

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL

**DYNAMIC BEHAVIOR OF ROTOR SYSTEMS WITH
BELLEVILLE WASHER-INDUCED SUPPORT NONLINEARITY**

MURAT KUTLU

**A THESIS OF MASTER OF SCIENCE
DEPARTMENT OF MECHANICAL ENGINEERING**

ADVISOR: ASSIST. PROF. DR. SAEED LOTFAN

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GEBZE TEKNİK ÜNİVERSİTESİ
LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

BELLEVİLE RONDELALARININ NEDEN OLDUĞU
DOĞRUSAL OLMAYAN DESTEĞİN ROTOR
SİSTEMLERİNİN DİNAMİK DAVRANIŞINA ETKİLERİ

MURAT KUTLU

YÜKSEK LİSANS TEZİ
MAKİNE MÜHENDİSLİĞİ ANABİLİM DALI

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*To my Family,
my Mother, my Father and my Brothers*

ABSTRACT

Support nonlinearity is a significant and often unavoidable feature in rotating machinery, with direct implications for system performance, stability, and long-term reliability. Such nonlinearities frequently arise at mechanical joints, particularly at bolted connections where components interface with structural supports. This study focuses on the numerical investigation of a single-disk flexible rotor system with support nonlinearity introduced through Belleville washers—conical spring elements commonly used in bolted joints. These washers exhibit a highly nonlinear force–deflection relationship, making them a suitable candidate for representing geometric and contact-induced nonlinearities in rotor–support interactions.

This research aims to model and simulate the dynamic behavior of a rotor system with nonlinear support conditions, and to assess the impact of geometric variations in Belleville washers on the steady-state response of the system. The rotor configuration considered in this study consists of a rigid disk mounted on a flexible shaft modelled using Euler–Bernoulli beam theory. The model also adds the effects of rotary inertia, gyroscopic terms, and mass imbalance to reflect realistic operating conditions. The governing equations of motion are derived in weak form using Hamilton’s Principle, allowing for a consistent energy-based formulation of the problem.

To solve the system dynamics, the transverse (in-plane and out-of-plane) displacements are approximated using the Rayleigh–Ritz method, and the resulting weak-form equations are discretized accordingly. The Harmonic Balance Method (HBM) is then employed to obtain approximate steady-state periodic solutions of the nonlinear differential equations. Frequency response curves are generated across a range of rotor speeds to assess resonance behavior, softening or hardening effects, and nonlinear shifts in response characteristics. Additionally, time responses are obtained to observe the evolution of transverse deflections and orbital behavior under different support conditions.

The results demonstrate that the presence and magnitude of nonlinear stiffness at the support, primarily governed by the geometry of the Belleville washers, have a substantial effect on the dynamic behavior of the rotor system. These effects include amplitude-dependent resonance shifts, changes in orbit shape, and nonlinear modifications to the frequency response. The study highlights the importance of incorporating nonlinear support models into the design and analysis of rotor systems, particularly in applications where bolted joints and conical washers are prevalent.

Keywords: Rotor dynamics, Nonlinear support, Belleville washer, Harmonic Balance Method (HBM), Rayleigh–Ritz method, Weak-form formulation

ÖZET

Doğrusal olmayan desteklerin davranışı, döner makinelerde önemli ve çoğu zaman kaçınılmaz bir özelliktir; bu durum sistemin performansı, kararlılığı ve uzun vadeli güvenilirliği üzerinde doğrudan etkiler yaratır. Bu tür doğrusal olmayan özellikler genellikle mekanik bağlantı noktalarında, özellikle de bileşenlerin yapısal desteklerle birleştiği cıvatalı bağlantılarda ortaya çıkar. Bu çalışma, Belleville pulları aracılığıyla doğrusal olmayan desteklerin ve tek diskli esnek bir rotor sisteminin sayısal incelemesine odaklanmaktadır. Konik yay elemanları olan Belleville pulları, genellikle cıvatalı bağlantılarda kullanılır ve kuvvet–yer değiştirme ilişkileri yüksek derecede doğrusal olmayan bir yapıya sahiptir. Bu nedenle, rotor–destek etkileşimindeki geometrik ve temas kaynaklı doğrusal olmayan davranışları temsil etmek için uygun bir adaydırlar.

Bu araştırmanın amacı, doğrusal olmayan destek koşullarına sahip bir rotor sisteminin dinamik davranışını modellemek ve simüle etmek, ayrıca Belleville pullarındaki geometrik değişimlerin sistemin kararlı hâl yanıtı üzerindeki etkilerini değerlendirmektir. Çalışmada ele alınan rotor konfigürasyonu, esnek bir mil üzerine merkezî olarak yerleştirilmiş katı bir diskten oluşmaktadır ve mil, Euler–Bernoulli kiriş teorisi kullanılarak modellenmiştir. Model, döner atalet, jiroskopik etkiler ve kütle dengesizliğini hesaba katarak gerçekçi çalışma koşullarını yansıtmaktadır. Hareket denklemleri, Hamilton Prensibi kullanılarak zayıf formda türetilmiş ve bu sayede probleme enerji temelli tutarlı bir yaklaşım sağlanmıştır.

Sistem dinamiklerini çözmek amacıyla, enine (düzlem içi ve düzlem dışı) yer değiştirmeler Rayleigh–Ritz yöntemiyle yaklaşık olarak ifade edilmiş, elde edilen zayıf form denklemleri buna uygun şekilde ayrıştırılmıştır. Doğrusal olmayan diferansiyel denklemlerin yaklaşık kararlı hâl periyodik çözümleri ise Harmonik Denge Yöntemi ile elde edilmiştir. Farklı rotor devir sayıları aralığında frekans yanıt eğrileri elde edilerek rezonans davranışı, yumuşama veya sertleşme etkileri ve doğrusal olmayan yanıt kaymaları incelenmiştir. Buna ek olarak, farklı destek koşulları altında enine yer değiştirme ve yörünge davranışının zamana bağlı evrimi gözlemlenmiştir.

Elde edilen sonuçlar, destek bölgesindeki doğrusal olmayan katılığın varlığı ve büyüklüğünün—özellikle Belleville pullarının geometrisine bağlı olarak—rotor sisteminin dinamik davranışı üzerinde önemli bir etkiye sahip olduğunu göstermektedir. Bu etkiler arasında genliğe bağlı rezonans kaymaları, yörünge şekillerinde değişiklikler ve frekans yanıtında doğrusal olmayan modifikasyonlar yer almaktadır. Çalışma, özellikle cıvatalı bağlantıların ve konik pulların yaygın olarak kullanıldığı uygulamalarda, rotor sistemlerinin tasarım ve analizinde doğrusal olmayan destek modellerinin dikkate alınmasının önemini vurgulamaktadır.

Anahtar Kelimeler: Rotor dinamiği, Doğrusal olmayan destek, Belleville pulu, Harmonik Denge Yöntemi (HBM), Rayleigh–Ritz yöntemi, Zayıf formülasyon

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I extend my heartfelt thanks to my mother and father, whose endless sacrifices, resilience, and unconditional love have been the foundation of all my achievements. Their tireless efforts and unwavering belief in me have shaped every success I have attained. It is with immense pride and gratitude that I dedicate this thesis to them.

I am also deeply grateful to my brothers, who have always been my role models. Their continuous support and belief in me have been a source of great encouragement throughout my life.

This thesis is not only the result of my own efforts but also of the support, inspiration, and sacrifices of all these wonderful people. I dedicate this work to them with love and appreciation.

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LIST OF SYMBOLS AND ABBREVIATIONS

A	: Static mode shapes of free-free beam
A_s	: Cross-sectional area of shaft
B	: Static mode shapes of free-free beam
C	: Global damping matrix
C_p	: Proportional damping matrix
C_m	: Modal damping matrix
c_0	: Viscous Damping at support at $x=0$
c_L	: Viscous Damping at support at $x=L$
d_0	: Outer diameter of shaft
d_i	: Inner diameter of shaft
D	: Total dissipation energy of system
D_0	: Outer diameter of disk
D_w	: Outer diameter of conical washer
d_w	: Inner diameter of conical washer
E	: Elastic modulus of shaft and disk
E_w	: Elastic modulus of conical washer
e	: Eccentricity of unbalance mass from shaft axis.
F_{unb}	: Global unbalance force vector
F_{nl}	: Global nonlinear force vector
G	: Global Gyroscopic Matrix
H	: Total number of harmonics
I	: Second moment of area
I_{Dx}	: Polar mass moment of inertia of disk x-axis
I_{Dy}	: Mass moment of inertia of disk about y-axis
I_{Dz}	: Mass moment of inertia of disk about z-axis
K	: Global stiffness matrix
k_0	: Linear Stiffness of Support at $x=0$
k_L	: Linear Stiffness of Support at $x=L$
k_c	: Cubic stiffness
k_q	: Quadratic stiffness
L	: Length of shaft
M	: Global mass matrix
m_u	: Unbalance mass on disk
m_0	: Mass of support at $x=0$
m_L	: Mass of support at $x=L$
m_d	: Mass of the disk
N_w	: Number of conical washers in series
N	: Total number of mode shape functions
NL	: Nonlinear
p	: Time function in v-direction (y-axis) for static mode
q	: Time function in w-direction (z-axis) for static mode
$\underline{q}$	: Vector of generalized coordinates
s	: Thickness of conical washer
U	: Total potential energy of system
U_d	: Potential energy of disk
U_s	: Potential energy of shaft
U_p	: Potential energy of pedestals (supports)

U_w	: Potential energy of conical washer
t_D	: Thickness of the disk
T	: Total kinetic energy of system
T_d	: Kinetic Energy of Disk
T_s	: Kinetic Energy of Shaft
T_u	: Kinetic Energy of unbalance mass
T_p	: Kinetic Energy of pedestals (supports)
T_i^v	: i-th time function in v-direction (y-axis)
T_i^w	: i-th time function in w-direction (z-axis)
v	: Transverse Displacement at y-axis
w	: Transverse Displacement at z-axis
V_i	: i-th shape function of beam
W_i	: i-th shape function of beam
x_D	: Position of disk along x-axis
x_0	: Position along x-axis at $x=0$
x_L	: Position along x-axis at $x=L$
θ	: Rotational deformation around at y-axis
φ	: Rotational deformation around at z-axis
Ω	: Rotor Speed
ρ	: Density of shaft and disk material
δ	: Variational operator
$\hat{\delta}$	: Direct delta function
δ_0	: Free (uncompressed) height of conical washer
δ_z	: Total conical washer axial compression in z-direction
P	: Force acting by coned spring washer
μ	: Friction coefficient
ζ	: Damping ratio
FEM	: Finite Element Method
NOFRF	: Nonlinear Output Frequency Response Functions
HBM	: Harmonic Balance Method
NMPC	: Nonlinear Model Predictive Control

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1. INTRODUCTION

1.1. Background

Rotating machinery is one of the foundations of modern mechanical systems, with applications such as energy, aerospace, automotive, and manufacturing. The smooth operation of these systems is crucial to their safety, efficiency, and productivity. Amongst all the factors that affect the dynamic behavior of rotating machinery, support nonlinearity is one of the most critical one as it introduces substantial impact on both the rotating components as well as their foundations. Therefore, recognizing and solving this problem is crucial to stability and reliability of these rotating systems.

The nonlinearity in components is caused by their design and material properties in the system. These nonlinearities typically seen in components at, such as, bolted joint connection, bearings, and elastomeric supports. Conical washers which are also known as Belleville washers, for instance, are commonly used in bolted joints because of their spring-back capability which provides a better joint strength under varying conditions. On the other hand, force-deflection behavior of these washers is highly nonlinear which introduces complexities to the dynamics response of the rotor systems. This often requires an extensive analysis to predict and mitigate undesirable effects.

Identification of the effect of support nonlinearity is important in designing rotating machinery in order to operate wide range of speeds and load conditions. The nonlinear dynamic response of such systems usually comprises amplitude jumps, bifurcations, and whirl speed changes which are not seen in linear systems. Exploration of these phenomena can assist engineers in developing design concepts to mitigate adverse effects and improve system performance.

In summary, the dynamic behavior of rotating machinery with nonlinear supports is complex but critical area of research. The interaction between structural properties, nonlinear supports and rotor operating conditions of the rotor plays an important role for the system's performance and reliability. This study aims to extend the understanding of these interactions and provide insights that can be applied to the optimization of the design of rotating machinery.

1.2. Literature Review

Nonlinear supports in rotor dynamics are increasingly investigated due to their engineering relevance under real-world machinery applications. Although the linear models have been utilized to analyze rotor dynamics, current engineering demands clearly show the limitations of linear assumptions. The majority of rotating machinery often operates under severe or fluctuating conditions, which highlights nonlinear effects such as amplitude-dependent frequencies, subharmonic resonances, chaotic vibration, and bifurcations. Understanding all these phenomena makes the nonlinear support modeling essential for accurate prediction of rotor-bearing systems, thereby ensuring safety, efficiency, and reliability in rotating machinery operations.

Literature reports various nonlinear supports commonly observed in rotor dynamics, particularly pedestal looseness [1]–[5], clearance fits [2],[6],[12], rub-impact interactions [3],[4],[21]–[23], and support asymmetry or misalignment [17],[29]. Nonlinearities can also arise from inherent material properties, geometric irregularities, operational loads, fluid-structure interactions, and frictional contacts. Pedestal looseness typically results from assembly imperfections or long-term fatigue, causing occasional contact and changing system stiffness and damping characteristics. Clearance fits and rub-impact interactions introduce nonlinear stiffness changes and damping variations, significantly affecting rotor responses. Advanced nonlinear supports such as gas bearings [28], squeeze film dampers [4], and magnetic bearings [29] further highlight the complexity and diversity of nonlinear sources within rotor-support systems.

Researchers have studied several modeling methodologies in order to analyze rotor systems including nonlinear supports. One of the oldest methodologies is lumped-parameter models which were widely used in early studies due to their simplicity and capability for fast simulations of local nonlinearities like clearance and backlash effects [6],[8],[18]. However, the increasing complexity of real systems required more refined approaches. Finite Element Models (FEM) appeared as critical tools for detailed analyses to capture nonlinearities arises from such as rotor-casing interactions, spatial variations in material properties, and complex geometry. High-fidelity FEM studies by Ma et al. [10] and Li et al. [13] integrated numerical simulations with experimental validations, considerably improving the fidelity of simulations. Hybrid

methods, integrating FEM with nonlinear output frequency response functions (NOFRFs) and sensitivity-based model updating [23],[24], have also been developed to improve accuracy and optimal design decisions.

In addition to structural modeling methods, various nonlinear dynamic analysis techniques have become essential. Among these techniques, the Harmonic Balance Method (HBM) is remarkably effective due to its ability to solve the periodic nonlinear differential equations. Rezvani [24] presented HBM for flexible rotor-bearing systems with nonlinear supports, and it simplifies steady-state analysis by converting nonlinear differential equations into algebraic equations. It has been successfully applied to study a range of nonlinear behaviors, including those in hydrodynamic bearings [25], squeeze film dampers [4], and systems with multiple support clearances [16]. Extensions of HBM, such as multi-harmonic approaches, have enabled more detailed exploration of nonlinearities, including subharmonic and super-harmonic resonances [30]. Additional analysis techniques, such as time-domain numerical integration methods, have been commonly incorporated to address transient and strongly nonlinear behaviors, providing detailed evolution of dynamic responses [7],[9],[11]. Bifurcation analysis has been widely used to explore stability limits, identify critical thresholds, and detect transitions to chaos in rotor systems [21],[22]. The invariant manifold approach has provided systematic frameworks to reduce the dimensionality of complex rotor-bearing dynamics, providing efficient and precise analysis of nonlinear behaviors [31]. Furthermore, nonlinear model predictive control (NMPC) has recently appeared as a robust methodology for analyzing rotor dynamics by considering real-time system nonlinearities and optimizing control performance [32].

Another significant method is the arc-length continuation technique, which is particularly effective in tracking solution through the limit points and bifurcations in nonlinear systems. This method allows for the systematic solution of complex response surfaces, including unstable and multi-valued solutions, which are often inaccessible through traditional continuation methods. For instance, von Groll and Ewins [34] applied the arc-length continuation method in conjunction with the harmonic balance method to analyze rotor/stator contact problems, successfully capturing complex dynamic behaviors such as jump phenomena and multiple stable solutions. Similarly, Akay et al. [9] incorporated numerical continuation techniques to investigate the

responses of nonlinear rotor systems, presenting the method's capability to discover complex dynamic features like internal resonances and asynchronous responses.

On the other hand, the specialized application of Belleville washers (conical disc springs) as nonlinear support elements has attracted significant in recent years. These washers exhibit distinct nonlinear load-deflection characteristics, presenting variable stiffness responses to applied loads. This nonlinear property provides effective management of substantial loads within limited spaces, making Belleville washers especially suitable for rotor applications, mainly at the supports' bolted joint connections. They are commonly used for tasks such as bearing preload adjustment, thermal expansion compensation at the bolted joints, and vibration dampening by contributing extensively to improved stability, reliability, and performance of rotor-bearing systems [33].

Overall, the literature highlights a critical transition from traditional linear assumptions to advanced nonlinear analyses in rotor dynamics, highlight the need of accurately predicting and modeling real-world nonlinear behaviors. It becomes crucial for predicting nonlinear response of the system in order to ensure safe performance and robust design of rotating machinery.

1.3. Motivation and Scope

The study of nonlinear support behavior in rotor dynamics has gained significant attention in recent years due to its significant impact on system behavior. There are several studies exists on nonlinear rotor dynamics, with particular attention given to the support nonlinearity. In these studies, the nonlinear behavior of the loose pedestal support is often modeled using cubic or piecewise nonlinear stiffness approximations. While these methods provide valuable insights, they may not fully capture the complex stiffness characteristics of real-world problems, leaving room for further exploration and refinement.

This thesis focuses on investigating the dynamic characteristics of a flexible shaft–disk rotor system with nonlinear support behavior introduced by Belleville washers. Belleville washers, widely used and modeled as spring elements in various mechanical systems, especially in the structural bolted joints to minimize the clamp load loss due to thermal effects, and exhibit a polynomial nonlinear force-deflection characteristic

that significantly influences the overall behavior of the rotor system. The novelty of this research lies in the use of a third-order polynomial stiffness to model the nonlinear behavior of the pedestal due to Belleville washers attached to the bolted joints of pedestal. This approach improves the resolution of the stiffness representation, providing more accurate analysis of its impact on rotor dynamics.

1.4. Thesis Outline

In this regard, in chapter 2, analytical modelling of the rotor systems is presented to derive the kinetic and potential energies of the rotor system. The weak-form of the governing differential equation of motion is then derived using Hamilton's Variational Principle in [36], a powerful tool in analytical mechanics. This principle guarantee that the derived equations satisfy the energy conservation laws of the system, providing a solid foundation for further analysis. In chapter 3, the in-plane and out-of-plane transverse displacements of the rotor are then expanded using the Rayleigh–Ritz method similar to Sadeghi et al [37]. This method allows for an efficient approximation of the system's displacement fields by discretizing the weak form into a set of ordinary differential equations. The formulation and structure of the submatrices that constitute the global system matrix are then presented. Additionally, the procedure for assembling these submatrices into the global mass, damping, gyroscopic, and stiffness matrices is presented in detail. In chapter 4, a convergence study on the number of mode shapes is conducted by using a benchmark Example 5.8.1 from Friswell et al [38] in order to determine the appropriate number of mode shapes required for accurate discretization. Furthermore, the discretized mathematical model is validated by incorporating and solving the same benchmark and the calculated natural frequencies are compared with reference values. In chapter 5, the effects of various washer geometries which result in distinct nonlinear stiffness properties are studied together with cubic and quadratic nonlinearity. Comparative analyses are performed to evaluate impact of these nonlinearities on the frequency responses of both linear and nonlinear rotor systems over a range of operating speeds.

2. ANALYTICAL MODELLING OF FLEXIBLE ROTOR SYSTEMS

2.1. Introduction

In this chapter, the analytical modeling of a rotor system under consideration is developed and then equation of motion is derived in weak-form by using variational method.

2.2. Model Development

This study focuses on a rotor configuration composed of a flexible shaft coupled with a rigid disk. The shaft-disk unit is mounted on flexible support structures, also referred as pedestals, positioned at both ends of the shaft, as illustrated in Figure 2.1. The shaft has a circular hollow profile, defined by its length L , outer diameter d_o , and inner diameter d_i , and its dynamic behavior is described through its kinetic and strain energy contributions. Since the disk is treated as rigid, its influence is accounted for only via its kinetic energy. It is located along the shaft at a position $x=x_D$, with an outer diameter D_o . A mass imbalance, represented by m_u , is also introduced on the disk, placed eccentrically by a radial distance e from the shaft's centerline.

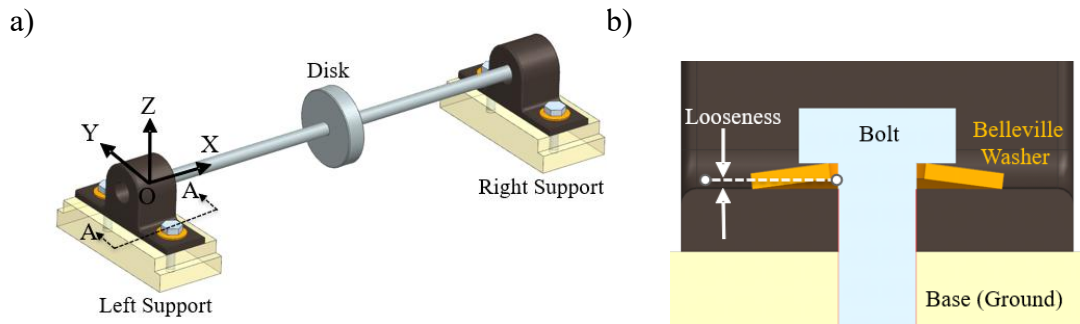


Figure 2.1: a) Single-disk rotor system on flexible support structures, b) Loose Bolt with Belleville washer at section A-A.

The schematic representation of the single-disc rotor system is also shown in Figure 2.2. The XYZ frame denotes the fixed reference frame, located at the left support. The variables $v(x,t)$ and $w(x,t)$ represent the transverse displacements of the rotor in the Z and Y directions, respectively. The rotational displacements about the Z and Y

axes are denoted by $\theta(x,t)$ and $\varphi(x,t)$, respectively. The symbol Ω represent the angular speed of the rotor.

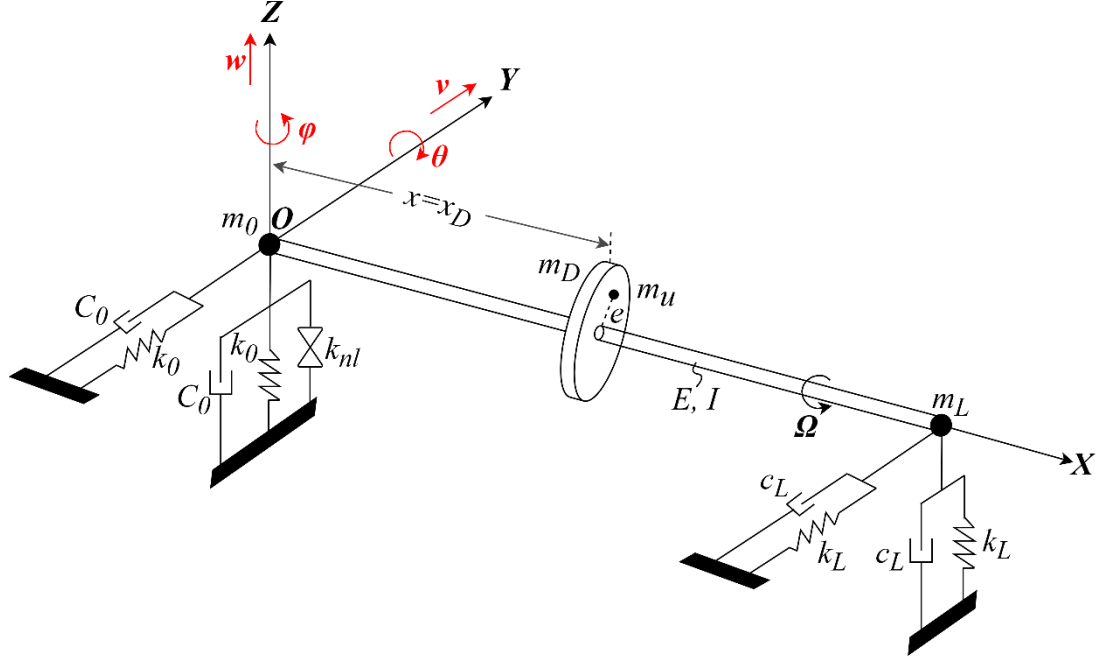


Figure 2.2: Schematic representation of single-disk rotor system.

2.2.1. Assumptions

In order to analyze the proposed rotor model within a clear framework, following assumptions has been made. Within the scope of the study, these assumptions have been carefully considered and they serve as the boundaries and context for the investigation.

1. As disk is assumed to be rigid, the potential energy U_d of the disk can be neglected (i.e., $U_d = 0$), and only the kinetic energy of disk T_d is considered.
2. The shaft is modelled as continuous system which has an infinite number of degrees of freedom. The vibration of such systems is governed by partial differential equations which involve variables that depend on time as well as the spatial coordinates.
 - a. Shaft is assumed to have a circular cross section and the Euler-Bernoulli beam theory is applied with added effects such as rotary inertia and gyroscopic effects.
 - b. The higher order large deformation of the shaft is ignored.

- c. The supports at both ends of the shaft are such that they allow the shaft to elongate, therefore there is no axial force acting dynamically to shaft.
- 3. Supports at the each end of the rotor are modelled as point masses m_0 at $x = x_0$ and m_L at $x = x_L$.
- 4. The left pedestal, located at $x = x_0$, is bolted to the ground using two bolts, one on each side. Each bolted joint incorporates a Belleville washer. The bolts at the left pedestal are assumed to be loose, with the looseness gap equal to the free height of the corresponding Belleville washers; thus, the looseness gap is fully filled by uncompressed washer. In contrast, the bolts at the right pedestal at $x = x_L$ are fully tightened, and no looseness or nonlinearity is introduced at that support. As a result, the nonlinear spring force generated by the Belleville washers acts only at the left pedestal and exclusively in the Z-direction.

2.2.2. Kinetic Energy of the System

The kinetic energy of the system is written for rigid disk, shaft and unbalance mass. Firstly, the kinetic energy of the rigid disk with mass center coinciding with the flexible rotor centerline is given by the expression below:

$$\begin{aligned}
 T_d = & \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{1}{2} m_D \left(\dot{v}(x,t)^2 + \dot{w}(x,t)^2 \right) \hat{\delta}(x - x_D) dx \\
 & + \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{1}{2} I_{Dy} \left(\dot{\theta}(x,t)^2 + \dot{\phi}(x,t)^2 \right) \hat{\delta}(x - x_D) dx \\
 & + \int_{-\frac{L}{2}}^{\frac{L}{2}} I_{Dx} \Omega \dot{\phi}(x,t) \theta(x,t) \hat{\delta}(x - x_D) dx
 \end{aligned} \tag{2.1}$$

where $\hat{\delta}$ is the direct delta function and $\dot{}$ is derivation with respect to time. Moreover, I_{Dx} is the mass (polar) moment of inertia and I_{Dy} shows the mass moment of inertia of disk about Y axis, which both can be calculated by:

$$I_{Dx} = \frac{1}{32} \rho \pi t_D (D_o^4 - d_o^4) \tag{2.2}$$

$$I_{Dy} = I_{Dz} = \frac{1}{64} \rho \pi t_D (D_o^4 - d_o^4) \tag{2.3}$$

The first two components in equation (2.1) are homogeneous quadratic functions of the generalized velocities, characteristic of a natural system. The first term corresponds to the translational kinetic energy influenced by the disk mass effect, while the second term represents a portion of the rotational kinetic energy arising from rotatory inertia. Both of these terms are inherently positive. In contrast, the third component is linear in the generalized velocities and is associated with the gyroscopic effect, which results from gyroscopic moments. Consequently, the kinetic energy formulation in equation (2.1) is classified as non-natural, highlighting the distinctive nature of rotor dynamics compared to conventional structural dynamics.

The second kinetic energy is of the shaft which is obtained by integrating the differential energy for an infinitesimal shaft element over the length of the shaft:

$$T_s = \frac{1}{2} \int_0^L \rho A_s (\dot{v}(x,t)^2 + \dot{w}(x,t)^2) dx + \frac{1}{2} \int_0^L \rho I (\dot{\theta}(x,t)^2 + \dot{\phi}(x,t)^2) dx + \int_0^L 2\rho I \Omega \dot{\phi}(x,t) \theta(x,t) dx \quad (2.4)$$

Similar to kinetic energy of the disk, the first term in (2.4) corresponds to the translational kinetic energy influenced by the shaft mass effect, while the second term represents a portion of the rotational kinetic energy arising from rotatory inertia. The third term represents the gyroscopic effect, which results from coriolis forces.

The unbalance mass will spin around a path as shaft and disk rotates and will generate a kinetic energy and this energy will excite the system. Thus, it needs to be considered in the system's kinetic energy formulation.

The kinetic energy of the spinning unbalance mass can be calculated by:

$$T_u = \int_0^L m_u \Omega e (\dot{v}(x,t) \cos \Omega t - \dot{w}(x,t) \sin \Omega t) \hat{\delta}(x - x_D) dx \quad (2.5)$$

Note that the kinetic energy of the unbalance mass is proportional to the rotational speed of the disk. As the rotational speed of the rotor increases, the kinetic energy due to unbalance mass also increases, thus the excitation force on the rotor also increases. Its impact on the dynamic response of rotor system is observed in chapter 5.

Even though, this study considers massless flexible supports (pedestals) in chapter 5, kinetic energy of the flexible supports, denoted by T_p respectively, is given below for completeness of the mathematical modelling.

$$T_p = \int_0^L \frac{1}{2} m_0 \left(\dot{v}(x,t)^2 + \dot{w}(x,t)^2 \right) \hat{\delta}(x-x_0) dx + \int_0^L \frac{1}{2} m_L \left(\dot{v}(x,t)^2 + \dot{w}(x,t)^2 \right) \hat{\delta}(x-x_L) dx \quad (2.6)$$

First term represents the kinetic energy of the left support at $x=0$, similarly second term represents the kinetic energy of the right support at $x=L$. Please note that, forces due the flexible supports only act to the rotor system at $x=0$ and $x=L$ thus these equations include the direct delta function.

2.2.3. Strain Energy of the System

The potential (strain) energy is written for shaft and flexible supports. Firstly, the potential (strain) energy of the rotating shaft element arises from elastic bending energy due to the bending moments: (w is in-plane and v is out-of-plane motion)

$$U_s = \frac{1}{2} \int_0^L EI \left(\frac{\partial^2 v(x,t)}{\partial x^2} \right)^2 dx + \frac{1}{2} \int_0^L EI \left(\frac{\partial^2 w(x,t)}{\partial x^2} \right)^2 dx \quad (2.7)$$

Secondly, the strain energy of flexible supports, denoted by U_p respectively, are calculated by:

$$U_p = \int_0^L \frac{1}{2} k_0 \left(v(x,t)^2 + w(x,t)^2 \right) \hat{\delta}(x-x_0) dx + \int_0^L \frac{1}{2} k_L \left(v(x,t)^2 + w(x,t)^2 \right) \hat{\delta}(x-x_L) dx \quad (2.8)$$

2.2.4. Strain Energy of Conical Spring (Belleville) Washer

In this particular scenario, the nonlinear conical spring washer, also known as a Belleville washer, as modeled in [34] is assumed to be attached to the left support bolted joints at $x=0$. As the bolt axis of the flexible supports is in Z-direction, the nonlinear force created by conical spring washer only applies in Z-direction.

The geometric characteristics of the conical spring washer are illustrated in Figure 2.3. In this figure, s is the thickness of Belleville washer, d_w is the inner diameter, D_w is the outer diameter, δ_0 is the free height, and P is the spring force of the Belleville washer.

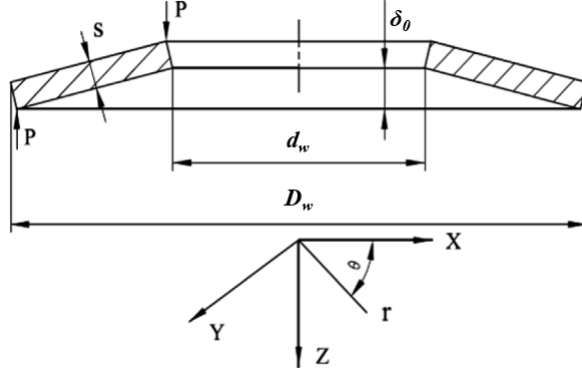


Figure 2.3: Analytical model of Belleville spring washer in [35].

The load/deflection relation of the single conical spring washer in [35] is given as:

$$P(\delta_z) = P_h \frac{\delta_z}{\delta_0} \left[\left(\frac{\delta_0}{s} \right)^2 \left(1 - \frac{\delta_z}{\delta_0} \right) \left(1 - \frac{\delta_z}{2\delta_0} \right) \alpha + 1 \right] \quad (2.9)$$

After reorganizing equation (2.9), 3rd order polynomial form may be expressed as:

$$P(\delta_z) = P_h (a_1 \delta_z + a_2 \delta_z^2 + a_3 \delta_z^3) \quad (2.10)$$

where the constants are as follows:

$$P_h = \frac{2\pi E_w s^3 \delta_0}{3(1-\mu^2) D_w^2} \left(\frac{C}{C-1} \right)^2 \ln(C) \quad (2.11)$$

$$a_1 = \frac{A_h + 1}{\delta_0} \quad (2.12)$$

$$a_2 = -\frac{3A_h}{2\delta_0^2} \quad (2.13)$$

$$a_3 = \frac{A_h}{2\delta_0^3} \quad (2.14)$$

$$A_h = \left(\frac{\delta_0}{s} \right)^2 \alpha \quad (2.15)$$

$$\alpha = \frac{6(1-\mu^2)}{(C-1)^2 \ln(C)} \left[\frac{1}{4} (C^2 - 1) - \frac{C^2}{C^2 - 1} \ln^2(C) \right] \quad (2.16)$$

$$C = \frac{D_w}{d_w} \quad (\text{diameter ratio}) \quad (2.17)$$

As seen in equation (2.10), the stiffness behavior of the conical washer is characterized by three polynomial coefficients, namely a_1 , a_2 and a_3 , which represent the linear, quadratic, and cubic stiffness contributions to the overall force–displacement relationship. Since the load/deformation relationship of the conical washer is known, the strain energy of the washer, denoted by U_w can be calculated as follow:

$$U_w = \int_0^{\delta_z} P(\delta_z) dz \quad (2.18)$$

In the proposed model, the total deformation of the conical spring washer is equal to the rotor deflection in Z-direction at $x = x_0$ as follows:

$$\delta_z = w(x, t) \hat{\delta}(x - x_0) \quad (2.19)$$

After inserting equation (2.19) into (2.18), U_w becomes;

$$U_w = \int_0^L P_h \left(\frac{a_1}{2} w(x, t)^2 + \frac{a_2}{3} w(x, t)^3 + \frac{a_3}{4} w(x, t)^4 \right) \hat{\delta}(x - x_0) dx \quad (2.20)$$

Equation (2.20) represents stain energy of the single conical spring washer. If there are multiple number of conical spring washer attached in series to the bolted joints, equation (2.20) shall be multiplied by number of conical springs in series. Note that the overall load-deflection behavior of stacked conical washers depend on the stacking arrangements. There are multiple stacking arrangements for conical spring washers, most common ones are series and nested arrangements. This study considers single conical spring washer at each bolted joint in interest unless otherwise specified.

2.2.5. Dissipation Energy of the System

In the current model, energy dissipation at the flexible supports is represented using a viscous damping approach. This means the resistive forces acting at the supports are assumed to depend linearly on the relative motion between the rotor and the pedestal. As a result, the energy lost due to damping is captured through a dissipation function, which quantifies the rate at which mechanical energy is converted into heat or other forms of energy loss through these velocity-dependent forces. Thus, the dissipation energy of the flexible supports may be expressed as:

$$D = \int_0^L \left[\frac{1}{2} c_0 (\dot{v}(x,t)^2 + \dot{w}(x,t)^2) + \frac{1}{2} c_L (\dot{v}(x,t)^2 + \dot{w}(x,t)^2) \right] \hat{\delta}(x - x_0) dx \quad (2.21)$$

2.2.6. Weak Form of Equation of Motion

Kinetic, strain and dissipation energies due to various sources on the rotor system have been derived for rotor-disk system in pervious sections. Total kinetic energy of the system can be written as summation of equations (2.1), (2.4), (2.5) and (2.6):

$$T = T_d + T_s + T_p + T_u \quad (2.22)$$

$$\begin{aligned} T = & \int_0^L \left(\frac{1}{2} m_D (\dot{v}(x,t)^2 + \dot{w}(x,t)^2) + \frac{1}{2} I_{Dy} (\dot{\theta}(x,t)^2 + \dot{\phi}(x,t)^2) \right) \hat{\delta}(x - x_D) dx \\ & + \left(\int_0^L \frac{1}{2} \rho A_s (\dot{v}(x,t)^2 + \dot{w}(x,t)^2) + \frac{1}{2} \rho I (\dot{\theta}(x,t)^2 + \dot{\phi}(x,t)^2) \right) dx \\ & + \int_0^L \frac{1}{2} m_0 (\dot{v}(x,t)^2 + \dot{w}(x,t)^2) \hat{\delta}(x - x_0) dx \\ & + \int_0^L \frac{1}{2} m_L (\dot{v}(x,t)^2 + \dot{w}(x,t)^2) \hat{\delta}(x - x_L) dx \\ & + \int_0^L m_u \Omega e (\dot{v}(x,t) \cos \Omega t - \dot{w}(x,t) \sin \Omega t) \hat{\delta}(x - x_D) dx \end{aligned} \quad (2.23)$$

Similarly, total potential energy of the rotor system can be written as a summation of equations (2.7), (2.8) and (2.20):

$$U = U_s + U_p + N_w U_w \quad (2.24)$$

Note that N_w represents the number of washers in series in the last term of equation (2.24). There are two bolted joints at the left pedestal and each bolted joint includes one conical spring washer. Therefore, there are two conical spring washers in the system that needs to be accounted for while calculating the total potential energy of contribution of the conical washers.

Total potential energy of the rotor system in equation (2.24) may be expressed in an open form as follow:

$$\begin{aligned}
U = & \frac{1}{2} \int_0^L EI \left(\frac{\partial^2 v(x,t)}{\partial x^2} \right)^2 dx + \frac{1}{2} \int_0^L EI \left(\frac{\partial^2 w(x,t)}{\partial x^2} \right)^2 dx \\
& + \int_0^L \frac{1}{2} k_0 (v(x,t)^2 + w(x,t)^2) \hat{\delta}(x-x_0) dx \\
& + \int_0^L \frac{1}{2} k_L (v(x,t)^2 + w(x,t)^2) \hat{\delta}(x-x_L) dx \\
& + N_w \times \int_0^L P_h \left(\frac{a_1}{2} w(x,t)^2 + \frac{a_2}{3} w(x,t)^3 + \frac{a_3}{4} w(x,t)^4 \right) \hat{\delta}(x-x_0) dx
\end{aligned} \tag{2.25}$$

There is only one dissipation energy source in the system so that the total dissipation energy of the system is equal to D given in equation (2.21).

In this study, the equation of motion for the rotor system is derived by applying the Extended Hamilton Principle, incorporating Rayleigh's dissipation function [36] as:

$$\delta A_{action} = - \int_{t_1}^{t_2} (\delta T - \delta U) dt + \int_{t_1}^{t_2} \frac{\partial D}{\partial \dot{q}_k} \delta q_k dt = 0, \quad \delta q_k = 0, \quad k = 1, 2, \dots, n; \quad t = t_1, t_2 \tag{2.26}$$

By substituting the expression for the total kinetic energy from equation (2.23) into equation (2.26), and applying the variational principle, the first term of the first integral becomes:

$$\int_{t_1}^{t_2} \delta T dt = \int_{t_1}^{t_2} \int_0^L \left\{ \begin{aligned} & m_D \dot{v} \delta \dot{v} \hat{\delta}(x-x_D) + m_D \dot{w} \delta \dot{w} \hat{\delta}(x-x_D) \\ & + I_{Dy} \dot{\theta} \delta \dot{\theta} \hat{\delta}(x-x_D) + I_{Dy} \dot{\phi} \delta \dot{\phi} \hat{\delta}(x-x_D) \\ & + I_{Dx} \Omega (\theta \delta \dot{\phi} + \dot{\phi} \delta \theta) \hat{\delta}(x-x_D) \\ & + \rho A_s (\dot{v} \delta \dot{v} + \dot{w} \delta \dot{w}) + \rho I (\dot{\theta} \delta \dot{\theta} + \dot{\phi} \delta \dot{\phi}) + 2 \rho I \Omega (\theta \delta \dot{\phi} + \dot{\phi} \delta \theta) \\ & + m_0 \dot{v} \delta \dot{v} \hat{\delta}(x-x_0) + m_0 \dot{w} \delta \dot{w} \hat{\delta}(x-x_0) \\ & + m_L \dot{v} \delta \dot{v} \hat{\delta}(x-x_L) + m_L \dot{w} \delta \dot{w} \hat{\delta}(x-x_L) \\ & + m_u \Omega e \{ \dot{v} \cos(\Omega t) - \dot{w} \sin(\Omega t) \} \hat{\delta}(x-x_D) \end{aligned} \right\} dx dt \tag{2.27}$$

Similarly, substituting the total potential energy from equation (2.25) into equation (2.26) and applying the variational principle yields the second term of first integral as:

$$\int_{t_1}^{t_2} \delta U dt = \int_{t_1}^{t_2} \int_0^L \left\{ \begin{aligned} & EI v_{,xx} \delta v_{,xx} + EI w_{,xx} \delta w_{,xx} \\ & + k_0 v \delta v \hat{\delta}(x-x_0) + k_0 w \delta w \hat{\delta}(x-x_0) \\ & + k_L v \delta v \hat{\delta}(x-x_L) + k_L w \delta w \hat{\delta}(x-x_L) \\ & + N_w \times P_h (a_1 w(x,t) + a_2 w(x,t)^2 + a_3 w(x,t)^3) \delta w \hat{\delta}(x-x_0) \end{aligned} \right\} dx dt \tag{2.28}$$

Finally, substituting the total dissipation energy from equation (2.21) into equation (2.26) and applying the variational principle yields the last integral as:

$$\int_{t_1}^{t_2} \frac{\partial D}{\partial \dot{q}_k} \delta q_k dt = \int_{t_1}^{t_2} \int_0^L \left(c_0 \dot{\delta} v \hat{\delta}(x-x_0) + c_0 \dot{\delta} w \hat{\delta}(x-x_0) + c_L \dot{\delta} v \hat{\delta}(x-x_L) + c_L \dot{\delta} w \hat{\delta}(x-x_L) \right) dx dt \quad (2.29)$$

By substituting equations (2.27), (2.28) and (2.29) into equation (2.26), the generalized form of the governing equations of motion for the system is obtained and can be expressed as follows:

$$\int_{t_1}^{t_2} \left(\begin{aligned} & - \left[\int_0^L \rho A_s \dot{v} dx + \int_0^L m_D \dot{v} \hat{\delta}(x-x_D) dx + \int_0^L m_u \Omega e \cos(\Omega t) \delta(x-x_D) dx \right. \\ & \quad \left. + \int_0^L m_0 \dot{v} \hat{\delta}(x-x_0) dx + \int_0^L m_L \dot{v} \hat{\delta}(x-x_L) dx \right] \delta \dot{v} \\ & - \left[\int_0^L \rho A_s \dot{w} dx + \int_0^L m_D \dot{w} \hat{\delta}(x-x_D) dx - \int_0^L m_u \Omega e \sin(\Omega t) \delta(x-x_D) dx \right. \\ & \quad \left. + \int_0^L m_0 \dot{w} \hat{\delta}(x-x_0) dx + \int_0^L m_L \dot{w} \hat{\delta}(x-x_L) dx \right] \delta \dot{w} \\ & - \left[\int_0^L I_{Dx} \Omega \dot{\phi} \hat{\delta}(x-x_D) dx + \int_0^L 2\rho I \Omega \dot{\phi} dx \right] \delta \dot{\theta} \\ & - \left[\int_0^L I_{Dy} \dot{\theta} \hat{\delta}(x-x_D) dx + \int_0^L \rho I \dot{\theta} dx \right] \delta \dot{\phi} \\ & - \left[\int_0^L \rho I \dot{\phi} dx + \int_0^L I_{Dy} \dot{\phi} \hat{\delta}(x-x_D) dx + \int_0^L I_{Dx} \Omega \theta \hat{\delta}(x-x_D) dx + \int_0^L 2\rho I \Omega \theta dx \right] \delta \dot{\phi} \\ & + \left[\int_0^L k_0 v \hat{\delta}(x-x_0) dx + \int_0^L c_0 \dot{v} \hat{\delta}(x-x_0) dx \right. \\ & \quad \left. + \int_0^L k_L v \hat{\delta}(x-x_L) dx + \int_0^L c_L \dot{v} \hat{\delta}(x-x_L) dx \right] \delta v \\ & + \left[\int_0^L k_0 w \hat{\delta}(x-x_0) dx + \int_0^L c_0 \dot{w} \hat{\delta}(x-x_0) dx \right. \\ & \quad \left. + \int_0^L k_L w \hat{\delta}(x-x_L) dx + \int_0^L c_L \dot{w} \hat{\delta}(x-x_L) dx \right. \\ & \quad \left. + \int_0^L N_w \times P_h \left(a_1 w(x,t) + a_2 w(x,t)^2 + a_3 w(x,t)^3 \right) \hat{\delta}(x-x_0) dx \right] \delta w \\ & + \int_0^L EI v_{,xx} dx \delta v_{,xx} + \int_0^L EI w_{,xx} dx \delta w_{,xx} \end{aligned} \right) dt = 0 \quad (2.30)$$

At this point, integration by parts can be systematically applied to the relevant terms within the generalized formulation of the governing equations of motion in equation (2.30), specifically those that contains the time derivatives acting on the variations of the generalized coordinates (i.e. $\delta\dot{q}_i$ and $\delta\ddot{q}_i$). This procedure assists to reduce the order of time derivatives acting on the variation terms, so that facilitating a more traceable numerical implementation. It is important to note that this operation is not performed on terms containing $\delta v_{,xx}$ and $\delta w_{,xx}$ as they do not involve time derivatives acting on the variations of the generalized coordinates.

An illustrative application of the integration by parts technique is provided below. In order to demonstrate this, consider the first term appearing in equation (2.30) below:

$$-\int_{t_2}^{t_1} \int_0^L \rho A_s \dot{v} \delta \dot{v} dx dt = -\int_0^L \left(\int_{t_1}^{t_2} \rho A_s \dot{v} \delta \dot{v} dt \right) dx \quad (2.31)$$

Note that $\int_0^L$ integral is taken outside as integration by parts is applied in time domain.

Setting integration by parts parameters as follows:

$$\begin{aligned} u = \rho A_s \dot{v} &\rightarrow du = \rho A_s \ddot{v} dt \\ dv = \delta \dot{v} dt &\rightarrow v = \delta v \end{aligned} \quad (2.32)$$

By inserting equation (2.32) into equation (2.31);

$$-\int_0^L \left(\int_{t_1}^{t_2} \rho A_s \dot{v} \delta \dot{v} dt \right) dx = -\int_0^L \left(\rho A_s \dot{v} \delta v \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \rho A_s \ddot{v} \delta v dt \right) dx \quad (2.33)$$

It is important to recognize that virtual displacements are arbitrary functions, provided that they satisfy the essential boundary conditions. Consequently, it is convenient to select virtual displacements $\delta q_i = 0$ that vanish at $t = t_1$ and $t = t_2$. Under this assumption, the first term on the right-hand side of equation (2.33) evaluates to zero. As a result, equation (2.33) effectively eliminates the time derivatives associated with the variation of the generalized coordinates. This simplification is useful in deriving a weak formulation that is well-suited for numerical implementation. The equivalence between the original and simplified expressions is demonstrated in equation (2.34) below.

$$-\int_0^L \left(\int_{t_1}^{t_2} \rho A_s \dot{v} \delta \dot{v} dt \right) dx = \int_0^L \int_{t_1}^{t_2} \rho A_s \ddot{v} \delta v dt dx \quad (2.34)$$

By systematically applying the integration by parts procedure to all terms involving time derivatives acting on the variations of the generalized coordinates, the governing equations of motion can be reformulated into the following variational representation:

$$\left(\begin{aligned} & \int_0^L \rho A_s \ddot{v} \delta v(x, t) dx + \int_0^L \rho A_s \ddot{w} \delta w(x, t) dx \\ & + \left(\int_0^L \rho I \ddot{\theta} \delta \theta(x, t) dx - \int_0^L 2 \rho I \Omega \dot{\phi} \delta \theta(x, t) dx \right) \\ & + \left(\int_0^L \rho I \ddot{\phi} \delta \phi(x, t) dx + \int_0^L 2 \rho I \Omega \dot{\theta} \delta \phi(x, t) dx \right) \\ & + (m_D \ddot{v}(x_D, t) - m_u \Omega^2 e \sin(\Omega t)) \delta v(x_D, t) \\ & + (m_D \ddot{w}(x_D, t) + m_u \Omega^2 e \cos(\Omega t)) \delta w(x_D, t) \\ & + (I_{Dy} \ddot{\theta}(x_D, t) - I_{Dx} \Omega \dot{\phi}(x_D, t)) \delta \theta(x_D, t) \\ & + (I_{Dy} \ddot{\phi}(x_D, t) + I_{Dx} \Omega \dot{\theta}(x_D, t)) \delta \phi(x_D, t) \\ & + (m_0 \ddot{v}(0, t) + c_0 \dot{v}(0, t) + k_0 v(0, t)) \delta v(0, t) \\ & + (m_L \ddot{v}(L, t) + c_L \dot{v}(L, t) + k_L v(L, t)) \delta v(L, t) \\ & + (m_0 \ddot{w}(0, t) + c_0 \dot{w}(0, t) + k_0 w(0, t)) \delta w(0, t) \\ & + (m_L \ddot{w}(L, t) + c_L \dot{w}(L, t) + k_L w(L, t)) \delta w(L, t) \\ & + [N_w \times P_h (a_1 w(0, t) + a_2 w(0, t)^2 + a_3 w(0, t)^3)] \delta w(0, t) \\ & + \int_0^L EI v_{,xx} \delta v_{,xx} dx + \int_0^L EI w_{,xx} \delta w_{,xx} dx \end{aligned} \right) dt = 0 \quad (2.35)$$

As seen in equation (2.35), the time derivatives acting on the variations of the generalized coordinates in equation (2.30) are eliminated. Equation (2.35) is called the **weak-form** of the governing equations of motion, which is significantly advantageous for numerical implementation, as it accommodates the incorporation of boundary conditions in a natural way and reduces the order of the derivatives required. Equation (2.35) serves as a main expression that is well-suited for subsequent discretization. In the following chapter, this formulation will be discretized using an appropriate numerical method, enabling the transformation of the continuous system into a finite-dimensional set of algebraic equations which are convenient for further computational analysis.

3. DISCRETIZATION OF THE GOVERNING DIFFERENTIAL EQUATION OF MOTION

In this chapter, the weak form of governing differential equation of motion derived in the previous section is discretized to enable numerical analysis. Through discretization process, continuous system is approximated by a finite set of algebraic equations which are more convenient for numerical analysis. This is achieved by applying Rayleigh-Ritz Method approximation to prepare the mathematical model for further numerical simulations and analyses.

3.1. Rayleigh–Ritz Approximation

The Rayleigh–Ritz method is a widely used and mathematically powerful approach utilized for the discretization of partial differential equations, particularly those governing the dynamic behavior of mechanical systems. In the context of the present study, this method is applied to discretize the equations of motion described in equation (2.35). The weak form of the governing differential equation of motions, which incorporates the essential boundary conditions and ensures variational consistency, is discretized in the next section.

In order to proceed with the discretization process, the transverse displacement are represented as a finite series of appropriately selected basis functions. These basis functions that are being derived in accordance with the Rayleigh–Ritz framework are selected to satisfy the geometric and physical boundary conditions of the problem. The trial (or approximating) functions that corresponds to the transverse displacements are expressed as:

$$v(x, t) = \sum_{i=1}^N V_i(x) T_i^v(t) + A(x)p(t) \quad (3.1)$$

$$w(x, t) = \sum_{i=1}^N W_i(x) T_i^w(t) + B(x)q(t) \quad (3.2)$$

In the above expressions, $V_i(x)$ and $W_i(x)$ denote the assumed functions, representing the mode shapes of a pinned–pinned beam defined over the domain $x = 0$ to $x = L$. The functions $T_i^v(t)$ and $T_i^w(t)$ are the corresponding time-dependent modal coordinates

associated with the transverse motion in the Y and Z directions, respectively. The terms $p(t)$ and $q(t)$ represent the generalized coordinates associated with static mode shapes $A(x)$ and $B(x)$, respectively. Lastly, N is the total number of model shape functions used. In this study, shape function for simply supported beam (pinned-pinned) is selected as follows:

$$V_i(x) = W_i(x) = \sin\left(\frac{i\pi x}{L}\right) \quad (3.3)$$

where i represents i -th mode shape functions used for transverse displacements.

In addition to that, $A(x)$ and $B(x)$ represent the static mode shapes, which describe the deformation profile of the beam in response to unit displacement and rotation applied at the free ends of the beam. These static deformations are required for capturing the boundary behavior of the structure together with simply supported beam mode shapes selected above. Static deflection of free-free beam under static loading is obtained through the solution of the following equations similar to [37]:

$$\frac{\partial^4 A(x)}{\partial x^4} = 0 \quad A(0) = 1 \quad A(L) = 1 \quad \frac{dA(0)}{dx} = 1 \quad \frac{dA(L)}{dx} = 1 \quad (3.4)$$

It is assumed that the general solution to the static differential equation takes the following form:

$$A(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \quad (3.5)$$

The static mode shape $A(x)$ is then determined by imposing the boundary conditions specified in equation (3.4) into the general solution given in equation (3.5). This procedure yields explicit expressions for the static deflection, which can be written as:

$$A(x) = \frac{2}{L^2} x^3 - \frac{3}{L} x^2 + x + 1 \quad (3.6)$$

It is crucial to emphasize that the inclusion of the static mode shape terms in equations (3.1) and (3.2) are essential for an accurate and physically consistent representation of the rotor's boundary conditions within the scope of this study. These terms ensure that equilibrium requirements at the boundaries are adequately satisfied. Moreover, due to the symmetric nature of the boundary conditions in the case of a free-free beam, it follows that $A(x) = B(x)$.

Similar to the representation of transverse displacements, the angular displacements are also approximated using a similar Rayleigh–Ritz-based expansion. These rotational degrees of freedom are represented as:

$$\varphi(x, t) = \frac{\partial v}{\partial x} \rightarrow \sum_{i=1}^N V_i'(x) T_i^v(t) + A'(x) p(t) \quad (3.7)$$

$$\theta(x, t) = -\frac{\partial w}{\partial x} \rightarrow -\left(\sum_{i=1}^N W_i'(x) T_i^w(t) + B'(x) q(t) \right) \quad (3.8)$$

In these equations $\varphi(x, t)$ and $\theta(x, t)$ correspond to the rotational displacements around the Z-axis and Y-axis, respectively. The notation ' indicates differentiation with respect to the spatial coordinate x , reflecting the spatial rate of change in the deformation fields. The use of derivative-based approximations for angular displacements is consistent with beam theory formulations, where rotations are related to the slope of the displacement field.

The Rayleigh–Ritz procedure applied herein offers a robust and systematic framework for approximating the displacement and rotation of the beam. It allows a consistent discretization of the system while preserving the essential physical characteristics imposed by boundary conditions and governing dynamics of the system.

3.2. Discretization of the Weak-Form

Following the derivation and representation of the displacements and rotations through appropriate trial functions based on the Rayleigh–Ritz method, the next step involves the discretization of the weak form of the governing equations. This process is a critical bridge between the analytical formulation of the physical system and its numerical implementation, promoting the construction of a system of algebraic equations that accurately captures the dynamics of the rotor system under investigation. By inserting approximated displacements and rotations in equations (3.1), (3.2), (3.7) and (3.8) into the weak form in equation (2.35), the matrix form equations of motion are obtained as:

$$[\mathbf{M}]\{\ddot{\mathbf{q}}\} + [\mathbf{C} + \mathbf{G}]\{\dot{\mathbf{q}}\} + [\mathbf{K}]\{\mathbf{q}\} + \{\mathbf{f}_{\text{unb}}\} + \{\mathbf{f}_{\text{nl}}\} = 0 \quad (3.9)$$

Here, $\mathbf{M}$, $\mathbf{C}$, $\mathbf{G}$ and $\mathbf{K}$ denote the global mass, damping, gyroscopic, and stiffness matrices, respectively; $\mathbf{q}$ represents the vector of generalized coordinates, $\mathbf{f}_{\text{unb}}$

denotes the global unbalance force vector, and $\mathbf{f}_{nl}$ corresponds to the global nonlinear force vector. Details of each matrices and vectors are presented below.

$$\{\underline{\mathbf{q}}\} = \begin{Bmatrix} \mathbf{T}_i^v \\ \mathbf{p} \\ \mathbf{T}_i^w \\ \mathbf{q} \end{Bmatrix} \quad i=1,2,\dots,N \quad (3.10)$$

where $\mathbf{T}^v = [T_1^v, T_2^v, \dots, T_N^v]^T$, $\mathbf{T}^w = [T_1^w, T_2^w, \dots, T_N^w]^T$, $\mathbf{p} = [p]$ and $\mathbf{q} = [q]$ respectively.

Unbalance and nonlinear force vectors may be expressed as follows:

$$\{\mathbf{f}_{unb}\} = \begin{Bmatrix} \mathbf{f}_{v_{unb},i}^t \\ \mathbf{f}_{v_{unb}}^p \\ \mathbf{f}_{w_{unb},i}^t \\ \mathbf{f}_{w_{unb}}^q \end{Bmatrix} \quad (3.11)$$

$$\{\mathbf{f}_{nl}\} = \begin{Bmatrix} \mathbf{f}_{v_{nl},i}^t \\ \mathbf{f}_{v_{nl}}^p \\ \mathbf{f}_{w_{nl},i}^t \\ \mathbf{f}_{w_{nl}}^q \end{Bmatrix} \quad (3.12)$$

The global mass $\mathbf{M}$, damping $\mathbf{C}$, gyroscopic $\mathbf{G}$, and stiffness $\mathbf{K}$ matrices may also be expressed in the following structured forms:

$$[\mathbf{M}] = \begin{bmatrix} [\bar{\mathbf{M}}_v^{tt}]_{NxN} & [\bar{\mathbf{M}}_v^{tp}]_{Nx1} & 0 & 0 \\ [\bar{\mathbf{M}}_v^{pt}]_{1xN} & [\bar{\mathbf{M}}_v^{pp}]_{1x1} & 0 & 0 \\ 0 & 0 & [\bar{\mathbf{M}}_w^{tt}]_{NxN} & [\bar{\mathbf{M}}_w^{tq}]_{Nx1} \\ 0 & 0 & [\bar{\mathbf{M}}_w^{qt}]_{1xN} & [\bar{\mathbf{M}}_w^{qq}]_{1x1} \end{bmatrix} \quad (3.13)$$

It is important to note that the only source of damping in the system arises from the support structures. Therefore, it is reasonable to adopt a proportional damping assumption for the rotor system as follows:

$$[\mathbf{C}_m] = 2\zeta\omega_1 [\mathbf{I}] \quad (3.14)$$

$$[\mathbf{C}_p] = [\mathbf{M}][\mathbf{C}_m] \quad (3.15)$$

where $\mathbf{C}_m$ is the diagonal modal damping matrix, ζ is the damping ratio, ω_1 is the first natural frequency of the linear system, $\mathbf{I}$ is the identity matrix of size of $(2N+2) \times (2N+2)$, and $\mathbf{C}_p$ is the proportional damping matrix for the rotor system.

Thus, global damping matrix $[\mathbf{C}]$ becomes:

$$[\mathbf{C}] = [\bar{\mathbf{C}}] + [\mathbf{C}_p] \quad (3.16)$$

where $\bar{\mathbf{C}}$ is the damping matrix of the support structures which may also be expressed in the following structured forms:

$$[\bar{\mathbf{C}}] = \begin{bmatrix} [\bar{\mathbf{C}}_{\mathbf{v}}^{\mathbf{tt}}]_{N \times N} & [\bar{\mathbf{C}}_{\mathbf{v}}^{\mathbf{tp}}]_{N \times 1} & 0 & 0 \\ [\bar{\mathbf{C}}_{\mathbf{v}}^{\mathbf{pt}}]_{1 \times N} & [\bar{\mathbf{C}}_{\mathbf{v}}^{\mathbf{pp}}]_{1 \times 1} & 0 & 0 \\ 0 & 0 & [\bar{\mathbf{C}}_{\mathbf{w}}^{\mathbf{tt}}]_{N \times N} & [\bar{\mathbf{C}}_{\mathbf{w}}^{\mathbf{tq}}]_{N \times 1} \\ 0 & 0 & [\bar{\mathbf{C}}_{\mathbf{w}}^{\mathbf{qt}}]_{1 \times N} & [\bar{\mathbf{C}}_{\mathbf{w}}^{\mathbf{qq}}]_{1 \times 1} \end{bmatrix} \quad (3.17)$$

Similarly, global gyroscopic matrix $\mathbf{G}$ may be expressed as:

$$[\mathbf{G}] = \begin{bmatrix} 0 & 0 & [\bar{\mathbf{G}}_{\theta}^{\mathbf{tt}}]_{N \times N} & [\bar{\mathbf{G}}_{\theta}^{\mathbf{tq}}]_{N \times 1} \\ 0 & 0 & [\bar{\mathbf{G}}_{\theta}^{\mathbf{qt}}]_{1 \times N} & [\bar{\mathbf{G}}_{\theta}^{\mathbf{qq}}]_{1 \times 1} \\ [\bar{\mathbf{G}}_{\varphi}^{\mathbf{tt}}]_{N \times N} & [\bar{\mathbf{G}}_{\varphi}^{\mathbf{tp}}]_{N \times 1} & 0 & 0 \\ [\bar{\mathbf{G}}_{\varphi}^{\mathbf{pt}}]_{1 \times N} & [\bar{\mathbf{G}}_{\varphi}^{\mathbf{pp}}]_{1 \times 1} & 0 & 0 \end{bmatrix} \quad (3.18)$$

Finally, global stiffness matrix $\mathbf{K}$ may be written as:

$$[\mathbf{K}] = \begin{bmatrix} [\bar{\mathbf{K}}_{\mathbf{v}}^{\mathbf{tt}}]_{N \times N} & [\bar{\mathbf{K}}_{\mathbf{v}}^{\mathbf{tp}}]_{N \times 1} & 0 & 0 \\ [\bar{\mathbf{K}}_{\mathbf{v}}^{\mathbf{pt}}]_{1 \times N} & [\bar{\mathbf{K}}_{\mathbf{v}}^{\mathbf{pp}}]_{1 \times 1} & 0 & 0 \\ 0 & 0 & [\bar{\mathbf{K}}_{\mathbf{w}}^{\mathbf{tt}}]_{N \times N} & [\bar{\mathbf{K}}_{\mathbf{w}}^{\mathbf{tq}}]_{N \times 1} \\ 0 & 0 & [\bar{\mathbf{K}}_{\mathbf{w}}^{\mathbf{qt}}]_{1 \times N} & [\bar{\mathbf{K}}_{\mathbf{w}}^{\mathbf{qq}}]_{1 \times 1} \end{bmatrix} \quad (3.19)$$

It is important to note that the global mass $\mathbf{M}$, damping $\mathbf{C}$, gyroscopic $\mathbf{G}$, and stiffness $\mathbf{K}$ matrices are of dimension $(2N+2) \times (2N+2)$.

Following the formulation of the governing equations of motion and the identification of the global system matrices, the subsequent step is to explicitly derive each submatrices. This process begins by substituting the approximated expressions for

transverse displacements and angular rotations in equations (3.1), (3.2), (3.6) and (3.7) into the weak form presented in equation (2.35).

An illustrative discretization application is provided below for the first term of the weak form:

$$\int_{t_1}^{t_2} \left(\int_0^L \rho A_s \ddot{v} \delta v(x, t) dx \right) dt \quad (3.20)$$

By inserting equation (3.1) of $v(x, t)$ into equation (3.20);

$$\int_{t_1}^{t_2} \left(\int_0^L \rho A_s \left(\underbrace{\sum_{i=1}^N V_i(x) \ddot{T}_i^v(t) + A(x) \ddot{p}(t)}_{\ddot{v}(x, t)} \right) \left(\underbrace{\sum_{j=1}^N V_j(x) \delta T_j^v(t) + A(x) \delta p(t)}_{\delta v(x, t)} \right) dx \right) dt \quad (3.21)$$

After expanding the multiplication of parentheses, the expression can be reformulated in more open form as follows:

$$\int_{t_1}^{t_2} \int_0^L \rho A_s \left(\sum_{i=1}^N \sum_{j=1}^N [V_i(x) V_j(x) \ddot{T}_i^v(t)] \delta T_j^v(t) + \sum_{i=1}^N [V_i(x) A(x) \ddot{T}_i^v(t)] \delta p(t) + \sum_{j=1}^N [A(x) V_j(x) \ddot{p}(t)] \delta T_j^v(t) + [A(x) A(x) \ddot{p}(t)] \delta p(t) \right) dx dt \quad (3.22)$$

Equation (3.22) may also be represented in an equivalent matrix form, as given below:

$$\int_{t_1}^{t_2} \left(\begin{bmatrix} \delta \mathbf{T}_j^v(t) & \delta p(t) \end{bmatrix} \underbrace{\begin{bmatrix} \int_0^L \rho A_s \sum_{i=1}^N \sum_{j=1}^N \{V_i(x) V_j(x)\} dx & \int_0^L \rho A_s \sum_{j=1}^N \{A(x) V_j(x)\} dx \\ \int_0^L \rho A_s \sum_{i=1}^N \{V_i(x) A(x)\} dx & \int_0^L \rho A_s [A(x) A(x)] dx \end{bmatrix}}_{[\mathbf{M}_v]} \begin{bmatrix} \ddot{\mathbf{T}}_i^v(t) \\ \ddot{p}(t) \end{bmatrix} \right) dt \quad (3.23)$$

The elements of matrix $\mathbf{M}_v$ may be expressed element-wise as follows:

$$m_{v,ij}^v = \int_0^L \rho A_s [V_i(x) V_j(x)] dx \quad i, j=1, 2, \dots, N \quad (3.24)$$

$$m_{v,il}^{tp} = \int_0^L \rho A_s [A(x) V_i(x)] dx \quad i=1, 2, \dots, N \quad (3.25)$$

$$m_{v,lj}^{pt} = \int_0^L \rho A_s [V_j(x) A(x)] dx \quad j=1, 2, \dots, N \quad (3.26)$$

$$m_{v,ll}^{pp} = \int_0^L \rho A_s [A(x) A(x)] dx \quad (3.27)$$

Finally, submatrix $\mathbf{M}_v$ may be expressed in the following structured form:

$$[\mathbf{M}_v] = \begin{bmatrix} [\mathbf{m}_v^{tt}] & [\mathbf{m}_v^{tp}] \\ [\mathbf{m}_v^{pt}] & [\mathbf{m}_v^{pp}] \end{bmatrix} \quad (3.28)$$

It is important to note that the discretization of each individual term in the weak form given in equation (2.35) is not fully detailed in this section, since it requires extensive elaboration. Instead, an illustrative example of discretization process has been provided above to demonstrate the procedure for discretizing representative terms and generating the corresponding submatrices.

3.3. Equation of Motion in Matrix Form

This section compiles all the submatrices obtained from the discretization process in the previous section to generate the global system matrices. The assembly process starts by assembling the submatrices that form the global mass matrix, as shown below:

$$[\bar{\mathbf{M}}_v^{tt}] = [\mathbf{m}_v^{tt}] + [\mathbf{m}_{v_d}^{tt}] + [\mathbf{m}_{v_0}^{tt}] + [\mathbf{m}_{v_L}^{tt}] + [\mathbf{m}_\varphi^{tt}] + [\mathbf{m}_{\varphi_d}^{tt}] \quad (3.29)$$

$$[\bar{\mathbf{M}}_v^{tp}] = [\mathbf{m}_v^{tp}] + [\mathbf{m}_{v_d}^{tp}] + [\mathbf{m}_{v_0}^{tp}] + [\mathbf{m}_{v_L}^{tp}] + [\mathbf{m}_\varphi^{tp}] + [\mathbf{m}_{\varphi_d}^{tp}] \quad (3.30)$$

$$[\bar{\mathbf{M}}_v^{pt}] = [\mathbf{m}_v^{pt}] + [\mathbf{m}_{v_d}^{pt}] + [\mathbf{m}_{v_0}^{pt}] + [\mathbf{m}_{v_L}^{pt}] + [\mathbf{m}_\varphi^{pt}] + [\mathbf{m}_{\varphi_d}^{pt}] \quad (3.31)$$

$$[\bar{\mathbf{M}}_v^{pp}] = [\mathbf{m}_v^{pp}] + [\mathbf{m}_{v_d}^{pp}] + [\mathbf{m}_{v_0}^{pp}] + [\mathbf{m}_{v_L}^{pp}] + [\mathbf{m}_\varphi^{pp}] + [\mathbf{m}_{\varphi_d}^{pp}] \quad (3.32)$$

$$[\bar{\mathbf{M}}_w^{tt}] = [\mathbf{m}_w^{tt}] + [\mathbf{m}_{w_d}^{tt}] + [\mathbf{m}_{w_0}^{tt}] + [\mathbf{m}_{w_L}^{tt}] + [\mathbf{m}_\theta^{tt}] + [\mathbf{m}_{\theta_d}^{tt}] \quad (3.33)$$

$$[\bar{\mathbf{M}}_w^{tq}] = [\mathbf{m}_w^{tq}] + [\mathbf{m}_{w_d}^{tq}] + [\mathbf{m}_{w_0}^{tq}] + [\mathbf{m}_{w_L}^{tq}] + [\mathbf{m}_\theta^{tq}] + [\mathbf{m}_{\theta_d}^{tq}] \quad (3.34)$$

$$\left[\bar{\mathbf{M}}_w^{qt} \right] = \left[\mathbf{m}_w^{qt} \right] + \left[\mathbf{m}_{w_D}^{qt} \right] + \left[\mathbf{m}_{w_0}^{qt} \right] + \left[\mathbf{m}_{w_L}^{qt} \right] + \left[\mathbf{m}_\theta^{qt} \right] + \left[\mathbf{m}_{\theta_D}^{qt} \right] \quad (3.35)$$

$$\left[\bar{\mathbf{M}}_w^{qq} \right] = \left[\mathbf{m}_w^{qq} \right] + \left[\mathbf{m}_{w_D}^{qq} \right] + \left[\mathbf{m}_{w_0}^{qq} \right] + \left[\mathbf{m}_{w_L}^{qq} \right] + \left[\mathbf{m}_\theta^{qq} \right] + \left[\mathbf{m}_{\theta_D}^{qq} \right] \quad (3.36)$$

The submatrices presented in equations (3.29) through (3.36) are then used to construct the global mass matrix which is previously defined in equation (3.13).

In a similar manner, the damping matrix of the support structures is assembled using the corresponding submatrices, as follows:

$$\left[\bar{\mathbf{C}}_v^{tt} \right] = \left[\mathbf{c}_{v_0}^{tt} \right] + \left[\mathbf{c}_{v_L}^{tt} \right] \quad (3.37)$$

$$\left[\bar{\mathbf{C}}_v^{tp} \right] = \left[\mathbf{c}_{v_0}^{tp} \right] + \left[\mathbf{c}_{v_L}^{tp} \right] \quad (3.38)$$

$$\left[\bar{\mathbf{C}}_v^{pt} \right] = \left[\mathbf{c}_{v_0}^{pt} \right] + \left[\mathbf{c}_{v_L}^{pt} \right] \quad (3.39)$$

$$\left[\bar{\mathbf{C}}_v^{pp} \right] = \left[\mathbf{c}_{v_0}^{pp} \right] + \left[\mathbf{c}_{v_L}^{pp} \right] \quad (3.40)$$

$$\left[\bar{\mathbf{C}}_w^{tt} \right] = \left[\mathbf{c}_{w_0}^{tt} \right] + \left[\mathbf{c}_{w_L}^{tt} \right] \quad (3.41)$$

$$\left[\bar{\mathbf{C}}_w^{tq} \right] = \left[\mathbf{c}_{w_0}^{tq} \right] + \left[\mathbf{c}_{w_L}^{tq} \right] \quad (3.42)$$

$$\left[\bar{\mathbf{C}}_w^{qt} \right] = \left[\mathbf{c}_{w_0}^{qt} \right] + \left[\mathbf{c}_{w_L}^{qt} \right] \quad (3.43)$$

$$\left[\bar{\mathbf{C}}_w^{qq} \right] = \left[\mathbf{c}_{w_0}^{qq} \right] + \left[\mathbf{c}_{w_L}^{qq} \right] \quad (3.44)$$

These damping submatrices, given in equations (3.37) through (3.44), are subsequently used to construct the damping matrix of the support structures, as defined in equation (3.17).

Similarly, the gyroscopic matrix is assembled using the corresponding submatrices, as follows:

$$\left[\bar{\mathbf{G}}_\varphi^{tt} \right] = \left[\mathbf{g}_\varphi^{tt} \right] + \left[\mathbf{g}_{\varphi_D}^{tt} \right] \quad (3.45)$$

$$\left[\bar{\mathbf{G}}_\varphi^{tp} \right] = \left[\mathbf{g}_\varphi^{tp} \right] + \left[\mathbf{g}_{\varphi_D}^{tp} \right] \quad (3.46)$$

$$\begin{bmatrix} \bar{\mathbf{G}}_{\varphi}^{\text{pt}} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_{\varphi}^{\text{pt}} \end{bmatrix} + \begin{bmatrix} \mathbf{g}_{\varphi_{\text{b}}}^{\text{pt}} \end{bmatrix} \quad (3.47)$$

$$\begin{bmatrix} \bar{\mathbf{G}}_{\varphi}^{\text{pp}} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_{\varphi}^{\text{pp}} \end{bmatrix} + \begin{bmatrix} \mathbf{g}_{\varphi_{\text{b}}}^{\text{pp}} \end{bmatrix} \quad (3.48)$$

$$\begin{bmatrix} \bar{\mathbf{G}}_{\theta}^{\text{tt}} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_{\theta}^{\text{tt}} \end{bmatrix} + \begin{bmatrix} \mathbf{g}_{\theta_{\text{b}}}^{\text{tt}} \end{bmatrix} \quad (3.49)$$

$$\begin{bmatrix} \bar{\mathbf{G}}_{\theta}^{\text{tq}} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_{\theta}^{\text{tq}} \end{bmatrix} + \begin{bmatrix} \mathbf{g}_{\theta_{\text{b}}}^{\text{tq}} \end{bmatrix} \quad (3.50)$$

$$\begin{bmatrix} \bar{\mathbf{G}}_{\theta}^{\text{qt}} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_{\theta}^{\text{qt}} \end{bmatrix} + \begin{bmatrix} \mathbf{g}_{\theta_{\text{b}}}^{\text{qt}} \end{bmatrix} \quad (3.51)$$

$$\begin{bmatrix} \bar{\mathbf{G}}_{\theta}^{\text{qq}} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_{\theta}^{\text{qq}} \end{bmatrix} + \begin{bmatrix} \mathbf{g}_{\theta_{\text{b}}}^{\text{qq}} \end{bmatrix} \quad (3.52)$$

These gyroscopic submatrices, given in equations (3.45) through (3.52), are subsequently used to construct the global gyroscopic matrix, as defined in equation (3.18).

Similarly, the stiffness matrix is assembled using the corresponding submatrices, as follows:

$$\begin{bmatrix} \bar{\mathbf{K}}_{\text{v}}^{\text{tt}} \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{\text{v}}^{\text{tt}} \end{bmatrix} + \begin{bmatrix} \mathbf{k}_{\text{v}_0}^{\text{tt}} \end{bmatrix} + \begin{bmatrix} \mathbf{k}_{\text{v}_L}^{\text{tt}} \end{bmatrix} \quad (3.53)$$

$$\begin{bmatrix} \bar{\mathbf{K}}_{\text{v}}^{\text{tp}} \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{\text{v}}^{\text{tp}} \end{bmatrix} + \begin{bmatrix} \mathbf{k}_{\text{v}_0}^{\text{tp}} \end{bmatrix} + \begin{bmatrix} \mathbf{k}_{\text{v}_L}^{\text{tp}} \end{bmatrix} \quad (3.54)$$

$$\begin{bmatrix} \bar{\mathbf{K}}_{\text{v}}^{\text{pt}} \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{\text{v}}^{\text{pt}} \end{bmatrix} + \begin{bmatrix} \mathbf{k}_{\text{v}_0}^{\text{pt}} \end{bmatrix} + \begin{bmatrix} \mathbf{k}_{\text{v}_L}^{\text{pt}} \end{bmatrix} \quad (3.55)$$

$$\begin{bmatrix} \bar{\mathbf{K}}_{\text{v}}^{\text{pp}} \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{\text{v}}^{\text{pp}} \end{bmatrix} + \begin{bmatrix} \mathbf{k}_{\text{v}_0}^{\text{pp}} \end{bmatrix} + \begin{bmatrix} \mathbf{k}_{\text{v}_L}^{\text{pp}} \end{bmatrix} \quad (3.56)$$

$$\begin{bmatrix} \bar{\mathbf{K}}_{\text{w}}^{\text{tt}} \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{\text{w}}^{\text{tt}} \end{bmatrix} + \begin{bmatrix} \mathbf{k}_{\text{w}_0}^{\text{tt}} \end{bmatrix} + \begin{bmatrix} \mathbf{k}_{\text{w}_L}^{\text{tt}} \end{bmatrix} \quad (3.57)$$

$$\begin{bmatrix} \bar{\mathbf{K}}_{\text{w}}^{\text{tq}} \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{\text{w}}^{\text{tq}} \end{bmatrix} + \begin{bmatrix} \mathbf{k}_{\text{w}_0}^{\text{tq}} \end{bmatrix} + \begin{bmatrix} \mathbf{k}_{\text{w}_L}^{\text{tq}} \end{bmatrix} \quad (3.58)$$

$$\begin{bmatrix} \bar{\mathbf{K}}_{\text{w}}^{\text{qt}} \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{\text{w}}^{\text{qt}} \end{bmatrix} + \begin{bmatrix} \mathbf{k}_{\text{w}_0}^{\text{qt}} \end{bmatrix} + \begin{bmatrix} \mathbf{k}_{\text{w}_L}^{\text{qt}} \end{bmatrix} \quad (3.59)$$

$$\begin{bmatrix} \bar{\mathbf{K}}_{\text{w}}^{\text{qq}} \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{\text{w}}^{\text{qq}} \end{bmatrix} + \begin{bmatrix} \mathbf{k}_{\text{w}_0}^{\text{qq}} \end{bmatrix} + \begin{bmatrix} \mathbf{k}_{\text{w}_L}^{\text{qq}} \end{bmatrix} \quad (3.60)$$

These stiffness submatrices, given in equations (3.53) through (3.60), are subsequently used to construct the global stiffness matrix, as defined in equation (3.19). Finally, unbalance force vector $\{\mathbf{f}_{\text{unb}}\}$ and nonlinear force vector $\{\mathbf{f}_{\text{nl}}\}$ can be constructed by using equations (3.11) and (3.12) respectively.

3.4. Harmonic Balance Method

The Harmonic Balance Method is a widely used technique for approximating the steady-state periodic response of nonlinear systems subjected to harmonic excitation. It is based on the assumption that the system response can be expressed as a truncated Fourier series. Considering a general nonlinear equation of motion:

$$\mathbf{M}\ddot{\mathbf{x}}(\mathbf{t}) + (\mathbf{C} + \Omega\mathbf{G})\dot{\mathbf{x}}(\mathbf{t}) + \mathbf{K}\mathbf{x}(\mathbf{t}) + \mathbf{f}_{\text{nl}}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f}(\mathbf{t}) \quad (3.61)$$

where $\mathbf{M}$, $\mathbf{C}$, $\mathbf{G}$ and $\mathbf{K}$ are the mass, damping, gyroscopic and stiffness matrices, respectively; $\mathbf{f}_{\text{nl}}$ represents the nonlinear forces, and $\mathbf{f}(\mathbf{t})$ is the external periodic excitation.

The steady-state response $\mathbf{x}(\mathbf{t})$ is approximated by a finite Fourier series:

$$x(t) = x_0 + \sum_{n=1}^H (x_n^c \cos(n\omega t) + x_n^s \sin(n\omega t)) \quad (3.62)$$

where x_0 is the bias component, x_n^c and x_n^s are the cosine and sine coefficients of the n-th harmonic, ω is frequency, and H is the number of harmonics considered.

Substituting the Fourier series expansions of $\dot{\mathbf{x}}(t)$ and $\ddot{\mathbf{x}}(t)$ into the original equation of motion and enforcing harmonic balance at each frequency component leads to a set of nonlinear algebraic equations for the Fourier coefficients.

The first and second derivatives are:

$$\dot{x}(t) = \sum_{n=1}^H (-n\omega x_n^c \sin(n\omega t) + n\omega x_n^s \cos(n\omega t)) \quad (3.63)$$

$$\ddot{x}(t) = \sum_{n=1}^H (-n^2\omega^2 x_n^c \cos(n\omega t) - n^2\omega^2 x_n^s \sin(n\omega t)) \quad (3.64)$$

After substituting these into the equation of motion and grouping the terms by harmonics (constant term, $\cos(n\omega t)$ and $\sin(n\omega t)$), the coefficients of each harmonic must individually satisfy the resulting algebraic equations. These equations are then solved, typically by an iterative method such as the Newton-Raphson algorithm.

The Harmonic Balance Method is particularly effective for capturing complex nonlinear cases such as amplitude-dependent frequencies, subharmonic resonances, and jump phenomena in the frequency response. However, its accuracy is dependent on the number of harmonics H included in the approximation, and convergence studies are often necessary to determine a suitable value for H .

In this study, the Harmonic Balance Method is employed to determine the steady-state response of the discretized rotor system with nonlinear supports. The NLvib tool presented in [39] for HBM is adapted and appropriately modified to accommodate the specific characteristics of the proposed rotor system and is used for solution of the steady-state response of the linear and nonlinear rotor systems using MATLAB R2022a software in next chapters.

4. VALIDATION OF MATHEMATICAL MODEL

This chapter presents the validation of the proposed discretized mathematical model for the rotor system by incorporating and solving benchmark Example 5.8.1 from Friswell et al. [38]. The properties of the rotor system used in benchmark example for validation process are summarized in Table 4.1. Validation is performed in two stages; first, a convergence study on the number of assumed mode shapes is conducted using a benchmark example to ensure that the discretization adequately captures the dynamic behavior without excessive computational cost. Second, the calculated natural frequencies are compared against a benchmark example to verify the accuracy of the frequency predictions by using proposed mathematical model. Particular attention is given to the agreement of the fundamental and lower-order modes, which are critical for rotor dynamic analyses. These steps establish the reliability and efficiency of the proposed modeling approach for nonlinear steady-state response analyses in subsequent chapter.

Table 4.1: Rotor properties of Friswell benchmark example 5.8.1 [38].

Parameters	Nomenclature	Value
d_o	Outer Diameter of Shaft [m]	0.080
d_i	Inner Diameter of Shaft [m]	0.030
D_o	Outer Diameter of Disk [m]	0.400
t_D	Thickness of Disk [m]	0.080
L	Length of Shaft [m]	1.2
E	Elastic Modulus Disk and Shaft [GN/m ²]	200
ρ	Density of Shaft and Disk Material [kg/m ³]	7800
k_0	Stiffness of Left Support at $x=0$ [N/m]	1.0×10^{12}
k_L	Stiffness of Right Support at $x=L$ [N/m]	1.0×10^{12}
Ω	Rotor Speed [rev/min]	5000

4.1. Convergence of Number of Mode Shapes

Convergence study on the number of assumed mode shapes N is conducted to systematically assess the discretization quality. The number of retained mode shapes directly influences the resolution and computational cost of the model, with higher values of N generally improving accuracy but increasing computational demands.

In this study, natural frequencies are computed for successive even values of N (specifically $N=8$, $N=10$, $N=12$ and $N=14$) and compared to reference $N=20$ frequencies to evaluate convergence behavior. A convergence error threshold of less than 4% is targeted to select an appropriate N value for further nonlinear analyses. Table 4.2 presents the frequencies and convergence errors observed.

Table 4.2: Convergence of number of mode shapes.

N=20 (reference)		N=8		N=10		N=12		N=14	
Frequency [Hz]		Frequency [Hz]	Error [%]	Frequency [Hz]	Error [%]	Frequency [Hz]	Error [%]	Frequency [Hz]	Error [%]
ω_{1b}	53.97	53.97	0.01	53.97	0.00	53.97	0.00	53.97	0.00
ω_{1f}	54.08	54.08	0.01	54.08	0.00	54.08	0.00	54.08	0.00
ω_{2b}	249.83	257.32	3.00	254.92	2.04	253.27	1.37	252.06	0.89
ω_{2f}	354.44	357.59	0.89	356.59	0.61	355.90	0.41	355.39	0.27
ω_{3b}	774.47	776.26	0.23	775.34	0.11	774.92	0.06	774.71	0.03
ω_{3f}	777.08	778.89	0.23	777.95	0.11	777.53	0.06	777.32	0.03
ω_{4b}	870.92	942.14	8.18	917.55	5.35	901.62	3.53	890.48	2.25
ω_{4f}	918.78	990.20	7.77	965.75	5.11	949.80	3.38	938.59	2.16
ω_{5b}	2368.18	2386.94	0.79	2377.09	0.38	2372.80	0.20	2370.62	0.10
ω_{5f}	2376.99	2395.95	0.80	2386.00	0.38	2381.66	0.20	2379.46	0.10
ω_{6b}	2467.62	2692.99	9.13	2608.69	5.72	2558.00	3.66	2524.14	2.29
ω_{6f}	2482.35	2709.75	9.16	2624.83	5.74	2573.69	3.68	2539.49	2.30

As can be seen in Table 4.2, the fundamental frequencies (ω_{1b} and ω_{1f}) converge rapidly, showing negligible errors (below 0.01%) even at $N=8$. The second modes (ω_{2b} and ω_{2f}) demonstrate consistent convergence, with errors decreasing to below 1.37% at $N=12$ and further to below 0.89% at $N=14$.

For higher modes, particularly ω_{4b} and ω_{4f} , convergence is slower, with errors initially exceeding 8% at lower N -values. However, errors decrease progressively, reaching less than 3.68% at $N=12$ and further below 2.30% at $N=14$ for the first six modes. Based on these observations, $N=12$ is selected as the discretization level which achieves the desired convergence accuracy while maintaining computational efficiency.

4.2. Frequency Validation

In addition to convergence of the number of modes shapes, the discretized mathematical model is validated by incorporating and solving benchmark example given in Table 4.1. It is noted that this example assumes the bearings are short and stiff, meaning that k_0 and k_L are set to very high values to represent rigid supports. Furthermore, the model does not include any internal shaft damping.

Based on these inputs, a comparison of the first four natural frequencies between the proposed model and the benchmark solution is performed to assess the accuracy of the discretized proposed rotor model. The validation of the discretized rotor model is performed by comparing the calculated natural frequencies with the benchmark values across different numbers of assumed mode shapes N .

Table 4.3: Validation of natural frequencies at rotor speed $\Omega=5000$ RPM.

Friswell Example 5.8.1		Proposed Model N=12		Proposed Model N=30		Proposed Model N=60		Proposed Model N=100	
Frequency [Hz]		Frequency [Hz]	Error [%]	Frequency [Hz]	Error [%]	Frequency [Hz]	Error [%]	Frequency [Hz]	Error [%]
ω_{1b}	53.62	53.97	0.66	53.97	0.65	53.97	0.65	53.97	0.65
ω_{1f}	53.72	54.08	0.66	54.08	0.66	54.08	0.66	54.08	0.66
ω_{2b}	242.69	253.27	4.36	248.05	2.21	246.22	1.46	245.48	1.15
ω_{2f}	349.63	355.90	1.79	353.68	1.16	352.88	0.93	352.56	0.84
ω_{3b}	731.74	774.92	5.90	774.38	5.83	774.35	5.82	774.35	5.82
ω_{3f}	733.85	777.53	5.95	776.99	5.88	776.96	5.87	776.95	5.87
ω_{4b}	780.79	901.62	15.48	856.15	9.65	841.79	7.81	836.16	7.09
ω_{4f}	828.65	949.80	14.62	903.72	9.06	888.99	7.28	883.18	6.58

Table 4.3 presents the results for $N=12$, $N=30$, $N=60$, and $N=100$. The first backward and forward whirl modes (ω_{1b} and ω_{1f}) are predicted with high accuracy, with errors consistently below 1% for all N values, providing that the discretized model accurately captures the fundamental dynamic behavior even at relatively low discretization levels. The second modes (ω_{2b} and ω_{2f}) also show good convergence behavior, with errors reducing from 4.36% and 1.79% at $N=12$ to 1.15% and 0.84% at $N=100$, respectively. While the third modes (ω_{3b} and ω_{3f}) exhibit only minor changes as N increases, the higher-order modes (ω_{4b} and ω_{4f}) show more substantial improvements, with errors decreasing from approximately 15% at $N=12$ to around 7% at $N=100$.

It should also be noted that the reference benchmark example 5.8.1 in Friswell does not account for rotary inertia effects, whereas the proposed rotor model incorporates rotary inertia contributions in the formulation. This difference leads to slight differences between the computed and reference natural frequencies, particularly at higher modes where rotary inertia becomes more influential. Despite this modeling difference, the proposed rotor model provides excellent agreement for the fundamental and second modes, validating its suitability for further nonlinear analyses.

5. NUMERICAL RESULTS

This chapter presents numerical results obtained from the proposed mathematical model to investigate the steady-state dynamic behavior of the rotor system. Following the validation of the mathematical model and discretization, a series of simulations are performed to study both linear and nonlinear system responses under various operating conditions. In these numerical simulations, the nonlinear differential equations of motion are solved by using the Harmonic Balance Method (HBM), which provides the efficient computation of steady-state periodic responses. Frequency response is calculated based on $0.5 \times (|\max| + |\min|)$ of time response and normalized by the shaft outer diameter d_o . The effects of different support characteristics, including nonlinear stiffness behaviors, are studied in detail.

5.1. Approximate Numerical Solutions to Steady-State Response

For all numerical investigations, a symmetric rotor with a single centrally located disk is considered. The system properties are based on the benchmark rotor model described in Friswell Example 5.8.1 [38] and are summarized in Table 5.1. In both linear and nonlinear cases studied, the disk remains positioned at the center of the shaft to preserve system symmetry.

Table 5.1: System properties of rotor model.

Parameters	Value	Unit	Parameters	Value	Unit
L	1.2	m	m_u	0.1	kg
E	2.00×10^{11}	Pa	e	0.150	m
d_o	0.080	m	m_θ	0	kg
d_i	0.030	m	m_L	0	kg
ρ	7800	kg/m ³	c_θ	0	N.s
D_o	0.400	m	c_L	0	N.s
td	0.080	m	Ω	5000	rev/min

Six different cases with varying conical washer geometries are analyzed to investigate the influence of support nonlinearity on the dynamic response of the rotor system. These cases are defined by variations in key geometric parameters of the conical washer, such as free height, thickness, and inner and outer diameters. These parameters significantly affect the stiffness behavior of the washer, as described in equations (2.9)

to (2.17). The selected configurations are intended to provide distinct nonlinear stiffness profiles, enabling a detailed analysis of their impact on system dynamics. The specific geometric properties corresponding to each washer configuration are listed in Table 5.2, serving as a clear reference for the comparative study. In addition to that damping ratio $\zeta=0.01$ is used for all linear and nonlinear studies in this chapter.

Table 5.2: Conical washer geometric parameters.

Parameters	Unit	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
E_w	Pa	2.06×10^{11}	2.06×10^{11}	2.06×10^{11}	2.06×10^{11}	2.06×10^{11}	2.06×10^{11}
D_w	m	0.065	0.085	0.085	0.120	0.120	0.120
d_w	m	0.040	0.040	0.040	0.080	0.100	0.100
s	m	0.0015	0.0015	0.002	0.002	0.002	0.002
f_0	m	0.002	0.002	0.002	0.003	0.003	0.002

5.1.1. Linear Rotor System

For linear system, case 2 conical washer geometry in Table 5.2 is considered. The support stiffnesses k_0 and k_L are assigned to the value of the linear stiffness term a_1 derived from using equation (2.10), while the quadratic and cubic stiffness coefficients of the washer are neglected (set to zero) to linearize the system. Accordingly, k_0 and k_L are determined based on the calculated linear stiffness contribution of the washer.

Firstly, the impact of the number of harmonics included in the approximation is studied for linear rotor system. Figure 5.1 and Figure 5.2 illustrate the frequency response curves of Case 2 for different harmonic numbers at both transverse directions. It can be observed that the change of the frequency response curve with different harmonics is negligible and frequency response curves overlay each other for $H=2$ to $H=5$. This analysis indicates that considering two harmonics ($H=2$) is sufficient to accurately capture the linear system behavior while minimizing the computational cost.

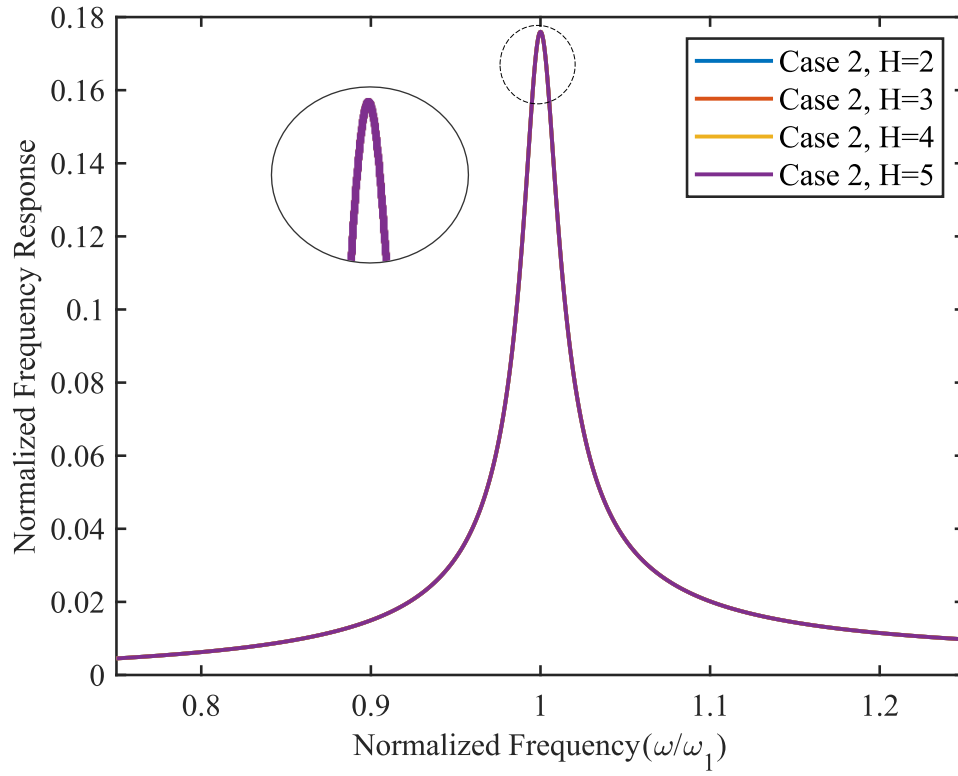


Figure 5.1: Case 2 (Linear): Normalized frequency response in Y-direction at $x=0.6$ when $\omega_1 = 194.69$ rad/s.

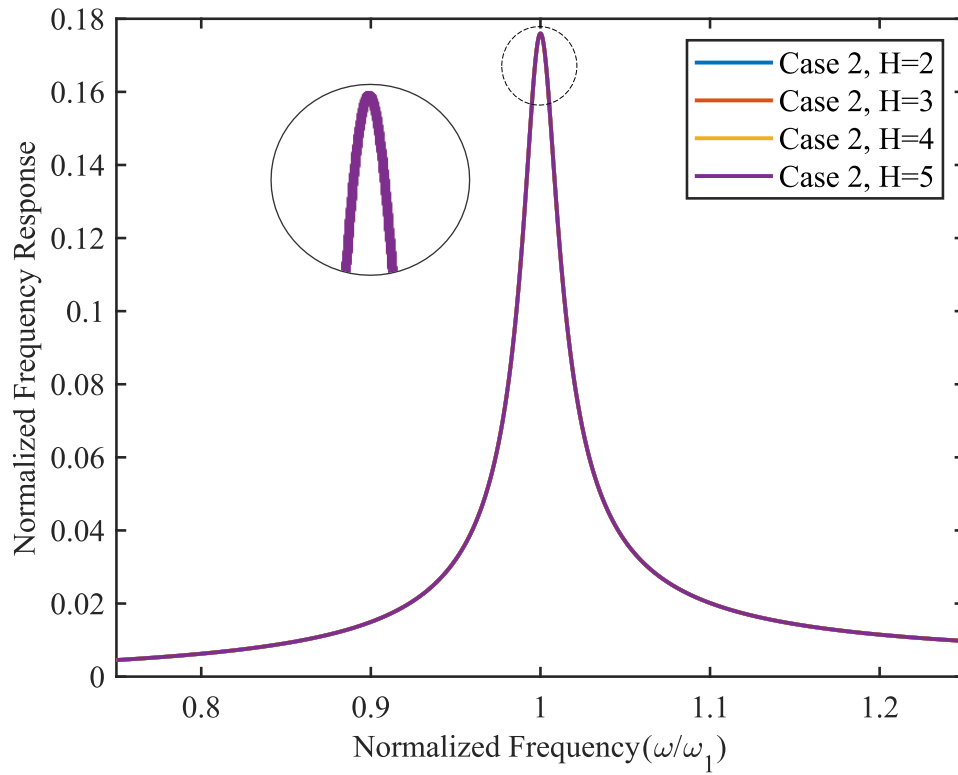


Figure 5.2: Case 2 (Linear): Normalized frequency response in Z-direction at $x=0.6$ when $\omega_1 = 194.69$ rad/s.

The first three natural frequencies of the linear rotor are listed in Table 5.3 below. The backward and forward waves separate as rotor speed increases where the gyroscopic effects become present in the system.

Table 5.3: Natural Frequencies for first three modes of linear rotor system.

	Mode 1		Mode 2		Mode 3	
	Backward	Forward	Backward	Forward	Backward	Forward
Nat. Frequency at 0 RPM, [Hz]	30.98	30.98	225.84	225.84	301.14	301.14
Nat. Frequency at 5000 RPM, [Hz]	30.97	30.99	225.24	226.45	253.27	355.90

Backward and forward waves of the first three modes are shown in Campbell plot in Figure 5.3. There is a crossing within operating range where the 1st mode intersects with the 1st engine order (1EO) around critical speed of 1859 RPM within operating range. In rotor dynamics, this condition is not allowed and this mode should be moved out of the operating range to prevent any failure of the rotor system.

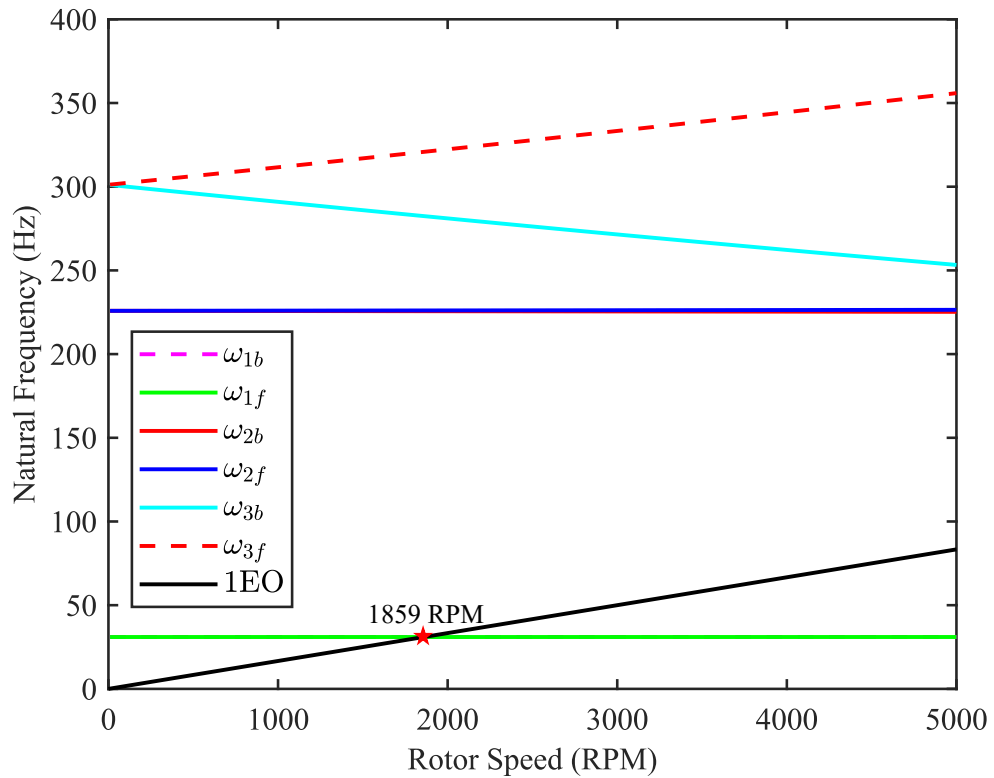


Figure 5.3: Campbell plot of linear rotor system, first three modes.

Figures 5.4 through 5.7 present the normalized steady-state responses of the rotor system modeled with a centrally located disk at $x=0.6$. Figure 5.4 shows the normalized frequency response curve of the disk, indicating a resonance peak at the normalized

frequency of unity which corresponds to the system's fundamental whirl frequency under linear conditions with harmonic number $H=2$. Since the system is symmetric in both transverse direction, normalized frequency responses of $V(x,t)$ and $W(x,t)$ overlays each other.

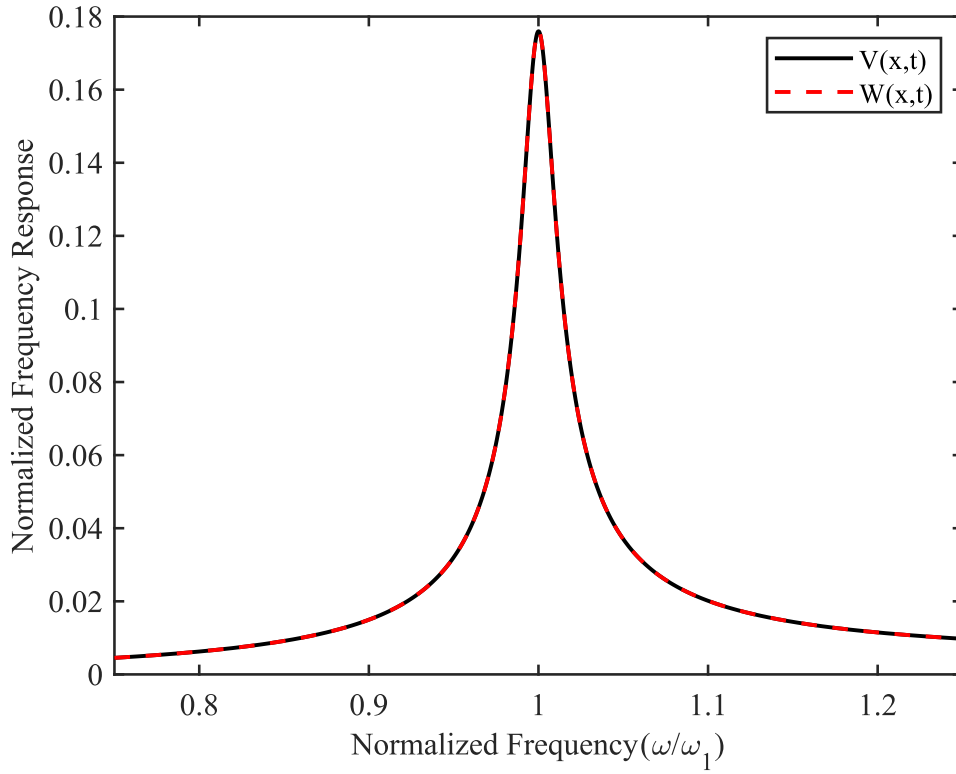


Figure 5.4: Case 2 (Linear): Normalized frequency response of $V(x,t)$ and $W(x,t)$ at $x=0.6$, $H=2$ and $\omega_1 = 194.69$ rad/s.

In Figure 5.5, the normalized transverse displacements $V(x,t)$ and $W(x,t)$ are plotted at the excitation frequency $\omega=194.69$ rad/s. The phase angle between transverse displacements can also be seen such that the phase angle is 90° since the linear system is symmetric in both transverse planes.

Figures 5.6 and 5.7 provide further information about the time-domain behavior by highlighting the normalized modal coordinates. Figure 5.6 shows the time responses of the first twelve modal coordinates of $T_i^v(t)$ corresponds to beam mode shapes along with generalized coordinate $p(t)$ of static mode in Y-direction, while Figure 5.7 presents the responses of the corresponding modal coordinates $T_i^w(t)$ and generalized coordinate $q(t)$ of static mode in Z-direction. In both cases, the dominant contribution to the shaft midpoint motion arises from the static mode shape and the first transverse vibration mode. The first dynamic mode shows the highest amplitude, while higher-

order modes, including the second, contribute negligibly. This is particularly true for even-numbered modes, which are suppressed at the midpoint due to nodal point of the simply supported boundary conditions. These results confirm that the response is largely governed by the static deflection and first mode, consistent with lightly damped linear rotor systems where lower modes dominate within typical operating speeds.

Moreover, shape (comparison) functions which are assumed to be the mode shapes of the simply supported beam in equation (3.3) are zero at the boundary conditions of the rotor. At this point, importance of incorporating static mode shapes $A(x)$ and $B(x)$ becomes visible since the boundary condition forces are transferred to the system by static mode shapes in equations (3.1) and (3.2). This can be clearly seen in Figures 5.6 and Figure 5.7 where $p(t)$ and $q(t)$ are primary contributor to the system response.

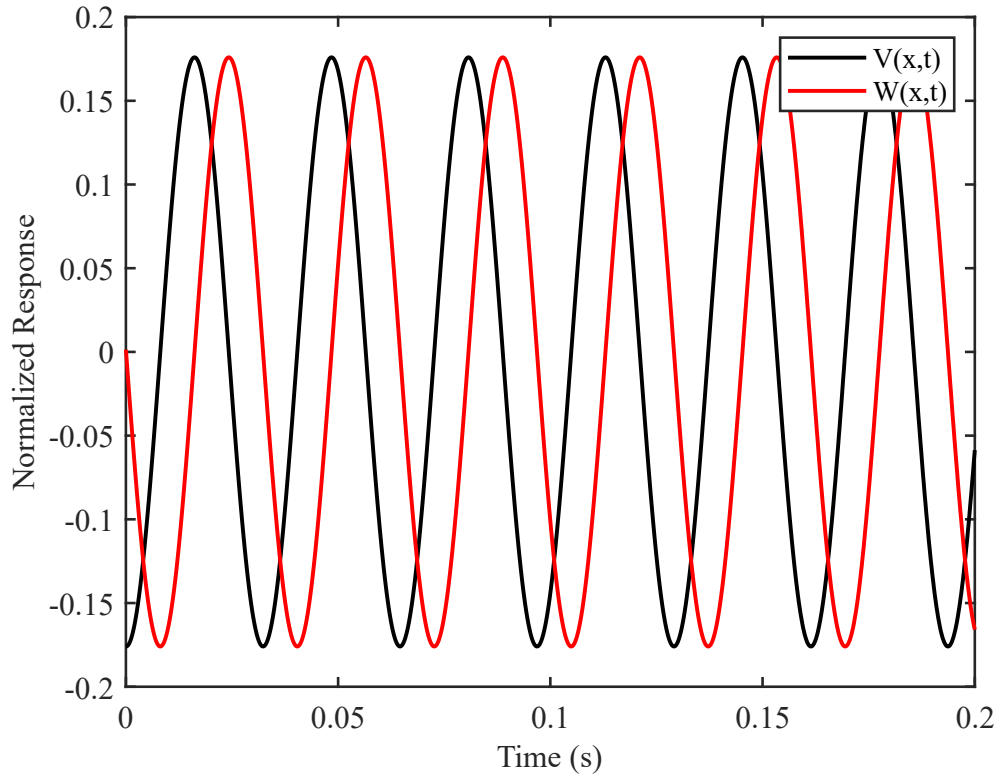


Figure 5.5: Normalized transverse displacements of the disk at $x=0.6$ when excitation frequency $\omega = \omega_1 = 194.69$ rad/s.

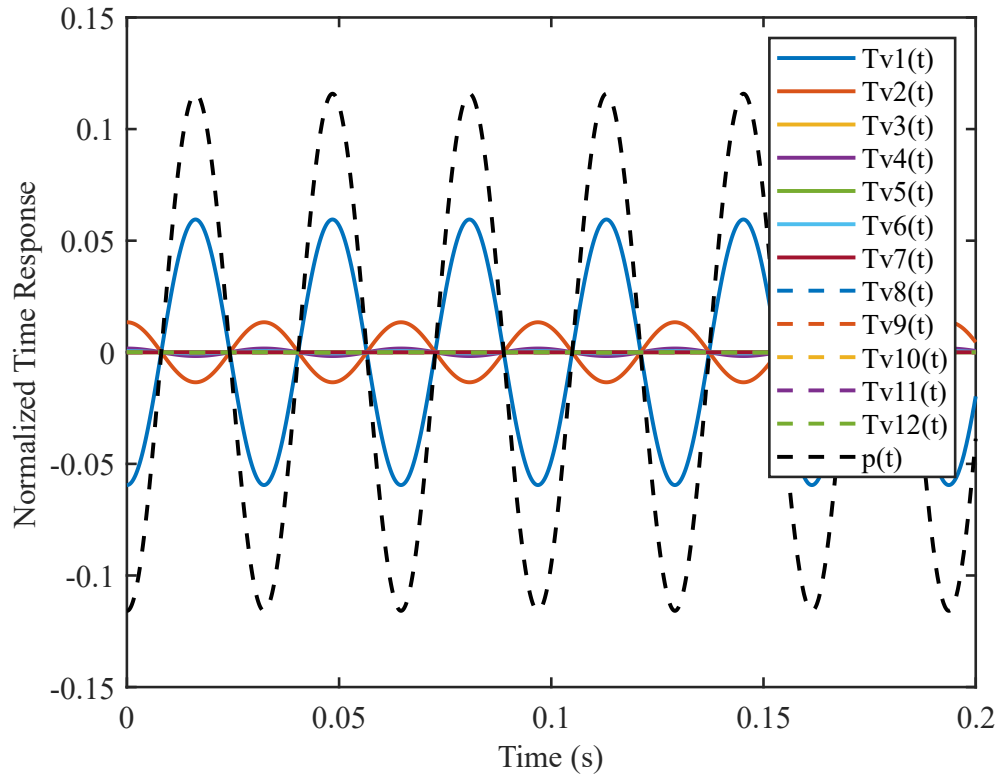


Figure 5.6: Normalized time responses of modal coordinates in Y-direction at $x=0.6$ and $\omega=\omega_1=194.69$ rad/s.

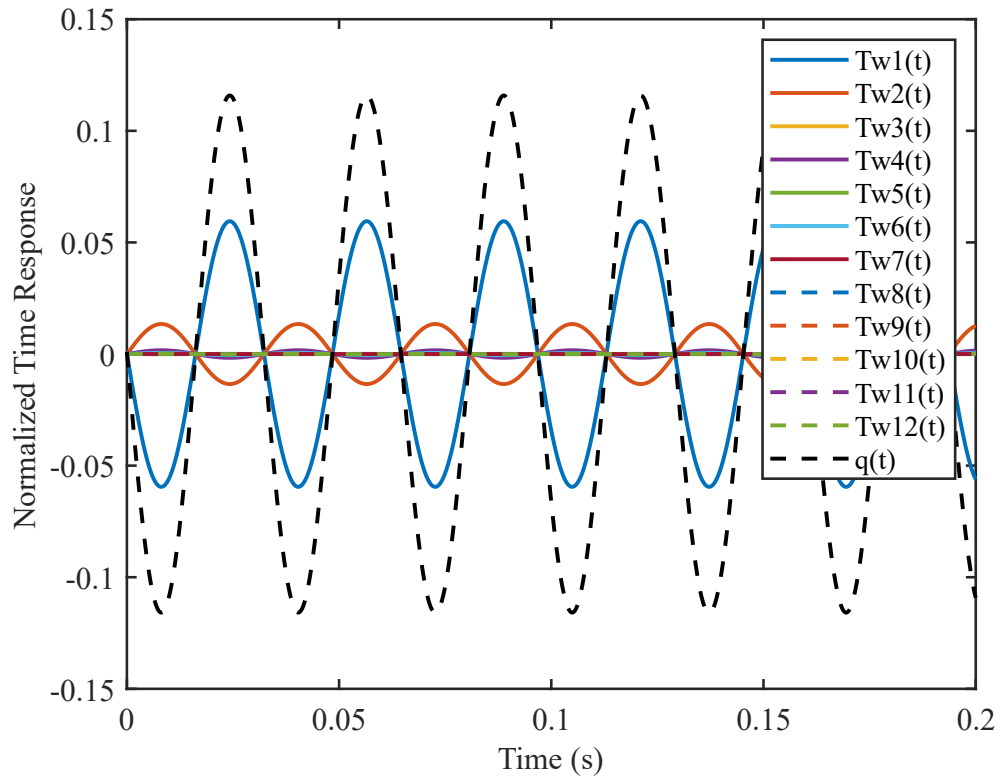


Figure 5.7: Normalized time responses of modal coordinates in Z-direction at $x=0.6$ and $\omega=\omega_1=194.69$ rad/s.

5.1.2. Full Belleville Washer Nonlinearity

This section focuses on the nonlinear dynamic behavior of the rotor system supported by conical washers which have third-order polynomial stiffness characteristics. Six different washer configurations presented at Table 5.2 are investigated and each design introduces varying degrees of nonlinearity into the support structure. The primary objective is to study how changes in the geometric properties of washer influence the steady-state response of the system across different operating speeds. The HBM is used to compute the steady-state solutions, capturing both linear and nonlinear effects within the frequency response and time-domain behavior. Through these studies, the sensitivity of the rotor system to different types of support nonlinearity being driven from washer geometry is analyzed.

Similar to the linear case, the impact of the number of harmonics included in the approximation initially is studied using the Case 1 washer geometry. Figure 5.8 presents the frequency response curves of Case 1 for different harmonic numbers. It can be observed that as the number of harmonics increases, the system response experiences slight changes, particularly between $H=2$ and $H=3$, where a minor shift in the response is observed. However, beyond $H=3$, no significant differences are observed in the system's frequency response. It indicates that considering three harmonics ($H=3$) is sufficient to accurately capture the essential features of the system behavior. Based on the harmonic convergence results presented in Figure 5.8, the number of harmonics is set to $H=3$ for the analysis of all subsequent cases listed in Table 5.2.

Following the harmonic convergence study, all six washer geometries are analyzed to evaluate their influence on the response of rotor system at $H=3$. Figure 5.9 presents the normalized frequency response overlays for each case where “NL” in legends represent nonlinear cases respectively. As shown in Figure 5.9, the system experiences different nonlinear behaviors depending on the washer geometry. From Case 1 to Case 2, the outer diameter of the washer is increased from 0.065 to 0.085 while other parameters remain constant, resulting in an enhanced hardening effect in the system response. It can be seen that system response is leaning towards right in the graph. From Case 2 to Case 3, the washer thickness is increased from 0.0015 to 0.002, which increased the softening behavior and causes a response to shift right in the graph.

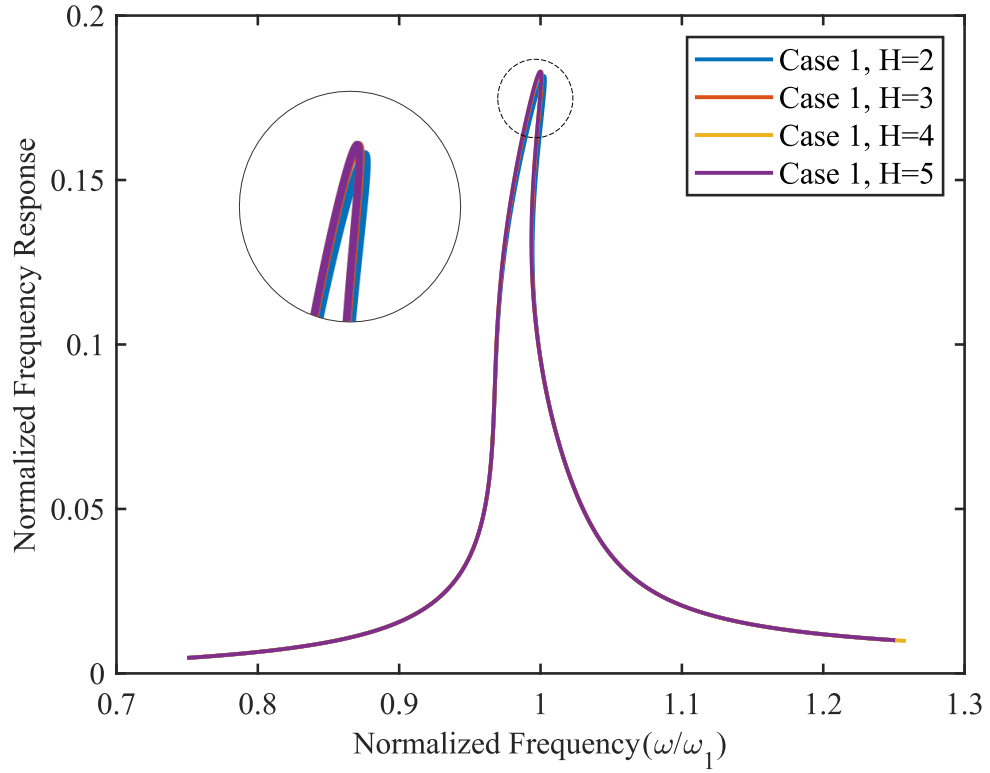


Figure 5.8: Normalized frequency responses in the Z-direction calculated $x=0.6$, corresponding to the first natural frequency $\omega_1=241.74$ rad/s.

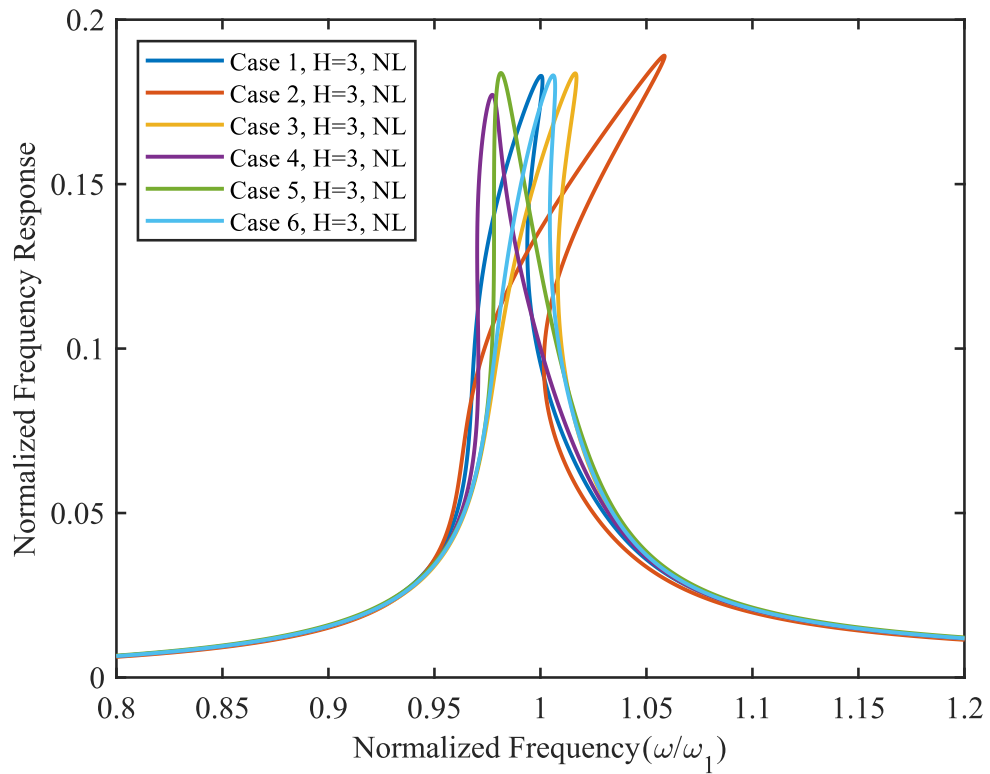


Figure 5.9: Normalized frequency responses in the Z-direction calculated at $x=0.6$, corresponding to the first natural frequency ω_1 of each distinct washer case.

Between Case 4 and Case 5, the inner diameter is increased from 0.080 to 0.100, while maintaining the other parameters. It is observed that increasing the inner diameter causes the cubic nonlinear term to dominate more rapidly compared to the quadratic term, thereby showing a hardening behavior in the frequency response relative to Case 4. Finally, from Case 5 to Case 6, the free height of the washer is reduced from 0.003 to 0.002 while keeping other parameters unchanged. A reduction in free height has a similar effect to increasing the inner diameter, further increasing the hardening characteristics of the system. As a result of these, it can easily be stated that the geometric parameters of washer have significant and distinct impact on nonlinear characteristics of the washer, which eventually impacts the nonlinear response of rotor system.

Figure 5.10 presents the normalized time responses of the transverse displacements $V(x,t)$ and $W(x,t)$ at $x=0.6$ for Cases 1 through 6, corresponding to different nonlinear conical washer configurations. Transverse displacements are computed at $\omega = \omega_1$ which vary slightly for each case depending on the stiffness characteristics of the washer. Across all configurations, the $V(x,t)$ response remains nearly harmonic and consistent, indicating that the dynamic behavior in the V-direction is largely unaffected by changes in the nonlinear support. In contrast, $W(x,t)$ shows increasing distortion and reduction in amplitude from Case 1 to Case 6. This suggests that the nonlinearity introduced by the washer geometry primarily influences the W-direction, leading to waveform deformation and reduced dynamic amplitude. The contrast between the two directions highlights the anisotropic effect of the support nonlinearity. While one transverse direction maintains a linear-like response, the other becomes increasingly nonlinear. These results emphasize the sensitivity of rotor system dynamics to support geometry and demonstrate how minor geometric variations can significantly alter the steady-state behavior in one direction while leaving the other mostly unchanged. Furthermore, this comparative analysis shows that while the first natural frequency $\omega = \omega_1$ sets the fundamental oscillation rate, the amplitude and waveform of the steady-state response in the W-direction are significantly shaped by the degree and type of support nonlinearity.

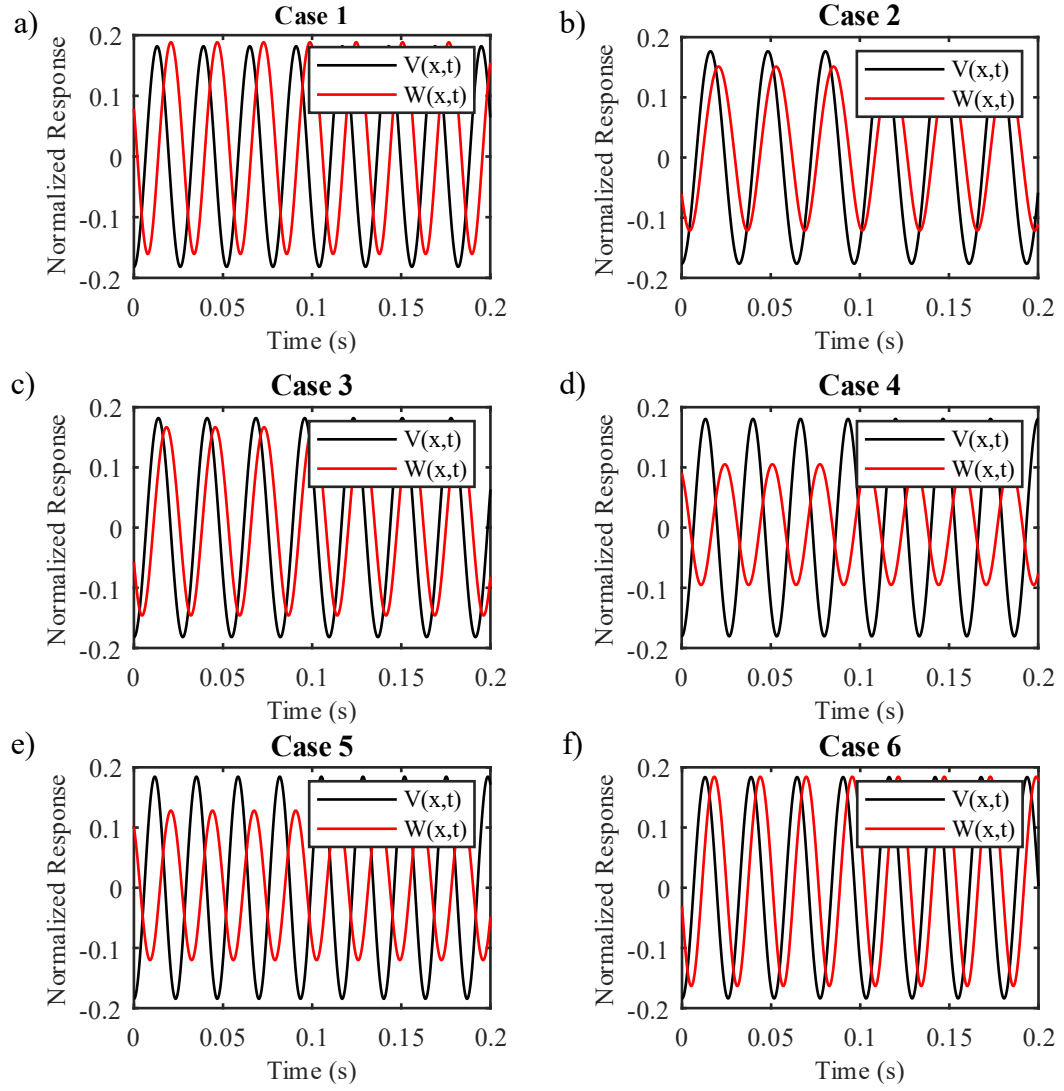


Figure 5.10: Normalized time responses at $x=0.6$, computed for each case at the first natural frequency $\omega=\omega_1$ specific to each washer configuration.

Similarly, Figure 5.11 also shows the normalized time responses of the transverse displacements, calculated at the excitation frequency $\omega=\omega_{\text{max_amplitude}}$, which corresponds to the frequency that the system exhibits its maximum dynamic amplification in the Z-direction. This selection is intentional, as the nonlinear behavior in Z-direction is most pronounced at this point. Configurations such as Case 2, Case 3, Case 4 and Case 5 exhibit relatively larger peak responses in Z-direction compared to Case 1 and Case 6, indicating a stronger nonlinear amplification effect. The observed amplitude variation across cases confirms that the system exhibits amplitude-dependent dynamics, which are not captured by conventional linear models and therefore requires a nonlinear modeling for accurate prediction.

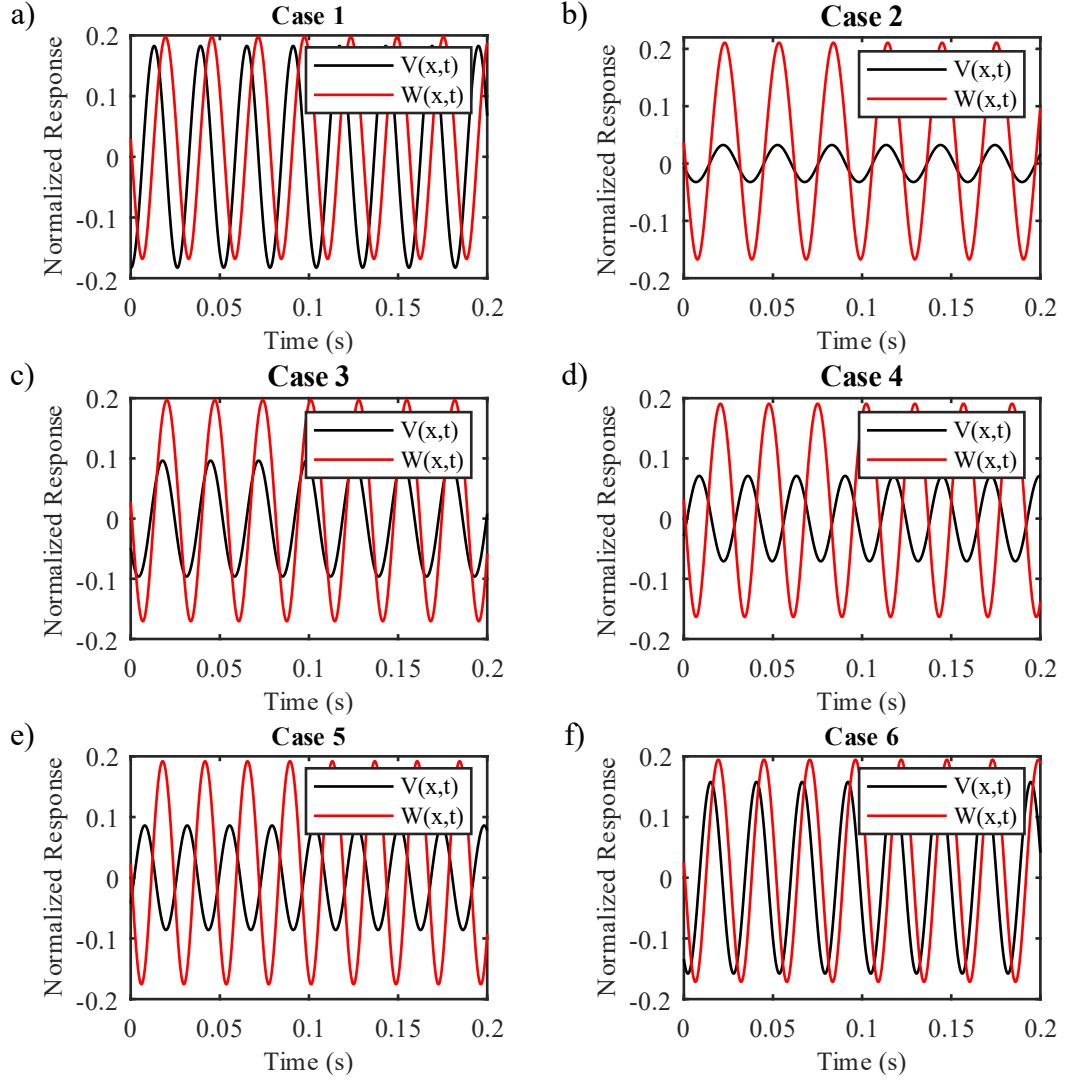


Figure 5.11: Normalized time responses at $x=0.6$, computed for each case at $\omega=\omega_{\max_amplitude}$ specific to each washer configuration.

Similar to the linear case, Figures 5.12 and 5.13 shows the normalized time responses of the modal coordinates for the rotor system incorporating the Case 2 washer geometry. As observed in the linear configuration, the shaft midpoint motion is primarily influenced by the static mode shape and the first transverse vibration mode. The first dynamic mode exhibits the highest amplitude, while the contributions of higher-order modes, including the second, remain negligible. This effect is particularly evident for even-numbered modes, which are suppressed at the midpoint due to nodal positions associated with simply supported boundary conditions. Overall, the nonlinear support introduced by the conical washer does not significantly change the modal dominance, and the system's behavior remains primarily governed by the static deflection and the first mode.

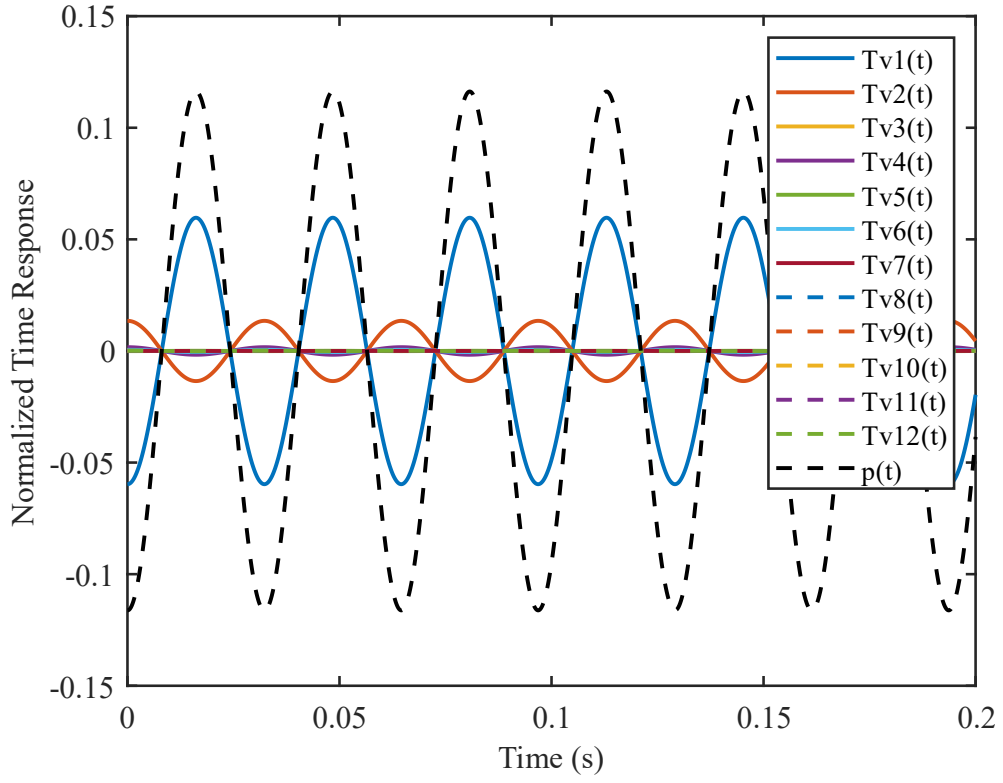


Figure 5.12: Normalized time responses of Case 2 in Y-direction at $x=0.6$ computed at $\omega=\omega_1=194.69$ rad/s.

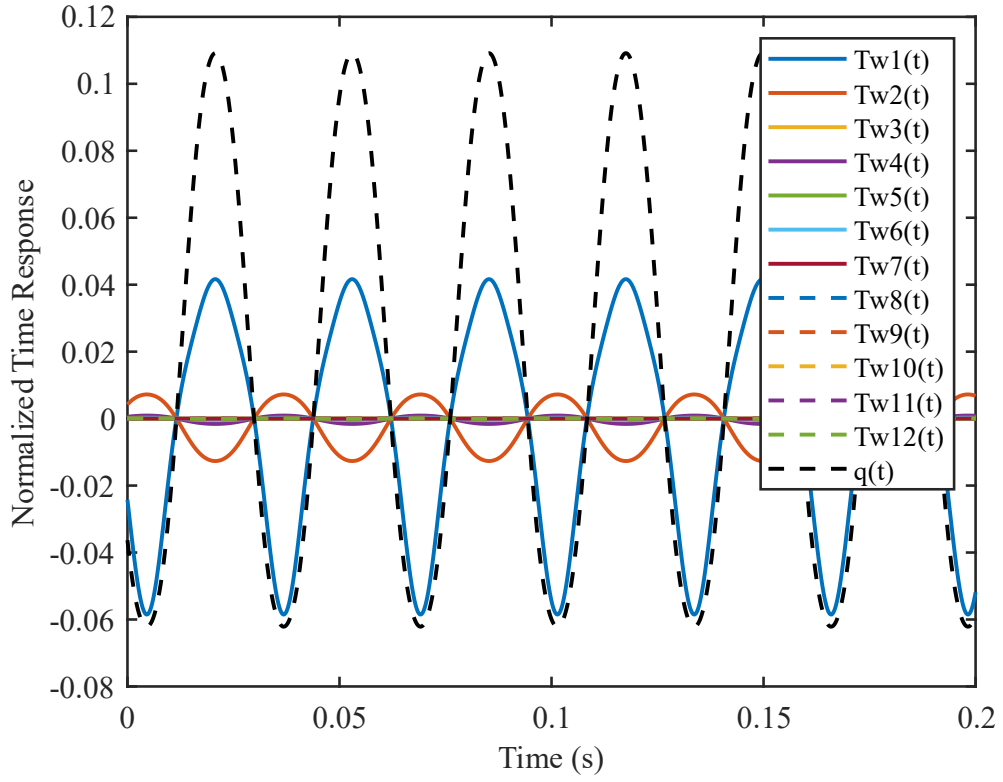


Figure 5.13: Normalized time responses of Case 2 in Z-direction at $x=0.6$ computed at $\omega=\omega_1=194.69$ rad/s.

Moreover, the effects of different washer configurations on the rotor system's time-domain response are also reflected in the disk orbit plots, as illustrated in Figure 5.14. As nonlinear stiffness is introduced, the orbits deviate from ideal circular paths, showing increasingly elliptical and asymmetric shapes. The variation in orbit geometry across cases highlights the sensitivity of the rotor's whirling motion to the geometric properties of the Belleville washers. Specifically, configurations with stronger nonlinear characteristics produce more distorted orbits, indicating amplitude-dependent stiffness effects. These findings also emphasize the role of support nonlinearity in shaping the global dynamic behavior of the rotor system.

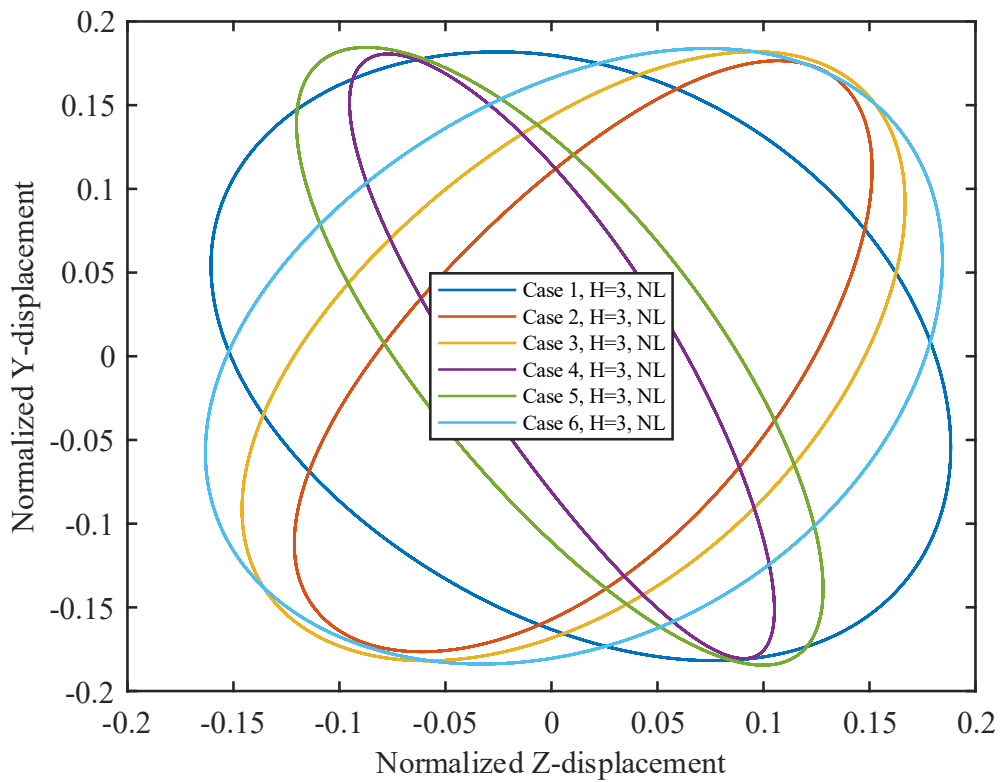


Figure 5.14: Normalized disk orbit at $x=0.6$, computed for each case at the first natural frequency $\omega=\omega_1$ specific to each washer configuration.

Finally, these findings demonstrate that variations in the geometric parameters of conical washers, specifically outer diameter, inner diameter, thickness, and free height, significantly impacts the system's nonlinear behavior due to the third-order polynomial force–deflection relationship defined in Equation (2.10). Moreover, changes in any geometric parameter simultaneously affect the linear, quadratic, and cubic stiffness coefficients which makes the prediction of resulting frequency response challenging since these stiffness coefficients of washer are coupled.

5.1.3. Cubic Stiffness Support Nonlinearity

In this section, the nonlinear element at the left support, located at $x=0$, is replaced by a cubic stiffness component to investigate its influence on the steady-state response of the rotor system. The right support at $x=L$ remains linear. In addition to the system parameters defined in Table 5.1, additional support properties, specifically the cubic stiffness coefficients k_c , are incorporated into the model and summarized in Table 5.4. The HBM with three harmonics ($H=3$) is employed to capture the nonlinear steady-state responses.

Table 5.4: Stiffness properties for cubic stiffness nonlinearity.

Parameters	Unit	Case 1	Case 2	Case 3
k_θ	N/m	1×10^6	1×10^6	1×10^6
k_L	N/m	1×10^6	1×10^6	1×10^6
k_c	N/m ³	1×10^{10}	2×10^{10}	4×10^{10}

The increase in amplitude with rotor speed observed in the frequency response curves in Figure 5.15 can be attributed to the combined effects of the unbalance force and the cubic nonlinearity at the support. As the rotor speed Ω increases, the unbalance force acting on the disk, which is proportional to Ω^2 , also increases, leading to larger excitation amplitudes. In the presence of cubic stiffness at the support, the system's effective stiffness becomes amplitude-dependent, with the cubic term becoming increasingly significant at higher displacement levels. As a result, the system experiences a hardening behavior, where the resonance frequency shifts to higher values as the amplitude grows. This interaction between the speed-dependent unbalance force and the amplitude-sensitive cubic nonlinearity explains both the increase in vibration amplitude and the characteristic hardening bending of the frequency response curves at higher rotor speeds.

Figure 5.16 shows the normalized orbit plots of the disk for different cubic stiffness cases at the left support, with the linear response included for comparison. As cubic stiffness increases from Case 1 to Case 3, the orbits become more elongated along the Y-direction and compressed in the Z-direction, deviating from the circular shape seen in the linear case. Case 1, with the lowest cubic stiffness, shows the slightest distortion, while higher cubic stiffness levels lead to more obvious changes.

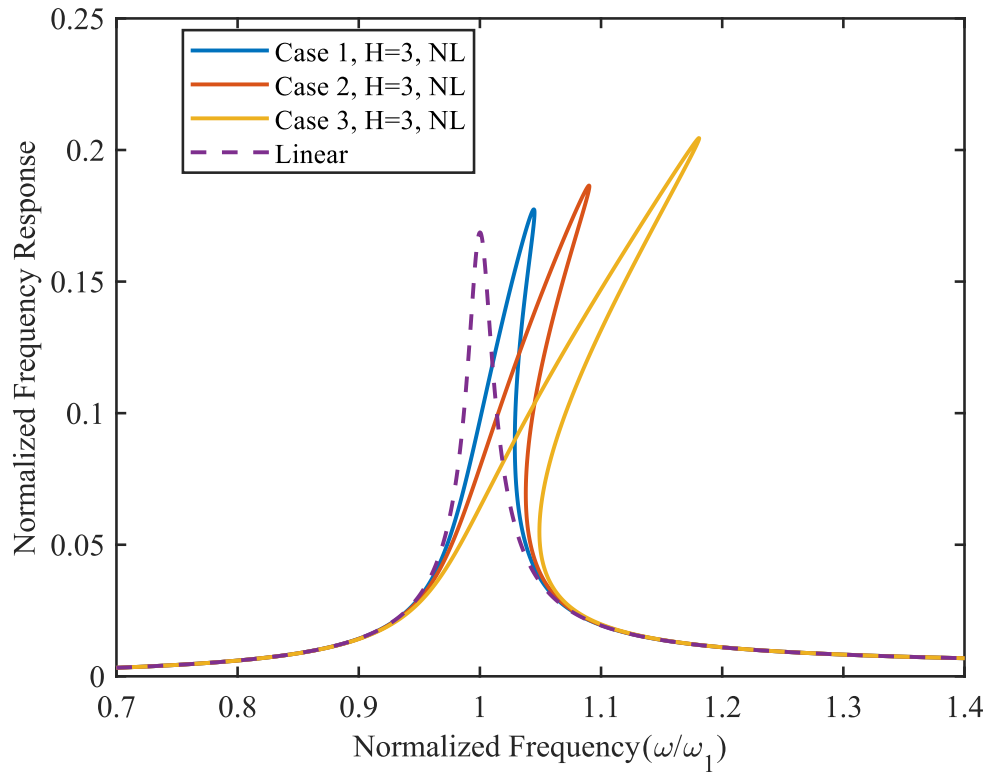


Figure 5.15: Normalized frequency responses in the Z-direction calculated at $x=0.6$, corresponding to the first natural frequency $\omega_1=132.50$ rad/s.

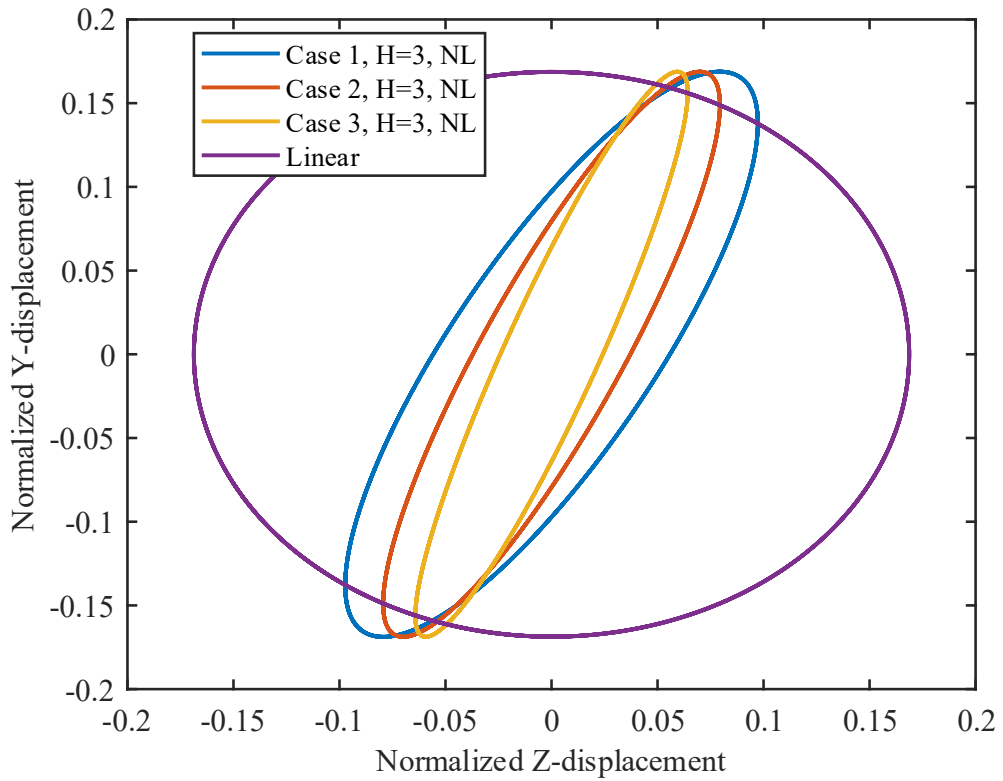


Figure 5.16: Normalized disk orbit at $x=0.6$ for each cubic stiffness cases, corresponding to the first natural frequency $\omega=\omega_1=132.50$ rad/s.

5.1.4. Quadratic Stiffness Support Nonlinearity

In this section, the nonlinear element at the left support, located at $x=0$, is replaced by a quadratic stiffness component to investigate its influence on the steady-state response of the rotor system. Similar to cubic stiffness case in Section 5.2.3, the right support at $x=L$ remains linear. In addition to the system parameters defined in Table 5.1, additional support properties, specifically the quadratic stiffness coefficients k_q , are incorporated into the model and summarized in Table 5.5. Similar to previous nonlinear studies, three harmonics ($H=3$) is employed to capture the nonlinear steady-state responses.

Table 5.5: Stiffness properties for quadratic stiffness nonlinearity.

Parameters	Unit	Case 1	Case 2	Case 3
k_0	N/m	1×10^6	1×10^6	1×10^6
k_L	N/m	1×10^6	1×10^6	1×10^6
k_q	N/m ²	1×10^8	2×10^8	4×10^8

Figure 5.17 shows the normalized frequency response curves of the disk for different cases with varying quadratic stiffness applied at the left support. The linear system response is also included for reference. As the rotor speed Ω decreases, the unbalance force acting on the disk, which is proportional to Ω^2 , also decreases, leading to smaller excitation amplitudes. As seen in the figure, the introduction of quadratic stiffness leads to a softening behavior, resulted in leftward shift of the resonance peak compared to the linear case. Among the nonlinear cases, Case 1, which has the smallest quadratic stiffness, shows a relatively minor deviation, while Case 3, with the largest quadratic stiffness, displays a much more noticeable softening effect. The responses become increasingly asymmetric and distorted as the quadratic stiffness increases, with the resonance amplitude decreasing and the system showing a broader frequency range of significant nonlinear response. The reduced resonance amplitude and wider response range characteristic of quadratic stiffness may serve as a damping strategy for vibration mitigation of rotor systems. These results confirm that quadratic nonlinearity at the support induces amplitude-dependent softening, substantially modifying the steady-state dynamic characteristics of the rotor system.

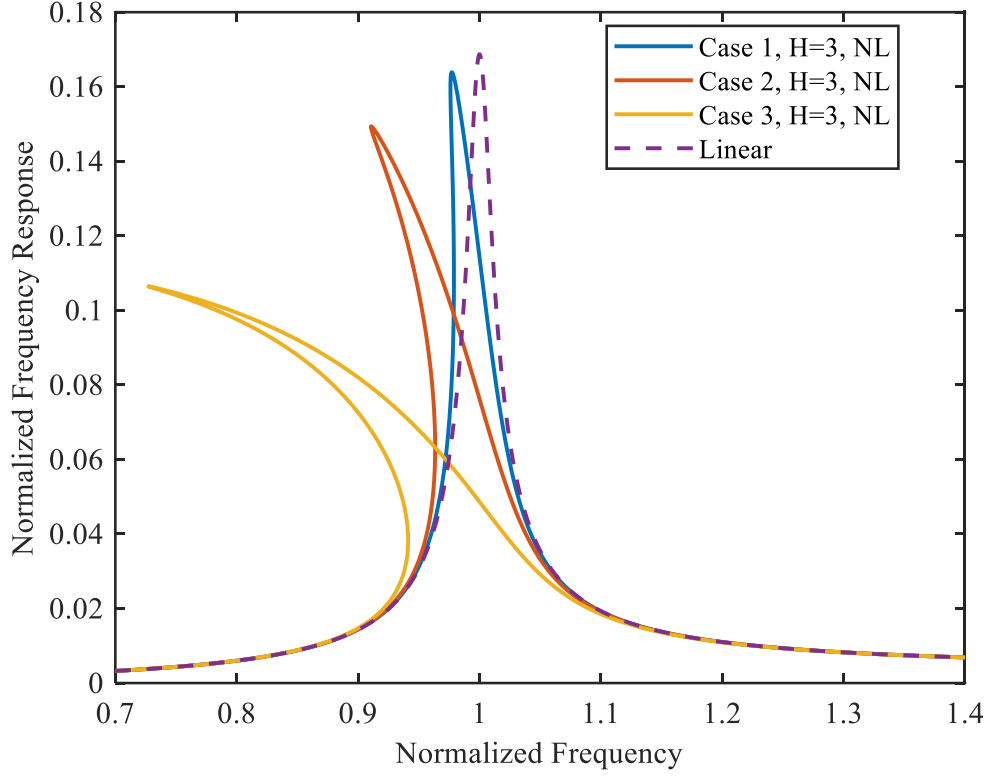


Figure 5.17: Normalized frequency responses in the Z-direction calculated at $x=0.6$, corresponding to the first natural frequency $\omega_1=132.50$ rad/s.

Finally, Figure 5.18 presents the normalized disk orbits at the axial position $x=0.6$ for three nonlinear cases with varying quadratic stiffness at the left support, along with the corresponding linear case for reference. These orbits are computed at the first natural frequency, $\omega=\omega_1=132.50$ rad/s, and provide insight into the influence of amplitude-dependent stiffness on the lateral disk motion in the Y–Z plane. The linear case yields a nearly circular orbit, as expected from a symmetric, uncoupled, and linear system excited at resonance. This serves as the baseline to assess the impact of quadratic stiffness nonlinearity. As the quadratic stiffness increases from Case 1 to Case 3, the orbit shape becomes progressively more distorted compared to the circular path of the linear case. The orbits exhibit increased elongation along the Y-direction and compression along the Z-direction, showing the amplitude-dependent softening introduced by the quadratic nonlinearity. Case 1 shows the smallest distortion from the linear orbit, while Case 3, with the strongest quadratic stiffness, displays the most noticeable distortion.

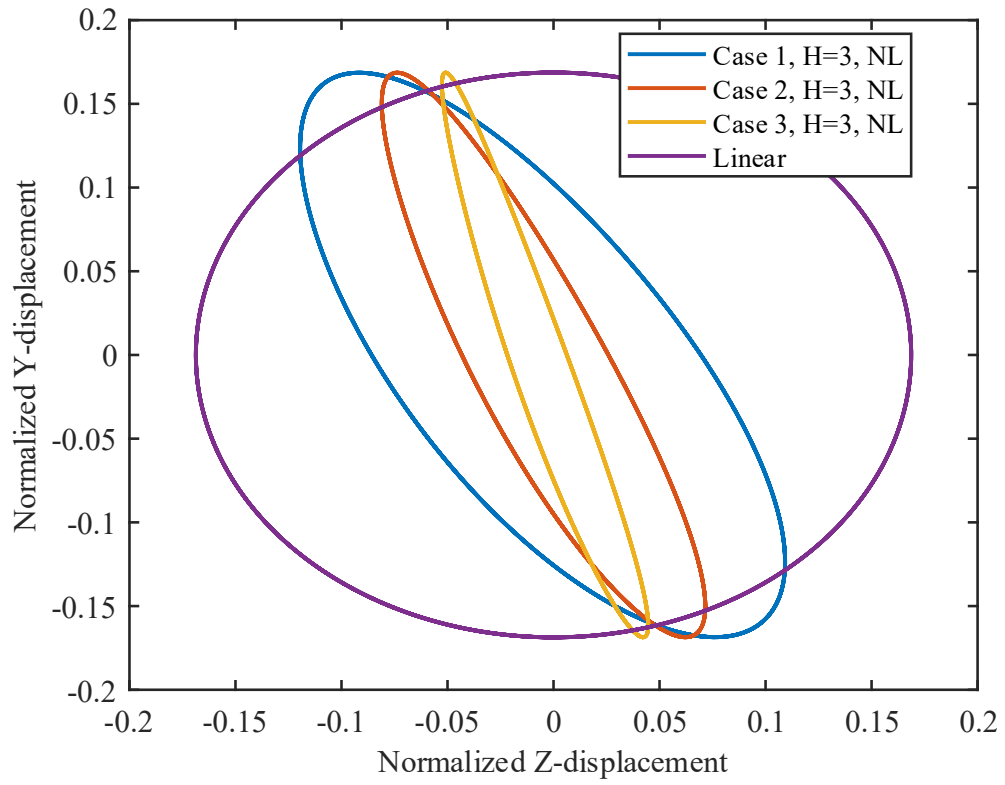


Figure 5.18: Normalized disk orbit at $x=0.6$ for each quadratic stiffness cases, corresponding to the first natural frequency $\omega=\omega_1=132.50$ rad/s.

6. CONCLUSION

This thesis has presented a comprehensive numerical investigation of a flexible single-disk rotor system supported by nonlinear elements, with a focus placed specifically on the nonlinear effects introduced by Belleville washers at the support. These conical spring elements were modeled using a third-order polynomial force–displacement relationship [35] in order to represent the inherent nonlinear stiffness characteristics arising from their geometry. Over the modeling approach presented in this thesis, the study has investigated how nonlinear support impacts the steady-state dynamic response of the rotor system.

The mathematical model derived in this thesis captures significant physical effects, namely rotary inertia, gyroscopic effects, and rotor mass imbalance, which are critical for accurately predicting the dynamic behavior of high-speed rotating systems. The governing equations of motion were derived using Hamilton’s Principle in weak-form, and the Rayleigh–Ritz method was applied to discretize the system. The Harmonic Balance Method was then used to obtain approximate steady-state solutions with computational efficiency while accurately capturing nonlinear dynamic features.

The study initially explored the sensitivity of the rotor system to geometric variations in the Belleville washer configurations. Parameters such as free height, outer and inner diameters, and thickness were varied across six different cases to study how they affect the system's dynamic response. It was observed that even minor geometric modifications led to significant changes in the linear, quadratic, and cubic term coefficients of the washer’s force–deflection behavior. As these stiffness coefficients in force-deflection equation are inherently coupled through the washer geometry, predicting the resulting nonlinear behavior becomes complex. This finding highlights the difficulty in isolating or tuning specific polynomial stiffness terms, such as the cubic or quadratic components, with geometric design alone, and points out the need for precise modeling and characterization during the design process.

Study also presented that these polynomial stiffness terms, particularly the quadratic and cubic components, affects the rotor's vibrational behavior in fundamentally different ways. Cubic nonlinearity resulted in hardening-type responses, with resonance peaks shifting to higher frequencies and system effective stiffness increases with amplitude. In contrast, quadratic nonlinearity introduced softening effects,

shifting the resonance to lower frequencies and reducing peak amplitudes. These behaviors were evident in both the frequency response curves and the orbit plots, which showed increasingly asymmetric and distorted trajectories as the strength of the nonlinearity increased. The combined effect of these nonlinearities, when changed simultaneously through washer geometry, is particularly challenging to predict or control rotor dynamics with single parameter change.

The numerical results also confirmed that dynamic response at the disk located at $x=L/2$ is strongly governed by the static deflection mode and the first mode of simply-supported beam, while even and higher-order mode shapes' contribution is negligible. This finding also validates the approximation of the transverse displacement through Rayleigh Ritz Method where the static mode shapes added to comparison functions to accurately represent the boundary conditions of the rotor system.

From an engineering perspective, the results of this thesis provide critical insights into the design and diagnostic of rotor systems with nonlinear support features. In practice, bolted joints with conical washers are common in turbomachinery, compressors, and high-speed rotating equipment, where loosening, preloading, or assembly tolerances can introduce non-negligible nonlinearities. The modeling strategy, solution framework, and analysis tools presented here offer a robust foundation for analyzing such systems and can be adapted to other forms of local nonlinearities, such as contact interfaces or clearance gaps.

Overall, this study provides the insight of support characteristics in rotor dynamic performance and demonstrates that conical spring elements such as Belleville washers can introduce substantial nonlinear effects that must be accounted for in modeling. The methods and findings presented here offer valuable insights for analysis and design of rotating systems where support nonlinearity plays a non-negligible role.

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PUBLICATIONS AND PRESENTATIONS FROM THE THESIS

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[2] M. Kutlu and S. Lotfan, “Dynamic behavior of rotor systems with support nonlinearities,” presented at the 12th Int. Eur. Congr. Adv. Stud. Basic Sci., Tirana, Albania, May 16–18, 2025.