

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL

**SOUND RADIATION IN A COAXIAL DUCT WITH A STEP
DISCONTINUITY ON THE INNER WALL**



ADİLE ULUCA

A THESIS OF MASTER OF SCIENCE
DEPARTMENT OF MATHEMATICS

ADVISOR: ASSOC. PROF. DR. HÜLYA ÖZTÜRK

MARCH 2024

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL

**SOUND RADIATION IN A COAXIAL DUCT WITH A
STEP DISCONTINUITY ON THE INNER WALL**

ADİLE ULUCA

A THESIS OF MASTER OF SCIENCE
DEPARTMENT OF MATHEMATICS

ADVISOR: ASSOC. PROF. DR. HÜLYA ÖZTÜRK

MARCH 2024

T.C.
GEBZE TEKNİK ÜNİVERSİTESİ
LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

**İÇ DUVARINDA ADIM SÜREKSİZLİĞİNE SAHİP OLAN
BİR KOAKSİYEL DALGA KILAVUZUNDA SES
DALGASININ YAYILIMI**

ADİLE ULUCA

YÜKSEK LİSANS TEZİ
MATEMATİK ANABİLİM DALI

DANIŞMAN: DOÇ. DR. HÜLYA ÖZTÜRK

MART 2024

MASTER of SCIENCE JURY APPROVAL FORM

A thesis submitted by Adile, Uluca, defended on 01/03/2024 before the jury formed with the 12/02/2024 date and 2024/13 numbered decision of the GTU Graduate Administration Board, has been accepted as a MASTER of SCIENCE thesis in the Department of Mathematics.

JURY

MEMBER

(THESIS ADVISOR) : Doç. Dr. Hülya ÖZTÜRK

MEMBER

: Doç. Dr. Ayşe SÖNMEZ

MEMBER

: Dr. Öğr. Üyesi Sait ERKOVAN

APPROVAL

Gebze Technical University Graduate Administration Board
...../...../..... date and/..... numbered decision.

SIGNATURE/SEAL

ABSTRACT

In this thesis, the propagation of plane acoustic waves in a coaxial waveguide with a step discontinuity in the inner cylinder is analyzed in detail using the Wiener-Hopf technique. First of all, in the first part, considering the geometry of the problem, appropriate boundary, continuity, separation and radiation conditions are determined for the boundary value problem. Under these conditions, a direct Fourier transform is applied to the Helmholtz equation given by plane acoustic waves and the problem is reduced to the solution of a Wiener-Hopf equation. For the solution of this Wiener-Hopf equation, classical factorization and decomposition operations are applied and, taking into account the analytical continuation principle and Liouville's theorem, the solution is expressed in terms of an infinite system of linear algebraic equations with infinitely many unknown coefficients. This system was solved by a certain truncation number, the details of which will be given in the numerical results section, and the desired solution of the problem was obtained. In the second part, using the inverse Fourier transform, explicit expressions for the reflection and transmission coefficients are obtained according to Jordan's lemma and the Residue theorem. Finally, in the numerical results section, some numerical evaluations are made by means of graphs obtained using MATLAB program. The effect of the truncation number on the reflection and transmission coefficients is investigated and the ideal truncation number for the system solution is determined. In addition, graphs are presented revealing the effects of parameters such as the inner and outer radii of the waveguide and different frequency values on the reflection and transmission coefficients.

Keywords: Wiener-Hopf Technique, Boundary Value Problem, Acoustic Waves.

ÖZET

Bu tez çalışmasında, iç silindirin de adım süreksizliğine sahip bir koaksiyel dalga kılavuzunda düzlemsel akustik dalgaların yayılımının analizi Wiener-Hopf tekniğı kullanarak detaylı bir şekilde incelenmiştir. Öncelikli olarak birinci bölümde, problemin geometrisi göz önünde bulundurularak ilgili sınır değ er problemi için uygun sınır, süreklilik, ayırıt ve radyasyon koş ulları belirlenmiştir. Belirlenen bu koş ullar altında düzlemsel akustik dalgaların sağ ladiğı Helmholtz denklemin e doğ rudan Fourier dönüş ümü uygulanmış ve problem bir Wiener-Hopf denkleminin çöz ümüne indirgenmiştir. Söz konusu Wiener-Hopf denkleminin çöz ümü için klasik faktörizasyon ve dekompozisyon iş lemleri uygulanmış ve analitik devam ilkesi ve Liouville teoreminin de dikkate alınmasıyla bu iş lemler sonucunda çöz üm, sonsuz sayıda bilinmeyen katsayısı olan sonsuz sayıda lineer cebirsel denklem sistemi cinsinden ifade edilmiştir. Bu sistem, detayları nümerik sonuçlar bölümünde verilecek olan, belirli bir kesme sayısında kesilerek çöz ülmüş ve problemin istenilen çöz ümüne ulaşılmıştır. İkinci bölümde, ters Fourier dönüş ümü kullanarak yansıma ve iletim katsayılarının açık ifadeleri Jordan lemması ve Rezidü teoremi uyarınca elde edilmiştir. Son olarak nümerik sonuçlar bölümünde, MATLAB programı kullanılarak elde edilen grafikler aracılığı ile bazı sayısal değ erlendirmelerde bulunulmuştur. Kesme sayısının yansıma ve iletim katsayılarına etkisi araştırılarak sistem çöz ümü için ideal kesme sayısı değ eri belirlenmiştir. Ayrıca, dalga kılavuzunun iç ve dış yarıçapları ve farklı frekans değ erleri gibi parametrelerin yansıma ve iletim katsayıları üzerindeki etkilerini ortaya çıkaran grafikler sunulmuştur.

Anahtar Kelimeler: Wiener-Hopf Tekniğı, Sınır Değ er Problemi, Akustik Dalgalar.

ACKNOWLEDGEMENTS

First of all, I would like to express my sincere gratitude to my invaluable advisor, Assoc. Prof. Dr. Hülya Öztürk, for her patience and sincerity throughout this journey.

I extend my deepest appreciation to my thesis jury members, Assoc. Prof. Dr. Ayşe Sönmez and PhD Instructor Sait Erkovan, for generously dedicating their time and expertise to evaluate my work.

Special thanks to my friend Kumriye Yıldız, whose support was instrumental in the beginning of my master's degree, and to my cousin Dr. Boran Uluca Yazgı, whose unwavering support I have always felt.

I would also like to thank research assistant Fatih Yetgin and my friends Nur Şirin and Hatice Nur Kolcu for their invaluable assistance in writing my thesis.

Finally, my endless gratitude goes to my dear family, who never ceased to believe in and support me under all circumstances.

TABLE OF CONTENTS

	<u>Page</u>
ABSTRACT	v
ÖZET	vi
ACKNOWLEDGEMENT	vii
TABLE OF CONTENTS	viii
LIST OF SYMBOLS AND ABBREVIATIONS	ix
LIST OF FIGURES	x
1. INTRODUCTION	1
2. FORMULATION OF THE PROBLEM	3
2.1. Derivation of The Wiener-Hopf Equation	4
2.2. Solution of the Wiener-Hopf Equation	14
2.3. Determination of the Expansion Coefficients	16
2.4. Analysis of the Fields	18
2.5. Factorization of the $N(\alpha)$ Function	20
2.6. Numerical Results	23
3. CONCLUSIONS	30
REFERENCES	31
BIOGRAPHY	33
PUBLICATIONS WITHIN THE SCOPE OF THESIS STUDY	34

LIST OF SYMBOLS AND ABBREVIATIONS

GTU	:	Gebze Technical University
k	:	Wave number
ω	:	Angular frequency
f	:	Frequency
Hz	:	Hertz
N	:	Truncation number



LIST OF FIGURES

	<u>Page</u>
Figure 2.1: The geometry of the problem.	3
Figure 2.2: Complex α -plane.	6
Figure 2.3: Magnitude of fields for $f=500$ Hz.	24
Figure 2.4: Magnitude of fields for $f=2000$ Hz.	25
Figure 2.5: Magnitude of the reflection coefficient for different a/c values.	26
Figure 2.6: Magnitude of the reflection coefficient for different b/a values.	27
Figure 2.7: Magnitude of the transmission coefficient for different a/c values.	28
Figure 2.8: Magnitude of the transmission coefficient for different b/a values.	29

1. INTRODUCTION

Noise pollution has become an important problem worldwide. Studies for the solution of this problem are increasing gradually. Studies to prevent the transmission of sound waves in coaxial waveguides are one of them. In these studies, dissipative and reactive silencers are used.

Dissipative silencers are systems created by covering the inside of the channel with sound-absorbing material to reduce noise. When sound waves pass through the silencer, the absorbing material absorbs sound energy and converts it into a small amount of heat. Thus, it reduces the noise level. Reactive silencers are silencers in which the sound is damped with perforated ducts and expansion chambers [1]. The most commonly used reactive silencer elements are sudden expansion chambers ([2–7]). As evident, both types of silencers have their advantages. At times, a combination of these two types of silencers can also be employed. Such systems are analyzed using the transfer matrix approach [8].

The first work on the subject belongs to Lord Rayleigh on acoustic waveguides. Rayleigh studied infinitely long cylindrical and rectangular guides with rigid walls [9]. Levine and Schwinger were the first scientists to use the Wiener-Hopf technique. They analytically analyzed the reflection coefficients for a semi-infinite rigid circular duct [10]. Later, these studies were continued by some researchers [11],[12]. Researcher Weinstein used the Wiener-Hopf technique to analyze the reflection coefficients for higher order modes [13]. Infinite channels lined with absorbent material attracted the attention of many researchers in the mid-20th century because they reduce the propagation of sound. Morse [14], Nayfeh, Kaiser and Telionis [15] and Lapin [16] are some of these researchers. Rawlins worked on semi-infinite ducts. In these studies, after obtaining the Wiener-Hopf equation, he transformed it into a system of infinite linear algebraic equations [17].

In recent years, Ozturk has studied the radiation of acoustic waves from a circumferential slot on a circular waveguide using the Wiener-Hopf technique [18]. In their next study, Ozturk and Tiryakioglu used the Wiener-Hopf technique for radiation of sound from a coaxial duct formed by a semi-infinite rigid outer cylinder and infinite inner cylinder

having different linings [19]. Demir and Büyükaksoy used the Modified Wiener-Hopf technique for transmission of sound waves in a cylindrical duct with an acoustically lined muffler [20].

In this work, the effect of step discontinuity on sound propagation along a coaxial duct is analyzed. The problem will be reduced to the solution of a Wiener-Hopf equation by applying a direct Fourier transform to the equation of Helmholtz provided by the plane sound waves. This reduced solution will require the solution of an infinite dimensional system of algebraic linear equations with infinite unknowns. This system of equations will be solved numerically to find the reflection and transmission coefficients.



2. FORMULATION OF THE PROBLEM

Consider the propagation of a harmonic plane sound wave with time dependence $\exp(-i\omega t)$ in a coaxial cylindrical waveguide with a step discontinuity on the inner wall, shown in Figure 2.1. All surfaces are rigid.

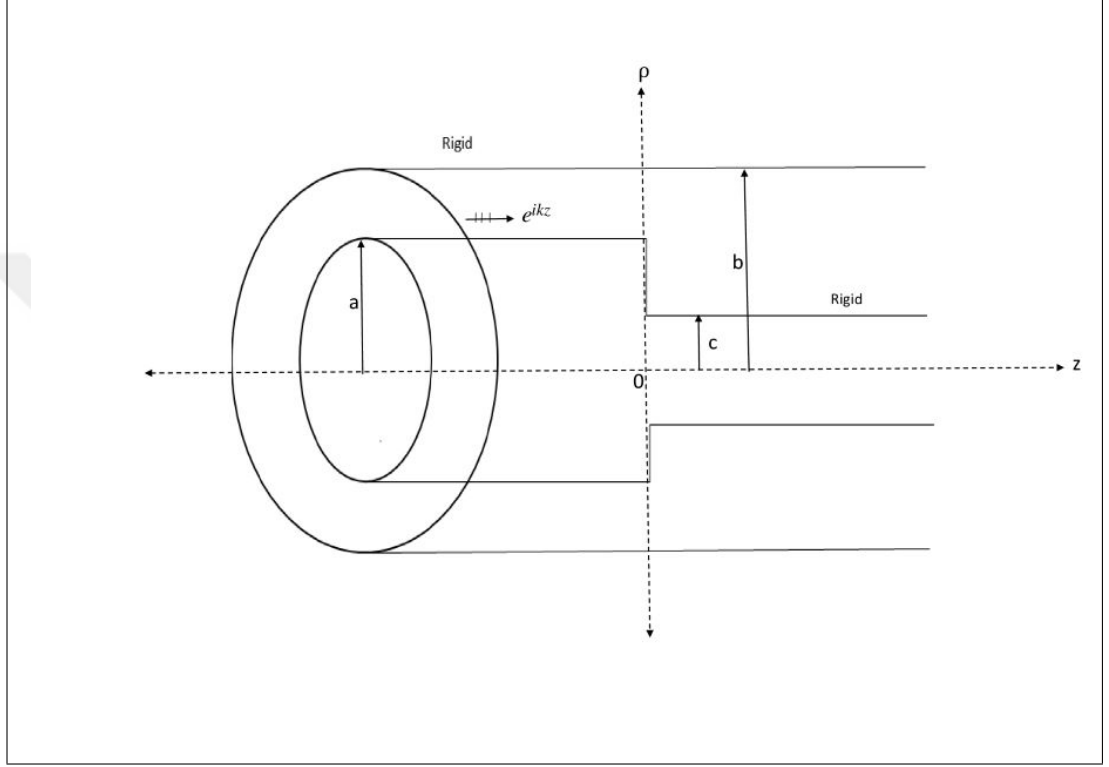


Figure 2.1: The geometry of the problem.

For analytical convenience's sake, the total field $u^T(\rho, z)$ can be expressed as

$$u^T(\rho, z) = \begin{cases} u_i(\rho, z) + u_1(\rho, z); & a < \rho < b, \quad z \in (-\infty, \infty), \\ u_2(\rho, z); & c < \rho < a, \quad z \in (0, \infty) \end{cases} \quad (2.1)$$

where the functions $u_1(\rho, z)$ and $u_2(\rho, z)$ denote the scattered fields in the specified regions. The incident field is taken to be

$$u_i(\rho, z) = e^{ikz}. \quad (2.2)$$

Boundary conditions, continuity conditions and edge conditions in their respective validity domains can be expressed as follows :

$$\frac{\partial}{\partial \rho} u_1(b, z) = 0, \quad z \in (-\infty, \infty) \quad (2.3)$$

$$\frac{\partial}{\partial \rho} u_1(a, z) = 0, \quad z \in (-\infty, 0) \quad (2.4)$$

$$\frac{\partial}{\partial \rho} u_2(c, z) = 0, \quad z \in (0, \infty) \quad (2.5)$$

$$\frac{\partial}{\partial z} u_1(\rho, 0) = 0, \quad c < \rho < a \quad (2.6)$$

$$\frac{\partial}{\partial \rho} u_1(a, z) = \frac{\partial}{\partial \rho} u_2(a, z), \quad z \in (0, \infty) \quad (2.7)$$

$$u_1(a, z) + u_i(a, z) = u_2(a, z), \quad z \in (0, \infty) \quad (2.8)$$

and

$$u^T(a, z) = O(z^{\frac{2}{3}}), \quad z \rightarrow 0 \quad (2.9)$$

$$\frac{\partial}{\partial \rho} u^T(a, z) = O(z^{-\frac{1}{3}}), \quad z \rightarrow 0 \quad (2.10)$$

2.1. Derivation of The Wiener-Hopf Equation

Sound waves satisfy the Helmholtz equation. The Helmholtz equation is expressed as

$$\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} + k^2 \right] u_j(\rho, z) = 0, \quad j = 1, 2 \quad (2.11)$$

in cylindrical coordinates. Due to the symmetry of the geometry of the problem we can write equation (2.11) independently of ϕ as follows :

$$\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial z^2} + k^2 \right] u_j(\rho, z) = 0, \quad j = 1, 2. \quad (2.12)$$

In the region $a < \rho < b$, the scattered field $u_1(\rho, z)$ satisfies the Helmholtz equation given by

$$\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial z^2} + k^2 \right] u_1(\rho, z) = 0. \quad (2.13)$$

By taking Fourier transform of (2.13) with respect to $z \in (-\infty, \infty)$, we obtain

$$\int_{-\infty}^{\infty} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u_1}{\partial \rho} \right) e^{i\alpha z} dz + \int_{-\infty}^{\infty} \frac{\partial^2 u_1}{\partial z^2} e^{i\alpha z} dz + \int_{-\infty}^{\infty} k^2 u_1 e^{i\alpha z} dz = 0. \quad (2.14)$$

Equation (2.14) has a solution under the following radiation and is unique :

$$u_1(\rho, z) = O(e^{\pm ikz}), \quad z \rightarrow \pm\infty \quad (2.15)$$

If $F(\rho, \alpha)$ is defined as :

$$F(\rho, \alpha) = \int_{-\infty}^{\infty} u_1(\rho, z) e^{i\alpha z} dz \quad (2.16)$$

as indicated in equation (2.14), the equation becomes :

$$\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + (k^2 - \alpha^2) \right] F(\rho, \alpha) = 0. \quad (2.17)$$

$F(\rho, \alpha)$ can also be written as

$$F(\rho, \alpha) = F_-(\rho, \alpha) + F_+(\rho, \alpha) \quad (2.18)$$

with $F_-(\rho, \alpha)$ and $F_+(\rho, \alpha)$ being defined by

$$F_{-}(\rho, \alpha) = \int_{-\infty}^0 u_1(\rho, z) e^{i\alpha z} dz \quad (2.19)$$

and

$$F_{+}(\rho, \alpha) = \int_0^{\infty} u_1(\rho, z) e^{i\alpha z} dz. \quad (2.20)$$

Note that the half range Fourier transforms, $F_{+}(\rho, \alpha)$ and $F_{-}(\rho, \alpha)$ are regular functions of α in the regions $Im\alpha > Im(-k)$ (upper half-plane) and $Im\alpha < Im(k)$ (lower half-plane) respectively. Solution of equation (2.17)

$$F(\rho, \alpha) = A(\alpha)J_0(K\rho) + B(\alpha)Y_0(K\rho), \quad (2.21)$$

with

$$K(\alpha) = \sqrt{k^2 - \alpha^2}. \quad (2.22)$$

In (2.21), $A(\alpha)$ and $B(\alpha)$ are spectral coefficients to be determined, while J_0 and Y_0 are Bessel functions. $K(\alpha)$ given by (2.22) is the square-root function which is defined in the complex α -plane cut as shown in Figure 2.2 such that $K(0) = k$.

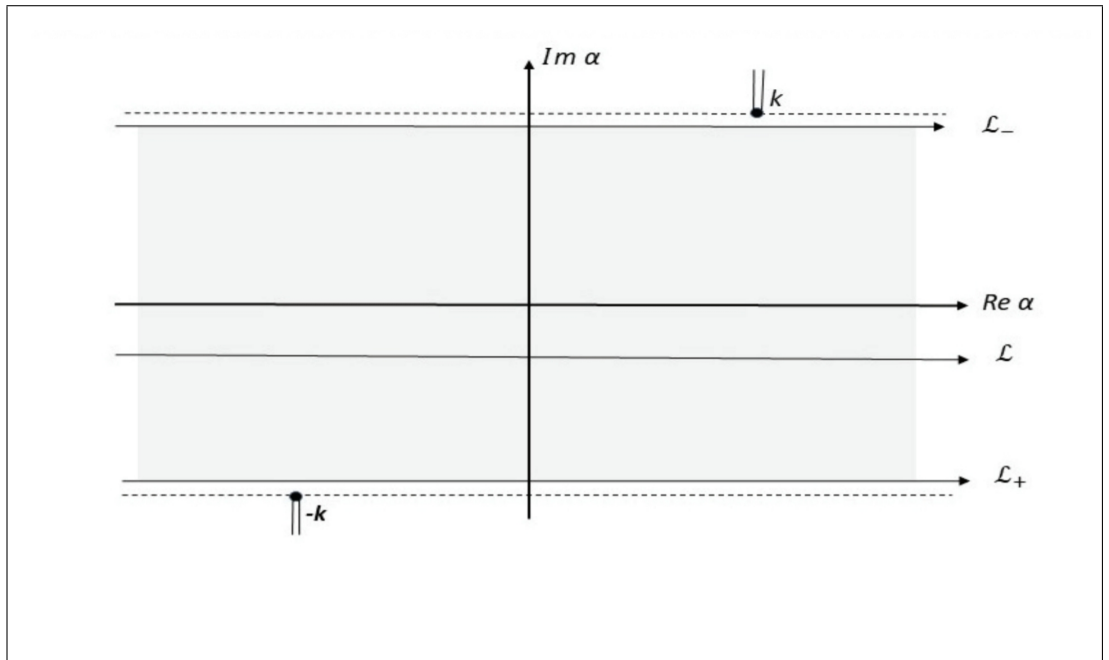


Figure 2.2: Complex α -plane.

Applying the Fourier transform of boundary conditions (2.3) and (2.4) yields

$$B(\alpha) = -A(\alpha) \frac{J_1(Kb)}{Y_1(Kb)} \quad (2.23)$$

and

$$F'_+(a, \alpha) = -A(\alpha)K J_1(Ka) - B(\alpha)K Y_1(Ka) \quad (2.24)$$

to give

$$A(\alpha) = F'_+(a, \alpha) \frac{Y_1(Kb)}{K[J_1(Kb)Y_1(Ka) - J_1(Ka)Y_1(Kb)]}. \quad (2.25)$$

In (2.24), the prime denotes the first-degree derivative with respect to ρ .

$$F(\rho, \alpha) = F'_+(a, \alpha) \frac{K[J_0(K\rho)Y_1(Kb) - J_1(Kb)Y_0(K\rho)]}{K^2 M(b, a, \alpha)} \quad (2.26)$$

is obtained by substituting the coefficients $A(\alpha)$ and $B(\alpha)$ in (2.21), where

$$M(\rho_1, \rho_2, \alpha) = J_1(K\rho_1)Y_1(K\rho_2) - J_1(K\rho_2)Y_1(K\rho_1). \quad (2.27)$$

On the other hand, the Helmholtz Equation's Fourier transform satisfied by $u_2(\rho, z)$ with respect to z , in the range of $z \in (0, \infty)$ gives right-sided non-homogeneous equation

$$\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + K^2 \right] G_+(\rho, \alpha) = -i\alpha f(\rho) \quad (2.28)$$

where $f(\rho)$ stands for

$$f(\rho) = u_2(\rho, 0) \quad (2.29)$$

and $G_+(\rho, \alpha)$ which is a regular function in the upper half complex α -plane, is defined by :

$$G_+(\rho, \alpha) = \int_0^\infty u_2(\rho, z) e^{i\alpha z} dz. \quad (2.30)$$

Non-homogeneous equation (2.28) has a homogeneous solution and a particular solution. We can write the solution of the non-homogeneous equation (2.28) as the sum of its homogeneous solution and its particular solution

$$G_+(\rho, \alpha) = G_+^h(\rho, \alpha) + G_+^p(\rho, \alpha). \quad (2.31)$$

First, let us apply the Green's function method for the particular solution. The Green's function related to (2.28) satisfies the equation :

$$\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + K^2(\alpha) \right] \mathcal{G}(\rho, t, \alpha) = \frac{1}{\rho} \delta(\rho - t), \quad \rho, t \in (c, a) \quad (2.32)$$

with the following conditions

$$\mathcal{G}(t+0, t, \alpha) = \mathcal{G}(t-0, t, \alpha), \quad (2.33)$$

$$\frac{\partial}{\partial \rho} \mathcal{G}(t+0, t, \alpha) - \frac{\partial}{\partial \rho} \mathcal{G}(t-0, t, \alpha) = \frac{1}{t}, \quad (2.34)$$

$$\frac{\partial}{\partial \rho} \mathcal{G}(c, t, \alpha) = 0, \quad (2.35)$$

and

$$\frac{\partial}{\partial \rho} \mathcal{G}(a, t, \alpha) = 0. \quad (2.36)$$

The general solution of this Green's function in (2.32) is in the form of

$$\mathcal{G}(\rho, t, \alpha) = \begin{cases} A_1(\alpha) J_0(K\rho) + B_1(\alpha) Y_0(K\rho), & c \leq \rho \leq t, \\ A_2(\alpha) J_0(K\rho) + B_2(\alpha) Y_0(K\rho), & t \leq \rho \leq a \end{cases} \quad (2.37)$$

where $A_1(\alpha)$, $A_2(\alpha)$, $B_1(\alpha)$ and $B_2(\alpha)$ are spectral coefficients. Considering the condition in (2.36)

$$B_2(\alpha) = -A_2(\alpha) \frac{J_1(Ka)}{Y_1(Ka)} \quad (2.38)$$

and considering the condition in (2.35)

$$B_1(\alpha) = -A_1(\alpha) \frac{J_1(Kc)}{Y_1(Kc)}. \quad (2.39)$$

Then, using equation (2.33),

$$A_2(\alpha) = A_1(\alpha) \left[\frac{J_0(Kt)Y_1(Kc) - J_1(Kc)Y_0(Kt)}{J_0(Kt)Y_1(Ka) - J_1(Ka)Y_0(Kt)} \right] \frac{Y_1(Ka)}{Y_1(Kc)} \quad (2.40)$$

is obtained. Finally, when condition (2.34) is applied,

$$A_1(\alpha) = \frac{\pi}{2} Y_1(Kc) \left[\frac{J_0(Kt)Y_1(Ka) - J_1(Ka)Y_0(Kt)}{J_1(Kc)Y_1(Ka) - J_1(Ka)Y_1(Kc)} \right] \quad (2.41)$$

is found. Hence, the solution of (2.32) under the conditions given above is

$$\mathcal{G}(\rho, t, \alpha) = \frac{1}{M(c, a, \alpha)} Q(\rho, t, \alpha), \quad (2.42)$$

where

$$Q(\rho, t, \alpha) = \frac{\pi}{2} \begin{cases} H(a, t, \alpha)H(c, \rho, \alpha), & c \leq \rho \leq t, \\ H(c, t, \alpha)H(a, \rho, \alpha), & t \leq \rho \leq a \end{cases} \quad (2.43)$$

and

$$H(\rho_1, \rho_2, \alpha) = J_0(K\rho_2)Y_1(K\rho_1) - J_1(K\rho_1)Y_0(K\rho_2). \quad (2.44)$$

The general solution of (2.28) can be written as

$$G_+(\rho, \alpha) = C(\alpha)J_0(K\rho) + D(\alpha)Y_0(K\rho) - \frac{i\alpha}{M(c, a, \alpha)} \int_c^a Q(t, \rho, \alpha) f(t) t dt \quad (2.45)$$

In (2.45), $C(\alpha)J_0(K\rho) + D(\alpha)Y_0(K\rho)$ comes from the homogeneous solution and $C(\alpha)$ and $D(\alpha)$ are spectral coefficients to be determined. If we consider the Fourier transform of boundary condition (2.5)

$$G'_+(c, \alpha) = 0 \quad (2.46)$$

and the derivation of (2.45) with respect to ρ yields

$$G'_+(\rho, \alpha) = -KC(\alpha)J_1(K\rho) - KD(\alpha)Y_1(K\rho) - \frac{i\alpha}{M(c, a, \alpha)} \int_c^a Q'(t, \rho, \alpha) f(t) t dt. \quad (2.47)$$

By setting $\rho = c$ in (2.47) and considering (2.46), it is obtained that

$$D(\alpha) = -C(\alpha) \frac{J_1(Kc)}{Y_1(Kc)}. \quad (2.48)$$

Applying the Fourier transform of continuity condition (2.7) yields

$$F'_+(a, \alpha) = G'_+(a, \alpha). \quad (2.49)$$

By substituting $\rho = a$ in (2.47) and considering (2.49), we obtain

$$C(\alpha) = K \frac{F'_+(a, \alpha) Y_1(Kc)}{K^2 M(c, a, \alpha)}. \quad (2.50)$$

If we write (2.48) and (2.50) in (2.45), the result is

$$G_+(\rho, \alpha) = \frac{1}{K^2 M(c, a, \alpha)} \left\{ K H(c, \rho, \alpha) F'_+(a, \alpha) - i\alpha K^2 \int_c^a Q(t, \rho, \alpha) f(t) t dt \right\}. \quad (2.51)$$

The left-hand side of (2.51) is regular function in the upper half plane then its right-hand side must also be a regular function. The regularity of the right-hand side of (2.51) may be violated by the presence of simple poles occurring at the zeros of $K^2 M(c, a, \alpha)$ lying in the upper half of the complex α -plane. Simple poles will be removed by making residues zero, namely at $\alpha_0 = k$ and $\alpha = \alpha_m$ ($m = 1, 2, \dots$) for

$$K^2(\alpha) M(c, a, \alpha) = (k^2 - \alpha^2) M(c, a, \alpha) = 0. \quad (2.52)$$

First, let's consider $\alpha = \alpha_m$ and mark it

$$K(\alpha_m) = K_m. \quad (2.53)$$

These poles can be removed by enforcing that their residues are zero, that is

$$K_m \cdot [J_0(K_m \rho) Y_1(K_m c) - J_1(K_m c) Y_0(K_m \rho)] F'_+(a, \alpha_m) - i\alpha_m K_m^2 \int_c^a Q(t, \rho, \alpha_m) f(t) t dt = 0 \quad (2.54)$$

and

$$F'_+(a, \alpha_m) = -\frac{2i\alpha_m}{\pi K_m a} \left[\frac{\int_c^a \frac{K_m \pi}{2} [J_0(K_m t) Y_1(K_m c) - J_1(K_m c) Y_0(K_m t)] f(t) t dt}{[J_0(K_m a) Y_1(K_m c) - J_1(K_m c) Y_0(K_m a)]} \right] \quad (2.55)$$

is obtained from equation (2.54). By substituting $\alpha_m = k$ in (2.55) obtains that

$$F'_+(a, +k) = -\frac{ik}{a} \int_c^a f(t) t dt. \quad (2.56)$$

Since $f(t)$ appearing in (2.55) is absolutely integrable function satisfying Dini conditions, it can be expanded into the Fourier-Bessel series as follows :

$$f(t) = \sum_{m=0}^{\infty} f_m \psi_m \quad (2.57)$$

with

$$\psi_m = \frac{K_m \pi}{2} H(c, t, \alpha_m) \quad (2.58)$$

where f_m and ψ_m are the normalized Bessel coefficients and the related set of orthogonal functions, respectively. This series expansion allows one to write

$$f_0 = \frac{1}{||\psi_0||^2} \int_c^a f(t) t dt \quad (2.59)$$

and

$$f_m = \frac{1}{||\psi_m||^2} \int_c^a \frac{K_m \pi}{2} H(c, t, \alpha_m) f(t) t dt, \quad (2.60)$$

use of the orthogonality integral given by :

$$||\psi_m||^2 = \int_c^a \frac{K_m^2 \pi^2}{4} [J_0(K_m t) Y_1(K_m c) - J_1(K_m c) Y_0(K_m t)]^2 t dt. \quad (2.61)$$

Equation (2.61) is evaluated as :

$$||\psi_m||^2 = \frac{J_1^2(K_m c) - J_1^2(K_m a)}{2J_1^2(K_m a)}. \quad (2.62)$$

When equation (2.62) is substituted in (2.60),

$$f_m = \frac{2J_1^2(K_m a)}{J_1^2(K_m c) - J_1^2(K_m a)} \int_c^a \frac{K_m \pi}{2} H(c, t, \alpha_m) f(t) t dt \quad (2.63)$$

is obtained. Also

$$f_0 = \frac{\int_c^a f(t)tdt}{\int_c^a tdt} = \frac{2}{a^2 - c^2} \int_c^a f(t)tdt \quad (2.64)$$

is found from (2.59). Then we substitute equation (2.63) into (2.55), we obtain

$$F'_+(a, \alpha_m) = -\frac{i\alpha_m}{\pi K_m a} \frac{[J_1^2(K_m c) - J_1^2(K_m a)]f_m}{J_1^2(K_m a)H(c, a, \alpha_m)} \quad (2.65)$$

and owing to (2.56) and (2.64) we write

$$F'_+(a, k) = -\frac{ik}{a} \left(\frac{a^2 - c^2}{2} \right) f_0. \quad (2.66)$$

As mentioned before, $f(t)$ can be represented as its Fourier-Bessel series

$$f(t) = f_0 + \sum_{m=1}^{\infty} \frac{K_m \pi}{2} [J_0(K_m t)Y_1(K_m c) - J_1(K_m c)Y_0(K_m t)]f_m. \quad (2.67)$$

Then, the Fourier transform of the continuity condition (2.8) allows to be write

$$F_+(a, \alpha) - \frac{1}{i(\alpha + k)} = G_+(a, \alpha) \quad (2.68)$$

If we consider (2.68) together with (2.18), (2.26) and (2.51) and write $\rho = a$, we obtain

$$\begin{aligned} & -\frac{2}{\pi a} \frac{M(c, b, \alpha)}{M(c, a, \alpha)M(b, a, \alpha)} \frac{F'_+(a, \alpha)}{K^2} - F_-(a, \alpha) - \frac{1}{i(\alpha + k)} \\ & = \frac{i\alpha}{a} \frac{1}{KM(c, a, \alpha)} \int_c^a [J_0(Kt)Y_1(Kc) - J_1(Kc)Y_0(Kt)]f(t)tdt. \end{aligned} \quad (2.69)$$

As a result, when the integrals of the products of the Bessel functions obtained after writing the series expansions of the $f(t)$ function are calculated, the following Wiener-Hopf equation, which is valid in the band $Im(-k) < Im(\alpha) < Im(k)$ is obtained:

$$\begin{aligned} \frac{2}{\pi a} \frac{1}{N(\alpha)} \frac{F'_+(a, \alpha)}{K^2} - F_-(a, \alpha) - \frac{1}{i(k + \alpha)} = -\frac{i\alpha}{(k^2 - \alpha^2)} f_0 \\ - \sum_{m=1}^{\infty} \frac{i\alpha}{a} \frac{1}{(\alpha^2 - \alpha_m^2)} \frac{J_1(K_m c)}{J_1(K_m a)} f_m \end{aligned} \quad (2.70)$$

where

$$\frac{1}{N(\alpha)} = -\frac{M(c, b, \alpha)}{M(c, a, \alpha)M(b, a, \alpha)}. \quad (2.71)$$

2.2. Solution of the Wiener-Hopf Equation

The equation of Wiener-Hopf will be solved by performing decomposition and factorization operations. To solve the equation of Wiener-Hopf we first factorize the kernel function $N(\alpha)$. This can be done by considering the methods given in [21] as $N(\alpha) = N_+(\alpha)N_-(\alpha)$. Here, $N_+(\alpha)$ and $N_-(\alpha) = N_+(-\alpha)$ are regular functions in the upper ($Im(\alpha) > Im(-k)$) and lower ($Im(\alpha) < Im(k)$) halves of the complex α -plane, respectively. Now, multiply all terms of Wiener-Hopf equation by $iN_-(\alpha)(k - \alpha)\sqrt{\frac{\pi a}{2}}$, we get

$$\begin{aligned} i\sqrt{\frac{2}{\pi a}} \frac{1}{N_+(\alpha)(k + \alpha)} F'_+(a, \alpha) - iN_-(\alpha)(k - \alpha)\sqrt{\frac{\pi a}{2}} F_-(a, \alpha) \\ - \frac{(k - \alpha)}{(k + \alpha)} N_-(\alpha)\sqrt{\frac{\pi a}{2}} = \frac{\alpha}{(k + \alpha)} N_-(\alpha)\sqrt{\frac{\pi a}{2}} f_0 \\ + \frac{\alpha}{a} N_-(\alpha)(k - \alpha)\sqrt{\frac{\pi a}{2}} \sum_{m=1}^{\infty} \frac{1}{(\alpha^2 - \alpha_m^2)} \frac{J_1(K_m c)}{J_1(K_m a)} f_m \end{aligned} \quad (2.72)$$

There are non-regular terms in the above equation. These terms will be written as the sum of functions that are regular on the upper and lower planes by applying decomposition. First, let's decompose the following term

$$g_1(\alpha) = -\frac{(k - \alpha)}{(k + \alpha)} N_-(\alpha)\sqrt{\frac{\pi a}{2}}. \quad (2.73)$$

This term has a simple pole at $\alpha = -k$. If we consider the residue for this pole, we find

$$g_1^+(\alpha) = \frac{1}{2\pi i} \int_{L_+} -\frac{(k - \tau)N_-(\tau)}{(k + \tau)(\tau - \alpha)} \sqrt{\frac{\pi a}{2}} d\tau = -\frac{N_+(k)2k}{(k + \alpha)} \sqrt{\frac{\pi a}{2}} \quad (2.74)$$

$$g_1^-(\alpha) = g_1(\alpha) - g_1^+(\alpha) \quad (2.75)$$

$$= -\sqrt{\frac{\pi a}{2}} \frac{1}{(k + \alpha)} [N_-(\alpha)(k - \alpha) - N_+(k)2k].$$

Now let's consider the function

$$g_2(\alpha) = \frac{\alpha}{(k + \alpha)} N_-(\alpha) \sqrt{\frac{\pi a}{2}} f_0. \quad (2.76)$$

There is a simple pole at $\alpha = -k$ in this function. As a result of the residue operation,

$$g_2^+(\alpha) = -\frac{N_+(k)k}{(k + \alpha)} \sqrt{\frac{\pi a}{2}} f_0 \quad (2.77)$$

$$g_2^-(\alpha) = \sqrt{\frac{\pi a}{2}} \frac{f_0}{(k + \alpha)} [N_-(\alpha)\alpha + N_+(k)k] \quad (2.78)$$

Finally, let's do decomposition for $\alpha = -\alpha_m$ singular points of the function

$$g_3(\alpha) = \frac{\alpha}{a} N_-(\alpha)(k - \alpha) \sqrt{\frac{\pi a}{2}} \sum_{m=1}^{\infty} \frac{1}{(\alpha^2 - \alpha_m^2)} \frac{J_1(K_m c)}{J_1(K_m a)} f_m. \quad (2.79)$$

We find

$$g_3^+(\alpha) = \sqrt{\frac{\pi a}{2}} \sum_{m=1}^{\infty} \frac{N_+(\alpha_m)(k + \alpha_m)}{2(\alpha_m + \alpha)} \frac{J_1(K_m c)}{a J_1(K_m a)} f_m \quad (2.80)$$

$$g_3^-(\alpha) = \sqrt{\frac{\pi a}{2}} \sum_{m=1}^{\infty} \frac{J_1(K_m c)}{a J_1(K_m a)} f_m \left[\frac{\alpha(k - \alpha)N_-(\alpha)}{(\alpha^2 - \alpha_m^2)} - \frac{(k + \alpha_m)N_+(\alpha_m)}{2(\alpha_m + \alpha)} \right]. \quad (2.81)$$

When we substitute the decomposition results we found into the Wiener-Hopf equation and arrange them, we obtain the equation below

$$\begin{aligned}
& i\sqrt{\frac{2}{\pi a}} \frac{1}{N_+(\alpha)(k+\alpha)} F'_+(a, \alpha) - \frac{N_+(k)2k}{(k+\alpha)} \sqrt{\frac{\pi a}{2}} + \frac{N_+(k)k}{(k+\alpha)} \sqrt{\frac{\pi a}{2}} f_0 \\
& - \sqrt{\frac{\pi a}{2}} \sum_{m=1}^{\infty} \frac{N_+(\alpha_m)(k+\alpha_m)}{2(\alpha_m+\alpha)} \frac{J_1(K_m c)}{a J_1(K_m a)} f_m = iN_-(\alpha)(k-\alpha) \sqrt{\frac{\pi a}{2}} F_-(a, \alpha) \\
& + \sqrt{\frac{\pi a}{2}} \frac{1}{(k+\alpha)} [N_-(\alpha)(k-\alpha) - N_+(k)2k] - \sqrt{\frac{\pi a}{2}} \frac{f_0}{(k+\alpha)} [N_-(\alpha)\alpha + N_+(k)k] \\
& - \sqrt{\frac{\pi a}{2}} \sum_{m=1}^{\infty} \frac{J_1(K_m c)}{a J_1(K_m a)} f_m \left[\frac{\alpha(k-\alpha)N_-(\alpha)}{(\alpha^2 - \alpha_m^2)} - \frac{(k+\alpha_m)N_+(\alpha_m)}{2(\alpha_m+\alpha)} \right]
\end{aligned} \tag{2.82}$$

Let B_+ and B_- represent the regions ($Im(\alpha) > Im(-k)$) and ($Im(\alpha) < Im(k)$), respectively. The left hand side of (2.82) is the sum of terms which are regular on region B_+ , while the right hand side is the sum of terms which are regular on region B_- . It is clear that this equality is valid on the intersection region B described by $B_+ \cap B_-$. Therefore, the right and left sides of the equation must be zero due to the principle of analytic continuation. By setting the right side equal to zero, we find the solution of the Wiener-Hopf equation as follows :

$$\begin{aligned}
F'_+(a, \alpha) = -\frac{i\pi a}{2} N_+(\alpha) & \left[2kN_+(k) - kf_0N_+(k) + \right. \\
& \left. \sum_{m=1}^{\infty} \frac{N_+(\alpha_m)(k+\alpha_m)(k+\alpha)}{2a(\alpha_m+\alpha)} \frac{J_1(K_m c)}{J_1(K_m a)} f_m \right].
\end{aligned} \tag{2.83}$$

2.3. Determination of the Expansion Coefficients

The function $F'_+(a, \alpha)$ obtained by solving the Wiener-Hopf equation has unknown terms f_0 and f_m . Let us now find these terms. If we first write the integral in equation (2.55), we get

$$\begin{aligned}
& \int_c^a \frac{K_m \pi}{2} [J_0(K_m t)Y_1(K_m c) - J_1(K_m c)Y_0(K_m t)] f(t) t dt \\
& = -\frac{\pi K_m a}{2i\alpha_m} [J_0(K_m a)Y_1(K_m c) - J_1(K_m c)Y_0(K_m a)] F'_+(a, \alpha_m).
\end{aligned} \tag{2.84}$$

When we substitute this equation in (2.84) into (2.63), we find

$$f_m = \frac{i\pi a K_m J_1^2(K_m a)}{\alpha_m [J_1^2(K_m c) - J_1^2(K_m a)]} [J_0(K_m a) Y_1(K_m c) - J_1(K_m c) Y_0(K_m a)] F'_+(a, \alpha_m). \quad (2.85)$$

Substituting $\alpha = \alpha_m$ in equation (2.83) gives

$$F'_+(a, \alpha_m) = i\pi a k N_+(\alpha_m) N_+(k) - \frac{i\pi a k}{2} N_+(\alpha_m) N_+(k) f_0 + \frac{i\pi}{4} N_+(\alpha_m) \sum_{n=1}^{\infty} \frac{N_+(\alpha_n)(k + \alpha_n)(k + \alpha_m)}{(\alpha_n + \alpha_m)} \frac{J_1(K_n c)}{J_1(K_n a)} f_n \quad (2.86)$$

If equation (2.86) is substituted into equation (2.85),

$$f_m = -\pi k a V(\alpha_m) N_+(\alpha_m) N_+(k) + \frac{\pi a k}{2} f_0 V(\alpha_m) N_+(\alpha_m) N_+(k) - \frac{\pi}{4} V(\alpha_m) N_+(\alpha_m) \sum_{n=1}^{\infty} \frac{N_+(\alpha_n)(k + \alpha_n)(k + \alpha_m)}{(\alpha_n + \alpha_m)} \frac{J_1(K_n c)}{J_1(K_n a)} f_n \quad (2.87)$$

where

$$V(\alpha_m) = \frac{\pi a K_m}{\alpha_m} \frac{J_1^2(K_m a)}{J_1^2(K_m c) - J_1^2(K_m a)} H(c, a, \alpha_m), \quad m = 1, 2, \dots \quad (2.88)$$

Now let us derive the integral from equation (2.56)

$$\int_c^a f(t) t dt = -\frac{a}{ik} F'_+(a, +k). \quad (2.89)$$

Substituting this equation in (2.64) gives

$$f_0 = -\frac{2}{a^2 - c^2} \frac{a}{ik} F'_+(a, +k). \quad (2.90)$$

Then, when we write $\alpha = k$ in (2.83) and substitute it in (2.90), we get

$$\left[1 + \frac{\pi a^2}{a^2 - c^2} N_+^2(k) \right] f_0 + \frac{\pi a}{a^2 - c^2} N_+(k) \sum_{m=1}^{\infty} S(\alpha_m) f_m = \frac{2\pi a^2}{a^2 - c^2} N_+^2(k) \quad (2.91)$$

where

$$S(\alpha_m) = N_+(\alpha_m) \frac{J_1(K_m c)}{J_1(K_m a)}. \quad (2.92)$$

Thus, the coefficients f_m and f_0 were found. It will also be discussed numerically in the next section.

2.4. Analysis of the Fields

The reflected field in the region $a < \rho < b$, $z < 0$, can be determined by solving the inverse Fourier transform $F_-(\rho, \alpha)$ and considering (2.18) and (2.26) we obtain

$$u_1(\rho, z) = \frac{1}{2\pi} \int_L K F'_+(a, \alpha) \frac{[J_0(K\rho)Y_1(Kb) - J_1(Kb)Y_0(K\rho)]}{K^2 M(b, a, \alpha)} e^{-i\alpha z} d\alpha - \frac{1}{2\pi} \int_L F_+(\rho, \alpha) e^{-i\alpha z} d\alpha \quad (2.93)$$

where L is a straight line parallel to the real α -axis, lying in the strip $Im(-k) < Im(\alpha) < Im(k)$. To apply Jordan's lemma due to the $e^{-i\alpha z}$ term, the L curve is closed from above. Since the $F_+(\rho, \alpha)$ will be regular in this closed region, its integral will be zero in accordance with the Residue theorem. The poles located in the upper half-plane are the point $\alpha = k$ from the K^2 function and the points $\alpha = \Upsilon_m$ given by

$$J_1(\chi_m b)Y_1(\chi_m a) - J_1(\chi_m a)Y_1(\chi_m b) = 0, \quad m = 1, 2, \dots \quad (2.94)$$

where

$$\Upsilon_m = \sqrt{k^2 - \chi_m^2}, \quad m = 1, 2, \dots \quad (2.95)$$

$$u_1(\rho, z) = \frac{\pi a^2}{2(a^2 - b^2)} N_+(k) \left[N_+(k)(2 - f_0) + \sum_{m=1}^{\infty} \frac{N_+(\alpha_m)}{a} \frac{J_1(K_m c)}{J_1(K_m a)} f_m \right] e^{-ikz} + \sum_{r=1}^{\infty} \frac{i F'_+(a, \Upsilon_r) [J_0(\chi_r \rho)Y_1(\chi_r b) - J_1(\chi_r b)Y_0(\chi_r \rho)] \chi_r}{\frac{d}{d\alpha} [(k^2 - \alpha^2) M(b, a, \alpha)]_{\alpha=\Upsilon_r}} e^{-i\Upsilon_r z} \quad (2.96)$$

is obtained from the residue contribution of these expressions. As a result, the reflection coefficient of the fundamental mode is as follows :

$$R_0 = \frac{\pi a^2}{2(a^2 - b^2)} N_+(k) \left[2N_+(k) - N_+(k)f_0 + \sum_{m=1}^{\infty} \frac{N_+(\alpha_m)}{a} \frac{J_1(K_m c)}{J_1(K_m a)} f_m \right]. \quad (2.97)$$

In the region $z > 0$ the inverse Fourier transform of $F_+(\rho, \alpha)$ gives the transmitted field, that is

$$u_1(\rho, z) = \frac{1}{2\pi} \int_L K F'_+(a, \alpha) \frac{[J_0(K\rho)Y_1(Kb) - J_1(Kb)Y_0(K\rho)]}{K^2 M(b, a, \alpha)} e^{-i\alpha z} d\alpha - \frac{1}{2\pi} \int_L F_-(\rho, \alpha) e^{-i\alpha z} d\alpha \quad (2.98)$$

By similar logic, if we close the curve L from below in a suitable way, the integral will be zero by the residue theorem since the function $F_-(\rho, \alpha)$ will be regular in this closed region. Also, substituting (2.83) into (2.98), $u_1(\rho, z)$ becomes as follows :

$$u_1(\rho, z) = \frac{1}{2\pi} \int_L \frac{i\pi a}{2} \frac{M(c, a, \alpha)}{N_-(\alpha)} \left[2kN_+(k) - kN_+(k)f_0 + \sum_{m=1}^{\infty} \frac{N_+(\alpha_m)(k + \alpha_m)(k + \alpha)}{2a(\alpha_m + \alpha)} \frac{J_1(K_m c)}{J_1(K_m a)} f_m \right] \times \left[\frac{K[J_0(K\rho)Y_1(Kb) - J_1(Kb)Y_0(K\rho)]}{K^2 M(c, b, \alpha)} \right] e^{-i\alpha z} d\alpha \quad (2.99)$$

The integral becomes equal to the sum of the residues related to the poles occurring at the simple zeros of $K^2 M(c, b, \alpha)$ lying in the lower half-plane. The zero of K^2 in the lower half-plane is $\alpha = -k$. Additionally, the function $[J_1(Kc)Y_1(Kb) - J_1(Kb)Y_1(Kc)]$ will be simple zeros $\alpha = -\mu_m$ points. Using the residue theorem,

$$\begin{aligned}
u_1(\rho, z) = & \sum_{r=1}^{\infty} \frac{[J_0(\lambda_r \rho)Y_1(\lambda_r b) - J_1(\lambda_r b)Y_0(\lambda_r \rho)]\lambda_r}{\frac{d}{d\alpha}[(k^2 - \alpha^2)M(c, b, \alpha)]_{\alpha=-\mu_r}} \\
& \times \frac{\pi a}{2} \frac{M(c, a, -\mu_r)}{N_+(\mu_r)} \left[2kN_+(k) - kN_+(k)f_0 \right. \\
& + \sum_{m=1}^{\infty} \frac{N_+(\alpha_m)(k + \alpha_m)(k - \mu_r)}{2a(\alpha_m - \mu_r)} \frac{J_1(K_m c)}{J_1(K_m a)} f_m \left. \right] e^{i\mu_r z} \\
& + \frac{c^2 - a^2}{2(b^2 - c^2)} (2 - f_0) e^{ikz}
\end{aligned} \tag{2.100}$$

is obtained. Consequently, the transmission coefficient of the fundamental mode is as follows :

$$T_0 = \frac{c^2 - a^2}{2(b^2 - c^2)} (2 - f_0). \tag{2.101}$$

2.5. Factorization of the $N(\alpha)$ Function

In this section, the factorization of the function

$$\begin{aligned}
N(\alpha) &= \frac{[J_1(Kc)Y_1(Ka) - J_1(Ka)Y_1(Kc)][J_1(Kb)Y_1(Ka) - J_1(Ka)Y_1(Kb)]}{[J_1(Kb)Y_1(Kc) - J_1(Kc)Y_1(Kb)]} \\
&= -\frac{M(c, a, \alpha)M(b, a, \alpha)}{M(c, b, \alpha)}
\end{aligned} \tag{2.102}$$

will be analyzed. $N(\alpha)$ has no branching singularity. It has zeros and poles. First, let's factorize for $M(c, a, \alpha) = M_1(\alpha) = M_1^+(\alpha)M_1^-(\alpha)$.

$$M_1^+(\alpha) = \sqrt{J_1(kc)Y_1(ka) - J_1(ka)Y_1(kc)} \exp(P(\alpha)) \prod_{m=1}^{\infty} \left(1 + \frac{\alpha}{\alpha_m} \right) e^{\frac{\alpha}{K_m}} \tag{2.103}$$

is written for

$$J_1(K_m c)Y_1(K_m a) - J_1(K_m a)Y_1(K_m c) = 0, \tag{2.104}$$

$$\alpha_m = \sqrt{k^2 - K_m^2}. \quad (2.105)$$

Here $P(\alpha)$ will be set to eliminate the asymptotic behavior of the infinite product. The asymptotic behavior of the Bessel functions for $|z| \rightarrow \infty$ is as follows :

$$J_n(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right) \quad (2.106)$$

$$Y_n(z) \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right) \quad (2.107)$$

Then the large zeros of the function (2.102) will have the behavior

$$\alpha_m = \sqrt{k^2 - \left(\frac{m\pi}{a-c}\right)^2} \sim m \frac{\pi i}{a-c} \quad (2.108)$$

since

$$\cos\left(Kc - \frac{3\pi}{4}\right) \sin\left(Ka - \frac{3\pi}{4}\right) - \cos\left(Ka - \frac{3\pi}{4}\right) \sin\left(Kc - \frac{3\pi}{4}\right) = \sin(K(a-c)). \quad (2.109)$$

Hence

$$K_m = \frac{im\pi}{a-c}. \quad (2.110)$$

The asymptotic behavior of the infinite product is obtained as

$$\prod_{m=1}^{\infty} \left(1 + \frac{\alpha}{\alpha_m}\right) e^{\frac{i\alpha(a-c)}{m\pi}} = \sqrt{\frac{i}{2\alpha(a-c)}} e^{-\frac{i\alpha(a-c)}{\pi}(1-C-\log\alpha+\log\pi-\log(a-c)+i\frac{\pi}{2})} \quad (2.111)$$

using the formula

$$\prod_{m=1}^{\infty} \left(1 + \frac{\alpha}{mc+a}\right) e^{-\frac{\alpha}{mc}} = \frac{\Gamma\left(\frac{a}{c}+1\right)}{\sqrt{2\pi}} \left(\frac{\alpha}{c}\right)^{-\frac{a}{c}} e^{\frac{\alpha}{c}(1-C-\log\frac{\alpha}{c})}. \quad (2.112)$$

Thus

$$M_1^+(\alpha) = \sqrt{J_1(kc)Y_1(ka) - J_1(ka)Y_1(kc)} e^{\frac{i\alpha(a-c)}{\pi}(1-C-\log\alpha+\log\pi-\log(a-c)+i\frac{\pi}{2})} \times \prod_{m=1}^{\infty} \left(1 + \frac{\alpha}{\alpha_m}\right) e^{\frac{i\alpha(a-c)}{m\pi}}. \quad (2.113)$$

Now let's factorize for $M(b, a, \alpha) = M_2(\alpha) = M_2^+(\alpha)M_2^-(\alpha)$ in a similar way.

$$M_2^+(\alpha) = \sqrt{J_1(kb)Y_1(ka) - J_1(ka)Y_1(kb)} e^{\frac{i\alpha(a-b)}{\pi}(1-C-\log\alpha+\log\pi-\log(a-b)+i\frac{\pi}{2})} \times \prod_{m=1}^{\infty} \left(1 + \frac{\alpha}{\Upsilon_m}\right) e^{\frac{i\alpha(a-b)}{m\pi}} \quad (2.114)$$

is written for

$$J_1(\chi_m b)Y_1(\chi_m a) - J_1(\chi_m a)Y_1(\chi_m b) = 0, \quad (2.115)$$

$$\Upsilon_m = \sqrt{k^2 - \chi_m^2}. \quad (2.116)$$

Finally, let's do factorization for $-M(c, b, \alpha) = M_3(\alpha) = M_3^+(\alpha)M_3^-(\alpha)$

$$M_3^+(\alpha) = \sqrt{J_1(kb)Y_1(kc) - J_1(kc)Y_1(kb)} e^{\frac{i\alpha(c-b)}{\pi}(1-C-\log\alpha+\log\pi-\log(c-b)+i\frac{\pi}{2})} \times \prod_{m=1}^{\infty} \left(1 + \frac{\alpha}{\mu_m}\right) e^{\frac{i\alpha(c-b)}{m\pi}} \quad (2.117)$$

is written for

$$J_1(\beta_m b)Y_1(\beta_m c) - J_1(\beta_m c)Y_1(\beta_m b) = 0, \quad (2.118)$$

$$\mu_m = \sqrt{k^2 - \beta_m^2}. \quad (2.119)$$

Moreover, considering the asymptotic behaviour of the Bessel functions,

$$N_{\pm}(\alpha) \sim (\pm\alpha)^{-\frac{1}{2}} \quad (2.120)$$

for $|\alpha| \rightarrow \infty$.

2.6. Numerical Results

In this section, numerical results of reflection and transmission coefficients for different problem parameters are obtained using MATLAB program. In these graphs, the amplitudes of the transmission and reflection coefficient are analysed for the fundamental mode. Firstly, the graphs of the change of reflection and transmission coefficient amplitudes according to the truncation number, N , are given. Figure 2.3 and Figure 2.4 show that there is no change in the coefficient amplitudes after $N \geq 5$. Considering this situation, $N = 40$ is taken in the graphs for the system created to find the coefficients.

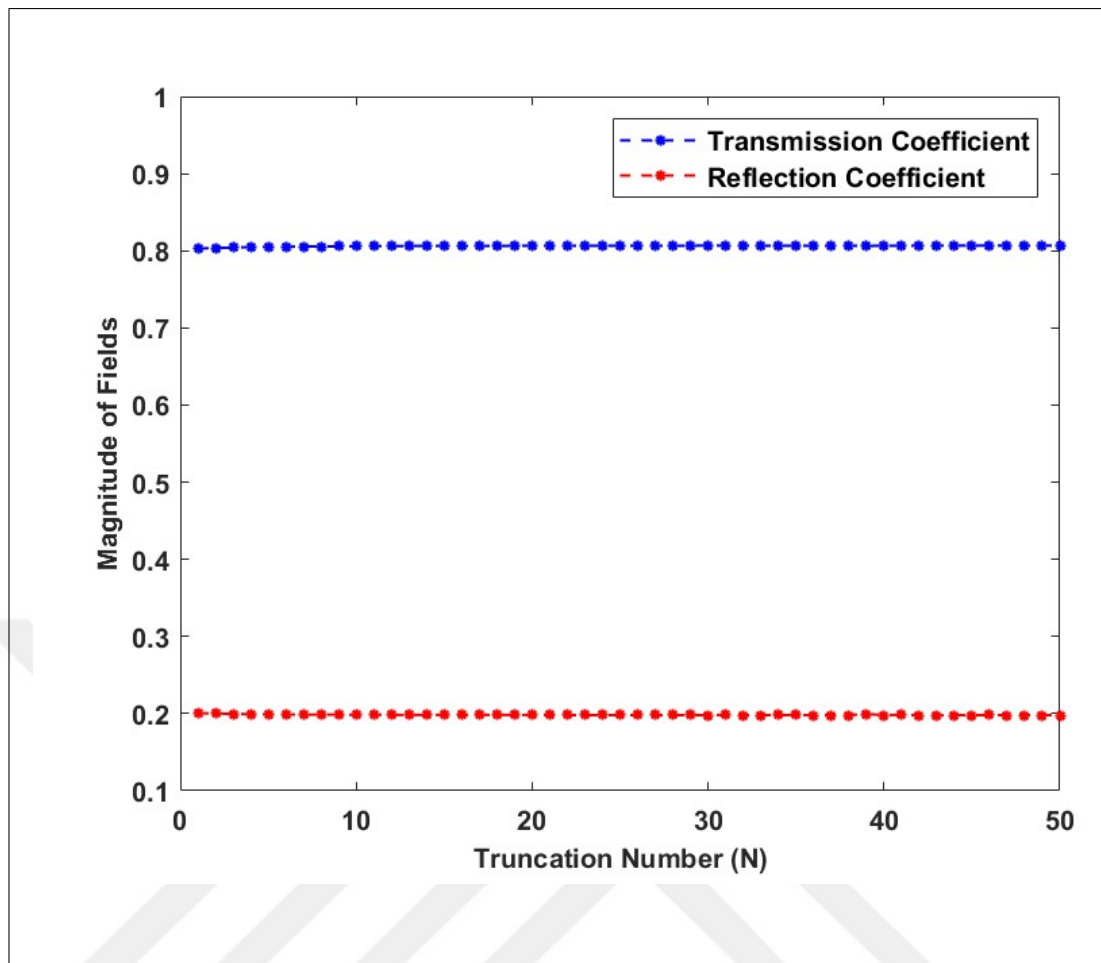


Figure 2.3: Magnitude of fields for $f=500$ Hz.

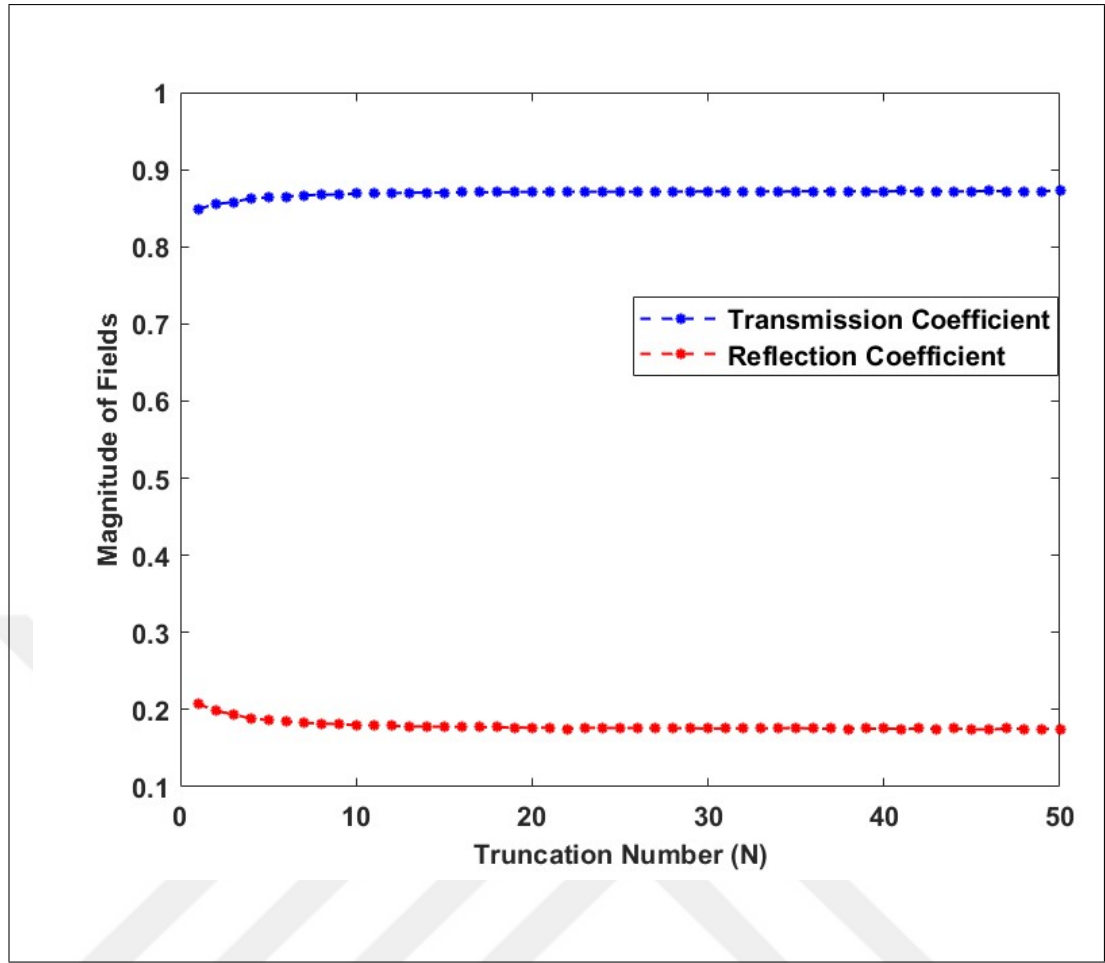


Figure 2.4: Magnitude of fields for $f=2000$ Hz.

Figure 2.5 and Figure 2.6 show the variation of the amplitude of the reflection coefficient with respect to frequency. The amplitude of the reflection coefficient decreased in both graphs as the frequency increased. Also, Figure 2.5 indicates that the reflection coefficient's amplitude is larger for increasing values of a/c . In Figure 2.6, the reflection coefficient's amplitude is smaller for increasing values of b/a .

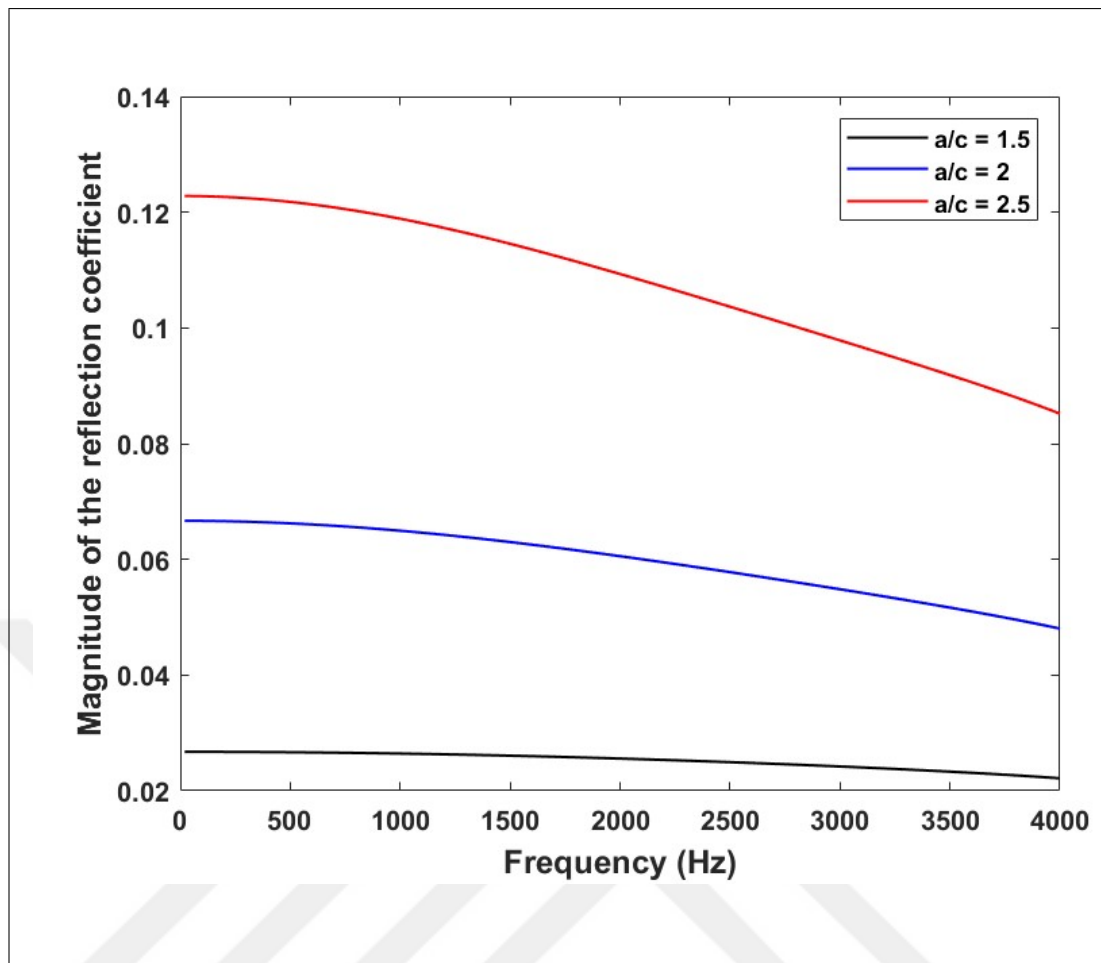


Figure 2.5: Magnitude of the reflection coefficient for different a/c values.

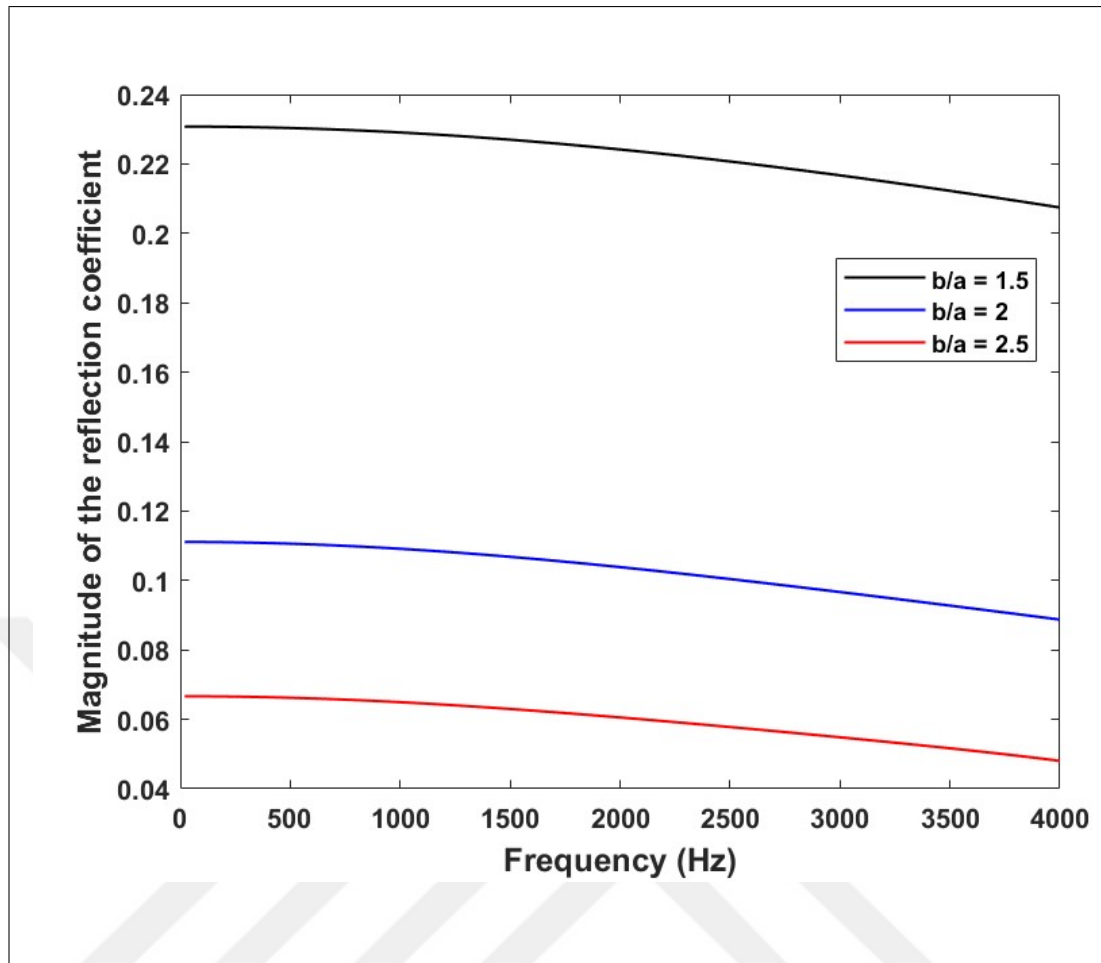


Figure 2.6: Magnitude of the reflection coefficient for different b/a values.

Figures 2.7 and 2.8 display how the transmission coefficient's amplitude varies with frequency. As can be seen in the graphs, as the frequency increases, the amplitude of the transmission coefficient also increases. Also, in Figure 2.7, the amplitude values of the transmission coefficient are smaller with increasing values of a/c . In Figure 2.8, the transmission coefficient's amplitude is larger for increasing values of b/a .

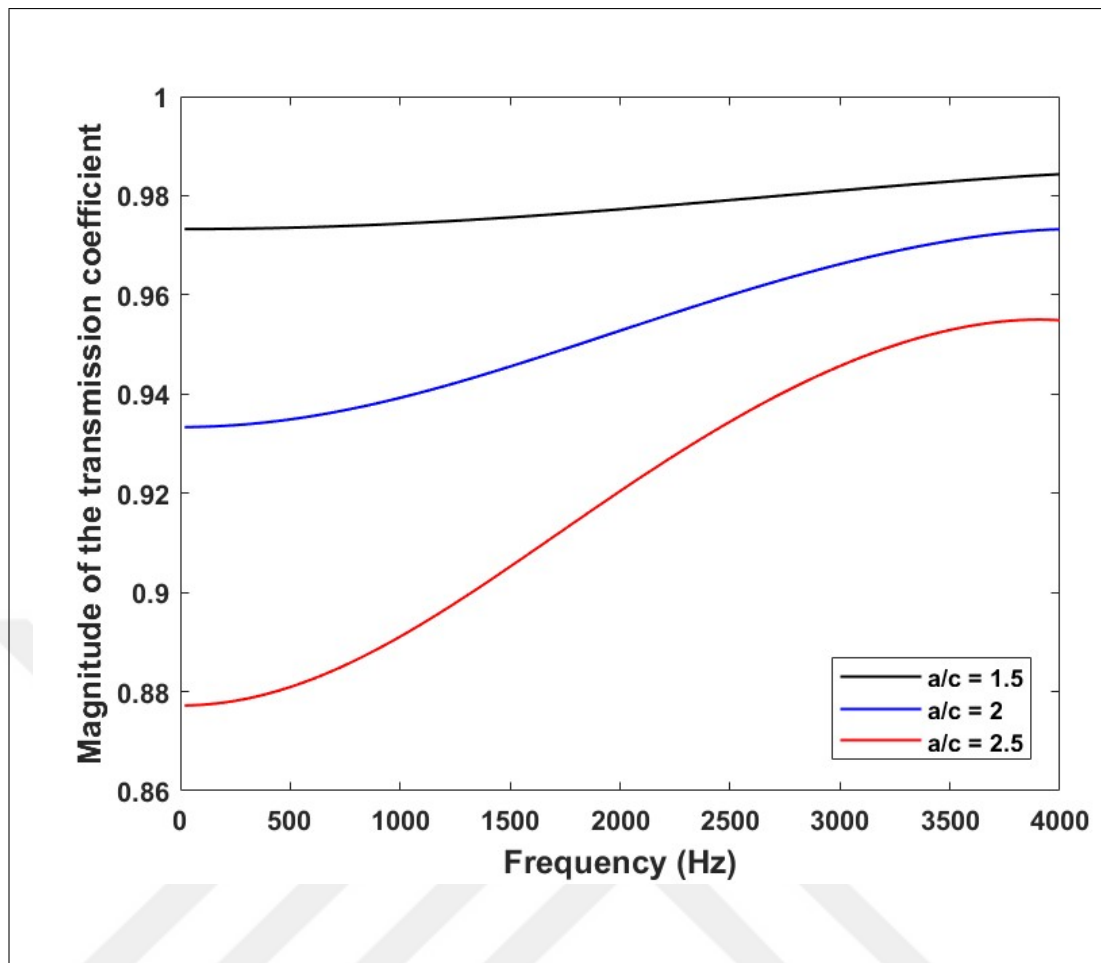


Figure 2.7: Magnitude of the transmission coefficient for different a/c values.

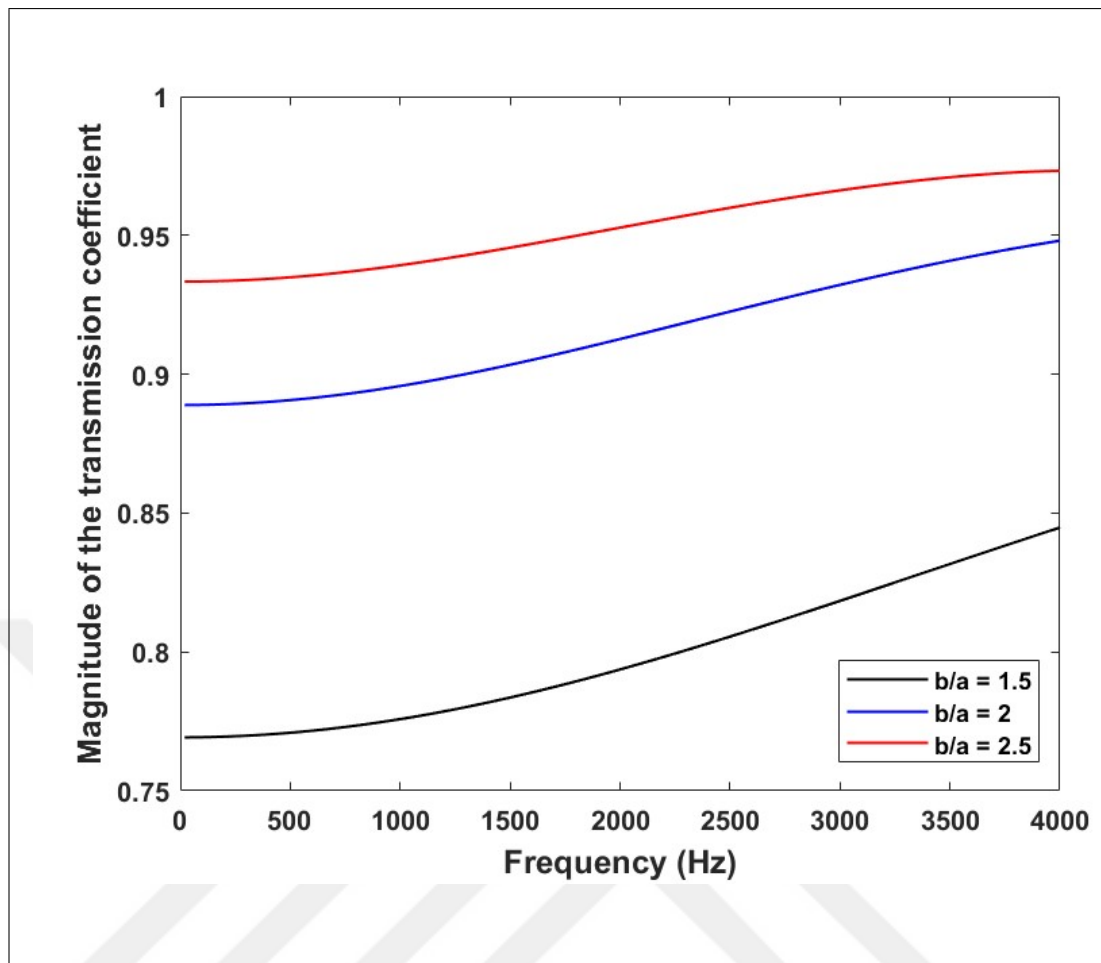


Figure 2.8: Magnitude of the transmission coefficient for different b/a values.

3. CONCLUSIONS

In this study, the propagation of plane acoustic waves in a coaxial waveguide with a step discontinuity on its inner cylinder is investigated by the Wiener-Hopf method. In the first part, the Wiener-Hopf equation is obtained by applying the direct Fourier transform to the Helmholtz equation provided by acoustic waves under boundary and continuity conditions. Decomposition and factorisation processes are then applied for this equation. As a result of these operations, the solution is an infinite number of linear solutions with an infinite number of unknown coefficients. This system is numerically calculated using MATLAB program and unknown coefficients are found. In the field analysis section, reflection and transmission coefficients are obtained using the inverse Fourier transform. In the numerical results section, the effect of frequency and radius on the transmission and reflection coefficients is directly observed with the graphs obtained.

REFERENCES

- [1] Peat K. S., (1991), "The acoustical impedance at the junction of an extended inlet or outlet duct", *Journal of Sound and Vibration*, 150, 101–110.
- [2] Munjal M. L., (1997), "Plane wave analysis of side inlet/outlet chamber mufflers with mean flow", *Journal of Applied Acoustics*, 52, 165–175.
- [3] Selamet A., Ji Z. L., (1999), "Acoustic attenuation performance of circular expansion chambers with extended inlet outlet", *Journal of Sound and Vibration*, 223, 197–212.
- [4] Selamet A., Ji Z. L., (2000), "Acoustic attenuation performance of circular expansion chambers with single-inlet and double-outlet", *Journal of Sound and Vibration*, 229, 3–19.
- [5] Auregan Y., Debray A., Strabinski R., (2001), "Low frequency sound propagation in a coaxial cylindrical duct: Application to sudden area expansions and to dissipative silencers", *Journal of Sound and Vibration*, 243, 461–473.
- [6] Boij S., Nilsson B., (2003), "Reflection of sound at area expansions in a flow duct", *Journal of Sound and Vibration*, 260, 477–498.
- [7] Rienstra S. W., Tester B. J., (2008), "An analytic Green's function for a lined circular duct containing uniform mean flow", *Journal of Sound and Vibration*, 317, 994–1016.
- [8] Beranek L. L., Vér I. L., (1992), "Noise and Vibration Control Engineering", John Wiley and Sons, New York, 289–395.
- [9] Rayleigh J. W. S., (1945), "The Theory of Sound", Dover, New York.
- [10] Levine H., Schwinger J., (1948), "On the radiation of sound from an unflanged circular pipe", *American Physical Society*, 73, 383–406.
- [11] Morse P. M., Feshbach H., (1953), "Methods of Theoretical Physics", McGraw Hill.
- [12] Noble B., (1958), "Methods based on The Wiener-Hopf Technique for the solution of partial differential equations", Pergamon Press Inc.
- [13] Weinstein L. A., (1969), "The Theory of Diffraction and The Factorization Method", The Golem Press, Boulder, Colorado.
- [14] Morse P. M., (1939), "The Transmission of Sound Inside Pipes", *The Journal of the Acoustical Society of America* 11, 205–210.
- [15] Neyfeh A. H., Kaiser J. E., Telionis D. P., (1975), "Acoustics of Aircraft Engine Duct Systems", *American Institute of Aeronautics and Astronautics* 13, 130–153.
- [16] Lapin A. D., (1975), "Sound Attenuation in Waveguides", *Soviet Physics Acoustics* 21, 215–222.

- [17] Rawlins A. D., (1978), “Radiation of sound from an unflanged rigid cylindrical duct with an acoustically absorbing internal surface”, *Proceedings of the Royal Society of London A*-361, 65–91.
- [18] Ozturk H., (2020), “Radiation of acoustic waves from a circumferential slot on a circular waveguide, accepted”, *TWMS Journal of Applied and Engineering Mathematics*, 10, 690–701.
- [19] Ozturk H., Tiryakioglu B., (2020), “Radiation of sound from a coaxial duct formed by a semi-infinite rigid outer cylinder and infinite inner cylinder having different linings”, *Archives of Acoustics*, 45, 655–662.
- [20] Demir A., Buyukaksoy A., (2003), “Transmission of Sound Waves in a Cylindrical Duct with an Acoustically Lined Muffler”, *International Journal of Engineering Science*, 41, 2411–2427.
- [21] Mittra R., Lee S. W., (1971), “Analytical Techniques in the Theory of Guided Waves”, The MacMillan Company.

BIOGRAPHY

Adile ULUCA completed her primary and secondary education in Istanbul. In 2015, she graduated from Gebze Technical University, Department of Mathematics. In 2021, she started her master's degree at Gebze Technical University Institute of Graduate School, Department of Mathematics.



PUBLICATIONS WITHIN THE SCOPE OF THESIS STUDY

[1] Uluca A. and Ozturk H., (2023) “Sound Wave Propagation Along a Coaxial Duct with a Step Discontinuity Via Wiener-Hopf Method”, International Symposium on Current Developments in Fundamental and Applied Mathematics Sciences, 2, 87.

