

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

**ON THE AUTOMORPHISM GROUP OF
THE ALGEBRA OF THE
SCALAR UPPER TRIANGULAR MATRICES**

ÖZKAY ÖZKAN
**A THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
DEPARTMENT OF MATHEMATICS**

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THESIS SUPERVISOR
ASSOC. PROF. DR. MUSTAFA AKKURT

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T.C.
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SUMMARY

For a commutative ring R with unity, the R -algebra of strictly upper triangular $n \times n$ matrices over R is denoted by $N_n(R)$, where n is a positive integer greater than 1. For the identity matrix I , $\alpha \in R$, $A \in N_n(R)$, the set of all elements $\alpha I + A$ is defined as the scalar upper triangular matrix algebra $ST_n(R)$ which is a subalgebra of the upper triangular matrix algebra $T_n(R)$. In this thesis, we investigate the R -algebra automorphisms of $ST_n(R)$. We extend the automorphisms of $N_n(R)$ to $ST_n(R)$ and classify all the automorphisms of $ST_n(R)$. We prove that there exist some automorphisms of $ST_n(R)$ which cannot be extended to any automorphism of $T_n(R)$.

Keywords: Upper Triangular Matrix Algebra, Strictly Upper Triangular Matrix Algebra , Automorphisms, Central Automorphisms.

ÖZET

Birimli ve deęişmeli bir R halkası üzerinde, kesin üst üçgensel $n \times n$ matris cebirleri, $n > 1$ bir tamsayı olmak üzere $N_n(R)$ ile gösterilir. I birim matris, $\alpha \in R$, $A \in N_n(R)$ olmak üzere $\alpha I + A$ biçiminde tanımlı tüm elemanların oluşturduğu küme, üst üçgensel matris cebri $T_n(R)$ 'ın bir alt cebridir. Bu alt cebire skaler üst üçgensel matris cebirleri denir ve bu cebir $ST_n(R)$ ile gösterilir. Bu tezde, $ST_n(R)$ cebrinin tüm otomorfizmaları araştırılmıştır. $N_n(R)$ cebrinin otomorfizmaları $ST_n(R)$ cebrinin otomorfizmalarına genişletilmiş ve $ST_n(R)$ cebrinin tüm otomorfizmaları sınıflandırılmıştır. $ST_n(R)$ cebrinin otomorfizmalarının $T_n(R)$ cebrinin otomorfizmalarına her koşulda genişletilemeyeceęi ispatlanmıştır.

Anahtar Kelimeler: Üst Üçgensel Matris Cebirleri, Kesin Üst Üçgensel Matris Cebirleri, Otomorfizmalar, Merkezil Otomorfizmalar.

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LIST of ABBREVIATIONS and ACRONYMS

<u>Abbreviations</u>	<u>Explanations</u>
<u>and Acronyms</u>	
R	: The Ring
S	: A Subring of The Ring R
I	: An Ideal of The Ring R
$\mathbb{Q}$	: The Rational Numbers
G	: The Group
$Z(G)$	: The Center of The Group G
$M_n(R)$	: The Ring of $n \times n$ Matrices Over The Ring R
$T_n(R)$	: Upper Triangular Matrix Ring Of The Ring R
$ST_n(R)$	: Strictly Upper Triangular Matrix Ring Of The Ring R
E_{ij}	: The Matrix Units
$Aut(R)$	: Automorphism Group of The Ring R
$Inn(R)$	: Inner Automorphism Group of The Ring R
$Cen(R)$	: Central Automorphism Group of The Ring R

1. INTRODUCTION

Let $T_n(R)$ be the R -algebra of $n \times n$ upper triangular matrices over R where R is a commutative ring with identity. The automorphisms of $T_n(R)$ were studied by several authors. It was stated in [Barker and Kezlan, 1987] that if R is an integral domain then every R -algebra automorphism of $T_n(R)$ is inner. In I.M. Isaac's review [Isaac, 1998], he pointed out that the hypothesis is still true in the case R is a commutative ring. Kezlan [Kezlan, 1990] confirmed Isaac's review and generalized the result of [Barker and Kezlan, 1987]. He has shown that every R -algebra automorphism of $T_n(R)$ is inner. Moreover, for a ring R with unity S.P. Coelho [Coelho, 1993] observed that every R -automorphism of $T_n(R)$ is inner.

For a commutative ring R , consider an associative R -algebra A . Barker and Kezlan proved in [Barker and Kezlan, 1990] that, in case every R -endomorphism of $T_n(A)$ is an automorphism then every R -algebra automorphism of $T_n(A)$ can be written as a composition of an inner automorphism and an induced R -automorphism of A . Provided A is a prime R -algebra, S. Jøndrup showed that the same result holds [Jøndrup, 1987]. If A is a simple artinian ring, finite dimensional over its center F , then every F -automorphism of $T_n(A)$ is inner [Jøndrup, 1991]. Moreover, all these results were generalized to rings where all idempotents are central [Jøndrup, 1995]. For a semiprime ring A and central subring R , every R -automorphism of $T_n(A)$ decomposes into a product of an inner automorphism and an automorphism defined by an automorphism of A .

M. Koppinen [Koppinen, 1996] guaranteed that such decompositions for A is also valid if R -algebra automorphism of A respects the subspace of strictly upper triangular matrices $N_n(A)$. At this point, an algebra automorphism θ of $T_n(A)$ satisfies

$$\theta(N_n(A)) = N_n(A) \tag{1.1}$$

whether A is semiprimitive or A contains no nonzero ideal I with $\bigcap_{m=1}^{\infty} I^m = 0$ or A is semiprime. He proved that this result is also true for the ring of infinite matrices indexed by $\mathbb{Z}^+$. If A is an indecomposable R -algebra, then in the ring of matrices

indexed by $\mathbb{Z}$, any R -automorphism fixing $N_n(A)$ decomposes into a product of an inner automorphism and an induced R -automorphism of A and an index shift of matrix indices.

In [Spiegel, 2001] and [Spiegel and O'Donnell, 1997], E. Spiegel extended both Koppinen's [Koppinen, 1995] and Kezlan's [Kezlan, 1990] results by considering the automorphisms of incidence algebra of a locally finite partially ordered set X over a ring R with identity.

In note [Cao and Wang, 2000], Cao and Wang explicitly determined the automorphism group of the R -algebra of strictly upper triangular $n \times n$ matrices over an arbitrary commutative ring. For $n \geq 4$, they stated that $Aut(N_n(R))$ consists of the semidirect product of inner automorphism group and central automorphism group of $N_n(R)$.

For a group G , there are some well known results about central automorphisms of finite groups. For infinite groups, it is more complicated. The set of all central automorphisms is very important in studying $Aut(G)$. If $\theta \in Cen(G)$ then the map

$$f_\theta(x) = x^{-1}\theta(x) \tag{1.2}$$

is a homomorphism of G into the center of G . Adney and Yen [Adney and Yen, 1965] showed that if a finite group G does not have an abelian direct factor, the correspondence $\theta \longrightarrow f_\theta$ is one-to-one between $Cen(G)$ and the group of all homomorphisms from G into $Z(G)$. If p is a prime, then a finite group G is called a p -group if $|G| = p^n$ for some $n \geq 0$. They proved that if p is odd and G has no abelian direct factors then $Cen(G)$ is also a p -group.

Sanders [Sanders, 1969] improved this result. He stated that if $p = 2$, it is a 2-group if and only if G has at most one cyclic direct factor of order 2^i for any i . Using this result, Jamali and Mousavi [Jamali and Mousavi, 2002] studied the center of $Cen(G)$ for a certain class of finite p -groups. As a basic fact, they also stated that the group of central automorphisms $Cen(G)$ of a finite group G induces the identity automorphism on $G/Z(G)$.

In this work, we generalize the result of [Cao and Wang, 2000] and determine why central automorphisms exist on $ST_n(R)$ but not on $T_n(R)$.

2. SOME BASIC DEFINITIONS AND PROPERTIES

For a group G , an isomorphism G onto itself is called an automorphism of G and the set of all automorphisms of G is denoted by $Aut(G)$. First, we introduce some important families of subgroups of an arbitrary group which provide us to define the relation between the automorphism groups of G .

Definition 2.1: [Dummit and Foote, 2004] Let G be a group and A be a nonempty subset of G .

$$C_G(A) = \{g \mid gag^{-1} = a, \text{ for all } a \in A\} \quad (2.1)$$

is the centralizer of A in G .

Definition 2.2: [Dummit and Foote, 2004] $Z(G) = \{g \in G \mid gx = xg, \text{ for all } x \in G\}$ defines the center of G .

For a ring $R(+, *)$ with multiplicative identity, we can define an automorphism of R in the same manner.

Definition 2.3: A function $\psi : R \rightarrow R$ is a ring automorphism if

- i) $\psi(a + b) = \psi(a) + \psi(b)$*
- ii) $\psi(a * b) = \psi(a) * \psi(b)$*
- iii) $\psi(1) = 1$*
- iv) ψ is bijective.*

Denote the automorphism group of R by $Aut(R)$. One of the most important examples of $Aut(G)$ is provided by conjugation by a fixed element in G . If $G = R$ is a ring then, g must be unit.

Definition 2.4: If $g \in R$ is an invertible element, then g determines an inner automorphism $\theta(u)$ given by

$$\theta_g(u) = g \cdot u \cdot g^{-1} \quad (2.2)$$

for each $u \in R$.

Inner automorphism group of R is a normal subgroup of automorphism group of R . Denote the inner automorphism group of R by $\text{Inn}(R)$.

Definition 2.5: [Dummit and Foote, 2004] Let G be a group and A be a nonempty subset of G .

$$N_G(A) = \{g \mid gAg^{-1} = A\} \quad (2.3)$$

is the normalizer of A in G .

Remark 2.1: [Dummit and Foote, 2004][Corollary 15] For any subgroup H of a group G , the quotient group $N_G(H)/C_G(H)$ is isomorphic to $\text{Inn}(G)$.

We can define an automorphism which is constructed by $g \in G$, using conjugation. If we restrict this automorphism to a subgroup H of G , we may get an automorphism of H which is not inner on H .

Example 2.1: Let G be the alternating group A_4 . The Klein 4-group is a normal subgroup of A_4 which is not contained in the center of A_4 . For any 3-cycle $g \in A_4$ such that conjugation by g restricted to V_4 is not an inner automorphism. For $g = (134)$, conjugation by (134) restricted to V_4 , we get $f : V_4 \longrightarrow V_4$ as

$$f_g(V_4) = g.V_4.g^{-1} = \{(id), (14)(23), (12)(34), (13)(24)\} \quad (2.4)$$

This is an automorphism of V_4 but not an inner automorphism.

Definition 2.6: [Jamali and Mousavi, 2002] An automorphism μ of a group G is a central automorphism if μ commutes with every automorphism in the group of inner automorphisms of G , or equivalently if $g^{-1}\mu(g)$ lies in the center $Z(G)$ of for any $g \in G$.

Denote the central automorphism group of G by $Cen(G)$. The central automorphism group $Cen(G)$ of a finite group is of great importance in studying the full automorphism group of G . The central automorphisms fix the commutator subgroup G' of G pointwise and form a normal subgroup of full automorphism group of G . A central automorphism of a ring R is a ring automorphism of R which fixes the center of R element wise.

Example 2.2: [Roggenkamp and Taylor, 1992] Define the field $\mathbb{F}_4 = \{0, 1, x, x + 1\}$ which is isomorphic to the field $\mathbb{Z}_2[x]/(x^2 + x + 1)$. In the ring of 3×3 matrices over $\mathbb{F}_4$, consider the following matrices:

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \quad (2.5)$$

$$U = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x+1 \\ 0 & 0 & 1 \end{pmatrix}; \quad (2.6)$$

$$D_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; D_2 = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; D_3 = \begin{pmatrix} 1 & 0 & x+1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.7)$$

Over the field $\mathbb{F}_4$, the matrices $B, C, U, V, D_1, D_2, D_3$ generate the group of upper triangular matrices with diagonal entries 1. Roggenkamp[Roggenkamp and Taylor, 1992] denote this group by H_0 and determines the center of H_0 as $Z(H_0) = \{1, D_1, D_2, D_3\}$. The automorphism σ is defined on H_0 as:

$$\sigma : H_0 \longrightarrow H_0$$

with $\sigma(B) = D_1.B$, $\sigma(C) = D_1.C$ and the rest of the generators are fixed. Roggenkamp[Roggenkamp and Taylor, 1992] prove that σ is a central automorphism on H_0 which is not inner.

Let R be a commutative ring with unity and $N_n(R)$ be the algebra of all $n \times n$ strictly upper triangular matrices over R , where n is a positive integer greater than 1.

Cao and Wang [Cao and Wang, 2000] show that we have central automorphisms on $N_n(R)$.

Definition 2.7: For unity matrix I ,

$$ST_n(R) = \{\alpha I + A : \alpha \in R, A \in N_n(R)\} \quad (2.8)$$

is the scalar upper triangular matrix algebra over R .

$ST_n(R)$ is a subalgebra of $T_n(R)$. Any $X \in ST_n(R)$ is of the form,

$$X = \alpha I + A = \alpha I + \sum_{i < j} a_{ij} E_{ij} \quad (2.9)$$

where $\alpha \in R$ and $A \in N_n(R)$. Our aim is to define the central automorphisms on $ST_n(R)$.

For the R -algebra $ST_n(R)$, the standard matrix units are the matrices E_{ij} with a 1 at (i, j) position and zero elsewhere. Clearly, the set of matrices $\{E_{ij} : 1 \leq i < j \leq n\}$ form a basis of $N_n(R)$.

For any matrix $X \in ST_n(R)$, let $S_1, S_2, \dots, S_{n-1}$ define the diagonals of $X = \alpha I + \sum_{i < j} a_{ij} E_{ij}$ as follows:

$$\begin{aligned} S_1 &= \{a_{12}, a_{23}, \dots, a_{n-1,n}\} \\ S_2 &= \{a_{13}, a_{24}, \dots, a_{n-2,n}\} \\ &\vdots \\ S_{n-1} &= \{a_{1n}\}. \end{aligned} \quad (2.10)$$

Cao and Wang [Cao and Wang, 2000] use the notation

$$M_r = \left\{ \sum_{j-i \geq r} a_{ij} E_{ij} \in N_n(R) \mid a_{ij} \in R \right\} \quad (2.11)$$

for $r = 1, \dots, n-1$, and $M_r = 0$ for $r \geq n$. Clearly, M_r defines an ideal of $N_n(R)$.

It is important to state that we can get the ideals M_r of [Cao and Wang, 2000] via

$$M_r = \sum_{i=r}^{n-1} S_i. \quad (2.12)$$

The following definitions will be used while classifying the automorphisms of $M_n(R)$.

Definition 2.8: An element g of a group G is called a torsion element of the group if it has a finite order. An element m of the R -module is called a torsion element if $rm = 0$ for some nonzero element $r \in R$.

Definition 2.9: An element r of a ring R is called nilpotent if there exists some positive integer n such that $r^n = 0$.

3. AUTOMORPHISMS OF $ST_N(R)$

First, we start with the classification of the automorphism groups of $M_n(R)$ since it is very important to emphasize the correlation between the terms used by G.P. Barker, T.P. Kezlan [Barker and Kezlan, 1987] and Y. Cao, J. Wang [Cao and Wang, 2000].

Definition 3.1: [Rotman, 2003] An F -algebra A over a field F is central simple if it is finite dimensional, simple (no two-sided ideals other than A and 0), and its center is F .

The well known examples of central simple algebras are the quaternions $\mathbb{H}$ over $\mathbb{R}$ and $M_n(F)$ over a field F .

Theorem 3.1: [Rotman, 2003][Theorem 9.121] Let A be a central simple F -algebra over a field F , and let B and B' be isomorphic simple F -subalgebras of A . If $\psi : B \longrightarrow B'$ is an isomorphism, then there exists a unit $u \in A$ with $\psi(b) = ubu^{-1}$ for all $b \in B$.

Since $M_n(F)$ over a field F is a central simple algebra, a well known conclusion of Skolem-Noether Theorem is that every automorphism of $M_n(F)$ over a field F is inner. For a commutative ring R with unity, I.M. Isaacs [Isaac, 1980] measured the failure of Skolem-Noether Theorem. He defined the Skolem-Noether group as

$$SN_n(R) = Aut(M_n(R))/Inn(M_n(R)). \quad (3.1)$$

Theorem 3.2: Let R be a commutative ring with unity and $n > 0$,

- i) If R is a unique factorization domain, then $SN_n(R) = 1$.*
- ii) If R is a Dedekind domain, then $SN_n(R)$ is isomorphic to the subgroup of the class group of R consisting of all the elements of order dividing n .*

For Dedekind domains, there is an example in [Isaac, 1980], to show that not every automorphism of $M_n(R)$ is inner.

Example 3.1: Let $R = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$. In $M_2(R)$, consider the matrix

$$B = \begin{pmatrix} 1 + \sqrt{-5} & -2 \\ -2 & 1 - \sqrt{-5} \end{pmatrix} \quad (3.2)$$

with the inverse,

$$B^{-1} = \frac{1}{2} \begin{pmatrix} 1 - \sqrt{-5} & 2 \\ 2 & 1 + \sqrt{-5} \end{pmatrix}. \quad (3.3)$$

It can be seen that $B^{-1} \notin M_2(R)$. But for any $A \in M_2(R)$, the product $B^{-1}AB \in M_2(R)$. Then $\theta(A) = B^{-1}AB$ defines an automorphism on $M_2(R)$. Assume that there exists $P \in M_2(R)$ such that $\theta(A) = P^{-1}AP$ for all $A \in M_2(R)$. Then we get $P^{-1}AP = B^{-1}AB$. It follows that,

$$(BP^{-1})A = A(BP^{-1}). \quad (3.4)$$

So BP^{-1} is in the center of $M_2(R)$ and it must be of the form λI for some $\lambda \in R$. If $BP^{-1} = \lambda I$ then $B = \lambda IP$.

Since $\det(B) = \det(\lambda P) = 2$ we get $2 = \lambda^2 \det(P)$. For invertible matrix $P \in M_2(R)$, we have $\det(P P^{-1}) = 1$, so $\det(P)$ is a unit of R . But the units of R are 1 and -1 , then $\lambda = \pm\sqrt{\pm 2}$. There is no such λ in R . Thus θ is not an inner automorphism.

G.P. Barker and T.P. Kezlan [Barker and Kezlan, 1987] use the following theorems to extend the result of I.M. Isaacs.

Theorem 3.3: [Barker and Kezlan, 1987] Let R be a commutative ring with 1, and assume that R has either no nonzero nilpotent elements or no 2-torsion. Then $C(\text{Inn}(M_n(R)))$ is trivial; in particular $Z(\text{Aut}(M_n(R)))$ is trivial.

Corollary 3.1: [Barker and Kezlan, 1987] If R is an integral domain, then $C(\text{Inn}(M_n(R)))$ is trivial; in particular $Z(\text{Aut}(M_n(R)))$ is trivial.

Remark 3.1: $C(\text{Inn}(M_n(R)))$ is the notation used for centralizer of inner automorphisms of the matrix algebra $M_n(R)$ [Barker and Kezlan, 1987]. Remember that an automorphism is central automorphism if it commutes with every

automorphism in the inner automorphisms group of G [Cao and Wang, 2000]. If $C(\text{Inn}(M_n(R)))$ is trivial then we have no central automorphism of $(M_n(R))$; in particular $Z(\text{Aut}(M_n(R)))$ is trivial. Thus, we know that the central automorphisms do not exist for the full $n \times n$ matrix algebra in the case R has either no nonzero nilpotent element or no 2-torsion element.

G.P. Barker and T.P. Kezlan [Barker and Kezlan, 1987] show that the hypothesis of the theorem cannot be removed.

Example 3.2: [Barker and Kezlan, 1987][Example b] For commutative ring $R = \mathbb{Z}_4$ with unity 1 ,

$$\theta : M_2(\mathbb{Z}_4) \longrightarrow M_2(\mathbb{Z}_4) \quad (3.5)$$

defines the map by

$$\theta(A) = CAC + 2(BA + AB) \quad (3.6)$$

for all $A \in M_2(\mathbb{Z}_4)$, where

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.7)$$

Using properties on B and C , Barker and Kezlan prove that θ preserves multiplication and addition, then it is an algebra endomorphism on $M_2(\mathbb{Z}_4)$. For any $A \in M_2(\mathbb{Z}_4)$, consider

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (3.8)$$

for any $a, b, c, d \in \mathbb{Z}_4$. To prove θ is one-to-one, they use $\theta(A) = 0$. Then

$$CAC = 2(BA + AB) \quad (3.9)$$

and using $C^2 = I$,

$$A = C(CAC)C = 2C(BA + AB)C. \quad (3.10)$$

On the right hand side of the equation, since $CB = -BC$ we obtain

$$BC = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ then } CBAC = \begin{pmatrix} c & -d \\ -a & b \end{pmatrix} \text{ and } CABC = \begin{pmatrix} b & -a \\ -d & c \end{pmatrix}. \quad (3.11)$$

Hence, we have

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 2(CBAC + CABC) = \begin{pmatrix} 2b + 2c & -2a - 2d \\ -2a - 2d & 2b + 2c \end{pmatrix}. \quad (3.12)$$

Comparing the entries of the equal matrices, it is obtained that $a = b = c = d = 0$. Therefore, $A = 0$ and θ is one-to-one.

By routine computation $\theta(C) = C$ and $\theta(B) = -B$, then for all $A \in M_2(\mathbb{Z}_4)$,

$$\theta(\theta(A)) = C\theta(A)C + 2(-B\theta(A)) + 2(\theta(A)(-B)) = C\theta(A)C - 2B\theta(A) - 2\theta(A)B. \quad (3.13)$$

So,

$$\theta(\theta(A)) = A. \quad (3.14)$$

Thus θ is onto and θ is an algebra automorphism of $M_2(\mathbb{Z}_4)$.

For any inner automorphism $\zeta \in \text{Inn}(M_2(\mathbb{Z}_4))$ and for any $A \in M_2(\mathbb{Z}_4)$ and an invertible matrix $D \in M_2(\mathbb{Z}_4)$, we know $\zeta = DAD^{-1}$. If we have $\theta\zeta = \zeta\theta$ for any $\zeta \in \text{Inn}(M_2(\mathbb{Z}_4))$, then $\theta \in \text{Cen}(\text{Inn}(M_2(\mathbb{Z}_4)))$. Thus, θ is a central automorphism.

Define the invertible matrix D as,

$$D = \begin{pmatrix} x & y \\ z & t \end{pmatrix}. \quad (3.15)$$

To verify $\theta\zeta = \zeta\theta$, Barker and Kezlan just use the basis e_{12} and e_{21} . Since similar computations can be used for e_{21} , they show $(\theta\zeta - \zeta\theta)(e_{12}) = 0$. To verify this equality, we need to show

$$(\theta\zeta - \zeta\theta)(e_{12}) = \begin{pmatrix} 2(x^2 + z^2 + xz + xt + yz) & 0 \\ 0 & 2(x^2 + z^2 + xz + xt + yz) \end{pmatrix}. \quad (3.16)$$

Since $\det(D) = xt - yz = \pm 1 \pmod{4}$, at least one of x and z must be $\pm 1 \pmod{4}$. In the case $x = \pm 1 \pmod{4}$, we get $x^2 + z^2 + xz = z^2 \pm z + 1 = \pm 1 \pmod{4}$. For the case $z = \pm 1 \pmod{4}$, we also get $x^2 + z^2 + xz = \pm 1 \pmod{4}$. As a result,

$$2(x^2 + z^2 + xz + xt + yz) = 0 \quad (3.17)$$

Therefore, $(\theta\zeta - \zeta\theta)(e_{12}) = 0$ and θ is a central automorphism on $M_2(\mathbb{Z}_4)$ which is not inner.

Now, we are going to classify the automorphisms of $ST_n(R)$. Before that, we have some theorems that used to determine the automorphisms of upper triangular matrix algebra. We know by the Skolem-Noether Theorem, if A is a simple Artinian algebra, finite dimensional over its center F , then all F -automorphisms of A are inner. S. Jøndrup prove the following result of this theorem for upper triangular matrix algebras.

Theorem 3.4: [Jøndrup, 1987][Theorem A] Let A be a simple artinian ring, finite dimensional over its center F . Every F -automorphisms of $T_n(A)$ is inner.

In [Barker and Kezlan, 1987], G.P. Barker and T.P. Kezlan study the algebra of upper triangular matrices over an integral domain.

Theorem 3.5: [Barker and Kezlan, 1987][Theorem 3] Let $T_n(R)$ be the algebra of upper triangular matrices over an integral domain R . Then every R -algebra automorphism of $T_n(R)$ is inner.

In I.M. Isaac's review [Isaac, 1980] suggested that the result of [Barker and Kezlan, 1987] could be generalized. Then, T. P. Kezlan confirmed the accuracy of Isaac's conjecture.

Theorem 3.6: [Kezlan, 1990] If R is any commutative ring with unity, then every R -algebra automorphism of $T_n(R)$ is inner.

As we noted before, to define an inner automorphism on a commutative ring R , we must have an invertible element. But, there are no invertible elements in $N_n(R)$, hence one cannot define an inner automorphism by the usual terminology. For a fixed

matrix $X \in T_n(R)$ with 1 on the main diagonal, by using the map $\Theta(Y) = XYX^{-1}$, Y.A. Cao and J. Wang [Cao and Wang, 2000] define an inner automorphism on $N_n(R)$.

As noted in [Cao and Wang, 2000], each M_r is invariant under any R -algebra automorphisms of $N_n(R)$. It is obvious that the center of the R -algebra $N_n(R)$ is $R \cdot E_{1n}$.

Lemma 3.1: [Jamali and Mousavi, 2002][Lemma 2.1] Let G be a group. Then an automorphism μ is central if and only if μ induces the identity on $G/Z(G)$.

In [Cao and Wang, 2000], we have

$$T_n^*(R) = \left\{ I + \sum_{i < j} a_{ij} E_{ij} \mid a_{ij} \in R \right\} \quad (3.18)$$

and

$$U(R) = \left\{ I + a_{1n} E_{1n} \mid a_{1n} \in R \right\}. \quad (3.19)$$

Using these definitions, we get

Lemma 3.2: [Cao and Wang, 2000][Lemma 2] $\text{Inn}(N_n(R)) \cong T_n^*(R)/U(R)$.

Thus, we can define central automorphism on $N_n(R)$. For $n \geq 4$, consider $c = (c_2, \dots, c_{n-2}) \in R^{n-3}$. Y.Cao and J. Wang [Cao and Wang, 2000] defined the *central automorphisms* of $N_n(R)$, $\mu_c : N_n(R) \rightarrow N_n(R)$ such that for any $A \in N_n(R)$

$$\mu_c(A) = A + \left(\sum_{k=2}^{n-2} a_{k,k+1} c_k \right) E_{1n}. \quad (3.20)$$

With respect to the basis of $N_n(R)$, automorphisms can be written as:

$$\mu_c(E_{i,i+1}) = E_{i,i+1} + c_i E_{1n} \quad (3.21)$$

for $2 \leq i \leq n-2$ and

$$\mu_c(E_{ij}) = E_{ij} \quad (3.22)$$

for $j \neq i + 1$.

The following property of inner automorphisms of $ST_n(R)$ is adapted form of [Cao and Wang, 2000, Lemma 2]. Let

$$U(R) = \{\alpha I + a_{1n}E_{1n} \mid a_{1n}, \alpha \in R\}. \quad (3.23)$$

To state the following lemma we need to determine the center of $ST_n(R)$. For α and $a_{ij} \in R$, we have $A = \alpha I + \sum_{i < j} a_{ij}E_{ij} \in ST_n(R)$. When $1 \leq k < l \leq n$ for any matrix unit $E_{kl} \in ST_n(R)$. We are going to determine which A 's are in the center of $ST_n(R)$, that is $A \cdot E_{kl} = E_{kl} \cdot A$.

$$(\alpha I + \sum_{i < j} a_{ij}E_{ij}) \cdot E_{kl} = E_{kl} \cdot (\alpha I + \sum_{i < j} a_{ij}E_{ij}) \quad (3.24)$$

If we simplify the equation, we get

$$(\sum_{i < j} a_{ij}E_{ij}) \cdot E_{kl} = E_{kl} \cdot (\sum_{i < j} a_{ij}E_{ij}). \quad (3.25)$$

Since $(\sum_{i < j} a_{ij}E_{ij}) \cdot E_{kl}$ is valid for the case $i < k$ and $E_{kl} \cdot (\sum_{i < j} a_{ij}E_{ij})$ is valid for $j > l$. Thus,

$$\sum_{i < k} a_{ik}E_{il} = \sum_{j > l} a_{lj}E_{kj}. \quad (3.26)$$

Then $a_{ij} = 0$ when $j - i < n - 1$. So A is in the center of $ST_n(R)$ in the case $A = \alpha I + a_{1n}E_{1n} \in U(R)$.

We can define the *inner automorphism* θ on $ST_n(R)$, with a given invertible matrix $B \in T_n(R)$ and for any $A \in ST_n(R)$ as

$$\theta(A) = BAB^{-1}. \quad (3.27)$$

Lemma 3.3: $\text{Inn}(ST_n(R)) \cong ST_n(R)/U(R)$.

Proof 3.3: Let $\vartheta : ST_n(R) \rightarrow Inn(ST_n(R))$ be a map with $\vartheta(A) = \theta(A)$ where $\theta(A)$ is an inner automorphism of $ST_n(R)$. This map is obviously surjective and $Ker(\vartheta)$ contains $U(R)$. On the other hand, for $A \in Ker(\vartheta)$ we get A is in the center of $ST_n(R)$. Thus $A = \alpha I + a_{1n}E_{1n} \in U(R)$. That is $Ker(\vartheta) \subseteq U(R)$. Hence $Ker(\vartheta) = U(R)$.

Now, we investigate the central automorphisms of $ST_n(R)$ in the classification. It is not difficult to see that, for $n \geq 4$,

$$\mu_c : ST_n(R) \rightarrow ST_n(R) \quad (3.28)$$

is an automorphism of $ST_n(R)$.

Remark 3.2: Consider an automorphism $\zeta \in Inn(ST_n(R))$, that is, for all $A \in ST_n(R)$ and a fixed invertible $B \in ST_n(R)$, we have $\zeta(A) = BAB^{-1}$. As we defined before, $C(Inn(ST_n(R)))$ is the centralizer of the inner automorphisms of $ST_n(R)$. Take any $\theta \in C(Inn(ST_n(R)))$, then we get

$$\theta\zeta(A) = \zeta\theta(A). \quad (3.29)$$

If we examine the right side of the equality, we see that

$$\theta\zeta(A) = \theta(BAB^{-1}) = \theta(B)\theta(A)\theta(B^{-1}). \quad (3.30)$$

On the left side,

$$\zeta\theta(A) = B\theta(A)B^{-1} \quad (3.31)$$

Thus, we have the equality,

$$\theta(B)\theta(A)\theta(B)^{-1} = B\theta(A)B^{-1}. \quad (3.32)$$

If we multiply this equation by $\theta(B^{-1})$ on the left side and B on the right side, it becomes

$$\theta(B)^{-1}B\theta(A) = \theta(A)\theta(B)^{-1}B. \quad (3.33)$$

Since this equality is true for any $A \in ST_n(R)$, then $\theta(B)^{-1}B$ is in the center of $ST_n(R)$.

To avoid the details of a tedious proof, we use Kezlan's [Kezlan, 1990] notation. Let

$$\theta : T_n(R) \longrightarrow T_n(R) \quad (3.34)$$

be an R -algebra automorphism of $T_n(R)$. Define

$$\theta(E_{kk}) = \begin{bmatrix} e_{ij}^{(k)} \end{bmatrix} \text{ for } k = 1, 2, \dots, n \quad (3.35)$$

$$\theta(E_{k,k+1}) = \begin{bmatrix} a_{ij}^{(k)} \end{bmatrix} \text{ for } k = 1, 2, \dots, n-1. \quad (3.36)$$

The results in [Kezlan, 1990] show that $a_{k,k+1}^{(k)}$ is a unit with $a_{k,k}^{(k)} = 0$, $e_{k,k}^{(k)} = 1$ and $e_{k,k}^{(q)} = 0$ for $q \neq k$.

Remark 3.3: In terms of the image of matrix units under θ , it is shown that, k^{th} diagonal entry of $\theta(E_{kk})$ is 1, while the other diagonal entries are 0. That is $\theta(E_{kk})$ is in the form $\theta(E_{kk}) = E_{kk} + J_k$ where J_k is strictly upper triangular. For $k = 1, \dots, n$ we use the fact that $\theta(E_{kk})$ is the idempotent, that is $\theta(E_{kk}) = \theta(E_{kk})\theta(E_{kk})$. Thus as a consequence of [Kezlan, 1990], $\theta(E_{kk})$ has the following form:

$$\theta(E_{kk}) = \begin{bmatrix} 0 & \dots & 0 & e_{1,k}^{(k)} & * & \cdot & * \\ \vdots & \dots & \vdots & \vdots & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & e_{k-1,k}^{(k)} & * & \cdot & * \\ 0 & \dots & 0 & 1 & e_{k,k+1}^{(k)} & \cdot & e_{k,n}^{(k)} \\ 0 & \dots & 0 & 0 & 0 & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdot & 0 & 0 & 0 & \cdot & 0 \end{bmatrix}. \quad (3.37)$$

The following proposition gives us the property of $\theta(E_{kk}) = \begin{bmatrix} e_{ij}^{(k)} \end{bmatrix}$.

Proposition 3.1: Assume that $\theta(E_{kk}) = \begin{bmatrix} e_{ij}^{(k)} \end{bmatrix}$ for $k = 1, \dots, n$. If $i < j$ then for $i = 1, \dots, n-1$

$$\sum_{k=i}^j e_{ij}^{(k)} = 0. \quad (3.38)$$

Proof 3.1: It was shown in [Kezlan, 1990] that

$$e_{kk}^{(p)} = \begin{cases} 1 & \text{if } p = k \\ 0 & \text{if } p \neq k. \end{cases} \quad (3.39)$$

Consider $(e_{ij}^{(1)}, e_{ij}^{(2)}, \dots, e_{ij}^{(r)})$ as an r -tuple and define r -tuple sum as $e_{ij}^{(1)} + e_{ij}^{(2)} + \dots + e_{ij}^{(r)}$. To use induction on r , we start with the case $r = 2$. Using the multiplication of matrix units E_{pp} and $E_{p+1,p+1}$ we get

$$E_{pp}E_{p+1,p+1} = 0. \quad (3.40)$$

Since $\theta(0) = 0$, in our notation this result can be viewed as:

$$\begin{bmatrix} e_{ij}^{(p)} \end{bmatrix} \begin{bmatrix} e_{ij}^{(p+1)} \end{bmatrix} = 0. \quad (3.41)$$

For $p = 1, \dots, n - 1$,

$$\left(e_{p,p+1}^{(p)} + e_{p,p+1}^{(p+1)} \right) E_{p,p+1} = 0. \quad (3.42)$$

Then we obtain $e_{p,p+1}^{(p)} + e_{p,p+1}^{(p+1)} = 0$ for all 2-tuples. Assume our claim is true for the sum of $(r - 1)$ -tuples. That is

$$\sum_{i=p}^{p+r-2} e_{p,p+r-2}^{(i)} = 0 \text{ for } p = 1, \dots, (n - r) + 2. \quad (3.43)$$

Use induction on p . For $p = 1$,

$$e_{1r}^{(1)} + e_{1r}^{(2)} + \dots + e_{1r}^{(r-1)} + e_{1r}^{(r)} = 0. \quad (3.44)$$

Now consider each case for $t = 1, \dots, (r - 1)$. We know that $E_{tt}E_{rr} = 0$. Apply the automorphism θ to $E_{tt}E_{rr}$, we get

$$\theta(E_{tt}E_{rr}) = 0. \quad (3.45)$$

For $t = 1, \dots, (r - 1)$ the following terms can be obtained. If $t = 1$,

$$\left(e_{1r}^{(1)} + e_{1r}^{(r)} + \sum_{i=2}^{r-1} e_{1i}^{(1)} e_{ir}^{(r)} \right) E_{1r} = 0. \quad (3.46)$$

For $1 < t \leq r - 1$,

$$\left(e_{1r}^{(t)} + \sum_{i=2}^{r-1} e_{1i}^{(t)} e_{ir}^{(r)} \right) E_{1r} = 0. \quad (3.47)$$

By using the sum

$$\theta(E_{11}E_{rr} + E_{22}E_{rr} + \dots + E_{(r-1),(r-1)}E_{rr}) = 0, \quad (3.48)$$

we get

$$e_{1r}^{(1)} + e_{1r}^{(2)} + \dots + e_{1r}^{(r-1)} + e_{1r}^{(r)} + \sum_{j=2}^{r-1} \left(\sum_{i=1}^{r-1} e_{1j}^{(i)} e_{jr}^{(r)} \right) = 0. \quad (3.49)$$

By the assumption $\sum_{j=2}^{r-1} \left(\sum_{i=1}^{r-1} e_{1j}^{(i)} e_{jr}^{(r)} \right) = 0$ for $j = 2, \dots, r-1$. Also the calculations on $\theta(E_{ii})$, in (3.3), can be used to deduce the result,

$$e_{1r}^{(1)} + e_{1r}^{(2)} + \dots + e_{1r}^{(r-1)} + e_{1r}^{(r)} = 0. \quad (3.50)$$

Assume that our assumption is true for $p = q$, that is

$$e_{q,q+r-1}^{(q)} + e_{q,q+r-1}^{(q+1)} + \dots + e_{q,q+r-1}^{(q+r-2)} + e_{q,q+r-1}^{(q+r-1)} = 0. \quad (3.51)$$

Now consider the case $p = q + 1$. We want to prove that

$$e_{q+1,q+r}^{(q+1)} + e_{q+1,q+r}^{(q+2)} + \dots + e_{q+1,q+r}^{(q+r-1)} + e_{q+1,q+r}^{(q+r)} = 0. \quad (3.52)$$

Apply the automorphism θ to $E_{tt}E_{q+1,q+1}$ for $t = 1, \dots, q$ we get

$$\theta(E_{tt}E_{q+1,q+1}) = 0. \quad (3.53)$$

For $t = q + 1$,

$$\left(e_{q+1,q+r}^{(q+1)} + e_{q+1,q+r}^{(q+r)} + \sum_{i=q+2}^{q+r-1} e_{q+1,i}^{(q+1)} e_{i,q+r}^{(q+r)} \right) E_{q+1,q+r} = 0. \quad (3.54)$$

For $q + 1 < t < q + r$,

$$\left(e_{q+1,q+r}^{(t)} + \sum_{i=t}^{q+r-1} e_{q+1,i}^{(t)} e_{i,q+r}^{(q+r)} \right) E_{q+1,q+r} = 0. \quad (3.55)$$

Consider the following sum:

$$\theta(E_{q+1,q+1}E_{q+r,q+r} + E_{q+2,q+2}E_{q+r,q+r} + \dots + E_{q+r-1,q+r-1}E_{q+r,q+r}) = 0, \quad (3.56)$$

then we have

$$e_{q+1,q+r}^{(q+1)} + e_{q+1,q+r}^{(q+2)} + \dots + e_{q+1,q+r}^{(q+r-1)} + e_{q+1,q+r}^{(q+r)} + \sum_{j=q+2}^{q+r-1} \left(\sum_{i=q+1}^{q+r-1} e_{q+1,i}^{(i)} e_{i,q+r}^{(q+r)} \right) = 0. \quad (3.57)$$

By the assumption we have $\sum_{j=q+2}^{q+r-1} \left(\sum_{i=q+1}^{q+r-1} e_{q+1,i}^{(i)} e_{i,q+r}^{(q+r)} \right) = 0$ and this implies

$$e_{q,q+r-1}^{(q)} + e_{q,q+r-1}^{(q+1)} + \dots + e_{q,q+r-1}^{(q+r-2)} + e_{q,q+r-1}^{(q+r-1)} = 0. \quad (3.58)$$

Thus the proof is completed.

Lemma 3.4: Let R be a commutative ring with identity and θ be an R -algebra automorphism of $ST_n(R)$. If $n = 2, 3$ then θ is an inner automorphism η_D for some $D \in T_n(R)$.

Proof 3.4: For the automorphism $\theta : ST_n(R) \rightarrow ST_n(R)$ as given in Remark 3.3, let

$$\theta(E_{kk}) = \begin{bmatrix} e_{ij}^{(k)} \end{bmatrix} \text{ for } k = 1, 2, \dots, n \quad (3.59)$$

$$\theta(E_{k,k+1}) = \begin{bmatrix} a_{ij}^{(k)} \end{bmatrix} \text{ for } k = 1, 2, \dots, n-1. \quad (3.60)$$

Recall that all the diagonal entries $a_{ii}^{(k)}$ of $\theta(E_{k,k+1})$ are zero and

$$e_{qq}^{(p)} = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases} \quad (3.61)$$

Also $a_{12}^{(1)}, a_{23}^{(2)}, \dots, a_{n-1,n}^{(n-1)}$ are all units.

For $n = 2$, consider $A \in ST_2(R)$ where

$$A = \begin{bmatrix} \alpha & b \\ 0 & \alpha \end{bmatrix} \quad (3.62)$$

for $\alpha, b \in R$. Every matrix $A \in ST_2(R)$ can be written as linear combination of matrix units. That is,

$$A = \alpha E_{11} + b E_{12} + \alpha E_{22}. \quad (3.63)$$

Applying θ to A , we get

$$\theta(A) = \alpha \theta(E_{11}) + b \theta(E_{12}) + \alpha \theta(E_{22}). \quad (3.64)$$

By using the notation of Remark 3.3,

$$\theta(\alpha E_{11}) = \begin{pmatrix} \alpha & \alpha e_{12}^{(1)} \\ 0 & 0 \end{pmatrix} \quad (3.65)$$

$$\theta(bE_{12}) = \begin{pmatrix} 0 & ba_{12}^{(1)} \\ 0 & 0 \end{pmatrix} \quad (3.66)$$

$$\theta(\alpha E_{22}) = \begin{pmatrix} 0 & \alpha e_{12}^{(1)} \\ 0 & \alpha \end{pmatrix}. \quad (3.67)$$

As a result,

$$\theta(A) = \begin{bmatrix} \alpha & \alpha e_{12}^{(1)} + ba_{12}^{(1)} + \alpha e_{12}^{(2)} \\ 0 & \alpha \end{bmatrix} \quad (3.68)$$

for some invertible element $a_{12}^{(1)} \in R$. By Proposition 3.1, we have that $e_{12}^{(1)} + e_{12}^{(2)} = 0$ and

$$\theta(A) = \begin{bmatrix} \alpha & ba_{12}^{(1)} \\ 0 & \alpha \end{bmatrix} \quad (3.69)$$

for some invertible element $a_{12}^{(1)} \in R$. Choosing $D = \begin{bmatrix} 1 & 0 \\ 0 & (a_{12}^{(1)})^{-1} \end{bmatrix} \in T_2(R)$, we can obtain that

$$\theta(A) = DAD^{-1} \quad (3.70)$$

which means that all the automorphisms of $ST_2(R)$ are inner.

For the case $n = 3$, we proceed with the steps similar to the case $n = 2$ in order to determine the invertible matrix B . Let $A = \begin{bmatrix} \alpha & b & c \\ 0 & \alpha & d \\ 0 & 0 & \alpha \end{bmatrix} \in ST_n(R)$. By using matrix units, we get

$$\theta(E_{11}) = \begin{pmatrix} 1 & e_{12}^{(1)} & e_{13}^{(1)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.71)$$

$$\theta(E_{22}) = \begin{pmatrix} 0 & e_{12}^{(2)} & e_{13}^{(2)} \\ 0 & 1 & e_{23}^{(2)} \\ 0 & 0 & 0 \end{pmatrix} \quad (3.72)$$

$$\theta(E_{33}) = \begin{pmatrix} 0 & 0 & e_{13}^{(3)} \\ 0 & 0 & e_{23}^{(3)} \\ 0 & 0 & 1 \end{pmatrix} \quad (3.73)$$

$$\theta(E_{12}) = \begin{pmatrix} 0 & a_{12}^{(1)} & a_{13}^{(1)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.74)$$

$$\theta(E_{23}) = \begin{pmatrix} 0 & 0 & a_{13}^{(2)} \\ 0 & 0 & a_{23}^{(2)} \\ 0 & 0 & 0 \end{pmatrix} \quad (3.75)$$

and we obtain

$$\theta(A) = \begin{bmatrix} \alpha & ba_{12}^{(1)} & ca_{12}^{(1)}a_{23}^{(1)} + ba_{13}^{(1)} + da_{13}^{(2)} + \alpha(e_{13}^{(1)} + e_{13}^{(2)} + e_{13}^{(3)}) \\ 0 & \alpha & da_{23}^{(2)} \\ 0 & 0 & \alpha \end{bmatrix}. \quad (3.76)$$

Similarly, it can be deduced from Proposition 3.1,

$$e_{13}^{(1)} + e_{13}^{(2)} + e_{13}^{(3)} = 0 \Rightarrow \alpha(e_{13}^{(1)} + e_{13}^{(2)} + e_{13}^{(3)}) = 0. \quad (3.77)$$

Hence,

$$\theta(A) = \begin{bmatrix} \alpha & ba_{12}^{(1)} & ca_{12}^{(1)}a_{23}^{(1)} + ba_{13}^{(1)} + da_{13}^{(2)} \\ 0 & \alpha & da_{23}^{(2)} \\ 0 & 0 & \alpha \end{bmatrix}. \quad (3.78)$$

It is now easy to define the invertible matrix $B \in T_3(R)$ as,

$$B = \begin{pmatrix} 1 & a_{13}^{(1)} \left(a_{12}^{(1)} a_{23}^{(2)} \right)^{-1} & 0 \\ 0 & \left(a_{12}^{(1)} \right)^{-1} & -a_{13}^{(1)} \left(a_{12}^{(1)} \right)^2 \left(a_{23}^{(2)} \right)^{-1} \\ 0 & 0 & \left(a_{12}^{(1)} \right)^{-1} \left(a_{23}^{(2)} \right)^{-1} \end{pmatrix} \quad (3.79)$$

so that

$$\theta(A) = \eta_B = BAB^{-1}. \quad (3.80)$$

We would like to give an example using the commutative ring $\mathbb{Z}$.

Example 3.3: Let $A \in T_3(\mathbb{Z})$ and $\theta(A) : T_3(\mathbb{Z}) \rightarrow T_3(\mathbb{Z})$ be an automorphism given by

$$\theta \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} a & 2d - 2a - b & 2b - 2a + c - 4d + 6f - 2e \\ 0 & d & 2f - 2d - e \\ 0 & 0 & f \end{pmatrix} \quad (3.81)$$

By the main theorem of [Kezlan, 1990], we know that θ must be an inner automorphism with some invertible matrix $B \in T_3(\mathbb{Z})$

$$\theta(A) = \eta_B = BAB^{-1}. \quad (3.82)$$

One can find the matrix B as

$$B = \begin{pmatrix} 1 & -2 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.83)$$

In order to verify the unity of the entry $a_{12}^{(1)}$ of $\theta(E_{12})$,

$$\theta(E_{12}) = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.84)$$

and the unity of the entry $a_{23}^{(2)}$ of $\theta(E_{23})$,

$$\theta(E_{23}) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.85)$$

In the following example, we are going to use almost the same matrices for $ST_3(\mathbb{Z})$ which includes construction of invertible matrix B of Proposition 3.1.

Example 3.4: Let $A \in ST_3(\mathbb{Z})$ and $\theta(A) : ST_3(\mathbb{Z}) \rightarrow ST_3(\mathbb{Z})$ be an automorphism given by

$$\theta \begin{pmatrix} a & b & c \\ 0 & a & e \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} a & -b & -2e + 2b + c \\ 0 & a & -e \\ 0 & 0 & a \end{pmatrix} \quad (3.86)$$

By the proposition 3.1, θ must be an inner automorphism with some invertible matrix $B \in T_3(\mathbb{Z})$

$$\theta(A) = \eta_B = BAB^{-1}.$$

Again we can get the matrix B as

$$B = \begin{pmatrix} 1 & -2 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.87)$$

Now for applying the steps of the proof of the proposition 3.1, we get the following results:

$$\theta(E_{11}) = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.88)$$

$$\theta(E_{22}) = \begin{pmatrix} 0 & 2 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.89)$$

$$\theta(E_{33}) = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.90)$$

Note that, we found $e_{13}^{(1)} = -2$, $e_{13}^{(2)} = -4$ and $e_{13}^{(3)} = 6$. Hence $e_{13}^{(1)} + e_{13}^{(2)} + e_{13}^{(3)} = 0$.

Moreover, in the example 3.3, we have evaluated $a_{12}^{(1)} = -1$, $a_{13}^{(1)} = 2$, $a_{13}^{(2)} = -2$, and $a_{23}^{(2)} = -1$ by using $\theta(E_{12})$ and $\theta(E_{23})$. All these results are useful to construct the invertible matrix B of Proposition 3.1 via

$$B = \begin{pmatrix} 1 & a_{13}^{(1)} \left(a_{12}^{(1)} a_{23}^{(2)} \right)^{-1} & 0 \\ 0 & \left(a_{12}^{(1)} \right)^{-1} & -a_{13}^{(1)} \left(a_{12}^{(1)} \right)^2 \left(a_{23}^{(2)} \right)^{-1} \\ 0 & 0 & \left(a_{12}^{(1)} \right)^{-1} \left(a_{23}^{(2)} \right)^{-1} \end{pmatrix} \quad (3.91)$$

as

$$B = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.92)$$

This result is also important to show that B is not unique.

Before stating the main theorem, let us define the diagonal matrix $D \in T_n(R)$ with diagonal entries from the set

$$\left\{ 1, \left(a_{12}^{(1)} \right)^{-1}, \left(a_{12}^{(1)} \cdot a_{23}^{(2)} \right)^{-1}, \dots, \left(a_{12}^{(1)} \cdot a_{23}^{(2)} \dots a_{n-1,n} \right)^{-1} \right\}. \quad (3.93)$$

This allows us to construct an invertible matrix D as:

$$D = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & \left(a_{12}^{(1)} \right)^{-1} & 0 & \dots \\ 0 & 0 & \left(a_{12}^{(1)} \cdot a_{23}^{(2)} \right)^{-1} & \dots \\ 0 & 0 & \dots & \dots \end{pmatrix}. \quad (3.94)$$

Now we can state the main theorem.

Theorem 3.7: Let θ be an R -algebra automorphism of $ST_n(R)$ and R be a commutative ring with identity. For $n \geq 4$,

$$\theta = \eta_D \mu_c \lambda_P \quad (3.95)$$

where η_D, λ_P are inner automorphisms and μ_c is a central automorphism of $ST_n(R)$.

Proof 3.7:

First, apply θ to each E_{ii} for $i = 1, \dots, (n-1)$. The S_2 diagonal only contains the elements $e_{i,i+1}^{(p)}$. Thus we obtain

$$\sum_{p=i}^{i+1} e_{i,i+1}^{(p)} E_{i,i+1} \text{ for } p = 1, \dots, n-1. \quad (3.96)$$

By Proposition 3.1, we can get

$$\sum_{p=i}^{i+1} e_{i,i+1}^{(p)} = 0. \quad (3.97)$$

A consequence of above sum allows us just to see the image of $E_{i,i+1}$ under θ .

We examine

$$\theta(E_{i,i+1}) = a_{i,i+1}^{(i)} E_{i,i+1} \text{ for } i = 1, \dots, n-1, \quad (3.98)$$

We want to show that $a_{p,p+1}^{(k)} = 0$ if $k \neq p$. On the contrary, assume that $a_{p,p+1}^{(k)} \neq 0$ and take $k = 1$ to get a contradiction. Apply θ to the equality $E_{p,p+1} E_{23} = 0$, we obtain

$$\theta(E_{p,p+1} E_{23}) = \theta(E_{p,p+1}) \theta(E_{23}) \quad (3.99)$$

$$\begin{bmatrix} a_{ij}^{(p)} \end{bmatrix} \begin{bmatrix} a_{ij}^{(2)} \end{bmatrix} = 0. \quad (3.100)$$

Consider the term $a_{12}^{(p)} a_{23}^{(2)} E_{13}$ on the left side of the above equality. Since $a_{12}^{(p)} a_{23}^{(2)} = 0$ and $a_{23}^{(2)}$ is a unit by [Kezlan, 1990], then $a_{12}^{(p)} = 0$. If $k > 1$, apply θ to the equality $E_{k,k+1} E_{p+1,p+2}$ to get

$$\begin{bmatrix} a_{ij}^{(k)} \end{bmatrix} \begin{bmatrix} a_{ij}^{(p+1)} \end{bmatrix} = 0. \quad (3.101)$$

Consider the term $a_{p,p+1}^{(k)} a_{p+1,p+2}^{(p+1)} E_{p,p+2} = 0$ on the left side of the above equality. Since $a_{p,p+1}^{(k)} a_{p+1,p+2}^{(p+1)} = 0$ and $a_{p+1,p+2}^{(p+1)}$ is a unit then $a_{p,p+1}^{(k)} = 0$ but, this is a contradiction. Now defining the diagonal matrix $D \in T_n(R)$ with diagonal entries from the set $\left\{ 1, (a_{12}^{(1)})^{-1}, (a_{12}^{(1)} \cdot a_{23}^{(2)})^{-1}, \dots, (a_{12}^{(1)} \cdot a_{23}^{(2)} \dots a_{n-1,n})^{-1} \right\}$ we have the following result on the diagonal S_2 ,

$$\eta_D^{-1} \theta(E_{i,i+1}) = E_{i,i+1} \text{ for } i = 1, \dots, n-1. \quad (3.102)$$

We are going to use induction on t to prove that there exist inner automorphisms η_{P_t} with $P_t \in T_n(R)$ and

$$\eta_{P_t}^{-1} \eta_D^{-1} \theta(E_{i,i+1}) = E_{i,i+1} \text{ for } i = 1, \dots, n-1 \text{ and } t = 2, \dots, n-1. \quad (3.103)$$

Assume that we have

$$\eta_{P_t}^{-1} \eta_D^{-1} \theta(E_{i,i+1}) = E_{i,i+1} \quad (3.104)$$

for $i = 1, \dots, n-1$, on the diagonals $S_1, \dots, S_t$. What about on the diagonal S_{t+1} ?

Apply θ to E_{ii} , we can obtain the sum

$$\sum_{p=i}^t e_{i,i+t}^{(p)} E_{i,i+t} \text{ for } i = 1, \dots, n-t+1. \quad (3.105)$$

By Proposition 3.1, we have

$$\sum_{p=i}^t e_{i,i+t}^{(p)} E_{i,i+t} = 0. \quad (3.106)$$

Then we just consider the rest, that is:

$$\eta_{P_t}^{-1} \eta_D^{-1} \theta(E_{i,i+1}) = E_{i,i+1} + \sum_{j=1}^{n-t} b_{j,i}^{(i)} E_{j,j+t}. \quad (3.107)$$

We want to show $b_{j,i}^{(i)} = 0$ if $j \neq i$ and $j \neq i-t+1$. Assume that $b_{p,q}^{(q)} \neq 0$ with $p \neq q$ with $p \neq q+1-t$. If $p < n-t$ then we apply $\eta_{P_t}^{-1} \eta_D^{-1} \theta$ to $E_{q,q+1} E_{p+t,p+t+1}$, we get $\eta_{P_t}^{-1} \eta_D^{-1} \theta(E_{q,q+1} E_{p+t,p+t+1}) = 0$.

Thus, we obtain the following results:

$$\left[E_{q,q+1} + \sum_{j=1}^{n-t} b_{j,q}^{(q)} E_{j,j+t} \right] \left[E_{p+t,p+t+1} + \sum_{j=1}^{n-t} b_{j,p+t}^{(p+t)} E_{j,j+t} \right] = 0 \quad (3.108)$$

and

$$b_{q+1,p+t}^{(p+t)} E_{q,p+t+1} + b_{p,q}^{(q)} E_{p,p+t+1} = 0. \quad (3.109)$$

Hence $b_{pq}^{(q)} = 0$, we get a contradiction. If $p = n - t$ then apply $\eta_{P_t}^{-1} \eta_D^{-1} \theta$ to $E_{p-1,p} E_{q,q+1}$ we get

$$\eta_{P_t}^{-1} \eta_D^{-1} \theta(E_{p-1,p} E_{q,q+1}) = 0$$

and

$$b_{pq}^{(q)} E_{p-1,p+t} + b_{q-t,p-1}^{(p-1)} E_{q-t,q+1} = 0. \quad (3.110)$$

Since $b_{pq}^{(q)} = 0$, we get a contradiction again. Hence, on the S_{t+1} diagonal we have

$$\eta_{P_t}^{-1} \eta_D^{-1} \theta(E_{i,i+1}) = E_{i,i+1} + b_{i-t+1,i}^{(i)} E_{i-t+1,i+1} + b_{i,i}^{(i)} E_{i,i+t}. \quad (3.111)$$

However, we want to get $\eta_{P_t}^{-1} \eta_D^{-1} \theta(E_{i,i+1}) = E_{i,i+1}$.

To prove this, we use induction again. Assume that there exists an inner automorphism $\lambda_{G_{k-1}}$ such that on the S_{t+1} diagonal:

$$\lambda_{G_{k-1}}^{-1} \eta_{P_t}^{-1} \eta_D^{-1} \theta(E_{i,i+1}) = E_{i,i+1} \text{ for } i = 1, \dots, k-1, \quad (3.112)$$

and

$$\lambda_{G_{k-1}}^{-1} \eta_{P_t}^{-1} \eta_D^{-1} \theta(E_{i,i+1}) = E_{i,i+1} + d_{i-t+1,i}^{(i)} E_{i-t+1,i+1} + d_{i,i}^{(i)} E_{i,i+t} \text{ for } i = k, \dots, n-1. \quad (3.113)$$

Setting $Z = I + d_{k-t+1,k}^{(k)} E_{k-t+1,k} - d_{k,k}^{(k)} E_{k,k+t}$ and $G_k = G_{k-1} Z$ then for $1 \leq i < k$ we have

$$\lambda_{G_k}^{-1} \eta_{P_t}^{-1} \eta_D^{-1} \theta(E_{i,i+1}) = E_{i,i+1} + \delta_{i,k-t} d_{k+1-t,k}^{(k)} E_{i,k} \quad (3.114)$$

on the S_{t+1} diagonal.

For $i = k - t$, applying $\lambda_{G_{k-1}}^{-1} \eta_{P_t}^{-1} \eta_D^{-1} \theta$ to the equality $E_{k-t, k-t+1} E_{k, k+1} = 0$, we obtain

$$d_{k+1-t, k}^{(k)} E_{k-t, k+1} = 0 \quad (3.115)$$

on the diagonal S_{t+1} .

Hence

$$d_{k+1-t, k}^{(k)} = 0. \quad (3.116)$$

We get

$$\lambda_{G_k}^{-1} \eta_{P_t}^{-1} \eta_D^{-1} \theta(E_{k, k+1}) = E_{k, k+1} \quad (3.117)$$

for $i = 1, \dots, k$. For $i > k$, as stated on [Cao and Wang, 2000], we have

$$\lambda_{G_k}^{-1} \eta_{P_t}^{-1} \eta_D^{-1} \theta(E_{i, i+1}) = E_{k, k+1} + (d_{i-t+1, i+1}^{(i)} + \delta_{k+t, i} d_{kk}^{(k)}) E_{i+1-t, i+1} + d_{ii}^{(i)} E_{i, i+t}. \quad (3.118)$$

Thus, there exists an inner automorphism $\lambda_{P_{n-1}}$ of $ST_n(R)$ such that

$$\lambda_{P_{n-1}}^{-1} \eta_D^{-1} \theta(E_{i, i+1}) = E_{i, i+1} \quad (3.119)$$

on the S_{n-1} diagonal. We want to know what happens on the S_n diagonal. Assume that, we have

$$\lambda_{P_{n-1}}^{-1} \eta_D^{-1} \theta(E_{i, i+1}) = E_{i, i+1} + c_i E_{1n} \text{ for } i = 1, \dots, n-1 \quad (3.120)$$

on the S_n diagonal. If we apply $\lambda_{P_{n-1}}^{-1} \eta_D^{-1} \theta$ to E_{ii} , we get

$$(e_{1n}^{(1)} + e_{1n}^{(2)} + \dots + e_{1n}^{(n-1)}) E_{1n}. \quad (3.121)$$

By Proposition 3.1, $\left(e_{1n}^{(1)} + e_{1n}^{(2)} + \dots + e_{1n}^{(n-1)}\right) = 0$ and if we take

$$Z = I + c_{n-1}E_{1,n-1} - c_1E_{2n} \quad (3.122)$$

as in [Cao and Wang, 2000], we get

$$\lambda_Z^{-1}\lambda_{p_{n-1}}^{-1}\eta_D^{-1}\theta(E_{i,i+1}) = E_{i,i+1} \text{ for } i = 1, n-1 \quad (3.123)$$

and

$$\lambda_Z^{-1}\lambda_{p_{n-1}}^{-1}\eta_D^{-1}\theta(E_{i,i+1}) = E_{i,i+1} + c_iE_{1n} \text{ for } i = 2, \dots, n-2. \quad (3.124)$$

Setting $c = (c_2, \dots, c_{n-2})$, we get

$$\mu_c^{-1}\lambda_Z^{-1}\lambda_{p_{n-1}}^{-1}\eta_D^{-1}\theta(E_{i,i+1}) = E_{i,i+1} \text{ for } i = 1, \dots, n. \quad (3.125)$$

So we have

$$\mu_c^{-1}\lambda_Z^{-1}\lambda_{p_{n-1}}^{-1}\eta_D^{-1}\theta = 1. \quad (3.126)$$

Thus

$$\theta = \eta_D\mu_c\lambda_P, \quad (3.127)$$

for some $P \in T_n(R)$.

This example illustrates some results of the main theorem.

Example 3.5: Define an automorphism $\theta : ST_4(\mathbb{Z}) \rightarrow ST_4(\mathbb{Z})$ as

$$\theta \begin{pmatrix} a & b & c & d \\ 0 & a & e & f \\ 0 & 0 & a & p \\ 0 & 0 & 0 & a \end{pmatrix} = \begin{pmatrix} a & -b & 2b - c - 2e & 2c - 5b + d + 2f - 4p + 4e \\ 0 & a & e & -2p - f - 2e \\ 0 & 0 & a & -p \\ 0 & 0 & 0 & a \end{pmatrix}. \quad (3.128)$$

Now in order to evaluate $a_{12}^{(1)}$, $a_{23}^{(2)}$ and $a_{34}^{(3)}$, we need to find $\theta(E_{i,i+1})$ for $i = 1, 2, 3$.

$$\theta(E_{12}) = \begin{pmatrix} 0 & -1 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.129)$$

$$\theta(E_{23}) = \begin{pmatrix} 0 & 0 & -2 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.130)$$

$$\theta(E_{34}) = \begin{pmatrix} 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.131)$$

Thus we get $a_{12}^{(1)} = -1$, $a_{23}^{(2)} = -1$ and $a_{34}^{(3)} = -1$, as expected the unitary elements of $\mathbb{Z}$. The diagonal matrix D can be constructed as:

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (a_{12}^{(1)})^{-1} & 0 & 0 \\ 0 & 0 & (a_{12}^{(1)} \cdot a_{23}^{(2)})^{-1} & 0 \\ 0 & 0 & 0 & (a_{12}^{(1)} \cdot a_{23}^{(2)} \cdot a_{34}^{(3)})^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (3.132)$$

Applying η_D^{-1} to $\theta(E_{i,i+1})$, we get $\eta_D^{-1}\theta(E_{i,i+1}) = E_{i,i+1}$ as stated in the proof of the theorem 3.7.

Remark 3.4: Let $N_n(R)$ be the strictly upper triangular matrix algebra over R , then

$$\text{Aut}(N_n(R)) \simeq \text{Aut}(ST_n(R)). \quad (3.133)$$

Example 3.6: Let $A \in ST_4(R)$ be defined as $\begin{bmatrix} a & b & c & d \\ 0 & a & e & f \\ 0 & 0 & a & r \\ 0 & 0 & 0 & a \end{bmatrix}$.

We can check that $\theta : ST_4(R) \rightarrow ST_4(R)$ defines an automorphism via

$$\theta \left(\begin{bmatrix} a & b & c & d \\ 0 & a & e & f \\ 0 & 0 & a & r \\ 0 & 0 & 0 & a \end{bmatrix} \right) = \begin{bmatrix} a & b & c & d - 3e \\ 0 & a & e & f \\ 0 & 0 & a & r \\ 0 & 0 & 0 & a \end{bmatrix}. \quad (3.134)$$

But θ is not an inner automorphism. Notice that $\text{Aut}(ST_n(R)) \not\subseteq \text{Aut}(T_n(R))$. Then there exists an automorphism of $ST_n(R)$ that cannot be extended to an automorphism of $T_n(R)$.

4. CONCLUSIONS

The classification problem for the scalar upper triangular matrices which is a subalgebra of upper triangular matrices is considered at this dissertation. The automorphism group of the upper triangular matrix algebras on a commutative ring R consists of inner automorphisms. The automorphism group of the scalar upper triangular matrix algebras on a commutative ring R is the semidirect product of inner automorphisms and the central automorphisms. The incidence algebras could be used to generalize this result to classify for a given subalgebra of the upper triangular matrix algebras. The automorphism group of incidence algebras has been classified for a commutative ring R . If we assign the same scalar for $f(x,x)$ for all x in a poset, we may get a different classification. In this case, the main problem is to determine the center of the incidence algebra for a given arbitrary poset.

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BIOGRAPHY

Özkay ÖZKAN was born in Balıkesir, Turkey in 1976. He received the B.Sc. degree in Mathematics from the Middle East Technical University. He received M.Sc degree in Mathematics from the Gebze Technical University. He has been working as a teacher in the Education of Ministry since 2003. Also, he has been a PhD student of Mathematics Department in Gebze Technical University Graduate School of Nature and Applied Science. His research interests includes incidence algebras, upper triangular matrices.

APPENDICES

Appendix : Publications

Özkan Ö., Akkurt M., “Automorphisms of a certain subalgebra of the upper triangular matrix algebra ”, Hacet. J. Math. Stat., (SCI-E) accepted, 1-9, 2019.