

**T.R.**  
**GEBZE TECHNICAL UNIVERSITY**  
**GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES**

**ON COMPLEX IRREDUCIBLE CHARACTERS OF  
FINITE GROUPS AND THEIR ZEROS**

**GÜLSEMİN ÇONOĞLU**  
**A THESIS SUBMITTED FOR THE DEGREE OF  
MASTER OF SCIENCE  
DEPARTMENT OF MATHEMATICS**

**GEBZE**  
**2021**

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**THESIS SUPERVISOR**  
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**GEBZE**  
**2021**

**T.C**  
**GEBZE TEKNİK ÜNİVERSİTESİ**  
**FEN BİLİMLERİ ENSTİTÜSÜ**

**SONLU GRUPLARIN KOMPLEKS**  
**İNDİRGENEMEZ KARAKTERLERİ VE**  
**SIFIRLARI**

**GÜLSEMİN ÇONOĞLU**  
**YÜKSEK LİSANS TEZİ**  
**MATEMATİK ANABİLİM DALI**

**DANIŞMANI**  
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**GEBZE**

**2021**

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|-----------------------------------------------------------------------------------|--------------------------------------|
|  | <b>YÜKSEK LİSANS JÜRİ ONAY FORMU</b> |
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GTU Fen Bilimleri Enstitüsü Yönetim Kurulu'nun 08/07/2021 tarih ve 2021/32 sayılı kararıyla oluşturulan jüri tarafından 27/07/2021 tarihinde tez savunma sınavı yapılan Gülsemin Çonoğlu'nun tez çalışması Matematik Anabilim Dalında YÜKSEK LİSANS tezi olarak kabul edilmiştir.

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### **ONAY**

Gebze Teknik Üniversitesi Fen Bilimleri Enstitüsü Yönetim Kurulu'nun  
..... tarih ve ...../..... sayılı kararı.

**İMZA/MÜHÜR**

## SUMMARY

For given a finite group  $G$ , an irreducible character of  $G$  is a map from  $G$  to the complex numbers satisfying some special conditions. A conjugacy class  $x$  of  $G$  is called a vanishing class if there exists an irreducible character of  $G$  such that  $\chi(x) = 0$ . The element  $x$  is called a zero of  $G$ . Actually, this vanishing class will be mentioned as a  $\chi$ -vanishing class with respect to the notation that will be used in this thesis. The relation between the set of  $G$ 's irreducible characters denoted by  $\text{Irr}(G)$  and the set of vanishing classes denoted by  $\text{Van}(G)$  has been considered by different groups of authors. In this thesis, we focus on the relation between the number of vanishing classes of a finite group and the group structure. In particular, we consider the non-solvable groups each of whose vanishing class sizes are divisible at most two prime numbers. Moreover, we give some results on odd-square-free sizes of vanishing classes of a non-abelian simple group  $G$ .

**Keywords:** Irreducible Characters, Conjugacy Classes, Odd-Square-Free, Vanishing Classes.

## ÖZET

Sonlu bir  $G$  grubu için,  $G$ 'nin bir indirgenemez karakteri, bazı özel koşulları sağlayan  $G$ 'den kompleks sayılara giden bir fonksiyondur. Eğer  $\chi(x) = 0$  olacak şekilde  $G$ 'nin bir indirgenemez karakteri varsa,  $x$  eşlenik sınıfına sıfırlanan sınıf denir. Bu  $x$  elemanına  $G$ 'nin sıfırı denir. Aslında, bu tezde kullanılacak notasyona göre, bu sıfırlanan sınıftan  $\chi$ -sıfırlanan sınıfı olarak bahsedilecektir.  $G$ 'nin  $\text{Irr}(G)$  ile gösterilen indirgenemez karakterler kümesi ile  $\text{Van}(G)$  ile gösterilen sıfırlanan sınıflar kümesi arasındaki ilişki farklı yazar grupları tarafından ele alınmıştır. Bu tezde, sonlu bir grubun sıfırlanan sınıflarının sayısı ile grubun yapısı arasındaki ilişkiye odaklanıyoruz. Özellikle, sıfırlanan sınıflarının her birinin boyutlarının en fazla iki asal sayıya bölünebilen çözilemeyen grupları ele alıyoruz. Ayrıca, değişmeli olmayan basit bir grup  $G$ 'nin sıfırlanan sınıflarının tek-kare-serbest boyutları üzerine bazı sonuçlar veriyoruz.

**Anahtar Kelimeler:** İndirgenemez Karakterler, Eşlenik Sınıfları, Tek-Kare-Serbest, Sıfırlanan Sınıflar.

## ACKNOWLEDGEMENTS

I would like to express my sincerest gratitude to my supervisor Roghayeh Hafezieh for her endless support, patience and motivation throughout the process of writing up this thesis. Without her constant patience and advices, I would not be able to write the thesis.

I would like to thank my professor Ayşe Berkman, who supported me with endless patience and understanding in my undergraduate education process, even in every turning point in my life. I felt most deeply my love and commitment to mathematics in the fruitful courses I took from her.

Also, I would like to thank İpek Tuvay and Nursel Erey for being the members of my thesis committee and their corrections on the first version of this thesis.

Finally, I wish to thank to my mother Günay Çonoğlu and my lovely friends Özlem Kurnaz, Mustafa Bayramoğlu, Aslıhan Gür, Şeyma Yaşar, Feyza Yazıcı, Burçe Seven and Hatice Düzen for their great support and encouragement. They have been always there for me. I am also grateful to Zeynep Begüm Kara and Aysu İrem Sönmez, who came into my life during the pandemic and supported me even though we were far away from each other.

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## LIST of ABBREVIATIONS and ACRONYMS

| <u>Abbreviations</u>   | <u>Explanations</u>                                          |
|------------------------|--------------------------------------------------------------|
| <u>and Acronyms</u>    |                                                              |
| $(m, n)$               | : Greatest common divisor of the integers $m$ and $n$ .      |
| $ G $                  | : Order of a group $G$ .                                     |
| $ x $                  | : Order of an element $x$ .                                  |
| $\text{Irr}(G)$        | : Set of irreducible characters of a group $G$ .             |
| $Z(G)$                 | : Center of a group $G$ .                                    |
| $\text{Con}(G)$        | : Set of conjugacy classes of a group $G$ .                  |
| $[G, G] = G'$          | : Commutator subgroup of a group $G$ .                       |
| $\text{Lin}(G)$        | : Set of linear characters of a group $G$ .                  |
| $ x^G $                | : Size of conjugacy class of an element $x$ in a group $G$ . |
| $\text{Aut}(G)$        | : Automorphism group of a group $G$ .                        |
| $[G:H]$                | : Index of a subgroup $H$ in a group $G$ .                   |
| $\text{Ker}(\psi)$     | : Kernel of a homomorphism $\psi$                            |
| $\text{Van}(G)$        | : Set of vanishing elements in a group $G$ .                 |
| $\langle a \rangle$    | : Subgroup generated by the element $a$ of a group $G$ .     |
| $C_G(x)$               | : Centralizer of the element $x$ in $G$ .                    |
| $\text{Im}(\psi)$      | : Image of a homomorphism $\psi$ .                           |
| $\langle a, b \rangle$ | : Inner product of $a$ and $b$ .                             |
| $\text{stab}_G(x)$     | : Stabilizer of the element $x$ in a group $G$               |
| $\text{orb}_G(x)$      | : Orbit of the element $x$ in a group $G$                    |

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# 1. INTRODUCTION

Let  $G$  be a finite group and  $Irr(G)$  be the set of irreducible characters of  $G$ . For years, the zeros of irreducible characters and character degrees of  $G$  are attracting the attention of some mathematicians to get some information about the group structure of  $G$ .

Burnside's Theorem, which has an important place in this area, asserts that for any irreducible character  $\chi$  whose degree is larger than 1, there exists  $g \in G$  such that  $\chi(g) = 0$  (Theorem 3.15 in [Isaacs, 1976]). This theorem inspires mathematicians who are studying this area to obtain further results. For instance, G. Malle, G. Navarro, J. B. Olsson proved that for a non-linear irreducible character  $\chi$  of  $G$  there exists a prime  $p$  and a  $p$ -element  $g \in G$  with the property that  $\chi(g) = 0$  [Malle et al., 2000]. Then, Navarro showed that for a normal subgroup  $N$  of  $G$ ,  $\chi_N$  is not an irreducible character of  $N$  if and only if there exists  $g \in G$  such that  $\chi(x) = 0$  for every  $x \in gN$  [Navarro, 2001].

Another perspective on Burnside's theorem concerns the set of vanishing classes of  $G$  denoted by  $Van(G)$ , that is  $\{g \in G \mid \chi(g) = 0 \text{ for some } \chi \in Irr(G)\}$ . The concept of such elements is introduced by I. M. Isaacs, G. Navarro and T. R. Wolf [Isaacs et al., 1999].  $Van(G)$  has attracted attention from some mathematicians over the past few years. As a consequence of this interest, there are a lot of studies about vanishing classes of finite groups. For example, E. M. Zhmud studied those groups which have exactly one vanishing class [Zhmud, 1979]. Also, J. Zhang, Z. Shen and J. Shi investigate the groups with  $|Van(G)| = 4$  and they get a characterization of finite non-abelian solvable groups vanishing on exactly 4 conjugacy classes [Zhang et al., 2014].

Studies on vanishing classes are separated into two cases depending on whether the groups are solvable or not. For example, M. Bianchi, R. D. Camina, M. L. Lewis and E. Pacifici in [Bianchi et al., 2020] showed that for given a finite supersolvable group  $G$ , let  $cs(G)$  be the set of conjugacy class sizes of  $G$ ,  $vcs(G)$  the set of conjugacy class sizes of the vanishing elements of  $G$  and let  $vcs(G) = \{s\}$ . Set  $\pi$  be the set of prime divisors of  $s$ , we have  $\pi = \pi(G/Z(G))$ , then up to an abelian direct factor of  $\pi'$ -order, one of the following holds:

- i) The number  $s$  is a power of a suitable prime number  $p$ ,  $G$  is a  $p$ -group and

$$cs(G) = \{s, 1\}.$$

- ii) For a suitable prime number  $p$ , we have  $G = NP$ , where  $N$  is a non-trivial abelian normal  $p$ -complement of  $G$ , and  $P$  is a Sylow  $p$ -subgroup of  $G$  such that  $|cs(P)| = 2$ ,  $Z(P) = C_P(N)$ , and  $P/Z(P)$  is an elementary abelian  $p$ -group.

Also, S. Gagola interested in solvable groups which have an irreducible character that vanishes on all but two conjugacy classes, and he proved that for a solvable group  $G$  which has an irreducible character vanishing on all but 2 conjugacy classes of  $G$ , there exists a unique prime number  $p$  for which  $O_p(G) \neq e_G$  [Gagola, 1983].

In this thesis, we will consider the vanishing elements of non-solvable groups. S. M. Robati considered non-solvable groups each of whose vanishing class sizes is divisible by at most two prime divisors [Robati, 2020]. As it will be discussed in this thesis, he proved that if  $G$  is a finite non-solvable group and each vanishing class size of  $G$  is divisible by at most two primes, then  $G/Sol(G)$  is isomorphic with  $PSL_2(4)$  or  $PSL_2(8)$ , where  $Sol(G)$  is solvable radical of  $G$ . In addition to all these studies, S. Dolfi, E. Pacifici and L. Sanus survey some results about zeros and vanishing elements of finite groups [Dolfi et al., 2018].

Throughout the thesis, we consider the effects of both sizes and numbers of vanishing classes on the group that we study. To do this, we first give some basic group-theoretical properties, and then we give some definitions on rings and modules. In section 3, we introduce the character theory of finite groups that will be used throughout the thesis.

Whenever we have a conjugacy class  $\alpha$  of a finite group  $G$ , if we can find an irreducible character  $\chi$  of  $G$  such that  $\chi(\alpha) = 0$ , then the conjugacy class  $\alpha$  is called a  $\chi$ -vanishing class of  $G$ . According to this definition, we specifically consider the number of  $\chi$ -vanishing classes of a finite group  $G$  and their influences on the group structure of  $G$ . For example, if the number of  $\chi$ -vanishing classes of a finite group  $G$  is a positive integer  $m$ , then the properties that we can say about the structure of  $G$  with related to the number  $m$  is given in section 4. In this part of the thesis, we follow D. Chillag's studies [Chillag, 1999].

In section 5, we see some properties of the sizes of the conjugacy classes of non-abelian simple groups. Additionally, we introduce some effects of the size of

vanishing classes of a finite group  $G$  on the group structure. And, in section 6 we give a result on non-solvable groups which is proven in [Robati, 2020].

Finally, in section 7, we make mention of the odd-square-free sizes of vanishing classes of non-abelian simple groups and we give some results on this topic.

## 2. PRELIMINARIES AND BASIC RESULTS

### 2.1. Groups, Subgroups, Homomorphisms

*Definition 2.1:* [Lovett, 2015] A group is a non-empty set  $G$  with a binary operation " $\cdot$ " (denoted by  $(G, \cdot)$ ) such that it satisfies the following properties:

G1) The operation is associative:  $(g \cdot h) \cdot k = g \cdot (h \cdot k)$  for all  $g, h, k \in G$ .

G2) There exists  $e_G \in G$  such that  $g \cdot e_G = e_G \cdot g = g$  for all  $g \in G$ .

G3) For all  $g \in G$ , there exists  $h \in G$  such that  $g \cdot h = h \cdot g = e_G$ .

We call  $(G, \cdot)$  is abelian if it satisfies that  $g \cdot h = h \cdot g$  for all  $g, h \in G$ .

*Example 2.1:* The pairs  $(\mathbb{Z}, +)$ ,  $(\mathbb{Q}, +)$ ,  $(\mathbb{R}, +)$  and  $(\mathbb{C}, +)$  are all abelian groups.

*Example 2.2:* Let  $X$  be a nonempty set. We define  $S_X$  as the set of all bijections from  $X$  to itself. We see that the pair  $(S_X, \circ)$  is a group under the operation  $\circ$  which is composition. We call  $S_X$  the symmetric group on  $X$  and if  $|X| = n$ , then we denote it by  $S_n$ . Note that  $|S_n| = n!$

*Example 2.3:* Let  $\mathbb{Z}_n$  denote the set  $\{0, 1, \dots, n-1\}$ . Then  $\mathbb{Z}_n$  is a group with respect to the addition modulo  $n$ .

*Definition 2.2:* [Dummit and Foote, 1991] Suppose that  $(F, +)$  and  $(F - \{e_F\}, \cdot)$  are both abelian groups and that  $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$  for all  $x, y, z \in F$ . Then  $F$  is called a field.

*Example 2.4:* Let  $n$  be a positive integer and let  $F$  be a field. The set of all  $n \times n$  invertible matrices with coefficients from  $F$  is a group under the multiplication of matrices. This group is called General Linear Group over  $F$  and is denoted by  $GL_n(F)$ .

*Definition 2.3:* [Lovett, 2015] Let  $(G, \cdot)$  be a group. A nonempty subset  $H$  of  $G$  is called a subgroup of  $G$  if  $(H, \cdot)$  is a group itself. If  $H$  is a subgroup of  $G$ , we write  $H \leq G$ .

Every group  $G$  has two trivial subgroups which are  $G$  itself and  $\{e_G\}$ .

*Example 2.5: (Special Linear Group)* Let  $F$  be a field. A subset of  $GL_n(F)$  defined by

$$SL_n(F) = \{A \in GL_n(F) : \det(A) = 1\} \quad (2.1)$$

is a subgroup of  $GL_n(F)$ .

From now on, we denote the groups without specifying the group operations except in necessary cases.

*Definition 2.4: [Lovett, 2015]* Let  $G$  be a group. If  $G$  has  $m$  elements, then we say that the order of  $G$  is  $m$  and we denote it by  $|G| = m$ .

*Proposition 2.1: [Lovett, 2015](One-Step Subgroup Criterion)* A nonempty subset  $H$  of a group  $G$  is a subgroup if and only if for all  $x, y \in H$ ,  $xy^{-1} \in H$ .

As a corollary of Proposition 2.1, it is easy to see that the intersection of two subgroups is a subgroup.

*Definition 2.5: [Rose, 2009]* Let  $G$  be a group and let  $g$  be an element of  $G$ . The set of powers of  $g$ , that is  $\langle g \rangle = \{g^t : t \in \mathbb{Z}\}$ , is called the cyclic subgroup of  $G$ . We say that  $g$  is a generator of the subgroup  $\langle g \rangle$  of  $G$ . In particular, if there exists  $x \in G$  such that  $\langle x \rangle = G$  then  $G$  is called a cyclic group.

*Example 2.6:* Let  $G = \mathbb{Z}$  and  $X = \{5\}$ , then  $\langle 5 \rangle$  is the subgroup of  $\mathbb{Z}$  consisting of the set of integers divisible by 5.

*Remark 2.1: [Bhattacharya et al., 1994]* Let  $G$  be a group. If  $G$  is finite cyclic group with  $n$  elements, then  $G \cong \mathbb{Z}_n$ . If  $G$  is infinite cyclic group, then  $G \cong \mathbb{Z}$ .

*Definition 2.6:* Let  $G$  be a group and let  $g \in G$ . The order of  $g$ , denoted by  $|g|$ , is defined by  $|g| = |\langle g \rangle|$ .

*Example 2.7:* Let  $G = \mathbb{Z}_6$  and  $g = 2 \in G$ . Then  $\langle g \rangle = \{0, 2, 4\}$ , so  $|g| = 3$ .

*Definition 2.7: [Lovett, 2015]* Let  $G$  be a group. A presentation of  $G$  is an expression of the following form,

$$G = \langle g_1, g_2, \dots, g_k \mid R_1 R_2 \cdots R_s \rangle \quad (2.2)$$

where each  $R_i$  is an equation in the elements  $g_1, g_2, \dots, g_k$ .

*Example 2.8:* Let  $G = \langle x \rangle$  and  $|x| = n$ . The presentation of  $G$  is  $G = \langle x \mid x^n = e_G \rangle$ .

*Definition 2.8:* [Lovett, 2015] Let  $G$  be a group and let  $H$  be a subgroup of  $G$ . Let  $g \in G$ . The set  $Hg = \{hg : h \in H\}$  is called a right coset of  $H$  in  $G$ . Similarly, the set  $gH = \{gh : h \in H\}$  is called a left coset of  $H$  in  $G$ . The number of right (left) cosets of  $H$  in  $G$  is called the index of  $H$  in  $G$  and is denoted by  $[G : H]$ .

*Theorem 2.1:* [Lovett, 2015](Lagrange's Theorem) Let  $G$  be a group and  $H \leq G$ . Then  $|H| \mid |G|$ . Moreover,  $\frac{|G|}{|H|} = [G : H]$ .

*Definition 2.9:* [Lovett, 2015] Let  $G$  be a group and  $H \leq G$ . The subgroup  $H$  is called normal in  $G$  if  $gH = Hg$  for all  $g \in G$ . If  $H$  is a normal subgroup of  $G$  then we write  $H \trianglelefteq G$ .

The subgroups  $\{e_G\}$  and  $G$  are normal in  $G$ . If  $G$  is abelian, then all subgroups of  $G$  are normal.

*Definition 2.10:* [Lovett, 2015] A group  $G$  is called a simple group if it has only trivial normal subgroups.

*Definition 2.11:* [Lovett, 2015] Let  $G$  be a group and let  $N \trianglelefteq G$ . Then the set  $G/N = \{gN : g \in G\}$  has a well-defined binary operation given by

$$(gN) \cdot (hN) = (gh)N \quad (2.3)$$

for  $g, h \in G$ . This operation implies  $G/N$  into a group, such a group is called the quotient group.

*Example 2.9:* The group  $(\mathbb{Z}/\langle n \rangle, +)$  is the quotient group of  $\mathbb{Z}$  by  $\langle n \rangle$ .

*Definition 2.12:* [Lovett, 2015] Let  $G$  be a group. The center of  $G$ , denoted by  $Z(G)$ , is defined as follows:

$$Z(G) = \{g \in G : gh = hg, \text{ for all } h \in G\} \quad (2.4)$$

It is straightforward to see that  $Z(G) \trianglelefteq G$ .

*Definition 2.13:* [Rose, 2009] Let  $G$  be a group. The commutator subgroup denoted by  $G'$  is defined as the subgroup generated by all commutator elements which are elements of the form  $a^{-1}b^{-1}ab$ , for any  $a, b \in G$ .

Moreover,  $G$  is called perfect if  $G' = G$ .

It is clear that a group  $G$  is abelian if and only if  $G' = \{e_G\}$ .

*Definition 2.14:* [Fite, 1902] Let  $G$  be a group. If there exists a normal abelian subgroup  $N$  of  $G$  such that the quotient  $G/N$  is abelian then  $G$  is called metabelian.

*Definition 2.15:* [Lovett, 2015] Let  $(G, \star)$  and  $(H, \bullet)$  be groups. A homomorphism from  $G$  to  $H$  is a map  $\varphi : G \rightarrow H$  such that for all  $g, h \in G$ ,  $\varphi(g \star h) = \varphi(g) \bullet \varphi(h)$ . The kernel of  $\varphi$  is defined as follows:

$$\text{Ker}(\varphi) = \{g \in G : \varphi(g) = e_H\} \quad (2.5)$$

The image of  $\varphi$  is defined as follows:

$$\text{Im}(\varphi) = \{\varphi(g) : g \in G\} \quad (2.6)$$

We can see that  $\text{Ker}(\varphi) \leq G$  and  $\text{Im}(\varphi) \leq H$ .

If  $\varphi$  is a one-to-one homomorphism, then it is called a monomorphism. If  $\varphi$  is an onto homomorphism, then it is called an epimorphism. If  $\varphi$  is both monomorphism and epimorphism, then it is called an isomorphism. In this case, we call  $H$  and  $G$  are isomorphic groups denoted by  $G \cong H$ .

*Example 2.10:* Let  $G$  be an abelian group. The function  $\varphi : G \rightarrow G$  defined as  $\varphi(x) = x^2$ , for each  $x \in G$  is a homomorphism.

*Example 2.11:* Let  $G$  be a group and let  $N \leq G$ . The function  $\phi : G \rightarrow G/N$  defined by  $\phi(g) = gN$  is called canonical projection of  $G$  onto  $G/N$ .

*Definition 2.16:* [Lovett, 2015] Let  $G$  be a group. An isomorphism  $\varphi : G \rightarrow G$  is called an automorphism of  $G$ . The set of all automorphisms of the group  $G$  is denoted by  $\text{Aut}(G)$ , which is itself a group under composition.

*Theorem 2.2:* [Lovett, 2015](First Isomorphism Theorem) Let  $G$  and  $H$  be groups. Suppose that  $\varphi : G \rightarrow H$  is a homomorphism. Then  $G/\text{Ker}(\varphi) \cong \text{Im}(\varphi)$ .

## 2.2. Group Actions and Frobenius Groups

*Definition 2.17: [Lovett, 2015] Let  $G$  be a group and let  $X$  be a nonempty set. We say that  $G$  acts on  $X$  if there exists a function  $\cdot : G \times X \rightarrow X$  with the following conditions:*

- i)  $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ , for all  $g_1, g_2 \in G$  and  $x \in X$ ;*
- ii)  $e_G \cdot x = x$ , for all  $x \in X$ .*

*Definition 2.18: [Lovett, 2015] Let  $G$  be a group and let  $\emptyset \neq X$  be any set. Suppose that  $G$  acts on  $X$ . Given  $x \in X$ , the stabilizer of  $x$  in  $G$  is a subgroup of  $G$ , which is defined as follows:*

$$\text{stab}_G(x) = \{g \in G : g \cdot x = x\} \quad (2.7)$$

*and the orbit of  $x$  in  $G$  is defined as follows:*

$$\text{orb}_G(x) = \{g \cdot x : g \in G\} \quad (2.8)$$

*Theorem 2.3: [Lovett, 2015](The Orbit-Stabilizer Theorem) Let  $G$  be a group and let  $X$  be a nonempty set. Suppose that  $G$  acts on  $X$ . Then, for each  $x \in X$ , we have that  $|\text{orb}_G(x)| = |G : \text{stab}_G(x)|$ , where  $|\text{orb}_G(x)|$  called the length of the orbit is the number of elements in  $\text{orb}_G(x)$ .*

*Example 2.12: Let  $G$  be a group. We say that  $G$  acts on itself via conjugation if we define*

$$g \cdot x = gxg^{-1} \quad (2.9)$$

*for all  $g, x \in G$ . We can see that the operation defined in (2.9) is an action:*

- i)  $x \cdot (g_1 \cdot g_2) = x \cdot (g_1 g_2 g_1^{-1}) = x g_1 g_2 (x g_1)^{-1} = (x g_1) \cdot g_2$  for all  $x, g_1, g_2 \in G$ ;*
- ii)  $e_G \cdot x = e_G x e_G^{-1} = x$  for all  $x \in G$ .*

Considering this action, the orbit of  $x$  is called the conjugacy class of  $x$  and it is denoted by  $x^G$  which is  $x^G = \{g^{-1}xg : g \in G\}$ . The stabilizer of  $x \in G$  is called the centralizer of  $x$  in  $G$  and it is denoted by  $C_G(x)$ , that is,

$$C_G(x) = \{g \in G : gxg^{-1} = x\} \quad (2.10)$$

One can see that  $C_G(x)$  is a subgroup of  $G$ .

*Corollary 2.1:* [Lovett, 2015] Let  $G$  be a group and  $x \in G$ . Then  $|x^G| = [G : C_G(x)]$ .

*Example 2.13:* Let  $G = \langle x, y \mid x^3 = y^2 = e_G, yx = x^2y \rangle$  (Dihedral group  $D_3$ ). Then,  $x^G = \{g^{-1}xg : g \in G\} = \{x, x^2\}$  and  $y^G = \{g^{-1}yg : g \in G\} = \{y, xy, x^2y\}$ . By Corollary 2.1 we can see that,  $|C_G(x)| = 3$  and  $|C_G(y)| = 2$ .

*Theorem 2.4:* [Lovett, 2015](Cauchy's Theorem) Let  $G$  be a group and let  $p$  be a prime number. If  $p \mid |G|$ , then  $G$  has an element of order  $p$ .

*Definition 2.19:* [Dummit and Foote, 1991] Let  $G$  be an abelian group and let  $p$  be a prime number. Then  $G$  is called an elementary abelian  $p$ -group if  $|g| = p$  for all  $e_G \neq g \in G$ .

*Theorem 2.5:* [Lovett, 2015](Normalizer-Centralizer Theorem) Let  $G$  be a group and let  $H \leq G$ . Then  $C_G(H) \trianglelefteq N_G(H)$  and  $N_G(H)/C_G(H)$  is isomorphic to a subgroup of  $\text{Aut}(H)$ .

*Definition 2.20:* [Alperin and Bell, 1995] Let  $F \in \{\mathbb{Z}_p, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ . Let  $G$  be a group acting on an  $F$ -vector space  $V$  (see Definition 2.34). The action of  $G$  on  $V$  is called linear if the action it satisfies the following two conditions:

- i)  $g \cdot (v + w) = g \cdot v + g \cdot w$  for all  $v, w \in V$  and  $g \in G$ ,
- ii)  $g \cdot (c \cdot v) = c (g \cdot v)$  for all  $c \in F$ ,  $v \in V$  and  $g \in G$ .

*Definition 2.21:* [Bhattacharya et al., 1994] Let  $G = S_n$  and let  $H$  be a subgroup of  $S_n$ .  $H$  is called a transitive permutation group if, for all  $i, j \in (1, 2, \dots, n)$ , there exists  $\sigma \in H$  such that  $\sigma(i) = j$ .

*Definition 2.22: [Isaacs, 1976] Let  $H$  be a non-trivial subgroup of  $G$ . Assume that  $H \cap H^g = \{e_G\}$  whenever  $g \in G - H$ . Then  $H$  is a Frobenius complement in  $G$ . A group which contains a Frobenius complement is called a Frobenius group. If  $G$  is a Frobenius group with a Frobenius complement  $H$ , then the set*

$$N = \left( G - \bigcup_{x \in G} H^x \right) \cup \{e_G\} \quad (2.11)$$

*is called Frobenius Kernel of  $G$ .*

*Example 2.14: Let  $G = D_{2q}$  where  $q$  is odd. Then*

$$G = \langle x, y : x^q = y^2 = e_G, yxy = x^{-1} \rangle \quad (2.12)$$

$$= \{e_G, x, x^{-1}, \dots, x^{q-1}, \dots, y, xy, \dots, x^{q-1}y\}. \quad (2.13)$$

*Let  $H = \langle y \rangle$ . We can see that  $H \cap H^g = \{e_G\}$  for all  $g \in G - H$  and hence by Definition 2.22, we see that  $G$  is a Frobenius group.*

### 2.3. Sylow Theorems

*Definition 2.23: [Lovett, 2015] Let  $G$  be a group and let  $p$  be a prime number.*

- i) If  $|G| = p^n$  for some  $n \in \mathbb{N}$  then  $G$  is called a  $p$ -group.*
- ii) If  $|G| = p^n m$ , where  $p \nmid m$ , then a subgroup of order  $p^n$  is called a Sylow  $p$ -subgroup of  $G$ .*
- iii) The set of Sylow  $p$ -subgroups of  $G$  is denoted by  $\text{Syl}_p(G)$  and the number of Sylow  $p$ -subgroups of  $G$  is denoted by  $n_p(G)$ .*

*Theorem 2.6: [Lovett, 2015] (Sylow Theorem) Let  $G$  be a group and let  $|G| = p^\alpha n$ , where  $\alpha \geq 1$ ,  $n > 1$  and  $p$  is a prime number such that  $p \nmid n$ . Then we have the following:*

- i) There exists a subgroup of  $G$  of order  $p^\alpha$ .*
- ii) If  $P_1$  and  $P_2$  are two Sylow  $p$ -subgroups of  $G$  then they are conjugate in  $G$ .*
- iii)  $n_p(G) = [G : N_G(P)] \equiv 1 \pmod{p}$*

*Remark 2.2:* [Lovett, 2015] As  $n_p(G) = [G : N_G(P)]$  is the number of distinct conjugates of  $P$  in  $G$ , we conclude that  $n_p(G) = 1$  if and only if  $P \trianglelefteq G$ .

*Example 2.15:* Let  $|G| = pq$  where  $p$  and  $q$  are primes, and  $p < q$ , then there exists a normal subgroup of order  $q$ . By Sylow theorem, we have  $n_q \equiv 1 \pmod{q}$  and  $n_q \mid p$ . This implies that  $n_q = 1$ , so by Remark 2.2,  $P \trianglelefteq G$ .

## 2.4. Nilpotent Groups

*Definition 2.24:* [Rose, 2009] Let  $G$  be a group and let  $K$  be a subgroup of  $G$ . A series from  $K$  to  $G$  is a finite sequence

$$K = H_0 \leq H_1 \leq \cdots \leq H_n = G \quad (2.14)$$

of subgroups of  $G$  where each  $H_i$  is a subgroup of its successor. If  $K = \{e_G\}$  then we say that the series in (2.14) is a series for  $G$ .

*Definition 2.25:* [Rose, 2009] Suppose  $G$  is a group with the following series

$$K = H_0 \leq H_1 \leq \cdots \leq H_n = G \quad (2.15)$$

The series (2.15) of  $G$  is called subnormal if each term  $H_i$  in the series is a normal subgroup of its successor;  $H_{i-1} \trianglelefteq H_i$  for  $i = 1, \dots, n$ . The series (2.15) is called normal if each term in the series is also normal in  $G$ ;  $H_i \trianglelefteq G$  for  $i = 0, \dots, n$ . In particular, it is called a composition series of  $G$  if it has no proper subnormal refinement. The factors of a composition series are called composition factors.

*Definition 2.26:* [Rose, 2009] A normal series of  $G$   $\{e_G\} = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G$  is called a central series of  $G$  if for each  $i$  in the range  $0 \leq i \leq n-1$ , we have

$$H_{i+1}/H_i \leq Z(G/H_i) \quad (2.16)$$

*Definition 2.27:* [Rose, 2009] Let  $G$  be a group.  $G$  is called nilpotent if  $G$  has a central series.

*Definition 2.28:* [Rose, 2009] Let  $G$  be a group. If  $G$  has a subnormal series all of whose factors are abelian, then  $G$  is called solvable.

## 2.5. Rings and Homomorphisms

*Definition 2.29: [Dummit and Foote, 1991] A ring  $R$  is a set together with two binary operations  $+$  (addition) and  $\times$  (multiplication) satisfying the following conditions:*

- i)  $(R, +)$  is an abelian group,*
- ii)  $(a \times b) \times c = a \times (b \times c)$  for all  $a, b, c \in R$ ,*
- iii)  $(a + b) \times c = (a \times c) + (b \times c)$  and  $a \times (b + c) = (a \times b) + (a \times c)$  for all  $a, b, c \in R$ .*

*The ring  $R$  is called commutative if the multiplication is commutative and the ring  $R$  is said to have an identity if there is an element  $1 \in R$  with*

$$1 \times a = a \times 1 = a \quad (2.17)$$

*for all  $a \in R$ .*

From now on, we denote  $ab$  rather than  $a \times b$  for all  $a, b \in R$ .

*Example 2.16: [Dummit and Foote, 1991] The ring of integers,  $\mathbb{Z}$ , under the usual operations of addition and multiplication is a commutative ring with identity (the integer 1).*

*Definition 2.30: [Dummit and Foote, 1991] A subring of the ring  $R$  is a subgroup of  $R$  that is closed under multiplication.*

*Example 2.17: [Dummit and Foote, 2004]  $2\mathbb{Z}$  is a subring of  $\mathbb{Z}$ , as is  $n\mathbb{Z}$  for any integer  $n$ .*

*Definition 2.31: [Dummit and Foote, 1991] Let  $R$  and  $S$  be two rings.*

- i) A ring homomorphism is a map  $\varphi : R \rightarrow S$  satisfying*

$$a) \quad \varphi(a + b) = \varphi(a) + \varphi(b) \text{ for all } a, b \in R \text{ and}$$

$$b) \quad \varphi(ab) = \varphi(a)\varphi(b) \text{ for all } a, b \in R.$$

- ii) A bijective ring homomorphism is called an isomorphism.*

*Example 2.18: [Dummit and Foote, 1991] The map  $\varphi : \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$  defined by sending an even integer to 0 and an odd integer to 1 is a ring homomorphism.*

*Definition 2.32: [Alperin and Bell, 1995] Let  $R$  be a ring and let  $G$  be a group. The group ring of  $G$  over  $R$ , denoted by  $RG$ , is the set of all finite formal  $R$  – linear combinations of elements of  $G$  with the following operations:*

*Without loss of generality, we assume that  $\alpha = \sum_{t \in G} \alpha_t t$  and that  $\beta = \sum_{t \in G} \beta_t t$ . The addition is defined as follows:*

$$\alpha + \beta = \sum_{t \in G} (\alpha_t + \beta_t) t \quad (2.18)$$

*And the multiplication is defined as follows:*

$$\cdot : RG \times RG \rightarrow RG \quad (2.19)$$

$$\left( \sum_{x \in G} \alpha_x x, \sum_{y \in G} \beta_y y \right) \mapsto \sum_{x \in G} \alpha_x x \cdot \sum_{y \in G} \beta_y y = \sum_{x \in G} \sum_{y \in G} \alpha_x \beta_y xy = \alpha_x \beta_y xy \quad (2.20)$$

The case where  $R = F$ , a field and  $G$  is a finite group is important for us.

*Definition 2.33: [Alperin and Bell, 1995] Suppose that  $F$  is a field and  $G$  is a finite group. Then  $FG$  is called the group algebra.*

*Definition 2.34: [Hoffman and Kunze, 2004] Let  $V$  be a nonempty set and  $F$  be a field.  $V$  is called a vector space over the field  $F$ , if there exists two operations as follows: a vector addition,*

$$+ : V \times V \rightarrow V \quad (2.21)$$

$$(\alpha, \beta) \mapsto \alpha + \beta \quad (2.22)$$

*and a scalar multiplication,*

$$\cdot : F \times V \rightarrow V \quad (2.23)$$

$$(c, \beta) \mapsto c\beta \quad (2.24)$$

which satisfy the followings:

- i)  $\alpha + \beta = \beta + \alpha$  for each  $\alpha, \beta \in V$ .
- ii)  $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$  for each  $\alpha, \beta, \gamma \in V$ .
- iii) There exists a unique zero vector  $0_V$  such that  $0_V + \alpha = \alpha + 0_V = \alpha$  for all  $\alpha \in V$ .
- iv) For all  $\alpha$  in  $V$ , there exists a unique vector  $-\alpha$  in  $V$  such that  $\alpha + (-\alpha) = 0_V$ .
- v)  $1 \cdot \alpha = \alpha$  for all  $\alpha$  in  $V$ .
- vi)  $(c_1 c_2) \alpha = c_1 (c_2 \alpha)$  where  $c_1, c_2 \in F$  and  $\alpha \in V$ .
- vii)  $c(\alpha + \beta) = c\alpha + c\beta$  where  $c \in F$  and  $\alpha, \beta \in V$ .
- viii)  $(c_1 + c_2) \alpha = c_1 \alpha + c_2 \alpha$  where  $c_1, c_2 \in F$  and  $\alpha \in V$ .

*Definition 2.35: [Alperin and Bell, 1991] Let  $F$  be a field. An  $F$ -algebra is a set  $A$  with a ring structure and an  $F$ -vector space structure that share the same addition operation, and with the additional property that  $(ca)b = c(ab) = a(cb)$  for any  $c \in F$  and  $a, b \in A$ .*

## 2.6. Modules

*Definition 2.36: [Dummit and Foote, 1991] Let  $R$  be a commutative ring with identity. An  $R$ -algebra is a ring  $A$  with identity together with a ring homomorphism  $f : R \rightarrow A$  mapping  $1_R$  to  $1_A$  such that the subring  $f(R)$  of  $A$  is contained in the center of  $A$ .*

*Definition 2.37: [Dummit and Foote, 1991] Let  $R$  be a ring. A left  $R$ -module is a set  $M$  with the following properties:*

- i)  $(M, +)$  is an abelian group,
- ii) There exists an action of  $R$  on  $M$  (that is, a map  $R \times M \rightarrow M$ ) denoted by  $rm$ , for all  $r \in R$  and for all  $m \in M$  which satisfies:
  - a)  $(r + s)m = rm + sm$  for all  $r, s \in R, m \in M$ ,
  - b)  $(rs)m = r(sm)$ , for all  $r, s \in R, m \in M$ , and
  - c)  $r(m + n) = rm + rn$ , for all  $r \in R, m, n \in M$ .

*If the ring  $R$  has 1 we impose the additional axiom:*

d)  $1m = m$ , for all  $m \in M$ .

We can see that if  $A$  is an  $R$ -algebra, then  $A$  has a left (or right)  $R$ -module structure defined by  $r \cdot a = a \cdot r = f(r)a$  where  $f(r)a$  is the multiplication in the ring  $A$ .

*Definition 2.38: [Dummit and Foote, 1991] Let  $R$  be a ring and let  $M$  be an  $R$ -module. If  $N$  is a subgroup of  $M$  and it is closed under the action of ring elements, then we say that  $N$  is an  $R$ -submodule of  $M$ .*

In particular, if we choose  $R = F$  in Definition 2.38 where  $F$  is a field, submodules are the same as subspaces. Every  $R$ -module  $M$  has the two submodules  $M$  and the trivial submodule  $0$ .

*Example 2.19: [Dummit and Foote, 2004] Let  $R = F$  be a field. Then, every vector space over  $F$  is an  $F$ -module and vice versa.*

Throughout this thesis we work with  $FG$ -algebras,  $FG$ -modules where  $F = \mathbb{C}$ .

### 3. CHARACTER THEORY

In this section we introduce some definitions and results on character theory. To define a character of a group  $G$ , we first define its representation.

#### 3.1. Characters

It should be mentioned that by  $GL_n(F)$  we mean the general linear group of degree  $n$  over  $F \in \{\mathbb{Z}_p, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ . From now on, we consider  $F = \mathbb{C}$  and work with the algebra  $\mathbb{C}G$ .

*Theorem 3.1: [Alperin and Bell, 1991] There exists some  $r \in \mathbb{N}$  and some  $f_1, \dots, f_r \in \mathbb{N}$  such that  $\mathbb{C}G \cong M_{f_1}(\mathbb{C}) \oplus \dots \oplus M_{f_r}(\mathbb{C})$  as  $\mathbb{C}$ -algebras such that  $r$  is the number of isomorphism classes of simple  $\mathbb{C}G$ -modules. If we let  $S_1, \dots, S_r$  be representatives of these  $r$  classes, then we can order the  $S_i$ 's so that  $\mathbb{C}G \cong f_1 S_1 \oplus \dots \oplus f_r S_r$  as  $\mathbb{C}G$ -modules, where  $\dim_{\mathbb{C}} S_i = f_i$  for each  $i$ . Moreover, any  $\mathbb{C}G$ -module can be written uniquely in a form  $a_1 S_1 \oplus \dots \oplus a_r S_r$ , where the  $a_i$  are non-negative integers.*

*Definition 3.1: [Alperin and Bell, 1991] The  $\mathbb{C}$ -dimensions  $f_1, \dots, f_r$  of the  $r$  simple  $\mathbb{C}G$ -modules are called the degrees of  $G$ . The trivial  $\mathbb{C}G$ -module  $\mathbb{C}$  is one-dimensional and hence simple, so  $G$  has at least one of its degrees equal to 1, and without loss of generality we set  $f_1 = 1$ .*

*Corollary 3.1: [Alperin and Bell, 1991] We have  $\sum_{i=1}^r f_i^2 = |G|$ .*

*Theorem 3.2: [Alperin and Bell, 1991] The number  $r$  of simple  $\mathbb{C}G$ -modules gives us the number of conjugacy classes of  $G$ .*

*Definition 3.2: [Alperin and Bell, 1995] Let  $U$  be a  $\mathbb{C}G$ -module which implies that  $U$  is a vector space over  $\mathbb{C}$  and  $G$  acts on  $U$  linearly. For each  $x \in G$ , define a map as follows:*

$$X_U : G \rightarrow GL_n(U) \tag{3.1}$$

$$g \mapsto X_U(g) \tag{3.2}$$

where

$$X_U(g) : U \rightarrow U \quad (3.3)$$

$$\alpha \mapsto \alpha^g \quad (3.4)$$

$X_U(g)$  is an invertible linear operator on  $U$  and  $n = \dim_{\mathbb{C}} U$ . Let  $\beta$  be a basis of  $U$  over  $\mathbb{C}$  then  $[X_U(g)]_{\beta}$  is an invertible  $n \times n$  matrix over  $\mathbb{C}$ .

Now, define a map  $\chi_U$  as follows:

$$\chi_U : G \rightarrow \mathbb{C} \quad (3.5)$$

$$g \mapsto \text{trace}([X_U(g)]_{\beta}) \quad (3.6)$$

$\chi_U$  is called the character of  $G$ . It is clear that  $\chi_U(e_G)$  gives us the dimension of  $U$  over  $\mathbb{C}$ . If  $g$  is conjugate to  $x$  in  $G$ , that is, there exists  $h \in G$  such that  $g = h x h^{-1}$ , then we can see that

$$\chi_U(g) = \text{trace}[X_U(h x h^{-1})]_{\beta} = \text{trace}[X_U(h) X_U(x) X_U(h)^{-1}]_{\beta} \quad (3.7)$$

Since the matrix  $[X_U(h) X_U(x) X_U(h)^{-1}]_{\beta}$  is conjugate of  $[X_U(x)]_{\beta}$  we see that these two matrices are similar and hence their traces are equal. This means that any character of  $G$  is constant on conjugacy classes.

*Notation 3.1:* [Alperin and Bell, 1991] From now on, we denote the characters of  $G$  with respect to the simple  $\mathbb{C}G$ -module  $S_i$  which appears in  $\mathbb{C}G \cong f_1 S_1 \oplus \cdots \oplus f_r S_r$  as  $\chi_i$  instead of writing  $\chi_{S_i}$ . With respect to this notation  $f_1$  denotes the degree of the trivial representation and  $\chi_1$  is the character of the trivial representation, we call  $\chi_1$  the principal character of  $G$ , and we have  $\chi_1(g) = 1$  for all  $g \in G$ . In particular,  $\chi_i$ 's are called the irreducible characters of  $G$ . The set of irreducible characters of  $G$  is denoted by  $\text{Irr}(G)$ . If  $U$  is one-dimensional  $\mathbb{C}G$ -module then  $\chi_U$  is called a linear character of  $G$ . Since one-dimensional modules are simple, it is clear that all linear characters are irreducible.

*Proposition 3.1:* [Alperin and Bell, 1991] Let  $U$  be a  $\mathbb{C}G$ -module, let  $\rho : G \rightarrow GL_n(U)$  be the representation corresponding to  $U$ , and let  $g \in G$  be of order  $n$ . Then:

- i)  $\rho(g)$  is diagonalizable.
- ii)  $\chi_U(g)$  equals the sum of the eigenvalues of  $\rho(g)$ .
- iii)  $\chi_U(g)$  is a sum of  $\chi_U(e_G)$   $n$ th roots of unity.
- iv)  $\chi_U(g^{-1}) = \overline{\chi_U(g)}$ .
- v)  $|\chi_U(g)| \leq \chi_U(e_G)$ .
- vi)  $\{x \in G \mid \chi_U(x) = \chi_U(e_G)\}$  is a normal subgroup of  $G$ .

We can define the characters of  $G$  as elements of a  $\mathbb{C}$ -vector space of functions from  $G$  to  $\mathbb{C}$ . Because, for given two characters  $\chi, \psi$  of  $G$  we can define new functions  $\chi + \psi$  and  $\chi\psi$  from  $G$  to  $\mathbb{C}$  by  $(\chi + \psi)(g) = \chi(g) + \psi(g)$  and  $(\chi\psi)(g) = \chi(g)\psi(g)$  for  $g \in G$ . Additionally, for given  $\lambda \in \mathbb{C}$ , we can define a new function  $\lambda\chi : G \rightarrow \mathbb{C}$  by  $(\lambda\chi)(g) = \lambda\chi(g)$ .

*Proposition 3.2: [Alperin and Bell, 1991] The irreducible characters of  $G$  are, as functions from  $G$  to  $\mathbb{C}$ , linearly independent over  $\mathbb{C}$ .*

*Definition 3.3: [Alperin and Bell, 1991] A class function on  $G$  is a function from  $G$  to  $\mathbb{C}$  whose value within any conjugacy class is constant. It is clear that characters of  $\mathbb{C}G$ -modules are class functions.*

The set of class functions on  $G$  forms a  $\mathbb{C}$ -vector space of dimension  $r$ , where  $r$  is the number of conjugacy classes of  $G$ .

*Proposition 3.3: [Alperin and Bell, 1991] Let  $G$  be a group. Then,  $\text{Irr}(G)$  form a basis for the space of class functions on  $G$ .*

*Definition 3.4: [Alperin and Bell, 1991] Inner product of class functions  $\alpha, \beta$  on  $G$  is defined as follows:*

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \alpha(g) \overline{\beta(g)} \quad (3.8)$$

*We list some properties of this inner product:*

- i)  $\langle \alpha, \alpha \rangle \geq 0$  for all  $\alpha$ , and  $\langle \alpha, \alpha \rangle = 0$  if and only if  $\alpha = 0$ .
- ii)  $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$  for all  $\alpha, \beta$ .

- iii)  $\langle \lambda \alpha, \beta \rangle = \lambda \langle \alpha, \beta \rangle$  for all  $\alpha, \beta$  and all  $\lambda \in \mathbb{C}$ .
- iv)  $\langle \alpha_1 + \alpha_2, \beta \rangle = \langle \alpha_1, \beta \rangle + \langle \alpha_2, \beta \rangle$  for all  $\alpha_1, \alpha_2, \beta$ .
- v)  $\langle \alpha, \lambda \beta \rangle = \bar{\lambda} \langle \alpha, \beta \rangle$  for all  $\alpha, \beta$  and all  $\lambda \in \mathbb{C}$ .
- vi)  $\langle \alpha, \beta_1 + \beta_2 \rangle = \langle \alpha, \beta_1 \rangle + \langle \alpha, \beta_2 \rangle$  for all  $\alpha, \beta_1, \beta_2$ .

*Theorem 3.3: [Isaacs, 1976] Let  $G$  be a group and let  $\chi \in \text{Irr}(G)$ . Then  $\chi(e_G) \mid |G|$ .*

*Theorem 3.4: [Isaacs, 1976] Let  $G$  be a group and let  $A \trianglelefteq G$ . Then  $\chi(e_G) \mid [G : A]$ .*

### 3.2. The Character Table

One of the most interesting problems in character theory is finding out the structure of a group by considering the character table of that group. In this part we present some information about the character table of a group.

*Definition 3.5: [Alperin and Bell, 1991] Let  $X = [\chi_i(g_j)]_{1 \leq i, j \leq r}$  where  $g_1, \dots, g_r$  are representatives of the  $r$  conjugacy classes of a group  $G$ , and we set  $g_1 = e_G$ . We write  $X$  in the form as follows:*

Table 3.2.1: Character Table of a group  $G$ .

|          |          |               |          |               |
|----------|----------|---------------|----------|---------------|
|          | 1        | $k_2$         | ...      | $k_r$         |
|          | 1        | $g_2$         | ...      | $g_r$         |
| $\chi_1$ | 1        | 1             | ...      | 1             |
| $\chi_2$ | $f_2$    | $\chi_2(g_2)$ | ...      | $\chi_2(g_r)$ |
| $\vdots$ | $\vdots$ | $\vdots$      | $\ddots$ | $\vdots$      |
| $\chi_r$ | $f_r$    | $\chi_r(g_2)$ | ...      | $\chi_r(g_r)$ |

where the  $f_i$  are the degrees of  $G$ , and  $k_i = [G : C_G(g_i)]$  is the size of the conjugacy class of  $g_i$  for each  $i$ . Then  $X$  is called the character table of  $G$ .

*Lemma 3.1: [Alperin and Bell, 1991] Let  $N \trianglelefteq G$ , and let  $U$  be a  $\mathbb{C}(G/N)$ -module. Then we say that  $U$  admits a canonical  $\mathbb{C}G$ -module structure, with a subspace of  $U$  being a  $\mathbb{C}G$ -submodule if and only if it is a  $\mathbb{C}(G/N)$ -submodule. If  $\phi$  is the character of the  $\mathbb{C}(G/N)$ -module  $U$ , then the character of the  $\mathbb{C}G$ -module  $U$  is  $\phi \circ \eta$ , where  $\eta : G \rightarrow G/N$  is the natural map.*

*Definition 3.6:* [Alperin and Bell, 1991] Given any character of  $G$ , say  $\chi$ , define  $K_\chi = \{x \in G \mid \chi(x) = \chi(e_G)\}$ . It can be seen that  $K_\chi \trianglelefteq G$ .  $K_\chi$  is called the kernel of  $\chi$ . In particular, for  $\chi_i \in \text{Irr}(G)$ , we simply denote  $K_{\chi_i}$  by  $K_i$ .

*Example 3.1:* Let  $G = \langle g \rangle \cong \mathbb{Z}_n$ . Then the character table of  $G$  is as follows:

$$\begin{array}{c|ccccc}
 & 1 & 1 & 1 & \dots & 1 \\
 & 1 & g & g^2 & \dots & g^{n-1} \\
 \hline
 \chi_1 & 1 & 1 & 1 & \dots & 1 \\
 \chi_2 & 1 & \lambda & \lambda^2 & \dots & \lambda^{n-1} \\
 \chi_3 & 1 & \lambda^2 & \lambda^4 & \dots & \lambda^{n-2} \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 \chi_n & 1 & \lambda^{n-1} & \lambda^{n-2} & \dots & \lambda
 \end{array} \tag{3.9}$$

where  $\lambda$  is a primitive  $n$ th root of unity.

By Example 3.1, we see that if  $G$  is an abelian group then all irreducible characters of  $G$  are linear.

*Example 3.2:* [Alperin and Bell, 1991] Let  $G$  be a group and let  $G'$  be the commutator subgroup of  $G$ . From the fact that  $G/G'$  is abelian we can say that all of its irreducible characters are linear. By Lemma 3.1, each linear character of  $G/G'$  can be lifted to a linear character of  $G$ . It is clear that if  $\psi$  and  $\psi'$  are distinct irreducible characters of  $G/G'$  then  $\psi \circ \pi$  and  $\psi' \circ \pi$  are distinct irreducible characters of  $G/G'$ . Now, let  $\chi$  be a linear character of  $G$ . Then  $\chi : G \rightarrow \mathbb{C}^*$  is a group homomorphism with kernel  $K_\chi$ . By First Isomorphism Theorem,  $G/K_\chi \cong H \leq \mathbb{C}^*$ , and hence  $G/K_\chi$  is abelian. Since  $G' \subseteq \text{Ker}(\chi) = K_\chi$  the map  $\theta : G/G' \rightarrow \mathbb{C}^*$  defined by  $\theta(gG') = \chi(g)$  is a linear character of  $G/G'$ . Hence every linear character of  $G$  is a lift of a linear character of  $G/G'$ .

### 3.3. Restriction and Induction

Let  $G$  be a group and  $H$  be a subgroup of  $G$ . If  $\chi \in \text{Irr}(G)$ , then we can obtain the character  $\chi_H$  of  $H$  by restriction. We have the following:

$$\chi_H = \sum_{\varphi \in \text{Irr}(H)} k_{\varphi} \varphi \quad (3.10)$$

where  $k_{\varphi} \in \mathbb{Z}$ . It is not hard to see that if  $\chi_H \in \text{Irr}(H)$ , then  $\chi \in \text{Irr}(G)$ . However, if  $\chi \in \text{Irr}(G)$ , then  $\chi_H$  is not an irreducible character of  $H$  in general. For the dual case of this fact, the following definition plays an important role.

*Definition 3.7: [Isaacs, 1976] Let  $H \leq G$  and let  $\varphi$  be a class function of  $H$ . Then  $\varphi^G$ , the induced class function on  $G$ , is given by*

$$\varphi^G(g) = \frac{1}{|H|} \sum_{x \in G} \varphi^o(xgx^{-1}) \quad (3.11)$$

where  $\varphi^o$  is defined by  $\varphi^o(h) = \varphi(h)$  if  $h \in H$  and  $\varphi^o(y) = 0$  if  $y \notin H$ .

We can see that  $\varphi^G$  is a class function on  $G$  and that  $\varphi^G(e_G) = [G : H] \cdot \varphi(e_H)$ .

*Lemma 3.2: [Isaacs, 1976] (Frobenius Reciprocity) Let  $H \leq G$  and suppose that  $\varphi$  is a class function on  $H$  and that  $\theta$  is a class function on  $G$ . Then*

$$\langle \varphi, \theta_H \rangle = \langle \varphi^G, \theta \rangle \quad (3.12)$$

*Corollary 3.2: [Isaacs, 1976] Let  $H \leq G$  and suppose  $\varphi$  is a character of  $H$ . Then  $\varphi^G$  is a character of  $G$ .*

*Corollary 3.3: [Isaacs, 1976] Let  $H \leq G$  and suppose  $\varphi \in \text{Irr}(H)$ . Then there exists  $\chi \in \text{Irr}(G)$  such that  $\chi_H \in \text{Irr}(H)$  such that  $\varphi$  is a constituent of  $\chi_H$ .*

Whenever we have a normal subgroup  $H$  of a group  $G$  it is easy to give information about  $\chi_H$  where  $\chi \in \text{Irr}(G)$ . If  $\theta$  is a class function of  $H \trianglelefteq G$  and  $g \in G$ , then we define  $\theta^g : H \rightarrow \mathbb{C}$  by  $\theta^g(h) = \theta(ghg^{-1})$  and we call that  $\theta^g$  is conjugate to  $\theta$  in  $G$ .

*Lemma 3.3: [Isaacs, 1976] Let  $H \trianglelefteq G$  and  $\varphi, \theta$  be class functions of  $H$  and  $x, y \in G$ . Then*

- i)  $\varphi^x$  is a class function;
- ii)  $(\varphi^x)^y = \varphi^{xy}$ ;
- iii)  $\langle \varphi^x, \theta^x \rangle = \langle \varphi, \theta \rangle$ ;
- iv)  $\langle \chi_H, \varphi^x \rangle = \langle \chi_H, \varphi \rangle$  for class functions  $\chi$  of  $G$ ;
- v)  $\varphi^x$  is a character if  $\varphi$  is.

*Remark 3.1: [Isaacs, 1976] Let  $G$  be group and let  $H \trianglelefteq G$ . If for any  $g \in G$ ,  $\theta^g \in \text{Irr}(H)$ , then we can see that  $\theta \in \text{Irr}(H)$  and vice versa. Hence,  $G/H$  permutes  $\text{Irr}(H)$  via conjugation action. Now, let  $\theta \in \text{Irr}(H)$ . The inertia group of  $\theta$  in  $G$  is defined as follows:*

$$I_G(\theta) = \{g \in G : \theta^g = \theta\} \quad (3.13)$$

*We can see that  $I_G(\theta)$  is a subgroup of  $G$  and  $[G : I_G(\theta)]$  is the size of  $G$ -orbit of  $\theta$  with respect to this action.*

*Theorem 3.5: [Isaacs, 1976](Clifford's Theorem) Let  $H \trianglelefteq G$  and let  $\chi \in \text{Irr}(G)$ . Let  $\theta$  be an irreducible constituent of  $\chi_H$  and suppose  $\theta = \theta_1, \theta_2, \dots, \theta_t$  are the distinct conjugates of  $\theta$  in  $G$ . Then*

$$\chi_H = e \sum_{i=1}^t \theta_i \quad (3.14)$$

*where  $e = \langle \chi_H, \theta \rangle$  and  $t = [G : I_G(\theta)]$ .*

Let  $H \trianglelefteq G$ ,  $\theta \in \text{Irr}(H)$  and set  $T = I_G(\theta)$ . Suppose that  $T = G$ . Since we mentioned that  $\theta^G$  is a character of  $G$ , we have that  $\theta^G = \sum e_i \chi_i$  where  $e_i \in \mathbb{Z}$  and for some  $\chi_i \in \text{Irr}(G)$ . By Clifford's Theorem and by Frobenius Reciprocity, we can see that  $(\chi_s)_H = e_s \theta$ . Therefore, we have that  $\chi_s(e_G) = e_s \theta(e_G)$  and the following

$$[G : H] \cdot \theta(e_G) = \theta^G(e_G) \quad (3.15)$$

$$= \sum e_s \chi_s(e_G) \quad (3.16)$$

$$= \sum e_s^2 \theta(e_G) \quad (3.17)$$

It is straight forward to see that  $[G : H] = \sum e_s^2$ . Also, it is remarkable that the integers  $e_s$  behave like character degrees of  $G/H$ . Since one of the character degrees is equal to 1 we can call this character  $e_1$ . In this case, we can see that  $(\chi_1)_H = \theta$  and we say that  $\theta$  is extendible to  $G$ . Also, we say that  $\chi$  extends to an irreducible character  $\theta$  of  $H$  if  $\chi_H = \theta$  for an irreducible character  $\chi$  of  $G$ .

*Theorem 3.6: [Isaacs, 1976] Let  $G$  be a group and let  $H \trianglelefteq G$ . Let  $\theta \in \text{Irr}(H)$ , and  $T = I_G(\theta)$ . Let*

$$A = \{\psi \in \text{Irr}(T) \mid \langle \psi_H, \theta \rangle \neq 0\}, \quad B = \{\chi \in \text{Irr}(G) \mid \langle \chi_H, \theta \rangle \neq 0\}. \quad (3.18)$$

*Then*

- i) If  $\psi \in A$ , then  $\psi^G$  is irreducible;*
- ii) The map  $\psi \mapsto \psi^G$  is a bijection of  $A$  onto  $B$ ;*
- iii) If  $\psi^G = \chi$ , with  $\psi \in A$ , then  $\psi$  is the unique irreducible constituent of  $\chi_T$  which lies in  $A$ ;*
- iv) If  $\psi^G = \chi$ , with  $\psi \in A$ , then  $\langle \psi_H, \theta \rangle = \langle \chi_H, \theta \rangle$ .*

*Definition 3.8: [Isaacs, 1976] Let  $G$  be a group and let  $\chi \in \text{Irr}(G)$ . If  $p$  does not divide  $\frac{|G|}{\chi(e_G)}$ , then  $\chi$  is called  $p$ -defect zero.*

*Lemma 3.4: [Isaacs, 1976] Let  $G$  be a group and let  $\chi \in \text{Irr}(G)$ . Suppose that  $p$  does not divide  $\frac{|G|}{\chi(e_G)}$  for some prime  $p$ . Then  $\chi(g) = 0$  whenever  $p \mid |g|$ .*

*Example 3.3: Let  $G = S_4$ .*

Table 3.3.1: Character Table of  $S_4$ .

|          | (1) | (12)(34) | (123) | (12) | (1234) |
|----------|-----|----------|-------|------|--------|
| $\chi_1$ | 1   | 1        | 1     | 1    | 1      |
| $\chi_2$ | 1   | 1        | 1     | -1   | -1     |
| $\chi_3$ | 2   | 2        | -1    | 0    | 0      |
| $\chi_4$ | 3   | -1       | 0     | 1    | -1     |
| $\chi_5$ | 3   | -1       | 0     | -1   | 1      |

We can see that  $\chi_5$  is 3-defect zero since  $3 \nmid \frac{|G|}{\chi_5(1)} = \frac{24}{3} = 8$ . Now, let  $p = 3$ . Since  $3 \nmid \frac{24}{3}$  whenever  $3 \mid |g|$ , we have that  $\chi(g) = 0$ . Hence, we can observe from the character table of  $G$  that  $\chi((123)) = 0$ .

*Definition 3.9:* Let  $G$  be group and let  $\chi \in \text{Irr}(G)$ . If  $\chi(g) = 0$ , then the conjugacy class of  $g$  is called a  $\chi$ -vanishing class.

*Example 3.4:* In Example 3.3, the conjugacy class of  $(123)$  is a  $\chi_5$ -vanishing class.

*Lemma 3.5:* [Brough, 1982] Let  $G$  be a group and let  $N$  be a non-trivial normal subgroup of  $G$ . Suppose that  $N$  has an irreducible character of  $p$ -defect zero. Then every element of  $N$  of order divisible by  $p$  is a vanishing element in  $G$ .

## 4. ZEROS OF CHARACTERS OF FINITE GROUPS

In this section, we basically discuss the relation between the conjugacy classes of a group  $G$  on which an irreducible character  $\chi$  vanishes and the centralizer size of an element of  $G$ . We start by claiming that we can find an element of  $G$  such that the centralizer size of such an element is bounded by a number which is some multiple of the number of conjugacy classes of  $G$  on which  $\chi$  vanishes. Unless otherwise stated, the group  $G$  will be considered as a finite non-abelian group throughout this section.

*Theorem 4.1: [Chillag, 1999] Let  $G$  be a group such that  $G \neq G'$ , let  $\chi$  be a non-linear irreducible character of  $G$ , in another word,  $\chi(e_G) \neq 1$  and let  $m$  be the number of  $\chi$ -vanishing classes. Then either we have an element  $g \in G$  such that  $|C_G(x)| \leq 2m$  or  $\varphi = \chi_{G'} \in \text{Irr}(G')$  and  $\varphi^G = \chi \sum \{\lambda \mid \lambda \in \text{Irr}(G), \lambda(e_G) = 1\}$ . In the second case, the set  $\{\chi\lambda \mid \lambda \in \text{Irr}(G), \lambda(e_G) = 1\}$  consists of exactly  $|G : G'|$  extensions of  $\varphi$ , which are all the extensions of  $\varphi$  and the character  $\varphi$  is not linear. In particular,  $G'' \neq e_G$ .*

*Proof :* Let  $C_1, C_2, \dots, C_m$  be the  $\chi$ -vanishing classes. We can see that  $\text{Lin}(G)$  acts on  $\text{Irr}(G)$  via multiplication as follows:

$$\varphi : \text{Lin}(G) \times \text{Irr}(G) \rightarrow \text{Irr}(G) \quad (4.1)$$

$$(\lambda, \chi) \mapsto \lambda\chi \quad (4.2)$$

The map  $\varphi$  is well defined since  $\lambda\chi$  is an irreducible character of  $G$ . Let  $O = \{\chi\lambda \mid \lambda \in \text{Lin}(G)\}$  be the orbit of  $\chi$  with respect to this action. Now, we have two cases according to the length of the orbit of  $\chi$ . As we know from the Orbit-Stabilizer Theorem,  $|O| \mid |\text{Lin}(G)|$ .

(i) If  $|O| < |\text{Lin}(G)|$ , then we may assume that  $\lambda\chi = \chi$  for some  $\lambda \in \text{Lin}(G) - \{1_G\}$ .

In other words, since  $|O| < |\text{Lin}(G)|$ , there exists  $\lambda_1, \lambda_2 \in \text{Lin}(G)$  such that  $\lambda_1 \neq \lambda_2$  and  $\lambda_1 \cdot \chi = \lambda_2 \cdot \chi$ . As  $\text{Lin}(G)$  is a group under multiplication  $\lambda_2^{-1}$  exists and  $\lambda_2^{-1} \lambda_1 \chi = \chi$ . Without loss of generality we set  $\lambda \doteq \lambda_2^{-1} \lambda_1$ . Let  $U = \bigcup_{i=1}^m C_i$  and let  $g \in G - U$ , then  $\chi(g) \neq 0$ . From the fact that  $\lambda\chi(g) = \lambda(g)\chi(g) = \chi(g)$  we deduce that if  $g \in G - U$ , then  $\lambda(g) = 1$ . Since  $\langle \lambda, 1_G \rangle \neq 0$  we get that

$\sum_{g \in G} \lambda(g) = 0$  so we can rewrite  $|G|$  as follows:

$$|G| = \sum_{g \in G} 1 - \sum_{g \in G} \lambda(g) = \sum_{g \in G} (1 - \lambda(g)) \quad (4.3)$$

$$= \sum_{x \in U} (1 - \lambda(x)) = \sum_{i=1}^m |C_i| (1 - \lambda(C_i)). \quad (4.4)$$

Now, let  $C \in \{C_1, C_2, \dots, C_m\}$  be the conjugacy class with maximal size. Then we observe that  $|G| \leq |C| \sum_{i=1}^m (1 - \lambda(C_i)) \leq 2m|C|$ . Taking an element  $c \in C$  we see that  $|C_G(c)| \leq 2m$ .

(ii) Now, we examine the case that  $|O| = |\text{Lin}(G)|$ . Since  $G' \trianglelefteq G$ , by Clifford's Theorem,

$$\chi_{G'} = e \sum_{i=1}^s \varphi_i \quad (4.5)$$

where  $\varphi_1, \varphi_2, \dots, \varphi_s$  are the distinct conjugates of  $\varphi = \varphi_1$  with  $\varphi \in \text{Irr}(G')$  and  $e = \langle \chi_{G'}, \varphi \rangle$ . By Frobenius Reciprocity,  $e = \langle \chi_{G'}, \varphi \rangle = \langle \chi, \varphi^G \rangle$ . Let  $\lambda \in \text{Lin}(G)$  and let  $x \in G'$ . Since  $\lambda$  is a homomorphism and  $x$  is generated by commutator elements, by setting  $x$  as  $aba^{-1}b^{-1}$ ,

$$\chi\lambda(aba^{-1}b^{-1}) = \chi(aba^{-1}b^{-1})\lambda(aba^{-1}b^{-1}) \quad (4.6)$$

$$= \chi(aba^{-1}b^{-1})\lambda(a)\lambda(b)\lambda(a^{-1})\lambda(b^{-1}) \quad (4.7)$$

$$= \chi(aba^{-1}b^{-1})\lambda(a)\lambda(a)^{-1}\lambda(b)\lambda(b)^{-1} \quad (4.8)$$

$$= \chi(aba^{-1}b^{-1}) \quad (4.9)$$

so we conclude that  $\chi\lambda(x) = \chi(x)$ .

Also, since  $\varphi^o(y) = 0$  for all  $y \in G - G'$  we have the following,

$$\varphi^G(h) = \frac{1}{|G'|} \sum_{x \in G} \varphi^o(xhx^{-1}) \quad (4.10)$$

$$= \frac{1}{|G'|} \sum_{x \in G} 0 = 0 \quad (4.11)$$

Thus, we get that  $\langle \varphi^G, \chi\lambda \rangle = \langle \varphi^G, \chi \rangle = e$ . Therefore,  $\chi\lambda$  is a constituent of  $\varphi^G$  with multiplicity  $e$ , for all  $\lambda \in \text{Lin}(G)$ . Now,  $O$  is a regular orbit under

$\text{Lin}(G)$ . Hence,  $\sigma = e\chi \sum_{\lambda \in \text{Lin}(G)} \lambda$  is a constituent of  $\varphi^G$ . If we consider the degrees as follows:

$$\chi(e_G) = \chi_{G'}(e_G) = e \sum_{i=1}^s \varphi(e_G) = e \cdot s \cdot \varphi(e_G) \quad (4.12)$$

$$\varphi^G(e_G) = |G:H| \cdot \varphi(e_G) = r \cdot \varphi(e_G) \quad (4.13)$$

Then we obtain the following:

$$\varphi^G(e_G) = r \cdot \varphi(e_G) \geq e \cdot \chi(e_G) \cdot r = e^2 \cdot s \cdot \varphi(e_G) \cdot r \quad (4.14)$$

This implies that  $e^2 \cdot s \leq 1$  and hence  $e = s = 1$ . Finally, we conclude that

$$\chi_{G'} = \varphi \in \text{Irr}(G') \quad (4.15)$$

Moreover,  $\sigma(e_G) = \chi(e_G) \cdot r = \varphi^G(e_G)$  and hence  $\varphi^G = \chi \sum_{\lambda \in \text{Lin}(G)} \lambda$ .

Now, since  $\lambda_{G'} = 1_{G'}$  for all  $\lambda \in \text{Lin}(G)$ , we see that  $(\chi\lambda)_{G'} = \chi^{G'} = \varphi$ . This implies that every element of the orbit  $O$  is an extension of  $\varphi$ . Lastly, if  $\theta \in \text{Irr}(G)$  is an extension of  $\varphi$ , then we have  $\langle \theta_{G'}, \varphi \rangle = \langle \theta, \varphi^G \rangle = 1$ . We are already given that  $\varphi^G$  is the sum of irreducible elements of  $O$ , so  $\theta \in O$ . In particular,  $\chi(e_G) = \varphi(e_G) > 1$ , and hence  $\varphi$  is non-linear character and so  $G'' \neq e_G$ .

**Corollary 4.1:** [Chillag, 1999] Let  $G$  be a metabelian group and let  $s$  be minimal size of a centralizer of an element of  $G$ . Suppose that  $A = \{\chi_1, \chi_2, \dots, \chi_l\}$  is the set of non-linear irreducible characters of  $G$ . Then for each  $i \in \{1, 2, \dots, l\}$ ,  $\chi_i$  vanishes on at least  $\frac{s}{2}$  conjugacy classes.

*Proof :* Let  $\chi_k$  be an element of  $A$  and let  $m$  be the number of  $\chi_k$ -vanishing classes. It is clear that there exists no  $g \in G$  such that  $|C_G(g)| < s$ . Since  $G$  is a metabelian group, we have that  $[G, G] \neq G$  and  $[G', G'] = e_G$ . Now, by Theorem 4.1, we see that  $s < |C_G(x)| \leq 2m$  for some  $x \in G$ . Hence, we conclude that  $s \leq 2m$  which implies that  $m \geq \frac{s}{2}$ .

**Lemma 4.1:** [Chillag, 1999] Let  $G$  be a group. Suppose that there exists an element  $x \in G$  such that  $|C_G(x)| = 2$ . Then  $G$  is a Frobenius group with complement  $H$  such that

$|H| = 2$  and an abelian kernel  $K$  such that  $|K| = m$  where  $m$  is odd.

*Proof :* Since  $G$  is non-abelian, we have that  $|G| > 2$ . By the hypothesis, there exists an element  $x \in G$  such that  $|C_G(x)| = 2$ . Since  $x \neq e_G$  and  $|x| = 2$ ,  $x$  must be contained in a Sylow 2-subgroup of  $G$ , say  $S$ . We claim that  $x$  is an element of  $Z(G)$ . We know that  $Z(S) \leq S$  is also 2-subgroup. By Cauchy Theorem, there exists an element  $y \in Z(S)$  such that  $|y| = 2$ . If  $x \notin Z(S)$  then the group  $\langle x, Z(S) \rangle$  has order at least 4 elements. On the other hand if  $\langle x, Z(S) \rangle \subseteq C_G(x)$ , since  $xy = yx$  for all  $y \in Z(S)$  we have that  $Z(S) \subseteq C_G(x)$ . Hence we conclude that  $\langle x, Z(S) \rangle \subseteq C_G(x)$ . However, this contradicts with the fact that  $|C_G(x)| = 2$ . Therefore,  $x \in Z(S)$ . Now, let  $s \in S$  then  $|s| = 2$ . We know that  $x \in Z(S)$ . Since  $x \in S$ , we have that  $xs = sx$  and  $s \in C_G(x) \supseteq S$ . This forces  $|S| = 2$ . Thus,  $S = C_G(x) \leq G$  and  $|S| = 2$ . In order to prove that  $G$  is a Frobenius group, we remember that we need to find a proper subgroup  $H$  of  $G$  such that  $H \cap H^g = \{e_G\}$  for all  $g \in G - H$ . In our case, we can see that  $S \cap S^g = \{e_G\}$  for all  $g \in G - S$ . Hence, we conclude that  $G$  is a Frobenius group with complement  $S$  such that  $|S| = 2$  and there exists a normal subgroup  $O$  which is the Frobenius Kernel.

Since we have Lemma 4.1, we can state the improved version of Theorem 4.1 as follows:

*Corollary 4.2:* [Chillag, 1999] Let  $G$  be a group such that  $G \neq G'$  and let  $\chi \in \text{Irr}(G)$ . Suppose that  $\chi$  is a non-linear and that  $\chi$  vanishes on exactly one conjugacy class of  $G$ . Then either  $G$  is a Frobenius group with complement of order 2 and an abelian odd-order kernel or  $\phi = \chi_{G'} \in \text{Irr}(G')$  and  $\phi^G = \chi \sum \{\lambda \mid \lambda \in \text{Irr}(G), \lambda(e_G) = 1\}$ . In the second case,  $\phi$  is not linear and  $G'' \neq e_G$ .

*Proof :* Since  $\chi$  vanishes on only one conjugacy class of  $G$  we see that  $m = 1$  in Theorem 4.1. Then we have the following two cases:

- (i) We can find  $g \in G$  such that  $|C_G(g)| = 2$  and by Lemma 4.1, we can see that  $G$  is a Frobenius group with a complement of order 2.
- (ii) By Theorem 4.1, when we can restrict  $\chi$  to the commutator subgroup we conclude that  $\phi = \chi_{G'} \in \text{Irr}(G')$ . Also, we have that  $\phi^G = \chi \sum \{\lambda \mid \lambda \in \text{Irr}(G), \lambda(e_G) = 1\}$ ,  $\phi$  is not linear and  $G'' \neq e_G$ .

*Example 4.1: Let  $G = S_3$ . Since  $G' = A_3$  we can see that  $G \neq G'$ . It is well-known fact that there are 3 distinct irreducible character of  $G$ . One of the characters is a non-linear irreducible character of  $G$  and it vanishes only one conjugacy class as we can see from the character table of  $G$ :*

Table 4.1: Character Table of  $S_3$ .

|          | (1) | (12) | (123) |
|----------|-----|------|-------|
| $\chi_1$ | 1   | 1    | 1     |
| $\chi_2$ | 1   | -1   | 1     |
| $\chi_3$ | 2   | 0    | -1    |

*As we can see from the character table of  $G$ ,  $\chi_3$  vanishes on exactly one conjugacy class whose representative element is (12). We can observe that  $G$  is a Frobenius group.*

*Theorem 4.2: [Chillag, 1999] Let  $G$  be a group such that  $G \neq G'$  and let  $n(\chi)$  be the number of  $\chi$ -vanishing classes. Let  $m = \max\{n(\chi) \mid \chi \in \text{Irr}(G)\}$ . Then we have all of the following:*

- i) If  $\chi \in \text{Irr}(G)$ , then  $\chi_{G'}$  is an irreducible character of  $G'$  and the set  $\{\chi\lambda \mid \lambda \in \text{Irr}(G), \lambda(e_G) = 1\}$  consists of exactly  $|G : G'|$  extensions of  $\chi_{G'}$ , which are all the extensions of  $\chi_{G'}$ .*
- ii) For any irreducible character  $\phi$  of  $G'$ , we can find an irreducible character  $\chi$  of  $G$  such that  $\chi_{G'} = \phi$ .*
- iii)  $G'' = G'$*
- iv) Every  $G'$  – conjugacy class is a  $G$  – conjugacy class.*
- v) Let  $\chi \in \text{Irr}(G)$  such that  $\chi(e_G) \neq 1$  then  $\chi$  vanishes on at least one conjugacy class outside  $G'$  and on at least one conjugacy class inside  $G'$ .*

To prove Theorem 4.2, we need a lemma and a theorem in the following:

*Lemma 4.2: [Isaacs, 1976](Brauer's Permutation Lemma) Let  $A$  be any group such that it acts on  $\text{Irr}(G)$  and on the set of conjugacy classes of  $G$ . Suppose that  $\chi(g) = \chi^a(g^a)$  for all  $\chi \in \text{Irr}(G)$ ,  $a \in A$  and  $g \in G$ , where  $g^a$  is an element of  $\text{cl}(g)^a$ . Then for each  $a \in A$ , the number of fixed irreducible characters of  $G$  is equal to the number of fixed classes.*

*Theorem 4.3: [Isaacs, 1976](Gallagher's Theorem) If  $N$  is a proper normal subgroup of  $G$  and  $G$  fixes each  $N$ -class of  $N$  then each non-linear irreducible character of  $G$  has zeros of  $N$ .*

*Proof :*

- i) *Let  $\chi$  be a non-linear irreducible character of  $G$ . Suppose that there exists an element  $x \in G$  such that  $|C_G(x)| \leq 2n(\chi)$ . We are given that  $2m < |C_G(x)|$ . This implies that  $n(\chi) > m$  which contradicts with the definition of  $m$ . So there exists no such  $x \in G$ . By Theorem 4.1,  $\chi_{G'} \in \text{Irr}(G')$  and  $\{\chi\lambda \mid \lambda \in \text{Irr}(G), \lambda(e_G) = 1\}$  consists of exactly  $|G : G'|$  extensions of  $\chi_{G'}$ , which are all the extensions of  $\chi_{G'}$ .*
- ii) *Let  $\varphi \in \text{Irr}(G')$ . We know that the induced character can be written as  $\varphi^G = a_1\chi_1 + a_2\chi_2 + \cdots + a_n\chi_n$  where  $\chi_i \in \text{Irr}(G)$ . There exists  $1 \leq j \leq n$  such that  $a_j \neq 0$  so  $\langle \varphi^G, \chi_j \rangle \neq 0$ . Let denote  $\chi_j$  by  $\chi$ . By Frobenius Reciprocity,  $\langle \varphi, \chi_{G'} \rangle \neq 0$ . Now, by part 1, we conclude that  $\chi_{G'} = \varphi$ .*
- iii) *Since we want to show that  $G'' = G'$  it is enough to show that  $|G' : G''| = 1$ . By Corollary 2.23 in [Isaacs, 1976] we know that  $|G' : G''| = |\text{Lin}(G')|$  and also,  $|G' : G''| = 1$ . It is easy to see that  $|\text{Lin}(G')| = 1$  if and only if  $\text{Lin}(G') = \{1_{G'}\}$ . Now, let  $\varphi \in \text{Lin}(G')$ . By part 2, we can conclude that  $\varphi = \chi_{G'}$  for some  $\chi \in \text{Irr}(G)$  so that  $\chi$  is linear. Then, we obtain that  $\chi_{G'} = 1_{G'}$ . and hence  $\text{Lin}(G') = \{1_{G'}\}$ .*
- iv) *By part 2, we know that for any  $\varphi \in \text{Irr}(G')$ , there exists  $\chi \in \text{Irr}(G)$  such that  $\chi_{G'} = \varphi$ . By Clifford's Theorem,  $\chi_{G'} = e \sum_{i=1}^t \varphi_i$ ,  $t = [G : I_G(\varphi)]$  is the distinct conjugates of  $\varphi$  in  $G$ . As  $\chi_{G'} = \varphi$ ,  $e = t = 1$ . In other words, the number of distinct conjugates of  $\varphi$  in  $G$  is 1. This means that for all  $g \in G$ ,  $\varphi^g = \varphi$  which implies that any  $g \in G$  fixes any  $\varphi \in \text{Irr}(G')$ , because  $\varphi$  was arbitrary. Any  $\varphi \in \text{Irr}(G')$  is  $G$ -invariant so any  $C \in \text{Con}(G')$  is  $G$ -invariant (By Brauer's Lemma). For all  $C \in \text{Con}(G')$  and for all  $g \in G$ ,  $C^g = C$ . There exists  $x \in G'$  such that  $C = x^{G'}$ . Since this is true for any  $g \in G$ , we have that  $x^{G'} = x^G$ .*

v) Since  $G' \neq G$  we can apply the Gallagher's Theorem.  $G$  fixes each  $G'$  – class because  $G$  fixes any irreducible character of  $G'$  and by Brauer's permutation lemma,  $G$  fixes any conjugacy class of  $G'$ . Then each non-linear irreducible character of  $G$  has zero off  $N$  so when we choose one  $\chi$  which is non-linear irreducible characters of  $G$ , by the Gallagher's Theorem, we know that there exists one element  $g$  (or the conjugacy class of  $g$ ) out of  $G'$  such that  $\chi(g) = 0$ . Now also we want to prove that there exists one element inside  $G'$ . But this is also true, because  $\varphi = \chi_{G'}$ , and  $\chi$  is non-linear so that  $\varphi$  is non-linear. By Burnside Theorem, we know that any non-linear irreducible character has a zero. So,  $G'$  has a zero inside  $G'$ .

**Corollary 4.3:** [Chillag, 1999] Let  $G$  be a group such that  $G \neq G' \neq G''$  and let  $n(\chi)$  be the number of  $\chi$ -vanishing classes for every  $\chi \in \text{Irr}(G)$ . Then we can find  $x \in G$  such that  $|C_G(x)| \leq 2m$  where  $m = \max\{n(\chi) \mid \chi \in \text{Irr}(G)\}$ .

*Proof :* Since  $G \neq G' \neq G''$  we cannot apply the theorem 4.2. By theorem 4.1, we can find such an element  $x \in G$  such that  $|C_G(x)| \leq 2m$  where  $m = \max\{n(\chi) \mid \chi \in \text{Irr}(G)\}$ .

**Proposition 4.1:** [Chillag, 1999] Let  $G$  be a group. Suppose that each  $\chi \in \text{Irr}(G)$  vanishes on at most one conjugacy class. Then  $G$  is a Frobenius group with a complement of order 2 and an abelian odd-order kernel.

*Proof :* Let  $\chi \in \text{Irr}(G)$ . Let  $n(\chi)$  be the number of  $\chi$ -vanishing classes of  $G$  and let  $m = \max\{n(\chi) \mid \chi \in \text{Irr}(G)\}$ . Now, we consider two cases which are  $G \neq G'$  and  $G = G'$ . First, suppose that  $G \neq G'$ . Since each  $\chi \in \text{Irr}(G)$  vanishes at most one conjugacy class,  $m = 1$ . If  $|C_G(x)| > 2$  for all  $x \in G$ , then by part 5 of Theorem 4.2, each non-linear irreducible character of  $G$  vanishes on at least two conjugacy classes. One of them is in  $G'$  and the other one is in  $G - G'$ . But this contradicts with the fact that  $m = 1$ . Hence  $|C_G(x)| = 2$  for some  $x \in G$ . Now, by Lemma 4.1 we can see that  $G$  is a Frobenius group with a complement of order 2 and an abelian odd-order kernel. Now, we will consider the case  $G = G'$ . We aim to show that this case is not possible. We can observe that  $G$  is not a solvable group since  $G = G'$ . Then, there exists a normal subgroup  $N$  of  $G$  such that  $X = G/N$  is a non-abelian simple group. Since  $X$  is non-abelian group there exists a non-linear irreducible character of  $X$ . Let  $\chi$  be a non-linear

irreducible character of  $X$ . By induction of characters, if  $\chi(xN) = \chi(yN) = 0$ , then  $\chi(x) = \chi(y) = 0$ . Since  $x$  and  $y$  are conjugate in  $G$ ,  $xN$  and  $yN$  are conjugate in  $X$  and hence if  $\theta$  is a non-linear irreducible character of  $X$ , then  $\theta$  vanishes on exactly one conjugacy class of  $X$ . To get a contradiction, we will show that there exists a non-linear irreducible character  $\chi$  of  $X$  such that it vanishes on at least two conjugacy classes. Now, we separate the rest of the proof into two cases. First, suppose that  $X$  has a 2-defect-zero. Then, there exists an irreducible character  $\chi$  of  $X$  such that  $2 \nmid \frac{|X|}{\chi(e_X)}$ . By Lemma 3.4, for all  $g \in X$ , such that  $2 \mid |g|$  we have that  $\chi(g) = 0$ . We claim that all elements of even order of  $X$  are conjugate in  $X$ . By Cauchy Theorem, there exists an element whose order is 2 since  $2 \mid |X|$ . Thus, we can find  $g_0 \in X$  such that  $|g_0| = 2$ . Let  $g \in X$  be element of even order. Then  $\chi(g_0) = \chi(g) = 0$  so  $g_0$  and  $g$  are conjugate in  $X$ . Let  $C = g_0^X$ . If  $g \in X$  and  $2 \mid |g|$ , then  $g \in C$ . Now, we claim that there exists no  $g \in X$  such that  $2 \mid |g|$  but  $|g| \neq 2$ . If not, then there exists  $g \in X$  such that  $|g| = 2m$  where  $m \neq 1$ . Since  $g \in C$ , there exists  $x \in X$  such that  $g = x^{-1}g_0x$ , so  $|g| = |g_0| = 2$ , a contradiction. Hence any element of even order in  $X$  is of order 2. Now, let  $S \in \text{Syl}_2(X)$ . Then  $S$  is an elementary abelian 2-group. Since  $S$  is unique subgroup with this property,  $S$  is normal in  $X$ . Let  $s \in S$ , then  $S \subseteq C_X(s)$ . Because  $S$  is abelian and  $s$  commutes with every element of  $X$ . Now, we want to show that  $S = C_X(S)$ . If  $t \in C_X(S) - S$  then either  $2 \mid |t|$  or  $2 \nmid |t|$ . If  $2 \mid |t|$ , then  $t \in S$ . If  $2 \nmid |t|$ , then  $t$  is an element of odd order. So, we have that  $s \in S$ ,  $|s| = 2$ ,  $t \in C_X(S)$  and  $|t| = m$  where  $m$  is odd and hence, there exists an element  $y$  with order  $2m$  which gives us a contradiction. Because we found an element whose order is a multiple of 2 but not 2. It can be seen in [Suzuki, 1963] that  $X \cong \text{PSL}_2(2^n)$  for some  $n$ . So  $X$  has an irreducible character vanishing on at least two conjugacy classes, a contradiction. So  $X$  does not have 2-defect zero. Thus, in [Willems, 1986] we can see that  $X$  is isomorphic with one of the following groups:  $A_n$ ,  $M_{12}$ ,  $M_{22}$ ,  $M_{24}$ ,  $J_2$ ,  $HS$ ,  $Suz$ ,  $Ru$ ,  $C1$ ,  $C3$ ,  $BM$ . In [Conway et al., 1984], we see that each of these groups as well as  $A_n$ ,  $n \leq 9$ , has an irreducible character vanishing on at least two conjugacy classes. Therefore,  $X = A_n$ ,  $n \geq 10$ . Then, as we discuss in section 5,  $X$  has an irreducible character  $\theta$  of degree  $n - 1$  with  $g \in G$  such that  $\theta(g)$  is the number of fixed points of  $g$  minus 1. Now, we set  $x = (1, 2, \dots, n-4)(n-3)(n-2)(n-1)(n)$  and  $y = (1, 2, \dots, n-6)(n-5, n-4, n-3, n-2, n-1)(n)$ . If  $n$  is odd number, then  $x$  and  $y$  are in  $A_n$ ,  $\theta(x) = \theta(y) = 0$  and they are not conjugate. Then,  $n$  is even and hence

$z = (1, 2, \dots, n-1)(n)$  and  $w = (1, 2, \dots, n-5)(n-4, n-3)(n-2, n-1)(n)$  are two elements which are not conjugates in  $A_n$  with  $\theta(z) = \theta(w) = 0$ , a contradiction.

**Theorem 4.4:** [Chillag, 1999] Let  $G$  be a group with non-trivial center and let  $C$  be a non-central conjugacy class of  $G$ . Let  $A = \{\theta_1, \theta_2, \dots, \theta_m\}$  be the set of irreducible characters of  $G$  vanishing on  $C$ . Then, either we can find an irreducible character  $\chi$  of  $G$  such that  $\frac{|G|}{\chi^2(e_G)} \leq 2m$  or  $|(cZ(G))^{G/Z(G)}| = |C|$  for  $c \in C$ . In the second case, we conclude that  $G/Z(G)$  is non-abelian group.

*Proof :* This proof is similar to the proof of Theorem 4.1. We define an action and then we examine two cases according to the orbit of the action. Now, since  $A$  is the set of all irreducible characters of  $G$  we have  $\theta_i(c) = 0$  for all  $i \in \{1, 2, \dots, m\}$  and for any  $c \in C$ . Let  $\text{Con}(G) = \{C_1, C_2, \dots, C_k\}$  be the set of all conjugacy classes of  $G$ . We claim that  $Z(G)$  acts on  $\text{Con}(G)$  via multiplication.

Define the action

$$\psi : \text{Con}(G) \times Z(G) \rightarrow \text{Con}(G) \quad (4.16)$$

$$(C, g) \mapsto C \cdot g = \{a_i \cdot g : 1 \leq i \leq k\} \quad (4.17)$$

For all  $g_1, g_2 \in Z(G)$  we have the following

$$C \cdot (g_1 g_2) = \{a \cdot (g_1 g_2) : a \in C\} \quad (4.18)$$

$$= \{(a \cdot g_1) \cdot g_2 : a \in C\} \quad (4.19)$$

$$= (C \cdot g_1) \cdot g_2 \quad (4.20)$$

Hence,  $\psi$  is an action of  $Z(G)$  on  $\text{Con}(G)$ . Now let  $O = \{zC : z \in Z(G)\}$  be the orbit of  $C \in \text{Con}(G)$  under the action  $\psi$ . We have two cases:  $|O| < |Z(G)|$  or  $|O| = |Z(G)|$ . Because, by Orbit-Stabilizer theorem, we know that  $|O| \mid |Z(G)|$  or  $|O| = |Z(G)|$ . First, suppose that  $|O| < |Z(G)|$ . In this case,  $C$  is fixed by some  $z \in Z(G) - \{e_G\}$ , and hence  $zC = C$ . For  $c \in C$ , since  $zc$  and  $c$  are conjugate, for all  $\chi \in \text{Irr}(G)$  we see that  $\chi(zc) = \chi(c)$ . Thus, we have that  $\chi(zc) = \lambda(z)\chi(c)$  and hence  $\chi(c) = \lambda(z)\chi(c)$  for all  $\chi \in \text{Irr}(G)$ . Let  $I = \text{Irr}(G) - A$ . So for  $c \in C$  and  $\phi \in I$ , we conclude that  $\phi(c) \neq 0$ . This implies that  $\langle z \rangle \subseteq \text{Ker}(\phi)$  for all  $\phi \in I$ . So we get that  $I \subseteq \text{Irr}(G / \langle z \rangle)$ .

Thus we have

$$|G| = \sum_{i=1}^t \chi_i^2(e_G) \quad (4.21)$$

$$= \sum_{i=1}^m \theta_i^2(e_G) + \sum_{\chi \in I} \chi^2(e_G) \quad (4.22)$$

$$\leq \sum_{i=1}^m \theta_i^2(e_G) + \sum_{\chi \in \text{Irr}(G/\langle z \rangle)} \chi^2(e_G). \quad (4.23)$$

Now let  $\theta \in A$  with maximal degree. Then we have the following

$$|G| \leq m \cdot \theta^2(e_G) + \frac{|G|}{|\langle z \rangle|} \quad (4.24)$$

$$\leq m \cdot \theta^2(e_G) + \frac{|G|}{2} \quad (4.25)$$

$$2 \cdot |G| \leq 2 \cdot m \cdot \theta^2(e_G) + |G| \quad (4.26)$$

Hence, we obtain that  $\frac{|G|}{\theta^2(e_G)} \leq 2m$ . Thus, the claim 1 holds.

Now, suppose that  $|O| = |Z(G)|$ . By Lemma 1.1 in [Chillag and Herzog, 1990], it is straight forward to see that  $|c^G| \geq |(cZ)^Z|$ . To obtain the inverse, we set  $c^G = \{c^{g_1}, c^{g_2}, \dots, c^{g_s}\}$  where  $\{g_1 = e_G, g_2, \dots, g_s\} \subseteq G$ . We claim that  $c^{g_1}Z, c^{g_2}Z, \dots, c^{g_s}Z$  are all distinct. To get contradiction, we suppose that  $c^{g_i}Z = c^{g_j}Z$  for some  $i \neq j$ . So  $c^{g_i} \in c^{g_j}Z$ . Then, there exists  $v \in Z$  such that  $c^{g_i} = c^{g_j}v$ . This implies that  $(cv)^{g_j} = g_j^{-1}cg_jg_j^{-1}vg_j = c^{g_j}v$  and so,  $c^{g_i} = c^{g_j}v = (cv)^{g_j}$ . Therefore, we get that  $c^G = cv^G = v(c^G)$ . This means that  $v$  fixes  $c^G$  implying that  $|O| < |Z|$  which contradicts with our assumption. Therefore, we obtain that  $|(cZ(G))^{G/Z(G)}| = |C|$ .

**Corollary 4.4:** [Chillag, 1999] Let  $G$  be a nilpotent group of class 2 and let  $g \in G - Z(G)$ . Set  $m$  be the number of irreducible characters of  $G$  vanishing on  $g$ . Then there exists  $\chi \in \text{Irr}(G)$  such that  $\frac{|G|}{\chi^2(e_G)} \leq 2m$ .

*Proof :* Since  $g \in G - Z(G)$  we have that  $|g^G| > 1$  and  $|(aZ(G))^{G/Z(G)}|$ . So we see that claim 2 of Theorem 4.4 does not hold and hence claim 1 of Theorem 4.4 holds.

**Example 4.2:** Let  $G = Q_8$ . Set  $g = i \in Q_8 - Z(Q_8)$ . The character table of  $Q_8$  is as follows:

Table 4.2: Character Table of  $Q_8$ .

|          | 1 | -1 | i  | j  | k  |
|----------|---|----|----|----|----|
| $\chi_1$ | 1 | 1  | 1  | 1  | 1  |
| $\chi_2$ | 1 | 1  | 1  | -1 | -1 |
| $\chi_3$ | 1 | 1  | -1 | 1  | -1 |
| $\chi_4$ | 1 | 1  | -1 | -1 | 1  |
| $\chi_5$ | 2 | -2 | 0  | 0  | 0  |

Since there exists only one irreducible character with  $\chi(g) = 0$ , we conclude that  $m = 1$  in Corollary 4.4. We can see that the claim in Corollary 4.4 holds since  $\frac{|G|}{\chi_5^2(1)} = \frac{8}{4} \leq 2 \cdot 1$ .

*Corollary 4.5: [Chillag, 1999] Let  $G$  be a group with non-trivial center. For each  $g \in G$  let  $n(g)$  be the number of irreducible characters of  $G$  vanishing on  $g$  and set  $m = \max\{n(g) \mid g \in G\}$ . Let  $\frac{|G|}{\chi^2(e_G)} > 2m$  for all  $\chi \in \text{Irr}(G)$ . Then for every  $g \in G$ , we conclude that  $|(gZ(G))^{G/Z(G)}| = |g^G|$ .*

*Proof :* Let  $g$  be an element of  $G$ . If  $g \in Z(G)$ , then  $g$  has a trivial conjugacy class and hence  $|g^G| = |(gZ(G))^{G/Z(G)}| = 1$ . Then, we may assume that  $g \in G - Z(G)$ . If there exists an irreducible character  $\chi$  in  $G$  such that  $\frac{|G|}{\chi^2(e_G)} \leq 2n(g)$ , then  $\frac{|G|}{\chi^2(e_G)} \leq 2m$  which contradicts with the assumption. Therefore, in Theorem 4.4, claim 1 does not hold. So claim 2 holds and we have that  $|g^G| = |(gZ(G))^{G/Z(G)}| > 1$ .

## 5. CONJUGACY CLASSES OF FINITE NON-SOLVABLE GROUPS

In this part we see some effects of the size of conjugacy classes of  $G$  on the structure of  $G$ . First, we see some properties of conjugacy classes of a finite simple groups.

*Remark 5.1: Let  $S$  be a non-abelian simple group. According to the classification of finite non-abelian simple groups in [Conway et al., 1984]  $S$  is one of the following:*

- i) An alternating group,  $A_n$  for  $n \geq 5$ ,*
- ii) An exceptional group of Lie type,*
- iii) A classical group of Lie type,*
- iv) One of the 26 sporadic groups.*

In the following three theorems we discuss the existence of a subgroup  $T$  of a simple group  $S$  where  $S$  is either a simple group of Lie type or one of the sporadic simple groups with the property  $C_S(t) = T$  for some  $t \in T - \{e_T\}$ .

*Theorem 5.1: [Babai et al., 2009]*

- i) Let  $S$  be an exceptional simple group of Lie type then there exists a subgroup  $T$  in  $S$  such that  $C_S(t) = T$  for some  $t \in T - \{e_T\}$ .*
- ii) Let  $S$  be a classical simple group of Lie type then there exists a subgroup  $T$  in  $S$  such that  $C_S(t) = T$  for some  $t \in T - \{e_T\}$ .*
- iii) Let  $S$  be one of the sporadic simple groups then  $S$  has a subgroup  $T$  such that  $C_S(t) = T$  for every  $t \in T - \{e_T\}$ .*

In Table 5.1 we see the exceptional simple groups of Lie type and their subgroups  $T$  which satisfies the property that  $C_S(t) = T$  for some  $t \in T$ . For example, we can see that if  $G = G_2(q)$  where  $q \geq 3$ , then  $G$  has a subgroup  $T$  such that  $|T| = q^2 + q + 1$ . In the last column, we can see some divisors of  $|t^s|$  for  $t \in T - \{e_T\}$ .

Table 5.1: Self-centralizer subgroup  $T$  of an exceptional simple group  $S$  of Lie type.

| S             | Condition              | $ T $                                   | Some divisors of $ t^S $    |
|---------------|------------------------|-----------------------------------------|-----------------------------|
| ${}^2B_2(q)$  | $q = 2^{2k+1} \geq 8$  | $q + \sqrt{2q} + 1$                     | $q(q-1)(q - \sqrt{2q} + 1)$ |
| ${}^3D_4(q)$  |                        | $q^4 - q^2 + 1$                         | $q(q^6 - 1)$                |
| $G_2(q)$      | $q \geq 3$             | $q^2 + q + 1$                           | $q(q^2 - 1)(q^2 + q + 1)$   |
| ${}^2G_2(q)$  | $q = 3^{2k+1} \geq 27$ | $q + \sqrt{3q} + 1$                     | $q(q^2 - 1)$                |
| $F_4(q)$      |                        | $q^4 + q^2 + 1$                         | $q(q^8 - 1)$                |
| ${}^2F_4(q)$  | $q = 2^{2k+1} \geq 8$  | $q^2 + \sqrt{2q^3} + q + \sqrt{2q} + 1$ | $q(q^4 - 1)$                |
| $E_6(q)$      |                        | $\frac{q^6 + q^3 + 1}{(3, q-1)}$        | $q(q^6 - 1)$                |
| ${}^2E_6(q)$  |                        | $\frac{q^6 - q^3 + 1}{(3, q+1)}$        | $q(q^6 - 1)$                |
| $E_7(q)$      |                        | $\frac{q^7 - 1}{(2, q-1)}$              | $q(q^6 - 1)$                |
| $E_8(q)$      |                        | $q^8 - q^4 + 1$                         | $q(q^8 - 1)$                |
| ${}^2F_4(q)'$ |                        | 13                                      | $2 \cdot 3 \cdot 5$         |

Also, in Table 5.2 we see the classical simple groups of Lie type and their subgroups  $T$  which satisfies the property that  $C_S(t) = T$  for some  $t \in T$ . For example, we can see that if  $G = PSL_n(q)$  where  $n = 2$  and  $q \geq 4$ , then  $G$  has a subgroup  $T$  such that  $|T| = \frac{q^2 - 1}{(q-1)(2, q-1)}$ . In the last column, we can see some divisors of  $|t^S|$  for  $t \in T - \{e_T\}$ .

Table 5.2: Self-centralizer subgroup  $T$  of a classical simple group  $S$  of Lie type.

| S                   | Condition                      | $ T $                                    | Some divisors of $ t^S $ |
|---------------------|--------------------------------|------------------------------------------|--------------------------|
| $PSL_n(q)$          | $n = 2; q \geq 4$              | $\frac{q^2 - 1}{(q-1)(2, q-1)}$          | $q(q-1)$                 |
| $PSL_n(q)$          | $n = 3; q \geq 3$              | $\frac{q^3 - 1}{(q-1)(3, q-1)}$          | $q(q^2 - 1)$             |
| $PSL_n(q)$          | $n \geq 4$                     | $\frac{q^n - 1}{(q-1)(n, q-1)}$          | $q(q^2 - 1)(q^3 - 1)$    |
| $PSU_n(q)$          | $n = 3; q > 2$                 | $\frac{q^3 + 1}{(q+1)(3, q+1)}$          | $q(q^2 - 1)$             |
| $PSU_n(q)$          | $n \geq 4$                     | $\frac{q^n + (-1)^{n-1}}{(q+1)(n, q+1)}$ | $q(q^2 - 1)(q^3 + 1)$    |
| $PSP_{2m}(q)$       | $m \geq 2; (m, q) \neq (2, 2)$ | $\frac{q^m - 1}{(2, q-1)}$               | $q(q^4 - 1)$             |
| $P\Omega_{2m+1}(q)$ | $m \geq 3, q \text{ odd}$      | $\frac{q^m + 1}{2}$                      | $q(q^4 - 1)$             |
| $P\Omega_{2m}^+(q)$ | $m \geq 4, m \equiv 0(2)$      | $\frac{(q^{m-1} + 1)(q+1)}{(2, q-1)^2}$  | $q(q^4 - 1)$             |
| $P\Omega_{2m}^+(q)$ | $m \geq 5, m \equiv 1(2)$      | $\frac{(q^{m-1} + 1)(q+1)}{(4, q-1)}$    | $q(q^4 - 1)$             |
| $P\Omega_{2m}^-(q)$ | $m \geq 4$                     | $\frac{q^m + 1}{(4, q^m + 1)}$           | $q(q^4 - 1)$             |

The lemma 5.1 plays an important role in the proofs of the next theorems.

*Lemma 5.1:* [Blackburn and Huppert, 1982] Suppose that  $p^a = q^b + 1$ , where  $p, q$  are primes and  $a, b$  are positive integers. Then either

- i)  $p = 2$ ,  $b = 1$ ,  $a$  is a prime and  $q = 2^a - 1$  is a Mersenne prime, or
- ii)  $q = 2$ ,  $a = 1$ ,  $b = 2^m$  and  $p = 2^{2^m} + 1$  is a Fermat prime, or
- iii)  $p^a = 9$ ,  $q^b = 8$ .

*Proposition 5.1: [Robati, 2020] If  $S$  is an exceptional simple group of Lie type, then it contains some conjugacy class  $t$  such that the size of  $t$  has at least three distinct prime divisors.*

*Proof :* Let  $q = 2^k$  where  $k$  is a non-negative integer. Since  $(q^2 + 1, q^2 - 1) = (q + \sqrt{2q} + 1, q - 1) = 1$  and  $(q, q^n - 1) = 1$  for every positive integer  $n$ , by Lemma 5.1 and Table 5.1 we can see that the claim holds.

*Example 5.1:* Let  $S = {}^2G_2(q)$  where  $q = 3^{2k+1} \geq 27$ . We know that  $S$  has a subgroup  $T$  with the property that  $C_S(t) = T$  for some  $t \in T$ . From Table 5.1, we can see that  $|T| = q + \sqrt{3q} + 1$  and some divisors of  $|t^S|$  are  $q \cdot (q^2 - 1)$ . Since  $q$  is a power of 3,  $|t^S|$  is divisible by 3. Now, since the term  $q^2 - 1$  is even, we can see that 2 is also a divisor of  $q^2 - 1$ . So far, we have found two distinct prime divisors of  $|t^S|$ . For the rest, suppose that  $q^2 - 1 = 2^m$  where  $m$  is a non-negative integer. Then we have  $(q + 1)(q - 1) = 2^m$  which implies that  $q - 1 = 2^c$  and  $q + 1 = 2^d$ . Then  $q + 1 = 2^c + 2 = 2 \cdot (2^{c-1} + 1) = 2^d$ , a contradiction. Therefore, we see that there exists a prime divisor of  $q^2 - 1$  which is not 2. Hence we obtain that there are 3 distinct prime divisors of  $|t^S|$ .

We will discuss the claim of Proposition 5.1 for classical groups of Lie type except for some exceptional cases. In order to progress we will start with some number theoretical properties for result.

*Remark 5.2:* Let  $q$  be a prime number. Using number theoretical properties, we can see that we have the following parts.

- i)  $(q, q^n - 1) = 1$  where  $n \in \mathbb{Z}_{>0}$
- ii)  $(q, q^2 - q + 1) = (q - 1, q^2 - q + 1) = 1$
- iii)  $q^3 + 1 = (q + 1) \cdot (q^2 - q + 1)$

*Remark 5.3:* In [Conway et al., 1984], we can see that there are 5 conjugacy classes with sizes 12, 15, 20 in  $PSL_2(4)$  and that there are 9 conjugacy classes with sizes 56, 63, 72 in  $PSL_2(8)$ . Herewith we can observe that the sizes of the conjugacy classes of these groups are divisible by at most 2 distinct primes.

*Proposition 5.2:* [Robati, 2020] Let  $S$  be a classical simple group and let  $\{t_1, \dots, t_n\}$  be representative elements of conjugacy classes of  $S$ . Then for each  $i$ ,  $|t_i^S|$  has at most two distinct prime divisors if and only if  $S \cong PSL_2(4)$  or  $PSL_2(8)$ .

*Proof :* Let  $S$  be a classical simple group. By Theorem 5.1, we know that there exists a subgroup  $T$  of  $S$  such that  $C_S(t) = T$  for some  $t \in T - \{e_T\}$ . By Remark 5.2 and Lemma 5.1, we can see that the claim holds for each group in Table 5.2 except the case  $S = PSL_n(q)$  for  $(n, q) = (2, q)$  and  $(3, 3)$  and  $PSU_n(q)$  for  $(n, q) = (3, 3)$ .

Now, let  $S = PSL_2(q)$  where  $q \geq 4$ . By the third table in [Bianchi et al., 2009], there exists a subgroup  $T'$  of  $S$  such that  $|T'| = \frac{q-1}{(2, q-1)}$  and  $T'$  satisfies the property that  $C_S(x) = T'$  for some  $x \in T'$ . By the size of conjugacy class formula and [Conway et al., 1984] for the orders of the simple group, we have the following:

$$|x^S| = [S : C_S(x)] = \frac{|S|}{|T'|} = \frac{\frac{|SL_2(q)|}{d}}{|T'|} = \frac{|SL_2(q)|}{q-1} = \frac{q \cdot (q^2 - 1)}{q-1} = q \cdot (q+1) \quad (5.1)$$

There exists one more subgroup  $T$  of  $S$  such that  $C_S(x') = T$  for some  $x' \in T$ .

$$|x'^S| = \frac{|S|}{|T|} = \frac{|SL_2(q)|}{d} \cdot \frac{d}{q+1} = q \cdot (q-1) \quad (5.2)$$

If each conjugacy class size of  $S$  has at most 2 prime divisors then  $q \cdot (q-1)$  and  $q \cdot (q+1)$  have at most 2 prime divisors. If  $q$  is odd then  $2 \mid (q-1, q+1)$  which implies that  $q-1$  and  $q+1$  are power of 2. Let  $q+1 = 2^k$  and let  $q-1 = 2^m$  where  $k, m \in \mathbb{Z}_{>0}$ . Then we have  $q = 2^k - 1 = 2^m + 1$ . So, we have that

$$2^k = 2^m + 2 = 2 \cdot (2^{m-1} + 1) \quad (5.3)$$

which gives a contradiction. Therefore,  $q$  cannot be an odd prime. By Lemma 5.1,

$q = 4$  or  $q = 8$ . In [Conway et al., 1984], we can see that the claim holds for  $PSL_2(4)$  and  $PSL_2(8)$ .

Considering the alternating group it is a well-known fact that  $A_5 \cong PSL_2(4)$ . To discuss the general case, we will consider the case  $n$  is at least 6 in the following theorem.

*Theorem 5.2: [Robati, 2020] Let  $S$  be an alternating group  $A_n$  where  $n \geq 6$ . Let  $x$  be an element of  $S$ . Then, we can find an irreducible character  $\chi$  of  $S$  such that  $\chi$  extends to  $Aut(S)$  and also a  $\chi$ -vanishing class  $x^S$  such that  $|x^S|$  is divisible by at least three distinct prime numbers.*

*Proof :* Suppose that  $S$  is an alternating group  $A_n$  where  $n \geq 7$ . Let  $\chi : S \rightarrow \mathbb{C}$  be a map such that for each  $x \in S$ ,  $\chi(x) = |Fix(x)| - 1$  where  $|Fix(x)|$  denotes the number of fixed points of  $x$ . We can see that the map  $\chi$  is an irreducible character of  $S$ . Because, if  $\alpha$  and  $\beta$  are conjugate in  $A_n$  then they have the same cycle structure, so the numbers of fixed points of  $\alpha$  and  $\beta$  are equal and hence we see that  $\chi$  is constant on conjugacy classes. This implies that  $\chi$  is an irreducible character of  $A_n$ . Now, we separate the proof into two cases for the rest. If  $n$  is even then we set

$$x = (1, \dots, n-1)(n). \quad (5.4)$$

We can see that  $x$  is a  $(n-1)$ -cycle which fixes the point  $n$ . Since  $x$  does not commute with any odd permutation in  $Sym(n)$ , we know that  $|x^{A_n}| = \frac{1}{2} \cdot |x^{S_n}|$ . Then we have the following:

$$|x^{A_n}| = \frac{1}{2} \cdot \frac{n!}{n-1} = \frac{1}{2} \cdot n \cdot (n-2)! = \frac{n \cdot (n-2)!}{2}. \quad (5.5)$$

We see that  $(n-2)! \cdot n$  divides  $|x^S|$  and hence we conclude that  $|x^S|$  is divisible by at least three distinct primes. For example, if  $n = 8$ , then  $|x^{A_8}| = \frac{8 \cdot 6!}{2} = 2^6 \cdot 3^2 \cdot 5$  so we can clearly see that we have three distinct primes that divide  $|x^{A_8}|$ .

We continue with the similar idea. If  $n$  is odd and  $n \geq 7$ , then we set

$$x = (1, \dots, n-3)(n-2, n-1)(n). \quad (5.6)$$

Since  $x$  commutes with the transposition  $(n-2, n-1)$  we have the following

$$|x^{A_n}| = |x^{S_n}| = \frac{n!}{2 \cdot (n-3)} \quad (5.7)$$

We see that  $(n-4)! \cdot n$  divides  $|x^S|$  and hence we conclude that  $|x^S|$  is divisible by at least three distinct primes for odd  $n \geq 7$ . In both cases, we can see that  $\chi(x) = 0$ .

In [Conway et al., 1984], we can see that there exists  $\chi_7 \in \text{Irr}(A_6)$  that extends to  $\text{Aut}(A_6)$  and also a  $\chi_7$ -vanishing class named  $4A$  whose size is divisible by 2, 3 and 5.

By now, we have seen that if  $S$  is an alternating group  $A_n$  where  $n \geq 6$  or a simple group of Lie type (except for  $S = \text{PSL}_2(4)$  and  $\text{PSL}_2(8)$ ), then  $S$  contains some class whose size is divisible by at least 3 distinct primes.

*Proposition 5.3:* [Robati, 2020] If  $S$  is a sporadic simple group, then it contains some conjugacy class  $t$  such that the size of  $t$  has at least three distinct primes.

*Proof :* By Theorem 5.1, we know that there exists a subgroup  $T$  of  $S$  such that  $C_S(t) = T$  for every  $t \in T - \{e_T\}$ . Since the size of each sporadic simple group has at least four prime divisors we see that the claim holds.

*Theorem 5.3:* [Robati, 2020] Let  $S$  be a simple group. If each conjugacy class size of  $S$  is divisible by at most 2 distinct primes then  $S \cong \text{PSL}_2(4)$  or  $\text{PSL}_2(8)$  and vice versa.

*Proof :* By Theorem 5.2, the claim is true for  $A_n$  where  $n \geq 6$  except  $n = 5$ . But it can be observed that there is an isomorphism between  $A_5$  and  $\text{PSL}_2(4)$  so the rest is clear by Proposition 5.1, 5.2, 5.3.

## 6. VANISHING CLASSES OF FINITE NON-SOLVABLE GROUPS

Let  $G$  be a group and let  $\chi \in \text{Irr}(G)$ . In this part, we see some effects of the size of  $\chi$ -vanishing classes of  $G$  on the structure of  $G$ . We start with the following Corollary 6.1 that will be very useful in the proof of next lemmas.

*Corollary 6.1: [Granville and Ono, 1996] If we are given a non-abelian simple group  $S$  and a prime  $p$  with the property that there exists no irreducible character  $\chi$  in  $S$  such that  $\chi$  is  $p$ -defect zero, then either  $p = 2$  or  $3$  and  $S$  is isomorphic with one of the these groups:  $M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_3, BM, A_n$  where  $n \geq 7$ .*

*Lemma 6.1: [Robati, 2020] Suppose that we are given a non-abelian simple group  $S$  and a prime number  $p$  such that there exists no irreducible character which is  $p$ -defect zero, then we obtain that  $\theta \in \text{Irr}(S)$  such that we can extend  $\theta$  to  $\text{Aut}(S)$ . Moreover, we can find a  $\theta$ -vanishing class  $x^S$  with the property that  $|x^S|$  contains at least three distinct primes.*

*Proof :* By Corollary 6.1 and considering the character tables of such groups from the [Conway et al., 1984] we can observe the existence of conjugacy classes  $x^S$  with their sizes that contains at least 3 distinct primes as well as we can find  $\theta \in \text{Irr}(S)$  such that  $x^S$  is a  $\theta$ -vanishing class and we can extend  $\theta$  to automorphism groups of such groups. In Table 6.1, we see some of the groups that mentioned in Lemma 6.1 and their irreducible characters that vanish on the classes which contains in their sizes at least 3 distinct primes.

Table 6.1: Some irreducible characters of groups in Corollary 6.1.

| Group    | Character   | Class |
|----------|-------------|-------|
| $M_{12}$ | $\chi_7$    | $8B$  |
| $M_{22}$ | $\chi_3$    | $5A$  |
| $M_{24}$ | $\chi_5$    | $12B$ |
| $J_2$    | $\chi_{10}$ | $10C$ |
| $HS$     | $\chi_7$    | $20A$ |
| $Suz$    | $\chi_3$    | $10B$ |
| $Ru$     | $\chi_2$    | $12A$ |
| $Co_1$   | $\chi_2$    | $23A$ |
| $Co_3$   | $\chi_9$    | $11A$ |
| $BM$     | $\chi_2$    | $70A$ |

In Theorem 5.2 we already showed the case  $S = A_n$  where  $n \geq 7$ . This fact completes the proof.

In order to obtain some results on the relation between the group structure and the size of vanishing classes of a non-solvable group we need two following propositions:

*Proposition 6.1: [Bianchi et al., 2007] Suppose that  $G$  is a group with a minimal normal subgroup  $N$  such that  $N = S_1 \times \cdots \times S_m$  where  $S_i$  is a non-abelian simple group  $S$  for all  $i \in \{1, \dots, m\}$ . Whenever  $\chi \in \text{Irr}(S)$  is extendible to  $\text{Aut}(S)$  we conclude that  $\chi \times \cdots \times \chi$  is also extendible to  $G$ .*

*Proposition 6.2: [Dolfi et al., 2010] If  $G$  is a group with two normal subgroups  $M$  and  $N$  such that they intersects trivially and if we have an element  $m$  in  $M \cap \text{Van}(G)$ , then we conclude that  $mn \in \text{Van}(G)$  for all  $n \in N$ .*

*Proof :* We need to find  $\phi \in \text{Irr}(G)$  with  $\phi(mn) = 0$ . First, let  $\chi \in \text{Irr}(G)$  such that  $\chi(m) = 0$  and let  $\alpha$  be an irreducible constituent of  $\chi_M$ . Then we have  $\langle \chi_M, \alpha \rangle \neq 0$  and since  $M$  is normal in  $G$ , by using Clifford's Theorem we have the following

$$0 = \chi_M(m) = e \sum_{i=1}^s \alpha^{g_i}(m) \quad (6.1)$$

where  $e = \langle \chi_M, \alpha \rangle \neq 0$ . Now we consider the irreducible character  $\alpha \times 1_N$  of  $MN$ . Since  $(\alpha \times 1_N)(mn) = \alpha(m)1_N(n) = \alpha(m)$  for all  $m \in M, n \in N$  we have that  $I_G(\alpha \times 1_N) = I_G(\alpha)$ . Let  $\gamma$  be an irreducible character of  $I_G(\alpha \times 1_N)$  lying over  $\alpha \times 1_N$  and we set  $\phi = \gamma^G$ . By Theorem 3.6 we obtain that  $\phi \in \text{Irr}(G)$ . For any  $n \in N$ , by Clifford's Theorem we have that

$$\phi(mn) = f \sum_{i=1}^s (\alpha \times 1_N)^{g_i}(mn) = f \sum_{i=1}^s \alpha^{g_i}(m) = 0. \quad (6.2)$$

where  $f = \langle \phi_{MN}, \alpha \times 1_N \rangle$ . Hence, we obtain that  $mn \in \text{Van}(G)$ .

*Theorem 6.1: [Robati, 2020] Let  $G$  be a non-solvable group. Suppose that the size of every vanishing class is divisible by at most two primes. Then  $G/\text{Sol}(G) \cong \text{PSL}_2(4)$  or  $\text{PSL}_2(8)$ .*

*Proof :* Let  $M = \text{Sol}(G)$  is the solvable radical of  $G$  and let  $N/M$  be normal minimal subgroup of  $G/N$ . Since  $N/M$  normal minimal, we have that  $N/M \cong S_1 \times \cdots \times S_n$  where each  $S_i$  is isomorphic to a non-abelian simple group, say  $S$ .

Now, we separate the proof into two cases whether  $S$  has an irreducible character of  $p$ -defect zero or does not.

First, we suppose that  $S$  does not have any irreducible character of  $p$ -defect zero. Then, by lemma 6.1, we can find a conjugacy class  $(xM)^S$  such that the size of  $(xM)^S$  has at least 3 distinct prime divisors and also there exists an irreducible character  $\theta$  of  $S$  which extends to  $\text{Aut}(S)$  such that  $\theta$  vanishes on the class  $(xM)^S$ . Since the conditions are satisfied in Proposition 6.1, the irreducible character  $\theta \times \cdots \times \theta$  of  $N/M$  extends to  $G/M$  and  $\theta \times \cdots \times \theta$  vanishes on  $(xM)^{G/M}$  such that  $|(xM)^G| \mid |(xM)^{G/M}|$ . This gives us a contradiction. Because, we are given that the vanishing class is divisible by at most 2 primes. Now, we may assume that for each prime  $p$ ,  $S$  has an irreducible character of  $p$ -defect zero. Since  $N/M \cong S_1 \times \cdots \times S_n$ , we have that  $\theta \times \cdots \times \theta$  is an irreducible character of  $p$ -defect zero of  $N/M$  for each prime  $p$ . By Lemma 3.5, we can see that every element of  $N/M$  is a vanishing element of  $G/M$  since for any prime  $p$  we can find an irreducible character which vanishes on every element that is divisible by  $p$ . By Theorem 5.3, we see that each  $S_i$  is isomorphic to  $\text{PSL}_2(4)$  or  $\text{PSL}_2(8)$  for  $1 \leq i \leq k$ . Because, by the hypothesis, every vanishing class of  $G$  has at most two different prime divisors so every vanishing class of  $G/M$  also has at most two prime divisors and hence this is also true for  $N/M$  which is isomorphic to  $S_1 \times \cdots \times S_n$ . Furthermore, if  $x = (x_1, \dots, x_k) \in N/M$  then  $x^{N/M} = x_1^{S_1} \times \cdots \times x_k^{S_k}$  and hence it is easy to see that each conjugacy class size of  $N/M$  has at most two different prime divisor if and only if  $k = 1$  and hence we obtain that  $N/M \cong \text{PSL}_2(4)$  or  $\text{PSL}_2(8)$ . For the uniqueness, let  $N_1/M$  and  $N_2/M$  be two normal minimal subgroups of  $G/M$  so we know that  $N_1/M$  and  $N_2/M$  are isomorphic to either  $\text{PSL}_2(4)$  or  $\text{PSL}_2(8)$ . Then by Proposition 6.2,  $N_1/M \times N_2/M$  is also a normal minimal subgroup which has some class size  $l$  such that  $l$  has at least 3 different prime divisors. This gives us a contradiction since we are working with the group  $G$  whose each vanishing class size has at most two prime divisors. So  $N/M$  is the unique normal minimal subgroup which is non-abelian simple. Thus we have two cases for  $G/M$  as follows:

- $PSL_2(4) \leq G/M \leq Aut(PSL_2(4)) \cong S_5$
- $PSL_2(8) \leq G/M \leq Aut(PSL_2(8)) \cong Ree(3)$ .

*By GAP we see that  $G/M$  is isomorphic to  $PSL_2(4)$  or  $PSL_2(8)$ .*

## 7. ON ODD-SQUARE-FREE SIZES OF VANISHING CLASSES OF NON-ABELIAN SIMPLE GROUPS

There have been several studies over the sizes of  $\chi$ -vanishing classes of  $G$ . For example, Lewis and White have determined all odd-square-free groups that satisfy the condition of being an almost simple group in 2011. Also, in 2019 Miraali and Robati have characterized non-solvable groups each of whose character degrees has at most two prime divisors. Additionally, in 2020, Robati has showed that if each vanishing class size of a non-solvable group  $G$  has at most two prime divisors, then  $G/\text{Sol}(G)$  is isomorphic to  $PSL_2(4)$  or  $PSL_2(8)$ . In this part, we study on odd-square-free sizes of vanishing classes of a finite non-abelian simple groups.

*Theorem 7.1: Let  $S = A_n$ ,  $n \geq 6$ . Then there exists an irreducible character  $\chi$  of  $S$  which extends to  $\text{Aut}(S)$  and also a  $\chi$ -vanishing class  $x^S$  of size divisible by the square of an odd prime.*

*Proof :* First, we suppose that  $S = A_n$ , for even  $n \geq 10$ . Let  $\chi : S \rightarrow \mathbb{C}$  be a map such that for each  $x \in S$ ,  $\chi(x) = |\text{Fix}(x)| - 1$  where  $|\text{Fix}(x)|$  denotes the number of fixed points of  $x$ . By Theorem 5.2, we see that  $\theta$  is in  $\text{Irr}(S)$  and it is extendible to  $\text{Aut}(A_n)$ .

Now, let  $\alpha = (1, 2, \dots, n-5)(n-4, n-3)(n-2, n-1)(n)$ . Since  $\alpha$  is an even permutation,  $\alpha \in A_n$ . It can be easily seen that  $|\text{Fix}(\alpha)| = 1$  and  $\theta(\alpha) = 0$ . Hence we have that  $\theta(\alpha^S) = 0$ . Since  $\alpha$  commutes with an odd permutation,  $|\alpha^S| = |\alpha^{S_n}|$ .

$$|\alpha^{S_n}| = \frac{n \cdot (n-1) \dots 7 \cdot 6}{n-5} \cdot \frac{5 \cdot 4 \cdot 3 \cdot 2}{2} = \frac{n \cdot (n-1) \dots 7 \cdot 6}{n-5} \cdot 5 \cdot 3. \quad (7.1)$$

If  $n \equiv 0 \pmod{3}$ , then  $3 \mid n$  and hence we can see that  $3^2 \mid |\alpha^S|$ . If  $n \equiv 1 \pmod{3}$ , then  $3 \mid n-1$  and hence we see that  $3^2 \mid |\alpha^S|$ . If  $n \equiv 2 \pmod{3}$ , then  $3 \mid n-2$ , so we obtain that  $3^2 \mid |\alpha^S|$ .

Let  $S = A_n$  where  $n \geq 7$  is odd and set  $\beta = (1, 2, \dots, n-3)(n-2, n-1)(n)$ . Since  $\beta$  is an even permutation,  $\beta \in A_n$ . It is clear that  $|\text{Fix}(\beta)| = 1$  and  $\theta(\beta) = 0$ . Since  $\beta$  commutes with an odd permutation, we have  $|\beta^S| = |\beta^{S_n}|$ .

- If  $n = 7$ , then  $\beta = (1, 2, 3, 4)(5, 6)$  and

$$|\beta^S| = \frac{7 \cdot 6 \cdot 5 \cdot 4}{4} \cdot \frac{3 \cdot 2}{2} = 7 \cdot 6 \cdot 5 \cdot 3. \quad (7.2)$$

- If  $n = 9$ , then  $\beta = (1, 2, 3, 4, 5, 6)(7, 8)$  and

$$|\beta^S| = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{6} \cdot \frac{3 \cdot 2}{2} = 9 \cdot 8 \cdot 7 \cdot 5 \cdot 4 \cdot 3. \quad (7.3)$$

If  $n \geq 11$  is odd then  $n - 3 \geq 8$  and hence  $n \equiv 0, 1$  or  $2 \pmod{3}$ . If  $n \equiv 0 \pmod{3}$  then  $3 \mid n$  and hence we can see that  $3^2 \mid |\beta^S|$ . If  $n \equiv 1 \pmod{3}$  then  $3 \mid n - 1$  and hence we see that  $3^2 \mid |\beta^S|$ . If  $n \equiv 2 \pmod{3}$  then  $3 \mid n - 2$  and so we see that  $3^2 \mid |\beta^S|$ .

It is easy to see that  $A_8$  satisfies the condition. For  $n = 6$ , there exists  $\chi_7 \in \text{Irr}(A_6)$  which extendible to  $\text{Aut}(A_6)$  and also it can be observed that  $4A$  is a  $\chi_7$ -vanishing class of  $A_6$  with  $|4A| = 90$ . It is clear that 90 is divisible by  $3^2$ .

In order to obtain some results on the vanishing classes of sporadic groups we need to mention one of the sporadic groups which is denoted by  $J_1$ . In [Conway et al., 1984], we can see that the sizes of the vanishing classes of  $J_1$  are not divisible by  $p^2$  where  $p$  is an odd prime. Considering this exceptional case, we have Theorem 7.2 as follows:

*Theorem 7.2: Let  $S$  be a sporadic simple group. All vanishing classes of  $S$  are odd-square-free if and only if  $S \cong J_1$ . The following groups have a vanishing class which is divisible by either  $3^2$  or  $5^2$ .*

Table 7.1: Some Sporadic groups and their odd-square-free vanishing classes.

| S        | Character   | $\alpha^S$ | $ \alpha^S $ is divisible by... |
|----------|-------------|------------|---------------------------------|
| $M_{12}$ | $\chi_7$    | 8B         | $3^2$                           |
| $M_{22}$ | $\chi_3$    | 5A         | $3^2$                           |
| $M_{24}$ | $\chi_5$    | 12B        | $3^2$                           |
| $J_2$    | $\chi_{10}$ | 10C        | $3^2$                           |
| $HS$     | $\chi_7$    | 20A        | $3^2$                           |
| $Ru$     | $\chi_7$    | 12A        | $5^2$                           |
| $Suz$    | $\chi_3$    | 10B        | $3^2$                           |
| $Co_3$   | $\chi_9$    | 11A        | $5^2$                           |
| $Co_1$   | $\chi_2$    | 23A        | $5^2$                           |
| $BM$     | $\chi_2$    | 70A        | $3^2$                           |

*Theorem 7.3: Let  $S$  be a non-abelian simple group. If each vanishing class size is odd-square-free, then  $S$  has an irreducible character of  $p$ -defect zero for each prime  $p$ .*

*Proof :* If not, by Corollary 6.1,  $p = 2$  or  $3$  and  $S$  is isomorphic with one of these groups:  $M_{12}$ ,  $M_{22}$ ,  $M_{24}$ ,  $HS$ ,  $J_2$ ,  $Suz$ ,  $Ru$ ,  $Co_1$ ,  $Co_3$ ,  $BM$  or  $A_n$  with  $n \geq 7$ . But as we saw, each of the groups contains a vanishing class whose size is divisible by  $3^2$  or  $5^2$ .

## 8. CONCLUSION

In this thesis we have studied on complex irreducible characters of finite groups and their zeros. Groups included in this thesis were non-abelian. As we expressed our goal in the introduction section, we focus on the sizes of vanishing classes of non-solvable finite groups. Since the vanishing classes of non-abelian finite simple are known we have studied on the sizes of these vanishing classes. Specially, we considered alternating groups  $A_n$ ,  $n \geq 6$ , Sporadic groups in Theorem 7.1 and 7.2. Also, we investigated the relation between the sizes of vanishing classes and the  $p$ -defect zeros of a finite non-abelian group and we stated a result about this in Theorem 7.3.

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