_C(\mathbb{R}^n)$, we define the norm by

$$D[A, \theta] = \|A\| = \sup_{\alpha \in A} \|\alpha\|,$$

where $\tilde{0}$ denotes the singleton containing the vector in $\mathbb{R}^n$.

Let $F : \Omega \rightarrow \mathbb{K}_C(\mathbb{R}^n)$ be a continuous mapping. Then the function

$$G(t) = G(t_0) + \int_{t_0}^t F(\zeta) d\zeta$$

is Hukuhara differentiable on Ω , and we have $D_H G(t) = F(t)$ for all $t \in \Omega$.

Furthermore, if F is Hukuhara differentiable and its derivative $D_H F$ is Bochner integrable, then for $t \in \Omega$ the fundamental theorem of calculus yields:

$$F(t) - F(t_0) = \int_{t_0}^t D_H F(\zeta) d\zeta.$$

Let $F, G : [0, 1] \rightarrow \mathbb{K}_C(\mathbb{R}^n)$ be integrable. Then their pointwise distance function

$$D[F(t), G(t)] : [0, 1] \rightarrow \mathbb{R}$$

is also integrable, and the following inequality holds:

$$D\left(\int_0^1 F(t) dt, \int_0^1 G(t) dt\right) \leq \int_0^1 D[F(t), G(t)] dt. \quad (2.2)$$

These results establish that, under appropriate conditions, the integration and differentiation of set-valued functions are inverse operations in the sense of Hukuhara. [0]

3. BASIC THEORY

This chapter introduces foundational results related to comparison techniques in the context of set-valued differential equations. These tools play a key role in analyzing the behavior and stability of such systems.

Our methodology relies on the concept of the Hukuhara derivative, which provides a natural extension of classical differentiation to set-valued functions. The theoretical framework is influenced by the seminal work of Lakshmikantham and Leela on differential inequalities [23], as well as the significant developments made by C. Yakar, who extended these ideas to multivalued systems [8].

These comparison principles allow for the estimation of solutions and the investigation of their qualitative features, such as boundedness and stability. Moreover, they form the basis for more advanced results presented in the subsequent chapters of this thesis.

3.1. Comparison Principles

We consider the initial value problem (IVP) for a set-valued differential equation of the form:

$$D_H U(t) = F(t, U(t)), \quad U(t_0) = U_0 \in \mathbb{K}_C(\mathbb{R}^n), \quad t_0 \geq 0, \quad (3.1)$$

where the mapping F belongs to the space of continuous functions

$$F \in C[\mathbb{R}_+ \times \mathbb{K}_C(\mathbb{R}^n), \mathbb{K}_C(\mathbb{R}^n)],$$

and $D_H U(t)$ denotes the Hukuhara derivative of $U(t)$.

Let $J = [t_0, t_0 + \alpha]$ with $\alpha > 0$, and assume that $U \in C[J, \mathbb{K}_C(\mathbb{R}^n)]$. Then U is said to be a solution of (3.1) on the interval J if it satisfies this equation at all $t \in J$.

Since $U(t)$ is continuously differentiable in the sense of Hukuhara, we can express it as

$$U(t) = U_0 + \int_{t_0}^t D_H U(s) ds, \quad t \in J. \quad (3.2)$$

Substituting the expression of $D_H U$ from (3.1) into (3.2), we obtain the equivalent

integral formulation:

$$U(t) = U_0 + \int_{t_0}^t F(s, U(s)) ds, \quad t \in J, \quad (3.3)$$

where the integral is interpreted as the Hukuhara integral.

Thus, a function U solves the differential problem (3.1) if and only if it satisfies the integral equation (3.3) on the interval J .

Theorem 3.1.1 : Let $F \in C[J \times \mathbb{K}_C(\mathbb{R}^n), \mathbb{K}_C(\mathbb{R}^n)]$, and suppose that for all $t \in J$ and $U(t), V(t) \in \mathbb{K}_C(\mathbb{R}^n)$, the following inequality holds:

$$D[F(t, U(t)), F(t, V(t))] \leq g(t, D[U(t), V(t)]), \quad (3.4)$$

where $g \in C[J \times \mathbb{R}_+, \mathbb{R}_+]$ is nondecreasing in its second argument for each fixed $t \in J$.

Assume further that the scalar differential equation

$$w' = g(t, w), \quad w(t_0) = w_0 \geq 0, \quad (3.5)$$

has a maximal solution $r(t, t_0, w_0)$ defined on J . Then, for any pair of solutions $U(t), V(t)$ of the set differential equation (3.1), corresponding to the initial conditions $U(t_0) = U_0$, $V(t_0) = V_0$, it follows that:

$$D[U(t), V(t)] \leq r(t, t_0, w_0), \quad \text{for all } t \in J, \quad (3.6)$$

provided that $D[U_0, V_0] \leq w_0$.

Proof. Let us define $m(t) = D[U(t), V(t)]$, so that $m(t_0) = D[U_0, V_0] \leq w_0$. Using the integral representation of the solutions and the properties of the Hausdorff metric, we have:

$$\begin{aligned} m(t) &= D \left[U_0 + \int_{t_0}^t F(s, U(s)) ds, V_0 + \int_{t_0}^t F(s, V(s)) ds \right] \\ &\leq D \left[\int_{t_0}^t F(s, U(s)) ds, \int_{t_0}^t F(s, V(s)) ds \right] + D[U_0, V_0]. \end{aligned}$$

By applying the sub-additivity of the Hausdorff metric and the assumption (3.4), it follows that:

$$\begin{aligned} m(t) &\leq m(t_0) + \int_{t_0}^t D[F(s, U(s)), F(s, V(s))] ds \\ &\leq m(t_0) + \int_{t_0}^t g(s, D[U(s), V(s)]) ds \\ &= m(t_0) + \int_{t_0}^t g(s, m(s)) ds. \end{aligned}$$

Using the result of the comparison of Theorem 1.9.2 in *Lakshmikantham and Leela* [29], we conclude:

$$m(t) \leq r(t, t_0, w_0), \quad \text{for all } t \in J.$$

□

Remark 1: It is worth noting that if one utilizes the theory of differential inequalities in place of integral inequalities, the requirement that $g(t, w)$ be monotonic in w as stated in *Theorem 3.1.1* can be removed. This observation forms the basis of the subsequent comparison result.

Theorem 3.1.2 : Assume that all conditions of *Theorem 3.1.1* are satisfied, except the requirement that $g(t, w)$ be nondecreasing in w . Then the conclusion (3.6) remains valid under these relaxed assumptions.

Proof. Let $h > 0$ be sufficiently small such that the Hukuhara differences $U(t+h) - U(t)$ and $V(t+h) - V(t)$ exist for all $t \in J$. Define $m(t) = D[U(t), V(t)]$. Then:

$$m(t+h) - m(t) = D[U(t+h), V(t+h)] - D[U(t), V(t)].$$

Applying the properties of the Hausdorff metric D , particularly inequality (2.1) from [0], we estimate:

$$\begin{aligned} D[U(t+h), V(t+h)] &\leq D[U(t+h), U(t) + hF(t, U(t))] \\ &\quad + D[U(t) + hF(t, U(t)), V(t) + hF(t, V(t))] \\ &\quad + D[V(t) + hF(t, V(t)), V(t+h)]. \end{aligned}$$

Now, dividing both sides by h and using the triangle inequality:

$$\begin{aligned} \frac{m(t+h) - m(t)}{h} &\leq \frac{1}{h} D[U(t+h), U(t) + hF(t, U(t))] \\ &\quad + \frac{1}{h} D[U(t) + hF(t, U(t)), V(t) + hF(t, V(t))] \\ &\quad + \frac{1}{h} D[V(t) + hF(t, V(t)), V(t+h)]. \end{aligned}$$

Taking the upper limit as $h \rightarrow 0^+$, we obtain the right derivative estimate:

$$\begin{aligned}
D^+m(t) &\leq \limsup_{h \rightarrow 0^+} \frac{1}{h} (m(t+h) - m(t)) \\
&\leq \limsup_{h \rightarrow 0^+} D \left[\frac{U(t+h) - U(t)}{h}, F(t, U(t)) \right] \\
&\quad + \limsup_{h \rightarrow 0^+} D \left[F(t, V(t)), \frac{V(t+h) - V(t)}{h} \right] \\
&\quad + D[F(t, U(t)), F(t, V(t))].
\end{aligned}$$

To justify this, we note that:

$$\begin{aligned}
D[U(t+h), U(t) + hF(t, U(t))] &= D[U(t) + Z(t, h), U(t) + hF(t, U(t))] \\
&= D[Z(t, h), hF(t, U(t))] \\
&= D[U(t+h) - U(t), hF(t, U(t))],
\end{aligned}$$

where $Z(t, h) = U(t+h) - U(t)$.

Now, using the fact that both U and V are solutions of (3.1), and applying the result of *Theorem 1.4.1* from [29], the desired inequality:

$$D[U(t), V(t)] \leq r(t, t_0, w_0), \quad t \in J,$$

follows. □

The next comparison result provides an estimate under even weaker conditions.

Theorem 3.1.3 : Suppose that $F \in C[J \times \mathbb{K}_C(\mathbb{R}^n), \mathbb{K}_C(\mathbb{R}^n)]$, and let the inequality

$$\begin{aligned}
\limsup_{h \rightarrow 0^+} \frac{1}{h} \left[D(U(t) + hF(t, U(t)), V(t) + hF(t, V(t))) \right. \\
\left. - D(U(t), V(t)) \right] \leq g(t, D(U(t), V(t)))
\end{aligned}$$

hold for all $t \in J$, where $U, V \in \mathbb{K}_C(\mathbb{R}^n)$ and $g \in C[J \times \mathbb{R}_+, \mathbb{R}]$. Let $r(t, t_0, w_0)$ denote the maximal solution of the scalar differential equation (3.5) on J . Then the conclusion of *Theorem 3.1.2*, namely the inequality (3.6), still holds.

Proof. We proceed analogously to the argument used in *Theorem 3.1.2*.

Let $m(t) = D[U(t), V(t)]$. Then, for sufficiently small $h > 0$, we have:

$$\begin{aligned} m(t+h) - m(t) &= D[U(t+h), V(t+h)] - D[U(t), V(t)] \\ &\leq D[U(t+h), U(t) + hF(t, U(t))] \\ &\quad + D[V(t+h), V(t) + hF(t, V(t))] \\ &\quad + D[U(t) + hF(t, U(t)), V(t) + hF(t, V(t))] - D[U(t), V(t)]. \end{aligned}$$

Dividing both sides by $h > 0$ and taking the upper limit as $h \rightarrow 0^+$, we obtain:

$$D^+m(t) = \limsup_{h \rightarrow 0^+} \frac{m(t+h) - m(t)}{h}.$$

Then:

$$\begin{aligned} D^+m(t) &\leq \limsup_{h \rightarrow 0^+} \frac{1}{h} [D(U(t) + hF(t, U(t)), V(t) + hF(t, V(t))) - D(U(t), V(t))] \\ &\quad + \limsup_{h \rightarrow 0^+} D\left(\frac{U(t+h) - U(t)}{h}, F(t, U(t))\right) \\ &\quad + \limsup_{h \rightarrow 0^+} D\left(F(t, V(t)), \frac{V(t+h) - V(t)}{h}\right). \end{aligned}$$

Using the assumption and the fact that $U(t)$ and $V(t)$ satisfy the differential equation (3.1), we get:

$$D^+m(t) \leq g(t, D[U(t), V(t)]) = g(t, m(t)), \quad t \in J.$$

Applying the comparison theorem (Theorem 1.4.1) from *Lakshmikantham and Leela* [29], the result follows. $\square$

Corollary 1: Let $F \in C[J \times \mathbb{K}_C(\mathbb{R}^n), \mathbb{K}_C(\mathbb{R}^n)]$, and suppose that at least one of the following conditions is satisfied:

- (i) $D[F(t, U(t)), \tilde{0}] \leq g(t, D[U(t), \tilde{0}])$,
- (ii) $\limsup_{h \rightarrow 0^+} \frac{1}{h} [D(U(t) + hF(t, U(t)), \tilde{0}) - D(U(t), \tilde{0})] \leq g(t, D[U(t), \tilde{0}])$.

If the initial condition satisfies $D[U_0, \tilde{0}] \leq w_0$, then the following estimate holds:

$$D[U(t), \tilde{0}] \leq r(t, t_0, w_0), \quad t \in J,$$

where $r(t, t_0, w_0)$ denotes the maximal solution of the scalar differential equation (3.5) on J .

Corollary 2: Consider the function $g(t, w) = \lambda(t)w$, where $\lambda(t)$ is a nonnegative continuous function. Then, under the assumptions of *Theorem 3.1.2*, we obtain the differential inequality:

$$m(t) \leq m(t_0) + \int_{t_0}^t \lambda(s)m(s) ds, \quad t \in J.$$

By applying Gronwall's inequality, this leads to the estimate:

$$m(t) \leq m(t_0) \exp \left(\int_{t_0}^t \lambda(s) ds \right), \quad t \in J,$$

which, in terms of the Hausdorff metric, yields:

$$D[U(t), V(t)] \leq D[U_0, V_0] \exp \left(\int_{t_0}^t \lambda(s) ds \right), \quad t \in J.$$

Corollary 3: Now consider the case $g(t, w) = -\lambda(t)w$, where $\lambda(t) \geq 0$ is continuous, as in *Corollary 2*. Then, under the setting of *Theorem 3.1.3*, we obtain:

$$D[U(t), V(t)] \leq D[U_0, V_0] \exp \left(- \int_{t_0}^t \lambda(s) ds \right), \quad t \in J.$$

In particular, if $\lambda(t) = \lambda > 0$ is a constant, this simplifies to:

$$D[U(t), V(t)] \leq D[U_0, V_0] e^{-\lambda(t-t_0)}, \quad t \in J.$$

Moreover, when $J = [t_0, \infty)$, it follows that

$$\lim_{t \rightarrow \infty} D[U(t), V(t)] = 0,$$

illustrating the asymptotic behavior of solutions and emphasizing the effectiveness of *Theorem 3.1.3*.

3.2. Local Existence and Uniqueness

Theorem 3.2.1 : Let the following conditions be satisfied:

- (i) The function $F \in C(\mathbb{R}_0, \mathbb{K}_C(\mathbb{R}^n))$ satisfies the bound $D[F(t, U), \tilde{0}] \leq M_0$, where

$$\mathbb{R}_0 = J \times B(U_0, b), \quad B(U_0, b) = \{U \in \mathbb{K}_C(\mathbb{R}^n) : D[U, U_0] \leq b\}.$$

(ii) Let $g \in C(J \times [0, 2b], \mathbb{R}_+)$, with $g(t, w) \leq M_1$, $g(t, 0) \equiv 0$, and $w(t) \equiv 0$ being the only solution to

$$w' = g(t, w), \quad w(t_0) = 0, \quad t \in J. \quad (3.7)$$

Moreover, $g(t, w)$ is decreasing in w for each fixed $t \in J$.

(iii) The inequality

$$D[F(t, U(t)), F(t, V(t))] \leq g(t, D[U(t), V(t)])$$

holds for all $(t, U(t), V(t)) \in \mathbb{R}_0$.

Then the sequence defined by successive approximations

$$U_{n+1}(t) = U_0 + \int_{t_0}^t F(s, U_n(s)) ds, \quad n = 0, 1, 2, \dots \quad (3.8)$$

exists and converges uniformly on the interval $J_0 = [t_0, t_0 + \eta)$, where

$$\eta = \min \left\{ \alpha, \frac{b}{M} \right\}, \quad M = \max\{M_0, M_1\}.$$

The limit function $U(t)$ is the unique solution of the IVP (3.1) on J_0 .

Proof. Using induction and the properties of the Hausdorff metric, we first observe:

$$D[U_{n+1}(t), U_0] \leq \int_{t_0}^t D[F(s, U_n(s)), \tilde{0}] ds \leq M_0(t - t_0) \leq M_0\alpha \leq b.$$

This shows that $\{U_n(t)\}$ remains within the ball $B(U_0, b)$, and thus the sequence is well-defined on J_0 .

Now define:

$$w_0(t) = M(t - t_0), \quad w_{n+1}(t) = \int_{t_0}^t g(s, w_n(s)) ds.$$

By monotonicity and boundedness of g , the sequence $\{w_n(t)\}$ is decreasing and bounded below by 0. Furthermore, $\{w_n(t)\}$ is equicontinuous and uniformly bounded, and by Ascoli–Arzelà theorem, converges uniformly to a function $w(t)$, which satisfies (3.7). By assumption (ii), we conclude $w(t) \equiv 0$ on J_0 .

We then verify by induction that:

$$D[U_{n+1}(t), U_n(t)] \leq w_n(t), \quad \forall n \in \mathbb{N}.$$

Let $v(t) = D[U_n(t), U_m(t)]$ for $n \leq m$. Using the recursive definition and Lipschitz-type condition (iii), we obtain:

$$D^+v(t) \leq g(t, v(t)) + 2g(t, w_{n-1}(t)).$$

Letting $r_n(t)$ solve:

$$r'_n(t) = g(t, r_n(t)) + 2g(t, w_{n-1}(t)), \quad r_n(t_0) = 0,$$

and noting $r_n(t) \rightarrow 0$ uniformly since $w_{n-1}(t) \rightarrow 0$, we deduce:

$$v(t) \leq r_n(t) \rightarrow 0 \Rightarrow U_n(t) \rightarrow U(t)$$

uniformly on J_0 , and $U(t)$ satisfies (3.1).

For uniqueness, let $V(t)$ be another solution with the same initial value. Letting $m(t) = D[U(t), V(t)]$, and using assumption (iii) and Theorem 3.1.1, we conclude:

$$m(t) \leq r(t, t_0, 0) \equiv 0 \Rightarrow U(t) \equiv V(t) \quad \text{on } J_0.$$

□

Lemma 3.1: Let $F \in C[J \times \mathbb{K}_C(\mathbb{R}^n), \mathbb{K}_C(\mathbb{R}^n)]$, and define

$$G(t, r) = \max \left\{ D[F(t, U(t)), \tilde{0}] : D[U(t), U_0] \leq r \right\}.$$

Suppose that $\bar{r}(t, t_0, 0)$ is the maximal solution of the scalar initial value problem

$$w' = G(t, w), \quad w(t_0) = 0, \quad t \in J.$$

Let $U(t) = U(t, t_0, U_0)$ denote the solution of the set-valued differential equation (3.2).

Then, for all $t \in J$, the following estimate holds:

$$D[U(t), U_0] \leq \bar{r}(t, t_0, 0).$$

Proof. Let us define $m(t) = D[U(t), U_0]$. By *Corollary 1*, we know that

$$D^+m(t) \leq D[D_H U(t), \tilde{0}] = D[F(t, U(t)), \tilde{0}].$$

Since $D[U(t), U_0] = m(t)$, it follows from the definition of $G(t, r)$ that

$$D[F(t, U(t)), \tilde{0}] \leq \max \{ D[F(t, U(t)), \tilde{0}] : D[U(t), U_0] \leq m(t) \} = G(t, m(t)).$$

Therefore, we conclude:

$$D^+m(t) \leq G(t, m(t)).$$

By applying the comparison result from Theorem 1.4.1 in Lakshmikantham and Leela [29], we obtain:

$$m(t) = D[U(t), U_0] \leq \bar{r}(t, t_0, 0), \quad t \in J,$$

which completes the proof. $\square$

Theorem 3.2.2 : Suppose that conditions (a), (b), and (c) in Theorem 3.2.1 are satisfied. Additionally, assume that the solutions $w(t, t_0, w_0)$ of the scalar differential equation (3.5), defined through (t_0, w_0) , depend continuously on both parameters t_0 and w_0 . Then, the corresponding solutions $U(t) = U(t, t_0, U_0)$ of the set-valued differential equation (3.1) are continuous with respect to the initial data (t_0, U_0) .

Proof. Let $U(t) = U(t, t_0, U_0)$ and $V(t) = V(t, t_0, V_0)$ be two solutions of (3.1) with initial values U_0 and V_0 , respectively, where $U_0, V_0 \in \mathbb{K}_C(\mathbb{R}^n)$. Define $m(t) = D[U(t), V(t)]$. Then, by Theorem 3.1.1, we obtain the estimate:

$$m(t) \leq r(t, t_0, D[U_0, V_0]), \quad t \in J.$$

Since the function $r(t, t_0, w_0)$ is continuous with respect to w_0 , and $D[U_0, V_0] \rightarrow 0$ as $U_0 \rightarrow V_0$, it follows that

$$\lim_{U_0 \rightarrow V_0} m(t) \leq \lim_{U_0 \rightarrow V_0} r(t, t_0, D[U_0, V_0]) = r(t, t_0, 0) \equiv 0.$$

Hence, $U(t, t_0, U_0) \rightarrow V(t, t_0, V_0)$ uniformly on J , which proves continuity of the solution with respect to the initial set U_0 .

To establish continuity in the initial time t_0 , let $U(t) = U(t, t_0, U_0)$ and $V(t) = V(t, \tau_0, V_0)$ be two solutions of (3.1) with $\tau_0 > t_0$, and $V_0 = U(\tau_0)$, i.e., the value of the first solution at time τ_0 . Define again $m(t) = D[U(t), V(t)]$. Then:

$$m(\tau_0) = D[U(\tau_0), V(\tau_0)] = D[U(\tau_0), V_0] = D[U(\tau_0), U(\tau_0)] = 0.$$

Using Lemma 3.1, we obtain:

$$m(\tau_0) \leq \bar{r}(\tau_0, t_0, 0),$$

and applying *Theorem 3.1.1* again, we deduce:

$$m(t) \leq r^*(t), \quad t \geq \tau_0,$$

where $r^*(t) = r^*(t, \tau_0, \bar{r}(\tau_0, t_0, 0))$ is the maximal solution of (3.5) satisfying the point $(\tau_0, \bar{r}(\tau_0, t_0, 0))$.

Due to the continuity of $\bar{r}$, we know that:

$$\lim_{\tau_0 \rightarrow t_0} \bar{r}(\tau_0, t_0, 0) = 0.$$

Hence,

$$\lim_{\tau_0 \rightarrow t_0} r^*(t, \tau_0, \bar{r}(\tau_0, t_0, 0)) = r^*(t, t_0, 0) \equiv 0.$$

uniformly on J .

This implies $m(t) \rightarrow 0$ uniformly as $\tau_0 \rightarrow t_0$ which proves the continuity of $U(t, t_0, U_0)$ is continuous with respect to the initial time t_0 .

The solution $U(t, t_0, U_0)$ of (3.1) is continuous with respect to both the initial value U_0 and the initial time t_0 . The proof is complete. □

3.3. Local Existence and Extremal Solutions

Theorem 3.3.1 : Let $\{U_n(t)\}$ be a sequence of set-valued functions defined on an interval $J \subset \mathbb{R}$, and assume that the sequence is both equicontinuous and uniformly bounded. Then there exists a subsequence $\{U_{n_k}(t)\}$ that converges uniformly on J to a continuous set-valued function $U(t)$.

It follows from *Theorem 3.3.1* that a local solution exists for the initial value problem (3.1).

Please see the proof of *Theorem 3.1.1* in the appendix.

Theorem 3.3.2 : Let $F \in C[\mathbb{R}_0, \mathbb{K}_C(\mathbb{R}^n)]$, where $\mathbb{R}_0 = J \times B[U_0, b]$ and

$$B[U_0, b] = \{U \in \mathbb{K}_C(\mathbb{R}^n) : D[U(t), U_0] \leq b\}.$$

Assume that $D[F(t, U(t)), \tilde{0}] \leq M$ holds for all $(t, U(t)) \in \mathbb{R}_0$. Then, there exists at least one solution $U(t)$ to the initial value problem (3.1) defined on the interval $J_0 = [t_0, t_0 + \alpha]$, where

$$\alpha = \min \left\{ \alpha, \frac{b}{M} \right\}, \quad \alpha > 0.$$

Proof. Let $U_0 \in C^1([t_0 - \delta, t_0], \mathbb{K}_C(\mathbb{R}^n))$ for some $\delta > 0$, satisfying $U_0(t_0) = U_0$, $D[U_0(t), U_0] \leq b$, and $D[D_H U_0(t), \tilde{0}] \leq M$.

Fix $0 < \epsilon \leq \delta$ and define the sequence $\{U_\epsilon(t)\}$ by

$$U_\epsilon(t) = \begin{cases} U_0, & \text{for } t \in [t_0 - \delta, t_0], \\ U_0 + \int_{t_0}^t F(s, U_\epsilon(s - \epsilon)) ds, & \text{for } t \in [t_0, t_0 + \alpha_1]. \end{cases} \quad (3.9)$$

Letting $\alpha_1 = \min\{\alpha, \epsilon\}$, and applying the triangle inequality along with properties (2.1) and (2.2) of the Hausdorff metric from [0], we obtain:

$$\begin{aligned} D[U_\epsilon(t), U_0] &= D \left[\int_{t_0}^t F(s, U_\epsilon(s - \epsilon)) ds, \tilde{0} \right] \\ &\leq \int_{t_0}^t D[F(s, U_\epsilon(s - \epsilon)), \tilde{0}] ds \\ &\leq M(t - t_0) \leq M\alpha_1 \leq b. \end{aligned}$$

Additionally, the function $U_\epsilon(t)$ satisfies

$$D_H U_\epsilon(t) \leq F(t, U_\epsilon(t - \epsilon)) \quad \text{for } t \in [t_0, t_0 + \alpha_1], \quad (3.10)$$

which implies that

$$D[D_H U_\epsilon(t), \tilde{0}] \leq M. \quad (3.11)$$

If $\alpha_1 < \alpha$, we define $\alpha_2 = \min\{\alpha_1, 2\epsilon\}$ and use (3.9) to extend $U_\epsilon(t)$ satisfying (3.10) and (3.11) on the interval $[t_0 - \delta, t_0 + \alpha_2]$. By continuing this extension process, we construct an a.e. differentiable function $U_\epsilon(t)$ satisfying (3.9)–(3.11) on $[t_0 - \delta, t_0 + \alpha]$.

Thus, the family $\{U_\epsilon(t)\}$ is equicontinuous and uniformly bounded. By applying *Theorem 3.3.1*, there exists a sequence $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ such that $U_{\epsilon_n}(t) \rightarrow U(t)$ uniformly on J_0 . Passing to the limit term-by-term in the integral form (3.9), we obtain

$$U(t) = U_0 + \int_{t_0}^t F(s, U(s)) ds, \quad t \in J_0,$$

which implies that $U(t)$ is a solution of (3.1) on J_0 .

To study the extremal solutions of (3.1), we introduce a partial order in the metric space $(\mathbb{K}_C(\mathbb{R}^n), D)$. Let $\mathbb{K}^0 \subset \mathbb{K}_C(\mathbb{R}^n)$ denote the collection of all compact convex subsets whose elements are nonnegative (resp., positive) vectors in $\mathbb{R}^n$; that is, for any $u \in U$,

we have $u_i \geq 0$ (resp., $u_i > 0$) for all $i = 1, \dots, n$. Then $\mathbb{K}^0$ forms the nonempty interior of a cone in $\mathbb{K}_C(\mathbb{R}^n)$, which induces a partial order on this space.

□

Definition 3.3.1 : Let $U, V \in \mathbb{K}_C(\mathbb{R}^n)$. If there exists a set $Z \in \mathbb{K}_C(\mathbb{R}^n)$ such that $Z \in \mathbb{K}(\mathbb{K}^0)$ and

$$U = V + Z,$$

then we say that $U \geq V$ (respectively, $U > V$ if $Z \in \mathbb{K}^0$). Similarly, we define the relations $U \leq V$ and $U < V$ in the analogous manner.

Definition 3.3.2 : Let $R(t)$ be a solution to the set-valued differential equation (3.1) defined on the interval J_0 . The function $R(t)$ is called a *maximal solution* if it dominates all other solutions on J_0 , that is,

$$U(t) \leq R(t), \quad \text{for all } t \in J_0,$$

for every solution $U(t)$ of (3.1) defined on the same interval.

Similarly, a solution $R(t)$ is said to be *minimal solution* of (3.1) if the reverse inequality

$$R(t) \leq U(t) \quad \text{for all } t \in J_0$$

.

Definition 3.3.3 : A solution $\rho(t)$ of the set differential equation (3.1), defined on the interval J_0 , is called a *minimal solution* if it is less than or equal to all other solutions in the sense of the partial order. That is,

$$U(t) \geq \rho(t), \quad \text{for all } t \in J_0,$$

for every solution $U(t)$ of (3.1) on J_0 .

Theorem 3.3.3 : Suppose the following conditions are satisfied:

- (i) The function $F \in C[\mathbb{R}_+ \times \mathbb{K}_C(\mathbb{R}^n), \mathbb{K}_C(\mathbb{R}^n)]$ is monotone nondecreasing in its second argument. That is, for all $t \in \mathbb{R}_+$, the inequality $U \leq V$ implies $F(t, U(t)) \leq F(t, V(t))$.
- (ii) The functions $V, W \in C^1[\mathbb{R}_+, \mathbb{K}_C(\mathbb{R}^n)]$ satisfy

$$D_H V(t) < F(t, V(t)) \quad \text{and} \quad D_H W(t) \geq F(t, W(t)), \quad \forall t \in \mathbb{R}_+. \quad (3.12)$$

(iii) The initial sets are strictly ordered: $V(t_0) < W(t_0)$.

Then the strict inequality

$$V(t) < W(t), \quad \forall t \geq t_0, \quad (3.13)$$

holds throughout the interval $[t_0, \infty)$.

Proof. Define $t_1 > t_0 \geq 0$ as the supremum of all $\delta > 0$ such that $V(t) < W(t)$ holds on the interval $[t_0, \delta)$. The existence of such a t_1 follows from the continuity of $V(t)$ and $W(t)$, and the initial condition $V(t_0) < W(t_0)$.

Assume, for contradiction, that $V(t_1) \leq W(t_1)$. By the monotonicity of $F(t, U)$ in U and the assumed differential inequalities, we have:

$$D_H V(t_1) < F(t_1, V(t_1)) \leq F(t_1, W(t_1)) \leq D_H W(t_1).$$

This chain of inequalities implies $V(t_1) < W(t_1)$.

Then, due to the continuity of both functions, there exists some $\eta > 0$ such that

$$V(t) < W(t), \quad \text{for all } t \in (t_1 - \eta, t_1).$$

This contradicts the definition of t_1 as the supremum. Hence, our assumption must be false, and the strict inequality $V(t) < W(t)$ must hold for all $t \geq t_0$, completing the proof. $\square$

Remark 2: It is worth noting that the conclusion of *Theorem 3.3.3* remains valid when the inequality assumptions in (3.12) are modified. In particular, the same result follows if the inequality

$$D_H V(t) \leq F(t, V(t)) \quad \text{and} \quad D_H W(t) > F(t, W(t)), \quad t \in \mathbb{R}_+ \quad (3.14)$$

holds, respectively.

Theorem 3.3.4 : Suppose the assumptions of *Theorem 3.3.2* are satisfied, and in addition, assume that $F(t, U(t))$ is nondecreasing with respect to $U(t)$ for each $t \in J$. Then, the initial value problem (3.1) admits both maximal and minimal solutions on the interval

$$J_0 = [t_0, t_0 + \alpha_0], \quad \text{where} \quad \alpha_0 = \min \left\{ \alpha, \frac{b}{2M + b} \right\}.$$

Proof. Let $\epsilon \in \mathbb{K}(\mathbb{K}^0)$ be such that $\|\epsilon\| \leq \frac{b}{2}$. For each positive integer N , consider the perturbed initial value problem:

$$D_H U(t) = F(t, U(t)) + \frac{\epsilon}{N}, \quad U(t_0) = U_0 + \frac{\epsilon}{N}. \quad (3.15)$$

Define the modified mapping $F_N(t, U(t)) = F(t, U(t)) + \frac{\epsilon}{N}$, which is continuous on the set $R_\epsilon = J \times B[U_0, \frac{b}{2}]$, where

$$B\left[U_0, \frac{b}{2}\right] = \left\{ U \in \mathbb{K}_C(\mathbb{R}^n) : D\left(U, U_0 + \frac{\epsilon}{N}\right) \leq \frac{b}{2} \right\}.$$

Observe that $D[F_N(t, U(t)), \tilde{0}] \leq M + \frac{b}{2}$ on R_ϵ . Therefore, by applying *Theorem 3.3.2*, the problem (3.15) admits a solution $U_N(t, \epsilon) \in \mathbb{K}_C(\mathbb{R}^n)$ on J_0 .

Now, for any $0 < \epsilon_2 < \epsilon_1 \leq \epsilon$, we have

$$\begin{aligned} U_N(t_0, \epsilon_2) &< U_N(t_0, \epsilon_1), \\ D_H U_N(t, \epsilon_2) &\leq F(t, U_N(t, \epsilon_2)) + \frac{\epsilon_2}{N}, \\ D_H U_N(t, \epsilon_1) &> F(t, U_N(t, \epsilon_1)) + \frac{\epsilon_2}{N}, \quad \text{on } J_0. \end{aligned}$$

Thus, by *Theorem 3.3.3*, we conclude that

$$U_N(t, \epsilon_2) < U_N(t, \epsilon_1), \quad \text{for all } t \in J_0.$$

Since the family $\{U_N(t, \epsilon)\}$ is equicontinuous and uniformly bounded on J_0 , it follows from *Theorem 3.3.1* that there exists a decreasing sequence $\left\{\frac{\epsilon}{N_k}\right\} \rightarrow 0$ such that the corresponding solutions converge uniformly. Hence, the limit

$$R(t) = \lim_{k \rightarrow \infty} U_{N_k}(t, \epsilon) \quad (3.16)$$

exists on J_0 , and clearly $R(t_0) = U_0$. Moreover, since F is uniformly continuous and $U_{N_k}(t, \epsilon) \rightarrow R(t)$ uniformly, we have

$$U_{N_k}(t, \epsilon) = U_0 + \frac{\epsilon}{N_k} + \int_{t_0}^t F(s, U_{N_k}(s, \epsilon)) ds \quad \Rightarrow \quad R(t) = U_0 + \int_{t_0}^t F(s, R(s)) ds,$$

showing that $R(t)$ satisfies (3.1). Hence, $R(t)$ is a maximal solution.

To demonstrate minimality, consider another solution $U(t)$ with initial condition $U(t_0) = U_0$. Then,

$$\begin{aligned} U(t_0) &= U_0 < U_0 + \frac{\epsilon}{N} = U_N(t_0, \epsilon), \\ D_H U(t) &< F(t, U(t)) + \frac{\epsilon}{N}, \\ D_H U_N(t, \epsilon) &\geq F(t, U_N(t, \epsilon)) + \frac{\epsilon}{N}. \end{aligned}$$

By again invoking *Theorem 3.3.3*, we find $U(t) < U_N(t, \epsilon)$ on J_0 , and since $U_N(t, \epsilon) \rightarrow R(t)$, we conclude $U(t) \leq R(t)$, confirming maximality.

A symmetric argument, based on the perturbed equation

$$D_H U(t) = F(t, U(t)) - \frac{\epsilon}{N}, \quad U(t_0) = U_0 - \frac{\epsilon}{N},$$

establishes the existence of a minimal solution $r(t)$, completing the proof. $\square$

Theorem 3.3.5: Suppose the assumptions of *Theorem 3.3.4* hold. Let $M \in C^1[J, \mathbb{K}_C(\mathbb{R}_+^n)]$ be a function satisfying

$$D_H M(t) \leq F(t, M(t)), \quad \text{for all } t \in J. \quad (3.17)$$

If the initial condition satisfies $M(t_0) \leq U_0$, then the comparison

$$M(t) \leq R(t), \quad t \in J_0,$$

holds, where $R(t)$ denotes the maximal solution of the initial value problem (3.1) on J_0 .

Proof. Let $U_N(t, \epsilon)$ denote the approximate solution to the perturbed problem (3.15), constructed in the proof of *Theorem 3.3.4*. By the assumption on the initial condition, we observe that

$$M(t_0) \leq U_0 < U_0 + \frac{\epsilon}{N} = U_N(t_0, \epsilon).$$

Furthermore, using the nondecreasing property of $F(t, \cdot)$ and the continuity of M and U_N , we can assert that there exists a subinterval of J_0 , say $[t_0, t_0 + \delta]$, on which

$$M(t) < U_N(t, \epsilon).$$

On this subinterval, since $F(t, \cdot)$ is nondecreasing, it follows that

$$D_H M(t) \leq F(t, M(t)) < F(t, U_N(t, \epsilon)) + \frac{\epsilon}{N} = D_H U_N(t, \epsilon).$$

Hence, the assumptions of *Theorem 3.3.3* are satisfied on $[t_0, t_0 + \delta]$, and by a standard continuation argument, we can conclude that

$$M(t) < U_N(t, \epsilon), \quad t \in J_0.$$

Taking the limit as $N \rightarrow \infty$, and using the convergence

$$\lim_{N \rightarrow \infty} U_N(t, \epsilon) = R(t),$$

established in (3.16), we obtain the desired result:

$$M(t) \leq R(t), \quad t \in J_0.$$

This completes the proof. □

4. STABILITY THEORY

4.1. Lyapunov-like Functions

We consider the initial value problem (IVP) for a set-valued differential equation of the form

$$D_H U(t) = F(t, U(t)), \quad U(t_0) = U_0 \in \mathbb{K}_C(\mathbb{R}^n), \quad (4.1)$$

where the mapping F belongs to the class $C[\mathbb{R}_+ \times \mathbb{K}_C(\mathbb{R}^n), \mathbb{K}_C(\mathbb{R}^n)]$.

To analyze the qualitative behavior of solutions to (4.1), it is often useful to employ a Lyapunov-like function that enables the transformation of the set-valued system into a scalar differential inequality. This auxiliary scalar equation serves as a comparison system, and under appropriate conditions on the Lyapunov-like function, stability properties of the original set-valued system can be inferred from the behavior of this scalar counterpart.

For this purpose, we also consider the differential equation

$$w' = g(t, w), \quad w(t_0) = w_0 \geq 0, \quad (4.2)$$

where $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$. This equation defines a scalar initial value problem. When it is used in the context of bounding the solutions of another system, it is referred to as a scalar comparison equation. Consequently, analyzing the qualitative behavior of this simpler comparison equation becomes sufficient, provided that appropriate conditions are satisfied for the associated Lyapunov-like function.

Theorem 4.1.1 : Let the following conditions be satisfied:

- (i) The function $\mathcal{V} \in C[\mathbb{R}_+ \times \mathbb{K}_C(\mathbb{R}^n), \mathbb{R}_+]$ is Lipschitz continuous in its second argument. That is, for all $t \in \mathbb{R}_+$ and $U, V \in \mathbb{K}_C(\mathbb{R}^n)$, we have

$$|\mathcal{V}(t, U) - \mathcal{V}(t, V)| \leq LD[U, V],$$

where $L > 0$ is the Lipschitz constant.

- (ii) Let $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$. For all $t \in \mathbb{R}_+$ and $U \in \mathbb{K}_C(\mathbb{R}^n)$, assume the upper right Dini derivative of $\mathcal{V}$ satisfies

$$D^+ \mathcal{V}(t, U) = \limsup_{h \rightarrow 0^+} \frac{1}{h} [\mathcal{V}(t+h, U + hF(t, U)) - \mathcal{V}(t, U)] \leq g(t, \mathcal{V}(t, U)).$$

Then, for any solution $U(t) = U(t, t_0, U_0)$ of the initial value problem (4.1) on the interval $[t_0, T)$, if

$$\mathcal{V}(t_0, U_0) \leq w_0,$$

it follows that

$$\mathcal{V}(t, U(t)) \leq r(t, t_0, w_0), \quad t \in [t_0, T), \quad (4.3)$$

where $r(t, t_0, w_0)$ is the maximal solution of the comparison equation (4.2).

Proof. Let $U(t)$ be a solution of (4.1), and define the function $m(t) = \mathcal{V}(t, U(t))$. Then $m(t_0) = \mathcal{V}(t_0, U_0) \leq w_0$. For small $h > 0$, we write

$$\begin{aligned} m(t+h) - m(t) &= \mathcal{V}(t+h, U(t+h)) - \mathcal{V}(t, U(t)) \\ &= [\mathcal{V}(t+h, U(t+h)) - \mathcal{V}(t+h, U(t) + hF(t, U(t)))] \\ &\quad + [\mathcal{V}(t+h, U(t) + hF(t, U(t))) - \mathcal{V}(t, U(t))]. \end{aligned}$$

Applying the Lipschitz condition from (a), we obtain

$$m(t+h) - m(t) \leq LD[U(t+h), U(t) + hF(t, U(t))] + \mathcal{V}(t+h, U(t) + hF(t, U(t))) - \mathcal{V}(t, U(t)).$$

Dividing by h and taking the upper limit as $h \rightarrow 0^+$, we get

$$D^+m(t) \leq D^+\mathcal{V}(t, U(t)) + L \limsup_{h \rightarrow 0^+} \frac{1}{h} D[U(t+h), U(t) + hF(t, U(t))].$$

Since $U(t)$ is a solution of (4.1), we have

$$\limsup_{h \rightarrow 0^+} \frac{1}{h} D[U(t+h), U(t) + hF(t, U(t))] = D[D_H U(t), F(t, U(t))] \equiv 0.$$

Hence, the inequality reduces to

$$D^+m(t) \leq g(t, m(t)), \quad m(t_0) \leq w_0,$$

which corresponds to a differential inequality. By applying standard comparison results, namely *Theorem 1.4.1* in [9], we obtain

$$m(t) \leq r(t, t_0, w_0), \quad t \in [t_0, T),$$

as claimed. □

Corollary 4: The function $g(t, w) = 0$ is admissible in the context of *Theorem 4.1.1*, and yields the following estimate:

$$\mathcal{V}(t, U(t)) \leq \mathcal{V}(t_0, U_0), \quad t \in [t_0, T].$$

This indicates that the Lyapunov-like function $\mathcal{V}$ is non-increasing along the solution.

Corollary 5: In *Theorem 4.1.1*, if the function $g(t, w)$ is taken as $g(t, w) = -\alpha w$ with $\alpha > 0$, then the following exponential decay estimate is obtained:

$$\mathcal{V}(t, U(t)) \leq \mathcal{V}(t_0, U_0) e^{-\alpha(t-t_0)}, \quad t \in [t_0, T].$$

This establishes that the Lyapunov-like function decreases exponentially over time.

Corollary 6: Suppose in *Theorem 4.1.1* that the inequality for the Dini derivative is refined as

$$D^+\mathcal{V}(t, U(t)) + c[w(t, U(t))] \leq g(t, \mathcal{V}(t, U(t))),$$

where $w \in C[\mathbb{R}_+ \times \mathbb{K}_C(\mathbb{R}^n), \mathbb{R}_+]$, c is a class- $\mathcal{K}$ function, and $g(t, w)$ is nondecreasing in w for each fixed $t \in [t_0, T]$. Then the following integral inequality holds:

$$\mathcal{V}(t, U(t)) + \int_{t_0}^t c[w(s, U(s))] ds \leq r(t, t_0, \mathcal{V}(t_0, U_0)), \quad t \in [t_0, T],$$

where $r(t, t_0, \mathcal{V}(t_0, U_0)) = r(t, t_0, w_0)$ is the maximal solution of the scalar comparison problem (4.2).

Proof. Define the function

$$\mathcal{H}(t, U(t)) = \mathcal{V}(t, U(t)) + \int_{t_0}^t c[w(s, U(s))] ds.$$

Then, using the assumptions and monotonicity of g , we compute the upper-right Dini derivative:

$$\begin{aligned} D^+\mathcal{H}(t, U(t)) &\leq D^+\mathcal{V}(t, U(t)) + c[w(t, U(t))] \\ &\leq g(t, \mathcal{V}(t, U(t))) \leq g(t, \mathcal{H}(t, U(t))). \end{aligned}$$

By applying *Theorem 4.1.1* to the function $\mathcal{H}(t, U(t))$, the bound (4.3) follows immediately. $\square$

4.2. Global Existence of Solutions

Theorem 4.2.1 : Suppose the following assumptions hold:

(i) The function $F: \mathbb{R}_+ \times \mathbb{K}_C(\mathbb{R}^n) \rightarrow \mathbb{K}_C(\mathbb{R}^n)$ is continuous, maps bounded subsets to bounded subsets, and the IVP (4.1) admits a local solution for each initial pair (t_0, U_0) , with $U_0 \in \mathbb{K}_C(\mathbb{R}^n)$ and $t_0 \geq 0$.

(ii) There exists a function $\mathcal{V} \in C[\mathbb{R}_+ \times \mathbb{K}_C(\mathbb{R}^n), \mathbb{R}_+]$ satisfying:

1. Lipschitz condition:

$$|\mathcal{V}(t, A) - \mathcal{V}(t, B)| \leq LD[A, B],$$

for some local constant $L > 0$, uniformly in $t \in \mathbb{R}_+$;

2. Let $\mathcal{V}(t, A) \rightarrow \infty$ as $D[A, \tilde{0}] \rightarrow \infty$, uniformly on $[0, T]$ for every $T > 0$, where $t \in \mathbb{R}_+$, $A \in \mathbb{K}_C(\mathbb{R}^n)$.

3. Comparison inequality:

$$D^+\mathcal{V}(t, A) \leq g(t, \mathcal{V}(t, A)),$$

where $g \in C[\mathbb{R}_+^2, \mathbb{R}]$, and $D^+\mathcal{V}(t, A)$ is the upper-right Dini derivative.

(iii) The scalar comparison equation

$$w' = g(t, w), \quad w(t_0) = w_0 > 0,$$

admits a unique maximal solution $r(t) = r(t, t_0, w_0)$ defined for all $t \geq t_0$.

Since the initial condition is positive $w_0 > 0$, the solution remains positive for all $t \geq t_0$, with $r(t) > 0$.

Then for every $U_0 \in \mathbb{K}_C(\mathbb{R}^n)$ such that $\mathcal{V}(t_0, U_0) \leq w_0$, the IVP (4.1) possesses a global solution $U(t)$ defined on $[t_0, \infty)$ that satisfies:

$$\mathcal{V}(t, U(t)) \leq r(t), \quad \forall t \geq t_0.$$

Proof. Let S denote the collection of all solutions $U(t)$ of (4.1) defined on intervals $I_U = [t_0, c_U) \subseteq \mathbb{R}_+$, such that:

$$\mathcal{V}(t, U(t)) \leq r(t), \quad \forall t \in I_U.$$

We define a partial order $\leq$ on S by: $U \leq V$ if $I_U \subseteq I_V$ and $U(t) = V(t)$ for all $t \in I_U$.

From assumption (a), a local solution $U(t)$ exists near t_0 , and by applying *Theorem 4.1.1*, we obtain:

$$\mathcal{V}(t, U(t)) \leq r(t), \quad \text{on its domain of definition.}$$

Thus $U \in S$, and $S \neq \emptyset$.

Let $\{U_\beta\}_\beta \subseteq S$ be a totally ordered chain. Define a function $\overline{U}(t)$ on the union of their domains by:

$$\overline{U}(t) = U_\beta(t), \quad \text{for } t \in I_{U_\beta}.$$

This is well-defined and belongs to S , so by Zorn's Lemma, there exists a maximal element $Z \in (S, \leq)$ defined on some interval $[t_0, c_Z)$. Suppose, to the contrary, that $c_Z < \infty$. Since $r(t)$ is defined and finite on $[t_0, \infty)$, and $\mathcal{V}(t, Z(t)) \leq r(t)$, the condition $\mathcal{V}(t, A) \rightarrow \infty$ as $D[A, \theta] \rightarrow \infty$, uniformly in $t \in [0, T]$, implies that $D[Z(t), \tilde{0}]$ is bounded.

Thus, by the continuity of F , there exists a constant $M > 0$ such that

$$D[F(t, Z(t)), \tilde{0}] \leq M, \quad \forall t \in [t_0, c_Z).$$

This implies Z is Lipschitz continuous on $[t_0, c_Z)$, and hence extends continuously to $t = c_Z$. Denote this extension by $Z_0(t)$. One can verify that Z_0 satisfies the integral form of (4.1), so $Z_0(t)$ is a solution on $[t_0, c_Z]$.

Again, using assumption (a), we can extend Z_0 beyond c_Z , contradicting the maximality of $Z \in S$. Therefore, $c_Z = \infty$, and the maximal solution $Z(t)$ is global.

Moreover, since $\mathcal{V}(t, Z(t)) \leq r(t)$, the conclusion follows.

Proof. Let S be the collection of all functions $U(t)$ defined on intervals $I_U = [t_0, c_U) \subseteq \mathbb{R}_+$, such that $U(t)$ is a solution of (4.1) and satisfies the inequality

$$\mathcal{V}(t, U(t)) \leq r(t), \quad \text{for all } t \in I_U.$$

Define a partial ordering $\leq$ on S as follows: for any $U, V \in S$, we write $U \leq V$ if $I_U \subseteq I_V$ and $U(t) = V(t)$ on I_U .

Due to assumption (a), there exists at least one local solution $U(t)$ of (4.1) defined near t_0 . Applying *Theorem 4.1.1* to this solution, we obtain

$$\mathcal{V}(t, U(t)) \leq r(t), \quad \text{on } I_U,$$

which ensures that $U \in S$, hence $S \neq \emptyset$.

Let $\{U_\beta\}_\beta \subset S$ be a totally ordered subset. Define a function $\bar{U}(t)$ on $\bigcup_\beta I_{U_\beta}$ by setting $\bar{U}(t) = U_\beta(t)$ whenever $t \in I_{U_\beta}$. Due to the compatibility condition from the ordering, this function is well-defined, and one can verify $\bar{U} \in S$. Thus, every chain in S has an upper bound.

By Zorn's Lemma, there exists a maximal element $Z \in S$, defined on an interval $[t_0, c_Z)$. Assume for contradiction that $c_Z < \infty$.

Given that $r(t)$ is defined and finite on $[t_0, \infty)$, and $\mathcal{V}(t, Z(t)) \leq r(t)$ holds, the uniform growth condition of $\mathcal{V}$ implies that the distance $D[Z(t), \tilde{0}]$ remains bounded on $[t_0, c_Z)$. Then, due to continuity of F , there exists $M > 0$ such that

$$D[F(t, Z(t)), \tilde{0}] \leq M, \quad \forall t \in [t_0, c_Z).$$

Hence, for any $t_1, t_2 \in [t_0, c_Z)$ with $t_1 \leq t_2$, the inequality

$$D[Z(t_2), Z(t_1)] \leq \int_{t_1}^{t_2} D[F(s, Z(s)), \tilde{0}] ds \leq M(t_2 - t_1)$$

shows that Z is Lipschitz continuous, and thus admits a continuous extension $Z_0(t)$ to $[t_0, c_Z]$.

This extension satisfies the integral form of (4.1), and due to continuity, also $\mathcal{V}(t, Z_0(t)) \leq r(t)$, $\forall t \in [t_0, c_Z]$.

Now, since local existence is assumed, there exists a solution $U_0(t)$ on $[c_Z, c_Z + \delta)$ with initial value $Z_0(c_Z)$. By defining

$$Z_1(t) = \begin{cases} Z_0(t), & t \in [t_0, c_Z], \\ U_0(t), & t \in [c_Z, c_Z + \delta), \end{cases}$$

we obtain an extended solution on $[t_0, c_Z + \delta)$, still satisfying the inequality $\mathcal{V}(t, Z_1(t)) \leq r(t)$. This contradicts the maximality of Z . Hence, $c_Z = \infty$, and the maximal solution is global. $\square$

In recent years, the qualitative analysis of differential equations has attracted considerable attention, especially due to the limitations of classical deterministic models in reflecting the complexity of real-world systems. Fields such as physics, biology, and economics often deal with uncertain or imprecise data, which cannot be adequately modeled using single-valued differential equations. To overcome these challenges, researchers have developed set-valued differential equations (SVDEs), a framework in which uncertainties are embedded directly into the dynamics of the system through set-valued maps.

This approach, first systematically analyzed by Lakshmikantham and others [30], has been further developed by many researchers including Coşkun Yakar [21], [2], [10], [8]. It has opened new directions in the study of stability theory, existence and uniqueness of solutions, and Lipschitz stability under perturbations.

□

4.3. Fundamental Definitions of Stability of Set Differential Equations

Let us consider the initial value problem for a set-valued differential equation given by

$$D_H U(t) = f(t, U(t)), \quad U(t_0) = U_0, \quad U_0 \in \mathbb{K}_C(\mathbb{R}^n), \quad t \geq t_0 \geq 0. \quad (4.4)$$

We also consider the same system starting at a different initial time:

$$D_H U(t) = f(t, U(t)), \quad U(\tau_0) = V_0, \quad V_0 \in \mathbb{K}_C(\mathbb{R}^n), \quad t \geq \tau_0 \geq t_0. \quad (4.5)$$

The perturbed system of (4.5) is given by

$$D_H V(t) = F(t, V(t)), \quad V(\tau_0) = V_0, \quad V_0 \in \mathbb{K}_C(\mathbb{R}^n), \quad t \geq \tau_0, \quad (4.6)$$

and the associated difference system is

$$D_H W(t) = H(t, W(t)), \quad W(\tau_0) = W_0 = V_0 - U_0, \quad W_0 \in \mathbb{K}_C(\mathbb{R}^n), \quad t \geq \tau_0. \quad (4.7)$$

Here, the functions f, F, H are assumed to be continuous, that is,

$$f, F, H \in C[\mathbb{R}_+ \times S_\rho, \mathbb{K}_C(\mathbb{R}^n)],$$

where

$$S_\rho = \left\{ U \in \mathbb{K}_C(\mathbb{R}^n) : D[U, \tilde{0}] < \rho < \infty \right\}.$$

Moreover, it is assumed that $f(t, \tilde{0}) \equiv \tilde{0}$ for all $t \geq 0$.

The system defined by (4.6) becomes a perturbed version of (4.5) when the right-hand side satisfies

$$F(t, V(t)) = f(t, V(t)) + R(t, V(t)),$$

where $R(t, V(t))$ is a perturbation term. Under these assumptions, trivial solutions exist for both (4.4) and (4.5) through the respective initial conditions (t_0, U_0) and (τ_0, V_0) [2], [21], [11]

Let $J_1 = [t_0, t_0 + T_1]$ and $J_2 = [\tau_0, \tau_0 + T_2]$. A function $U \in C^{0,1}[J_1, \mathbb{K}_C(\mathbb{R}^n)]$ is said to be a solution of (4.4) on J_1 if it satisfies the equation for all $t \in J_1$. Similarly, the functions $U, V, W \in C^{0,1}[J_2, \mathbb{K}_C(\mathbb{R}^n)]$ are considered solutions of (4.5), (4.6), and (4.7), respectively, on J_2 if they satisfy the corresponding equations for all $t \in J_2$.

Given that the mappings $U(t), V(t), W(t)$ are continuously Hukuhara differentiable, they admit the integral representations:

$$U(t) = U_0 + \int_{t_0}^t D_H U(s) ds, \quad t \in J_1, \quad (4.8)$$

$$U(t) = U_0 + \int_{\tau_0}^t D_H U(s) ds, \quad t \in J_2, \quad (4.9)$$

$$V(t) = V_0 + \int_{\tau_0}^t D_H V(s) ds, \quad t \in J_2, \quad (4.10)$$

$$W(t) = W_0 + \int_{\tau_0}^t D_H W(s) ds, \quad t \in J_2. \quad (4.11)$$

Accordingly, the equivalent integral formulations of the initial value problems (4.4)–(4.7) are given as follows:

$$U(t) = U_0 + \int_{t_0}^t f(s, U(s)) ds, \quad t \in J_1, \quad (4.12)$$

$$U(t) = U_0 + \int_{\tau_0}^t f(s, U(s)) ds, \quad t \in J_2, \quad (4.13)$$

$$V(t) = V_0 + \int_{\tau_0}^t F(s, V(s)) ds, \quad t \in J_2, \quad (4.14)$$

$$W(t) = W_0 + \int_{\tau_0}^t H(s, W(s)) ds, \quad t \in J_2. \quad (4.15)$$

Each of the above integrals is taken in the Hukuhara sense. The functions $U(t), V(t), W(t) \in \mathbb{K}_C(\mathbb{R}^n)$ are solutions of the respective differential equations (4.4)–(4.7) if and only if they satisfy the corresponding integral equations (4.12)–(4.15) [11, 12, 2].

Definition 4.3.1 : The solution $V(t, \tau_0, V_0)$ of the system (4.6) through (τ_0, V_0) for $t \geq \tau_0, \tau_0 \in \mathbb{R}_+$ is said to be the initial time difference.

(S_1) *stable* with respect to the solution $\bar{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, where $U(t, t_0, U_0)$ is any solution of system (4.4) for $t \geq \tau_0 \geq t_0 \geq 0, t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0$ if and only if given any $\epsilon > 0$ there exist $\delta_1 = \delta_1(\epsilon, \tau_0) > 0$ and $\delta_2 = \delta_2(\epsilon, \tau_0) > 0$ such that

$$D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] < \epsilon$$

provided that $D[V_0 - U_0, \tilde{0}] < \delta_1$ and $d_s[\tau_0 - t_0, 0] < \delta_2$ for $t \geq \tau_0$;

(S_2) *uniformly stable* with respect to the solution $U(t - \eta, t_0, U_0)$ if δ_1, δ_2 in (S_1) are independent of $\tau_0 \in \mathbb{R}_+$;

(S_3) *attractive* with respect to the solution $\bar{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, where $U(t, t_0, U_0)$ is any solution of system (4.4) for $t \geq \tau_0 \geq t_0 \geq 0, t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0$ if and only if given any $\epsilon > 0$ and $\tau_0 \in \mathbb{R}_+$ there exist $\delta_1 = \delta_1(\tau_0) > 0, \delta_2 = \delta_2(\tau_0) > 0$ and a $T = T(\epsilon, \tau_0) > 0$ such that

$$D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] < \epsilon \text{ provided that}$$

$$D[V_0 - U_0, \tilde{0}] < \delta_1 \text{ and } d_s[\tau_0 - t_0, 0] < \delta_2 \text{ for } t \geq \tau_0 + T;$$

(S_4) *uniformly attractive* with respect to the solution $\bar{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, where $U(t, t_0, U_0)$ is any solution of the system (4.4) for $t \geq \tau_0 \geq t_0 \geq 0, t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0$ if $\delta_1 > 0, \delta_2 > 0$ and a $T > 0$ in (S_3) are independent of $\tau_0 \in \mathbb{R}_+$;

(S_5) *asymptotically stable* with respect to the solution $\bar{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$ if (S_1) and (S_3) hold simultaneously. Or, if (S_1) holds and there exist $\gamma_1(\tau_0) > 0$ and $\gamma_2(\tau_0) > 0$ such that

$$\lim_{t \rightarrow \infty} D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] = 0$$

for all $V(t, \tau_0, V_0)$ and $U(t - \eta, t_0, U_0)$ with $D[V_0 - U_0, \tilde{0}] < \gamma_1$ and $d_s[\tau_0 - t_0] < \gamma_2$ for each $t \geq \tau_0$;

(S_6) *uniformly asymptotically stable* with respect to the solution

$\bar{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$ if (S_2) and (S_4) hold simultaneously. Or, if (S_2) holds, then for any $\epsilon > 0$, there exist positive constants $\gamma_1 = \gamma_1(\epsilon)$, $\gamma_2 = \gamma_2(\epsilon)$, and $T(\epsilon)$ such that

$$D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] < \epsilon$$

provided that $D[V_0 - U_0, \tilde{0}] < \gamma_1$ and $ds[\tau_0 - t_0, 0] < \gamma_2$ for each $t \geq \tau_0 + T(\epsilon)$;

(S_7) *exponentially asymptotically stable* with respect to the solution

$\bar{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$ if there exists an estimate such that

$$D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] \leq D[V_0 - U_0, \tilde{0}] \exp[-\alpha(t - \tau_0)]$$

for $\alpha > 0$ and all $t \geq t_0$, as discussed in [13], [14], [15], [2].

Definition 4.3.2 : A function $\phi \in C[[0, \rho), \mathbb{R}_+]$ is said to belong to the class $\mathcal{K}$ if it satisfies $\phi(0) = 0$ and is strictly increasing in r . If, in addition, $\rho = \infty$ and $\phi(r) \rightarrow \infty$ as $r \rightarrow \infty$, then ϕ belongs to the class $\mathcal{K}_\infty$. Furthermore, a function $a \in C[\mathbb{R}_+^2, \mathbb{R}_+]$ is said to belong to the class $\mathcal{CK}$ if for each $t \in \mathbb{R}_+$, the mapping $a(t, \cdot) \in \mathcal{K}$ [2].

Definition 4.3.3 : For a real-valued function $\mathcal{V} \in C[\mathbb{R}_+ \times \mathbb{K}(\mathbb{R}^n), \mathbb{R}_+]$

$$D^+\mathcal{V}(t, U) \equiv \limsup_{h \rightarrow 0^+} \frac{1}{h} [\mathcal{V}(t + h, U + hf(t, U)) - \mathcal{V}(t, U)]$$

$$D^-\mathcal{V}(t, U) \equiv \liminf_{h \rightarrow 0^-} \frac{1}{h} [\mathcal{V}(t + h, U + hf(t, U)) - \mathcal{V}(t, U)]$$

for $(t, U) \in \mathbb{R}_+ \times \mathbb{K}_C(\mathbb{R}^n)$ [2].

Definition 4.3.4 : For a real-valued function $\mathcal{V} \in C[\mathbb{R}_+ \times \mathbb{K}_C(\mathbb{R}^n), \mathbb{R}_+]$ we define the generalized derivatives as follows

$$D_*^+\mathcal{V}(t, V - \bar{U}) = \limsup_{h \rightarrow 0^+} \frac{1}{h} [\mathcal{V}(t + h, V - \bar{U} + h(F(t, V) - \bar{f}(t, \bar{U}))) - \mathcal{V}(t, V - \bar{U})]$$

$$D_*^-\mathcal{V}(t, V - \bar{U}) = \liminf_{h \rightarrow 0^-} \frac{1}{h} [\mathcal{V}(t + h, V - \bar{U} + h(F(t, V) - \bar{f}(t, \bar{U}))) - \mathcal{V}(t, V - \bar{U})]$$

for $(t, U) \in \mathbb{R}_+ \times \mathbb{K}_C(\mathbb{R}^n)$ where $V(t, \tau_0, V_0)$ is the solution of the system (4.6) and $\bar{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, where $U(t, t_0, U_0)$ is any solution of the system (4.16) for $t \geq \tau_0 \geq t_0 \geq 0$, $t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0$ [2].

4.4. Classical Notion of Stability of Set Differential Equation

Let $V(t, \tau_0, V_0)$ be a solution of the set differential equation

$$D_H V(t) = f(t, V(t)), \quad V(\tau_0) = V_0, \quad (4.16)$$

where $V_0 \in \mathbb{K}_C(\mathbb{R}^n)$, $t \geq \tau_0$, and $\tau_0 \in \mathbb{R}_+$. The function f is assumed to be continuous, $f \in C[\mathbb{R}_+ \times S_\rho, \mathbb{K}_C(\mathbb{R}^n)]$, where the set S_ρ is defined by

$$S_\rho = \left\{ V \in \mathbb{K}_C(\mathbb{R}^n) : D[V, \tilde{0}] < \rho < \infty \right\}.$$

Suppose further that $f(t, \tilde{0}) = \tilde{0}$ for all $t \in \mathbb{R}_+$. Then the null solution $V = 0$ satisfies the equation

$$D_H V(t) = f(t, V(t)), \quad V(\tau_0) = V_0$$

with initial data $V_0 \in \mathbb{K}_C(\mathbb{R}^n)$ and $t \geq \tau_0$, $\tau_0 \in \mathbb{R}_+$, through the initial point $(t_0, 0)$. [9], [2]

Definition 4.4.1 : The null solution $V = 0$ of the system (4.16) is called *stable* if, for every $\epsilon > 0$ and each $\tau_0 \in \mathbb{R}_+$, there exist two positive functions $\delta_1 = \delta_1(\epsilon, \tau_0)$ and $\delta_2 = \delta_2(\epsilon, \tau_0)$, both continuous in τ_0 for any fixed ϵ , such that the following condition is satisfied:

$$D[V(t, \tau_0, V_0), \tilde{0}] < \epsilon \quad \text{whenever} \quad D[V_0, \tilde{0}] < \delta_1 \quad \text{and} \quad d_s[\tau_0, 0] < \delta_2 \quad \text{for all } t \geq \tau_0. \quad (4.17)$$

Moreover, if δ_1 and δ_2 can be chosen independently of $\tau_0 \in \mathbb{R}_+$, then the null solution $V = 0$ is said to be *uniformly stable* [2].

Definition 4.4.2 : Let $V(t, \tau_0, V_0)$ and $V(t, \tau_0, U_0)$ be two solutions of the system (4.16), defined for $t \geq \tau_0 \in \mathbb{R}_+$. The solution $V(t, \tau_0, V_0)$ is said to be *stable with respect to* $V(t, \tau_0, U_0)$ if, for every $\epsilon > 0$, there exist continuous functions $\delta_1 = \delta_1(\epsilon, \tau_0) > 0$ and $\delta_2 = \delta_2(\epsilon, \tau_0) > 0$ such that

$$D[V(t, \tau_0, V_0) - V(t, \tau_0, U_0), \tilde{0}] < \epsilon \quad \text{whenever} \quad D[V_0 - U_0, \tilde{0}] < \delta_1 \quad \text{and} \quad d_s[\tau_0, 0] < \delta_2, \quad (4.18)$$

for all $t \geq \tau_0$. If these bounds δ_1 and δ_2 are independent of the initial time τ_0 , then $V(t, \tau_0, V_0)$ is said to be *uniformly stable with respect to* $V(t, \tau_0, U_0)$.

A classical technique in the analysis of stability is to examine the difference between two solutions. Define the function

$$W(t, \tau_0, V_0 - U_0) = V(t, \tau_0, V_0) - V(t, \tau_0, U_0),$$

which satisfies the set differential equation

$$D_H W(t, \tau_0, V_0 - U_0) = f(t, W(t, \tau_0, V_0 - U_0) + V(t, \tau_0, U_0)) - f(t, V(t, \tau_0, U_0)). \quad (4.19)$$

The expression on the right-hand side can be regarded as a new function $\tilde{f}$ defined by

$$\tilde{f}(t, W) = f(t, W(t, \tau_0, V_0 - U_0) + V(t, \tau_0, U_0)) - f(t, V(t, \tau_0, U_0)),$$

so that equation (4.19) takes the form

$$D_H W(t, \tau_0, V_0 - U_0) = \tilde{f}(t, W(t, \tau_0, V_0 - U_0)).$$

If $V_0 = U_0$, then clearly $W(t, \tau_0, \tilde{0}) \equiv \tilde{0}$ and under the assumption $f(t, \tilde{0}) = \tilde{0}$, the function $W(t)$ satisfies the null equation. In this case, the solution $V(t, \tau_0, U_0)$ of (4.16) corresponds to the null solution of the auxiliary equation governed by $\tilde{f}$. Thus, the problem of stability analysis can be reduced to the examination of the null solution of the variation equation (4.19), as discussed in [2].

Nevertheless, such a transformation cannot be applied in the case of initial time difference (ITD) stability due to the explicit dependence on the initial time, which prevents this simplification. For further details on this method, see [2].

4.5. A Reformulated Perspective on (ITD) Stability for Set Differential Equations

Let $U(t, \tau_0, V_0)$ be a solution of (4.17), and define

$$\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0),$$

where $U(t, t_0, U_0)$ is a solution of (4.16) for $t \geq \tau_0 \geq 0$. We now introduce a transformation similar to (4.19) to examine the behavior under initial time difference (ITD) [13, 14].

Define the difference between the two solutions as

$$W(t, \tau_0, V_0 - U_0) = U(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \quad t \geq \tau_0.$$

Differentiating both sides using the Hukuhara derivative, we obtain

$$D_H W(t, \tau_0, V_0 - U_0) = D_H U(t, \tau_0, V_0) - D_H U(t - \eta, t_0, U_0) = \tilde{f}(\eta; t, W(t, \tau_0, V_0 - U_0)),$$

where $\tilde{f}$ reflects the effect of the shift $\eta = \tau_0 - t_0$ and system nonlinearity [15, 12].

Notably, even if $V_0 = U_0$, the function $W(t, \tau_0, \tilde{0})$ is not identically zero,

$$W(t, \tau_0, \tilde{0}) \not\equiv \tilde{0}.$$

Therefore, the trajectory $\tilde{U}(t, \tau_0, U_0)$ cannot be regarded as the null solution of the transformed system.

Classical stability methods focusing on the null solution are not sufficient for ITD stability. Thus, one must use alternative techniques like comparison results or Lyapunov-type functions specific to ITD [2, 10].

4.6. Theorems on Initial Time Difference Comparisons

Theorem 4.6.1 : Let $f, F \in C(J \times \mathbb{K}_C(\mathbb{R}^n), \mathbb{K}_C(\mathbb{R}^n))$ and suppose the following conditions hold:

(i) For all $t \in J$ and $\tilde{U}, V \in \mathbb{K}_C(\mathbb{R}^n)$, we have

$$\limsup_{h \rightarrow 0^+} \frac{1}{h} \left[D(V - \tilde{U} + h(F(t, V) - \tilde{f}(t, \tilde{U})), \tilde{0}) - D(V - \tilde{U}, \tilde{0}) \right] \leq G(t, D(V - \tilde{U}, \tilde{0})).$$

(ii) Let $r(t) = r(t, \tau_0, z_0)$ be the maximal solution of the scalar initial value problem or comparison equations

$$z' = G(t, z), \quad z(\tau_0) = z_0 \geq 0, \quad t \in J. \quad (4.20)$$

Then, if $U(t)$ and $V(t)$ are solutions of the initial value problems

$$D_H U(t) = f(t, U(t)), \quad U(t_0) = U_0, \quad t \geq t_0 \geq 0,$$

and

$$D_H V(t) = F(t, V(t)), \quad V(\tau_0) = V_0, \quad t \geq \tau_0 \geq t_0,$$

then the following estimate holds:

$$D \left(V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0} \right) \leq r(t, \tau_0, z_0), \quad t \in J,$$

provided that $D(V_0 - U_0, \tilde{0}) \leq z_0$, where $\eta = \tau_0 - t_0$.

Proof. The proof follows by applying a comparison result for scalar differential inequalities in the Hausdorff metric setting. Condition (i) ensures that the distance between the two set-valued solutions, when perturbed by the respective right-hand side functions f and F , is controlled by the scalar function G . This allows us to apply a scalar comparison result as in [14] and [10].

Furthermore, due to the difference in initial times t_0 and τ_0 , we use a formulation that takes into account an initial time difference (ITD), as established in [16]. By constructing the scalar comparison solution $r(t)$ to the auxiliary problem (4.20), and using the continuity and monotonicity properties of the Hausdorff metric, we obtain the desired upper bound:

$$D \left(V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0} \right) \leq r(t, \tau_0, z_0),$$

for all $t \in J$, provided that $D(V_0 - U_0, \tilde{0}) \leq z_0$, where $\eta = \tau_0 - t_0$.

We define the scalar function

$$m(t) = D \left(V(t) - \tilde{U}(t), \tilde{0} \right),$$

and aim to estimate its upper right-hand Dini derivative $D^+ m(t)$. For a small increment $h > 0$, the difference $m(t + h) - m(t)$ can be expressed as

$$m(t + h) - m(t) = D \left(V(t + h) - \tilde{U}(t + h), \tilde{0} \right) - D \left(V(t) - \tilde{U}(t), \tilde{0} \right).$$

Using the triangle inequality and properties of the Hausdorff metric, we estimate

$$\begin{aligned} m(t + h) - m(t) &\leq D(V(t + h) - \tilde{U}(t + h), V(t) - \tilde{U}(t)) \\ &\quad + h(F(t, V(t)) - f(t, \tilde{U}(t))) \\ &\quad + D(h(F(t, V(t)) - f(t, \tilde{U}(t))), h(F(t, \tilde{U}(t)) - f(t, \tilde{U}(t)))) \\ &\quad - D(V(t) - \tilde{U}(t), \tilde{0}). \end{aligned}$$

Dividing both sides by h and taking the limit superior as $h \rightarrow 0^+$, we obtain:

$$\begin{aligned}
D^+m(t) &= \limsup_{h \rightarrow 0^+} \frac{1}{h} [m(t+h) - m(t)] \\
&\leq \limsup_{h \rightarrow 0^+} \frac{1}{h} \left(D \left(V(t) - \tilde{U}(t) + h(F(t, V(t)) - f(t, \tilde{U}(t))), \tilde{0} \right) \right. \\
&\quad \left. - D \left(V(t) - \tilde{U}(t), \tilde{0} \right) \right) \\
&\quad + \limsup_{h \rightarrow 0^+} D \left(\frac{V(t+h) - \tilde{U}(t+h)}{h} - \frac{V(t) - \tilde{U}(t)}{h}, \right. \\
&\quad \left. F(t, V(t)) - f(t, \tilde{U}(t)) \right).
\end{aligned}$$

By the continuity of the involved mappings and the structure of the right-hand sides, we conclude that

$$D^+m(t) \leq G(t, D(V(t, t_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0})) = G(t, m(t)), \quad t \in I.$$

This proof is based on the approach originally developed by Coşkun Yakar and M.D. Shaw in [2], with reformulated expressions tailored to the present context. $\square$

Corollary 7: Suppose $f, F \in C[J \times \mathbb{K}_C(\mathbb{R}^n), \mathbb{K}_C(\mathbb{R}^n)]$, and one of the following holds:

(i)

$$D(F(t, V) - \tilde{f}(t, \tilde{U}), \tilde{0}) \leq G(t, D(V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), V_0 - U_0));$$

(ii)

$$\begin{aligned}
\limsup_{h \rightarrow 0^+} \frac{1}{h} \left[D(V - \tilde{U} + h(F(t, V) - \tilde{f}(t, \tilde{U})), \tilde{0}) \right. \\
\left. - D(V - \tilde{U}, \tilde{0}) \right] \leq G(t, D(V - \tilde{U}, \tilde{0}))
\end{aligned}$$

where $G \in C[J \times \mathbb{R}_+, \mathbb{R}]$, and $\tilde{f}$ is defined by

$$\tilde{f}(t, \tilde{U}) = f(t, \tilde{U} + U(t - \eta, t_0, U_0)) - f(t, U(t - \eta, t_0, U_0)).$$

Then, under the condition $D(V_0 - U_0, \tilde{0}) \leq z_0$, the estimate

$$D(V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}) \leq r(t, \tau_0, z_0), \quad t \in J$$

holds, where $r = r(t, \tau_0, z_0)$ is the maximal solution of the scalar comparison equation (4.20). Please see the proof of the *Corollary 7* in [2].

4.7. Stability Criteria of the Set Differential Equations

Theorem 4.7.1 : Assume the following conditions hold:

- (i) The set-valued functions $f, F \in C[\mathbb{R}_+ \times S(\rho), \mathbb{K}_C(\mathbb{R}^n)]$, and $f(t, \tilde{0}) \equiv \tilde{0}$ for all $t \geq 0$.

Furthermore, for all $h > 0$, $t \in \mathbb{R}_+$, and $U, V \in S(\rho)$, the following inequality is satisfied:

$$\limsup_{h \rightarrow 0^+} \frac{1}{h} \left[D \left(V - \tilde{U} + h(F(t, V) - \tilde{f}(t, \tilde{U})), \tilde{0} \right) - D(V - \tilde{U}, \tilde{0}) \right] \leq G(t, D(V - \tilde{U}, \tilde{0})),$$

where $\tilde{f}(t, \tilde{U}) = f(t, \tilde{U} + U(t - \eta, t_0, U_0)) - f(t, U(t - \eta, t_0, U_0))$.

- (i) The function $G \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$ satisfies the condition:

$$G(t, 0) \equiv 0, \quad \forall t \in \mathbb{R}_+.$$

Then, the stability properties -such as classical stability, asymptotic stability, and uniform stability- of the null solution $z(t) = 0$ of the scalar differential equation

$$z' = G(t, z), \quad z(\tau_0) = 0,$$

imply the corresponding stability properties, interpreted in the sense of initial time difference of any solution $V(t, \tau_0, V_0)$ of the set differential equation

$$D_H V(t) = F(t, V(t)), \quad V(\tau_0) = V_0,$$

with respect to the function

$$\bar{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0),$$

where $U(t, t_0, U_0)$ is any solution of the set differential equation

$$D_H U(t) = f(t, U(t)), \quad U(t_0) = U_0.$$

Proof. The result can be established by a direct application of the comparison principle based on the generalized upper Dini derivative of the inequality presented in case (i) of the theorem. The inequality stated in part (i) guarantees that the difference between

the corresponding set-valued solutions can be bounded above by a scalar comparison function. The assumption $G(t, 0) = 0$ ensures the null solution is a valid baseline for comparison. This approach aligns with the techniques developed in [13, 14, 10], particularly for set-valued differential systems under time perturbations.

For completeness, a detailed version of the proof is also provided below. $\square$

Let $V(t, \tau_0, V_0)$ be a solution of the set differential equation (4.18), and suppose that it is stable with respect to the shifted solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, where $U(t, t_0, U_0)$ solves equation (4.16). Here, $\eta = \tau_0 - t_0$ denotes the initial time difference. To establish the result, it is sufficient to show that the null solution of the scalar comparison equation (4.20) is stable.

Assume that for any $\epsilon > 0$ and fixed $\tau_0 \in \mathbb{R}_+$, there exists a constant $\delta = \delta(\tau_0, \epsilon) > 0$ such that

$$z(t, \tau_0, z_0) < \epsilon \quad \text{whenever} \quad 0 \leq z_0 < \delta, \quad \text{for all } t \geq \tau_0,$$

where $z(t, \tau_0, z_0)$ satisfies the scalar initial value problem (4.20).

Suppose, contrary to the claim, that $V(t, \tau_0, V_0)$ is not stable with respect to $\tilde{U}(t, \tau_0, U_0)$. Then there exist initial data V_0, U_0 , and a time $t_1 > \tau_0$ such that the Hausdorff distance between $V(t, \tau_0, V_0)$ and $\tilde{U}(t, \tau_0, U_0)$ reaches ϵ at $t = t_1$, that is,

$$D[V(t_1, \tau_0, V_0) - U(t_1 - \eta, t_0, U_0), \tilde{0}] = \epsilon, \quad \text{while} \quad D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] \leq \epsilon < \rho.$$

for all $t \in [\tau_0, t_1]$. Assume further that the initial difference satisfies

$$D[V_0 - U_0, \tilde{0}] < \delta_1, \quad d_s[\tau_0 - t_0, 0] < \delta_2,$$

for suitable $\delta_1, \delta_2 > 0$. Let us define $\delta = \min\{\delta_1, \delta_2\}$.

Using the additive structure of the differential, we can define

$$W(t, t_0, V_0 - U_0) = V(t, \tau_0, V_0) - V(t, \tau_0, U_0),$$

so that

$$D_H W(t, t_0, V_0 - U_0) = D_H V(t, \tau_0, V_0) - D_H V(t, \tau_0, U_0).$$

Now, by applying *Corollary 7* and the scalar inequality given in (4.18), we conclude that

$$D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] \leq r(t, \tau_0, d_s[z_0, 0]) < \epsilon, \quad \text{for } t \in [\tau_0, t_1],$$

which contradicts the earlier assertion. Hence, the null solution of (4.18) is indeed stable.

The special cases of Theorem 4.7.1 are presented as Lemma A.2 and Lemma A.3 in Appendix A. The detailed proofs of these results are not included here; for the complete arguments, please refer to [10].

Theorem 4.7.2 : Assume that there exists a Lyapunov-like function $\mathcal{V}(t, U) \in C[\mathbb{R}_+ \times \mathbb{K}_C(\mathbb{R}^n), \mathbb{R}_+]$ satisfying the following conditions:

(i) $\mathcal{V}(t, U)$ is positive definite and decrescent, and satisfies the Lipschitz condition

$$|\mathcal{V}(t, U) - \mathcal{V}(t, V)| \leq LD[U, V], \quad \forall (t, U), (t, V) \in \mathbb{R}_+ \times S_\rho$$

for some constant $L > 0$.

Furthermore, the upper Dini derivative along the solutions $V(t)$ and $\tilde{U}(t) = U(t - \eta, t_0, U_0)$ satisfies

$$D_*^+ \mathcal{V}(t, V(t) - \tilde{U}(t)) \leq G(t, \mathcal{V}(t, V(t) - \tilde{U}(t))),$$

for some continuous function $G \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$.

Additionally, the function $\mathcal{V}$ is bounded below and above by $\mathcal{K}$ -class functions a and b , respectively:

$$a(D[U(t), U_0]) \leq \mathcal{V}(t, U(t)) \leq b(D[U(t), U_0]).$$

(ii) Let $r(t) = r(t, \tau_0, z_0)$ be the maximal solution of the scalar comparison differential equation

$$z' = G(t, z), \quad z(\tau_0) = z_0, \quad t \geq \tau_0, \quad (4.21)$$

and suppose that $G(t, 0) = 0$.

Then, the stability properties of the null solution of (4.20) are inherited by the solution $V(t, \tau_0, V_0)$ of the perturbed system (4.6) with respect to the shifted reference solution

$$\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0),$$

where $U(t, t_0, U_0)$ is any solution of the unperturbed system (4.4); please see the details of the proof of the theorem in [2], [13], [15], [17].

5. NONLINEAR VARIATION OF PARAMETERS WITH INITIAL TIME DIFFERENCE AND A COMPARISON RESULT IN TERMS OF LYAPUNOV-LIKE FUNCTIONS

5.1. Definition

We begin by examining a class of set differential systems with initial time differences. Consider the following initial value problems defined on,

$$D_H U(t) = f(t, U(t)), \quad U(t_0) = U_0, \quad U_0 \in \mathbb{K}_C(\mathbb{R}^n), \quad t \geq t_0 \geq 0, \quad (5.1)$$

$$D_H U(t) = f(t, U(t)), \quad U(\tau_0) = V_0, \quad V_0 \in \mathbb{K}_C(\mathbb{R}^n), \quad t \geq \tau_0 \geq t_0, \quad (5.2)$$

and its perturbed version:

$$D_H V(t) = F(t, V(t)), \quad V(\tau_0) = V_0, \quad V_0 \in \mathbb{K}_C(\mathbb{R}^n), \quad t \geq \tau_0. \quad (5.3)$$

Here, f and F are continuous mappings from $\mathbb{R}_+ \times \mathbb{R}^n$ into $\mathbb{K}_C(\mathbb{R}^n)$. Equation (5.3) is interpreted as a perturbation of (5.2) in cases where $F(t, V(t)) = f(t, V(t)) + R(t, V(t))$, with $R(t, V(t))$ representing the perturbation term that may model uncertainties or external influences.

This formulation establishes a framework for investigating the influence of initial time differences and perturbations on the qualitative behavior of solutions. The results developed in the subsequent sections utilize comparison techniques and Lyapunov-like functions to characterize the stability and variation of such systems [13], [15], [2].

Theorem 5.1.1 : Let the function f be continuously differentiable with respect to its second argument, that is, $f \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{K}_C(\mathbb{R}^n))$. Suppose $U(t, t_0, U_0)$ is the unique solution of system (5.1) for $t \geq t_0$. We define,

$$H(t, t_0, U_0) = \frac{\partial f(t, U(t, t_0, U_0))}{\partial U}.$$

Then the following properties hold:

- (i) The partial derivative $\Phi(t, t_0, U_0) = \frac{\partial U(t, t_0, U_0)}{\partial U_0}$ exists and satisfies the variation equation

$$Z' = H(t, t_0, U_0)Z, \quad (5.4)$$

together with the initial condition $\Phi(t_0, t_0, U_0) = \mathcal{I}_n$.

- (ii) The derivative of the solution with respect to the initial time, $\frac{\partial U(t, t_0, U_0)}{\partial t_0}$, also exists and satisfies the same linear system (5.4), along with the identity

$$\frac{\partial U(t, t_0, U_0)}{\partial t_0} + \Phi(t, t_0, U_0)f(t_0, U_0) = 0. \quad (5.5)$$

Equation (5.4) represents the variational equation associated with the differential system (5.1).

Proof. For part (a), let $h \in \mathbb{R}$ be a small scalar and define $e_k \in \mathbb{R}^n$ as the unit vector in the k -th direction. Set

$$U(t, h) = U(t, t_0, U_0 + he_k),$$

so that $\lim_{h \rightarrow 0} U(t, h) = U_0(t)$ uniformly on compact intervals. Then

$$D_H[U(t, h) - U_0(t)] = f(t, U(t, h)) - f(t, U_0(t)).$$

Using the integral form of the *Mean Value Theorem* :

$$D_H[U(t, h) - U_0(t)] = \int_0^1 \frac{\partial f(t, sU(t, h) + (1-s)U_0(t))}{\partial U} ds \cdot [U(t, h) - U_0(t)].$$

Define

$$U_h(t) = \frac{U(t, h) - U_0(t)}{h}, \quad h \neq 0,$$

which satisfies

$$Z' = H(t, t_0, U_0, h)Z, \quad Z(t_0) = e_k,$$

where

$$H(t, t_0, U_0, h) = \int_0^1 \frac{\partial f(t, sU(t, h) + (1-s)U_0(t))}{\partial U} ds.$$

Due to the continuity of $\frac{\partial f}{\partial U}$, as $h \rightarrow 0$ we have

$$H(t, t_0, U_0, h) \rightarrow H(t, t_0, U_0),$$

uniformly. Thus, $U_h(t) \rightarrow \Phi(t, t_0, U_0)$, which solves

$$Z' = H(t, t_0, U_0)Z, \quad Z(t_0) = e_k.$$

For part (b), define

$$\check{U}_h(t) = \frac{U(t, t_0 + h, U_0) - U(t, t_0, U_0)}{h}, \quad h \neq 0.$$

By uniqueness, we have

$$U(t, t_0 + h, U_0) = U(t, t_0, U(t_0, t_0 + h, U_0)).$$

Then,

$$h\check{U}_h(t) = U(t, t_0, U(t_0, t_0 + h, U_0)) - U(t, t_0, U_0).$$

Using differentiability with respect to initial data:

$$h\check{U}_h(t) = \left[\frac{\partial U(t, t_0, U_0)}{\partial U_0} + O(1) \right] \cdot [U(t_0, t_0 + h, U_0) - U_0].$$

By applying the *Mean Value Theorem* to each component, $k = 1, \dots, n$, there exists $\theta_k \in (0, 1)$ such that

$$U_k(t_0, t_0 + h, U_0) - U_{0,k} = -hf_k(t_0 + \theta_k h, U(t_0 + \theta_k h, t_0 + h, U_0)).$$

As $h \rightarrow 0$, we have

$$f_k(t_0 + \theta_k h, U(t_0 + \theta_k h, t_0 + h, U_0)) \rightarrow f_k(t_0, U_0).$$

Thus,

$$\check{U}_h(t) = - \left[\frac{\partial U(t, t_0, U_0)}{\partial U_0} + O(1) \right] \cdot [f(t_0, U_0) + O(1)],$$

which gives

$$\frac{\partial U(t, t_0, U_0)}{\partial t_0} + \Phi(t, t_0, U_0)f(t_0, U_0) = 0.$$

□

5.2. Generalized Variation of Parameters with Initial Time Difference

Theorem 5.2.1 : Suppose the function f is continuous on $\mathbb{R}_+ \times \mathbb{R}^n$ and continuously differentiable with respect to U , the partial derivative $\frac{\partial f}{\partial U}$ exists and is continuous. Then, the initial value problems (5.1) and (5.2) possess unique solutions, denoted by $U(t, t_0, U_0)$ for $t \geq t_0$ and $U(t, \tau_0, V_0)$ for $t \geq \tau_0$, respectively. These solutions satisfy the following integral identity:

$$\begin{aligned} U(t, \tau_0, V_0) &= U(t - \eta, t_0, U_0) + (\tau_0 - t_0) \\ &\quad \times \int_0^1 [f(u(s), U(u(s), v(s), w(s))) \\ &\quad - \Phi(u(s), v(s), w(s))f(v(s), w(s))] ds \\ &\quad + \left(\int_0^1 \Phi(u(s), v(s), w(s)) ds \right) (V_0 - U_0), \end{aligned} \quad (5.6)$$

where $t \geq \tau_0 \geq 0$, $t_0 \geq 0$, and $\eta = \tau_0 - t_0 \geq 0$. The auxiliary functions are given by:

$$u(s) = st + (1 - s)(t - \eta), \quad v(s) = s\tau_0 + (1 - s)t_0, \quad w(s) = sV_0 + (1 - s)U_0,$$

for all $s \in [0, 1]$.

Proof. Define the trajectory function $p(s) = U(u(s), v(s), w(s))$, where U solves the this problem,

$$D_H U = f(u, U), \quad U(v) = w, \quad u \geq v,$$

with $s \in [0, 1]$ and $t \geq \tau_0$. Applying the chain rule to differentiate $p(s)$ yields

$$\begin{aligned} \frac{dp(s)}{ds} &= \frac{\partial U(u(s), v(s), w(s))}{\partial u} (\tau_0 - t_0) + \frac{\partial U(u(s), v(s), w(s))}{\partial v} (\tau_0 - t_0) \\ &\quad + \frac{\partial U(u(s), v(s), w(s))}{\partial w} (V_0 - U_0). \end{aligned} \quad (5.7)$$

Using the results from *Theorem 5.1.1*, specifically the identities in (5.4) and (5.5), the differential of $p(s)$ can be written as

$$\begin{aligned} dp(s) &= [[f(u(s), U(u(s), v(s), w(s))) - \Phi(u(s), v(s), w(s))f(v(s), w(s))] \times (\tau_0 - t_0) \\ &\quad + \Phi(u(s), v(s), w(s))(V_0 - U_0)] ds. \end{aligned}$$

Integrating both sides with respect to s over the interval $[0, 1]$ yields:

$$\begin{aligned}
U(t, \tau_0, V_0) &= U(t - \eta, t_0, U_0) + (\tau_0 - t_0) \\
&\times \int_0^1 \left[f(u(s), U(u(s), v(s), w(s))) \right. \\
&\quad \left. - \Phi(u(s), v(s), w(s))f(v(s), w(s)) \right] ds \\
&+ \left(\int_0^1 \Phi(u(s), v(s), w(s)) ds \right) (V_0 - U_0).
\end{aligned} \tag{5.8}$$

Finally, substituting the explicit expressions of $u(s)$, $v(s)$, and $w(s)$ completes the proof. $\square$

Theorem 5.2.2 : Suppose that the function $f(t, U)$ and its partial derivative $\frac{\partial f(t, U)}{\partial U}$ exist and are continuous on $J \times \mathbb{R}^n$. Then, the unique solutions $U(t, \tau_0, V_0)$ to the system (5.2) and $V(t, \tau_0, V_0)$ to the system (5.3) satisfy the following integral relation:

$$\begin{aligned}
V(t, \tau_0, V_0) &= U(t - \eta, t_0, U_0) + (\tau_0 - t_0) \\
&\times \int_0^1 \left[f(u(s), U(u(s), v(s), w(s))) \right. \\
&\quad \left. - \Phi(u(s), v(s), w(s))f(v(s), w(s)) \right] ds \\
&+ \left(\int_0^1 \Phi(u(s), v(s), w(s)) ds \right) (V_0 - U_0) \\
&+ \int_{\tau_0}^t \Phi(t, \sigma, V(\sigma))R(\sigma, V(\sigma)) d\sigma
\end{aligned}$$

for all $t \geq \tau_0 \geq 0$, where $\Phi(t, \tau_0, V_0) = \frac{\partial U(t, \tau_0, V_0)}{\partial V_0}$.

Proof. Let $U(t, \tau_0, V_0)$ be the solution of the system (5.2). Then, by applying the variation of parameters formula (5.8), we have:

$$\begin{aligned}
V(t, \tau_0, V_0) &= U(t - \eta, t_0, U_0) + (\tau_0 - t_0) \\
&\times \int_0^1 \left[f(u(s), U(u(s), v(s), w(s))) \right. \\
&\quad \left. - \Phi(u(s), v(s), w(s))f(v(s), w(s)) \right] ds \\
&+ \left(\int_0^1 \Phi(u(s), v(s), w(s)) ds \right) (V_0 - U_0) \\
&+ \int_{\tau_0}^t \Phi(t, \sigma, V(\sigma))R(\sigma, V(\sigma)) d\sigma.
\end{aligned}$$

Let us now define the auxiliary function $u_*(\sigma) = U(t, \sigma, V(\sigma))$.

Then $v_*(t) = V(t) = V(t, \tau_0, V_0)$ and $v_*(\tau_0) = V_0 = U(t, \tau_0, V_0)$. Differentiating $u_*(\sigma)$ with respect to σ and applying the chain rule gives:

$$\begin{aligned} \frac{du_*(\sigma)}{d\sigma} &= \frac{\partial U(t, \sigma, V(\sigma))}{\partial \tau_0} + \frac{\partial U(t, \sigma, V(\sigma))}{\partial V_0} \cdot D_H V(\sigma) \\ &= \Phi(t, \sigma, V(\sigma)) [-f(\sigma, V(\sigma)) + D_H V(\sigma)], \end{aligned}$$

which implies

$$\frac{du_*(\sigma)}{d\sigma} = \Phi(t, \sigma, V(\sigma)) \cdot R(\sigma, V(\sigma)),$$

since $R(\sigma, V(\sigma)) = D_H V(\sigma) - f(\sigma, V(\sigma))$.

Integrating both sides with respect to σ over $[\tau_0, t]$, we obtain:

$$V(t, \tau_0, V_0) = U(t, \tau_0, V_0) + \int_{\tau_0}^t \Phi(t, \sigma, V(\sigma)) R(\sigma, V(\sigma)) d\sigma,$$

which establishes the desired relation. The rest of the integral structure follows directly from Theorem 5.2.1 and equation (5.6). This completes the proof, consistent with the stability approach given in Yakar [24]. □

6. LIPSCHITZ STABILITY

6.1. Definition

Consider the following set-valued differential systems:

$$D_H U(t) = f(t, U(t)), \quad U(t_0) = U_0, \quad U_0 \in \mathbb{K}_C(\mathbb{R}^n), \quad t \geq t_0 \geq 0, \quad (6.1)$$

$$D_H U(t) = f(t, U(t)), \quad U(\tau_0) = V_0, \quad V_0 \in \mathbb{K}_C(\mathbb{R}^n), \quad t \geq \tau_0 \geq t_0. \quad (6.2)$$

Let us now introduce a perturbed version of system (6.2) given by:

$$D_H V(t) = F(t, V(t)), \quad V(\tau_0) = V_0, \quad V_0 \in \mathbb{K}_C(\mathbb{R}^n), \quad t \geq \tau_0, \quad (6.3)$$

where the functions f and F are assumed to be elements of the space $C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{K}_C(\mathbb{R}^n)]$.

A special case of the perturbed system (6.3) is obtained when

$$F(t, V(t)) = f(t, V(t)) + R(t, V(t)),$$

with $R(t, V(t))$ representing a perturbation term.

Assume that f is a continuous function and that its partial derivative $\frac{\partial f}{\partial U}$ exists and is continuous on the domain $\mathbb{R}_+ \times \mathbb{R}^n$. Let $U(t, t_0, U_0)$ denote the solution of the system (6.1) for $t \geq t_0$.

Then, the derivative

$$\Phi(t, t_0, U_0) = \frac{\partial U(t, t_0, U_0)}{\partial U_0}$$

exists and is the *fundamental matrix solution* of the variational equation

$$Z' = \frac{\partial f(t, U(t, t_0, U_0))}{\partial U} Z, \quad Z(t_0) = I. \quad (6.4)$$

Furthermore, the derivative of the solution with respect to the initial time t_0 also exists and satisfies

$$\frac{\partial U(t, t_0, U_0)}{\partial t_0} + \Phi(t, t_0, U_0) f(t_0, U_0) = 0, \quad t \geq t_0. \quad (6.5)$$

Definition 6.1.1: Let $U(t - \eta, t_0, U_0)$ be a solution of the system (6.1) passing through the initial point (t_0, U_0) , where $t \geq \tau_0$ and $\eta = \tau_0 - t_0 \geq 0$. The solution is said to

be *Lipschitz stable (LS) with an initial time difference* with respect to the null solution $U(t, t_0, U_0) \equiv \tilde{0}$ of (6.1), if there exist constants $L = L(\tau_0) > 0$ and $\delta > 0$ such that

$$\|U(t - \eta, t_0, U_0)\| \leq L \|U_0\|, \quad \text{for all } t \geq \tau_0 \text{ whenever } \|U_0\| \leq \delta.$$

If the constant L is independent of the initial shifted time τ_0 , then the solution $U(t - \eta, t_0, U_0)$ is said to be *uniformly Lipschitz stable (uniform LS) with an initial time difference* with respect to the null solution $U(t, t_0, U_0) \equiv \tilde{0}$.

Definition 6.1.2 : The solution $U(t - \eta, t_0, U_0)$ of the system (6.1) through (t_0, U_0) for $t \geq \tau_0$ is said to be *globally (LS) with an initial time difference* concerning the null solution $U(t, t_0, U_0) \equiv \tilde{0}$ of (6.1) for $t \geq t_0$, where $U(t, t_0, U_0)$ is any solution of the system (6.1) and $\eta = \tau_0 - t_0 \geq 0$ if and only if there exist $L = L(\tau_0) > 0$ and $\delta > 0$ such that

$$\|U(t - \eta, t_0, U_0)\| \leq L \|U_0\| \quad \text{for } t \geq \tau_0 \text{ and } \|U_0\| < \infty.$$

If L is independent of τ_0 , then the solution $U(t - \eta, t_0, U_0)$ of the system (6.1) is *globally uniformly (LS) with an initial time different* concerning the null solution $U(t, t_0, U_0) \equiv \tilde{0}$.

Definition 6.1.3 : The solution $U(t - \eta, \tau_0, U_0)$ of the system (6.1) through (t_0, U_0) for $t \geq \tau_0$ is said to be *(LS) in variation with an initial time difference* with respect to the null solution $U(t, t_0, U_0) \equiv \tilde{0}$ of (6.1) for $t \geq t_0$, where $U(t, t_0, U_0)$ is any solution of the system (6.1) and $\eta = \tau_0 - t_0 \geq 0$, if and only if there exist $L = L(\tau_0) > 0$ and $\delta > 0$ such that

$$\|\Phi(t - \eta, t_0, U_0)\| \leq L \quad \text{for } t \geq \tau_0 \geq t_0 \geq 0 \text{ provided } \|U_0\| \leq \delta,$$

where $\Phi(t - \eta, t_0, U_0)$ is the characteristic solution of (6.1) such that $\Phi(t_0, t_0, U_0) = \mathcal{I}$, and $\mathcal{I}$ is the $n \times n$ identity matrix. If L is independent of τ_0 , then the solution $U(t - \eta, t_0, U_0)$ of the system (6.1) is *uniformly (LS) in variation with an initial time difference* concerning the null solution $U(t, t_0, U_0) \equiv \tilde{0}$.

Definition 6.1.4 : The solution $U(t - \eta, t_0, U_0)$ of the system (6.1) through (t_0, U_0) for $t \geq \tau_0$ is said to be *globally (LS) in variation with an initial time difference* with respect to the null solution $U(t, t_0, U_0) \equiv \tilde{0}$ of (6.1) for $t \geq t_0$, where $U(t, t_0, U_0)$ is

any solution of the system (6.1) and $\eta = \tau_0 - t_0 \geq 0$ if and only if there exists an $L = L(\tau_0) > 0$ such that

$$\|\Phi(t - \eta, t_0, U_0)\| \leq L \quad \text{for } t \geq \tau_0 \geq t_0 \geq 0 \text{ provided } \|U_0\| < \infty.$$

If L is independent of τ_0 , then the solution $U(t - \eta, t_0, U_0)$ of the system (6.1) is *globally uniformly (LS) in variation with an initial time difference* with respect to the null solution $U(t, t_0, U_0) \equiv \tilde{0}$.

Definition 6.1.5 : The solution $V(t, \tau_0, V_0)$ of the system (6.3) through (τ_0, V_0) is said to be *(ITDLS)* concerning the solution $U(t - \eta, t_0, U_0)$ for $t \geq \tau_0$, where $U(t, t_0, U_0)$ is any solution of the system (6.1) and $\eta = \tau_0 - t_0 \geq 0$, if and only if there exists an $L = L(\tau_0) > 0$ such that

$$\|V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)\| \leq L [\|V_0 - U_0\| + |\tau_0 - t_0|] \quad \text{for } t \geq \tau_0.$$

If L is independent of τ_0 , then the solution $V(t, \tau_0, V_0)$ of the system in (6.3) is *(ITD) uniformly (LS)* concerning the solution $U(t - \eta, t_0, U_0)$.

Definition 6.1.6 : The solution $V(t, \tau_0, V_0)$ of the system (6.3) through (τ_0, V_0) is said to be *(ITDLS) in variation* with respect to the solution $U(t - \eta, t_0, U_0)$ for $t \geq \tau_0$, where $U(t, t_0, U_0)$ is any solution of the system (6.1) and $\eta = \tau_0 - t_0 \geq 0$ if and only if

(i) There exists an $L = L(\tau_0) > 0$ such that

$$\|V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)\| \leq L [\|V_0 - U_0\| + |\tau_0 - t_0|].$$

(ii) For each $\alpha > 0$, there exists an $N > 0$ such that the $\Phi(t, \tau_0, V_0)$ solution of the variation system (6.4) satisfies

$$\|\Phi(t, \tau_0, V_0)\| \leq N \quad \text{for all } t \geq \tau_0 \text{ and } \|V_0\| \leq \alpha.$$

If L is independent of τ_0 , then the solution $V(t, \tau_0, V_0)$ of the system (6.3) is *(ITD) uniformly (LS) in variation* with respect to the solution $U(t - \eta, t_0, U_0)$.

The notions of Lipschitz stability (LS), global LS, LS in variation, and initial time difference Lipschitz stability (ITDLS), as presented in *Definitions 6.1.1–6.1.6*, are based on the general theory of nonlinear differential systems under perturbations and initial

time shifts. Fundamental contributions to stability theory were introduced in [28] and later extended to set-valued systems and set-valued differential equations in works such as [22, 31, 24]. Stability criteria under perturbations and time shifts were further developed in [13, 2], while comparison results involving Lyapunov-like functions can be found in [8]. Some of the variation techniques employed in this thesis are also influenced by the methods proposed in [18], which investigate set-valued differential equations via generalized Lyapunov methods. Accordingly, the stability definitions adopted here combine and generalize these earlier studies within the framework of set-valued systems with initial time differences.

6.2. Lipschitz Stability of Set Differential System with Initial Time Difference (ITD)

Consider the following set-valued differential systems:

$$D_H U = f(t, U), \quad U(t_0) = U_0 \quad \text{for } t \geq t_0 \geq 0, \quad t_0 \in \mathbb{R}_+, \quad (6.6)$$

$$D_H \bar{U} = f(t - \eta, \bar{U}), \quad \bar{U}(\tau_0) = U_0 \quad \text{for } t \geq \tau_0 \geq t_0 > 0, \quad t_0 \in \mathbb{R}_+, \quad (6.7)$$

where $\bar{U}(t) = U(t - \eta, t_0, U_0)$ for $t \geq t_0$, and the initial time shift is given by $\eta = \tau_0 - t_0$.

Assume that the function f belongs to the class $C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$, and satisfies $f(t, 0) = 0$ for all $t \in \mathbb{R}_+$. Under this condition, the trivial set solution $U = \tilde{0}$ becomes a solution to the system (6.6), and hence it serves as the null solution to the initial value problem defined by (6.6) through the point $(t_0, 0)$. Here, $U(t, t_0, U_0)$ denotes the solution of (6.6) satisfying $U(t_0, t_0, U_0) = U_0$. On the other hand, if $f(t, 0) \neq 0$, then $U = \tilde{0}$ ceases to be a solution of the system (6.6).

Theorem 6.2.1: The following stability properties are equivalent:

- (i) The solution $\bar{U}(t, \tau_0, U_0)$ of the system (6.7) is globally uniformly Lipschitz stable (LS) with initial time difference (ITD) in variation with respect to the null solution $U(t, t_0, U_0) \equiv \tilde{0}$ of the system (6.6).
- (ii) The solution $\bar{U}(t, \tau_0, U_0)$ of (6.7) is uniformly (LS) with ITD in variation with respect to the null solution $U(t, t_0, U_0) \equiv \tilde{0}$ of the system (6.6).
- (iii) The solution $\bar{U}(t, \tau_0, U_0)$ of (6.7) is globally uniformly Lipschitz stable (LS) with ITD in variation with respect to the null solution $U(t, t_0, U_0) \equiv \tilde{0}$ of the system

(6.6).

(iv) The solution $\bar{U}(t, \tau_0, U_0)$ of system (6.7) is uniformly Lipschitz stable (LS) with ITD with respect to the null solution $U(t, t_0, U_0) \equiv \tilde{0}$ of the system (6.6).

(v) The solution $\bar{U}(t, \tau_0, U_0)$ of system (6.7) is ITD uniformly stable in variation with respect to the null solution $U(t, t_0, U_0) \equiv \tilde{0}$ of (6.6).

Proof. (i)(i) $\Rightarrow$ (ii): This implication follows directly by the definitions given in *Definition 6.1.3* and *Definition 6.1.4*.

(ii)(ii) $\Rightarrow$ (iii): Consider the fundamental matrix solution $\Phi(t, t_0, U_0)$ of system (6.6) such that $\Phi(t_0, t_0, U_0) = \mathcal{I}$. The shifted matrix $\Phi(t - \eta, t_0, U_0)$ corresponds to $\frac{\partial \bar{U}(t, t_0, U_0)}{\partial U_0}$. Since Φ is independent of U_0 , there exists a constant $L = L(\tau_0) > 0$ and $\delta > 0$ such that

$$\|U(t - \eta, t_0, U_0)\| = \|\Phi(t - \eta, t_0, U_0)\| \cdot \|U_0\| \leq L\|U_0\|,$$

for all $t \geq \tau_0 \geq t_0 \geq 0$, assuming $\|U_0\| < \infty$.

(iii)(iii) $\Rightarrow$ (iv): This implication is immediate from the definitions 6.1.2 and 6.1.3.

(iv)(iv) $\Rightarrow$ (v): Suppose that the solution $\bar{U}(t, \tau_0, U_0)$ of system (6.7) is ITD uniformly stable with respect to the null solution $U(t, t_0, U_0) \equiv \tilde{0}$ of (6.6). Then, there exist constants $L = L(\tau_0) > 0$ and $\delta_1 > 0$ such that

$$\|\Phi(t - \eta, t_0, U_0)\| \leq L, \quad \text{whenever } \|U_0\| \leq \delta_1 \text{ and } t \geq \tau_0 \geq t_0 \geq 0.$$

For any $\epsilon > 0$, choose $\delta = \min \left\{ \delta_1, \frac{\epsilon}{L} \right\}$. Then,

$$\|U(t - \eta, t_0, U_0)\| \leq \epsilon, \quad \text{whenever } \|U_0\| \leq \delta.$$

(v)(v) $\Rightarrow$ (i): Suppose that $\bar{U}(t, \tau_0, U_0)$ is uniformly stable with ITD and globally stable in variation. Then, for a given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|V(t - \eta, t_0, U_0)\| \leq \epsilon = L\|U_0\|, \quad \text{whenever } \|U_0\| \leq \delta,$$

for all $t \geq \tau_0 \geq t_0 \geq 0$. Consequently,

$$\|U(t - \eta, t_0, U_0)\| = \|\Phi(t - \eta, t_0)\| \cdot \|U_0\| \leq L\|U_0\|,$$

which implies that

$$\|\Phi(t - \eta, t_0, U_0)\| \leq L, \quad \text{for all } \|U_0\| < \infty.$$

□

Corollary 8: The following stability properties are equivalent for the unperturbed system (6.6). These equivalences represent the special case of Theorem 6.2.1 when $\eta = \tau_0 - t_0 = 0$:

- (i) The solution $U(t, t_0, U_0)$ is globally uniformly Lipschitz stable in variation with respect to the null solution $U(t) \equiv \tilde{0}$.
- (ii) The solution $U(t, t_0, U_0)$ is uniformly Lipschitz stable in variation with respect to the null solution $U(t) \equiv \tilde{0}$.
- (iii) The solution $U(t, t_0, U_0)$ is globally uniformly Lipschitz stable with respect to the null solution $U(t) \equiv \tilde{0}$.
- (iv) The solution $U(t, t_0, U_0)$ is uniformly Lipschitz stable with respect to the null solution $U(t) \equiv \tilde{0}$.
- (v) The solution $U(t, t_0, U_0)$ is uniformly stable with respect to the null solution $U(t) \equiv \tilde{0}$.

Theorem 6.2.2. The following statements are equivalent:

- (i) The (ITD) stability properties of the system (6.7), as listed in *Theorem 6.2.1*, imply the corresponding stability properties of the system (6.6).
- (ii) The stability properties of the system (6.6), as listed in *Corollary 8*, imply the corresponding (ITD) stability properties of the system (6.7).

Proof. (i) This implication is directly established by *Corollary 8*, where the (ITD) stability properties of the linear unperturbed system (6.7) with respect to the null solution imply the corresponding stability properties of the system (6.6).

(ii) We now show the converse: that the stability properties of the linear unperturbed system (6.6) imply the corresponding (ITD) stability properties of the system (6.7). Assume that item (ii) of *Corollary 8* holds.

Consider the solution $\Phi(t, t_0, U_0)$ of system (6.6), where

$$\Phi(t_0, t_0, U_0) = \mathcal{I}, \quad \text{and} \quad \Phi(t, t_0, U_0) = \frac{\partial U(t, t_0, U_0)}{\partial U_0},$$

so that $\Phi(t, t_0, U_0)$ is independent of U_0 . Since condition (ii) holds, by *Definition 6.1.3* there exist constants $L > 0$ and $\delta > 0$ such that

$$\|\Phi(t, t_0, U_0)\| \leq L, \quad \text{for all } t \geq t_0 \geq 0 \text{ and } \|U_0\| \leq \delta.$$

For the (ITD) system (6.7), note that $t - \eta \geq t_0$ since $t \geq \tau_0 \geq t_0 \geq 0$ and $\eta = \tau_0 - t_0 \geq 0$. Then it follows that

$$\|\Phi(t - \eta, t_0, U_0)\| \leq L, \quad \text{for } t \geq \tau_0 \geq t_0 \geq 0 \text{ and } \|U_0\| \leq \delta,$$

which establishes the desired (ITD) stability property for system (6.7).

□

6.3. Lipschitz Stability Criteria with ITD

Theorem 6.3.1: Assume that systems (6.1) and (6.2) admit unique solutions $U(t, t_0, U_0)$ through (t_0, U_0) for $t \geq t_0$ and $U(t, \tau_0, V_0)$ through (τ_0, V_0) for $t \geq \tau_0$, respectively. Define the time shift $\eta = \tau_0 - t_0 \geq 0$, and let $U(t - \eta, t_0, U_0)$ be well-defined for $t \geq \tau_0$. Suppose the following hold:

(i) There exist real-valued functions $\mu, \lambda \in C[[0, \infty), \mathbb{R}_+]$ such that

$$\int_{\tau_0}^{\infty} \mu(s) ds = K(\tau_0), \quad \int_{\tau_0}^{\infty} \lambda(s) ds = L(\tau_0).$$

(ii) The function f is Lipschitz continuous in both time and state variables, satisfying

$$\begin{aligned} & \|f(t, X(t, \tau_0, V_0 - U_0) + U(t - \eta, t_0, U_0)) - f(t - \eta, U(t - \eta, t_0, U_0))\| \\ & \leq \lambda(t) \|X(t)\| + \frac{|\eta|}{K(\tau_0)} \mu(t) \end{aligned}$$

where $X(t, \tau_0, V_0 - U_0) = U(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)$.

Then, the solution $U(t, \tau_0, V_0)$ of (6.2) is ITD-Lipschitz stable (ITDLS) with respect to $U(t - \eta, t_0, U_0)$.

Proof. Consider the difference between the two solutions:

$$X(t, \tau_0, V_0 - U_0) = U(t, \tau_0, V_0) - U(t - \eta, t_0, U_0). \quad (6.8)$$

Differentiating both sides with respect to t gives

$$\begin{aligned} D_H X(t, \tau_0, X_0) &= F(t, X; \eta) \\ &= f(t, X(t, \tau_0, V_0 - U_0) + U(t - \eta, t_0, U_0)) \\ &\quad - f(t - \eta, U(t - \eta, t_0, U_0)). \end{aligned}$$

Using the triangle inequality, we estimate:

$$\begin{aligned} \|F(t, X; \eta)\| &\leq \left\| f(t, X(t, \tau_0, U_0 - V_0) + U(t - \eta, t_0, U_0)) \right. \\ &\quad \left. - f(t, U(t - \eta, t_0, U_0)) \right\| \\ &\quad + \left\| f(t, U(t - \eta, t_0, U_0)) - f(t - \eta, U(t - \eta, t_0, U_0)) \right\|. \end{aligned} \quad (6.9)$$

Applying the Lipschitz-type condition from assumption (ii), we obtain

$$\|F(t, X; \eta)\| \leq \lambda(t) \|X(t)\| + \frac{|\eta|}{K(\tau_0)} \mu(t). \quad (6.10)$$

Now, integrate the expression (6.8) from τ_0 to t and applying the norm, we obtain:

$$\begin{aligned} X(t, \tau_0, V_0 - U_0) &= X(\tau_0) + \int_{\tau_0}^t F(s, X(s); \eta) ds, \\ \|X(t, \tau_0, V_0 - U_0)\| &\leq \|V_0 - U_0\| + \int_{\tau_0}^t \|F(s, X(s); \eta)\| ds. \end{aligned}$$

Substituting the estimate from (6.10) gives:

$$\begin{aligned} \|X(t, \tau_0, V_0 - U_0)\| &\leq \|V_0 - U_0\| + \int_{\tau_0}^t \lambda(s) \|X(s)\| ds \\ &\quad + \frac{|\eta|}{K(\tau_0)} \int_{\tau_0}^t \mu(s) ds. \end{aligned} \quad (6.11)$$

Let $z(t) = \|X(t, \tau_0, X_0)\|$. Then inequality (6.11) becomes:

$$z(t) \leq z(\tau_0) + \frac{|\eta|}{K(\tau_0)} \int_{\tau_0}^t \mu(s) ds + \int_{\tau_0}^t \lambda(s) z(s) ds. \quad (6.12)$$

Using assumption (i), we rewrite this as

$$z(t) \leq z(\tau_0) + |\eta| + \int_{\tau_0}^t \lambda(s) z(s) ds. \quad (6.13)$$

Applying Gronwall's inequality to (6.13), we obtain the estimate:

$$z(t) \leq (z(\tau_0) + |\eta|) e^{L(\tau_0)}. \quad (6.14)$$

Recalling the definition of $X(t)$ in (6.8), we conclude that:

$$\|U(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)\| \leq e^{L(\tau_0)} [\|V_0 - U_0\| + |\tau_0 - t_0|].$$

Thus, for any constant $M \geq e^{L(\tau_0)}$, the inequality

$$\|U(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)\| \leq M [\|V_0 - U_0\| + |\tau_0 - t_0|]$$

holds, which completes the proof by the definition of ITDLS as given in *Definition 6.1.5*. $\square$

Example 1: Consider the scalar set differential system with an initial time difference:

$$D_H U(t) = -a(t)U(t), \quad U(t_0) = U_0, \quad (6.15)$$

$$D_H U(t) = -a(t - \eta)U(t), \quad U(\tau_0) = V_0, \quad (6.16)$$

where $\eta = \tau_0 - t_0$ and $a(t) = \frac{1}{t+1}$ for $t \geq 0$.

The function $a(t)$ is positive, continuous, and integrable on any interval $[c, \infty)$ with $c > -1$.

Define the following functions:

$$\lambda(t) = a(t) = \frac{1}{t+1}, \quad \mu(t) = |a(t) - a(t - \eta)| = \left| \frac{1}{t+1} - \frac{1}{t+1-\eta} \right|.$$

We now solve equations (6.15) and (6.16). The solution of equation (6.15) is:

$$U(t) = U_0 \cdot \exp \left(- \int_{t_0}^t a(s) ds \right) = U_0 \cdot \exp \left(- \ln \left(\frac{t+1}{t_0+1} \right) \right) = U_0 \cdot \frac{t_0+1}{t+1}.$$

Similarly, the solution of equation (6.16) becomes:

$$V(t) = V_0 \cdot \exp \left(- \int_{\tau_0}^t a(s - \eta) ds \right) = V_0 \cdot \exp \left(- \int_{t_0}^{t-\eta} a(s) ds \right),$$

which yields:

$$V(t) = V_0 \cdot \frac{t_0+1}{t-\eta+1}.$$

Then, the difference between the solutions is:

$$|V(t) - U(t - \eta)| = \left| \frac{t_0+1}{t-\eta+1} \cdot V_0 - \frac{t_0+1}{t-\eta+1} \cdot U_0 \right| = \frac{t_0+1}{t-\eta+1} \cdot |V_0 - U_0|.$$

This expression shows that the difference decays as $t \rightarrow \infty$, and the bound depends linearly on $|V_0 - U_0|$, verifying the ITD-Lipschitz stability.

Note: This example shows that the conclusion of the theorem may still hold even in the absence of its stated assumptions. Hence, the assumptions are sufficient but not necessary.

Note: In this example we show that the conclusion of the related Lipschitz stability theorem with initial time difference (ITD) may still hold even if certain assumptions—particularly those regarding the integrability of the perturbation functions—are not satisfied over the entire domain. Specifically, for the choice $a(t) = \frac{1}{t+1}$, while $\lambda(t) = a(t)$ is integrable on any interval $[c, \infty)$ with $c > -1$, the function

$$\mu(t) = |a(t) - a(t - \eta)| = \left| \frac{1}{t+1} - \frac{1}{t+1-\eta} \right|$$

may not always be integrable on $[\tau_0, \infty)$ depending on the value of η . Similarly, the auxiliary function

$$\alpha(t) = \frac{|\eta|}{K(\tau_0)(t+1)}$$

may also fail to be integrable over the same domain if $K(\tau_0)$ vanishes or if η is unbounded. Despite this, the explicit decay of the solution difference

$$|V(t) - U(t - \eta)| = \frac{t_0 + 1}{t - \eta + 1} \cdot |V_0 - U_0|$$

still verifies the stability behavior expected from the theorem. Hence, this shows that the assumptions in the theorem are sufficient but not necessary.

Theorem 6.3.2: Lipschitz stability with initial time difference (ITD) implies stability with (ITD).

Proof. We proceed using the definition of Lipschitz stability with ITD as given in Definition 6.1.5.

Assume that the solution $V(t, \tau_0, V_0)$ of the system (6.3), passing through the point (τ_0, V_0) , is (ITDLS) with respect to the solution $U(t - \eta, t_0, U_0)$ for $t \geq \tau_0$, where $U(t, t_0, U_0)$ is a solution of the system (6.1) and $\eta = \tau_0 - t_0 \geq 0$.

By the definition of (ITDLS), there exists a constant $L = L(\tau_0) > 0$ such that

$$\|V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)\| \leq L [\|V_0 - U_0\| + |\tau_0 - t_0|], \quad \text{for all } t \geq \tau_0.$$

Let $\epsilon > 0$ be arbitrary. Choose

$$\|V_0 - U_0\| < \frac{\epsilon}{2L} = \delta_1(\epsilon, \tau_0), \quad \text{and} \quad |\tau_0 - t_0| < \frac{\epsilon}{2L} = \delta_2(\epsilon, \tau_0)$$

so that

$$\|V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)\| < \epsilon, \quad \text{for all } t \geq \tau_0.$$

Hence, the solution $V(t, \tau_0, V_0)$ of system (6.3) is (ITD) stable with respect to $U(t - \eta, t_0, U_0)$.

Moreover, if the constant L is independent of τ_0 , then so are δ_1 and δ_2 , and the result holds uniformly with respect to the initial time. $\square$

Theorem 6.3.3: Stability with initial time difference (ITD) does not necessarily imply Lipschitz stability with ITD.

Proof. To illustrate this, we consider the special case where $V_0 = U_0$ and $\tau_0 = t_0$. A counterexample addressing this situation is provided in [32], specifically in Example 1.4. $\square$

Theorem 6.3.4 : Let the solution $U(t, \tau_0, V_0)$ of the system (6.2) be ITD-Lipschitz stable with respect to the solution $U(t - \eta, t_0, U_0)$ of the system (6.1) for $t \geq \tau_0$, and suppose that the system (6.3) admits a unique solution $V(t, \tau_0, V_0)$ passing through (τ_0, V_0) . Assume $\eta = \tau_0 - t_0 > 0$ or equivalently $\|V_0 - U_0\| > 0$. Furthermore, suppose that the partial derivative

$$\Phi(t, \tau_0, V_0) = \frac{\partial U(t, \tau_0, V_0)}{\partial V_0}$$

exists, with $\Phi(\tau_0, \tau_0, V_0) = \mathcal{I}$, where $\mathcal{I}$ denotes the identity matrix. Assume the following conditions hold:

(i) For any $\alpha > 0$, there exists a constant $\mu > 0$ such that:

$$\|\Phi(t, s, V(s))\| \leq \mu \quad \text{and} \quad \|R(s, V(s))\| \leq \lambda(s)\|V(s)\| \quad \text{whenever} \\ \|V(s)\| \leq \alpha,$$

(ii) There exists a function $\lambda(s) \in C([0, \infty), \mathbb{R}_+)$ such that

$$\int_{\tau_0}^{\infty} \lambda(s) ds = \gamma(\tau_0) < \infty.$$

Then, the solution $V(t, \tau_0, V_0)$ of system (6.3) is (ITDLS) in variation with respect to the solution $U(t - \eta, t_0, U_0)$ of system (6.1).

Proof. Consider the unique solution $V(t, \tau_0, V_0)$ of (6.3) with initial data (τ_0, V_0) . Then we have the variation representation:

$$\begin{aligned} V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0) &= X(t, \tau_0, V_0 - U_0) \\ &+ \int_{\tau_0}^t \Phi(t, s, V(s)) R(s, V(s)) ds. \end{aligned} \quad (6.17)$$

Taking norms on both sides of (6.17) and applying the triangle inequality, we get:

$$\begin{aligned} \|V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)\| &\leq \|X(t, \tau_0, V_0 - U_0)\| \\ &+ \int_{\tau_0}^t \|\Phi(t, s, V(s))\| \cdot \|R(s, V(s))\| ds \end{aligned}$$

Using estimate (6.14) for $\|X(t, \tau_0, V_0 - U_0)\|$, we obtain:

$$\begin{aligned} \|V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)\| &\leq (\|V_0 - U_0\| + |\eta|) e^{L(\tau_0)} \\ &+ \int_{\tau_0}^t \|\Phi(t, s, V(s))\| \cdot \|R(s, V(s))\| ds \end{aligned}$$

Define $M^*(t) = \|V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)\|$. Then, by assumption (i), we have:

$$M^*(t) \leq (\|V_0 - U_0\| + |\eta|) e^{L(\tau_0)} + \mu \int_{\tau_0}^t \lambda(s) \|V(s)\| ds + \mu \int_{\tau_0}^t \lambda(s) \|U(s - \eta, t_0, U_0)\| ds.$$

Using the triangle inequality again and assuming $\|U_0\| \leq \alpha$, we estimate:

$$\begin{aligned} M^*(t) &\leq (\|V_0 - U_0\| + |\eta|) e^{L(\tau_0)} + \alpha \mu^2 \int_{\tau_0}^t \lambda(s) ds \\ &+ \mu \int_{\tau_0}^t \lambda(s) M^*(s) ds \end{aligned} \quad (6.18)$$

Now, applying assumption (ii) in (6.18), we get:

$$M^*(t) \leq (\|V_0 - U_0\| + |\eta|) e^{L(\tau_0)} + \alpha \mu^2 \gamma(\tau_0) + \int_{\tau_0}^t \lambda^*(s) M^*(s) ds,$$

where we define $\lambda^*(t) = \mu \lambda(t)$. Applying Gronwall's inequality, we obtain:

$$\begin{aligned} M^*(t) &\leq [(\|V_0 - U_0\| + |\eta|) e^{L(\tau_0)} + \alpha \mu^2 \gamma(\tau_0)] \\ &\times \exp \left(\int_{\tau_0}^t \lambda^*(s) ds \right) \end{aligned} \quad (6.19)$$

Using the bound $\int_{\tau_0}^t \lambda^*(s) ds \leq \mu\gamma(\tau_0)$, we arrive at:

$$M^*(t) \leq [(\|V_0 - U_0\| + |\eta|)e^{L(\tau_0)} + \alpha\mu^2\gamma(\tau_0)] e^{\mu\gamma(\tau_0)}.$$

We now select a constant $M > 0$ such that:

$$[(\|V_0 - U_0\| + |\eta|)e^{L(\tau_0)} + \alpha\mu^2\gamma(\tau_0)] \times e^{\mu\gamma(\tau_0)} \leq M(\|V_0 - U_0\| + |\eta|).$$

Therefore, we conclude:

$$\|V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)\| \leq M(\|V_0 - U_0\| + |\eta|).$$

This completes the proof that the solution $V(t, \tau_0, V_0)$ of system (6.3) is (ITDLS) in variation with respect to $U(t - \eta, t_0, U_0)$, in the sense of Definition 6.1.6. $\square$

Example 2 : Consider the scalar set differential equation:

$$D_H U(t) = -aU(t), \quad U(t_0) = U_0,$$

where $U(t) \subset \mathbb{R}$ is a compact convex set for each $t \geq t_0$, $a > 0$ is a constant, and D_H denotes the Hukuhara derivative.

Now, consider the perturbed system with an different initial time and position:

$$D_H V(t) = -aV(t) + R(t, V(t)), \quad V(\tau_0) = V_0,$$

where the perturbation is given by $R(t, V(t)) = \lambda(t)V(t)$, with $\lambda(t)$ continuous and satisfying

$$0 \leq \lambda(t) \leq \frac{1}{1+t^2}, \quad \int_{\tau_0}^{\infty} \lambda(s) ds < \infty.$$

Assume $\|V_0 - U_0\| > 0$ and $\eta = \tau_0 - t_0$. Since the homogeneous system has exponential decay due to the negative coefficient $-a$, and the perturbation satisfies the boundedness and integrability conditions, the hypotheses of Theorem 6.3.4 are satisfied.

Hence, the system is ITD Lipschitz stable. That is, the solutions $U(t)$ and $V(t)$ satisfy the inequality:

$$\|V(t) - U(t - \eta)\| \leq C(\|V_0 - U_0\| + |\eta|),$$

for all $t \geq \tau_0$, where $C > 0$ is a constant depending on a , $\lambda(t)$, and the decay rate.

The perturbed solution $V(t)$ remains close to the time-shifted unperturbed solution $U(t - \eta)$, with the difference bounded linearly in terms of the initial state deviation and time delay.

See [19, 2] for related ITD Lipschitz stability results in set-valued systems.

7. CONCLUSION

In this thesis, it has been demonstrated that a set differential equation can be stable without satisfying the conditions for Lipschitz stability. However, if such a system is Lipschitz stable, it must also be stable. This indicates that Lipschitz stability represents a stricter form of stability within the framework of set-valued differential equations.

To support this result, several theoretical elements were developed. The classical definitions of stability were reviewed and adapted to set-valued systems. Local existence, uniqueness, and extremal solution properties were established through comparison methods. Furthermore, new notions of stability involving initial time difference (ITD) were introduced, and corresponding theorems were provided.

In particular, the idea of Lipschitz stability was carefully formulated for set-valued differential equations, with ITD. By comparing these concepts, it became clear that Lipschitz stability imposes more restrictive conditions than standard stability. The techniques based on the variation of parameters contributed to a clear and systematic understanding of the relationship between these two stability notions.

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BIOGRAPHY

I, Tuba Satılmış, received my undergraduate degree in Mathematics from Uludağ University in 2002. Later, I completed a non-thesis master's program at Eskişehir Osmangazi University in 2008.

PUBLICATIONS WITHIN THE SCOPE OF THESIS STUDY

Satılmış T., (2024), "Stability of perturbed set differential equations related to unperturbed set differential equations with initial time difference," 7th International Hybrid Conference on Mathematical Advances and Applications, Istanbul, Turkey, May 8–11.

Satılmış T., (2024), "Stability of perturbed set differential equations involving causal operators in regard to their unperturbed ones considering different initial conditions", 8th Graduate Research Symposium, Gebze Technical University, Kocaeli, Turkey, May 30–31.

Satılmış T., (2024) "Stability of perturbed set differential equations involving causal operators in regard to their unperturbed ones considering different initial conditions", Workshop on Nonlinear Systems IV, Gebze Technical University, Kocaeli, Turkey, January 2–4.

Satılmış T., (2025), "Variation of parameters (VPs) and Lipschitz stability (LS) of set-valued differential systems with initial time difference (ITD)", 8th International Hybrid Conference on Mathematical Advances and Applications, Istanbul, Turkey, May 7–9.

Satılmış T., (2025) "Lipschitz stability in set-valued differential equations", 9th Graduate Research Symposium, Gebze Technical University, Kocaeli, Turkey, May 26–27.

APPENDICES

Arzelà–Ascoli Theorem: [Used in *Theorem 3.2.1*]

Let X be a compact metric space and $\{f_n\}$ be a uniformly bounded, equicontinuous sequence of real-valued functions on X . Then $\{f_n\}$ has a subsequence that converges uniformly on X to a continuous function f on X .

Proof. Since X is a compact metric space, it is separable. As the sequence is equicontinuous, each f_n belongs to $C(X)$, and thus $\{f_n\}$ has a subsequence that converges pointwise on all of X to a real-valued function f .

For notational convenience, assume that the entire sequence $\{f_n\}$ converges pointwise on X (to avoid repeated reference to the subsequence). Therefore, for each $x \in X$, the sequence $\{f_n(x)\}$ is a Cauchy sequence of real numbers. We use this and the equicontinuity of the sequence to show that $\{f_n\}$ is a Cauchy sequence in $C(X)$.

Let $\epsilon > 0$. By the uniform equicontinuity of $\{f_n\}$ on X , there exists a $\delta > 0$ such that for all $n \in \mathbb{N}$,

$$|f_n(U) - f_n(V)| < \frac{\epsilon}{3} \text{ for all } U, V \in X \text{ such that } \rho(U, V) < \delta.$$

Since X is a compact metric space, it is totally bounded. Therefore, there exists a finite set of points $x_1, x_2, \dots, x_k \in X$ such that X is covered by $\{B(x_i, \delta)\}_{i=1}^k$. For $1 \leq i \leq k$, the sequence $\{f_n(x_i)\}$ is a Cauchy sequence of real numbers (since it converges to $f(x_i)$). So, there exists an index $N \in \mathbb{N}$ such that

$$|f_n(x_i) - f_m(x_i)| < \frac{\epsilon}{3} \text{ for } 1 \leq i \leq k \text{ and for all } n, m \geq N.$$

Since $\{B(x_i, \delta)\}_{i=1}^k$ covers X , for any $x \in X$ there exists an i with $1 \leq i \leq k$ such that $\rho(x, x_i) < \delta$. Therefore, for $n, m \geq N$, by the triangle inequality:

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f_m(x_i)| + |f_m(x_i) - f_m(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Thus, $\{f_n\}$ is uniformly Cauchy. Since $C(X)$ is complete, $\{f_n\}$ converges uniformly on X to some function f .

A uniform limit of a sequence of continuous functions is continuous, so the limit function f is continuous on X , as claimed. $\square$

Volterra Integral Equation:

In mathematics, the Volterra integral equation is a special type of integral equation. These equations are divided into two groups, referred to as the first and the second kind. A linear Volterra equation of the first kind is

$$f(t) = \int_a^t K(t, s)x(s) ds$$

where f is a given function and x is the unknown function to be solved.

A linear Volterra equation of the second kind is

$$x(t) = f(t) + \int_a^t K(t, s)x(s) ds$$

In operator theory and Fredholm theory, the corresponding operators are called Volterra operators. A useful method to solve such equations, the Adomian decomposition method, is due to George Adomian.

A linear Volterra integral equation is a convolution equation if

$$x(t) = f(t) + \int_{t_0}^t K(t-s)x(s) ds$$

The function K in the integral is called the kernel. Such equations can be analyzed and solved using Laplace transform techniques. For a weakly singular kernel of the form

$$K(t, s) = (t^2 - s^2)^{-\alpha} \quad \text{with} \quad 0 < \alpha < 1,$$

a Volterra integral equation of the first kind can conveniently be transformed into a classical Abel integral equation.

Differentiability:

We define the notion of the derivative of a function $F : \mathbb{K}_C(\mathbb{R}^n) \rightarrow \mathbb{K}_C(\mathbb{R}^n)$.

Let $e_k = (e_k^1, e_k^2, \dots, e_k^n)$ be the vector such that $e_k^j = 0$ for $k \neq j$, and $e_k^k = 1$.

Let $F \in C[\mathbb{K}_C(\mathbb{R}^n), \mathbb{K}_C(\mathbb{R}^n)]$, $U_0 \in \mathbb{K}_C(\mathbb{R}^n)$, and let $h > 0$ be sufficiently small.

Then the function F is said to be Hukuhara differentiable at U_0 if there exists

$$D_H F(U_0) \in 2^{\mathbb{K}_C(\mathbb{R}^n)}$$

such that the limits

$$\lim_{h \rightarrow 0^+} \frac{F(U_0 + he_k) - F(U_0)}{h} \quad \text{and} \quad \lim_{h \rightarrow 0^+} \frac{F(U_0) - F(U_0 - he_k)}{h}$$

exist in the topology of $(\mathbb{K}_C(\mathbb{R}^n), D)$ and are equal to $D_H F(U_0)$.

Clearly, implicit in the definition of $D_H F(U_0)$ is the existence of the Hukuhara differences:

$$F(U_0 + he_k) - F(U_0) \quad \text{and} \quad F(U_0) - F(U_0 - he_k).$$

The expression $D_H F(U_0)$ can be treated as a semi-linear map from $\mathbb{K}_C(\mathbb{R}^n)$ into $\mathbb{K}_C(\mathbb{R}^n)$, i.e., $D_H F(U_0)Z \in \mathbb{K}_C(\mathbb{R}^n)$ whenever $Z \in \mathbb{K}_C(\mathbb{R}^n)$.

Also, for any $Z \in 2^{\mathbb{K}_C(\mathbb{R}^n)}$, we have $D_H F(U_0)Z \in 2^{\mathbb{K}_C(\mathbb{R}^n)}$.

Hence,

$$D_H F(U_0) : \mathbb{K}_C(\mathbb{R}^n) \rightarrow 2^{\mathbb{K}_C(\mathbb{R}^n)}$$

for every $U \in \mathbb{K}_C(\mathbb{R}^n)$.

Since the elements of $\mathbb{K}_C(\mathbb{R}^n)$ are compact, convex subsets of $\mathbb{R}^n$, we can represent $D_H F(U_0)$ in the following suitable form:

$$D_H F(U_0) = \left[\frac{F(U_0)}{\partial u_0} : u_0 \in U_0 \subset \mathbb{K}_C(\mathbb{R}^n) \right] \in 2^{\mathbb{K}_C(\mathbb{R}^n)}$$

where

$$\frac{F(U_0)}{\partial u_0} = \left(\frac{F(U_0)}{\partial u_{01}}, \dots, \frac{F(U_0)}{\partial u_{0n}} \right)$$

and each partial derivative $\frac{F(U_0)}{\partial u_{0i}} \in \mathbb{K}_C(\mathbb{R}^n)$ for each i .

Lemma A.1:

Let $f \in C[J \times D, \mathbb{R}^n]$, where D is an open and convex subset of $\mathbb{R}^n$, and suppose that the partial derivative $\frac{\partial f}{\partial x}$ is continuous on $\mathbb{R}_+ \times D$. Then, for all $x_1, x_2 \in D$ and $t \in J$, we have:

$$f(t, x_1) - f(t, x_2) = \int_0^1 \left[\frac{\partial f(t, sx_2 + (1-s)x_1)}{\partial x} \right] ds \cdot (x_1 - x_2)$$

Proof:

Define the function

$$F(s) = f(t, sx_2 + (1-s)x_1), \quad \text{for } 0 \leq s \leq 1.$$

Since D is convex and $x_1, x_2 \in D$, the argument $sx_2 + (1-s)x_1$ lies in D for all $s \in [0, 1]$. Therefore, $F(s)$ is well-defined and differentiable.

By the chain rule, we compute the derivative:

$$\frac{dF}{ds} = \left[\frac{\partial f(t, sx_2 + (1-s)x_1)}{\partial x} \right] \cdot (x_2 - x_1).$$

Then integrating over $s \in [0, 1]$ yields:

$$F(1) - F(0) = \int_0^1 \frac{dF}{ds} ds = \int_0^1 \left[\frac{\partial f(t, sx_2 + (1-s)x_1)}{\partial x} \right] ds \cdot (x_2 - x_1).$$

Note that $F(1) = f(t, x_2)$ and $F(0) = f(t, x_1)$. Thus,

$$f(t, x_2) - f(t, x_1) = \int_0^1 \left[\frac{\partial f(t, sx_2 + (1-s)x_1)}{\partial x} \right] ds \cdot (x_2 - x_1),$$

which gives the desired result (by rearranging terms). □

Corollary:

Let $A(t)$ be a continuous $n \times n$ matrix defined on an interval J , and suppose that every solution $x(t)$ of the system

$$x' = A(t)x$$

is bounded for $t \geq t_0$. Let $U(t)$ denote the fundamental matrix solution of this system with $U(t_0) = I$. Then $U(t)$ is bounded for $t \geq t_0$ if and only if

$$\operatorname{Re} \left[\int_{t_0}^t \operatorname{tr} A(s) ds \right]$$

is bounded from below.

Alekseev's Formula:

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function with continuous partial derivatives with respect to the second variable.

Denote by $x(t, t_0, x_0)$ the unique solution of the initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0,$$

and by $y(t, t_0, x_0)$ the unique solution of the perturbed problem

$$x' = f(t, x) + F(t, x), \quad y(t_0) = x_0.$$

Then the following relation holds:

$$y(t, t_0, x_0) = x(t, t_0, x_0) + \int_{t_0}^t \frac{\partial x(t, s, y(s, t_0, x_0))}{\partial x_0} F(s, y(s, t_0, x_0)) ds.$$

Definition:

A family of functions $F = \{f(u)\}$ defined on some set $E \subset \mathbb{R}^n$ is said to be *equicontinuous* if, for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$, independent of $f \in F$ and $u_1, u_2 \in E$, such that

$$\|f(u_1) - f(u_2)\| < \epsilon \quad \text{whenever} \quad \|u_1 - u_2\| < \delta.$$

Zorn's Lemma: (Used in Theorem 4.2.1)

Let $(P, \leq)$ be a partially ordered set such that every chain (i.e., totally ordered subset) of P has an upper bound in P . Then P contains at least one maximal element.

Note that Zorn's Lemma is equivalent to the Axiom of Choice and is frequently employed in analysis and abstract existence proofs to establish the existence of maximal elements, especially in cases where a constructive method is not available.

Application in This Thesis:

In the proof of Theorem 4.2.1, we consider a chain $\{U_\beta\}_\beta$ in the partially ordered set $(S, \leq)$. By defining a uniquely determined mapping $\mathcal{V}$ that agrees with each U_β on its domain, and by proving that $\mathcal{V} \in S$, it is shown that this chain has an upper bound in S .

Therefore, by applying Zorn's Lemma, we deduce the existence of a maximal element $Z \in S$. This maximal element is fundamental in the construction of a global solution. The proof is complete once it is shown that the corresponding maximal existence interval satisfies $c_Z = \infty$.

Lemma A.2:

Let $f, F \in C[J \times \mathbb{K}_C(\mathbb{R}^n), \mathbb{K}_C(\mathbb{R}^n)]$ and define

$$G(t, r) = \max_{D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), V_0 - U_0] \leq r} D[F(t, V) - \tilde{f}(t, \tilde{U}), \tilde{0}]$$

where $G(t, r) \in C[J \times \mathbb{R}_+, \mathbb{R}_+]$. Assume that $r^*(t, \tau_0, \|V_0 - U_0\|)$ is the maximal solution of the scalar differential equation

$$z' = G(t, z), \quad z(\tau_0) = \|V_0 - U_0\| \quad \text{for } t \geq \tau_0 \in J.$$

Here,

- $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, where $U(t, t_0, U_0)$ is any solution of the set differential equation (4.4),
- $V(t, \tau_0, V_0)$ is the solution of (4.6),
- $\eta = \tau_0 - t_0$,

•and $t \geq \tau_0 \geq 0, t_0 \in \mathbb{R}_+$.

Then for all $t \geq \tau_0 \in J$, the following estimate holds:

$$D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)] \leq r^*(t, \tau_0, D[V_0 - U_0, \tilde{0}]).$$

Lemma A.3:

Let $\mathcal{V}(t, \tilde{U}) \in C[\mathbb{R}_+ \times \mathbb{K}_C(\mathbb{R}^n), \mathbb{R}_+]$ and assume that $\mathcal{V}(t, \tilde{U})$ is locally Lipschitz in $\tilde{U}$.

Assume also that the upper Dini derivative of $\mathcal{V}$ satisfies

$$D_*^+ \mathcal{V}(t, V - \bar{U}) = \limsup_{h \rightarrow 0^+} \frac{1}{h} \left[\mathcal{V}(t + h, V - \bar{U} + h(F(t, V) - \tilde{f}(t, \bar{U}))) - \mathcal{V}(t, V - \bar{U}) \right] \leq G(t, \mathcal{V}(t, V - \bar{U}))$$

for all $(t, \bar{U}), (t, V) \in \mathbb{R}_+ \times \mathbb{K}_C(\mathbb{R}^n)$, where $G(t, z) \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$.

Let $r(t) = r(t, \tau_0, z_0)$ denote the maximal solution of the scalar differential equation

$$z' = G(t, z), \quad z(\tau_0) = z_0 \geq 0, \quad t \geq \tau_0.$$

Assume that $\bar{U}(t) := U(t - \eta, t_0, U_0)$, where $U(t, t_0, U_0)$ is any solution of the set differential equation (4.16), and that $V(t) := V(t, \tau_0, V_0)$ is any solution of the set differential equation (4.6) for $t \geq \tau_0 \geq 0$, with $\eta := \tau_0 - t_0$ and $t_0, \tau_0 \in \mathbb{R}_+$.

If the initial condition satisfies

$$\mathcal{V}(\tau_0, V_0 - U_0) \leq z_0,$$

then the following estimate holds for all $t \geq \tau_0$:

$$\mathcal{V}(t, V(t) - \bar{U}(t)) \leq r(t).$$

Lemma A.4

Let $f \in C[\mathbb{R}_+ \times \mathcal{D}, \mathbb{K}_C(\mathbb{R}^n)]$, where $\mathcal{D} \subset \mathbb{R}^n$ is an open convex set. Suppose that the partial derivative $\frac{\partial f}{\partial U}$ exists and is continuous on $\mathbb{R}_+ \times \mathcal{D}$. Then for all $U_1, U_2 \in \mathcal{D}$ and any fixed $t \in \mathbb{R}_+$, the following identity holds:

$$f(t, U_1) - f(t, U_2) = \int_0^1 \frac{\partial f(t, sU_2 + (1-s)U_1)}{\partial U} ds \cdot (U_2 - U_1).$$

Proof. Define the auxiliary function $K(s) := f(t, sU_2 + (1-s)U_1)$ for $s \in [0, 1]$. Since $\mathcal{D}$ is convex, the point $sU_2 + (1-s)U_1$ lies in $\mathcal{D}$ for all $s \in [0, 1]$. Hence, $K(s)$ is well-defined and differentiable on $[0, 1]$.

Applying the chain rule, we compute the derivative of $K(s)$ with respect to s :

$$\frac{dK}{ds} = \frac{\partial f(t, sU_2 + (1-s)U_1)}{\partial U} \cdot (U_2 - U_1).$$

Integrating both sides over $[0, 1]$ gives:

$$K(1) - K(0) = \int_0^1 \frac{dK}{ds} ds = \int_0^1 \frac{\partial f(t, sU_2 + (1-s)U_1)}{\partial U} \cdot (U_2 - U_1) ds.$$

Note that $K(1) = f(t, U_2)$ and $K(0) = f(t, U_1)$. Thus,

$$f(t, U_2) - f(t, U_1) = \int_0^1 \frac{\partial f(t, sU_2 + (1-s)U_1)}{\partial U} \cdot (U_2 - U_1) ds.$$

Rewriting the left-hand side completes the proof:

$$f(t, U_1) - f(t, U_2) = \int_0^1 \frac{\partial f(t, sU_2 + (1-s)U_1)}{\partial U} ds \cdot (U_2 - U_1).$$

□

Theorem 3.3.1 Let $\{U_n(t)\}$ be a sequence of set-valued functions (multimappings) defined on an interval $J \subset \mathbb{R}$, and assume that the sequence is both equicontinuous and uniformly bounded. Then there exists a subsequence $\{U_{n_k}(t)\}$ that converges uniformly on J to a continuous set-valued function $U(t)$.

Proof. Let each $U_n : J \rightarrow \mathbb{K}_C(\mathbb{R}^n)$ be a set-valued map with compact convex values. Let H denote the Hausdorff metric on $\mathbb{K}_C(\mathbb{R}^n)$. Since $\mathbb{K}_C(\mathbb{R}^n)$ is a complete metric space under H , we proceed using standard compactness arguments.

Uniform boundedness: By assumption, there exists a compact set $K \subset \mathbb{R}^n$ such that for all $n \in \mathbb{N}$ and $t \in J$, we have

$$U_n(t) \subset K.$$

Thus, the image of each U_n lies in a compact subset of $\mathbb{K}_C(\mathbb{R}^n)$, implying relative compactness in the Hausdorff metric.

Equicontinuity: For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $t_1, t_2 \in J$ with $|t_1 - t_2| < \delta$, and for all $n \in \mathbb{N}$,

$$H(U_n(t_1), U_n(t_2)) < \varepsilon.$$

Pointwise relative compactness: Fix any $t \in J$. Since the sequence $\{U_n(t)\} \subset \mathbb{K}_C(\mathbb{R}^n)$ is uniformly bounded and takes values in a compact subset, it is relatively compact in the Hausdorff metric. Thus, for each $t \in J$, one can extract a subsequence $\{U_{n_k}(t)\}$ that converges pointwise on a dense subset $J \cap \mathbb{Q} \subset J$.

Uniform convergence: Using the equicontinuity and density of rationals, this pointwise convergence extends to uniform convergence on all of J . That is,

$$\sup_{t \in J} H(U_{n_k}(t), U(t)) \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

for some $U : J \rightarrow \mathbb{K}_C(\mathbb{R}^n)$.

Continuity of the limit function: Since each U_{n_k} is continuous and the convergence is uniform in the Hausdorff metric, the limit function $U(t)$ is also continuous.

There exists a uniformly convergent subsequence whose limit is a continuous set-valued function. This completes the proof. □