

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

NONUNIFORM FINITE DIFFERENCE TIME DOMAIN METHOD
WITH CHEBYSHEV GRIDS
(CHEBYSHEV FDTD METHOD)

OSMAN SAİD BİŞKİN
A THESIS SUBMITTED FOR THE DEGREE OF
MASTER OF SCIENCE
DEPARTMENT OF ELECTRONICS ENGINEERING

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T.C.
GEBZE TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

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ÜNİFORM OLMAYAN ZAMAN UZAYI
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(CHEBYSHEV ZUSF METODU)

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PROF. DR. SERKAN AKSOY

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SUMMARY

In this thesis, numerical analysis of Nonuniform Finite Difference Time Domain (NU FDTD) method is investigated in detail. A nonuniform numerical dispersion equation (NU NDE) is formulated after extraction of the NU FDTD update equations in three-dimensional Cartesian coordinates.

Two numerical examples of an empty cavity and a partially loaded cavity are considered for the accuracy analyses of a uniform mesh, a NU Chebyshev-Gauss-Lobatto (NU CGL) mesh (This mesh is used, for the first time, in the NU FDTD solution) and a NU Linear mesh. Specially, a single- and multi-mode cavity excitations are considered by using monochromatic and wideband pulse sources for two different Courant-Friedrichs-Lewy (CFL) numbers.

In the numerical accuracy evaluations, signals in time domain, resonance peaks in frequency domain and field distributions in space domain are used for comparisons with an analytical solution. Specially, a semi-analytical solution is used for the partially loaded cavity.

The numerical results for the empty cavity show that the uniform mesh gives the most accurate results. The NU CGL mesh comes second and the NU Linear mesh gives the worst results. However, the numerical results for the partially loaded cavity show that the NU CGL mesh gives the most accurate results. The uniform mesh comes second and the NU Linear mesh gives the worst results. All these numerical results also correspond to the results of the numerical dispersion equation.

The main conclusion of this thesis is that additional researches are necessary for better understanding of the NU FDTD numerical solution of the partially loaded cavity problem for different meshes since the effect of the material inhomogeneity (different permittivities) on the solution accuracy has not been known yet.

Key Words: Numerical Dispersion Equation (NDE), Chebyshev-Gauss-Lobatto (CGL) Points, Linear Mesh, PEC, Partially Dielectric Filled Cavity, Resonance Frequency.

ÖZET

Bu tezde, Üniform Olmayan Zaman Uzayı Sonlu Farklar (ÜO ZUSF) yönteminin sayısal analizi detaylı olarak incelenmiştir. Üç boyutlu Kartezyen koordinatlarda ÜO ZUSF güncelleme denklemlerinin çıkarılmasından sonra, üniform olmayan sayısal dispersiyon denklemi (ÜO SDD) elde edilmiştir.

Üniform ızgara, ÜO Chebyshev-Gauss-Lobatto (ÜO CGL) ızgarası (Bu ızgara ÜO ZUSF çözümünde ilk kez kullanılmıştır) ve ÜO Lineer ızgaranın doğruluk analizleri için boş kavite ve kısmen dolu kavite için incelemeler yapılmıştır. Özellikle, iki farklı CFL katsayısı için monokromatik ve geniş bant darbe kaynakları kullanılarak kaviteler tek-modlu ve çok-modlu olarak uyarılmıştır.

Sayısal doğruluk değerlendirmeleri için analitik çözümle karşılaştırma amaçlı zaman uzayında sinyaller, frekans uzayında rezonans tepe noktaları ve konumsal alan dağılımları kullanılmıştır. Özellikle, kısmen dolu kavite için yarı analitik çözüm kullanılmıştır.

Boş kavitenin sayısal sonuçları, üniform ızgaranın en doğru sonuçları verdiğini göstermektedir. ÜO CGL ızgarası ikinci sırada gelirken, ÜO Lineer ızgarası en kötü sonuçları vermektedir. Bununla birlikte, kısmen dolu kavitenin sayısal sonuçları, ÜO CGL ızgarasının en doğru sonucu verdiğini göstermektedir. Üniform ızgara ikinci sırada gelirken, ÜO Lineer ızgara en kötü sonucu vermektedir. Tüm bu sayısal sonuçlar, sayısal dispersiyon denkleminin sonuçlarını direkt doğrulamaktadır.

Bu tezin ana sonucu, farklı ızgaralar için kısmen dolu kavite probleminde ÜO ZUSF sayısal çözümünün daha iyi anlaşılması için ek araştırmalara ihtiyaç duyulduğunu ortaya çıkartmasıdır. Bunun ana sebebi homojen olmayan malzemelerin (farklı geçirgenliklerin) ZUSF çözüm doğruluğu üzerindeki etkisinin henüz bilinmemesinden kaynaklanmaktadır.

Anahtar Kelimeler: Sayısal Dispersiyon Denklemi (SDD), Chebyshev-Gauss-Lobatto (CGL) Noktaları, Lineer Izgara, MEİ, Kısmen Dielektrik Dolu Kavite, Rezonans Frekansı.

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LIST of ABBREVIATIONS and ACRONYMS

<u>Abbreviations</u>	<u>Explanations</u>
<u>and Acronyms</u>	
∇	: Nabla operator
j	: Imaginary number ($\sqrt{-1}$)
H	: Magnetic field vector (A/m)
E	: Electric field vector (V/m)
J	: Current density vector (A/m ²)
ρ	: Charge density (C/m ³)
ε	: Dielectric permittivity (F/m)
ε_0	: Dielectric permittivity of free space (F/m)
μ	: Magnetic permeability (H/m)
μ_0	: Magnetic permeability of free space (H/m)
σ	: Conductivity (S)
c	: Velocity of light (m/s)
ω	: Angular frequency (rad/s)
f	: Frequency (Hz)
λ	: Wavelength (m)
k	: Wave number (rad/m)
k_N	: Numerical wave number (rad/m)
t	: Time (sec)
a	: Length on the direction of x (m)
b	: Length on the direction of y (m)
h	: Length on the direction of z (m)
N_x	: Number of cells on the direction of x
N_y	: Number of cells on the direction of y
N_z	: Number of cells on the direction of z
Δx	: Unit cell size on the direction of x (m)
Δy	: Unit cell size on the direction of y (m)
Δz	: Unit cell size on the direction of z (m)
n	: Time step number

Δt	: Unit time step (s)
m	: Meter
sec	: Second
Hz	: Hertz
ABC	: Absorbing Boundary Condition
CFL	: Courant-Friedrichs-Levy
FDTD	: Finite Difference Time Domain
FEM	: Finite Element Method
NDE	: Numerical Dispersion Equation
NU FDTD	: Nonuniform Finite Difference Time Domain
PCS	: Personal Communications System
PDFC	: Partially Dielectric Filled Cavity
PML	: Perfectly Matched Layer
U FDTD	: Uniform Finite Difference Time Domain

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1. INTRODUCTION

1.1. Overview

Finite Difference Time Domain (FDTD) is an iterative numerical method based on finite differences of field derivatives. Its aim is to provide an efficient solution for complex electromagnetic problems.

The operational principle of the FDTD method is depicted in Figure 1.1. Accordingly, first, initial conditions are defined. Then, boundary conditions are applied in an iterative algorithm (n) in which electric and magnetic fields are updated for a leap-frog manner. In addition to the fields, absorbing boundary conditions (iteratively) are updated if they are necessary in the computational problem. The algorithm is run until a predefined time steps (N). After the time iteration is over, full time domain data is calculated for evaluations.

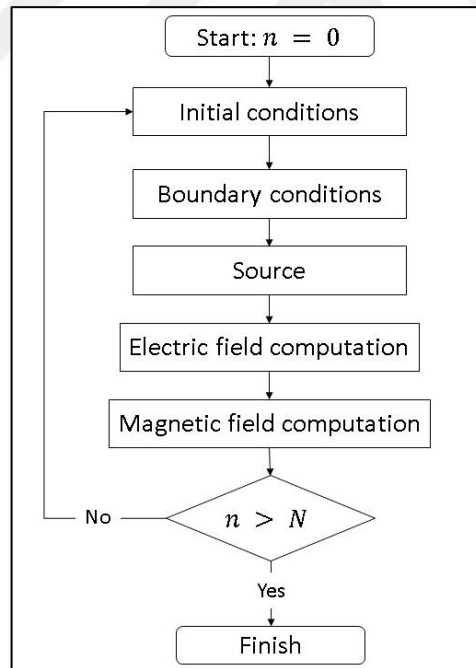


Figure 1.1: The operational principle of the FDTD method.

In this thesis, two different FDTD methods based on their mesh types are considered. These are U FDTD (Uniform Finite Difference Time Domain) method and NU FDTD (Nonuniform Finite Difference Time Domain) method. The U FDTD

method uses uniform (equidistant or constant) mesh and time intervals both in spatial domain and time domain. Therefore, its update equations are iterated uniformly.

The NU FDTD method uses nonuniform (non-constant) mesh interval in the spatial domain. In fact, the NU FDTD method principally differs from the U FDTD since its update equations are not specialized by using uniform unit spatial step [Perala et al., 1992], [Sangary, 1995].

1.2. Historical Development of The NU FDTD Method

In computationally large-scale problems, it is not feasible to use the classical U FDTD method due to unacceptable memory requirements and very long-time iterations. To overcome this problem, the NU FDTD method is proposed as an alternative pathway of using the NU mesh in the spatial domain. This method satisfying the similar stability limit can supply acceptable computational sources since the mesh number is considerably reduced for solution of the complex problems. Additionally, problems containing interfaces of material discontinuity or sharp changes require fine meshing. Simulating all domains with the fine meshes is extremely burdensome and computationally expensive.

To the best of author's knowledge, first publications of the NU meshes for finite differences are given in [Tikhonov and Samarskii, 1962], [Tikhonov and Samarskii, 1963] and [Brown, 1962] before [Yee, 1966]. In fact, the authors of these studies just declared the possibility of the NU finite difference approximations. However, no numerical calculation is performed until some years later.

The error analysis of the NU meshes are the most important step in order to determine their effectiveness. In terms of the error analysis, it is understood that nonuniform (nonsquare) grids with a highest density in the region of maximum change (such as boundaries) produce minimum overall error in the finite difference solutions [Crowder and Dalton, 1971].

The NU finite difference method is often designed for linear increasing or decreasing meshes. In other words, the linear mesh has a constant mesh ratio which increase or decrease at a constant rate. This mesh has a real-positive valued grading factor. Therefore, it is easy to implement for complex structures [Choi and Hoefer, 1987].

A detailed convergence analysis of the NU FDTD method is performed by investigating the effect of the mesh nonuniformity on the accuracy of the scheme [Monk and Süli, 1994]. It is proved that the local truncation error is only first order accurate at the NU mesh points [Mezzanotte and Roselli, 1995], [Heinrich et al., 1996]. Fortunately, finite difference method (similar to finite volume scheme) is a local algorithm in which the nonuniformity does not affect the accuracy at entire domain (or in global level). In other words, it is proved that the NU FDTD method is only first order accurate locally.

- Numerical Dispersion Equation for the NU FDTD Method

It is not trustable to a new or a modified technique's accuracy without analyzing its NDE. For this aim, firstly, some NU mesh NDE is extracted in [Svigelj and Mittra, 1995]. But, their approaches are legitimately problematic since the extracted NDE is valid only for two grids which can almost simulate no geometries.

In the literature, for the first time, the NDE for the NU meshes is completely extracted in [Christ et al., 2002]. This paper also mentions that the drawback of the NU FDTD meshes is the local loss of the second order accuracy in the central difference calculations. Although the algorithm retains its global second order accuracy, the local first order errors do influence the simulation outcome, depending on the cell size and the grading ratio. Therefore, using the NU NDE, the phase velocity corrections are proposed but the numerical examples (uniform or linear mesh) are given not well enough for the validation purposes. A new concept of the numerical phase velocity for the NU mesh is proposed as an effective phase velocity equivalence for the NU meshes since there is no single-phase velocity value as in the uniform cases.

- Absorbing Boundary Conditions (ABCs) in the NU FDTD Method

The choice of absorbing boundary conditions plays a critical role on the accuracy of the NU FDTD method. The NU finite difference is studied with MUR's ABC [Daniel and Railton, 1991]. It is claimed, for the first order MUR's ABC simulations that the NU finite differences reduce the computation time, drastically [Daniel and Railton, 1991], [Paul et al., 1991], [Brankovic et al., 1991].

The effect of the different boundary conditions on the accuracy of the NU FDTD method is also investigated. For this aim, a Perfectly Matched Layer (PML) ABC (evaluated as exact ABC) in the NU FDTD method is applied to solve a narrow iris

waveguide problem. The authors concluded that if the “exact” boundary condition is not applied in the simulation, the NU FDTD is shown to be degraded to the first order accuracy [Navarro et al., 1996].

Application of the second order MUR with PML comparison is also performed for the NU FDTD method. They are validated with the exact solutions. Therefore, the PML and MUR ABC are efficiently applied in the NU FDTD method [Liu et al., 1996].

- Improvement Studies of the NU FDTD Method

Some researchers studied adaptive or continuously variable mesh sizes. The variable grading is utilized with improving the accuracy from the first order to the second order by an analytical cancellation [Xiao et al., 1992].

A study using a continuously variable mesh is tried to eliminate the first order terms of the NU finite differences by using a phase shifting [Xiao and Vahldheck, 1995]. It is also proved that a wavelet-based adaptive gridding algorithm gives highly accurate results [Sun and Tam, 2001].

A new second order accurate NU FDTD method is proposed by introducing a three-point differencing for an abrupt changing mesh or a nonuniform mesh. The comparisons between this method and the first order accurate NU FDTD method show that the new method is more accurate but it requires more computational memory and time. Although its stability analysis has very complex formulation, it has a conventional stability criterion. Its numerical dispersion equation is not extracted. Therefore, its accuracy is only performed over numerical experiments [Falconer, 1997]. The treatment of the first order error is achieved and the second order accuracy is provided by utilizing three neighboring fields instead of the standard two neighboring fields used in the central finite differences [Falconer and Tripathi, 1997]. However, the problem of extra computational complexity must be taken into account. The other approach to treat the first order concept is attempted by using an interpolation scheme for the calculation of magnetic fields [Jiang and Arai, 1998], [Tupynamba and Omar, 1998], [Kermani and Ramahi, 2006]. Additionally, the NU FDTD method is modified where an error averaging procedure is attempted to cancel out the nonphysical reflection from the NU meshes [Yu and Mittra, 1999].

- Different Applications of the NU FDTD Method

A rectangular varying NU grid spacing is used for seismic problems [Moczo, 1989].

Two different antennas are introduced for the NU FDTD simulation example as an application in Personal Communications System (PCS). It is shown that the antennas performance can be accurately modeled by the nonuniform mesh with a PML boundary condition [Sangary, 1995]. Furthermore, step-index, multimode, graded-index multimode and single-mode step index structures are analyzed by using the NU FDTD method [Shen et al., 1998]. The errors, the computational memory, and the computational time of the NU mesh are also compared with a subgridding case in [Wiart et al., 1999]. The NU FDTD method is also used for the analysis of cellular phones [Bernardi et al., 2001]. Some expertized validations of the NU FDTD method are performed by IE3D (MoM), HFSS (FEM) and EMPIRE (FDTD) in [Yu et al., 2002]. A sandwiched-type coplanar waveguide with adjacent vias in PCBs, a central-metallic-strip for wireless communications, mode-locked vertical-extended-cavity semiconductor lasers, an elevator with one mobile phone user and a mobile phone, a wideband hybrid antenna, an aeroplane, a complex ring structure and a dielectric cube buried in infinite ground structures of applications are also analyzed, respectively [Chen et al., 2002], [Fu and Chen, 2002], [Kreuter and Witzigmann, 2008], [Simba et al., 2009], [Berens et al., 2016], [Zhang et al., 2017].

- The Other Types of the NU Meshes

Functional meshes can be a valuable alternative due to their simplicity. The logarithmic functions can also be used for the NU mesh generation [Mufti, 1978], [Gao and Gandhi, 1992]. Moreover, the functional meshes allow a further reduction of the minimum cell and applicability to higher frequencies [Gao and Gandhi, 1992].

The use of the logarithmically expanding grid FDTD algorithm is developed in [Tinniswood et al., 1998], [Tinniswood et al., 1999].

On the other hand, the arbitrary meshes are studied in the literature [Xiao et al., 1993], [Monk and Süli, 1994], [Shen et al., 1999], [Namiki and Ito, 2001].

The trigonometric meshes are also used such as Chebyshev-Gauss-Lobatto (CGL) collocation points [Gamet et al., 1999]. Lately, similar to the CGL meshes with a minimum mesh interval and a multiplication coefficient, more flexibility is obtained. This can create some trouble that the analytical extreme point location can be moved

with it. This may cause in undesired results. Therefore, two fundamental rules (restrictions) are proposed for the mesh type. One is the derivative of the mesh function cannot be larger than two, second is the summation of the mesh intervals must be equal to the total length of the problem space. Moreover, it is analytically found that the total mesh number cannot be an arbitrary number. However, there is no numerical error analyses are given [Jiménez-Mejía and Herrera-Murcia, 2015].

A variable mesh finite difference method is also implemented with investigation of its convergence properties. This new method is applied for solution of some types of differential equations [Chong, 1978], [Agrawal and Peckover, 1980].

Other different type of meshes are also studied but their detailed accuracy analyses are still remained absent [Sundqvist and Veronis, 1970], [de Rivas, 1972], [Manteuffel and White, 1986], [Armenta and Sarris, 2008].

1.3. The Purpose and the Scope of This Thesis

In the literature, there are no sufficient and full numerical detailed analyses of the NU FDTD method. Therefore, the numerical dispersion behavior of different mesh types is not well known even though some accuracy analyses are attempted. To eliminate this lack of knowledge in the literature, the NU FDTD method is detailly analyzed by extracting its numerical dispersion and validating it with the two numerical examples.



2. UNIFORM FINITE DIFFERENCES

2.1. One-Dimensional Finite Differences

There are three different approximations of finite differences techniques for the calculation of numerical derivatives such as forward, backward and central differences as shown in Figure 2.1.

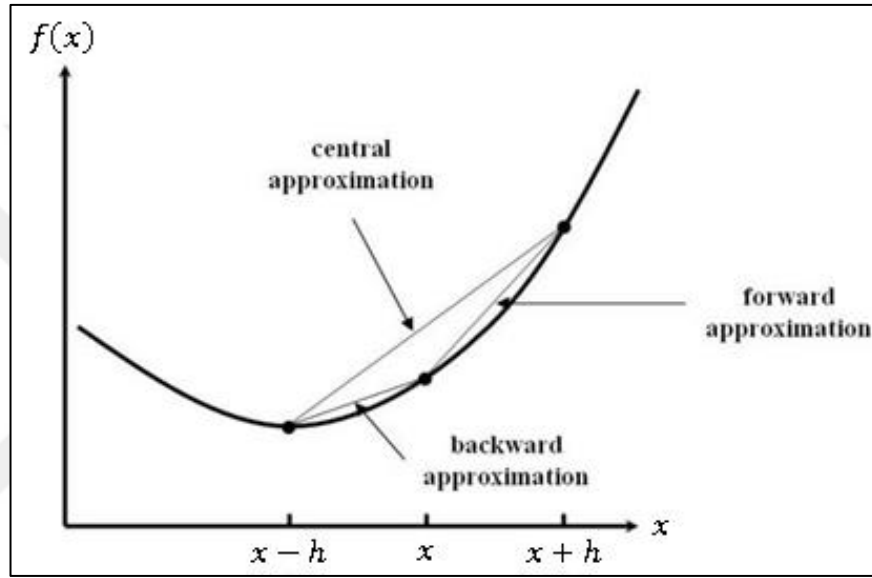


Figure 2.1: Types of the finite difference derivatives.

To apply these techniques to a differentiable function, the function should be initially discretized. Therefore, a one-dimensional $f(x)$ function is split by Δx_i unit cells with i number of pieces as

$$x_i = i\Delta x \Rightarrow f(x) = f(x_i) = f(i\Delta x). \quad (2.1)$$

Then, Taylor series expansion of $f(x)$ around $x_i = ih$ ($\Delta x = h$), the central difference derivation is found as

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{h^2}{3!} f'''(x) + \dots \quad (2.2)$$

where $f'(x)$ with a second order error $O(h^2)$ becomes [Aksoy, 2022]

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2). \quad (2.3)$$

2.2. Application to Field Components

As an example, let us consider the update equation for the magnetic field. In this way, the forward difference is

$$H_y(x+h) = H_y(x) + hH'_y(x) + \frac{h^2}{2}H''_y(x) + \frac{h^3}{6}H'''_y(x) + \frac{h^4}{24}H''''_y(x) + \dots \quad (2.4)$$

and the backward difference is

$$H_y(x-h) = H_y(x) - hH'_y(x) + \frac{h^2}{2}H''_y(x) - \frac{h^3}{6}H'''_y(x) + \frac{h^4}{24}H''''_y(x) + \dots \quad (2.5)$$

Subtracting the second equation from the first equation

$$H_y(x+h) - H_y(x-h) = 2hH'_y(x) + \frac{h^3}{3}H'''_y(x) \quad (2.6)$$

and rearranging it as

$$\frac{H_y(x+h) - H_y(x-h)}{2h} = H'_y + \frac{h^2}{6}H'''_y(x) \quad (2.7)$$

leading to

$$\left. \frac{\partial H_y}{\partial x} \right|_i = H'_y(x) + \underbrace{\frac{h^2}{6}}_{O(h^2)} H'''_y(x). \quad (2.8)$$

Then, considering the update equation for the electric field, the forward difference is

$$E_z(x+h) = E_z(x) + \frac{h}{4}E'_z(x) + \frac{h^2}{32}E''_z(x) + \frac{h^3}{384}E'''_z(x) + \frac{h^4}{6144}E''''_z(x) + \dots \quad (2.9)$$

and the backward difference is

$$E_z(x-h) = E_z(x) - \frac{h}{4}E'_z(x) + \frac{h^2}{32}E''_z(x) - \frac{h^3}{384}E'''_z(x) + \frac{h^4}{6144}E''''_z(x) + \dots \quad (2.10)$$

Subtracting the second from the first equation

$$E_z(x+h) - E_z(x-h) = 2hE'_z(x) + \frac{h^3}{384}E'''_z(x) \quad (2.11)$$

leading to

$$\left. \frac{\partial E_z}{\partial x} \right|_{i+\frac{1}{2}} = \frac{E_z(x+h) - E_z(x-h)}{2h} = E'_z(x) + \underbrace{\frac{h^2}{384}}_{O(h^2)} E'''_z(x). \quad (2.12)$$

Here, in the case of the central differences, it can be concluded that the first order numerical derivatives of the magnetic field and the electric field are second order accurate.

3. UNIFORM FINITE DIFFERENCES TIME DOMAIN METHOD

In a simple lossless medium, Maxwell's equations are given

$$\begin{aligned}\nabla \times \mathbf{E}(\mathbf{r}, t) &= -\mu \frac{\partial \mathbf{H}(\mathbf{r}, t)}{\partial t} \\ \nabla \times \mathbf{H}(\mathbf{r}, t) &= \varepsilon \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} + \mathbf{J}(\mathbf{r}, t) \\ \nabla \cdot \mathbf{E}(\mathbf{r}, t) &= \frac{\rho(\mathbf{r}, t)}{\varepsilon} \\ \nabla \cdot \mathbf{H}(\mathbf{r}, t) &= 0\end{aligned}\tag{3.1}$$

where ∇ is Nabla operator, $\mathbf{E}(\mathbf{r}, t)$ [V/m], $\mathbf{H}(\mathbf{r}, t)$ [A/m], $\mathbf{J}(\mathbf{r}, t)$ [A/m²] and $\rho(\mathbf{r}, t)$ [C/m³] are electric field strength vector, magnetic field strength vector, current density vector and volumetric charge density, respectively. ε [F/m] and μ [H/m] are permittivity and permeability of the medium, respectively.

To apply the Finite Difference Time Domain (FDTD) method for Maxwell's equations, only first two Maxwell's equations are used since the last two Maxwell's equations are not solved. They are simply initial conditions and will be automatically introduced in the FDTD algorithm by locations of the electric and magnetic fields in the FDTD unit cell [Taflove and Hagness, 2005].

The FDTD method is based on numerical equivalence of spatial and temporal derivatives in Maxwell's equations via central, backward, and forward difference approximation technique. Mostly, the central difference technique is preferred due to its higher accuracy [Yee, 1966]. The spatial and the temporal derivatives of Maxwell's equation are solved iteratively in the time domain using the FDTD method. In the classical FDTD method, the numerical derivatives are extracted from a Taylor series expansion with the second order accuracy.

3.1. Update Equations

The FDTD algorithm requires the electric and the magnetic fields to be updated at staggered half-time steps. The explicit update algorithm means that the new values of the field components depend only on the values at previous time steps. Thus, the fields are discretized in the temporal and the spatial domains over each of the field components such as $f(x, y, z, t) = f(i\Delta x, j\Delta y, k\Delta z, n\Delta t) = f(i, j, k, n) = f_{i,j,k}^n$.

In the Cartesian coordinates, the first and the second Maxwell's equations becomes

$$\begin{aligned}\frac{\partial E_x}{\partial t} &= \frac{1}{\varepsilon} \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) & \frac{\partial H_x}{\partial t} &= \frac{1}{\mu} \left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \right) \\ \frac{\partial E_y}{\partial t} &= \frac{1}{\varepsilon} \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) & \frac{\partial H_y}{\partial t} &= \frac{1}{\mu} \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \\ \frac{\partial E_z}{\partial t} &= \frac{1}{\varepsilon} \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) & \frac{\partial H_z}{\partial t} &= \frac{1}{\mu} \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right).\end{aligned}\tag{3.2}$$

The discretized field components in the space and the time are located in the FDTD unit cell as shown in Figure 3.1 (Yee cell). The cell locations are expressed with (i, j, k) terms along (x, y, z) directions, respectively. In this unit cell, the spatial and the temporal discretization (n) of the electric and the magnetic field components are shifted relative to each other. In the time discretization, the magnetic and the electric field components are placed on the integer and half values of the time steps, respectively.

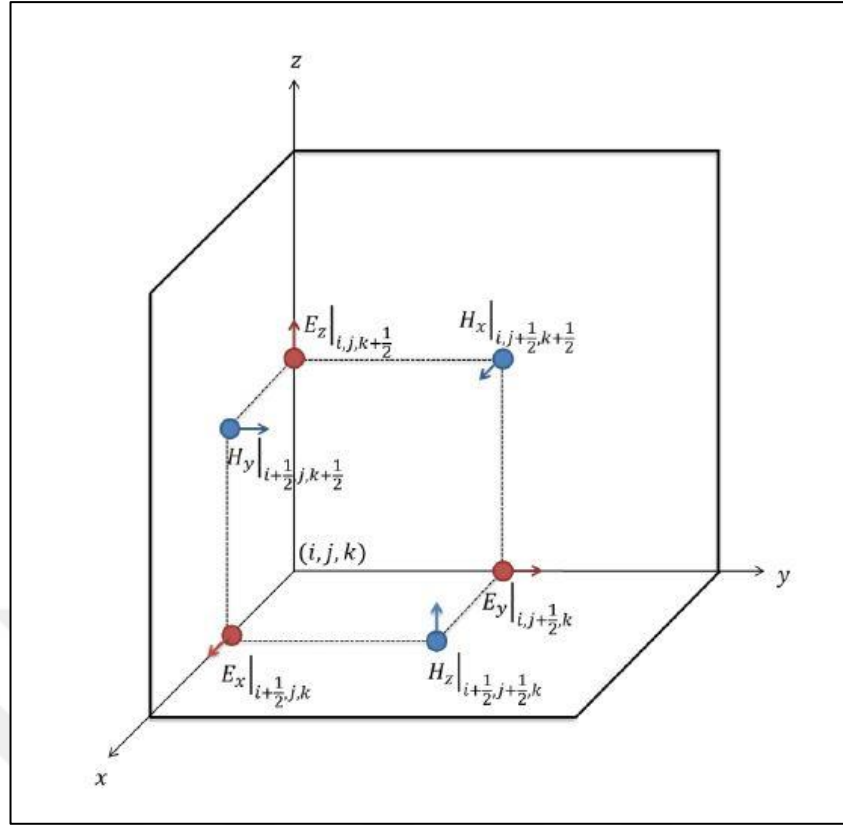


Figure 3.1: The locations of the field components in the unit Yee cell.

Using the central differences for the temporal domain and the spatial domain, the update equations of the magnetic field components are [Taflove and Hagness, 2005]

$$\begin{aligned}
 H_x|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} &= H_x|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} \\
 &+ \frac{\Delta t}{\mu} \left[\frac{E_y|_{i,j+\frac{1}{2},k+1}^n - E_y|_{i,j+\frac{1}{2},k}^n}{\Delta z} - \frac{E_z|_{i,j+1,k+\frac{1}{2}}^n - E_z|_{i,j,k+\frac{1}{2}}^n}{\Delta y} \right] \quad (3.3)
 \end{aligned}$$

$$\begin{aligned}
 H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} &= H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n-\frac{1}{2}} \\
 &+ \frac{\Delta t}{\mu} \left[\frac{E_x|_{i+\frac{1}{2},j,k+1}^n - E_x|_{i+\frac{1}{2},j,k}^n}{\Delta z} - \frac{E_z|_{i+1,j,k+\frac{1}{2}}^n - E_z|_{i,j,k+\frac{1}{2}}^n}{\Delta x} \right] \quad (3.4)
 \end{aligned}$$

$$H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} = H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n-\frac{1}{2}} + \frac{\Delta t}{\mu} \left[\frac{E_y|_{i+1,j+\frac{1}{2},k}^n - E_y|_{i,j+\frac{1}{2},k}^n}{\Delta x} - \frac{E_x|_{i+\frac{1}{2},j+1,k}^n - E_x|_{i+\frac{1}{2},j,k}^n}{\Delta y} \right] \quad (3.5)$$

and the update equations for the electric field components are

$$E_x|_{i+\frac{1}{2},j,k}^{n+1} = E_x|_{i+\frac{1}{2},j,k}^n + \frac{\Delta t}{\varepsilon} \left[\frac{H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - H_z|_{i+\frac{1}{2},j-\frac{1}{2},k}^{n+\frac{1}{2}}}{\Delta y} - \frac{H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - H_y|_{i+\frac{1}{2},j,k-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta z} \right] \quad (3.6)$$

$$E_y|_{i,j+\frac{1}{2},k}^{n+1} = E_y|_{i,j+\frac{1}{2},k}^n + \frac{\Delta t}{\varepsilon} \left[\frac{H_x|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - H_x|_{i,j+\frac{1}{2},k-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta z} - \frac{H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - H_z|_{i-\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}}}{\Delta x} \right] \quad (3.7)$$

$$E_z|_{i,j,k+\frac{1}{2}}^{n+1} = E_z|_{i,j,k+\frac{1}{2}}^n + \frac{\Delta t}{\varepsilon} \left[\frac{H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - H_y|_{i-\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta x} - \frac{H_x|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - H_x|_{i,j-\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta y} \right] \quad (3.8)$$

3.2. Numerical Dispersion Equation

Dispersion is a phenomenon that the phase velocity of the electromagnetic wave is frequency dependent. On the other hand, numerical dispersion is a dispersion caused by the meshing in the FDTD method. The numerical dispersion depends on

- Wavelength,
- Direction of wave propagation in mesh,

- Sort of mesh and discretization.

In this manner, the numerical dispersion can cause some undesired situations:

- Non-physical results such as fake refractions,
- Insufficient accuracy for phase canceling in multiple scatterings,
- Numerical anisotropy.

To analyze the effect of the numerical dispersion, the numerical dispersion relation for the three-dimensional (3D) FDTD in the Cartesian coordinates is given

$$\frac{1}{c^2 \Delta t^2} \sin^2 \left(\frac{\omega \Delta t}{2} \right) = \frac{1}{\Delta x^2} \sin^2 \left(\frac{k_{Nx} \Delta x}{2} \right) + \frac{1}{\Delta y^2} \sin^2 \left(\frac{k_{Ny} \Delta y}{2} \right) + \frac{1}{\Delta z^2} \sin^2 \left(\frac{k_{Nz} \Delta z}{2} \right) \quad (3.9)$$

where k_{Nx} , k_{Ny} and k_{Nz} show the numerical wavenumbers on the directions of x , y and z , respectively. When $k^2 = k_{Nx}^2 + k_{Ny}^2 + k_{Nz}^2$, a dispersionless solution is ensured in which $k = \omega/c$ is an analytical wavenumber. Therefore, to reduce the numerical dispersion error, Δx , Δy and Δz should be chosen as small as possible [Taflove and Hagness, 2005].

3.3. Numerical Stability

All numerical solutions must be stable to not cause unstable/wrong solutions. In order to maintain the stability over the spatial and temporal discretization, the FDTD method must satisfy a CFL stability condition. For the stable FDTD solution in 3D Cartesian coordinates, the unit time step (Δt) must satisfy the following condition

$$\Delta t \leq \frac{1}{c \sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}}} \quad (3.10)$$

where c is velocity of the light [Chu et al., 1991].

3.4. PEC Boundary Condition

Many electromagnetic problems are generally enclosed types. Therefore, the FDTD method must run in the finite spaces. To model these problems with the FDTD method, the finite problem space can be terminated with fully reflecting boundaries. This boundary condition is named as Perfect Electric Conductor (PEC) that it is easy for programming and leads to no heavy computational burden. In this thesis, a PEC type boundary condition is used for modelling the resonator walls.



4. NONUNIFORM FINITE DIFFERENCES

Nonuniform (NU) finite differences for the numerical derivatives can be formulated by using a series expansion. In general, only spatial domain differences are nonuniformly implemented where time differences are already uniform. In this chapter, the one-dimensional spatial (x) numerical NU derivatives of the two field components at different positions as $i\Delta x$ and $(i + 1/2)\Delta x$ are formulated since we want to give a preliminary study for the extraction of the full NU update equations given in next section.

4.1. General First Order Derivate

A one-dimensional NU finite difference is defined over the mesh intervals

$$x_i = \sum_{k=0}^i \Delta x_k \quad (4.1)$$

where Δx_k is a NU spatial interval. For simplicity, let us define two new variables

$$h_+ = \Delta x_{i+1} \quad ; \quad h_- = \Delta x_{i-1} \quad (4.2)$$

which represent the forward and the backward NU spatial intervals. Therefore, the forward difference is

$$f(x + h_+) = f(x) + h_+ f'(x) + \frac{h_+^2}{2} f''(x) + \frac{h_+^3}{6} f'''(x) + \dots \quad (4.3)$$

and the backward differences

$$f(x - h_-) = f(x) - h_- f'(x) + \frac{h_-^2}{2} f''(x) - \frac{h_-^3}{6} f'''(x) + \dots \quad (4.4)$$

Now let us calculate of the first order derivative. For this aim, the forward and the backward equations are subtracted

$$f(x + h_+) - f(x - h_-) = h_+ f'(x) + h_- f'(x) + \frac{h_+^2}{2} f''(x) - \frac{h_-^2}{2} f''(x) \quad (4.5)$$

where bracketing the first and the second order derivatives

$$f(x + h_+) - f(x - h_-) = (h_+ + h_-) f'(x) + \frac{h_+^2 - h_-^2}{2} f''(x) \quad (4.6)$$

and taking the first order derivative to the left-hand side, the equation becomes

$$(h_+ + h_-) f'(x) = f(x + h_+) - f(x - h_-) - \frac{h_+^2 - h_-^2}{2} f''(x). \quad (4.7)$$

Dividing both side of the equation by the $(h_+ + h_-)$ term and simplifying for the last term, the final result is

$$f'(x) = \frac{f(x + h_+) - f(x - h_-)}{h_+ + h_-} - \frac{(h_+ - h_-)(h_+ + h_-)}{2(h_+ + h_-)} f''(x) \quad (4.8)$$

where it is shown that the first order derivative of the NU mesh is first order accurate $O(h)$ as

$$f'(x) = \frac{f(x + h_+) - f(x - h_-)}{h_+ + h_-} - \underbrace{\frac{h_+ - h_-}{2} f''(x)}_{O(h)} \quad (4.9)$$

Specially for the uniform grid $(h_+ = h_- = h)$ it is known that the accuracy is second order $O(h^2)$ as

$$f'(x) = \frac{f(x + h) - f(x - h)}{2h} + \underbrace{\frac{h^2}{6} f'''(x)}_{O(h^2)} \quad (4.10)$$

4.2. Application to Spatial Derivatives of Field Components

The one-dimensional NU derivatives of $\partial H_y / \partial x$ at $i\Delta x$ and $\partial E_z / \partial x$ at $(i + 1/2)\Delta x$ are given as examples. The two different spatial positions are applied since the field components are located at different grid points in Yee cell for the full NU update equations.

- Calculation of $\partial H_y / \partial x$ Derivative

Let us first calculate the forward difference for H_y at $x + \Delta x_{i+\frac{1}{2}}$

$$\begin{aligned} H_y\left(x + \Delta x_{i+\frac{1}{2}}\right) &= H_y(x) + \Delta x_{i+\frac{1}{2}} H'_y(x) + \frac{\Delta x_{i+\frac{1}{2}}^2}{2} H''_y(x) \\ &\quad + \frac{\Delta x_{i+\frac{1}{2}}^3}{6} H'''_y(x) + \frac{\Delta x_{i+\frac{1}{2}}^4}{24} H''''_y(x) + \dots \end{aligned} \quad (4.11)$$

and the backward difference for H_y at $x - \Delta x_{i-\frac{1}{2}}$ is

$$\begin{aligned} H_y\left(x - \Delta x_{i-\frac{1}{2}}\right) &= H_y(x) - \Delta x_{i-\frac{1}{2}} H'_y(x) + \frac{\Delta x_{i-\frac{1}{2}}^2}{2} H''_y(x) \\ &\quad - \frac{\Delta x_{i-\frac{1}{2}}^3}{6} H'''_y(x) + \frac{\Delta x_{i-\frac{1}{2}}^4}{24} H''''_y(x) + \dots \end{aligned} \quad (4.12)$$

subtracting the second equation from the first equation

$$\begin{aligned} H_y\left(x + \Delta x_{i+\frac{1}{2}}\right) - H_y\left(x - \Delta x_{i-\frac{1}{2}}\right) &= \left(\Delta x_{i+\frac{1}{2}} + \Delta x_{i-\frac{1}{2}}\right) H'_y(x) \\ &\quad + \frac{\Delta x_{i+\frac{1}{2}}^2 - \Delta x_{i-\frac{1}{2}}^2}{2} H''_y(x) + \frac{\Delta x_{i+\frac{1}{2}}^3 + \Delta x_{i-\frac{1}{2}}^3}{6} H'''_y(x) \\ &\quad + \frac{\Delta x_{i+\frac{1}{2}}^4 - \Delta x_{i-\frac{1}{2}}^4}{24} H''''_y(x) \end{aligned} \quad (4.13)$$

with the rearrangement as

$$\begin{aligned}
\frac{H_y\left(x + \Delta x_{i+\frac{1}{2}}\right) - H_y\left(x - \Delta x_{i+\frac{1}{2}}\right)}{\Delta x_{i+\frac{1}{2}} + \Delta x_{i-\frac{1}{2}}} &= H'_y(x) + \frac{\Delta x_{i+\frac{1}{2}}^2 - \Delta x_{i-\frac{1}{2}}^2}{2\left(\Delta x_{i+\frac{1}{2}} + \Delta x_{i-\frac{1}{2}}\right)} H''_y(x) \\
&+ \frac{\Delta x_{i+\frac{1}{2}}^3 + \Delta x_{i-\frac{1}{2}}^3}{6\left(\Delta x_{i+\frac{1}{2}} + \Delta x_{i-\frac{1}{2}}\right)} H'''_y(x) + \frac{\Delta x_{i+\frac{1}{2}}^4 - \Delta x_{i-\frac{1}{2}}^4}{24\left(\Delta x_{i+\frac{1}{2}} + \Delta x_{i-\frac{1}{2}}\right)} H''''_y(x)
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
\frac{H_y\left(x + \Delta x_{i+\frac{1}{2}}\right) - H_y\left(x - \Delta x_{i+\frac{1}{2}}\right)}{\Delta x_{i+\frac{1}{2}} + \Delta x_{i-\frac{1}{2}}} &= \\
&H'_y(x) + \frac{\left(\Delta x_{i+\frac{1}{2}} + \Delta x_{i-\frac{1}{2}}\right)\left(\Delta x_{i+\frac{1}{2}} - \Delta x_{i-\frac{1}{2}}\right)}{2\left(\Delta x_{i+\frac{1}{2}} + \Delta x_{i-\frac{1}{2}}\right)} H''_y(x)
\end{aligned} \tag{4.15}$$

$$\frac{H_y\left(x + \Delta x_{i+\frac{1}{2}}\right) - H_y\left(x - \Delta x_{i+\frac{1}{2}}\right)}{\Delta x_{i+\frac{1}{2}} + \Delta x_{i-\frac{1}{2}}} = H'_y(x) + \frac{\Delta x_{i+\frac{1}{2}} - \Delta x_{i-\frac{1}{2}}}{2} H''_y(x) \tag{4.16}$$

and using the following equalities

$$\Delta x_{i+\frac{1}{2}} + \Delta x_{i-\frac{1}{2}} = \frac{\Delta x_i + \Delta x_{i-1}}{2} ; \quad \Delta x_{i+\frac{1}{2}} - \Delta x_{i-\frac{1}{2}} = \frac{\Delta x_i - \Delta x_{i-1}}{2} \tag{4.17}$$

leading to

$$2 \frac{H_y\left(x + \Delta x_{i+\frac{1}{2}}\right) - H_y\left(x - \Delta x_{i+\frac{1}{2}}\right)}{\Delta x_i + \Delta x_{i-1}} = H'_y(x) + \underbrace{\frac{\Delta x_i - \Delta x_{i-1}}{4}}_{O(\Delta x_i - \Delta x_{i-1})} H''_y(x) \tag{4.18}$$

Then, the NU derivative of H_y is found as

$$\begin{aligned}
\left. \frac{\partial H_y}{\partial x} \right|_i &= 2 \frac{H_y \left(x + \Delta x_{i+\frac{1}{2}} \right) - H_y \left(x - \Delta x_{i+\frac{1}{2}} \right)}{\Delta x_i + \Delta x_{i-1}} \\
&= H'_y(x) + \underbrace{\frac{\Delta x_i - \Delta x_{i-1}}{4}}_{O(\Delta x_i - \Delta x_{i-1})} H''_y(x).
\end{aligned} \tag{4.19}$$

- Calculation of $\partial E_z / \partial x$ Derivative

Let us first calculate the forward difference for E_z at $x + \Delta x_i/2$

$$\begin{aligned}
E_z \left(x + \frac{\Delta x_i}{2} \right) &= E_z(x) + \frac{\Delta x_i}{2} E'_z(x) \\
&+ \frac{\Delta x_i^2}{8} E''_z(x) + \frac{\Delta x_i^3}{48} E'''_z(x) + \frac{\Delta x_i^4}{384} E''''_z(x) + \dots
\end{aligned} \tag{4.20}$$

and the backward difference for E_z at $x - \Delta x_i/2$ is

$$\begin{aligned}
E_z \left(x - \frac{\Delta x_i}{2} \right) &= E_z(x) - \frac{\Delta x_i}{2} E'_z(x) \\
&+ \frac{\Delta x_i^2}{8} E''_z(x) - \frac{\Delta x_i^3}{48} E'''_z(x) + \frac{\Delta x_i^4}{384} E''''_z(x) + \dots
\end{aligned} \tag{4.21}$$

subtracting the second from the first equation

$$E_z \left(x + \frac{\Delta x_i}{2} \right) - E_z \left(x - \frac{\Delta x_i}{2} \right) = \Delta x_i E'_z(x) + \frac{\Delta x_i^3}{24} E'''_z(x) \tag{4.22}$$

Then, the NU derivative of E_z is found as

$$\left. \frac{\partial E_z}{\partial x} \right|_{i+\frac{1}{2}} = \frac{E_z \left(x + \frac{\Delta x_i}{2} \right) - E_z \left(x - \frac{\Delta x_i}{2} \right)}{\Delta x_i} = E'_z(x) + \underbrace{\frac{\Delta x_i^2}{24}}_{O(\Delta x_i^2)} E'''_z(x) \tag{4.23}$$

where it can be concluded that the first order derivative for the electric field is second order accurate ($O(\Delta x_i^2)$) whereas the first order derivative for the magnetic

field is first order accurate ($O(\Delta x_i - \Delta x_{i-1})$) due to the loss of central difference concept.

4.3. Mesh Types

There are many NU mesh types in which most of them are different linear meshes with a different constant grow ratio. The linear mesh is rather simple comparing to the other NU meshes. In this thesis, two different NU meshes are considered:

- Chebyshev-Gauss-Lobatto (CGL) Mesh (for the first time in the NU FDTD),
- Linear Mesh.

4.3.1. Chebyshev-Gauss-Lobatto (CGL) Mesh

The NU CGL mesh points for 3D space are

$$x_i = 1 - \cos\left(\pi \frac{i}{N_x}\right), \quad i = 0 \dots N_x \quad (4.24)$$

$$y_j = 1 - \cos\left(\pi \frac{j}{N_y}\right), \quad j = 0 \dots N_y \quad (4.25)$$

$$z_k = 1 - \cos\left(\pi \frac{k}{N_z}\right), \quad k = 0 \dots N_z \quad (4.26)$$

where i, j, k are mesh indices and N_x, N_y, N_z are the total mesh numbers along x, y, z axes, respectively [Chiang et al., 2005].

The NU CGL meshes for the 3D space is shown in Figure 4.1. It is seen that the mesh points are denser at the boundaries whereas they are coarser at the middle parts of the problem domain. This property is important for accuracy analysis which will be performed in the following section.

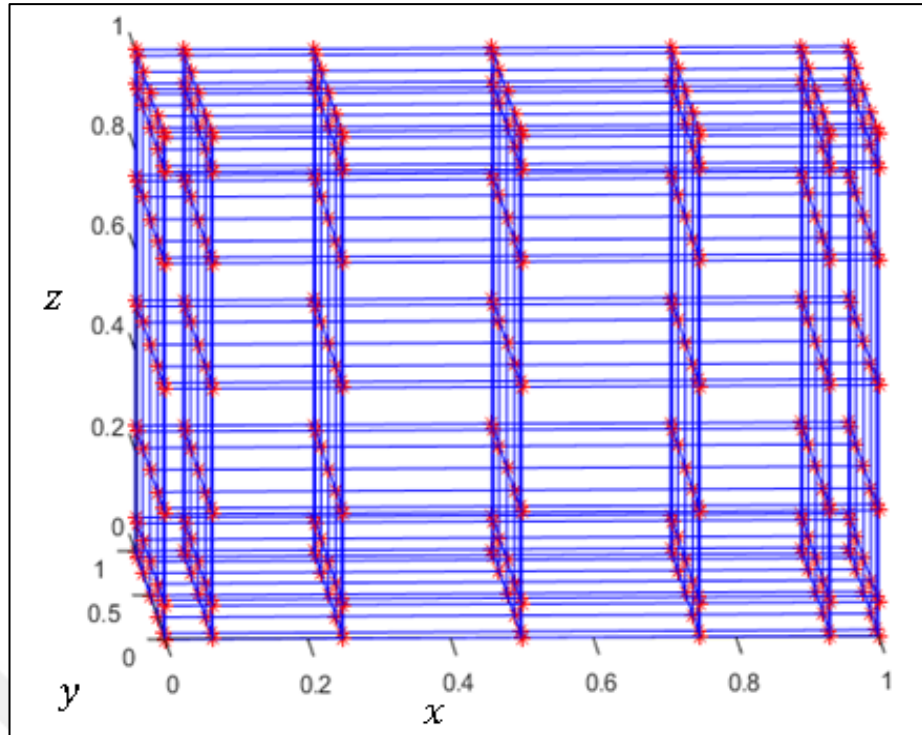


Figure 4.1: The 3D NU CGL mesh demonstration.

A reduction to the two-dimensional space of the NU CGL mesh points is also shown in Figure 4.2.

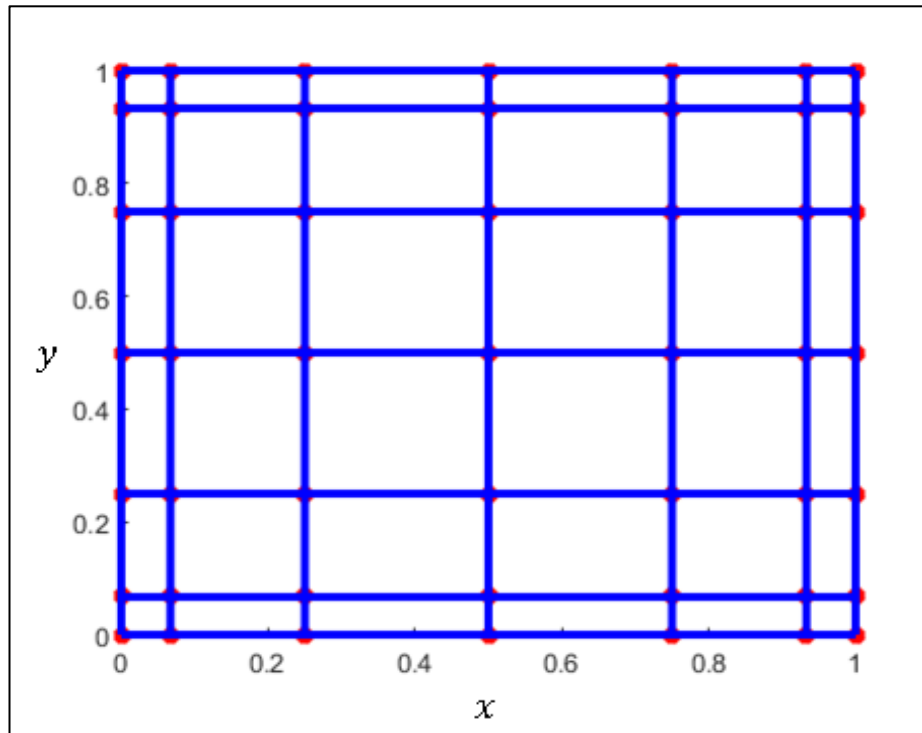


Figure 4.2: The two-dimensional NU CGL mesh demonstration.

4.3.2. Linear Mesh

The linear mesh is the most common type that their generation is rather easy. By referring to (45), (53) and (61), in the case of the constants $\alpha_\tau = \alpha$, $\beta_\tau = \beta$ and $\gamma_\tau = \gamma$, the mesh ratios become [Blottner and Roache, 1971], [Choi and Hoefer, 1987]

$$\alpha = \frac{\Delta x_i}{\Delta x_{i-1}} \quad ; \quad \beta = \frac{\Delta y_j}{\Delta y_{j-1}} \quad ; \quad \gamma = \frac{\Delta z_k}{\Delta z_{k-1}} \quad (4.27)$$

where i, j, k are the mesh number indices, respectively.

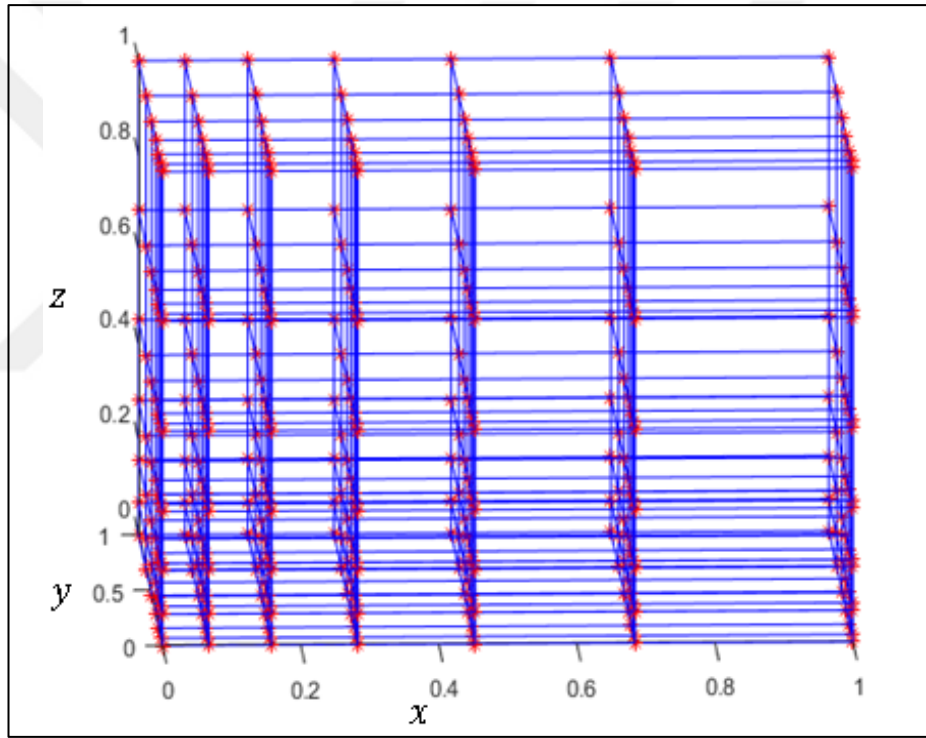


Figure 4.3: The 3D NU Linear mesh demonstration.

The NU Linear mesh increases at a constant growing ratio as shown in Figure 4.3. In fact, it is not desired that the mesh is coarser in one side of the domain whereas it is denser in the other side of the domain. The global application of this mesh form is not preferable because of weak boundary modelling along the coarse edges. However, its local application along all the boundaries of the objects is generally used in the commercial software programs [Jiménez-Mejía and Herrera-Murcia, 2015], [XFDTD, 2022]. The two-dimensional reduction of this mesh is shown in Figure 4.4.

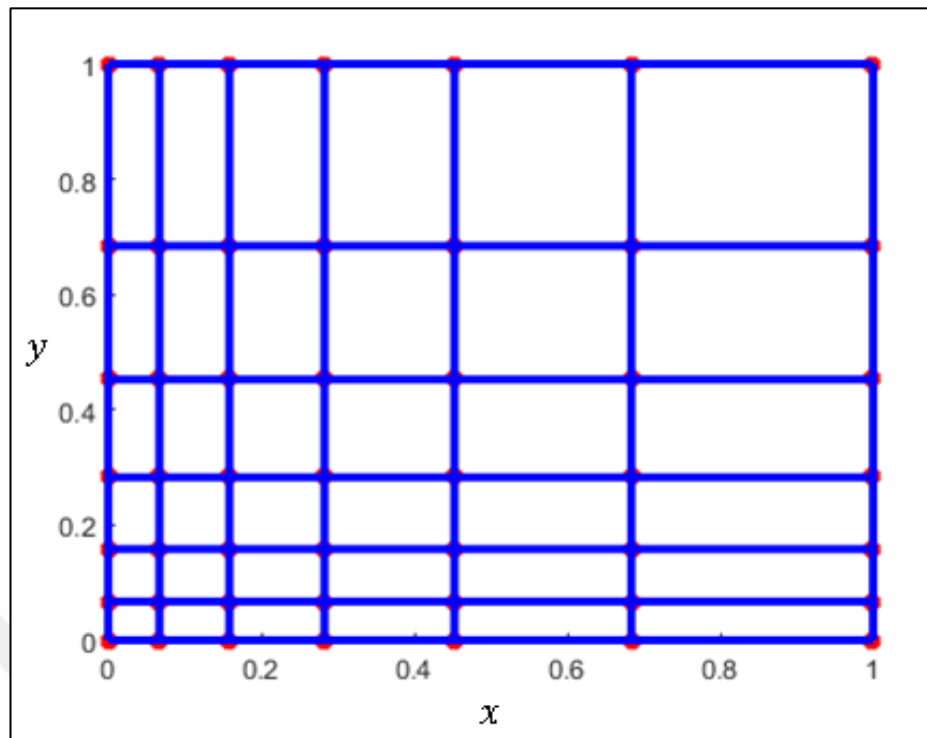


Figure 4.4: The two-dimensional NU Linear mesh demonstration.

5. NONUNIFORM FINITE DIFFERENCE TIME DOMAIN METHOD

Nonuniform Finite Difference Time Domain (NU FDTD) method is based on the usage of spatial NU meshes. This is the main difference between the U FDTD method and the NU FDTD method. The time derivatives are calculated as similar to the U FDTD method. In the NU FDTD method, new update equations must be developed since the nonuniformity in the mesh changes the whole algorithm.

5.1. Update Equations

In a 3D lossless and simple medium, Maxwell's equations are

$$\epsilon \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} \Rightarrow \begin{cases} \epsilon \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \epsilon \frac{\partial E_y}{\partial t} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \epsilon \frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{cases} \quad (5.1)$$

$$\mu \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E} \Rightarrow \begin{cases} \mu \frac{\partial H_x}{\partial t} = \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \\ \mu \frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \\ \mu \frac{\partial H_z}{\partial t} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \end{cases} \quad (5.2)$$

Now, step by step, the NU update equations of every field component are extracted in the following sections.

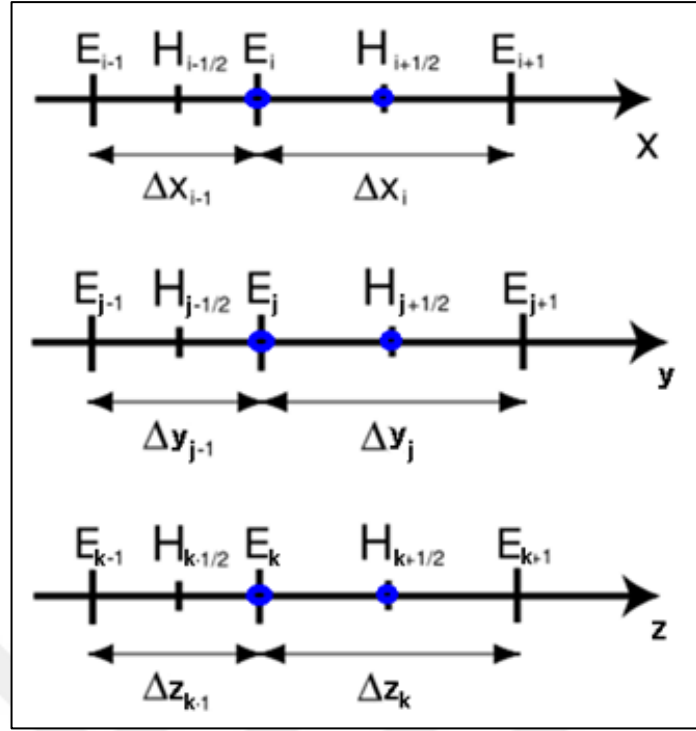


Figure 5.1: NU mesh demonstrations for the three dimensions.

- NU Update Equation for E_x Component

Let us discretize first equation in (5.1) at $i + 1/2, j, k$ and $n + 1/2$

$$\varepsilon \frac{\partial E_x}{\partial t} \Big|_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}} = \frac{\partial H_z}{\partial y} \Big|_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}} - \frac{\partial H_y}{\partial z} \Big|_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}}. \quad (5.3)$$

The left-hand side of above equation is treated as

$$\varepsilon \frac{\partial E_x}{\partial t} \Big|_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}} = \varepsilon \frac{E_x \Big|_{i+\frac{1}{2},j,k}^{n+1} - E_x \Big|_{i+\frac{1}{2},j,k}^n}{\Delta t}. \quad (5.4)$$

The first and the second terms at the right-hand side of (5.3) is treated as

$$\frac{\partial H_z}{\partial y} \Big|_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}} = \frac{H_z \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - H_z \Big|_{i+\frac{1}{2},j-\frac{1}{2},k}^{n+\frac{1}{2}}}{y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}} = \frac{H_z \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - H_z \Big|_{i+\frac{1}{2},j-\frac{1}{2},k}^{n+\frac{1}{2}}}{(\Delta y_j + \Delta y_{j-1})/2} \quad (5.5)$$

$$\left. \frac{\partial H_y}{\partial z} \right|_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}} = \frac{H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - H_y|_{i+\frac{1}{2},j,k-\frac{1}{2}}^{n+\frac{1}{2}}}{z_{k+\frac{1}{2}} - z_{k-\frac{1}{2}}} = \frac{H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - H_y|_{i+\frac{1}{2},j,k-\frac{1}{2}}^{n+\frac{1}{2}}}{(\Delta z_k + \Delta z_{k-1})/2} \quad (5.6)$$

where the denominator apparently includes a new equation for the spatial mesh. Along the x, y, z meshes half points are not logically defined. Thus, using Figure 5.1, one can derive the mesh point differences logically [Sundqvist and Veronis, 1970], [Chris et al., 2002]. For the x direction, the formulations are

$$\left. \begin{aligned} x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} &= \frac{1}{2}(x_{i+1} + x_i) - \frac{1}{2}(x_i + x_{i-1}) = \frac{1}{2}(x_{i+1} - x_{i-1}) \\ \Delta x_i + \Delta x_{i-1} &= (x_{i+1} - x_i) + (x_i - x_{i-1}) = x_{i+1} - x_{i-1} \end{aligned} \right\} \quad (5.7)$$

$$\Rightarrow x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = \frac{1}{2}(\Delta x_i + \Delta x_{i-1}).$$

For the y direction, the formulations are

$$\left. \begin{aligned} y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} &= \frac{1}{2}(y_{j+1} + y_j) - \frac{1}{2}(y_j + y_{j-1}) = \frac{1}{2}(y_{j+1} - y_{j-1}) \\ \Delta y_j + \Delta y_{j-1} &= (y_{j+1} - y_j) + (y_j - y_{j-1}) = y_{j+1} - y_{j-1} \end{aligned} \right\} \quad (5.8)$$

$$\Rightarrow y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} = \frac{1}{2}(\Delta y_j + \Delta y_{j-1}).$$

For the z direction, the formulations are

$$\left. \begin{aligned} z_{k+\frac{1}{2}} - z_{k-\frac{1}{2}} &= \frac{1}{2}(z_{k+1} + z_k) - \frac{1}{2}(z_k + z_{k-1}) = \frac{1}{2}(z_{k+1} - z_{k-1}) \\ \Delta z_k + \Delta z_{k-1} &= (z_{k+1} - z_k) + (z_k - z_{k-1}) = z_{k+1} - z_{k-1} \end{aligned} \right\} \quad (5.9)$$

$$\Rightarrow z_{k+\frac{1}{2}} - z_{k-\frac{1}{2}} = \frac{1}{2}(\Delta z_k + \Delta z_{k-1}).$$

Thus, the update equation for E_x is calculated as

$$\begin{aligned} & \varepsilon \frac{E_x|_{i+\frac{1}{2},j,k}^{n+1} - E_x|_{i+\frac{1}{2},j,k}^n}{\Delta t} \\ &= \frac{H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - H_z|_{i+\frac{1}{2},j-\frac{1}{2},k}^{n+\frac{1}{2}}}{(\Delta y_j + \Delta y_{j-1})/2} - \frac{H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - H_y|_{i+\frac{1}{2},j,k-\frac{1}{2}}^{n+\frac{1}{2}}}{(\Delta z_k + \Delta z_{k-1})/2} \end{aligned} \quad (5.10)$$

where extracting the most forward term in time, the E_x update equation becomes

$$\begin{aligned} E_x|_{i+\frac{1}{2},j,k}^{n+1} &= E_x|_{i+\frac{1}{2},j,k}^n \\ &+ \frac{\Delta t}{\varepsilon} \left(\frac{H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - H_z|_{i+\frac{1}{2},j-\frac{1}{2},k}^{n+\frac{1}{2}}}{\Delta y_j/2 + \Delta y_{j-1}/2} - \frac{H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - H_y|_{i+\frac{1}{2},j,k-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta z_k/2 + \Delta z_{k-1}/2} \right). \end{aligned} \quad (5.11)$$

- NU Update Equation for E_y Component

Let us discretize second equation in (5.1) at $i, j + 1/2, k$ and $n + 1/2$

$$\varepsilon \frac{\partial E_y}{\partial t} \Big|_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}} = \frac{\partial H_x}{\partial z} \Big|_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}} - \frac{\partial H_z}{\partial x} \Big|_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}}. \quad (5.12)$$

The left-hand side of above equation is treated as

$$\varepsilon \frac{\partial E_y}{\partial t} \Big|_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}} = \varepsilon \frac{E_y|_{i,j+\frac{1}{2},k}^{n+1} - E_y|_{i,j+\frac{1}{2},k}^n}{\Delta t}. \quad (5.13)$$

The first and the second terms at the right-hand side of (5.12) is treated as

$$\frac{\partial H_x}{\partial z} \Big|_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}} = \frac{H_x|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - H_x|_{i,j+\frac{1}{2},k-\frac{1}{2}}^{n+\frac{1}{2}}}{z_{k+\frac{1}{2}} - z_{k-\frac{1}{2}}} = \frac{H_x|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - H_x|_{i,j+\frac{1}{2},k-\frac{1}{2}}^{n+\frac{1}{2}}}{(\Delta z_k + \Delta z_{k-1})/2} \quad (5.14)$$

$$\frac{\partial H_z}{\partial x} \Big|_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}} = \frac{H_z \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - H_z \Big|_{i-\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} = \frac{H_z \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - H_z \Big|_{i-\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}}}{(\Delta x_i + \Delta x_{i-1})/2}. \quad (5.15)$$

Thus, the update equation for E_y is calculated as

$$\begin{aligned} \varepsilon \frac{E_y \Big|_{i,j+\frac{1}{2},k}^{n+1} - E_y \Big|_{i,j+\frac{1}{2},k}^n}{\Delta t} \\ = \frac{H_x \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - H_x \Big|_{i,j+\frac{1}{2},k-\frac{1}{2}}^{n+\frac{1}{2}}}{(\Delta z_k + \Delta z_{k-1})/2} - \frac{H_z \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - H_z \Big|_{i-\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}}}{(\Delta x_i + \Delta x_{i-1})/2}. \end{aligned} \quad (5.16)$$

Then, extracting the most forward term in time, it results as

$$\begin{aligned} E_y \Big|_{i,j+\frac{1}{2},k}^{n+1} = E_y \Big|_{i,j+\frac{1}{2},k}^n \\ + \frac{\Delta t}{\varepsilon} \left(\frac{H_x \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - H_x \Big|_{i,j+\frac{1}{2},k-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta z_k/2 + \Delta z_{k-1}/2} - \frac{H_z \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - H_z \Big|_{i-\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}}}{\Delta x_i/2 + \Delta x_{i-1}/2} \right). \end{aligned} \quad (5.17)$$

- NU Update Equation for E_z Component

Let us discretize third equation in (5.1) at $i, j, k + 1/2$ and $n + 1/2$

$$\varepsilon \frac{\partial E_z}{\partial t} \Big|_{i,j,k+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{\partial H_y}{\partial x} \Big|_{i,j,k+\frac{1}{2}}^{n+\frac{1}{2}} - \frac{\partial H_x}{\partial y} \Big|_{i,j,k+\frac{1}{2}}^{n+\frac{1}{2}}. \quad (5.18)$$

The left-hand side of above equation is treated as

$$\varepsilon \frac{\partial E_z}{\partial t} \Big|_{i,j,k+\frac{1}{2}}^{n+\frac{1}{2}} = \varepsilon \frac{E_z \Big|_{i,j,k+\frac{1}{2}}^{n+1} - E_z \Big|_{i,j,k+\frac{1}{2}}^n}{\Delta t}. \quad (5.19)$$

The first and the second terms at the right-hand side of (5.18) is treated as

$$\frac{\partial H_y}{\partial x} \Big|_{i,j,k+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{H_y \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - H_y \Big|_{i-\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} = \frac{H_y \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - H_y \Big|_{i-\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}}}{(\Delta x_i + \Delta x_{i-1})/2} \quad (5.20)$$

$$\frac{\partial H_x}{\partial y} \Big|_{i,j,k+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{H_x \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - H_x \Big|_{i,j-\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}}{y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}} = \frac{H_x \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - H_x \Big|_{i,j-\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}}{(\Delta y_j + \Delta y_{j-1})/2}. \quad (5.21)$$

Thus, the update equation E_z is calculated as

$$\begin{aligned} \frac{E_z \Big|_{i,j,k+\frac{1}{2}}^{n+1} - E_z \Big|_{i,j,k+\frac{1}{2}}^n}{\varepsilon \Delta t} &= \frac{H_y \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - H_y \Big|_{i-\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}}}{(\Delta x_i + \Delta x_{i-1})/2} - \frac{H_x \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - H_x \Big|_{i,j-\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}}{(\Delta y_j + \Delta y_{j-1})/2}. \end{aligned} \quad (5.22)$$

Then, extracting the most forward term in time, it results as

$$\begin{aligned} E_z \Big|_{i,j,k+\frac{1}{2}}^{n+1} &= E_z \Big|_{i,j,k+\frac{1}{2}}^n \\ &+ \frac{\Delta t}{\varepsilon} \left(\frac{H_y \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - H_y \Big|_{i-\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta x_i/2 + \Delta x_{i-1}/2} - \frac{H_x \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - H_x \Big|_{i,j-\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta y_j/2 + \Delta y_{j-1}/2} \right). \end{aligned} \quad (5.23)$$

- NU Update Equation for H_x Component

Let us discretize first equation in (5.2) at $i, j + 1/2, k + 1/2$ and n

$$\mu \frac{\partial H_x}{\partial t} \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^n = \frac{\partial E_y}{\partial z} \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^n - \frac{\partial E_z}{\partial y} \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^n. \quad (5.24)$$

The left-hand side of above equation is treated as

$$\mu \frac{\partial H_x}{\partial t} \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^n = \mu \frac{H_x \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - H_x \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}}}{\Delta t}. \quad (5.25)$$

The first and the second terms at the right-hand side of (5.24) is treated as

$$\frac{\partial E_y}{\partial z} \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^n = \frac{E_y \Big|_{i,j+\frac{1}{2},k+1}^n - E_y \Big|_{i,j+\frac{1}{2},k}^n}{z_{k+1} - z_k} = \frac{E_y \Big|_{i,j+\frac{1}{2},k+1}^n - E_y \Big|_{i,j+\frac{1}{2},k}^n}{\Delta z_k} \quad (5.26)$$

$$\frac{\partial E_z}{\partial y} \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^n = \frac{E_z \Big|_{i,j+1,k+\frac{1}{2}}^n - E_z \Big|_{i,j,k+\frac{1}{2}}^n}{y_{j+1} - y_j} = \frac{E_z \Big|_{i,j+1,k+\frac{1}{2}}^n - E_z \Big|_{i,j,k+\frac{1}{2}}^n}{\Delta y_j}. \quad (5.27)$$

Thus, the update equation for H_x is calculated as

$$\begin{aligned} \mu \frac{H_x \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - H_x \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}}}{\Delta t} \\ = \frac{E_y \Big|_{i,j+\frac{1}{2},k+1}^n - E_y \Big|_{i,j+\frac{1}{2},k}^n}{\Delta z_k} - \frac{E_z \Big|_{i,j+1,k+\frac{1}{2}}^n - E_z \Big|_{i,j,k+\frac{1}{2}}^n}{\Delta y_j}. \end{aligned} \quad (5.28)$$

Then, extracting the most forward term in time, it results as

$$\begin{aligned} H_x \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} &= H_x \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} \\ &+ \frac{\Delta t}{\mu} \left(\frac{E_y \Big|_{i,j+\frac{1}{2},k+1}^n - E_y \Big|_{i,j+\frac{1}{2},k}^n}{\Delta z_k} - \frac{E_z \Big|_{i,j+1,k+\frac{1}{2}}^n - E_z \Big|_{i,j,k+\frac{1}{2}}^n}{\Delta y_j} \right). \end{aligned} \quad (5.29)$$

- NU Update Equation for H_y Component

Let us discretize second equation in (5.2) at $i + 1/2, j, k + 1/2$ and n

$$\mu \frac{\partial H_y}{\partial t} \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}}^n = \frac{\partial E_z}{\partial x} \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}}^n - \frac{\partial E_x}{\partial z} \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}}^n. \quad (5.30)$$

The left-hand side of above equation is treated as

$$\mu \frac{\partial H_y}{\partial t} \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}}^n = \mu \frac{H_y \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - H_y \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n-\frac{1}{2}}}{\Delta t}. \quad (5.31)$$

The first and the second terms at the right-hand side of (5.30) is treated as

$$\frac{\partial E_z}{\partial x} \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}}^n = \frac{E_z \Big|_{i+1,j,k+\frac{1}{2}}^n - E_z \Big|_{i,j,k+\frac{1}{2}}^n}{x_{i+1} - x_i} = \frac{E_z \Big|_{i+1,j,k+\frac{1}{2}}^n - E_z \Big|_{i,j,k+\frac{1}{2}}^n}{\Delta x_i} \quad (5.32)$$

$$\frac{\partial E_x}{\partial z} \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}}^n = \frac{E_x \Big|_{i+\frac{1}{2},j,k+1}^n - E_x \Big|_{i+\frac{1}{2},j,k}^n}{z_{k+1} - z_k} = \frac{E_x \Big|_{i+\frac{1}{2},j,k+1}^n - E_x \Big|_{i+\frac{1}{2},j,k}^n}{\Delta z_k}. \quad (5.33)$$

Thus, the update equation for H_y is calculated as

$$\begin{aligned} \mu \frac{H_y \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - H_y \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n-\frac{1}{2}}}{\Delta t} \\ = \frac{E_z \Big|_{i+1,j,k+\frac{1}{2}}^n - E_z \Big|_{i,j,k+\frac{1}{2}}^n}{\Delta x_i} - \frac{E_x \Big|_{i+\frac{1}{2},j,k+1}^n - E_x \Big|_{i+\frac{1}{2},j,k}^n}{\Delta z_k}. \end{aligned} \quad (5.34)$$

Then, extracting the most forward term in time results as

$$\begin{aligned} H_y \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} &= H_y \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n-\frac{1}{2}} \\ &+ \frac{\Delta t}{\mu} \left(\frac{E_z \Big|_{i+1,j,k+\frac{1}{2}}^n - E_z \Big|_{i,j,k+\frac{1}{2}}^n}{\Delta x_i} - \frac{E_x \Big|_{i+\frac{1}{2},j,k+1}^n - E_x \Big|_{i+\frac{1}{2},j,k}^n}{\Delta z_k} \right). \end{aligned} \quad (5.35)$$

- NU Update Equation for H_z Component

Let us discretize third equation in (5.2) at $i + 1/2, j + 1/2, k$ and n

$$\mu \frac{\partial H_z}{\partial t} \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}^n = \frac{\partial E_x}{\partial y} \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}^n - \frac{\partial E_y}{\partial x} \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}^n. \quad (5.36)$$

The left-hand side of above equation is treated as

$$\mu \frac{\partial H_z}{\partial t} \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}^n = \mu \frac{H_z \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - H_z \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n-\frac{1}{2}}}{\Delta t}. \quad (5.37)$$

The first and the second terms at the right-hand side of (5.36) is treated as

$$\frac{\partial E_x}{\partial y} \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}^n = \frac{E_x \Big|_{i+\frac{1}{2},j+1,k}^n - E_x \Big|_{i+\frac{1}{2},j,k}^n}{y_{j+1} - y_j} = \frac{E_x \Big|_{i+\frac{1}{2},j+1,k}^n - E_x \Big|_{i+\frac{1}{2},j,k}^n}{\Delta y_j} \quad (5.38)$$

$$\frac{\partial E_y}{\partial x} \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}^n = \frac{E_y \Big|_{i+1,j+\frac{1}{2},k}^n - E_y \Big|_{i,j+\frac{1}{2},k}^n}{x_{i+1} - x_i} = \frac{E_y \Big|_{i+1,j+\frac{1}{2},k}^n - E_y \Big|_{i,j+\frac{1}{2},k}^n}{\Delta x_i}. \quad (5.39)$$

Thus, the update equation for H_z is calculated as

$$\begin{aligned} \mu \frac{H_z \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - H_z \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n-\frac{1}{2}}}{\Delta t} \\ = \frac{E_x \Big|_{i+\frac{1}{2},j+1,k}^n - E_x \Big|_{i+\frac{1}{2},j,k}^n}{\Delta y_j} - \frac{E_y \Big|_{i+1,j+\frac{1}{2},k}^n - E_y \Big|_{i,j+\frac{1}{2},k}^n}{\Delta x_i}. \end{aligned} \quad (5.40)$$

Then, extracting the most forward term in time, it results as

$$\begin{aligned} H_z \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+1} &= H_z \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}^n \\ &+ \frac{\Delta t}{\mu} \left(\frac{E_x \Big|_{i+\frac{1}{2},j+1,k}^n - E_x \Big|_{i+\frac{1}{2},j,k}^n}{\Delta y_j} - \frac{E_y \Big|_{i+1,j+\frac{1}{2},k}^n - E_y \Big|_{i,j+\frac{1}{2},k}^n}{\Delta x_i} \right) \end{aligned} \quad (5.41)$$

where space-time discretization is made using central difference with an extension of nonuniformity. Finally, the NU electric field update equations are

$$E_x|_{i+\frac{1}{2},j,k}^{n+1} = E_x|_{i+\frac{1}{2},j,k}^n + \frac{\Delta t}{\varepsilon} \left(\frac{H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - H_z|_{i+\frac{1}{2},j-\frac{1}{2},k}^{n+\frac{1}{2}}}{\Delta y_j/2 + \Delta y_{j-1}/2} - \frac{H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - H_y|_{i+\frac{1}{2},j,k-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta z_k/2 + \Delta z_{k-1}/2} \right) \quad (5.42)$$

$$E_y|_{i,j+\frac{1}{2},k}^{n+1} = E_y|_{i,j+\frac{1}{2},k}^n + \frac{\Delta t}{\varepsilon} \left(\frac{H_x|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - H_x|_{i,j+\frac{1}{2},k-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta z_k/2 + \Delta z_{k-1}/2} - \frac{H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - H_z|_{i-\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}}}{\Delta x_i/2 + \Delta x_{i-1}/2} \right) \quad (5.43)$$

$$E_z|_{i,j,k+\frac{1}{2}}^{n+1} = E_z|_{i,j,k+\frac{1}{2}}^n + \frac{\Delta t}{\varepsilon} \left(\frac{H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - H_y|_{i-\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta x_i/2 + \Delta x_{i-1}/2} - \frac{H_x|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - H_x|_{i,j-\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta y_j/2 + \Delta y_{j-1}/2} \right) \quad (5.44)$$

and the NU magnetic field update equations are

$$H_x|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} = H_x|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} + \frac{\Delta t}{\mu} \left(\frac{E_y|_{i,j+\frac{1}{2},k+1}^n - E_y|_{i,j+\frac{1}{2},k}^n}{\Delta z_k} - \frac{E_z|_{i,j+1,k+\frac{1}{2}}^n - E_z|_{i,j,k+\frac{1}{2}}^n}{\Delta y_j} \right) \quad (5.45)$$

$$H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} = H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n-\frac{1}{2}} + \frac{\Delta t}{\mu} \left(\frac{E_z|_{i+1,j,k+\frac{1}{2}}^n - E_z|_{i,j,k+\frac{1}{2}}^n}{\Delta x_i} - \frac{E_x|_{i+\frac{1}{2},j,k+1}^n - E_x|_{i+\frac{1}{2},j,k}^n}{\Delta z_k} \right) \quad (5.46)$$

$$H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} = H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n-\frac{1}{2}} + \frac{\Delta t}{\mu} \left(\frac{E_x|_{i+\frac{1}{2},j+1,k}^n - E_x|_{i+\frac{1}{2},j,k}^n}{\Delta y_j} - \frac{E_y|_{i+1,j+\frac{1}{2},k}^n - E_y|_{i,j+\frac{1}{2},k}^n}{\Delta x_i} \right). \quad (5.47)$$

5.2. Numerical Dispersion Equation

Dispersion is a phenomenon that the phase velocity of the electromagnetic wave is a function of frequency. On the other hand, the numerical dispersion is caused by the FDTD meshing. In this manner, the numerical dispersion must be taken into account to suppress undesired and unphysical results.

To find the NU NDE, a plane wave assumption is utilized in a discrete form as

$$\begin{aligned} H_x|_{i,j,k}^n &= H_{x0} e^{-j(k_{Nx}x_i + k_{Ny}y_j + k_{Nz}z_k - \omega n \Delta t)} \\ H_y|_{i,j,k}^n &= H_{y0} e^{-j(k_{Nx}x_i + k_{Ny}y_j + k_{Nz}z_k - \omega n \Delta t)} \\ H_z|_{i,j,k}^n &= H_{z0} e^{-j(k_{Nx}x_i + k_{Ny}y_j + k_{Nz}z_k - \omega n \Delta t)} \\ E_x|_{i,j,k}^n &= E_{x0} e^{-j(k_{Nx}x_i + k_{Ny}y_j + k_{Nz}z_k - \omega n \Delta t)} \\ E_y|_{i,j,k}^n &= E_{y0} e^{-j(k_{Nx}x_i + k_{Ny}y_j + k_{Nz}z_k - \omega n \Delta t)} \\ E_z|_{i,j,k}^n &= E_{z0} e^{-j(k_{Nx}x_i + k_{Ny}y_j + k_{Nz}z_k - \omega n \Delta t)} \end{aligned} \quad (5.48)$$

where $k_N = \sqrt{k_{Nx}^2 + k_{Ny}^2 + k_{Nz}^2}$ is the numerical wave number.

- NU NDE for H_x Component

Let us reconsider the H_x update equation as

$$\begin{aligned}
& \mu \frac{H_x|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - H_x|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}}}{\Delta t} \\
&= \frac{E_y|_{i,j+\frac{1}{2},k+1}^n - E_y|_{i,j+\frac{1}{2},k}^n}{\Delta z_k} - \frac{E_z|_{i,j+1,k+\frac{1}{2}}^n - E_z|_{i,j,k+\frac{1}{2}}^n}{\Delta y_j}.
\end{aligned} \tag{5.49}$$

Now, let us substitute the plane wave forms into the above equation

$$\begin{aligned}
& \frac{\mu H_{x0}}{\Delta t} e^{-j(k_{Nx}x_i + k_{Ny}y_{j+\frac{1}{2}} + k_{Nz}z_{k+\frac{1}{2}} - \omega(n+\frac{1}{2})\Delta t)} - \\
& \frac{\mu H_{x0}}{\Delta t} e^{-j(k_{Nx}x_i + k_{Ny}y_{j+\frac{1}{2}} + k_{Nz}z_{k+\frac{1}{2}} - \omega(n-\frac{1}{2})\Delta t)} = \\
& \frac{E_{y0}}{\Delta z_k} e^{-j(k_{Nx}x_i + k_{Ny}y_{j+\frac{1}{2}} + k_{Nz}z_{k+1} - \omega n\Delta t)} - \\
& \frac{E_{y0}}{\Delta z_k} e^{-j(k_{Nx}x_i + k_{Ny}y_{j+\frac{1}{2}} + k_{Nz}z_k - \omega n\Delta t)} - \\
& \frac{E_{z0}}{\Delta y_j} e^{-j(k_{Nx}x_i + k_{Ny}y_{j+1} + k_{Nz}z_{k+\frac{1}{2}} - \omega n\Delta t)} - \\
& \frac{E_{z0}}{\Delta y_j} e^{-j(k_{Nx}x_i + k_{Ny}y_j + k_{Nz}z_{k+\frac{1}{2}} - \omega n\Delta t)}
\end{aligned} \tag{5.50}$$

and simplifying it by using $e^{-j(k_{Nx}x_i + k_{Ny}y_j + k_{Nz}z_k - \omega n\Delta t)}$

$$\begin{aligned}
& \frac{\mu H_{x0}}{\Delta t} e^{-j(k_{Ny}(y_{j+\frac{1}{2}} - y_j) + k_{Nz}(z_{k+\frac{1}{2}} - z_k) - \frac{1}{2}\omega\Delta t)} - \\
& \frac{\mu H_{x0}}{\Delta t} e^{-j(k_{Ny}(y_{j+\frac{1}{2}} - y_j) + k_{Nz}(z_{k+\frac{1}{2}} - z_k) + \frac{1}{2}\omega\Delta t)} = \\
& E_{y0} \frac{e^{-j(k_{Ny}(y_{j+\frac{1}{2}} - y_j) + k_{Nz}(z_{k+1} - z_k))} - e^{-jk_{Ny}(y_{j+\frac{1}{2}} - y_j)}}{\Delta z_k} - \\
& E_{z0} \frac{e^{-j(k_{Ny}(y_{j+1} - y_j) + k_{Nz}(z_{k+\frac{1}{2}} - z_k))} - e^{-jk_{Nz}(z_{k+\frac{1}{2}} - z_k)}}{\Delta y_j}.
\end{aligned} \tag{5.51}$$

Then, the exponential forms are collected that

$$\mu H_{x0} e^{-j\left(k_{Ny}\frac{\Delta y_j}{2} + k_{Nz}\frac{\Delta z_k}{2}\right)} \frac{e^{j\omega\frac{\Delta t}{2}} - e^{-j\omega\frac{\Delta t}{2}}}{\Delta t} =$$

$$E_{y0} \frac{e^{-j\left(k_{Ny}\frac{\Delta y_j}{2} + k_{Nz}\frac{\Delta z_k}{2}\right)} - e^{-jk_{Ny}\frac{\Delta y_j}{2}}}{\Delta z_k} - E_{z0} \frac{e^{-j\left(k_{Ny}\frac{\Delta y_j}{2} + k_{Nz}\frac{\Delta z_k}{2}\right)} - e^{-jk_{Nz}\frac{\Delta z_k}{2}}}{\Delta y_j} \quad (5.52)$$

where it can be arranged in the way of trigonometric relations

$$\mu H_{x0} e^{-j\left(k_{Ny}\frac{\Delta y_j}{2} + k_{Nz}\frac{\Delta z_k}{2}\right)} \frac{e^{j\omega\frac{\Delta t}{2}} - e^{-j\omega\frac{\Delta t}{2}}}{\Delta t}$$

$$= E_{y0} e^{-j\left(k_{Ny}\frac{\Delta y_j}{2} + k_{Nz}\frac{\Delta z_k}{2}\right)} \frac{e^{-jk_{Nz}\frac{\Delta z_k}{2}} - e^{jk_{Nz}\frac{\Delta z_k}{2}}}{\Delta z_k} \quad (5.53)$$

$$- E_{z0} e^{-j\left(k_{Ny}\frac{\Delta y_j}{2} + k_{Nz}\frac{\Delta z_k}{2}\right)} \frac{e^{-jk_{Ny}\frac{\Delta y_j}{2}} - e^{jk_{Ny}\frac{\Delta y_j}{2}}}{\Delta y_j}.$$

Finally, the NU NDE for H_x is found as

$$\mu H_{x0} \frac{\sin\left(\omega\frac{\Delta t}{2}\right)}{\Delta t} = E_{z0} \frac{\sin\left(k_{Ny}\frac{\Delta y_j}{2}\right)}{\Delta y_j} - E_{y0} \frac{\sin\left(k_{Nz}\frac{\Delta z_k}{2}\right)}{\Delta z_k}. \quad (5.54)$$

- NU NDE for H_y Component

Let us reconsider the H_y update equation as

$$\mu \frac{H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n-\frac{1}{2}}}{\Delta t}$$

$$= \frac{E_z|_{i+1,j,k+\frac{1}{2}}^n - E_z|_{i,j,k+\frac{1}{2}}^n}{\Delta x_i} - \frac{E_x|_{i+\frac{1}{2},j,k+1}^n - E_x|_{i+\frac{1}{2},j,k}^n}{\Delta z_k} \quad (5.55)$$

and the plane wave forms are inserted into above equation that

$$\begin{aligned}
& \frac{\mu H_{y0}}{\Delta t} e^{-j\left(k_{Nx}x_{i+\frac{1}{2}}+k_{Ny}y_j+k_{Nz}z_{k+\frac{1}{2}}-\omega\left(n+\frac{1}{2}\right)\Delta t\right)} - \\
& \frac{\mu H_{y0}}{\Delta t} e^{-j\left(k_{Nx}x_{i+\frac{1}{2}}+k_{Ny}y_j+k_{Nz}z_{k+\frac{1}{2}}-\omega\left(n-\frac{1}{2}\right)\Delta t\right)} = \\
& \frac{E_{z0}e^{-j\left(k_{Nx}x_{i+1}+k_{Ny}y_j+k_{Nz}z_{k+\frac{1}{2}}-\omega n\Delta t\right)} - E_{z0}e^{-j\left(k_{Nx}x_i+k_{Ny}y_j+k_{Nz}z_{k+\frac{1}{2}}-\omega n\Delta t\right)}}{\Delta x_i} \\
& - \frac{E_{x0}e^{-j\left(k_{Nx}x_{i+\frac{1}{2}}+k_{Ny}y_j+k_{Nz}z_{k+1}-\omega n\Delta t\right)} - E_{x0}e^{-j\left(k_{Nx}x_{i+\frac{1}{2}}+k_{Ny}y_j+k_{Nz}z_k-\omega n\Delta t\right)}}{\Delta z_k}
\end{aligned} \tag{5.56}$$

and simplifying it by using $e^{-j(k_{Nx}x_i+k_{Ny}y_j+k_{Nz}z_k-\omega n\Delta t)}$

$$\begin{aligned}
& \frac{\mu H_{y0}}{\Delta t} e^{-j\left(k_{Nx}\left(x_{i+\frac{1}{2}}-x_i\right)+k_{Nz}\left(z_{k+\frac{1}{2}}-z_k\right)-\omega\frac{\Delta t}{2}\right)} - \\
& \frac{\mu H_{y0}}{\Delta t} e^{-j\left(k_{Nx}\left(x_{i+\frac{1}{2}}-x_i\right)+k_{Nz}\left(z_{k+\frac{1}{2}}-z_k\right)+\omega\frac{\Delta t}{2}\right)} = \\
& \frac{E_{z0}e^{-j\left(k_{Nx}(x_{i+1}-x_i)+k_{Nz}\left(z_{k+\frac{1}{2}}-z_k\right)\right)} - E_{z0}e^{-jk_{Nz}\left(z_{k+\frac{1}{2}}-z_k\right)}}{\Delta x_i} - \\
& \frac{E_{x0}e^{-j\left(k_{Nx}\left(x_{i+\frac{1}{2}}-x_i\right)+k_{Nz}(z_{k+1}-z_k)\right)} - E_{x0}e^{-jk_{Nx}\left(x_{i+\frac{1}{2}}-x_i\right)}}{\Delta z_k}.
\end{aligned} \tag{5.57}$$

Then, rearranging the equation

$$\begin{aligned}
& \mu H_{y0} e^{-j\left(k_{Nx}\frac{\Delta x_i}{2}+k_{Nz}\frac{\Delta z_k}{2}\right)} \frac{e^{j\omega\frac{\Delta t}{2}} - e^{-j\omega\frac{\Delta t}{2}}}{\Delta t} \\
& = E_{z0} e^{-j\left(k_{Nx}\frac{\Delta x_i}{2}+k_{Nz}\frac{\Delta z_k}{2}\right)} \frac{e^{-jk_{Nx}\frac{\Delta x_i}{2}} - e^{jk_{Nx}\frac{\Delta x_i}{2}}}{\Delta x_i} \\
& - E_{x0} e^{-j\left(k_{Nx}\frac{\Delta x_i}{2}+k_{Nz}\frac{\Delta z_k}{2}\right)} \frac{e^{-jk_{Nz}\frac{\Delta z_k}{2}} - e^{jk_{Nz}\frac{\Delta z_k}{2}}}{\Delta z_k}
\end{aligned} \tag{5.58}$$

Finally, the NU NDE for H_y is found as

$$\mu H_{y0} \frac{\sin\left(\omega \frac{\Delta t}{2}\right)}{\Delta t} = E_{x0} \frac{\sin\left(k_{Nz} \frac{\Delta z_k}{2}\right)}{\Delta z_k} - E_{z0} \frac{\sin\left(k_{Nx} \frac{\Delta x_i}{2}\right)}{\Delta x_i}. \quad (5.59)$$

- NU NDE for H_z Component

Let us reconsider the H_z update equation as

$$\begin{aligned} & \mu \frac{H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n-\frac{1}{2}}}{\Delta t} \\ &= \frac{E_x|_{i+\frac{1}{2},j+1,k}^n - E_x|_{i+\frac{1}{2},j,k}^n}{\Delta y_j} - \frac{E_y|_{i+1,j+\frac{1}{2},k}^n - E_y|_{i,j+\frac{1}{2},k}^n}{\Delta x_i} \end{aligned} \quad (5.60)$$

when the plane wave forms are inserted into above equation that

$$\begin{aligned} & \frac{\mu H_{z0}}{\Delta t} e^{-j\left(k_{Nx}x_{i+\frac{1}{2}}+k_{Ny}y_{j+\frac{1}{2}}+k_{Nz}z_k-\omega\left(n+\frac{1}{2}\right)\Delta t\right)} - \\ & \frac{\mu H_{z0}}{\Delta t} e^{-j\left(k_{Nx}x_{i+\frac{1}{2}}+k_{Ny}y_{j+\frac{1}{2}}+k_{Nz}z_k-\omega\left(n-\frac{1}{2}\right)\Delta t\right)} = \\ & \frac{E_{x0} e^{-j\left(k_{Nx}x_{i+\frac{1}{2}}+k_{Ny}y_{j+1}+k_{Nz}z_k-\omega n\Delta t\right)} - E_{x0} e^{-j\left(k_{Nx}x_{i+\frac{1}{2}}+k_{Ny}y_j+k_{Nz}z_k-\omega n\Delta t\right)}}{\Delta y_j} \\ & - \frac{E_{y0} e^{-j\left(k_{Nx}x_{i+1}+k_{Ny}y_{j+\frac{1}{2}}+k_{Nz}z_k-\omega n\Delta t\right)} - E_{y0} e^{-j\left(k_{Nx}x_i+k_{Ny}y_{j+\frac{1}{2}}+k_{Nz}z_k-\omega n\Delta t\right)}}{\Delta x_i} \end{aligned} \quad (5.61)$$

and simplifying it by using $e^{-j(k_{Nx}x_i+k_{Ny}y_j+k_{Nz}z_k-\omega n\Delta t)}$

$$\begin{aligned} & \frac{\mu H_{z0}}{\Delta t} e^{-j\left(k_{Nx}\left(x_{i+\frac{1}{2}}-x_i\right)+k_{Ny}\left(y_{j+\frac{1}{2}}-y_j\right)-\omega\frac{\Delta t}{2}\right)} - \\ & \frac{\mu H_{z0}}{\Delta t} e^{-j\left(k_{Nx}\left(x_{i+\frac{1}{2}}-x_i\right)+k_{Ny}\left(y_{j+\frac{1}{2}}-y_j\right)+\omega\frac{\Delta t}{2}\right)} = \\ & \frac{E_{x0} e^{-j\left(k_{Nx}\left(x_{i+\frac{1}{2}}-x_i\right)+k_{Ny}\left(y_{j+1}-y_j\right)\right)} - E_{x0} e^{-jk_{Nx}\left(x_{i+\frac{1}{2}}-x_i\right)}}{\Delta y_j} - \end{aligned} \quad (5.62)$$

$$\frac{E_{y0} e^{-j\left(k_{Nx}(x_{i+1}-x_i)+k_{Ny}\left(y_{j+\frac{1}{2}}-y_j\right)\right)} - E_{y0} e^{-jk_{Ny}\left(y_{j+\frac{1}{2}}-y_j\right)}}{\Delta x_i}$$

where a compact version is obtained by using the exponentials

$$\begin{aligned} \mu H_{z0} e^{-j\left(k_{Nx}\frac{\Delta x_i}{2}+k_{Ny}\frac{\Delta y_j}{2}\right)} \frac{e^{j\omega\frac{\Delta t}{2}} - e^{-j\omega\frac{\Delta t}{2}}}{\Delta t} \\ = E_{x0} e^{-j\left(k_{Nx}\frac{\Delta x_i}{2}+k_{Ny}\frac{\Delta y_j}{2}\right)} \frac{e^{-jk_{Ny}\frac{\Delta y_j}{2}} - e^{jk_{Ny}\frac{\Delta y_j}{2}}}{\Delta y_j} \\ - E_{y0} e^{-j\left(k_{Nx}\frac{\Delta x_i}{2}+k_{Ny}\frac{\Delta y_j}{2}\right)} \frac{e^{-jk_{Nx}\frac{\Delta x_i}{2}} - e^{jk_{Nx}\frac{\Delta x_i}{2}}}{\Delta x_i}. \end{aligned} \quad (5.63)$$

Finally, the NU NDE for H_z is found as

$$\mu H_{z0} \frac{\sin\left(\omega\frac{\Delta t}{2}\right)}{\Delta t} = E_{y0} \frac{\sin\left(k_{Nx}\frac{\Delta x_i}{2}\right)}{\Delta x_i} - E_{x0} \frac{\sin\left(k_{Ny}\frac{\Delta y_j}{2}\right)}{\Delta y_j}. \quad (5.64)$$

- NU NDE for E_x Component

Let us reconsider the E_x update equation as

$$\begin{aligned} \varepsilon \frac{E_x|_{i+\frac{1}{2},j,k}^{n+1} - E_x|_{i+\frac{1}{2},j,k}^n}{2\Delta t} \\ = \frac{H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - H_z|_{i+\frac{1}{2},j-\frac{1}{2},k}^{n+\frac{1}{2}}}{\Delta y_j + \Delta y_{j-1}} - \frac{H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - H_y|_{i+\frac{1}{2},j,k-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta z_k + \Delta z_{k-1}} \end{aligned} \quad (5.65)$$

where the plane wave forms are inserted into above equation that

$$\begin{aligned} \frac{\varepsilon E_{x0}}{2\Delta t} e^{-j\left(k_{Nx}x_{i+\frac{1}{2}}+k_{Ny}y_j+k_{Nz}z_k-\omega(n+1)\Delta t\right)} - \\ \frac{\varepsilon E_{x0}}{2\Delta t} e^{-j\left(k_{Nx}x_{i+\frac{1}{2}}+k_{Ny}y_j+k_{Nz}z_k-\omega n\Delta t\right)} = \end{aligned} \quad (5.66)$$

$$\begin{aligned}
& \frac{H_{z0}}{\Delta y_j + \Delta y_{j-1}} e^{-j\left(k_{Nx}x_{i+\frac{1}{2}} + k_{Ny}y_{j+\frac{1}{2}} + k_{Nz}z_k - \omega\left(n+\frac{1}{2}\right)\Delta t\right)} - \\
& \frac{H_{z0}}{\Delta y_j + \Delta y_{j-1}} e^{-j\left(k_{Nx}x_{i+\frac{1}{2}} + k_{Ny}y_{j-\frac{1}{2}} + k_{Nz}z_k - \omega\left(n+\frac{1}{2}\right)\Delta t\right)} - \\
& \frac{H_{y0}}{\Delta z_k + \Delta z_{k-1}} e^{-j\left(k_{Nx}x_{i+\frac{1}{2}} + k_{Ny}y_j + k_{Nz}z_{k+\frac{1}{2}} - \omega\left(n+\frac{1}{2}\right)\Delta t\right)} - \\
& \frac{H_{y0}}{\Delta z_k + \Delta z_{k-1}} e^{-j\left(k_{Nx}x_{i+\frac{1}{2}} + k_{Ny}y_j + k_{Nz}z_{k-\frac{1}{2}} - \omega\left(n+\frac{1}{2}\right)\Delta t\right)}
\end{aligned}$$

and simplifying it by using $e^{-j(k_{Nx}x_i + k_{Ny}y_j + k_{Nz}z_k - \omega n \Delta t)}$

$$\begin{aligned}
& \varepsilon \frac{E_{x0} e^{-j\left(k_{Nx}\left(x_{i+\frac{1}{2}} - x_i\right) - \omega \Delta t\right)} - E_{x0} e^{-j\left(k_{Nx}\left(x_{i+\frac{1}{2}} - x_i\right)\right)}}{2\Delta t} = \\
& \frac{H_{z0}}{\Delta y_j + \Delta y_{j-1}} e^{-j\left(k_{Nx}\left(x_{i+\frac{1}{2}} - x_i\right) + k_{Ny}\left(y_{j+\frac{1}{2}} - y_j\right) - \omega \frac{\Delta t}{2}\right)} - \\
& \frac{H_{z0}}{\Delta y_j + \Delta y_{j-1}} e^{-j\left(k_{Nx}\left(x_{i+\frac{1}{2}} - x_i\right) + k_{Ny}\left(y_{j-\frac{1}{2}} - y_j\right) - \omega \frac{\Delta t}{2}\right)} - \\
& \frac{H_{y0}}{\Delta z_k + \Delta z_{k-1}} e^{-j\left(k_{Nx}\left(x_{i+\frac{1}{2}} - x_i\right) + k_{Nz}\left(z_{k+\frac{1}{2}} - z_k\right) - \omega \frac{\Delta t}{2}\right)} - \\
& \frac{H_{y0}}{\Delta z_k + \Delta z_{k-1}} e^{-j\left(k_{Nx}\left(x_{i+\frac{1}{2}} - x_i\right) + k_{Nz}\left(z_{k-\frac{1}{2}} - z_k\right) - \omega \frac{\Delta t}{2}\right)}.
\end{aligned} \tag{5.67}$$

Then, the NU mesh intervals are applied as

$$\begin{aligned}
& \varepsilon E_{x0} e^{-jk_{Nx}\frac{\Delta x_i}{2}} \frac{e^{j\omega \Delta t} - 1}{2\Delta t} \\
& = H_{z0} \frac{e^{-j\left(k_{Nx}\frac{\Delta x_i}{2} + k_{Ny}\frac{\Delta y_j}{2} - \omega \frac{\Delta t}{2}\right)} - e^{-j\left(k_{Nx}\frac{\Delta x_i}{2} + k_{Ny}\frac{\Delta y_{j-1}}{2} - \omega \frac{\Delta t}{2}\right)}}{\Delta y_j + \Delta y_{j-1}} \\
& - H_{y0} \frac{e^{-j\left(k_{Nx}\frac{\Delta x_i}{2} + k_{Nz}\frac{\Delta z_k}{2} - \omega \frac{\Delta t}{2}\right)} - e^{-j\left(k_{Nx}\frac{\Delta x_i}{2} + k_{Nz}\frac{\Delta z_{k-1}}{2} - \omega \frac{\Delta t}{2}\right)}}{\Delta z_k + \Delta z_{k-1}}
\end{aligned} \tag{5.68}$$

where the exponential terms are converted that

$$\begin{aligned}
& \varepsilon E_{x0} e^{-jk_{Nx} \frac{\Delta x_i}{2}} e^{j\omega \frac{\Delta t}{2}} \frac{e^{j\omega \frac{\Delta t}{2}} - e^{-j\omega \frac{\Delta t}{2}}}{2\Delta t} \\
& = H_{z0} e^{-jk_{Nx} \frac{\Delta x_i}{2}} e^{j\omega \frac{\Delta t}{2}} \frac{e^{-jk_{Ny} \frac{\Delta y_j}{2}} - e^{-jk_{Ny} \frac{\Delta y_{j-1}}{2}}}{\Delta y_j + \Delta y_{j-1}} \\
& \quad - H_{y0} e^{-jk_{Nx} \frac{\Delta x_i}{2}} e^{j\omega \frac{\Delta t}{2}} \frac{e^{-jk_{Nz} \frac{\Delta z_k}{2}} - e^{-jk_{Nz} \frac{\Delta z_{k-1}}{2}}}{\Delta z_k + \Delta z_{k-1}}.
\end{aligned} \tag{5.69}$$

Finally, the NU NDE for E_x is found as

$$\begin{aligned}
& \varepsilon E_{x0} \frac{j \sin\left(\omega \frac{\Delta t}{2}\right)}{\Delta t} \\
& = H_{z0} \frac{e^{-jk_{Ny} \frac{\Delta y_j}{2}} - e^{-jk_{Ny} \frac{\Delta y_{j-1}}{2}}}{\Delta y_j + \Delta y_{j-1}} - H_{y0} \frac{e^{-jk_{Nz} \frac{\Delta z_k}{2}} - e^{-jk_{Nz} \frac{\Delta z_{k-1}}{2}}}{\Delta z_k + \Delta z_{k-1}}.
\end{aligned} \tag{5.70}$$

- NU NDE for E_y Component

Let us reconsider the E_y update equation as

$$\begin{aligned}
& \varepsilon \frac{E_y|_{i,j+\frac{1}{2},k}^{n+1} - E_y|_{i,j+\frac{1}{2},k}^n}{2\Delta t} \\
& = \frac{H_x|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - H_x|_{i,j+\frac{1}{2},k-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta z_k + \Delta z_{k-1}} - \frac{H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - H_z|_{i-\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}}}{\Delta x_i + \Delta x_{i-1}}
\end{aligned} \tag{5.71}$$

where the plane wave forms are inserted into above equation that

$$\begin{aligned}
& \frac{\varepsilon E_{y0}}{2\Delta t} e^{-j\left(k_{Nx}x_i + k_{Ny}y_{j+\frac{1}{2}} + k_{Nz}z_k - \omega(n+1)\Delta t\right)} - \\
& \frac{\varepsilon E_{y0}}{2\Delta t} e^{-j\left(k_{Nx}x_i + k_{Ny}y_{j+\frac{1}{2}} + k_{Nz}z_k - \omega n\Delta t\right)} = \\
& \frac{H_{x0}}{\Delta z_k + \Delta z_{k-1}} e^{-j\left(k_{Nx}x_i + k_{Ny}y_{j+\frac{1}{2}} + k_{Nz}z_{k+\frac{1}{2}} - \omega\left(n+\frac{1}{2}\right)\Delta t\right)} -
\end{aligned} \tag{5.72}$$

$$\begin{aligned}
& \frac{H_{x0}}{\Delta z_k + \Delta z_{k-1}} e^{-j\left(k_{Nx}x_i + k_{Ny}y_{j+\frac{1}{2}} + k_{Nz}z_{k-\frac{1}{2}} - \omega\left(n+\frac{1}{2}\right)\Delta t\right)} - \\
& \frac{H_{z0}}{\Delta x_i + \Delta x_{i-1}} e^{-j\left(k_{Nx}x_{i+\frac{1}{2}} + k_{Ny}y_{j+\frac{1}{2}} + k_{Nz}z_k - \omega\left(n+\frac{1}{2}\right)\Delta t\right)} - \\
& \frac{H_{z0}}{\Delta x_i + \Delta x_{i-1}} e^{-j\left(k_{Nx}x_{i-\frac{1}{2}} + k_{Ny}y_{j+\frac{1}{2}} + k_{Nz}z_k - \omega\left(n+\frac{1}{2}\right)\Delta t\right)}
\end{aligned}$$

and simplifying it by using $e^{-j(k_{Nx}x_i + k_{Ny}y_j + k_{Nz}z_k - \omega n\Delta t)}$

$$\begin{aligned}
& \varepsilon \frac{E_{y0} e^{-j\left(k_{Ny}\left(y_{j+\frac{1}{2}} - y_j\right) - \omega\Delta t\right)} - E_{y0} e^{-j\left(k_{Ny}\left(y_{j+\frac{1}{2}} - y_j\right)\right)}}{2\Delta t} = \\
& \frac{H_{x0}}{\Delta z_k + \Delta z_{k-1}} e^{-j\left(k_{Ny}\left(y_{j+\frac{1}{2}} - y_j\right) + k_{Nz}\left(z_{k+\frac{1}{2}} - z_k\right) - \omega\frac{\Delta t}{2}\right)} - \\
& \frac{H_{x0}}{\Delta z_k + \Delta z_{k-1}} e^{-j\left(k_{Ny}\left(y_{j+\frac{1}{2}} - y_j\right) + k_{Nz}\left(z_{k-\frac{1}{2}} - z_k\right) - \omega\frac{\Delta t}{2}\right)} - \\
& \frac{H_{z0}}{\Delta x_i + \Delta x_{i-1}} e^{-j\left(k_{Nx}\left(x_{i+\frac{1}{2}} - x_i\right) + k_{Ny}\left(y_{j+\frac{1}{2}} - y_j\right) - \omega\frac{\Delta t}{2}\right)} - \\
& \frac{H_{z0}}{\Delta x_i + \Delta x_{i-1}} e^{-j\left(k_{Nx}\left(x_{i-\frac{1}{2}} - x_i\right) + k_{Ny}\left(y_{j+\frac{1}{2}} - y_j\right) - \omega\frac{\Delta t}{2}\right)}
\end{aligned} \tag{5.73}$$

where using the NU mesh intervals, the exponentials are replaced by the trigonometric equivalencies

$$\begin{aligned}
& \varepsilon E_{y0} e^{-jk_{Ny}\frac{\Delta y_j}{2}} e^{j\omega\frac{\Delta t}{2}} \frac{e^{j\omega\frac{\Delta t}{2}} - e^{-j\omega\frac{\Delta t}{2}}}{2\Delta t} \\
& = H_{x0} e^{-jk_{Ny}\frac{\Delta y_j}{2}} e^{j\omega\frac{\Delta t}{2}} \frac{e^{-jk_{Nz}\frac{\Delta z_k}{2}} - e^{-jk_{Nz}\frac{\Delta z_{k-1}}{2}}}{\Delta z_k + \Delta z_{k-1}} \\
& - H_{z0} e^{-jk_{Ny}\frac{\Delta y_j}{2}} e^{j\omega\frac{\Delta t}{2}} \frac{e^{-jk_{Nx}\frac{\Delta x_i}{2}} - e^{-jk_{Nx}\frac{\Delta x_{i-1}}{2}}}{\Delta x_i + \Delta x_{i-1}}.
\end{aligned} \tag{5.74}$$

Finally, the NU NDE for E_y is found as

$$\begin{aligned}
& \varepsilon E_{y0} \frac{j \sin\left(\omega \frac{\Delta t}{2}\right)}{\Delta t} \\
&= H_{x0} \frac{e^{-jk_{Nz} \frac{\Delta z_k}{2}} - e^{-jk_{Nz} \frac{\Delta z_{k-1}}{2}}}{\Delta z_k + \Delta z_{k-1}} - H_{z0} \frac{e^{-jk_{Nx} \frac{\Delta x_i}{2}} - e^{-jk_{Nx} \frac{\Delta x_{i-1}}{2}}}{\Delta x_i + \Delta x_{i-1}}.
\end{aligned} \tag{5.75}$$

- NU NDE for E_z Component

Let us reconsider the E_z update equation as

$$\begin{aligned}
& \varepsilon \frac{E_z|_{i,j,k+\frac{1}{2}}^{n+1} - E_z|_{i,j,k+\frac{1}{2}}^n}{2\Delta t} \\
&= \frac{H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - H_y|_{i-\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta x_i + \Delta x_{i-1}} - \frac{H_x|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - H_x|_{i,j-\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta y_j + \Delta y_{j-1}}
\end{aligned} \tag{5.76}$$

where the plane wave forms are inserted into above equation that

$$\begin{aligned}
& \frac{\varepsilon E_{z0}}{2\Delta t} e^{-j\left(k_{Nx}x_i + k_{Ny}y_j + k_{Nz}z_{k+\frac{1}{2}} - \omega(n+1)\Delta t\right)} - \\
& \frac{\varepsilon E_{z0}}{2\Delta t} e^{-j\left(k_{Nx}x_i + k_{Ny}y_j + k_{Nz}z_{k+\frac{1}{2}} - \omega n\Delta t\right)} = \\
& \frac{H_{y0}}{\Delta x_i + \Delta x_{i-1}} e^{-j\left(k_{Nx}x_{i+\frac{1}{2}} + k_{Ny}y_j + k_{Nz}z_{k+\frac{1}{2}} - \omega\left(n+\frac{1}{2}\right)\Delta t\right)} - \\
& \frac{H_{y0}}{\Delta x_i + \Delta x_{i-1}} e^{-j\left(k_{Nx}x_{i-\frac{1}{2}} + k_{Ny}y_j + k_{Nz}z_{k+\frac{1}{2}} - \omega\left(n+\frac{1}{2}\right)\Delta t\right)} - \\
& \frac{H_{x0}}{\Delta y_j + \Delta y_{j-1}} e^{-j\left(k_{Nx}x_i + k_{Ny}y_{j+\frac{1}{2}} + k_{Nz}z_{k+\frac{1}{2}} - \omega\left(n+\frac{1}{2}\right)\Delta t\right)} - \\
& \frac{H_{x0}}{\Delta y_j + \Delta y_{j-1}} e^{-j\left(k_{Nx}x_i + k_{Ny}y_{j-\frac{1}{2}} + k_{Nz}z_{k+\frac{1}{2}} - \omega\left(n+\frac{1}{2}\right)\Delta t\right)}
\end{aligned} \tag{5.77}$$

and simplifying it by using $e^{-j(k_{Nx}x_i + k_{Ny}y_j + k_{Nz}z_k - \omega n\Delta t)}$

$$\begin{aligned}
& \varepsilon \frac{E_{z0} e^{-j\left(k_{Nz}\left(z_{k+\frac{1}{2}}-z_k\right)-\omega\Delta t\right)} - E_{z0} e^{-j\left(k_{Nz}\left(z_{k+\frac{1}{2}}-z_k\right)\right)}}{2\Delta t} = \\
& \frac{H_{y0}}{\Delta x_i + \Delta x_{i-1}} e^{-j\left(k_{Nx}\left(x_{i+\frac{1}{2}}-x_i\right)+k_{Nz}\left(z_{k+\frac{1}{2}}-z_k\right)-\omega\frac{\Delta t}{2}\right)} - \\
& \frac{H_{y0}}{\Delta x_i + \Delta x_{i-1}} e^{-j\left(k_{Nx}\left(x_{i-\frac{1}{2}}-x_i\right)+k_{Nz}\left(z_{k+\frac{1}{2}}-z_k\right)-\omega\frac{\Delta t}{2}\right)} - \\
& \frac{H_{x0}}{\Delta y_j + \Delta y_{j-1}} e^{-j\left(k_{Ny}\left(y_{j+\frac{1}{2}}-y_j\right)+k_{Nz}\left(z_{k+\frac{1}{2}}-z_k\right)-\omega\frac{\Delta t}{2}\right)} - \\
& \frac{H_{x0}}{\Delta y_j + \Delta y_{j-1}} e^{-j\left(k_{Ny}\left(y_{j-\frac{1}{2}}-y_j\right)+k_{Nz}\left(z_{k+\frac{1}{2}}-z_k\right)-\omega\frac{\Delta t}{2}\right)}.
\end{aligned} \tag{5.78}$$

Using the NU mesh intervals, the equation is simplified as

$$\begin{aligned}
& \varepsilon E_{z0} e^{-jk_{Nz}\frac{\Delta z_k}{2}} \frac{e^{j\omega\Delta t} - 1}{2\Delta t} \\
& = H_{y0} \frac{e^{-j\left(k_{Nx}\frac{\Delta x_i}{2}+k_{Nz}\frac{\Delta z_k}{2}-\omega\frac{\Delta t}{2}\right)} - e^{-j\left(k_{Nx}\frac{\Delta x_{i-1}}{2}+k_{Nz}\frac{\Delta z_k}{2}-\omega\frac{\Delta t}{2}\right)}}{\Delta x_i + \Delta x_{i-1}} \\
& - H_{x0} \frac{e^{-j\left(k_{Ny}\frac{\Delta y_j}{2}+k_{Nz}\frac{\Delta z_k}{2}-\omega\frac{\Delta t}{2}\right)} - e^{-j\left(k_{Ny}\frac{\Delta y_{j-1}}{2}+k_{Nz}\frac{\Delta z_k}{2}-\omega\frac{\Delta t}{2}\right)}}{\Delta y_j + \Delta y_{j-1}}
\end{aligned} \tag{5.79}$$

where the exponential terms can be converted to the trigonometric equivalencies as

$$\begin{aligned}
& \varepsilon E_{z0} e^{-jk_{Nz}\frac{\Delta z_k}{2}} e^{j\omega\frac{\Delta t}{2}} \frac{e^{j\omega\frac{\Delta t}{2}} - e^{-j\omega\frac{\Delta t}{2}}}{2\Delta t} \\
& = H_{y0} e^{-jk_{Nz}\frac{\Delta z_k}{2}} e^{j\omega\frac{\Delta t}{2}} \frac{e^{-jk_{Nx}\frac{\Delta x_i}{2}} - e^{-jk_{Nx}\frac{\Delta x_{i-1}}{2}}}{\Delta x_i + \Delta x_{i-1}} \\
& - H_{x0} e^{-jk_{Nz}\frac{\Delta z_k}{2}} e^{j\omega\frac{\Delta t}{2}} \frac{e^{-jk_{Ny}\frac{\Delta y_j}{2}} - e^{-jk_{Ny}\frac{\Delta y_{j-1}}{2}}}{\Delta y_j + \Delta y_{j-1}}.
\end{aligned} \tag{5.80}$$

Finally, the NU NDE for E_z is found as

$$\begin{aligned}
& \varepsilon E_{z0} \frac{j \sin\left(\omega \frac{\Delta t}{2}\right)}{\Delta t} \\
&= H_{y0} \frac{e^{-jk_{Nx} \frac{\Delta x_i}{2}} - e^{-jk_{Nx} \frac{\Delta x_{i-1}}{2}}}{\Delta x_i + \Delta x_{i-1}} - H_{x0} \frac{e^{-jk_{Ny} \frac{\Delta y_j}{2}} - e^{-jk_{Ny} \frac{\Delta y_{j-1}}{2}}}{\Delta y_j + \Delta y_{j-1}}.
\end{aligned} \tag{5.81}$$

Now, we have the six equations with the six unknowns. It means that this system of equations can be solved by a matrix operation. For this, let us rewrite the equations as

$$\mu H_{x0} \frac{\sin\left(\omega \frac{\Delta t}{2}\right)}{\Delta t} = E_{z0} \frac{\sin\left(k_{Ny} \frac{\Delta y_j}{2}\right)}{\Delta y_j} - E_{y0} \frac{\sin\left(k_{Nz} \frac{\Delta z_k}{2}\right)}{\Delta z_k} \tag{5.82}$$

$$\mu H_{y0} \frac{\sin\left(\omega \frac{\Delta t}{2}\right)}{\Delta t} = E_{x0} \frac{\sin\left(k_{Nz} \frac{\Delta z_k}{2}\right)}{\Delta z_k} - E_{z0} \frac{\sin\left(k_{Nx} \frac{\Delta x_i}{2}\right)}{\Delta x_i} \tag{5.83}$$

$$\mu H_{z0} \frac{\sin\left(\omega \frac{\Delta t}{2}\right)}{\Delta t} = E_{y0} \frac{\sin\left(k_{Nx} \frac{\Delta x_i}{2}\right)}{\Delta x_i} - E_{x0} \frac{\sin\left(k_{Ny} \frac{\Delta y_j}{2}\right)}{\Delta y_j} \tag{5.84}$$

$$\begin{aligned}
& \varepsilon E_{x0} \frac{j \sin\left(\omega \frac{\Delta t}{2}\right)}{\Delta t} \\
&= H_{z0} \frac{e^{-jk_{Ny} \frac{\Delta y_j}{2}} - e^{-jk_{Ny} \frac{\Delta y_{j-1}}{2}}}{\Delta y_j + \Delta y_{j-1}} - H_{y0} \frac{e^{-jk_{Nz} \frac{\Delta z_k}{2}} - e^{-jk_{Nz} \frac{\Delta z_{k-1}}{2}}}{\Delta z_k + \Delta z_{k-1}}
\end{aligned} \tag{5.85}$$

$$\begin{aligned}
& \varepsilon E_{y0} \frac{j \sin\left(\omega \frac{\Delta t}{2}\right)}{\Delta t} \\
&= H_{x0} \frac{e^{-jk_{Nz} \frac{\Delta z_k}{2}} - e^{-jk_{Nz} \frac{\Delta z_{k-1}}{2}}}{\Delta z_k + \Delta z_{k-1}} - H_{z0} \frac{e^{-jk_{Nx} \frac{\Delta x_i}{2}} - e^{-jk_{Nx} \frac{\Delta x_{i-1}}{2}}}{\Delta x_i + \Delta x_{i-1}}
\end{aligned} \tag{5.86}$$

$$\begin{aligned}
& \varepsilon E_{z0} \frac{j \sin\left(\omega \frac{\Delta t}{2}\right)}{\Delta t} \\
&= H_{y0} \frac{e^{-jk_{Nx} \frac{\Delta x_i}{2}} - e^{-jk_{Nx} \frac{\Delta x_{i-1}}{2}}}{\Delta x_i + \Delta x_{i-1}} - H_{x0} \frac{e^{-jk_{Ny} \frac{\Delta y_j}{2}} - e^{-jk_{Ny} \frac{\Delta y_{j-1}}{2}}}{\Delta y_j + \Delta y_{j-1}}.
\end{aligned} \tag{5.87}$$

Then, a 6×6 matrix can be solved as below

$$\begin{aligned} & \varepsilon \frac{j \sin\left(\omega \frac{\Delta t}{2}\right)}{\Delta t} E_{x0} + 0E_{y0} + 0E_{z0} + \\ & 0H_{x0} - \frac{e^{-jk_{Nz} \frac{\Delta z_k}{2}} - e^{-jk_{Nz} \frac{\Delta z_{k-1}}{2}}}{\Delta z_k + \Delta z_{k-1}} H_{y0} + \frac{e^{-jk_{Ny} \frac{\Delta y_j}{2}} - e^{-jk_{Ny} \frac{\Delta y_{j-1}}{2}}}{\Delta y_j + \Delta y_{j-1}} H_{z0} = 0 \end{aligned} \quad (5.88)$$

$$\begin{aligned} & 0E_{x0} + \varepsilon \frac{j \sin\left(\omega \frac{\Delta t}{2}\right)}{\Delta t} E_{y0} + 0E_{z0} + \\ & \frac{e^{-jk_{Nz} \frac{\Delta z_k}{2}} - e^{-jk_{Nz} \frac{\Delta z_{k-1}}{2}}}{\Delta z_k + \Delta z_{k-1}} H_{x0} + 0H_{y0} - \frac{e^{-jk_{Nx} \frac{\Delta x_i}{2}} - e^{-jk_{Nx} \frac{\Delta x_{i-1}}{2}}}{\Delta x_i + \Delta x_{i-1}} H_{z0} = 0 \end{aligned} \quad (5.89)$$

$$\begin{aligned} & 0E_{x0} + 0E_{y0} + \varepsilon \frac{j \sin\left(\omega \frac{\Delta t}{2}\right)}{\Delta t} E_{z0} - \\ & \frac{e^{-jk_{Ny} \frac{\Delta y_j}{2}} - e^{-jk_{Ny} \frac{\Delta y_{j-1}}{2}}}{\Delta y_j + \Delta y_{j-1}} H_{x0} + \frac{e^{-jk_{Nx} \frac{\Delta x_i}{2}} - e^{-jk_{Nx} \frac{\Delta x_{i-1}}{2}}}{\Delta x_i + \Delta x_{i-1}} H_{y0} + 0H_{z0} = 0 \end{aligned} \quad (5.90)$$

$$\begin{aligned} & 0E_{x0} - \frac{\sin\left(k_{Nz} \frac{\Delta z_k}{2}\right)}{\Delta z_k} E_{y0} + \frac{\sin\left(k_{Ny} \frac{\Delta y_j}{2}\right)}{\Delta y_j} E_{z0} + \\ & \mu \frac{\sin\left(\omega \frac{\Delta t}{2}\right)}{\Delta t} H_{x0} + 0H_{y0} + 0H_{z0} = 0 \end{aligned} \quad (5.91)$$

$$\begin{aligned} & \frac{\sin\left(k_{Nz} \frac{\Delta z_k}{2}\right)}{\Delta z_k} E_{x0} + 0E_{y0} - \frac{\sin\left(k_{Nx} \frac{\Delta x_i}{2}\right)}{\Delta x_i} E_{z0} + \\ & 0H_{x0} + \mu \frac{\sin\left(\omega \frac{\Delta t}{2}\right)}{\Delta t} H_{y0} + 0H_{z0} = 0 \end{aligned} \quad (5.92)$$

$$\begin{aligned} & -\frac{\sin\left(k_{Ny} \frac{\Delta y_j}{2}\right)}{\Delta y_j} E_{x0} + \frac{\sin\left(k_{Nx} \frac{\Delta x_i}{2}\right)}{\Delta x_i} E_{y0} + 0E_{z0} + \end{aligned} \quad (5.93)$$

$$0H_{x0} + 0H_{y0} + \mu \frac{\sin\left(\omega \frac{\Delta t}{2}\right)}{\Delta t} H_{z0} = 0.$$

Solving the matrix of $Ax = B$ in commercial software for [MATHEMATICA, 2020]

$$x = [E_{x0} \ E_{y0} \ E_{z0} \ H_{x0} \ H_{y0} \ H_{z0}]^T \quad (5.94)$$

where A is the coefficients of x and B is the right-hand side matrix. Then, one can find

$$\begin{aligned} E_{z0} \frac{1}{c^2 \Delta t^2} \sin^2\left(\frac{\omega \Delta t}{2}\right) = \\ E_{z0} \frac{j}{\Delta x_i (\Delta x_i + \Delta x_{i-1})} \sin\left(\frac{k_{Nx} \Delta x_i}{2}\right) \left(e^{-\frac{jk_{Nx} \Delta x_i}{2}} - e^{\frac{jk_{Nx} \Delta x_{i-1}}{2}}\right) + \\ E_{z0} \frac{j}{\Delta y_j (\Delta y_j + \Delta y_{j-1})} \sin\left(\frac{k_{Ny} \Delta y_j}{2}\right) \left(e^{-\frac{jk_{Ny} \Delta y_j}{2}} - e^{\frac{jk_{Ny} \Delta y_{j-1}}{2}}\right) + \\ E_{z0} \frac{j}{\Delta z_k (\Delta z_k + \Delta z_{k-1})} \sin\left(\frac{k_{Nz} \Delta z_k}{2}\right) \left(e^{-\frac{jk_{Nz} \Delta z_k}{2}} - e^{\frac{jk_{Nz} \Delta z_{k-1}}{2}}\right) \end{aligned} \quad (5.95)$$

then, cancelling the E_{z0} magnitudes; finally, the 3D NU NDE is found as

$$\begin{aligned} \frac{1}{c^2 \Delta t^2} \sin^2\left(\frac{\omega \Delta t}{2}\right) = \\ \frac{j}{\Delta x_i (\Delta x_i + \Delta x_{i-1})} \sin\left(\frac{k_{Nx} \Delta x_i}{2}\right) \left(e^{-\frac{jk_{Nx} \Delta x_i}{2}} - e^{\frac{jk_{Nx} \Delta x_{i-1}}{2}}\right) + \\ \frac{j}{\Delta y_j (\Delta y_j + \Delta y_{j-1})} \sin\left(\frac{k_{Ny} \Delta y_j}{2}\right) \left(e^{-\frac{jk_{Ny} \Delta y_j}{2}} - e^{\frac{jk_{Ny} \Delta y_{j-1}}{2}}\right) + \\ \frac{j}{\Delta z_k (\Delta z_k + \Delta z_{k-1})} \sin\left(\frac{k_{Nz} \Delta z_k}{2}\right) \left(e^{-\frac{jk_{Nz} \Delta z_k}{2}} - e^{\frac{jk_{Nz} \Delta z_{k-1}}{2}}\right). \end{aligned} \quad (5.96)$$

This formula is the same as in [Christ et al, 2002]. To reduce the numerical dispersion error, Δx_i , Δy_j and Δz_k should be chosen minimum as much as possible in which the NU mesh intervals will approach to the uniform mesh intervals for approximating to the dispersionless solution.

An alternative formulation and extraction of the NDE for the uniform mesh, the NU CGL mesh and the NU Linear mesh are given in Appendix.

5.3. Numerical Stability

All the numerical solutions must be stable. To maintain the stability in the sense of the spatial discretization and the temporal discretization, the NU FDTD method must satisfy the CFL stability condition. Within this scope, for the numerical stability of the NU FDTD method in the 3D Cartesian coordinates, the unit time step (Δt) must satisfy the following condition

$$\Delta t \leq \frac{1}{c_{max} \sqrt{\frac{1}{\Delta x_{min}^2} + \frac{1}{\Delta y_{min}^2} + \frac{1}{\Delta z_{min}^2}}} \quad (5.97)$$

where Δx_{min} , Δy_{min} and Δz_{min} are the minimum NU mesh intervals and c_{max} is the maximum phase velocity [Remis, 2006].

5.4. PEC Boundary Condition

The implementation of the Perfect Electric Conductor (PEC) boundary condition in the NU FDTD method is the same as the U FDTD method. This is important in this thesis since the walls of the cavities considered as numerical examples are assumed to be the PEC material.

6. NUMERICAL EXAMPLES

A rectangular empty cavity and a rectangular partially dielectric filled cavity (PDFC) are used as numerical examples in 3D Cartesian coordinates. First, the general 3D NDE of the homogenous medium is extracted for the NU FDTD method. In the numerical analyses, it is used for the empty cavity problem, but cannot be used for the PDFC problem. In this sense, using the three different types of the NU meshes, the numerical results are compared with validations in MATLAB environment [MATLAB, 2018].

6.1. Empty Cavity

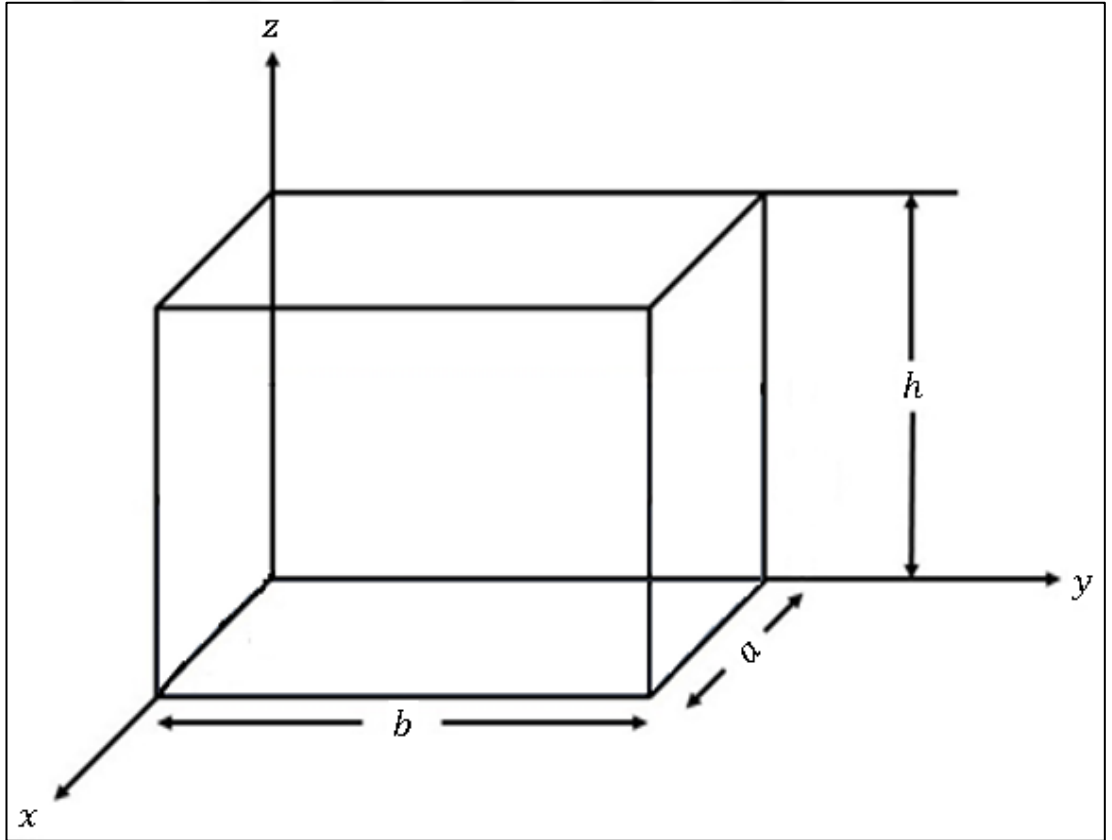


Figure 6.1: The 3D empty rectangular cavity resonator.

In Figure 6.1, the geometry of the considered 3D empty cavity resonator is shown where a , b and h are its dimensions along the directions of x , y and z , respectively.

First of all, let us analyze this problem, analytically. This analysis is generally based on a solution of a sourceless Helmholtz equation as

$$\Delta u(x, y, z) + k^2 u(x, y, z) = 0 \quad (6.1)$$

where $u(x, y, z)$ is one of the field components and $k = \omega/c$ is the wavenumber in the lossless cavity. It is known that the mode-based resonance frequency formula is given by [Balanis, 1989]

$$f = f_{mnp} = \frac{c}{2\pi} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{h}\right)^2} \quad (6.2)$$

where $m, n, p = 0, 1, 2, \dots$ are the mode numbers and c is the light velocity. Let us assume that a TM_{mnp}^z mode ($H_z = 0, E_z \neq 0$) is excited as [Balanis, 1989]

$$H_x = A \left(\frac{n\pi}{b}\right) \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \cos\left(\frac{p\pi}{h}z\right) \quad (6.3)$$

$$H_y = -A \left(\frac{m\pi}{a}\right) \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \cos\left(\frac{p\pi}{h}z\right) \quad (6.4)$$

$$H_z = 0 \quad (6.5)$$

$$E_x = -\frac{A}{i\omega\epsilon} \left[\left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{h}\right)\right] \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \sin\left(\frac{p\pi}{h}z\right) \quad (6.6)$$

$$E_y = -\frac{A}{i\omega\epsilon} \left[\left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{h}\right)\right] \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \sin\left(\frac{p\pi}{h}z\right) \quad (6.7)$$

$$E_z = -\frac{A}{i\omega\epsilon} \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \cos\left(\frac{p\pi}{h}z\right) \quad (6.8)$$

where the unknown coefficient A can be found by using a normalization condition of the total energy concept [Aksoy and Tretyakov, 2002].

In this part, the numerical dispersion equation is solved using both the uniform mesh and the NU mesh for the 3D empty cavity. The numerical phase velocities are extracted from the solution of the 3D NDE by using the root finding algorithm. The numerical results are shown for two different Courant factors of $S = 1.0$ and $S = 0.5$ with a comparison of the analytical solution of the empty cavity.

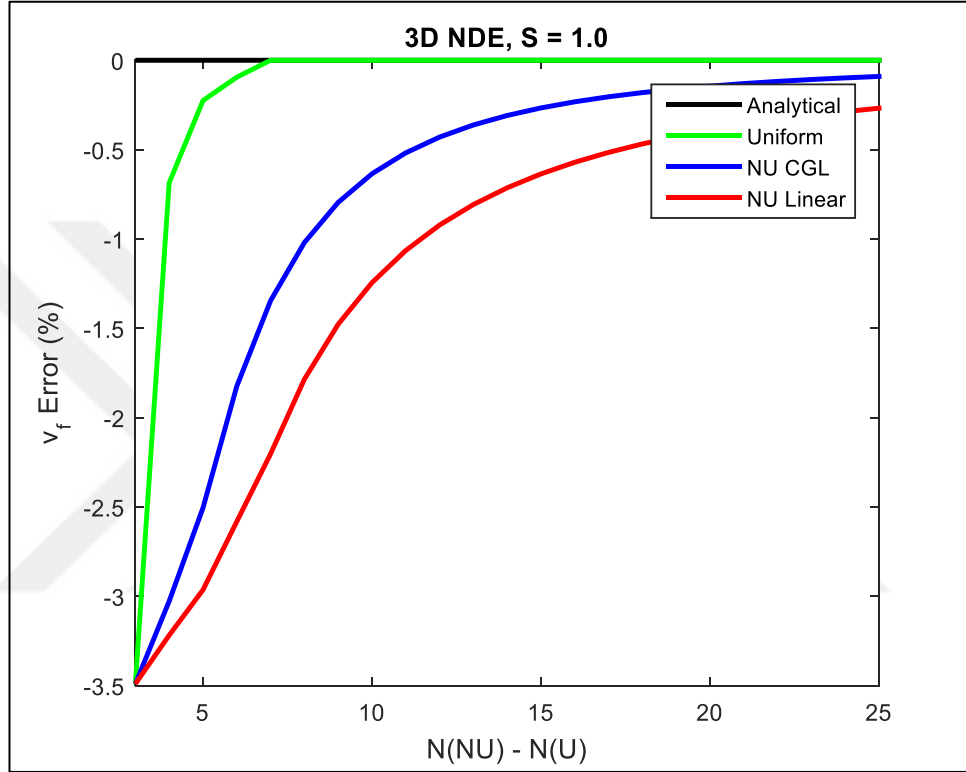


Figure 6.2: The comparison of the numerical phase velocity errors for $S = 1.0$.

In Figure 6.2, the 3D NU NDE is solved for the calculation of the numerical phase velocity errors for $S = 1.0$. The comparisons show that the uniform mesh gives the most accurate results versus the total number of mesh nodes. The NU CGL mesh gives better results than the NU Linear mesh.

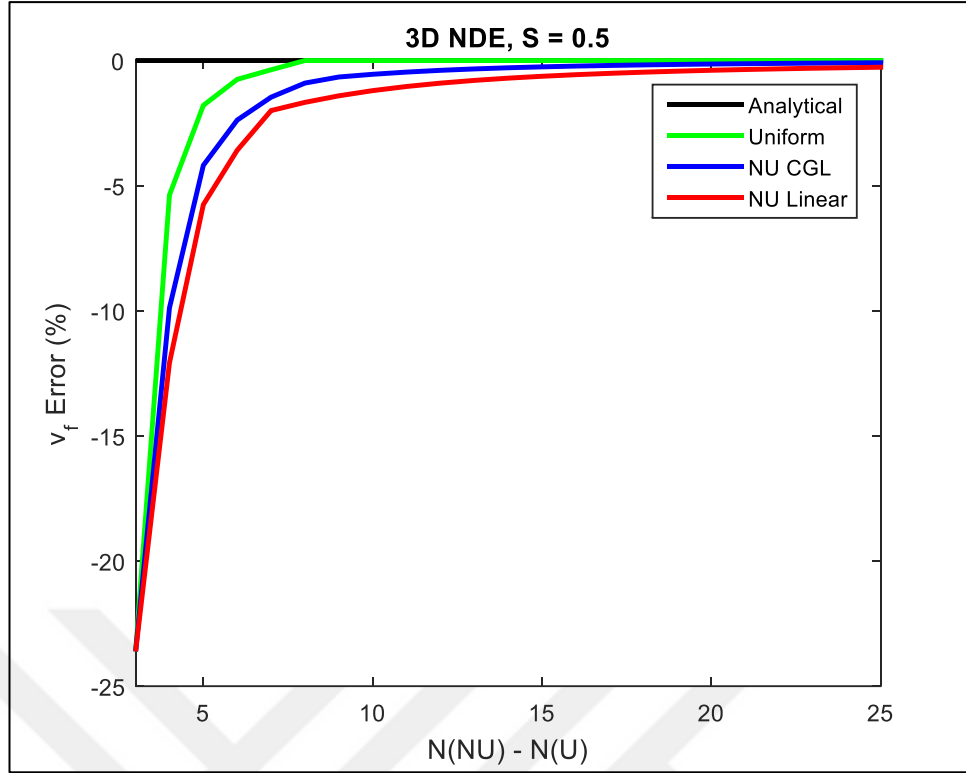


Figure 6.3: The comparison of the numerical phase velocity errors for $S = 0.5$.

In Figure 6.3, the numerical phase velocity errors are compared for $S = 0.5$. The uniform mesh again keeps its superiority over the NU meshes. The NU CGL mesh again comes second. The NU Linear mesh has the worst accuracy.

6.1.1. Single-Mode Excitation

A monochromatic sinusoidal soft source is used for the single-mode (dominant) TM_{110}^z excitation as

$$E_z|_{x_s, y_s, z_s}^{n+1} = E_z|_{x_s, y_s, z_s}^n + \sin(2\pi f n \Delta t) \quad (6.9)$$

where x_s, y_s, z_s show the source location along the x, y, z axes, respectively. The optimum source location is found for the desired cavity mode [Başaran et al., 2002]. The observation point is chosen at $x_s = a/2, y_s = b/2, z_s = h/4$. In the considered FDTD solution, the lengths of the cavity are set to $a = b = h = 1$ m. In the calculations, 100000 time steps and $\Delta x_{min} = \lambda/20$ spatial unit step are used for all the

TM_{mnp}^z modes. The values of the proper resonance frequencies are calculated, analytically. The numerical results are given for the uniform, the NU CGL and the NU Linear meshes.

6.1.1.1. Field Distributions

The field distributions of the cavity modes can be used for the validation. It is performed by comparing the FDTD calculated and analytical field distributions. For the empty cavity, the field distributions of the modes are analytically well-known including the source location effect of the cavity [Başaran et al., 2002].

In Figure 6.4, the TM_{110}^z field distribution of the E_z electric field at xy plane is demonstrated for the three different types of the meshes. For the U FDTD solution, a magic time step ($S = 1.0$) is applied that it corresponds to the analytical solution. The numerical results show that the NU CGL mesh shows the correct field distribution comparing to the U FDTD solution. But, the field amplitude is changed due to the NU meshing. The NU Linear mesh shows the incorrect cavity mode distribution. The reason for this its continuous expansion along all the problem geometry. Therefore, using of the NU Linear mesh is not convenient in the sense of global mean. It must be applied all the problem boundaries, locally. Specially, it is worth mentioning that the NU CGL mesh and the NU Linear mesh have the first order local truncation error while the uniform mesh has the second order error.

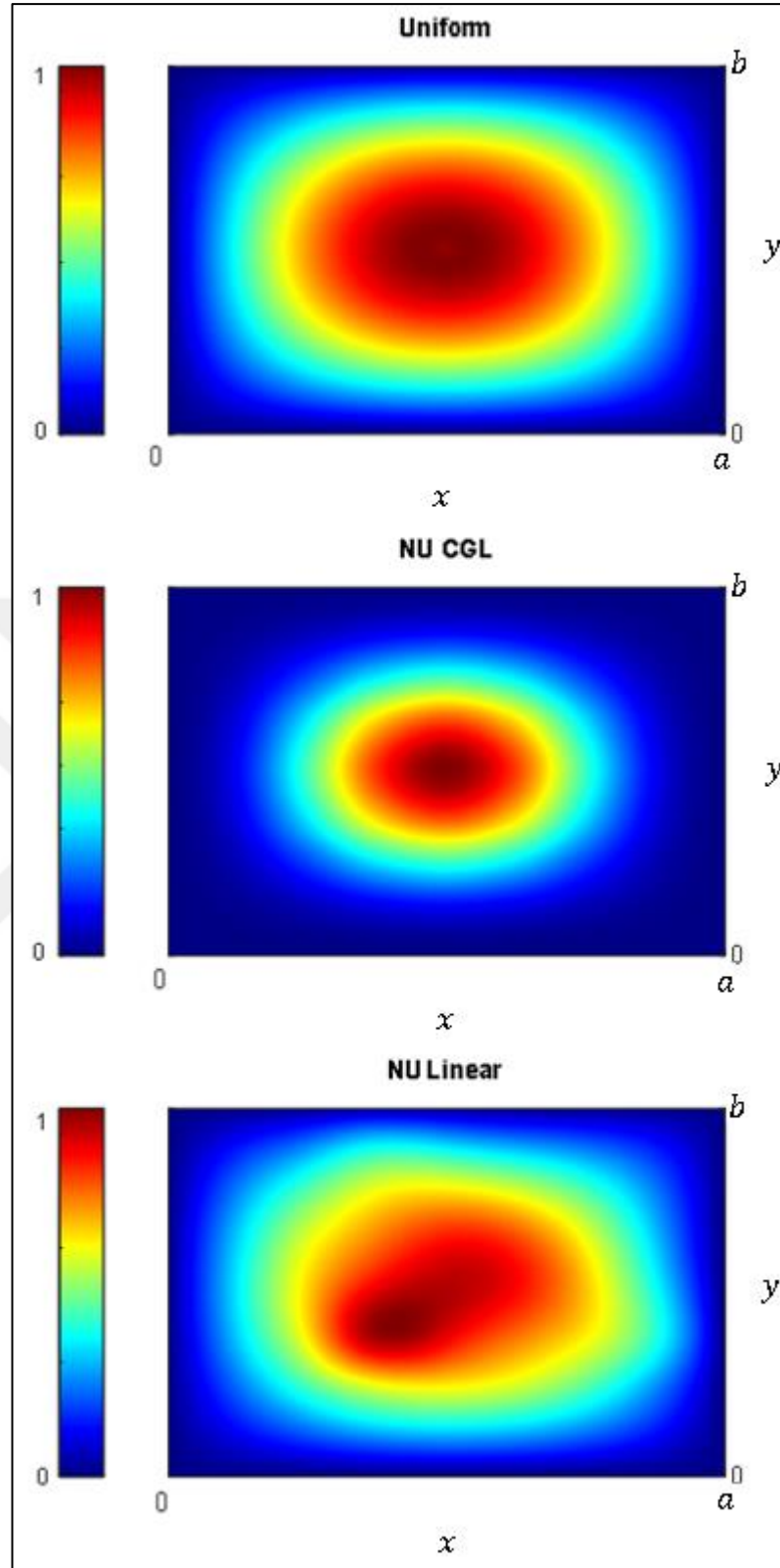


Figure 6.4: The normalized E_z electric field distributions of the TM_{110}^z cavity mode at the xy plane for the uniform, the NU CGL and the NU Linear meshes ($S = 1.0$).

6.1.1.2. Time Domain Results

A time domain analytical solution of a rectangular cavity with a fully filled lossy material is known [Aksoy and Tretyakov, 2002]. When limiting that solution to the lossy medium ($\sigma \rightarrow 0$), the time part of the lossless field expression is given by

$$E_z = \frac{A}{2} \frac{1 - e^{-\gamma t}}{\gamma \sqrt{1 - (\gamma/\omega_n)^2}} \sin(\omega_n t) \Rightarrow \lim_{\sigma \rightarrow 0} E_z = \frac{A}{2} t \sin(\omega_n t) \quad (6.10)$$

where A is the proper mode constant, $\gamma = \sigma/2\varepsilon_0\varepsilon$ and $\omega_n = \omega_{resonance}$ is the angular resonance frequency. This solution means that the E_z magnitude goes to infinity when the time goes to infinity. Therefore, the continuous increase in magnitude of the time domain signal means more accurate numerical simulation. In Figure 6.5, the time responses of the three different types of the FDTD meshes are shown for $S = 1.0$.

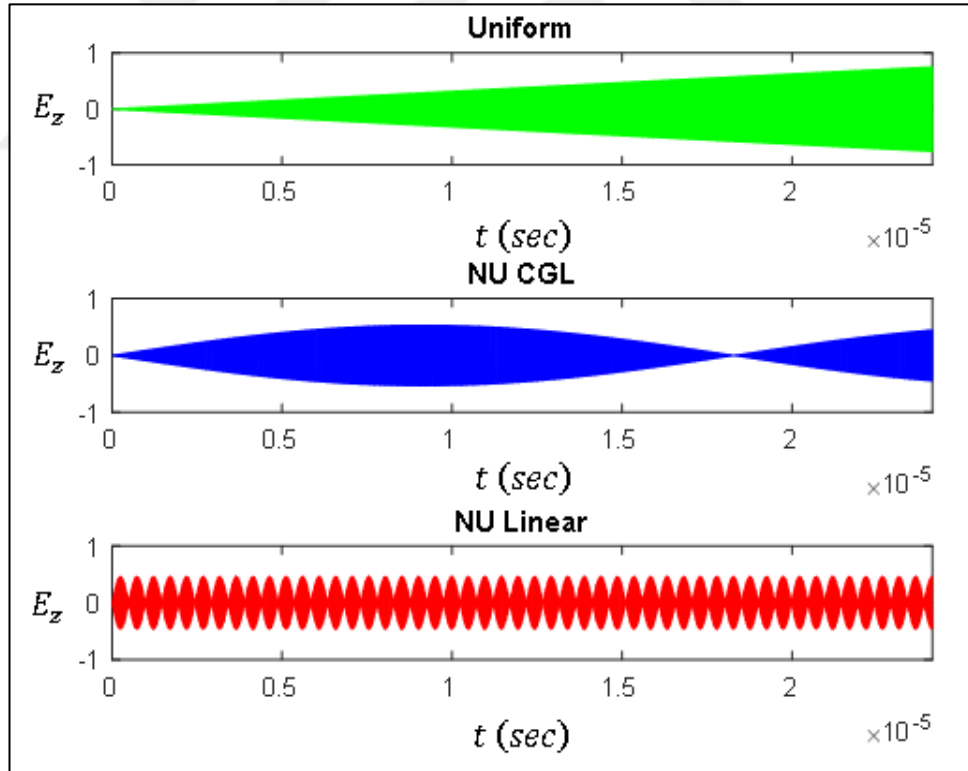


Figure 6.5: The time variations of the normalized E_z electric field for the uniform, the NU CGL and the NU Linear meshes for $S = 1.0$.

The earlier closing signal intervals in time domain are related to the different numerical errors of the solutions. The results show that the NU CGL mesh gives the best results referencing to the U FDTD solution ($S = 1.0$). However, the NU Linear mesh has very big deficiency. The accuracy levels of the different meshes also completely correspond to the NDE analysis given before.

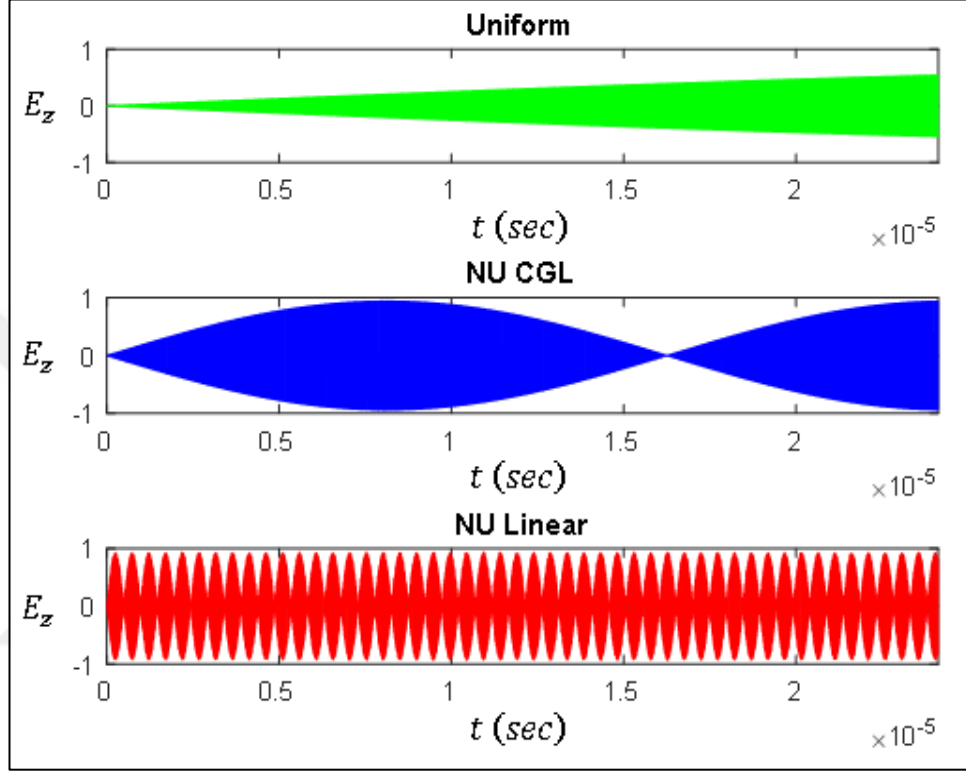


Figure 6.6: The time variations of the normalized E_z electric field for the uniform, the NU CGL and the NU Linear meshes for $S = 0.5$.

From this point of view, the same problem is solved for $S = 0.5$. In Figure 6.6, the time domain result of the normalized E_z electric field is demonstrated by the uniform, the NU CGL and the NU Linear meshes. In order to perform faithful comparisons, the same time step (Δt) corresponding to the minimum spatial step (Δx_{min}) are used in all the simulations. The results show that the NU CGL mesh again gives the best results referencing to the U FDTD solution. However, the NU Linear mesh has very big deficiency. The accuracy levels of the different meshes again completely correspond to the NDE analysis given before.

The results of these two examples can be summarized that

- the NU CGL and the NU Linear mesh solutions differ from each other since the structures of the meshes are completely different.
- the NU CGL mesh gives better accuracy than the NU Linear mesh for the same Δt , Δx_{min} and N_x .
- the two NU meshes give worse results than the uniform mesh since they have first-order truncation errors.
- the worst results are obtained from the NU Linear meshes.

Specially, it is worth noting that the NU FDTD method does not have a magic time step. Therefore, the same Δt , Δx_{min} and N_x are used for the faithful comparison.

6.1.2. Multi-Mode Excitations

The resonance behavior of the cavity is commonly validated from the resonance peaks calculated in frequency domain. The time results are converted to the frequency domain by using the Fast Fourier Transform (FFT). To perform this, let us excite the cavity with a wideband pulse point soft source as

$$E_z|_{x_s, y_s, z_s}^{n+1} = E_z|_{x_s, y_s, z_s}^n + \frac{e^{-\frac{(n\Delta t)^2}{2\tau^2}}}{\sqrt{2\pi\tau}} \quad (6.11)$$

where x_s, y_s, z_s are the source location and $\tau = 9.43 \times 10^{-9}$. The source is located in the cavity based on the calculations in [Başaran et al., 2002] while the observation point is located at center of xy plane and $z = h/10$.

In Figure 6.7, the resonance peaks of the six cavity modes are shown for the uniform mesh of $S = 1.0$. Some degenerated modes are also shown in the result.

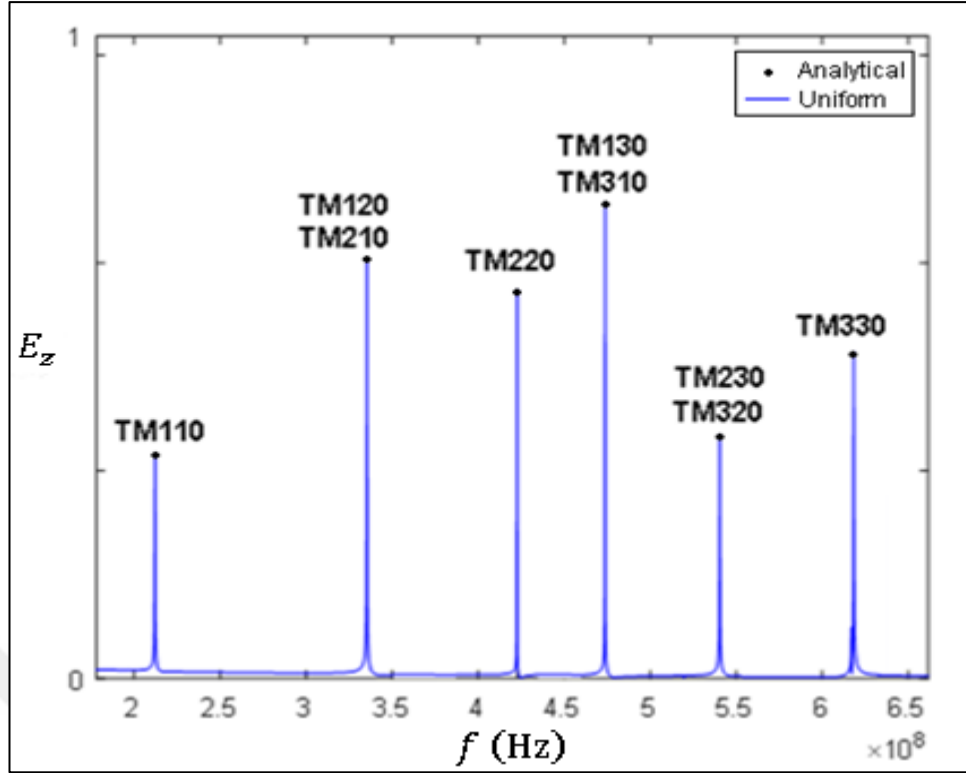


Figure 6.7: The normalized E_z electric field resonance peaks of the six different cavity modes for the uniform mesh of $S = 1.0$.

The FDTD calculated peak values of the resonance frequencies corresponds to the analytical well-known resonance frequency formula given before.

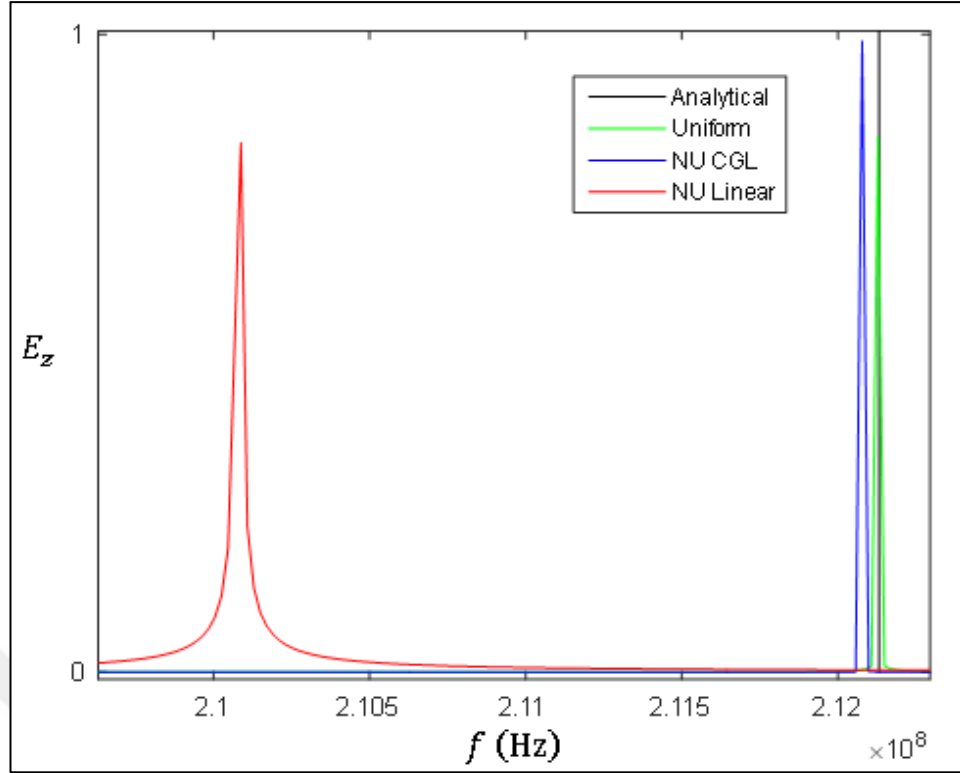


Figure 6.8: The resonance peaks of the normalized E_z electric field for the TM_{110}^z mode ($S = 1.0$).

Specially, in Figure 6.8, the frequency-domain numerical results of the dominant mode are compared for the different meshes. The uniform mesh exactly corresponds to the analytical one since $S = 1.0$. The resonance peaks calculated by the NU CGL mesh is closer to the analytical one. The results show that the NU CGL mesh gives better results than the NU Linear mesh.

Table 6.1: Relative errors of the resonance peaks ($\times 10^8$ Hz) for the six cavity modes.

Modes	Analytical Result (Exact)	UNIFORM (%)	NU CGL (%)	NU Linear (%)
TM_{110}^z	2.1213	0.029363724	0.103569626	1.504922357
$\text{TM}_{210}^z/\text{TM}_{120}^z$	3.3541	0.022357541	1.621445159	3.900485039
TM_{220}^z	4.2426	0.071067365	0.519554148	1.698169599
$\text{TM}_{310}^z/\text{TM}_{130}^z$	4.7434	0.110934852	0.183109780	2.413235335
$\text{TM}_{320}^z/\text{TM}_{230}^z$	5.4083	0.022307874	0.512306893	3.103541205
TM_{330}^z	6.3639	0.104544394	1.315397714	1.315397714

In Table 6.1, the numerical results of the resonance frequencies for the six cavity modes are compared in the sense of the relative errors calculated by using the uniform, the NU CGL and the NU Linear meshes ($S = 1.0$). The solution is run until 100000 time steps. This section's conclusion can be drawn that the U FDTD gives the best results whereas the NU CGL mesh comes second. But the NU Linear mesh error is not simply acceptable for the resonance frequency calculation of the empty cavity. These results are in good agreement with the field distribution results given in previous subsection.



6.2. Partially Dielectric Filled Cavity

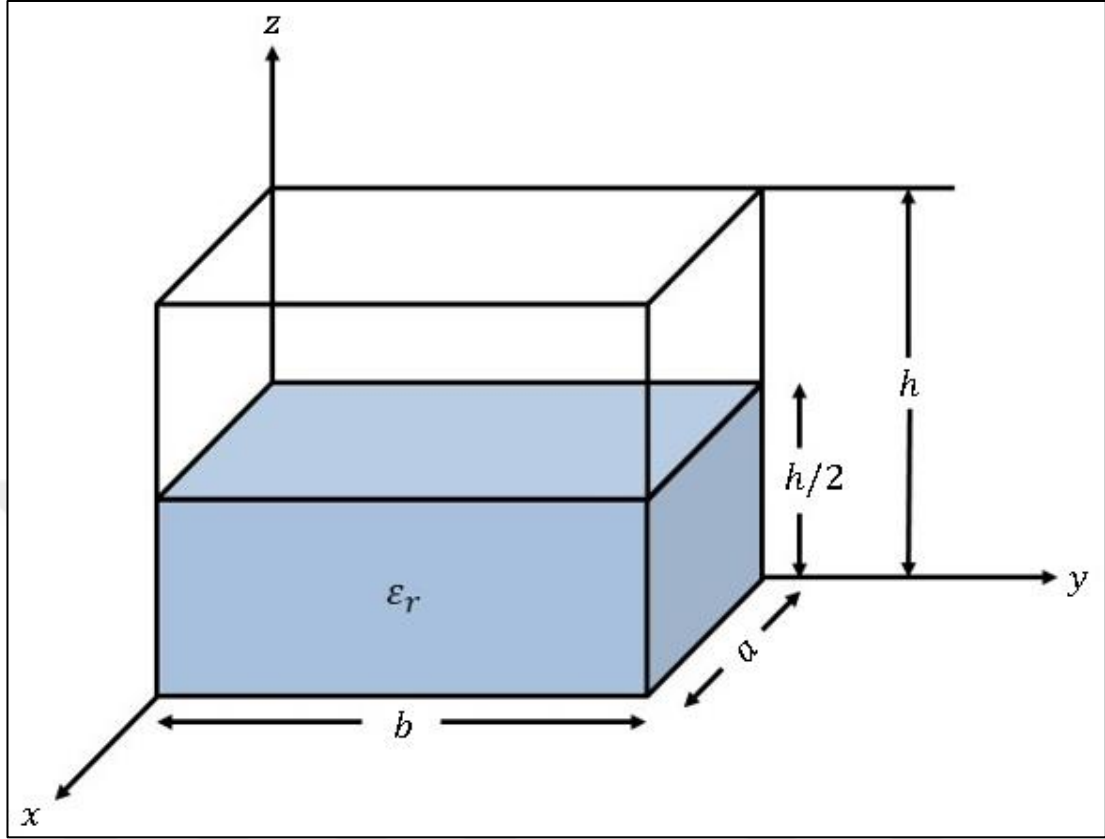


Figure 6.9: A partially dielectric filled cavity (PDFC) structure.

The considered resonator is partially loaded by a lossless dielectric material as shown in Figure 6.9. The dielectric filled part and the empty/air filled part of the cavity are demonstrated in terms of the axes lengths as

$$0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq h/2 \Rightarrow \text{Dielectric} \quad (6.12)$$

$$0 \leq x \leq a, 0 \leq y \leq b, h/2 \leq z \leq h \Rightarrow \text{Empty.} \quad (6.13)$$

A semi-analytical solution of the partially dielectric filled cavity (PDFC) is found by extracting transcendental equation of two different mediums in which a transverse resonance method is used for a partially filled rectangular waveguide [Balanis, 1989]. As an example, the dispersion equation for the Transverse Magnetic

(TM_{mnp}^z) modes where m, n, p are the mode numbers is given by [Ertay et al., 2017], [Li et al., 2020], [Bişkin et al., 2021]

$$\begin{aligned}
 -\varepsilon_r k_{zo} \tan(k_{zo}[h - z]) &= k_{zr} \tan(k_{zr}z) \\
 k_{zo} &= \sqrt{k_o^2 - k_{xo}^2 - k_{yo}^2} \quad ; \quad k_{zr} = \sqrt{k_r^2 - k_{xr}^2 - k_{yr}^2}
 \end{aligned} \tag{6.14}$$

where $k_o^2 = \omega^2 \varepsilon_0 \mu_0$ and $k_r^2 = \omega^2 \varepsilon_r \varepsilon_0 \mu_0$ are the wavenumbers in the empty and the dielectric filled region, respectively. This dispersion equation has to be solved at the dielectric interface $z = h/2$. Thus, the calculation of the PDFC resonance frequencies is only numerically possible by using a root finding algorithm due to the transcendental nature of the dispersion equation. The semi-analytical solution for the resonance frequencies is important since it is an efficient tool for the validation of the proposed numerical solutions.

6.2.1. Difficulty with p Mode Number

Difficulty with the p mode number arises in the PDFC problem since there is no certain knowledge of the third dimension (along z axis) in the semi-analytical solution. In fact, the 3D problem is reduced to the two-dimensional problem by using Transmission Line Method. Therefore, the analytical value of the p mode parameter cannot be found explicitly since equation (6.14) does not have any dependency on the p value. This leads to the less information for the mode distributions along the dielectric filling direction. The field distributions can only give the information about the value of the p mode number but there is no certain logic on it. The field distributions demonstrate that no integer values for the value of the p mode parameter are possible for the PDFC problems.

6.2.2. Single-Mode Excitations

The PDFC is partially filled by a dielectric material of a relative permittivity $\epsilon_r = 9$. It is excited by using a soft point monochromatic source with the proper resonance frequency. To excite the TM_{11p}^z cavity mode, the efficient source location that is found by cut-and-try procedure [Kaneda et al., 1997]. The observation point is located at the center bottom of the cavity at $z = h/10$. The FDTD simulation is run up to 1.85×10^{-6} sec.

6.2.2.1. Field Distributions

In Figure 6.10, the normalized E_z electric field distributions are shown at the xz plane of the cavity ($S = 1.0$). Three types of meshes in the NU FDTD algorithm are applied for observing the field distributional differences. All the numerical results show an apparent discontinuity at the interface of the two media. The NU Linear mesh completely breaks down the field distribution. The NU CGL mesh gives an acceptable field distribution with the different amplitudes comparing to the U FDTD result. This is related to the different numerical behavior of the NU CGL meshes comparing to the U FDTD meshes.

In Figure 6.11, the field distributions at yz plane are also shown with the same concept ($S = 1.0$). Since the cavity mode is symmetric along the x and the y axes, the results are mostly similar to the previous results. The NU Linear mesh again shows break down at the symmetry of the field distribution whereas the NU CGL mesh shows good results comparing to the uniform mesh. The NU CGL mesh gives better results than the NU Linear mesh since the NU CGL is more accurate at the boundaries. The reason for differences between the uniform mesh and the NU mesh is that the NU meshes have the first-order accuracy while the uniform meshes have the second order accuracy.

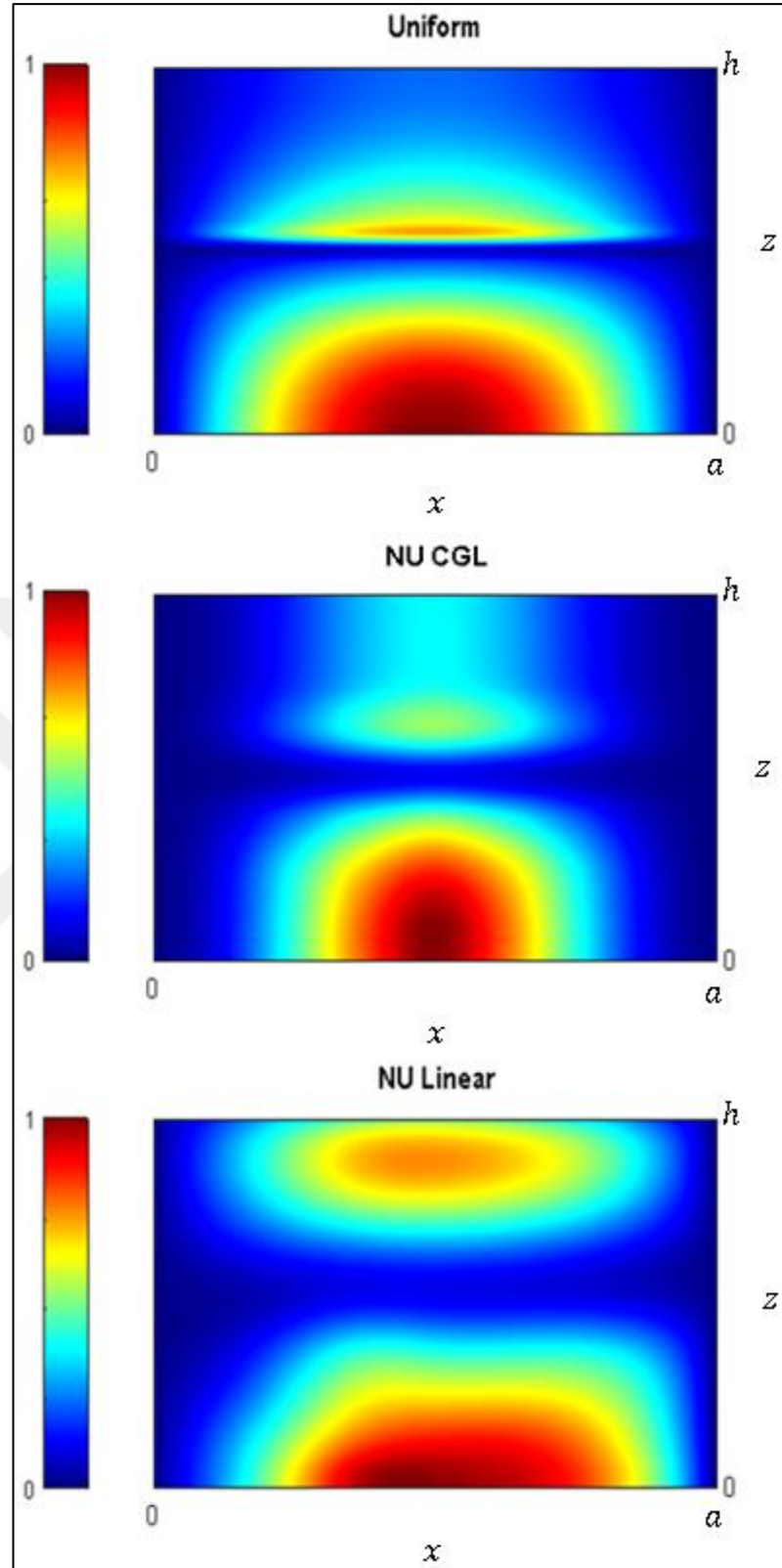


Figure 6.10: The normalized E_z electric field distribution of the TM_{11p}^z cavity mode at the xz plane ($S = 1.0$).

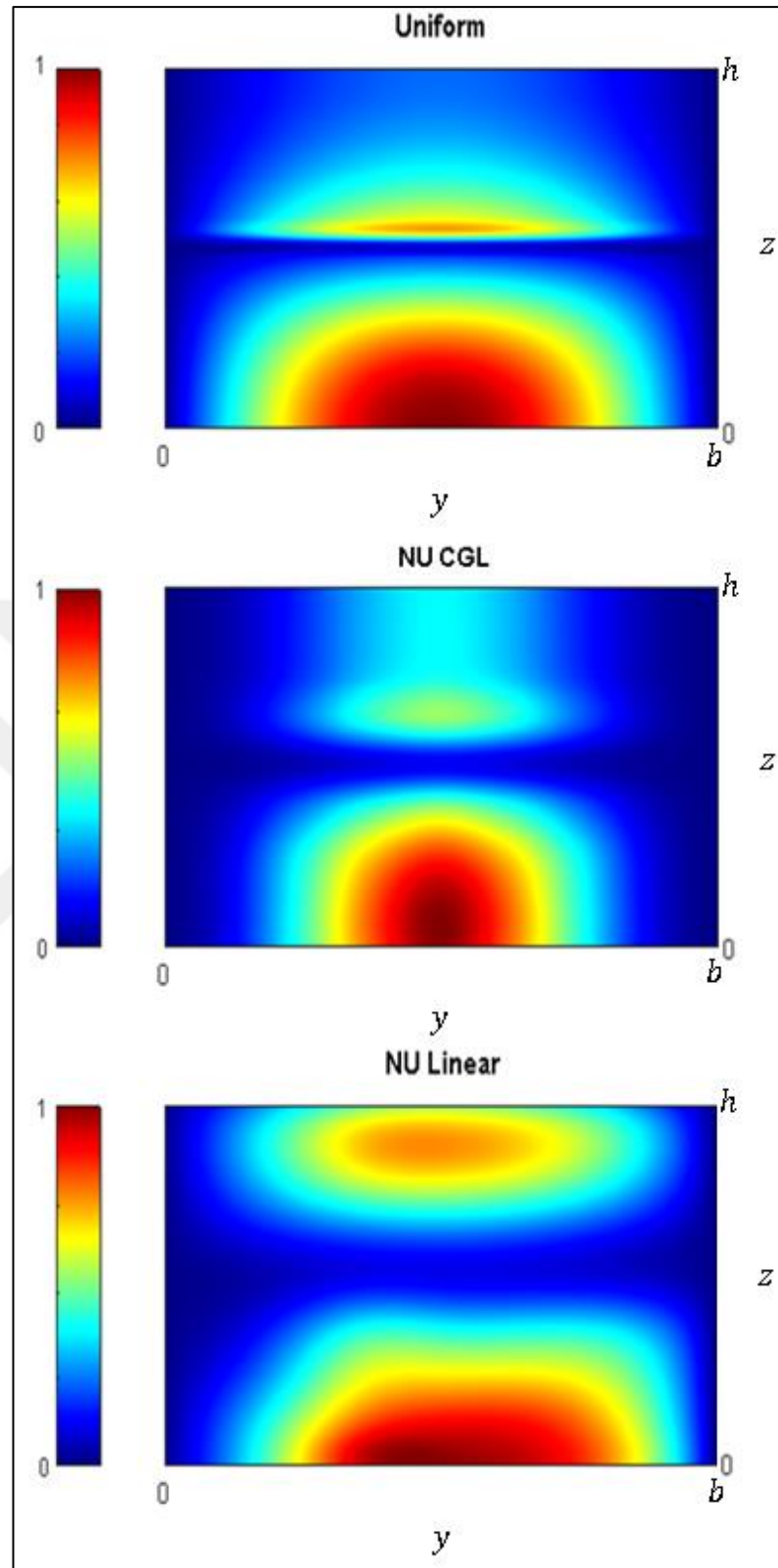


Figure 6.11: The normalized E_z electric field distribution of the TM_{11p}^z cavity mode at the yz plane ($S = 1.0$).

6.2.2.2. Time Domain Results

The comparisons in the time domain are important step to understand the performance of different meshes. The PDFC is efficiently excited in resonance when the source frequency is equal to the resonance frequency.

The FDTD results for the uniform mesh, the NU CGL mesh and the NU Linear mesh algorithms are calculated. In Figure 6.12, the magic time step of $S = 1.0$ is developed for the smallest NU mesh cell.

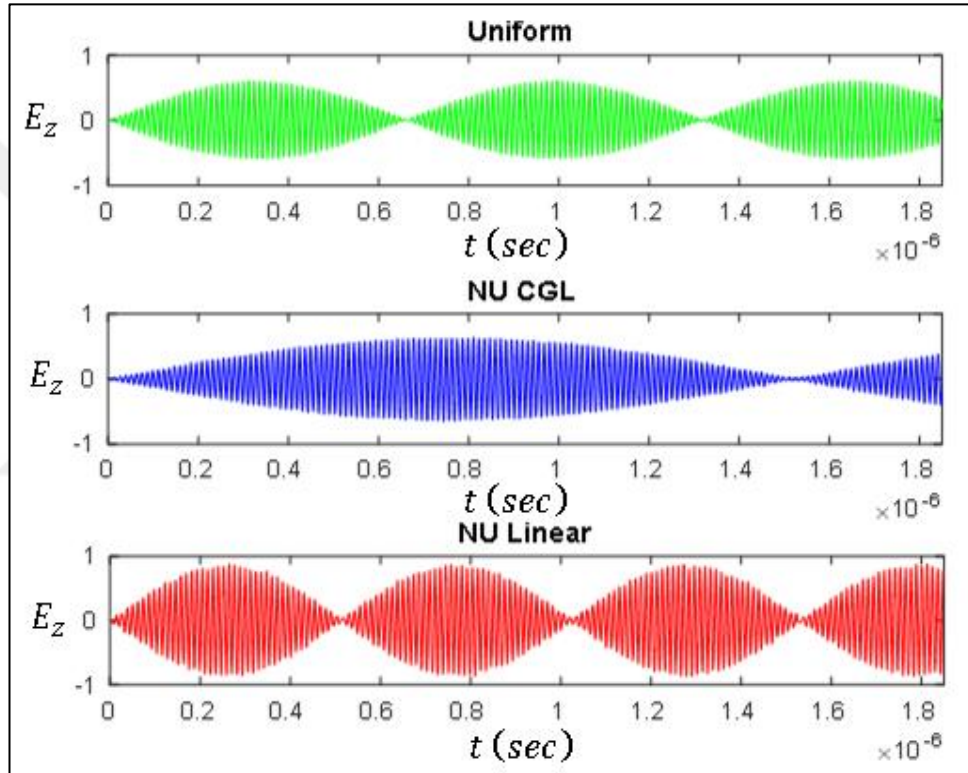


Figure 6.12: The normalized E_z electric field for the TM_{11p}^z cavity for $S = 1.0$.

In Figure 6.13, the expansion of the time signals shows the resonance behavior of the cavity. The Courant number is set to $S = 0.5$ corresponding the time step is $\Delta t = 9.25 \times 10^{-12}$ sec. To minimize the numerical dispersion errors, $\Delta x_{min} = \lambda/20 = 9.6$ mm is applied. The NU CGL mesh and the NU Linear mesh have $N_x = 17$ corresponding to $\Delta x_{min} = \lambda/20$ whereas the uniform mesh algorithm has $N_x = 105$. The difference is a must since the comparison must be done faithfully. This is performed by choosing the equal Δt and the equal $\Delta x = \Delta x_{min}$. The numerical results show that the uniform mesh gives worse results than the NU CGL mesh. The NU CGL

mesh gives the best results. However, it is still debatable that the expansion of the time signal closes soon. The U FDTD method still protects its superiority over the NU Linear mesh even though the medium is partially dielectric filled. The NU Linear mesh gives the worst results since the time domain signal is closed very soon. Therefore, it is necessary to investigate additional first order errors due to the inhomogeneous medium. This certainly needs using of extra treatment techniques for the dielectric boundary in which the mesh nonuniformity and the material inhomogeneity can treat or worsen the accuracy of the problem. Their combined effect on the solution accuracy is planned to investigate in the future.

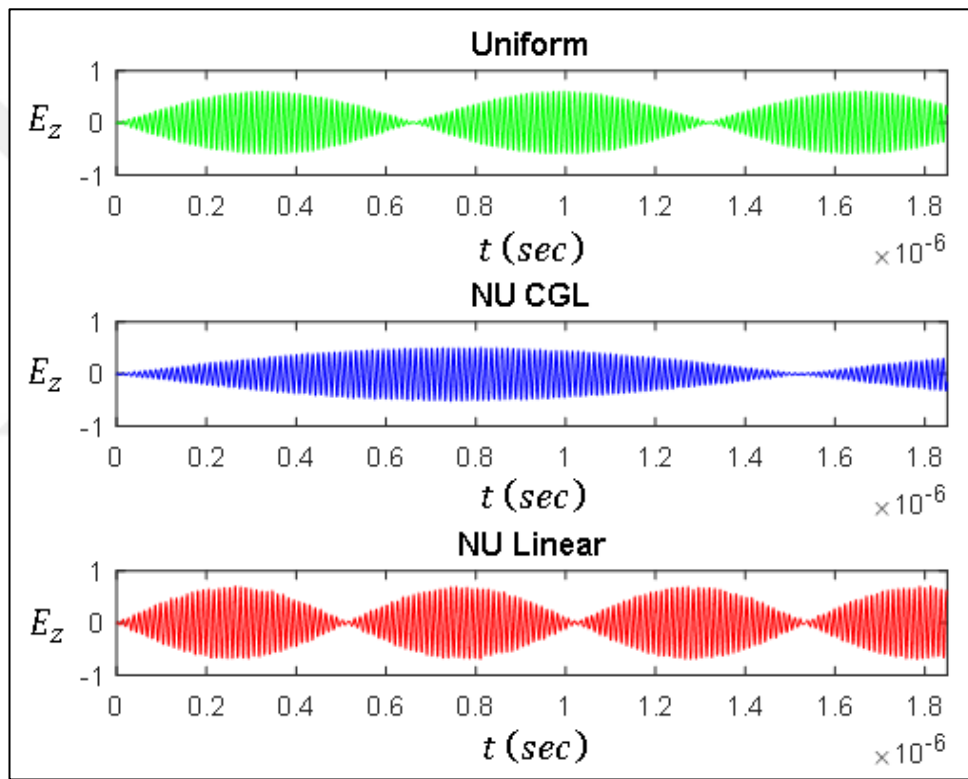


Figure 6.13: The normalized E_z electric field for the TM_{11p}^z cavity mode for $S = 0.5$.

6.2.3. Multi-Mode Excitations

Using the broadband pulse source (the same as in the previous section), the multi-modes (five modes) of the PDFC are excited. There are also some degenerated modes.

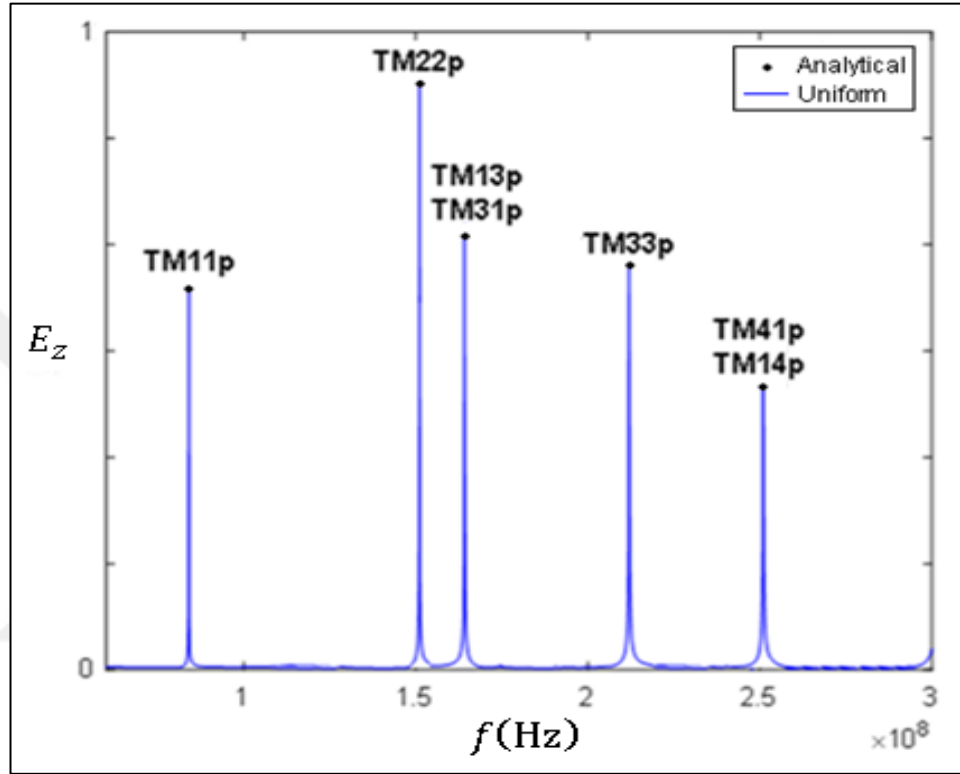


Figure 6.14: The normalized E_z electric field resonance peaks of the five different cavity modes (with degenerated ones) for $S = 1.0$ (The U FDTD result).

In Figure 6.14, the U FDTD calculated resonance frequencies of the PDFC multi-modes are shown for $S = 1.0$. In the simulation, $\Delta x_{min} = \lambda/20$ uniform mesh is used where the magic time step is $\Delta t = 5 \times 10^{-12}$ sec. 100000 time steps are applied for obtaining the accurate solution. The results are compared with the results found from the solution of the transcendental equation (dots in Figure 6.14). It is clear that the FDTD solution accurately predicts the multi-resonance peaks of the PDFC resonator.

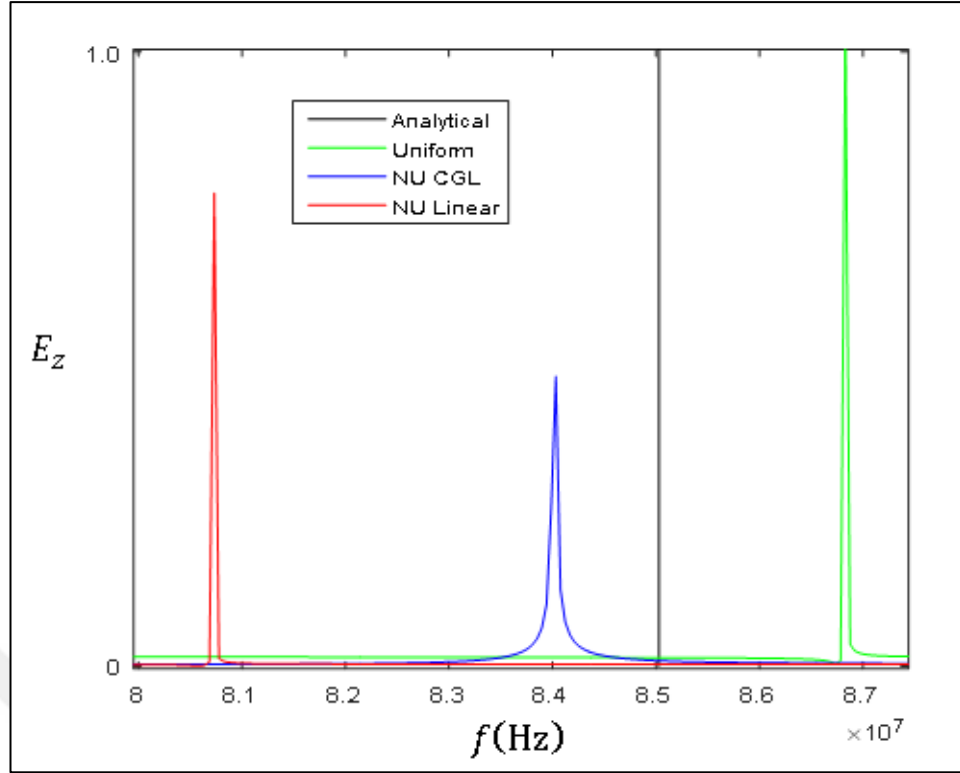


Figure 6.15: The normalized E_z electric field resonance peaks of the TM_{11p}^z cavity mode for $S = 1.0$.

In Figure 6.15, the uniform mesh, the NU CGL mesh and the NU Linear mesh are implemented in the FDTD calculations. The time domain results are validated in the frequency-domain results. In the calculations, 100000 time steps with $S = 1.0$ is applied. In fact, the magic time step condition is broken down due to both the mesh nonuniformity (the NU meshes) and material inhomogeneity (partially filling). The uniform unit spatial step $\Delta x = \Delta x_{min} = \lambda/20$ is utilized for the discretization. One can see that the uniform and the NU CGL mesh give the resonance peaks close to the analytical one. They give better outcomes than the NU Linear mesh. Importantly, the NU CGL mesh (single-domain) is more accurate than the uniform mesh. This higher order accuracy is obtained in the NU CGL mesh since the mesh symmetry is provided. Since the NU Linear mesh has non-symmetric collocation points in the global mean, its mesh distribution is not well supported for this problem. The accuracies of these results completely corresponds to the time domain results.

Table 6.2: Relative errors of the resonance peaks ($\times 10^8$ Hz) for the five cavity modes.

Modes	Analytical Result (Exact)	UNIFORM (%)	NU CGL (%)	NU Linear (%)
TM_{11p}^z	0.8503	2.118412193	1.171518691	4.946383619
TM_{22p}^z	1.4947	0.573404999	0.824300825	1.082924159
TM_{31p}^z/TM_{13p}^z	1.6537	0.656200827	1.740603812	1.740603812
TM_{33p}^z	2.1760	0.659805115	0.434091007	3.496142242
TM_{41p}^z/TM_{14p}^z	2.5316	0.204620149	0.906869895	1.345246698

In Table 6.2, the analytical results with the relative errors for the five different cavity modes are listed. The relative errors for the resonance peaks are calculated for the three different mesh types. All the resonance peaks are computed with a point pulse source and the simulation runs until 100000 time steps with the $\Delta x_{min} = \lambda/20$ spatial unit step. The results also correspond that the NU CGL meshes give the better results while the U FDTD gives also acceptable results. But, the NU Linear meshes gives the worst results.

7. DISCUSSIONS AND FUTURE WORKS

In this thesis, the numerical analyses of the NU FDTD method are with three steps: i) the derivations of the NU FDTD update equations, ii) the derivation of the NU FDTD NDE, iii) the validation of the numerical findings with two examples of the empty cavity and the partially dielectric filled cavity (PDFC).

The general NU FDTD update equations show that the electric field formulation has the first order accuracy whereas the magnetic field formulation has the second order accuracy. This is due to the fact that the spatial finite difference derivatives in the NU FDTD are calculated by using unequal spatial distances. It corresponds to two grid components rather than one in the denominators of the electric and magnetic update equations, respectively. The correspondence of the extracted NU NDE to the uniform mesh case is also shown.

The NU FDTD solution of the empty cavity and the PDFC are performed in the 3D Cartesian coordinates. The monochromatic and pulse type point sources are used for the excitation of the single- and multi-modes. The field distributions, the time domain signals and the frequency domain resonance peaks of the different cavity modes are calculated for the accuracy evaluations. The effects of the three different mesh types on the accuracy of the numerical calculations are investigated for the two different CFL numbers (two different time steps) of the two examples.

The empty cavity results show that the solution of the U FDTD at the magic time step almost gives the same results as the analytical solution. The NU CGL mesh is an appropriate choice for the empty cavity whereas the NU Linear mesh is not acceptable since it does not have appropriate collocation points for the proper modeling of the boundaries in the global mean. The NU Linear mesh also doesn't have symmetric distributions of the collocation points. Therefore, the boundary conditions cannot also be geometrically applied in a simple manner. The NU Linear mesh has only the better accuracy when its mesh ratio is close to one, corresponding to the uniform mesh.

The PDFC results show that the time domain signals are markedly different from each other for the different mesh types. The NU CGL mesh gives more accurate results than all the other types of the meshes. It means that the accuracy for the calculation of the PDFC resonance frequencies is improved by using the NU CGL meshes. However, the expected accuracy level referencing to the analytical solution is not achieved

because the combination of both mesh nonuniformity (the NU meshes) and the material inhomogeneity (different permittivities) strongly effect to the accuracy of the calculations. Therefore, different scenarios give different accuracy levels. The uniform meshes has also great accuracy but the NU Linear mesh is not an acceptable chose in the global mean. Specially, the results of the NU Linear mesh are relatively improved comparing to the empty cavity case. It can be concluded that performing the accurate numerical FDTD solutions for the PDFC problem is not a simple task. The material inhomogeneity in the cavity inherently creates a discontinuity at the interface between the two dielectric mediums. Therefore, the accuracy of the numerical solution is not only affected from the first order accuracy of the NU finite differences but also affected from the improper modeling of the material inhomogeneity. This may additionally lead to extra errors in the calculations.

In the numerical examples of the empty cavity, it is proved that the results of the numerical experiments completely correspond to the results of the NDE. This is not given for the PDFC since the extraction of the PDFC NDE is planned as a future work. As a general result, applications of the NU Linear mesh does not give good accuracy. Therefore, it must be applied locally at every boundary in the problem space for better accuracy. Specially, the NU CGL meshes inherently satisfies these geometrical requirements. However, its main disadvantage is that its multi-domain application is a must for the complex problems. This means that its global application is more difficult due to requirement of additional continuity conditions between every single-domain of the NU CGL boundary.

The planned future works are:

- Development of a multi-domain NU FDTD method for properly evaluating geometrical inhomogeneity effect such as object shape, object orientation, etc.
- Investigation of the accuracy levels in the sense of different material inhomogeneities such as magnetic, dispersive and/or lossy dielectric in the loaded cavity.
- Application of new interface treatment approaches such as characteristic variables,
- Performing the NU mesh accuracy analyses for open problems.

REFERENCES

- Agrawal A. K., Peckover R. S., (1980), “Nonuniform grid generation for boundary-layer problems”, *Computer Physics Communications*, 19 (2), 171-178.
- Aksoy S., (2022), “Zaman Uzayı Sonlu Farklar, ZUSF (Finite Difference Time Domain, FDTD) Yöntemi – Ders Notları”, Gebze Technical University, Gebze, Kocaeli (Web Book).
- Aksoy S., Tretyakov O. A., (2002), “Study of a time variant cavity system”, *Journal of Electromagnetic Waves and Applications*, 16 (11), 1535-1553.
- Armenta R. B., Sarris C. D., (2008), “A general method for introducing nonuniform grids into high-order FDTD methods: Formulation and microwave applications”, *IEEE MTT-S International Microwave Symposium Digest*, 1365-1368, Atlanta, GA, USA, 15-20 June.
- Balanis C. A., (1989), “Advanced Engineering Electromagnetics”, 1st Edition, John Wiley & Sons.
- Başaran E., Aksoy S., Ergin A. A., (2002), “The investigation of source location effect for resonant modes of a cubic cavity”, 3. International Conference on Mathematical & Computational Applications, 41-47, Konya, TÜRKİYE, 4-6 September.
- Bişkin O. S., Saydam T., Aksoy S., (2021), “The averaging effect on resonant frequency calculations of a partially filled microwave cavity using FDTD method”, *Workshop on (IEEE) Microwave Theory and Techniques in Wireless Communications (MTTW)*, 129-133, Riga, LATVIA, 7-8 October.
- Berens M. K., Flintoft I. D., Dawson J. F., (2016), “Structured mesh generation: Open-source automatic nonuniform mesh generation for FDTD simulation”, *IEEE Antennas and Propagation Magazine*, 58 (3), 45-55.
- Bernardi P., Cavagnaro M., Pisa S., Piuze E., (2001), “A graded-mesh FDTD code for the study of human exposure to cellular phones equipped with helical antennas”, *Aces Journal, Computational Bioelectromagnetics*, 16(2), 90-96.
- Blottner F. G., Roache P. J., (1971), “Nonuniform mesh systems”, *Journal of Computational Physics*, 8, 498-499.
- Brankovic V. J., Krupezevic D. V., Arndt F., (1992), “An efficient two-dimensional graded mesh finite-difference time-domain algorithm for shielded or open waveguide structures”, *IEEE Transactions on Microwave Theory and Techniques*, 40 (12), 2272-2277.

Brown R. R., (1962), "Numerical solution of boundary value problems using nonuniform grids", *Journal of the Society for Industrial and Applied Mathematics*, 10 (3), 475–495.

Chen Y., Yang M., Sun K., Mittra R., (2002), "Application of the non-uniform FDTD algorithm on an auto-adjustable mesh to the analysis of scattering parameters of a sandwiched-type coplanar waveguide with adjacent vias in PCBs", *Int. J. Electronics*, 89 (10), 791–800.

Chiang P., Yang C., Wu C., Teng C., Chang H., (2005), "Application of pseudospectral methods to optical waveguide mode solvers", *Integrated Photonics Research and Applications/Nanophotonics for Information Systems, Technical Digest (Optica Publishing Group)*, Paper IMG4, San Diego, CA, USA, 10–13 April.

Choi D. H., Hoefer W. J. R., (1987), "A graded mesh FD-TD algorithm for eigenvalue problems", *17th European Microwave Conference*, 413-417, Rome, ITALY, 07-11 September.

Chong T. H., (1978), "A variable mesh finite difference method for solving a class of parabolic differential equations in one space variable", *SIAM Journal on Numerical Analysis*, 15 (4), 835–857.

Christ A., Fröhlich J., Kuster N., (2002), "Correction of numerical phase velocity errors in nonuniform FDTD meshes", *IEICE Transactions on Communications*, E85 (B-12), 2904–2915.

Chu S. T., Huang W.P., Chaudhuri S.K., (1991), "Simulation and analysis of waveguide based optical integrated circuits", *Computer Physics Communications*, 68, 451-484.

Crowder H. J., Dalton C., (1971), "Errors in the use of nonuniform mesh systems", *Journal of Computational Physics*, 7, 32-45.

Daniel E. M., Railton C. J., (1991), "An improved second order radiating boundary condition for use with non-uniform grids in the finite difference time domain method", *21st European Microwave Conference*, 547-552, Stuttgart, GERMANY, 09-12 September.

Daniel E. M., Railton C. J., (1991), "Fast finite difference time domain analysis of microstrip patch antennas", *IEEE Antennas and Propagation Society Symposium Digest*, 1, 414-417, Ontario, CANADA, 24 June.

de Rivas E. K., (1972), "On the use of nonuniform grids in finite-difference equations", *Journal of Computational Physics*, 10 (2), 202-210.

Ertay A. O., Köseoğlu E., Şimşek S., Demiralay B., (2017), “On the study of comparative analysis of dielectric loaded effects for rectangular cavity resonators”, 4. International Electromagnetic Compatibility Conference (EMC Turkey), 1-6, Ankara, TÜRKİYE, 24-27 September.

Falconer M. C., (1997), “Three Dimensional Electromagnetic FDTD Simulation of General Lossy Structures with Nonuniform Grid Spacing”, Ph.D. Thesis, Oregon State University.

Falconer M. C., Tripathi V., (1997), “Second order nonuniform grid spacing in the FDTD technique”, IEEE MTT-S International Microwave Symposium Digest, 3, 1539-1542, Denver, CO, USA, 08-13 June.

Fu J., Chen Y., (2002), “Development of a memory-dynamic NU-FDTD Maxwell’s solver for wireless communications applications”, Proceedings IEEE Southeast, 201-204, Columbia, SC, USA, 5-7 April.

Gao B. Q., Gandhi O. P., (1992), “An expanding-grid algorithm for the finite-difference time-domain method (EM deposition in human body)”, IEEE Transactions on Electromagnetic Compatibility, 34 (3), 277-283.

Heinrich W., Beilenhoff K., Mezzanotte P., Roselli L., (1996), “Optimum mesh grading for finite-difference method”, IEEE Transactions on Microwave Theory and Techniques, 44 (9), 1569-1574.

Jiang H., Arai H., (1998), “3D FDTD analysis by using non-uniform mesh”, International Conference on Microwave and Millimeter Wave Technology ICMMT’98, 947-950, Beijing, CHINA, 18-20 August.

Jiang H., Arai H., (1998), “Fast and efficient FDTD analysis using non-uniform mesh for small antenna”, IEEE Antennas and Propagation Society International Symposium, 2, 1242-1245, Atlanta, GA, USA, 21-26 June.

Jiménez-Mejía E., Herrera-Murcia J., (2015), “Validation of a non-uniform meshing algorithm for the 3D-FDTD method by means of a two-wire crosstalk experimental set-up”, Ingeniería e Investigación, 35 (1), 98-103.

Kaneda N., Houshmand B., Itoh T., (1997), “FDTD analysis of dielectric resonators with curved surfaces”, IEEE Transactions on Microwave Theory and Techniques, 45 (9), 1645-1649.

Kermani M. H., Ramahi O. M., (2006), “The complementary derivatives method: a second-order accurate interpolation scheme for non-uniform grid in FDTD simulation”, IEEE Microwave and Wireless Components Letters, 16 (2), 60-62.

Kreuter P., Witzigmann B, (2008), “Combined analytical-finite difference time-domain full wave simulation of mode-locked vertical-extended-cavity semiconductor lasers”, J. Opt. Soc. Am. B, 25, 1118-1126.

Li H., Hussain I., Wang Y., Cao Q., (2020), “Smart-mesh strategy in DGTD method for partially filled cavity with uncertain interface parameters”, Radioengineering, 29 (1), 101-108.

Liu Z., Sanagavarapu A., Aubrey, T. A., (1996), “EM scattering using nonuniform mesh FDTD, PML and Mur’s ABC”, Electromagnetics, 16, 341-358.

Manteuffel T. A., White A. B., (1986), “The numerical solution of second-order boundary value problems on nonuniform meshes”, Mathematics of Computation, 47 (176), 511–535.

MATHEMATICA 13, (2021), ©Wolfram, USA.

MATLAB, (2018), Natick, Massachusetts: The MathWorks Inc., USA.

Mezzanotte P., Roselli L., Huber C., Zscheile H., Heinrich W., (1995), “On the accuracy of the finite-difference method using mesh grading”, IEEE MTT-S International Microwave Symposium, 2, 781-784, Orlando, FL, USA, 16-20 May.

Moczo P., (1989), “Finite-difference technique for SH-waves in 2-D media using irregular grids-application to the seismic response problem”, Geophysical Journal International, 99 (2), 321–329.

Monk P., Süli E., (1994), “A convergence analysis of Yee’s scheme on nonuniform grids”, SIAM Journal on Numerical Analysis, 31 (2), 393–412.

Monk P., Suli E., (1994), “Error estimates for Yee’s method on non-uniform grids”, IEEE Transactions on Magnetics, 30 (5), 3200-3203.

Mufti I. R., (1978), “A practical approach to finite-difference resistivity modeling”, Geophysics, 43 (5), 930–942.

Namiki T., Ito K., (2001), “Accuracy improvement technique applied to non-uniform FDTD cells using high-order implicit scheme”, IEEE Antennas and Propagation Society International Symposium, 1, 56-59, Boston, MA, USA, 8-13 July.

Navarro E. A., Sangary N. T., Litva J., (1996), "Some considerations on the accuracy of the nonuniform FDTD method and its application to waveguide analysis when combined with the perfectly matched layer technique", IEEE Transactions on Microwave Theory and Techniques, 44 (7), 1115-1124.

Paul D. L., Daniel E. M., Railton C. J., (1991), "Fast finite difference time domain method for the analysis of planar microstrip circuits", 21st European Microwave Conference, 303-308, Stuttgart, GERMANY, 09-12 September.

Perala R. A., Rudolph T. H., McKenna P. M., (1992), "Application of the time domain three-dimensional finite difference method to a wide variety of EMC problems", IEEE Regional Symposium on Electromagnetic Compatibility, 3.3.1/1-3.3.1/9, Santa Clara, CA, USA, 19-23 August.

Pozar D. M., (2012), "Microwave Engineering", 4rd Edition, John Wiley & Sons Inc.

Remis R. F., (2006), "Stability of FDTD on nonuniform grids for Maxwell's equations in lossless media", Journal of Computational Physics, 218 (2), 594-606.

Sangary N. T., Navarro E. A., Wu C., Litva J., (1995), "Application of non-uniform FDTD method to PCS antennas", IEEE Antennas and Propagation Society International Symposium, 2, 1148-1151, San Antonio, TX, USA, 16-21 June.

Shen G., Chen Y., Mittra R., (1998), "An efficient non-uniform FDTD method for analysis of propagation characteristics of optical fiber waveguides", IEEE Antennas and Propagation Society International Symposium, 3, 1488-1491, Atlanta, GA, USA, 21-26 June.

Shen G., Chen Y., Mittra R., (1999), "A nonuniform FDTD technique for efficient analysis of propagation characteristics of optical-fiber waveguides", IEEE Transactions on Microwave Theory and Techniques, 47 (3), 345-349.

Simba A. Y., Hikage T., Watanabe S., Nojima T., (2009), "Specific absorption rates of anatomically realistic human models exposed to RF electromagnetic fields from mobile phones used in elevators", IEEE Transactions on Microwave Theory and Techniques, 57 (5), 1250-1259.

Sundqvist H., Veronis G., (1970), "A simple finite-difference grid with nonconstant intervals", Tellus, 22, 26-31.

Sun M. K., Tam W. Y., (2001), "Analysis of 2-D scattering problems using a novel nonuniform gridding FDTD method", Microwave and Optical Technology Letters, 28 (6), 430-432.

Svigelj J., Mittra R., (1995), "Grid dispersion error using the nonuniform orthogonal finite difference time domain method", *Microwave and Optical Technology Letters*, 10 (4), 199-201.

Taflove A., Hagness S. C., (2005), "Computational Electrodynamics: The Finite-Difference Time-Domain Method", 3rd Edition, Artech House.

Tikhonov A. N., Samarskii A. A., (1962), "Homogeneous difference schemes of a high degree of accuracy on non-uniform nets", *USSR Computational Mathematics & Mathematical Physics*, 1 (3), 465-486.

Tikhonov A. N., Samarskii A., (1963), "Homogeneous difference schemes on non-uniform nets", *USSR Computational Mathematics and Mathematical Physics*, 2, 927-953.

Tinniswood A., Lazzi G., Gandhi O. P., (1998), "The use of the expanding grid FDTD technique for simulation of CAD-derived personal wireless telephones", *IEEE Antennas and Propagation Society International Symposium*, 1, 508-511, Atlanta, GA, USA, 21-26 June.

Tinniswood A., Lazzi G., Gandhi O. P., (1999), "The use of expanding grid FDTD method for simulation of CAD-derived personal wireless telephones", *Microw. Opt. Technol. Lett.*, 22 (1), 24-29.

Tupynamba R. C., Omar A., (1998), "Comparison between FDTD graded grids", *IEEE MTT-S International Microwave Symposium Digest*, 2, 905-908, Baltimore, MD, USA, 07-12 June.

Wiat J., Chaillou S., Altman Z., Drago S. P. (1999), "Calculation of the power deposited in tissues close to a handset antenna using a non-uniform FDTD", *Proceedings of the Second World Congress for Electricity and Magnetism in Biology and Medicine*, Bologna, Plenum Press, Boston, MA, USA.

XFDTD, (2022), "3D Electromagnetic Simulator", ©Remcom, USA.

Xiao S., Vahldieck R., Jin H., (1992), "A fast two-dimensional FDTD full-wave analyzer with adaptive mesh size", *IEEE MTT-S Microwave Symposium Digest*, 2, 783-786, Albuquerque, NM, USA, 1-5 June.

Xiao S., Vahldieck R., (1993), "An improved 2D-FDTD algorithm for hybrid mode analysis of quasi-planar transmission lines", *IEEE MTT-S International Microwave Symposium Digest*, 1, 421-424, Atlanta, GA, USA, 14-18 June.

Xiao S., Vahldheck R., (1995), “A fast FDTD analysis of guided wave structures using a continuously variable mesh with second order accuracy”, *Journal of the IETE*, 41 (1), 3-14.

Yee K. S., (1966), “Numerical solution of initial boundary value problems involving Maxwell’s equations in isotropic media”, *IEEE Transactions on Antennas and Propagation*, 14 (3), 302-307.

Yu W., Mittra R., (1999), “A technique for improving the accuracy of the nonuniform finite-difference time-domain algorithm”, *IEEE Transactions on Microwave Theory and Techniques*, 47 (3), 353-356.

Yu W., Mittra R., (1999), “Electromagnetic scattering of underground object using nonuniform mesh FDTD and PML”, *Microwave and Optical Technology Letters*, 21, 151-156.

Yu W., Yeo J., Mittra R., (2002), “Application of non-uniform and conformal FDTD techniques to the analysis of a class of planar antennas”, *IEEE Antennas and Propagation Society International Symposium*, 2, 674-677, San Antonio, TX, USA, 16-21 June.

Zhang M., Liao C., Xiong X.-Z., Xu X., (2017), “Scattering analysis of buried objects by using FDTD with nonuniform meshes”, *Progress in Electromagnetics Research M*, 54, 83-90.

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APPENDIX

Alternative Formulation of the 3D NDE

For the alternative formulation of the 3D NDE, the mesh intervals must be rewritten in each direction. It is given along the x direction as

$$\Delta x_1 = \alpha_1 \Delta x_0 \Rightarrow \Delta x_2 = \alpha_2 \Delta x_1 = \alpha_1 \alpha_2 \Delta x_0 \Rightarrow \Delta x_i = \prod_{\tau=1}^i \alpha_{\tau} \Delta x_0 \quad (1)$$

and along the y direction, it is given as

$$\Delta y_1 = \beta_1 \Delta y_0 \Rightarrow \Delta y_2 = \beta_2 \Delta y_1 = \beta_1 \beta_2 \Delta y_0 \Rightarrow \Delta y_j = \prod_{\tau=1}^j \beta_{\tau} \Delta y_0 \quad (2)$$

and along the z direction, that is given as

$$\Delta z_1 = \gamma_1 \Delta z_0 \Rightarrow \Delta z_2 = \gamma_2 \Delta z_1 = \gamma_1 \gamma_2 \Delta z_0 \Rightarrow \Delta z_k = \prod_{\tau=1}^k \gamma_{\tau} \Delta z_0. \quad (3)$$

Substituting Δx_i , Δx_{i-1} , Δy_j , Δy_{j-1} , Δz_k and Δz_{k-1} given in equations (4.1), (4.17), (4.27) into the 3D NDE

$$\begin{aligned} & \frac{1}{c^2 \Delta t^2} \sin^2 \left(\frac{\omega \Delta t}{2} \right) \\ &= \frac{j}{\Delta x_i (\Delta x_i + \Delta x_{i-1})} \sin \left(\frac{k_{Nx} \Delta x_i}{2} \right) \left(e^{-\frac{jk_{Nx} \Delta x_i}{2}} - e^{\frac{jk_{Nx} \Delta x_{i-1}}{2}} \right) \\ &+ \frac{j}{\Delta y_j (\Delta y_j + \Delta y_{j-1})} \sin \left(\frac{k_{Ny} \Delta y_j}{2} \right) \left(e^{-\frac{jk_{Ny} \Delta y_j}{2}} - e^{\frac{jk_{Ny} \Delta y_{j-1}}{2}} \right) \\ &+ \frac{j}{\Delta z_k (\Delta z_k + \Delta z_{k-1})} \sin \left(\frac{k_{Nz} \Delta z_k}{2} \right) \left(e^{-\frac{jk_{Nz} \Delta z_k}{2}} - e^{\frac{jk_{Nz} \Delta z_{k-1}}{2}} \right) \end{aligned} \quad (4)$$

inserting $\Delta x_i, \Delta x_{i-1}, \Delta y_j, \Delta y_{j-1}, \Delta z_k$ and Δz_{k-1} into the above equation

$$\begin{aligned}
& \frac{1}{c^2 \Delta t^2} \sin^2 \left(\frac{\omega \Delta t}{2} \right) = \\
& \frac{j}{\Delta x_0 \prod_{\tau=1}^i \alpha_\tau (\Delta x_0 \prod_{\tau=1}^i \alpha_\tau + \Delta x_0 \prod_{\tau=1}^{i-1} \alpha_\tau)} \\
& \sin \left(k_{Nx} \frac{\Delta x_0 \prod_{\tau=1}^i \alpha_\tau}{2} \right) \left(e^{-\frac{jk_{Nx} \Delta x_0 \prod_{\tau=1}^i \alpha_\tau}{2}} - e^{-\frac{jk_{Nx} \Delta x_0 \prod_{\tau=1}^{i-1} \alpha_\tau}{2}} \right) \\
& + \frac{j}{\Delta y_0 \prod_{\tau=1}^j \beta_\tau (\Delta y_0 \prod_{\tau=1}^j \beta_\tau + \Delta y_0 \prod_{\tau=1}^{j-1} \beta_\tau)} \\
& \sin \left(k_{Ny} \frac{\Delta y_0 \prod_{\tau=1}^j \beta_\tau}{2} \right) \left(e^{-\frac{jk_{Ny} \Delta y_0 \prod_{\tau=1}^j \beta_\tau}{2}} - e^{-\frac{jk_{Ny} \Delta y_0 \prod_{\tau=1}^{j-1} \beta_\tau}{2}} \right) \\
& + \frac{j}{\Delta z_0 \prod_{\tau=1}^k \gamma_\tau (\Delta z_0 \prod_{\tau=1}^k \gamma_\tau + \Delta z_0 \prod_{\tau=1}^{k-1} \gamma_\tau)} \\
& \sin \left(k_{Nz} \frac{\Delta z_0 \prod_{\tau=1}^k \gamma_\tau}{2} \right) \left(e^{-\frac{jk_{Nz} \Delta z_0 \prod_{\tau=1}^k \gamma_\tau}{2}} - e^{-\frac{jk_{Nz} \Delta z_0 \prod_{\tau=1}^{k-1} \gamma_\tau}{2}} \right)
\end{aligned} \tag{5}$$

and after simplification, the general multiplication form of the 3D NDE is found as

$$\begin{aligned}
& \frac{1}{c^2 \Delta t^2} \sin^2 \left(\frac{\omega \Delta t}{2} \right) = \\
& \frac{j}{\Delta x_0^2 (\prod_{\tau=1}^i \alpha_\tau)^2 (1 + \alpha_i^{-1})} \sin \left(\frac{k_{Nx} \Delta x_0 \prod_{\tau=1}^i \alpha_\tau}{2} \right) \\
& \left(e^{-\frac{jk_{Nx} \Delta x_0 \prod_{\tau=1}^i \alpha_\tau}{2}} - e^{-\frac{jk_{Nx} \Delta x_0 \prod_{\tau=1}^{i-1} \alpha_\tau}{2}} \right) + \\
& \frac{j}{\Delta y_0^2 (\prod_{\tau=1}^j \beta_\tau)^2 (1 + \beta_j^{-1})} \sin \left(k_{Ny} \frac{\Delta y_0 \prod_{\tau=1}^j \beta_\tau}{2} \right) \\
& \left(e^{-\frac{jk_{Ny} \Delta y_0 \prod_{\tau=1}^j \beta_\tau}{2}} - e^{-\frac{jk_{Ny} \Delta y_0 \prod_{\tau=1}^{j-1} \beta_\tau}{2}} \right) + \\
& \frac{j}{\Delta z_0^2 (\prod_{\tau=1}^k \gamma_\tau)^2 (1 + \gamma_k^{-1})} \sin \left(k_{Nz} \frac{\Delta z_0 \prod_{\tau=1}^k \gamma_\tau}{2} \right) \\
& \left(e^{-\frac{jk_{Nz} \Delta z_0 \prod_{\tau=1}^k \gamma_\tau}{2}} - e^{-\frac{jk_{Nz} \Delta z_0 \prod_{\tau=1}^{k-1} \gamma_\tau}{2}} \right).
\end{aligned} \tag{6}$$

• **3D NDE for the Uniform Mesh ($\alpha = 1$)**

Let us consider the mesh intervals for the uniform mesh ($\alpha = 1$) as

$$\Delta x = \Delta x_i = \Delta x_{i-1} ; \quad \Delta y = \Delta y_j = \Delta y_{j-1} ; \quad \Delta z = \Delta z_k = \Delta z_{k-1} \quad (7)$$

substituting them to the general 3D NDE given in equation (5.96) and by using the relations of $-2j \sin(k_{Nx}\Delta x/2) = e^{-jk_{Nx}\Delta x/2} - e^{jk_{Nx}\Delta x/2}$, $-2j \sin(k_{Ny}\Delta y/2) = e^{-jk_{Ny}\Delta y/2} - e^{jk_{Ny}\Delta y/2}$ and $-2j \sin(k_{Nz}\Delta z/2) = e^{-jk_{Nz}\Delta z/2} - e^{jk_{Nz}\Delta z/2}$, this can be written as

$$\begin{aligned} \frac{1}{c^2 \Delta t^2} \sin^2 \left(\frac{\omega \Delta t}{2} \right) &= \frac{j}{2 \Delta x^2} \sin \left(\frac{k_{Nx} \Delta x}{2} \right) \left(-2j \sin \left(\frac{k_{Nx} \Delta x}{2} \right) \right) \\ &+ \frac{j}{2 \Delta y^2} \sin \left(\frac{k_{Ny} \Delta y}{2} \right) \left(-2j \sin \left(\frac{k_{Ny} \Delta y}{2} \right) \right) \\ &+ \frac{j}{2 \Delta z^2} \sin \left(\frac{k_{Nz} \Delta z}{2} \right) \left(-2j \sin \left(\frac{k_{Nz} \Delta z}{2} \right) \right). \end{aligned} \quad (8)$$

Then, the uniform 3D NDE is found as

$$\begin{aligned} \frac{1}{c^2 \Delta t^2} \sin^2 \left(\frac{\omega \Delta t}{2} \right) &= \frac{1}{\Delta x^2} \sin^2 \left(\frac{k_{Nx} \Delta x}{2} \right) + \frac{1}{\Delta y^2} \sin^2 \left(\frac{k_{Ny} \Delta y}{2} \right) + \frac{1}{\Delta z^2} \sin^2 \left(\frac{k_{Nz} \Delta z}{2} \right). \end{aligned} \quad (9)$$

• **3D NDE for the NU CGL Mesh**

- Multiplication Concept

Using the NU CGL mesh points given in equations (4.24), (4.25) and (4.26), the NU mesh intervals are written as

$$\begin{aligned}\Delta x_i &= \cos\left(\pi \frac{i}{N_x}\right) - \cos\left(\pi \frac{i+1}{N_x}\right) \\ &= 2 \times \sin\left(\frac{\pi}{2N_x}(2i+1)\right) \times \sin\left(\frac{\pi}{2N_x}\right), \quad i = 0 \dots N_x - 1\end{aligned}\tag{10}$$

$$\begin{aligned}\Delta y_j &= \cos\left(\pi \frac{j}{N_y}\right) - \cos\left(\pi \frac{j+1}{N_y}\right) \\ &= 2 \times \sin\left(\frac{\pi}{2N_y}(2j+1)\right) \times \sin\left(\frac{\pi}{2N_y}\right), \quad j = 0 \dots N_y - 1\end{aligned}\tag{11}$$

$$\begin{aligned}\Delta z_k &= \cos\left(\pi \frac{k}{N_z}\right) - \cos\left(\pi \frac{k+1}{N_z}\right) \\ &= 2 \times \sin\left(\frac{\pi}{2N_z}(2k+1)\right) \times \sin\left(\frac{\pi}{2N_z}\right), \quad k = 0 \dots N_z - 1.\end{aligned}\tag{12}$$

The minimum mesh intervals are defined by using the NU CGL mesh formula as

$$\Delta x_{min} = \Delta x_0|_{i=0} = \Delta x_{N_x-1} = 2 \sin^2\left(\frac{\pi}{2N_x}\right)\tag{13}$$

$$\Delta y_{min} = \Delta y_0|_{j=0} = \Delta y_{N_y-1} = 2 \sin^2\left(\frac{\pi}{2N_y}\right)\tag{14}$$

$$\Delta z_{min} = \Delta z_0|_{k=0} = \Delta z_{N_z-1} = 2 \sin^2\left(\frac{\pi}{2N_z}\right).\tag{15}$$

Additionally, the NU mesh ratios must be developed over Δx_{min} , Δy_{min} and Δz_{min} as follow

$$\begin{aligned}
\Delta x_i &= \underbrace{2 \sin^2 \left(\frac{\pi}{2N_x} \right)}_{\Delta x_0} \prod_{\tau=1}^i \frac{\overbrace{\sin \left(\frac{\pi}{2N_x} (2\tau + 1) \right)}^{\alpha_\tau}}{\sin \left(\frac{\pi}{2N_x} (2\tau - 1) \right)} \\
&= 2 \sin^2 \left(\frac{\pi}{2N_x} \right) \left(\frac{\sin \left(\frac{\pi}{2N_x} 3 \right)}{\sin \left(\frac{\pi}{2N_x} \right)} \times \frac{\sin \left(\frac{\pi}{2N_x} 5 \right)}{\sin \left(\frac{\pi}{2N_x} 3 \right)} \right.
\end{aligned} \tag{16}$$

$$\begin{aligned}
&\times \frac{\sin \left(\frac{\pi}{2N_x} 7 \right)}{\sin \left(\frac{\pi}{2N_x} 5 \right)} \times \dots \times \frac{\sin \left(\frac{\pi}{2N_x} (2i + 1) \right)}{\sin \left(\frac{\pi}{2N_x} (2i - 1) \right)} \Bigg) \\
&= 2 \sin^2 \left(\frac{\pi}{2N_x} \right) \frac{\sin \left(\frac{\pi}{2N_x} (2i + 1) \right)}{\sin \left(\frac{\pi}{2N_x} \right)} \\
&= 2 \times \sin \left(\frac{\pi}{2N_x} (2i + 1) \right) \times \sin \left(\frac{\pi}{2N_x} \right)
\end{aligned} \tag{17}$$

$$\begin{aligned}
\Delta y_j &= \underbrace{2 \sin^2 \left(\frac{\pi}{2N_y} \right)}_{\Delta y_0} \prod_{\tau=1}^j \frac{\overbrace{\sin \left(\frac{\pi}{2N_y} (2\tau + 1) \right)}^{\beta_\tau}}{\sin \left(\frac{\pi}{2N_y} (2\tau - 1) \right)} \\
&= 2 \sin^2 \left(\frac{\pi}{2N_y} \right) \left(\frac{\sin \left(\frac{\pi}{2N_y} 3 \right)}{\sin \left(\frac{\pi}{2N_y} \right)} \times \frac{\sin \left(\frac{\pi}{2N_y} 5 \right)}{\sin \left(\frac{\pi}{2N_y} 3 \right)} \right.
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\times \frac{\sin \left(\frac{\pi}{2N_y} 7 \right)}{\sin \left(\frac{\pi}{2N_y} 5 \right)} \times \dots \times \frac{\sin \left(\frac{\pi}{2N_y} (2j + 1) \right)}{\sin \left(\frac{\pi}{2N_y} (2j - 1) \right)} \Bigg)
\end{aligned}$$

$$= 2 \sin^2 \left(\frac{\pi}{2N_y} \right) \frac{\sin \left(\frac{\pi}{2N_y} (2j+1) \right)}{\sin \left(\frac{\pi}{2N_y} \right)} \quad (19)$$

$$= 2 \times \sin \left(\frac{\pi}{2N_y} (2j+1) \right) \times \sin \left(\frac{\pi}{2N_y} \right)$$

$$\Delta z_k = \underbrace{2 \sin^2 \left(\frac{\pi}{2N_z} \right)}_{\Delta z_0} \prod_{\tau=1}^k \frac{\overbrace{\sin \left(\frac{\pi}{2N_z} (2\tau+1) \right)}^{\beta_\tau}}{\sin \left(\frac{\pi}{2N_z} (2\tau-1) \right)} \quad (20)$$

$$= 2 \sin^2 \left(\frac{\pi}{2N_z} \right) \left(\frac{\sin \left(\frac{\pi}{2N_z} 3 \right)}{\sin \left(\frac{\pi}{2N_z} \right)} \times \frac{\sin \left(\frac{\pi}{2N_z} 5 \right)}{\sin \left(\frac{\pi}{2N_z} 3 \right)} \right.$$

$$\left. \times \frac{\sin \left(\frac{\pi}{2N_z} 7 \right)}{\sin \left(\frac{\pi}{2N_z} 5 \right)} \times \dots \times \frac{\sin \left(\frac{\pi}{2N_z} (2k+1) \right)}{\sin \left(\frac{\pi}{2N_z} (2k-1) \right)} \right)$$

$$= 2 \sin^2 \left(\frac{\pi}{2N_z} \right) \frac{\sin \left(\frac{\pi}{2N_z} (2k+1) \right)}{\sin \left(\frac{\pi}{2N_z} \right)} = 2 \times \sin \left(\frac{\pi}{2N_z} (2k+1) \right) \times \sin \left(\frac{\pi}{2N_z} \right). \quad (21)$$

Then, α_i , β_j and γ_k can be formulated by using the NU CGL mesh formulas as

$$\alpha_i = \frac{\Delta x_i}{\Delta x_{i-1}} = \frac{2 \sin \left(\frac{\pi}{2N_x} (2i+1) \right) \sin \left(\frac{\pi}{2N_x} \right)}{2 \sin \left(\frac{\pi}{2N_x} (2i-1) \right) \sin \left(\frac{\pi}{2N_x} \right)} \Rightarrow \alpha_i = \frac{\Delta x_i}{\Delta x_{i-1}} \quad (22)$$

$$= \frac{\sin \left(\frac{\pi}{2N_x} (2i+1) \right)}{\sin \left(\frac{\pi}{2N_x} (2i-1) \right)} \Rightarrow \Delta x_i = \alpha_i \Delta x_{i-1}$$

$$\begin{aligned}
\beta_j = \frac{\Delta y_j}{\Delta y_{j-1}} &= \frac{2 \sin\left(\frac{\pi}{2N_y}(2j+1)\right) \sin\left(\frac{\pi}{2N_y}\right)}{2 \sin\left(\frac{\pi}{2N_y}(2j-1)\right) \sin\left(\frac{\pi}{2N_y}\right)} \Rightarrow \beta_j = \frac{\Delta y_j}{\Delta y_{j-1}} \\
&= \frac{\sin\left(\frac{\pi}{2N_y}(2j+1)\right)}{\sin\left(\frac{\pi}{2N_y}(2j-1)\right)} \Rightarrow \Delta y_j = \beta_j \Delta y_{j-1}
\end{aligned} \tag{23}$$

$$\begin{aligned}
\gamma_k = \frac{\Delta z_k}{\Delta z_{k-1}} &= \frac{2 \sin\left(\frac{\pi}{2N_z}(2k+1)\right) \sin\left(\frac{\pi}{2N_z}\right)}{2 \sin\left(\frac{\pi}{2N_z}(2k-1)\right) \sin\left(\frac{\pi}{2N_z}\right)} \Rightarrow \gamma_k = \frac{\Delta z_k}{\Delta z_{k-1}} \\
&= \frac{\sin\left(\frac{\pi}{2N_z}(2k+1)\right)}{\sin\left(\frac{\pi}{2N_z}(2k-1)\right)} \Rightarrow \Delta z_k = \gamma_k \Delta z_{k-1}.
\end{aligned} \tag{24}$$

Considering the 3D NDE for the general multiplication form of the mesh ratio

$$\begin{aligned}
&\frac{1}{c^2 \Delta t^2} \sin^2\left(\frac{\omega \Delta t}{2}\right) = \\
&\frac{j}{\Delta x_0^2 (\prod_{\tau=1}^i \alpha_\tau)^2 (1 + \alpha_i^{-1})} \sin\left(\frac{k_{Nx} \Delta x_0 \prod_{\tau=1}^i \alpha_\tau}{2}\right) \\
&\quad \left(e^{-\frac{jk_{Nx} \Delta x_0 \prod_{\tau=1}^i \alpha_\tau}{2}} - e^{\frac{jk_{Nx} \Delta x_0 \prod_{\tau=1}^{i-1} \alpha_\tau}{2}} \right) + \\
&\frac{j}{\Delta y_0^2 (\prod_{\tau=1}^j \beta_\tau)^2 (1 + \beta_j^{-1})} \sin\left(k_{Ny} \frac{\Delta y_0 \prod_{\tau=1}^j \beta_\tau}{2}\right) \\
&\quad \left(e^{-\frac{jk_{Ny} \Delta y_0 \prod_{\tau=1}^j \beta_\tau}{2}} - e^{\frac{jk_{Ny} \Delta y_0 \prod_{\tau=1}^{j-1} \beta_\tau}{2}} \right) + \\
&\frac{j}{\Delta z_0^2 (\prod_{\tau=1}^k \gamma_\tau)^2 (1 + \gamma_k^{-1})} \sin\left(k_{Nz} \frac{\Delta z_0 \prod_{\tau=1}^k \gamma_\tau}{2}\right) \\
&\quad \left(e^{-\frac{jk_{Nz} \Delta z_0 \prod_{\tau=1}^k \gamma_\tau}{2}} - e^{\frac{jk_{Nz} \Delta z_0 \prod_{\tau=1}^{k-1} \gamma_\tau}{2}} \right).
\end{aligned} \tag{25}$$

Then, substituting α_τ , β_τ and γ_τ for each term

$$\begin{aligned}
\prod_{\tau=1}^i \alpha_\tau &= \prod_{\tau=1}^i \frac{\overbrace{\sin\left(\frac{\pi}{2N_x}(2\tau+1)\right)}^{\alpha_\tau}}{\sin\left(\frac{\pi}{2N_x}(2\tau-1)\right)} \\
&= \frac{\sin\left(\frac{\pi}{2N_x}3\right)}{\sin\left(\frac{\pi}{2N_x}\right)} \times \frac{\sin\left(\frac{\pi}{2N_x}5\right)}{\sin\left(\frac{\pi}{2N_x}3\right)} \times \frac{\sin\left(\frac{\pi}{2N_x}7\right)}{\sin\left(\frac{\pi}{2N_x}5\right)} \times \dots \\
&\quad \times \frac{\sin\left(\frac{\pi}{2N_x}(2i+1)\right)}{\sin\left(\frac{\pi}{2N_x}(2i-1)\right)} = \frac{\sin\left(\frac{\pi}{2N_x}(2i+1)\right)}{\sin\left(\frac{\pi}{2N_x}\right)} \\
&\Rightarrow \left(\prod_{\tau=1}^i \alpha_\tau\right)^2 = \frac{\sin^2\left(\frac{\pi}{2N_x}(2i+1)\right)}{\sin^2\left(\frac{\pi}{2N_x}\right)}
\end{aligned} \tag{26}$$

$$\begin{aligned}
\prod_{\tau=1}^j \beta_\tau &= \prod_{\tau=1}^j \frac{\overbrace{\sin\left(\frac{\pi}{2N_y}(2\tau+1)\right)}^{\beta_\tau}}{\sin\left(\frac{\pi}{2N_y}(2\tau-1)\right)} \\
&= \frac{\sin\left(\frac{\pi}{2N_y}3\right)}{\sin\left(\frac{\pi}{2N_y}\right)} \times \frac{\sin\left(\frac{\pi}{2N_y}5\right)}{\sin\left(\frac{\pi}{2N_y}3\right)} \times \frac{\sin\left(\frac{\pi}{2N_y}7\right)}{\sin\left(\frac{\pi}{2N_y}5\right)} \times \dots \\
&\quad \times \frac{\sin\left(\frac{\pi}{2N_y}(2j+1)\right)}{\sin\left(\frac{\pi}{2N_y}(2j-1)\right)} = \frac{\sin\left(\frac{\pi}{2N_y}(2j+1)\right)}{\sin\left(\frac{\pi}{2N_y}\right)} \\
&\Rightarrow \left(\prod_{\tau=1}^j \beta_\tau\right)^2 = \frac{\sin^2\left(\frac{\pi}{2N_y}(2j+1)\right)}{\sin^2\left(\frac{\pi}{2N_y}\right)}
\end{aligned} \tag{27}$$

$$\begin{aligned}
\prod_{\tau=1}^k \gamma_{\tau} &= \prod_{\tau=1}^k \frac{\overbrace{\sin\left(\frac{\pi}{2N_z}(2\tau+1)\right)}^{\gamma_{\tau}}}{\sin\left(\frac{\pi}{2N_z}(2\tau-1)\right)} \\
&= \frac{\sin\left(\frac{\pi}{2N_z}3\right)}{\sin\left(\frac{\pi}{2N_z}\right)} \times \frac{\sin\left(\frac{\pi}{2N_z}5\right)}{\sin\left(\frac{\pi}{2N_z}3\right)} \times \frac{\sin\left(\frac{\pi}{2N_z}7\right)}{\sin\left(\frac{\pi}{2N_z}5\right)} \times \dots \\
&\quad \times \frac{\sin\left(\frac{\pi}{2N_z}(2k+1)\right)}{\sin\left(\frac{\pi}{2N_z}(2k-1)\right)} = \frac{\sin\left(\frac{\pi}{2N_z}(2k+1)\right)}{\sin\left(\frac{\pi}{2N_z}\right)} \\
&\Rightarrow \left(\prod_{\tau=1}^i \gamma_{\tau}\right)^2 = \frac{\sin^2\left(\frac{\pi}{2N_z}(2k+1)\right)}{\sin^2\left(\frac{\pi}{2N_z}\right)}
\end{aligned} \tag{28}$$

and for the lagging terms

$$\begin{aligned}
\prod_{\tau=1}^{i-1} \alpha_{\tau} &= \prod_{k=1}^{i-1} \frac{\overbrace{\sin\left(\frac{\pi}{2N_x}(2\tau+1)\right)}^{\alpha_{\tau}}}{\sin\left(\frac{\pi}{2N_x}(2\tau-1)\right)} \\
&= \frac{\sin\left(\frac{\pi}{2N_x}3\right)}{\sin\left(\frac{\pi}{2N_x}\right)} \times \frac{\sin\left(\frac{\pi}{2N_x}5\right)}{\sin\left(\frac{\pi}{2N_x}3\right)} \times \frac{\sin\left(\frac{\pi}{2N_x}7\right)}{\sin\left(\frac{\pi}{2N_x}5\right)} \times \dots \\
&\quad \times \frac{\sin\left(\frac{\pi}{2N_x}(2(i-1)+1)\right)}{\sin\left(\frac{\pi}{2N_x}(2(i-1)-1)\right)} = \frac{\sin\left(\frac{\pi}{2N_x}(2i-1)\right)}{\sin\left(\frac{\pi}{2N_x}\right)}
\end{aligned} \tag{29}$$

$$\begin{aligned}
\prod_{k=1}^{j-1} \beta_k &= \prod_{k=1}^{j-1} \frac{\overbrace{\sin\left(\frac{\pi}{2N_y}(2k+1)\right)}^{\beta_k}}{\sin\left(\frac{\pi}{2N_y}(2k-1)\right)} \\
&= \frac{\sin\left(\frac{\pi}{2N_y}3\right)}{\sin\left(\frac{\pi}{2N_y}\right)} \times \frac{\sin\left(\frac{\pi}{2N_y}5\right)}{\sin\left(\frac{\pi}{2N_y}3\right)} \times \frac{\sin\left(\frac{\pi}{2N_y}7\right)}{\sin\left(\frac{\pi}{2N_y}5\right)} \times \dots \\
&\quad \times \frac{\sin\left(\frac{\pi}{2N_y}(2(j-1)+1)\right)}{\sin\left(\frac{\pi}{2N_y}(2(j-1)-1)\right)} = \frac{\sin\left(\frac{\pi}{2N_y}(2j-1)\right)}{\sin\left(\frac{\pi}{2N_y}\right)}
\end{aligned} \tag{30}$$

$$\begin{aligned}
\prod_{\tau=1}^{k-1} \gamma_\tau &= \prod_{\tau=1}^{k-1} \frac{\overbrace{\sin\left(\frac{\pi}{2N_z}(2\tau+1)\right)}^{\gamma_\tau}}{\sin\left(\frac{\pi}{2N_z}(2\tau-1)\right)} \\
&= \frac{\sin\left(\frac{\pi}{2N_z}3\right)}{\sin\left(\frac{\pi}{2N_z}\right)} \times \frac{\sin\left(\frac{\pi}{2N_z}5\right)}{\sin\left(\frac{\pi}{2N_z}3\right)} \times \frac{\sin\left(\frac{\pi}{2N_z}7\right)}{\sin\left(\frac{\pi}{2N_z}5\right)} \times \dots \\
&\quad \times \frac{\sin\left(\frac{\pi}{2N_z}(2(k-1)+1)\right)}{\sin\left(\frac{\pi}{2N_z}(2(k-1)-1)\right)} = \frac{\sin\left(\frac{\pi}{2N_z}(2k-1)\right)}{\sin\left(\frac{\pi}{2N_z}\right)}
\end{aligned} \tag{31}$$

and for the inverse terms

$$\alpha_i^{-1} = \left(\frac{\sin\left(\frac{\pi}{2N_x}(2i+1)\right)}{\sin\left(\frac{\pi}{2N_x}(2i-1)\right)} \right)^{-1} = \frac{\sin\left(\frac{\pi}{2N_x}(2i-1)\right)}{\sin\left(\frac{\pi}{2N_x}(2i+1)\right)} \tag{32}$$

$$\beta_j^{-1} = \left(\frac{\sin\left(\frac{\pi}{2N_y}(2j+1)\right)}{\sin\left(\frac{\pi}{2N_y}(2j-1)\right)} \right)^{-1} = \frac{\sin\left(\frac{\pi}{2N_y}(2j-1)\right)}{\sin\left(\frac{\pi}{2N_y}(2j+1)\right)} \quad (33)$$

$$\gamma_k^{-1} = \left(\frac{\sin\left(\frac{\pi}{2N_z}(2k+1)\right)}{\sin\left(\frac{\pi}{2N_z}(2k-1)\right)} \right)^{-1} = \frac{\sin\left(\frac{\pi}{2N_z}(2k-1)\right)}{\sin\left(\frac{\pi}{2N_z}(2k+1)\right)}. \quad (34)$$

Using the Δx_0 , Δy_0 and Δz_0 in (13), (14) and (15)

$$\Delta x_0 = 2 \sin^2\left(\frac{\pi}{2N_x}\right) ; \quad \Delta y_0 = 2 \sin^2\left(\frac{\pi}{2N_y}\right) ; \quad \Delta z_0 = 2 \sin^2\left(\frac{\pi}{2N_z}\right). \quad (35)$$

Therefore, the 3D NDE as a complex-valued form becomes

$$\begin{aligned} & \frac{1}{c^2 \Delta t^2} \sin^2\left(\frac{\omega \Delta t}{2}\right) = \\ & \frac{j}{4 \sin^4\left(\frac{\pi}{2N_x}\right) \frac{\sin^2\left(\frac{\pi}{2N_x}(2i+1)\right)}{\sin^2\left(\frac{\pi}{2N_x}\right)} \left(1 + \frac{\sin\left(\frac{\pi}{2N_x}(2i-1)\right)}{\sin\left(\frac{\pi}{2N_x}(2i+1)\right)} \right)} \\ & \times \sin\left(k_{N_x} 2 \sin^2\left(\frac{\pi}{2N_x}\right) \frac{\sin\left(\frac{\pi}{2N_x}(2i+1)\right)}{2 \sin\left(\frac{\pi}{2N_x}\right)} \right) \\ & \times \left(e^{-jk_{N_x} 2 \sin^2\left(\frac{\pi}{2N_x}\right) \frac{\sin\left(\frac{\pi}{2N_x}(2i+1)\right)}{2 \sin\left(\frac{\pi}{2N_x}\right)}} - e^{jk_{N_x} 2 \sin^2\left(\frac{\pi}{2N_x}\right) \frac{\sin\left(\frac{\pi}{2N_x}(2i-1)\right)}{2 \sin\left(\frac{\pi}{2N_x}\right)}} \right) \end{aligned} \quad (36)$$

$$\begin{aligned}
& + \frac{j}{4 \sin^4 \left(\frac{\pi}{2N_y} \right) \frac{\sin^2 \left(\frac{\pi}{2N_y} (2j+1) \right)}{\sin^2 \left(\frac{\pi}{2N_y} \right)} \left(1 + \frac{\sin \left(\frac{\pi}{2N_y} (2j-1) \right)}{\sin \left(\frac{\pi}{2N_y} (2j+1) \right)} \right) \\
& \quad \times \sin \left(k_{N_y} 2 \sin^2 \left(\frac{\pi}{2N_y} \right) \frac{\sin \left(\frac{\pi}{2N_y} (2j+1) \right)}{2 \sin \left(\frac{\pi}{2N_y} \right)} \right) \\
& \quad \times \left(e^{-jk_{N_y} 2 \sin^2 \left(\frac{\pi}{2N_y} \right) \frac{\sin \left(\frac{\pi}{2N_y} (2j+1) \right)}{2 \sin \left(\frac{\pi}{2N_y} \right)}} - e^{jk_{N_y} 2 \sin^2 \left(\frac{\pi}{2N_y} \right) \frac{\sin \left(\frac{\pi}{2N_y} (2j-1) \right)}{2 \sin \left(\frac{\pi}{2N_y} \right)}} \right) \\
& + \frac{j}{4 \sin^4 \left(\frac{\pi}{2N_z} \right) \frac{\sin^2 \left(\frac{\pi}{2N_z} (2k+1) \right)}{\sin^2 \left(\frac{\pi}{2N_z} \right)} \left(1 + \frac{\sin \left(\frac{\pi}{2N_z} (2k-1) \right)}{\sin \left(\frac{\pi}{2N_z} (2k+1) \right)} \right) \\
& \quad \times \sin \left(k_{N_z} 2 \sin^2 \left(\frac{\pi}{2N_z} \right) \frac{\sin \left(\frac{\pi}{2N_z} (2k+1) \right)}{2 \sin \left(\frac{\pi}{2N_z} \right)} \right) \\
& \quad \times \left(e^{-jk_{N_z} 2 \sin^2 \left(\frac{\pi}{2N_z} \right) \frac{\sin \left(\frac{\pi}{2N_z} (2k+1) \right)}{2 \sin \left(\frac{\pi}{2N_z} \right)}} - e^{jk_{N_z} 2 \sin^2 \left(\frac{\pi}{2N_z} \right) \frac{\sin \left(\frac{\pi}{2N_z} (2k-1) \right)}{2 \sin \left(\frac{\pi}{2N_z} \right)}} \right).
\end{aligned}$$

After some mathematical operations, the 3D NDE becomes

$$\begin{aligned}
& \frac{1}{c^2 \Delta t^2} \sin^2 \left(\frac{\omega \Delta t}{2} \right) = \\
& \frac{j}{4 \sin^2 \left(\frac{\pi}{2N_x} \right) \sin^2 \left(\frac{\pi}{2N_x} (2i+1) \right) \left(1 + \frac{\sin \left(\frac{\pi}{2N_x} (2i-1) \right)}{\sin \left(\frac{\pi}{2N_x} (2i+1) \right)} \right)} \quad (37)
\end{aligned}$$

$$\begin{aligned}
& \times \sin \left(k_{N_x} \sin \left(\frac{\pi}{2N_x} \right) \sin \left(\frac{\pi}{2N_x} (2i+1) \right) \right) \\
& \times \left(e^{-jk_{N_x} \sin \left(\frac{\pi}{2N_x} \right) \sin \left(\frac{\pi}{2N_x} (2i+1) \right)} - e^{jk_{N_x} \sin \left(\frac{\pi}{2N_x} \right) \sin \left(\frac{\pi}{2N_x} (2i-1) \right)} \right) \\
& + \frac{j}{4 \sin^2 \left(\frac{\pi}{2N_y} \right) \sin^2 \left(\frac{\pi}{2N_y} (2j+1) \right) \left(1 + \frac{\sin \left(\frac{\pi}{2N_y} (2j-1) \right)}{\sin \left(\frac{\pi}{2N_y} (2j+1) \right)} \right)} \\
& \times \sin \left(k_{N_y} \sin \left(\frac{\pi}{2N_y} \right) \sin \left(\frac{\pi}{2N_y} (2j+1) \right) \right) \\
& \times \left(e^{-jk_{N_y} \sin \left(\frac{\pi}{2N_y} \right) \sin \left(\frac{\pi}{2N_y} (2j+1) \right)} - e^{jk_{N_y} \sin \left(\frac{\pi}{2N_y} \right) \sin \left(\frac{\pi}{2N_y} (2j-1) \right)} \right) \\
& + \frac{j}{4 \sin^2 \left(\frac{\pi}{2N_z} \right) \sin^2 \left(\frac{\pi}{2N_z} (2k+1) \right) \left(1 + \frac{\sin \left(\frac{\pi}{2N_z} (2k-1) \right)}{\sin \left(\frac{\pi}{2N_z} (2k+1) \right)} \right)} \\
& \times \sin \left(k_{N_z} \sin \left(\frac{\pi}{2N_z} \right) \sin \left(\frac{\pi}{2N_z} (2k+1) \right) \right) \\
& \times \left(e^{-jk_{N_z} \sin \left(\frac{\pi}{2N_z} \right) \sin \left(\frac{\pi}{2N_z} (2k+1) \right)} - e^{jk_{N_z} \sin \left(\frac{\pi}{2N_z} \right) \sin \left(\frac{\pi}{2N_z} (2k-1) \right)} \right)
\end{aligned}$$

where using the trigonometrical equivalencies, the 3D NDE becomes

$$\begin{aligned}
& \frac{1}{c^2 \Delta t^2} \sin^2 \left(\frac{\omega \Delta t}{2} \right) = \\
& \frac{j}{4 \sin^2 \left(\frac{\pi}{2N_x} \right) \sin^2 \left(\frac{\pi}{2N_x} (2i+1) \right) \left(1 + \frac{\sin \left(\frac{\pi}{2N_x} (2i-1) \right)}{\sin \left(\frac{\pi}{2N_x} (2i+1) \right)} \right)} \\
& \times \sin \left(k_{N_x} \sin \left(\frac{\pi}{2N_x} \right) \sin \left(\frac{\pi}{2N_x} (2i+1) \right) \right)
\end{aligned} \tag{38}$$

$$\begin{aligned}
& \times \left(e^{-jk_{N_x} \sin\left(\frac{\pi}{2N_x}\right) \sin\left(\frac{\pi}{2N_x}(2i+1)\right)} - e^{jk_{N_x} \sin\left(\frac{\pi}{2N_x}\right) \sin\left(\frac{\pi}{2N_x}(2i-1)\right)} \right) + \\
& \frac{j}{4 \sin^2\left(\frac{\pi}{2N_y}\right) \sin^2\left(\frac{\pi}{2N_y}(2j+1)\right) \left(1 + \frac{\sin\left(\frac{\pi}{2N_y}(2j-1)\right)}{\sin\left(\frac{\pi}{2N_y}(2j+1)\right)} \right)} \\
& \times \sin\left(k_{N_y} \sin\left(\frac{\pi}{2N_y}\right) \sin\left(\frac{\pi}{2N_y}(2j+1)\right)\right) \\
& \times \left(e^{-jk_{N_y} \sin\left(\frac{\pi}{2N_y}\right) \sin\left(\frac{\pi}{2N_y}(2j+1)\right)} - e^{jk_{N_y} \sin\left(\frac{\pi}{2N_y}\right) \sin\left(\frac{\pi}{2N_y}(2j-1)\right)} \right) + \\
& \frac{j}{4 \sin^2\left(\frac{\pi}{2N_z}\right) \sin^2\left(\frac{\pi}{2N_z}(2k+1)\right) \left(1 + \frac{\sin\left(\frac{\pi}{2N_z}(2k-1)\right)}{\sin\left(\frac{\pi}{2N_z}(2k+1)\right)} \right)} \\
& \times \sin\left(k_{N_z} \sin\left(\frac{\pi}{2N_z}\right) \sin\left(\frac{\pi}{2N_z}(2k+1)\right)\right) \\
& \times \left(e^{-jk_{N_z} \sin\left(\frac{\pi}{2N_z}\right) \sin\left(\frac{\pi}{2N_z}(2k+1)\right)} - e^{jk_{N_z} \sin\left(\frac{\pi}{2N_z}\right) \sin\left(\frac{\pi}{2N_z}(2k-1)\right)} \right).
\end{aligned}$$

Then, the final version of the 3D NDE is found as

$$\begin{aligned}
& \frac{1}{c^2 \Delta t^2} \sin^2\left(\frac{\omega \Delta t}{2}\right) = \\
& \frac{j}{4 \sin^2\left(\frac{\pi}{2N_x}\right) \sin^2\left(\frac{\pi}{2N_x}(2i+1)\right) \left(1 + \frac{\sin\left(\frac{\pi}{2N_x}(2i-1)\right)}{\sin\left(\frac{\pi}{2N_x}(2i+1)\right)} \right)} \\
& \times \sin\left(k_{N_x} \sin\left(\frac{\pi}{2N_x}\right) \sin\left(\frac{\pi}{2N_x}(2i+1)\right)\right) \\
& \times \left(e^{-jk_{N_x} \sin\left(\frac{\pi}{2N_x}\right) \sin\left(\frac{\pi}{2N_x}(2i+1)\right)} - e^{jk_{N_x} \sin\left(\frac{\pi}{2N_x}\right) \sin\left(\frac{\pi}{2N_x}(2i-1)\right)} \right)
\end{aligned} \tag{39}$$

$$\begin{aligned}
& + \frac{j}{4 \sin^2 \left(\frac{\pi}{2N_y} \right) \sin^2 \left(\frac{\pi}{2N_y} (2j+1) \right) \left(1 + \frac{\sin \left(\frac{\pi}{2N_y} (2j-1) \right)}{\sin \left(\frac{\pi}{2N_y} (2j+1) \right)} \right)} \\
& \quad \times \sin \left(k_{N_y} \sin \left(\frac{\pi}{2N_y} \right) \sin \left(\frac{\pi}{2N_y} (2j+1) \right) \right) \\
& \quad \times \left(e^{-jk_{N_y} \sin \left(\frac{\pi}{2N_y} \right) \sin \left(\frac{\pi}{2N_y} (2j+1) \right)} - e^{jk_{N_y} \sin \left(\frac{\pi}{2N_y} \right) \sin \left(\frac{\pi}{2N_y} (2j-1) \right)} \right) \\
& + \frac{j}{4 \sin^2 \left(\frac{\pi}{2N_z} \right) \sin^2 \left(\frac{\pi}{2N_z} (2k+1) \right) \left(1 + \frac{\sin \left(\frac{\pi}{2N_z} (2k-1) \right)}{\sin \left(\frac{\pi}{2N_z} (2k+1) \right)} \right)} \\
& \quad \times \sin \left(k_{N_z} \sin \left(\frac{\pi}{2N_z} \right) \sin \left(\frac{\pi}{2N_z} (2k+1) \right) \right) \\
& \quad \times \left(e^{-jk_{N_z} \sin \left(\frac{\pi}{2N_z} \right) \sin \left(\frac{\pi}{2N_z} (2k+1) \right)} - e^{jk_{N_z} \sin \left(\frac{\pi}{2N_z} \right) \sin \left(\frac{\pi}{2N_z} (2k-1) \right)} \right).
\end{aligned}$$

- Direct Extraction

Substituting the NU mesh intervals Δx_i , Δx_{i-1} , Δy_j , Δy_{j-1} , Δz_k and Δz_{k-1} of the NU CGL mesh points given in equations (4.24), (4.25) and (4.26) into the 3D NDE, it is found that

$$\begin{aligned}
& \frac{1}{c^2 \Delta t^2} \sin^2 \left(\frac{\omega \Delta t}{2} \right) = \\
& \frac{j \sin \left(\frac{k_N \Delta x_i}{2} \right)}{4 \sin^2 \left(\frac{\pi}{2N_x} \right) \sin^2 \left(\frac{\pi}{2N_x} (2i+1) \right) \left(1 + \frac{\sin \left(\frac{\pi}{2N_x} (2i-1) \right)}{\sin \left(\frac{\pi}{2N_x} (2i+1) \right)} \right)} \\
& \quad \times \left(e^{-jk_{N_x} \sin \left(\frac{\pi}{2N_x} (2i+1) \right) \sin \left(\frac{\pi}{2N_x} \right)} - e^{jk_{N_x} \sin \left(\frac{\pi}{2N_x} (2(i-1)+1) \right) \sin \left(\frac{\pi}{2N_x} \right)} \right)
\end{aligned} \tag{40}$$

$$\begin{aligned}
& + \frac{j \sin\left(\frac{k_{N_y} \Delta y_j}{2}\right)}{4 \sin^2\left(\frac{\pi}{2N_y}\right) \sin^2\left(\frac{\pi}{2N_y}(2j+1)\right) \left(1 + \frac{\sin\left(\frac{\pi}{2N_y}(2j-1)\right)}{\sin\left(\frac{\pi}{2N_y}(2j+1)\right)}\right)} \\
& \times \left(e^{-jk_{N_y} \sin\left(\frac{\pi}{2N_y}(2j+1)\right) \sin\left(\frac{\pi}{2N_y}\right)} - e^{jk_{N_y} \sin\left(\frac{\pi}{2N_y}(2(j-1)+1)\right) \sin\left(\frac{\pi}{2N_y}\right)} \right) \\
& + \frac{j \sin\left(\frac{k_{N_z} \Delta z_k}{2}\right)}{4 \sin^2\left(\frac{\pi}{2N_z}\right) \sin^2\left(\frac{\pi}{2N_z}(2k+1)\right) \left(1 + \frac{\sin\left(\frac{\pi}{2N_z}(2k-1)\right)}{\sin\left(\frac{\pi}{2N_z}(2k+1)\right)}\right)} \\
& \times \left(e^{-jk_{N_z} \sin\left(\frac{\pi}{2N_z}(2k+1)\right) \sin\left(\frac{\pi}{2N_z}\right)} - e^{jk_{N_z} \sin\left(\frac{\pi}{2N_z}(2(k-1)+1)\right) \sin\left(\frac{\pi}{2N_z}\right)} \right)
\end{aligned}$$

Then, multiplying the term in the denominator, the final form of the 3D NDE is found for the NU CGL mesh points

$$\begin{aligned}
& \frac{1}{c^2 \Delta t^2} \sin^2\left(\frac{\omega \Delta t}{2}\right) = \\
& \frac{j \sin\left(k_{N_x} \sin\left(\frac{\pi}{2N_x}\right) \sin\left(\frac{\pi}{2N_x}(2i+1)\right)\right)}{4 \sin^2\left(\frac{\pi}{2N_x}\right) \sin^2\left(\frac{\pi}{2N_x}(2i+1)\right) \left(1 + \frac{\sin\left(\frac{\pi}{2N_x}(2i-1)\right)}{\sin\left(\frac{\pi}{2N_x}(2i+1)\right)}\right)} \\
& \times \left(e^{-jk_{N_x} \sin\left(\frac{\pi}{2N_x}\right) \sin\left(\frac{\pi}{2N_x}(2i+1)\right)} - e^{jk_{N_x} \sin\left(\frac{\pi}{2N_x}\right) \sin\left(\frac{\pi}{2N_x}(2(i-1)+1)\right)} \right)
\end{aligned} \tag{41}$$

$$\begin{aligned}
& + \frac{j \sin \left(k_{N_y} \sin \left(\frac{\pi}{2N_y} \right) \sin \left(\frac{\pi}{2N_y} (2j+1) \right) \right)}{4 \sin^2 \left(\frac{\pi}{2N_y} \right) \sin^2 \left(\frac{\pi}{2N_y} (2j+1) \right) \left(1 + \frac{\sin \left(\frac{\pi}{2N_y} (2j-1) \right)}{\sin \left(\frac{\pi}{2N_y} (2j+1) \right)} \right)} \\
& \times \left(e^{-jk_{N_y} \sin \left(\frac{\pi}{2N_y} \right) \sin \left(\frac{\pi}{2N_y} (2j+1) \right)} - e^{jk_{N_y} \sin \left(\frac{\pi}{2N_y} \right) \sin \left(\frac{\pi}{2N_y} (2(j-1)+1) \right)} \right) \\
& + \frac{j \sin \left(k_{N_z} \sin \left(\frac{\pi}{2N_z} \right) \sin \left(\frac{\pi}{2N_z} (k+1) \right) \right)}{4 \sin^2 \left(\frac{\pi}{2N_z} \right) \sin^2 \left(\frac{\pi}{2N_z} (2k+1) \right) \left(1 + \frac{\sin \left(\frac{\pi}{2N_z} (2k-1) \right)}{\sin \left(\frac{\pi}{2N_z} (2k+1) \right)} \right)} \\
& \times \left(e^{-jk_{N_z} \sin \left(\frac{\pi}{2N_z} \right) \sin \left(\frac{\pi}{2N_z} (2k+1) \right)} - e^{jk_{N_z} \sin \left(\frac{\pi}{2N_z} \right) \sin \left(\frac{\pi}{2N_z} (2(k-1)+1) \right)} \right).
\end{aligned}$$

In fact, this is the same as the NDE extracted by using the ratio concept, before.

• **3D NDE for the NU Linear Mesh**

- The Mesh Formula for the x Direction

For the constant $\alpha_\tau = \alpha$, the ratio is

$$\alpha = \frac{\Delta x_i}{\Delta x_{i-1}} \quad (42)$$

where the mesh formula can be written

$$\Delta x_i = \underbrace{\Delta x_{min}}_{\Delta x_0} \alpha^i \quad ; \quad \Delta x_{i-1} = \underbrace{\Delta x_{min}}_{\Delta x_0} \alpha^{i-1} \quad (43)$$

and the NU mesh ratio becomes

$$\alpha_i = \frac{\Delta x_i}{\Delta x_{i-1}} \Rightarrow \Delta x_i = \alpha_i \Delta x_{i-1}. \quad (44)$$

The general multiplication form is given

$$\Delta x_1 = \alpha_1 \Delta x_0 \Rightarrow \Delta x_2 = \alpha_2 \Delta x_1 = \alpha_1 \alpha_2 \Delta x_0 \Rightarrow \Delta x_i = \prod_{\tau=1}^i \alpha_\tau \Delta x_0. \quad (45)$$

Substituting the relation of the constant mesh ratio into (44) as

$$\alpha_\tau = \alpha \Rightarrow \alpha = \frac{\Delta x_i}{\Delta x_{i-1}} \Rightarrow \Delta x_i = \alpha \Delta x_{i-1} \quad (46)$$

which can bring a new way of representing the general multiplication mesh ratio as

$$\Delta x_1 = \alpha \Delta x_0 \Rightarrow \Delta x_2 = \alpha \Delta x_1 = \alpha \times \alpha \times \Delta x_0 = \alpha^2 \Delta x_0 \quad (47)$$

$$\Delta x_i = \prod_{\tau=1}^i \alpha_\tau \Delta x_0 = \overbrace{\alpha \times \alpha \times \dots \times \alpha}^{\alpha^i} \Delta x_0 = \alpha^i \Delta x_0. \quad (48)$$

Specially, the NU mesh can be represented in terms of the minimum mesh interval

$$\Delta x_i = \Delta x_0 \alpha^i \quad ; \quad \Delta x_{i-1} = \Delta x_0 \alpha^{i-1}. \quad (49)$$

- The Mesh Formula for the y Direction

For the constant $\beta_\tau = \beta$, the ratio is

$$\beta = \frac{\Delta y_j}{\Delta y_{j-1}} \quad (50)$$

where the mesh formula for the y direction is

$$\Delta y_j = \frac{\Delta y_{min}}{\Delta y_0} \beta^j \quad ; \quad \Delta y_{j-1} = \frac{\Delta y_{min}}{\Delta y_0} \beta^{j-1} \quad (51)$$

and the NU mesh ratio can be written

$$\beta_j = \frac{\Delta y_j}{\Delta y_{j-1}} \Rightarrow \Delta y_j = \beta_j \Delta y_{j-1}. \quad (52)$$

The general multiplication form becomes

$$\Delta y_1 = \beta_1 \Delta y_0 \Rightarrow \Delta y_2 = \beta_2 \Delta y_1 = \beta_1 \beta_2 \Delta y_0 \Rightarrow \Delta y_j = \prod_{\tau=1}^j \beta_\tau \Delta y_0. \quad (53)$$

Substituting the relation of the constant mesh ratio into (52)

$$\beta_k = \beta \Rightarrow \beta = \frac{\Delta y_j}{\Delta y_{j-1}} \Rightarrow \Delta y_j = \beta \Delta y_{j-1} \quad (54)$$

which the general multiplication form of the mesh can be reformulated as

$$\Delta y_1 = \beta \Delta y_0 \Rightarrow \Delta y_2 = \beta \Delta y_1 = \beta \times \beta \times \Delta y_0 = \beta^2 \Delta y_0 \quad (55)$$

$$\Rightarrow \Delta y_j = \prod_{\tau=1}^j \beta_{\tau} \Delta y_0 = \overbrace{\beta \times \beta \times \dots \times \beta}^{\beta^j} \Delta y_0 = \beta^j \Delta y_0. \quad (56)$$

Specially, the NU mesh can be represented in terms of the minimum constant mesh interval

$$\Delta y_j = \beta^j \Delta y_0 \quad ; \quad \Delta y_{j-1} = \beta^{j-1} \Delta y_0. \quad (57)$$

- The Mesh Formula for the z Direction

For the constant $\gamma_{\tau} = \gamma$

$$\frac{\Delta z_k}{\Delta z_{k-1}} = \gamma \quad (58)$$

where the mesh formula for the z direction is

$$\Delta z_k = \underbrace{\Delta z_{min}}_{\Delta z_0} \gamma^k \quad ; \quad \Delta z_{k-1} = \underbrace{\Delta z_{min}}_{\Delta z_0} \gamma^{k-1} \quad (59)$$

and the NU mesh ratio is defined with the NU mesh intervals

$$\gamma_k = \frac{\Delta z_k}{\Delta z_{k-1}} \Rightarrow \Delta z_k = \gamma_k \Delta z_{k-1}. \quad (60)$$

The general multiplication form can be found as

$$\Delta z_1 = \gamma_1 \Delta z_0 \Rightarrow \Delta z_2 = \gamma_2 \Delta z_1 = \gamma_1 \gamma_2 \Delta z_0 \Rightarrow \Delta z_k = \prod_{\tau=1}^k \gamma_{\tau} \Delta z_0. \quad (61)$$

Substituting the relation of the constant mesh ratio into (60)

$$\gamma_k = \gamma \Rightarrow \gamma = \frac{\Delta z_k}{\Delta z_{k-1}} \Rightarrow \Delta z_k = \gamma \Delta z_{k-1} \quad (62)$$

$$\Delta z_1 = \gamma \Delta z_0 \Rightarrow \Delta z_2 = \gamma \Delta z_1 = \gamma \times \gamma \times \Delta z_0 = \gamma^2 \Delta z_0 \quad (63)$$

where one gets the general multiplication form to a special multiplication form which is simpler than the general case as

$$\Delta z_k = \prod_{\tau=1}^k \gamma_{\tau} \Delta z_0 = \overbrace{\gamma \times \gamma \times \dots \times \gamma}^{\gamma^k} \Delta z_0 = \gamma^k \Delta z_0. \quad (64)$$

Then, using all the findings, one can basically write the NU mesh intervals as

$$\Delta z_k = \Delta z_0 \gamma^k \quad ; \quad \Delta z_{k-1} = \Delta z_0 \gamma^{k-1} \quad (65)$$

$$\Delta x_i = \Delta x_0 \alpha^i \quad ; \quad \Delta x_{i-1} = \Delta x_0 \alpha^{i-1} \quad (66)$$

$$\Delta y_j = \Delta y_0 \beta^j \quad ; \quad \Delta y_{j-1} = \Delta y_0 \beta^{j-1}. \quad (67)$$

Substituting Δx_i , Δx_{i-1} , Δy_j , Δy_{j-1} , Δz_k and Δz_{k-1} into the 3D NDE, the 3D NDE for the NU Linear mesh is found to be

$$\begin{aligned} & \frac{1}{c^2 \Delta t^2} \sin^2 \left(\frac{\omega \Delta t}{2} \right) = \\ & \frac{j}{\Delta x_0 \alpha^i (\Delta x_0 \alpha^i + \Delta x_0 \alpha^{i-1})} \sin \left(\frac{k_{Nx} \Delta x_0 \alpha^i}{2} \right) \left(e^{-\frac{jk_{Nx} \Delta x_0 \alpha^i}{2}} - e^{\frac{jk_{Nx} \Delta x_0 \alpha^{i-1}}{2}} \right) + \\ & \frac{j}{\Delta y_0 \beta^j (\Delta y_0 \beta^j + \Delta y_0 \beta^{j-1})} \sin \left(\frac{k_{Ny} \Delta y_0 \beta^j}{2} \right) \left(e^{-\frac{jk_{Ny} \Delta y_0 \beta^j}{2}} - e^{\frac{jk_{Ny} \Delta y_0 \beta^{j-1}}{2}} \right) + \\ & \frac{j}{\Delta z_0 \gamma^k (\Delta z_0 \gamma^k + \Delta z_0 \gamma^{k-1})} \sin \left(\frac{k_{Nz} \Delta z_0 \gamma^k}{2} \right) \left(e^{-\frac{jk_{Nz} \Delta z_0 \gamma^k}{2}} - e^{\frac{jk_{Nz} \Delta z_0 \gamma^{k-1}}{2}} \right). \end{aligned} \quad (68)$$

A more compact 3D NDE formulation over the minimum intervals can also be represented. For this aim, first of all, let us consider the mesh formulas given below in Figure as

$$\Delta x_i = \Delta x_{min} \alpha^i \quad ; \quad \Delta x_{i-1} = \Delta x_{min} \alpha^{i-1} \quad (69)$$

$$\Delta y_j = \Delta y_{min} \beta^j \quad ; \quad \Delta y_{j-1} = \Delta y_{min} \beta^{j-1} \quad (70)$$

$$\Delta z_k = \Delta z_{min} \gamma^k \quad ; \quad \Delta z_{k-1} = \Delta z_{min} \gamma^{k-1}. \quad (71)$$

Substituting them into the general 3D NU NDE of Maxwell's equations given in [Chris et al., 2002], a more compact 3D NDE formulation is found as

$$\begin{aligned} \frac{1}{c^2 \Delta t^2} \sin^2 \left(\frac{\omega \Delta t}{2} \right) &= \frac{j}{\Delta x_{min} \alpha^i (\Delta x_{min} \alpha^i + \Delta x_{min} \alpha^{i-1})} \sin \left(\frac{k_{Nx} \Delta x_{min} \alpha^i}{2} \right) \\ &\quad \times \left(e^{-\frac{jk_{Nx} \Delta x_{min} \alpha^i}{2}} - e^{\frac{jk_{Nx} \Delta x_{min} \alpha^{i-1}}{2}} \right) \\ &+ \frac{j}{\Delta y_{min} \beta^j (\Delta y_{min} \beta^j + \Delta y_{min} \beta^{j-1})} \sin \left(\frac{k_{Ny} \Delta y_{min} \beta^j}{2} \right) \\ &\quad \times \left(e^{-\frac{jk_{Ny} \Delta y_{min} \beta^j}{2}} - e^{\frac{jk_{Ny} \Delta y_{min} \beta^{j-1}}{2}} \right) \\ &+ \frac{j}{\Delta z_{min} \gamma^k (\Delta z_{min} \gamma^k + \Delta z_{min} \gamma^{k-1})} \sin \left(\frac{k_{Nz} \Delta z_{min} \gamma^k}{2} \right) \\ &\quad \times \left(e^{-\frac{jk_{Nz} \Delta z_{min} \gamma^k}{2}} - e^{\frac{jk_{Nz} \Delta z_{min} \gamma^{k-1}}{2}} \right). \end{aligned} \quad (72)$$

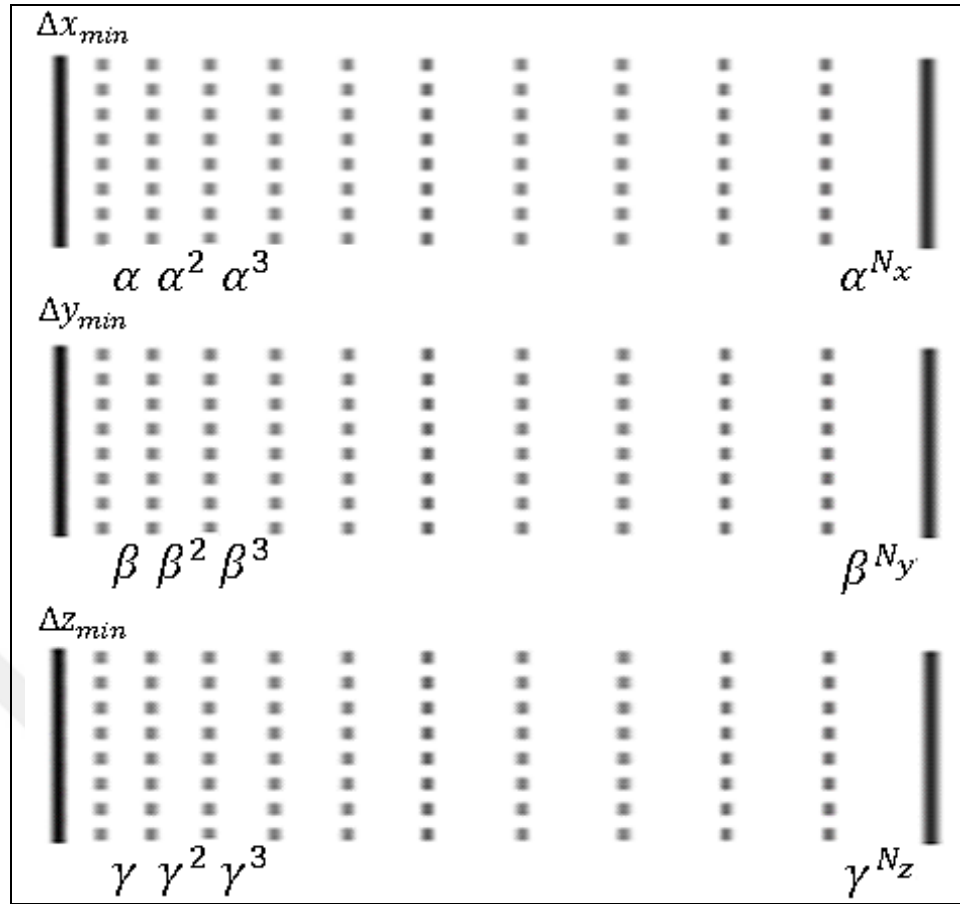


Figure 1: The NU Linear mesh demonstrations for the x , y and z directions.