T.R.

GEBZE TECHNICAL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

USE OF SPECIAL FUNCTIONS IN ENGINEERING; A DISTINGUISHED METHOD: "EVOLUTIONARY APPROACH TO ELECTROMAGNETICS THEORY"

BETÜL ÖZBAY

A THESIS SUBMITTED FOR THE DEGREE OF

MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

GEBZE

2015

T.R.

GEBZE TECHNICAL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

USE OF SPECIAL FUNCTIONS IN ENGINEERING; A DISTINGUISHED METHOD: "EVOLUTIONARY APPROACH TO ELECTROMAGNETICS THEORY"

BETÜL ÖZBAY

A THESIS SUBMITTED FOR THE DEGREE OF

MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

THESIS SUPERVISOR

ASSOC. PROF. DR. TAMER KOŞAN

II. THESIS SUPERVISOR

ASSIST. PROF. DR. EMRE EROĞLU

GEBZE

2015

T.C.
GEBZE TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

MÜHENDİSLİKTE ÖZEL FONKSİYONLAR KULLANIMI; SEÇKİN YÖNTEM: "ELEKTROMANYETİK TEORİYE EVRİMSEL YAKLAŞIMLAR"

BETÜL ÖZBAY
YÜKSEK LİSANS TEZİ
MATEMATİK ANABİLİM DALI

DANIŞMANI
DOÇ. DR. TAMER KOŞAN
II. DANIŞMANI
YRD. DOÇ. DR. EMRE EROĞLU

GEBZE
2015

GEBZE TEKNİK ÜNİVERSİTESİ	**YÜKSEK LİSANS JÜRİ ONAY FORMU**

GTÜ Fen Bilimleri Enstitüsü Yönetim Kurulu'nun 29../..06../.2015 tarih ve 2015../..41.... sayılı kararıyla oluşturulan jüri tarafından 21/07/2015 tarihinde tez savunma sınavı yapılan Betül ÖZBAY 'ın tez çalışması Matematik Anabilim Dalında YÜKSEK LİSANS tezi olarak kabul edilmiştir.

JÜRİ

ÜYE

(TEZ DANIŞMANI) : Doç. Dr. Tamer KOŞAN

ÜYE : .Doç. Dr. Kevser KÖKLÜ

ÜYE : Yrd.Doç.Dr. Yücel ENGİNER

ONAY

Gebze Teknik Üniversitesi Fen Bilimleri Enstitüsü Yönetim Kurulu'nun

....../....../......... tarih ve ..../........ sayılı kararı.

İMZA/MÜHÜR

SUMMARY

In this study, the problem of electromagnetic field force which are produced by given source function which has arbitrary time dependency in a waveguide that is electric conductor surface is examined by an analytical time domain method called Evolutionary Approach to Electromagnetics (EAE).

A complete set of TE and TM waveguide modes is o achieved in time-domain. Every field component of the modes is product of two elements: First one is a vector function of transverse waveguide coordinates which corresponds to a modal basis problem. It is specified via Dirichlet and Neumann boundary eigenvalue problems. These vector functions are distributions of the modal force lines in the waveguide cross-section. The second one is a scalar function corresponds to a time-dependent modal amplitude problem. This is obtained as the solution of Klein-Gordon equation depend on the waveguide's longitudinal coordinate and time. The problem of time-domain signal propagation in the waveguide is worked out in compliance with a causality principle. The graphical results are shown changes of amplitudes for the waveguide time-domain waveguide modes are represented via the first kind Bessel functions and Airy functions.

Keywords: Electromagnetic theory, Maxwell equations, time-domain, evolutionary equations, modal basis, modal amplitude, Bessel function, Airy function.

ÖZET

Bu çalışmada; mükemmel iletken yüzeylerden oluşmuş bir dalga kılavuzunda herhangi bir formda olan uyarıcı kaynak fonksiyonlarının oluşturduğu elektromagnetik alanların çözülmesi problemi "Elektromagnetik Teoriye Evrimsel Yaklaşım" (EAE) adı verilen analitik bir zaman uzayı yöntemi ile incelenmiştir.

TE ve TM dalga kılavuzu modlarının bir tam kümesi zaman-uzayında elde edilmiştir. Modların her bir alan bileşeni iki etkenin ürünüdür: Birincisi modal baz problemine karşılık gelen enlemsel dalga kılavuzu koordinatlarının bir vektör fonksiyonudur. Bu fonksiyon Dirichlet ve Neumann sınır-özdeğer problemi yoluyla belirlenir. Bu vektör fonksiyonları dalga kılavuzu kesit-alanı içerisinde modal kuvvet yollarıdır. İkincisi ise zaman-bağımlı modal genlik problemine karşılık gelen skalar bir fonksiyondur. Bu fonksiyon Klein-Gordon denkleminin dalga kılavuzunun eksenel koordinat ve zamana bağlı çözümü olarak elde edilir. Dalga kılavuzunda zaman-uzayı sinyal yayılması problemi nedensellik prensibine bağlı kalınarak çözülmüştür. Genlik değişimleri için sayısal sonuçlar, Bessel fonksiyonları ve Airy fonksiyonları yardımıyla ortaya konulmuştur.

Anahtar kelimeler: Electromanyetik teori, Maxwell denklemleri, zaman-uzayı, evrimsel denklemler, modal baz, modal genlik, Bessel fonksiyonu, Airy fonksiyonu.

ACKNOWLEDGEMENTS

I wish to express my warm thanks to Assoc. Prof. Tamer KOŞAN for his helping to my master education and this research.

I would like to express my deep and sincere gratitude to Asst. Prof. Emre EROĞLU, who not only shared his profound scientific knowledge with me but also taught me great lessons of life. His support, suggestions and encouragement gave me the drive and will to complete this work.

I am grateful to my family, especially to my mother, for their love and support.

TABLE of CONTENTS

LIST of ABBREVIATIONS and ACRONYMS

Abbreviations and Acronyms		**Explanations**
r	:	Location vectors (m)
t	:	Time (s)
E_m	:	The electric field vector (V / m)
H_m	:	Magnetic field vectors (A / m)
X_m	:	$col(E_m(r,t), H_m(r,t))$ vector consisting of vector E_m and H_m
$H(t)$	:	Heaviside step function
F	:	waveform vector induce to field
∂t	:	Partial derivative operator with respect to time (1/s)
E(r)	:	3-component vector as "electical" element of base
H(r)	:	3-component vector as "magnetic" element of base
$X(r)$	:	6-component vector comprising as $col(E(r), H(r))$
G, $\Im$	:	Operators are separated from Maxwell's equations
ε_0	:	Dielectric constant of the space (F/m)
μ_o	:	Magnetic constant of the space (H/m)
c	:	Speed of light in space (m/s) $(c = 1/\sqrt{\varepsilon_0 \mu_0})$
∂_n	:	Normal derivatives
κ_n	:	Eigenvalues of the Dirichlet problem
ϕ_n	:	Eigenfunctions of the Dirichlet problem
υ_m	:	Eigenvalues of the Neumann problem
ψ_n	:	Eigenfunctions of the Neumann problems
e_m	:	$E_m(r,t)$'s time-dependent modal amplitude.
h_m	:	$H_m(r,t)$'s time-dependent modal amplitude.
$\delta(r - r_s)$	:	Dirac delta-function
TE	:	Transverse Electric
TM	:	Transverse Magnetic

LIST of FIGURES

1.INTRODUCTION

1.1. Subjects and Importance

There are three canonical boundary-value problems using evolutionary approach (time derivative preservation) to the electromagnetic theory, with given the initial conditions. These problems called Cavity (Box) problem, Waveguide Problem and External (outer) Problem, research for the electromagnetic field strength. In this thesis, Waveguide Problem is studied in a subspace of Euclidean space and is a canonical problem. The methods separating the equations from Maxwell operator used in cavity problem is applied. The problem is solved in a section of hollow random-sectional area of waveguide which is assumed to be connected multiple but geometrically regular along the axis Oz. Waveguide surface is excellent conductor of electricity.

Investigation of electromagnetic field at finite boundary-value problems is a fundamental problem at electromagnetic theory. The problem is solved structures called cavity or rezonator. In classical time-harmonic electromagnetic theory, only it is possible theoretically to investigate event started $t = -\infty$ continued until $t = +\infty$. However, in the real situation the researchers used oscillations called "Forced Oscillations". "Forced" the term refers to that electromagnetic field is induced by a source. Electromagnetic fields only emerge and oscillate after source starts to work. These event have initial value as $t = 0$ [Erden, 2009].

The dissolution of the two main methods of time-domain studies is widely known. The most frequently exercised is Finite Difference Time Domain – FDTD-method which solved using numerical analysis, numerical quantity are analyzed and determined after each data. The used methods are applications of numerical analysis and diversity provided by these applications [Taflove and Hagness, 2005]. Provided data throughout the oscillation and to provide healthy result by comparing the results of the data recorded and the start-process-outcome observations provided the abundance of data in the metod. Wealth of metod is abundance and comparability of computer data. [Erden, 2009].

The second method is analytical. In this method, the Fourier or Laplace Integral Transformation is used. Causality principle is one of the principles should be underlined in this method becaused of integrals studied in the range of $(-\infty,+\infty)$. Physics and mathematics should be considered together. Mathematical results must meet also physical requirements.

Evolutionary Approach to Electromagnetic Theory (EAET) [Tretyakov, 1993] that is an analytical method meet this requirements. Method study analitically solutions of time-domain electromagnetic fields. Differential evolution equations is studied in the analytical examination. (When time derivative are protected, these equations are evolution equations.)

In recent history with the EAE method, in the perfectly conducting waveguide surface and sectional area of the this cavity and waveguide, stimulation of electromagnetic fields with sinusoidal sign which has initial time and transference throughout waveguide was investigated. [Aksoy ve Tretyakov, 2002], [Aksoy ve Tretyakov, 2004], [Aksoy et al, 2005].

1.2. The Purpose and Content of The Thesis

The aim of the thesis is to solve analytically the system of Maxwell’s equations by the evolution equations and Laplacian that ∂t is protected in time-domain. By decomposing Electromagnetic fields from Maxwell's equations will be solved in the time-domain. The solution obtained is analyzed along the waveguide. The results are analyzed in the following order in the thesis;

In section 2; an overview of the time-domain mode TE-TM was made, solution of the Dirichlet and Neumann problems were discussed, throughout the wave quide for signal transfer with the stimulating source, complete analytical solutions of real-valued eigenvector functions with the second degree derivatives (harmonic) of coordinates and time were given. Maxwell's equations were solved by applying the initial conditions, the solution obtained using border values and depending on the causality principle.

In section 3; general informations about Bessel's differential equation (function) was given, how to solve modal amplitude with the help of Bessel function was handled. Invariant solution of the Klein-Gordon equation in the Lorentz transformation was studied, the Klein-Gordon wave equation dissolved with variable parsing method and this solution was obtained as a product Bessel function with exponential function. A complete-function family of this solution has been obtained in the zero-infinite interval. Then the solution was applied to the Heaviside function.

The problems which have been studied previously [Aksoy ve Tretyakov, 2004] is applied in the space integers sentence and semi-integer sentences were given together. Index of Bessel fonctions is taken $a = s + 1/2 \quad s = 0,1,2...$ and a series solutions of complete and spherical Bessel functions family is given. Finally, a separate discussion is made about Bessel differential equation and Bessel functions.

In this chapter, the problem of amplitude is discussed and the solutions is shown with help of spherical and cylindrical Bessel fonctions. The equation obtained as a result of this approach is presented before applying the boundary conditions and normalizing, this discussions will be examined in further studies.

1.3. Historical Development

The discussions in this thesis, were made by the Evolutionary Approach to Electromagnetic Theory (EAE). The basic principle of EAE is to reach a solution using 3 values. These are the electromagnetic fields obtained by protecting ∂_t time derivative from Maxwell equations, the amplitude obtained from Klein-Gordon equation, potentials obtained from Helmholtz equation.

After acception that the electromagnetic theory is based on Maxwell's equations, most of the problems has been developed in frequency (Fourier) domain. When using the Fourier transform in the solution of electromagnetic problems, time was not accepted as independent variable. The results of the time-domain of electromagnetic problems were investigated with this way which amplitude of fields of determined the effective frequency is degraded to time-domain by using Fourier inverse transform. [Erden, 2009]

However; EAE method has enabled to calculate evolutions of time differantial equation is called evolution equations by matematicians in the range of $t_0 = 0$, $t_1 = t$ by protecting time derivative (∂_t). The difference of method is observability of behavior of electromagnetic fields that calculated by applying appropriate initial conditions and appropriate boundary conditions. Therefore, methosds Evolutionary Approach was called. Modal amplitudes that obtained from the Klein-Gordon equation is depend on time.(and on axial coordinate z). The solutions that obtained from equation, is depend on time, in addition, it is a series solution for this differential equation values of α in the $(0,\infty)$ range.

This idea was put forward by O.A. Tretyakov in 1980 and was published in the magazine of Russian scientific. [Tretyakov, 1986] [Tretyakov, 1989]. The English version of the method was first published in the 90s. [Tretyakov, 1993], [Tretyakov, 1994]. In recent publications give an idea about the various applications of the method EAE. [Aksoy ve Tretyakov, 2003], [Aksoy ve Tretyakov, 2004], [Aksoy, 2005].

EAE method is based on two main ideas. When Publications is examined chronologically, pioneers of the idea of working electromagnetic problems in the time domain is seen in the 1940s and 1950s. J. C. Slater [Slater, 1946], G. V. Kisun'ko [Kisun'ko, 1949], K. Kurokawa [Kurokawa, 1958] ve R. Müller [Müller, 1961] are the first persons who presented areas as a series of eigenmode. Later published the work of J. Van Bladel contains this idea in detail [Van Bladel, 1985]. This idea has been proposed for the complex amplitude of the field and has been continued to apply until now. ∂_t and $i\omega$ is replaced by applying Fourier transformation to Maxwell's equations in the time domain. In general conditions, nonlinear constitutive equations has been made linear. Therefore, there is only one way: linearization electromagnetic problems in the frequency domain. However, even if the Fourier transform in the linear electromagnetic problems is not as easy as it looks. [Erden, 2009]. An in-depth review of the encountered problems in this regard is available in P. Hillion article. [Hillion, 1993]

The second important idea is that achieving eigenmode solutions by keeping ∂_t in Maxwell's equations. Modal amplitude as coefficients in this series should be depend on time. As a result, the time differential evolutionary equation family should occur for the modal amplitudes. These equations family opens another way to the development of evolutionary approach to electromagnetic theory in time-domain. Constitutive equations actually have the differential form that contains the time derivative and converting to this form to algebraic form begins with solving motion equation that has force term that is harmonic sign. Recently, H. F. Harmuth reminded this fact in his publication [Harmuth, 1993]. Differential forms is a natural for EAE method for time differantial constitutive equations and it is easy to combine the constitutive equations with evolution equations. [Erden, 2009].

EAE can realize in different versions. Each of these begins with the decomposition of self-identical as heuristic or only symmetrical linear mathematical operator from Maxwell's equations. This second-mentioned acts according to the space variable. So that the separation of self-identification operator from the Maxwell equations gives an operator eigenvalues equation. This eigensolution of eigenvalue equation forms the basis of the chosen solution space. As a result, sought solutions can be presented in terms of eigenvectors series. This has the meaning physically the solution of modal area and modal amplitude depends on time. After that, the evolution equations is derived for the modal amplitudes. [Tretyakov, 1993], [Erden, 2009].

EAE method has practical examples for cavity problem. [Tretyakov, 1993].Firstly, self-identical operator that applied to an empty cavity separated from the Maxwell equations. Then, the cavity can be filled with the desired material. Dispersive or loss material is available. [Erden, 2009].

Electromagnetic fields in the waveguide was obtained in the expansion of the modal base element. Base of these elements are vector functions of the coordinates. Every element in the series has a modal amplitude scale factor. This amplitudes are functions that is required in terms of time dependencies. In general, base elements are represented by the vector boundary eigenvalue problem for Laplace operator. This vector problems are transformed into Dirichlet and Neumann boundary-value problems for the Laplace operator at all of the special occasion that can be applied to the separation of variables. The time differential system of ordinary differential equations is derived from Maxwell's equations for the modal amplitudes. The system is supported with appropriate initial conditions. This problem is the Cauchy problem. Thus, modal amplitudes are obtained as simple convolution integrals. Source sign may be a random function that can be integrated. For Example application, solution is calculated as the product of an exponential function and Bessel function. These is guaranteed to satisfy the causality principle. [Erden, 2009].

2. TIME-DOMAIN MODES PROBLEM

2.1.Introduction

Analysis of the signal transfer through the waveguide consists of two main parts. The first is the presentation of the modes of the time-domain waveguide in the modal base. The latter is presented with the dissolution of the Klein-Gordon equation. The solution of Klein-Gordon equation shows time-dependent modal amplitude providing the signal transfer. The solution of the Klein-Gordon equation is obtained while maintaining causality principle. This solutions is evolutionary expressed as a series.

The signal transfer problems dissolved as open along waveguides. The signal is presented as combination that spread by the Heaviside function that has a derivative terms with τ and ξ.

The problem of signal propagation and transmission in a hollow waveguide is considered as a time-domain analytical method. Waveguide is geometrically homogeneous, regular along the axis of Oz, ıts sectional-area is sufficiently smooth surface and closed single-linked. Any possible angle of this surface does not exceed from π degree. The waveguide surface is an excellent conductor. Complete-TE and TM waveguide modes are obtained directly in the time-domain. Each modal field is taken as the sum of the vector extension of the parts of transverse and longitudial. Each extension consists of two active. One is element of the waveguide and modal base is which vector founction of transverse waveguide coordinates, the other is modal amplitude is which scalar founction of t time and transverse coordinate z. All elements of the modal base is determined with the help of two scalar potentials. These are eigensoluntion of normalized Dirichlet and the Neumann boundary-value problems for the Laplacian. Each element of modal base provides suitable boundary conditions on the waveguide surface. Modal amplitude is solutions of evolutionary partial differential equations system according to time (and the longitudinal coordinate z)

3-component of the position vector R and ∇ operator, surface of the waveguide-sectional area (S) and the z-axis,

$$\mathrm{R} = \mathrm{r} + \mathrm{z}z,\ \nabla = \nabla_{\perp} + \mathrm{z}\partial_z \tag{2.1}$$

z is a unit vector directed along the Oz axis, r is 2-component vector at S section area of the waveguide and part of transverse of ∇ and $\nabla_{\perp}$. Differential operator $\nabla_{\perp}$ just works trasverse waveguide coordinates. 3-component electromagnetic field strength vector E and H, each is presented as the sum of two-component and one-component vectors as follows.

$$\begin{cases} \mathrm{E}_m(\mathrm{R,t}) = \mathrm{E(r,z,t)} + \mathrm{z}E_z(\mathrm{r,z,t}) \\ \mathrm{H}_m(\mathrm{R,t}) = \mathrm{H(r,z,t)} + \mathrm{z}\,H_z(\mathrm{r,z,t}) \end{cases} \tag{2.2}$$

For E electric and H magnetic field vectors the following vectorial Maxwell's equations will be solved.

$$\begin{cases} \nabla\times\mathrm{E(R,t)} = -\mu_0\partial_t\mathrm{H}(\mathrm{R},t) \\ \nabla\times\mathrm{H(R,t)} = \varepsilon_0\partial_t\mathrm{E}(\mathrm{R},t) \end{cases} \tag{2.3}$$

In addition to (2.4), the following scalar Maxwell equations will also be used.

$$\begin{cases} \nabla\cdot\mathrm{E(R,t)}=0 \\ \nabla\cdot\mathrm{H(R,t)}=0 \end{cases} \tag{2.4}$$

The equations (2.4) and (2.5) are valid in the waveguide, except the surface. Assuming that the surface of the waveguide has property of excellent electical conductivity property and the components of the fields are exposing to the following boundary conditions.

$$(\mathrm{n}\cdot\mathrm{H})_{|L} = 0,\ (\mathrm{l}\cdot\mathrm{E})_{|L} = 0,\ (\mathrm{z}\cdot\mathrm{E})_{|L} = 0 \tag{2.5}$$

The Maxwell's equations are hyperbolic partial differential equations, so the equations (2.3) should be subjected to the given initials conditions.

The Maxwell's equations in (2.3) can be decomped as their transverse and longitudinal parts using the above data. Then the equation (2.4) will be added to them.

$$\begin{cases} \nabla\times \mathrm{E(R,t)}=[(\nabla_{\perp}+\mathrm{z}\partial_z)\times(\mathrm{E}+\mathrm{zE}_z)] \\ \nabla\times \mathrm{H(R,t)}=[(\nabla_{\perp}+\mathrm{z}\partial_z)\times(\mathrm{H}+\mathrm{zH}_z)] \end{cases} \tag{2.6}$$

To find left-hand sides of the above equations, it is introduced a three-component vector fields $\mathcal{A}$ and calculate the curl of the vector as shown below.

$$\nabla\times\ \mathcal{A}\ =[(\nabla_{\perp}+\mathrm{z}\partial_z)\times(\mathrm{A}+\mathrm{zA}_z)] \tag{2.7}$$

$$\nabla\times\mathcal{A}=(\nabla_{\perp}+\mathrm{A})+(\nabla_{\perp}+\mathrm{zA}_z)+[\mathrm{z}\partial_z\times\mathrm{A}]+[\mathrm{z}\partial_z\times\mathrm{zA}_z] \tag{2.8}$$

It is evident that $[\mathrm{z}\partial_z\times\mathrm{zA}_z]=0, (\nabla_{\perp}+\mathrm{zA}_z)=(\nabla_{\perp}\mathrm{A}_z+\mathrm{z})=(\nabla_{\perp}\times\mathrm{z})\mathrm{A}_z$ and $[\mathrm{z}\partial_z\times\mathrm{A}]=\partial_z[\mathrm{z}\times\mathrm{A}]$, thus

$$\nabla\times\mathcal{A}=[(\nabla_{\perp}+\mathrm{z}\partial_z)\times(\mathrm{A}+\mathrm{zA}_z)]=[(\nabla_{\perp}+\mathrm{A})+(\nabla_{\perp}+\mathrm{z})\mathrm{A}_z]+\partial_z[\mathrm{z}\times\mathrm{A}] \tag{2.9}$$

is obtained. We now can apply this result to the tranverse and longitudinal parts of the Maxwell's equations as given below;

$$\begin{cases} \nabla\times \mathrm{E(R,t)}=[(\nabla_{\perp}+\mathrm{z}\partial_z)\times(\mathrm{E}+\mathrm{zE}_z)] \\ \qquad =[(\nabla_{\perp}\times\mathrm{E})+(\nabla_{\perp}+\mathrm{z})\mathrm{E}_z]+\partial_z[\mathrm{z}\times\mathrm{E}]=-\mu_0\partial_t\mathrm{H}-z\mu_0\partial_t\mathrm{H}_z \\ \nabla\times \mathrm{H(R,t)}=[(\nabla_{\perp}+\mathrm{z}\partial_z)\times(\mathrm{H}+\mathrm{zH}_z)] \\ \qquad =[(\nabla_{\perp}\times\mathrm{H})+(\nabla_{\perp}+\mathrm{z})\mathrm{H}_z]+\partial_z[\mathrm{z}\times\mathrm{H}]=\varepsilon_0\partial_t\mathrm{E}-\mathrm{z}\varepsilon_0\partial_t\mathrm{E}_z \end{cases} \tag{2.10}$$

By making use of a well-known vector identity $A(B\times C)=B(C\times A)$, the scalar multiplications of above equations by the unit vector z yield;

$$\begin{cases} \mathrm{z}.[\nabla\times \mathrm{E(R,t)}]=\mathrm{z}.[\nabla_{\perp}\times \mathrm{E}]=-\nabla_{\perp}.[\mathrm{z}\times \mathrm{E}]=-\mu_0\partial_t \mathrm{H}_z \\ \mathrm{z}.[\nabla\times \mathrm{H(R,t)}]=\mathrm{z}.[\nabla_{\perp}\times \mathrm{H}]=\nabla_{\perp}.[\mathrm{H}\times \mathrm{z}]=\varepsilon_0\partial_t \mathrm{E}_z \end{cases} \quad (2.11)$$

The protection of $[\nabla\times \mathrm{E(R,t)}]$ and $[\nabla\times \mathrm{H(R,t)}]$ onto S can be written as follows;

$$\begin{cases} [\nabla_{\perp}\times \mathrm{z}]\mathrm{E}_z+\partial_z[\mathrm{z}\times \mathrm{E}]=-\mu_0\partial_t \mathrm{H} \\ [\nabla_{\perp}\times \mathrm{z}]\mathrm{H}_z+\partial_z[\mathrm{z}\times \mathrm{H}]=\varepsilon_0\partial_t \mathrm{E} \end{cases} \quad (2.12)$$

Utilizing the identity $A\times B\times C = B(A.C)-C(A.B)$ known as triple product expansion or Lagrange's Formula, the vector multiplications of above eqautions by the unit vector z yield;

$$\begin{cases} \nabla_{\perp}\mathrm{E}_z=\mu_0\partial_t[\mathrm{H}\times \mathrm{z}]+\partial_z \mathrm{E} \\ \nabla_{\perp}\mathrm{H}_z=\mu_0\partial_t[\mathrm{z}\times \mathrm{E}]+\partial_z \mathrm{H} \end{cases} \quad (2.13)$$

In a similar fashion, the scalar Maxwell's equations can also be rewritten by using the transverse and longitudinal decompositions of nabla operator and of the fields as shown below;

$$\begin{cases} \nabla.\mathrm{E(R,t)}=[(\nabla_{\perp}+\mathrm{z}\partial_z).(\mathrm{E}+\mathrm{zE}_z)]=\nabla_{\perp}.\mathrm{E}+\partial_z \mathrm{E}_z=0 \\ \nabla.\mathrm{H(R,t)}=[(\nabla_{\perp}+\mathrm{z}\partial_z).(\mathrm{H}+\mathrm{zH}_z)]=\nabla_{\perp}.\mathrm{H}+\partial_z \mathrm{H}_z=0 \end{cases} \quad (2.14)$$

After the decomposition, the Maxwell's equations given in (2.3) and (2.4) can be grouped into two systems of equations: the system just contain E_z,

$$\nabla_{\perp}\mathrm{E}_z=\mu_0\partial_t[\mathrm{E}\times \mathrm{z}]+\partial_z \mathrm{E} \quad (2.15a)$$

$$\varepsilon_0\partial_t \mathrm{E}_z=\nabla_{\perp}.[\mathrm{H}\times \mathrm{z}] \quad (2.15b)$$

$$\partial_z \mathrm{E}_z=-\nabla_{\perp}.\mathrm{E} \quad (2.15c)$$

With the system just contain H_z component,

$$\nabla_\perp H_z = \varepsilon_0 \partial_t [z \times E] + \partial_z H \tag{2.16a}$$

$$\mu_0 \partial_t H_z = \nabla_\perp .[z \times E] \tag{2.16b}$$

$$\partial_z H_z = -\nabla_\perp .H \tag{2.16c}$$

are obtained in the form of transverse-longitudinal decompositions without any restriction. It can be seen that the equation (2.4) is written in (2.15c) and (2.16c). From the first couple of the boundary conditions (2.5).

$$\begin{cases} n.H\big|_L = 0 \Rightarrow n.(H + zH_z)\big|_L = 0 \\ l.E\big|_L = 0 \Rightarrow l.(E + zE_z)\big|_L = 0 \end{cases} \tag{2.17}$$

Can be written. From the last boundary condition in (2.5),

$$z.E\big|_L = 0 \Rightarrow z.(E + zE_z)\big|_L = 0 \Rightarrow E_z\big|_L = 0 \tag{2.18}$$

are obtained. When the condition, $E_z\big|_L = 0$ is applied to (2.15b) and (2.15c);

$$(\nabla_\perp [H \times z])\big|_L = 0,\ (\nabla_\perp .E)\big|_L = 0 \tag{2.19}$$

equalities are obtained.

2.2. TE Time-Domain Modes

TE modes are determined by $E_z(r,z,t) = 0$ condition. Substituting this condition in (2.32), $\nabla_\perp .[H \times z] = 0$ and $-\nabla_\perp .E = 0$ are obtained from (2.15b) and (3.6c).

$$\begin{cases} \nabla_\perp.[\mathrm{H}\times\mathrm{z}] = -\nabla_\perp.[\mathrm{z}\times\mathrm{H}] = -[\nabla_\perp\times\mathrm{z}].\mathrm{H} = 0 \\ \nabla_\perp.\mathrm{E} = 0 \end{cases} \tag{2.20}$$

vector identities $A\times B = -B\times A,\ A.(B\times C) = C.(A\times B), A.B = B.A$ are used in the first line, respectively to get identity 0=0.

In order to obtain the identity $[\nabla_\perp\times\nabla_\perp\psi(\mathbf{r})] = 0$, the transverse components of electric and magnetic field strength vectors can be written as;

$$\begin{cases} \mathrm{E(r,z,t)} = V(z,t)[\nabla_\perp\psi(\mathrm{r})\times\mathrm{z}] \\ \mathrm{H(r,z,t)} = I(z,t)\nabla_\perp\psi(\mathrm{r}) \end{cases} \tag{2.21}$$

The functions $\psi(\mathrm{r})$, $V(z,t)$ and $I(z,t)$ have found later. According to the system (2.16), H_z have been considered as $H_z = A(z,t)\psi(\mathbf{r})$. The unknown function $A(z,t)$ have also been found later. When H_z and the field vectors E and H in (2.21) are substituted in (2.16b) and (2.16c).

$$\begin{cases} -\mu_0\psi(\mathrm{r})\partial_t A(z,t) = -V(z,t)\nabla_\perp.[\mathrm{z}\times[\nabla_\perp\psi(\mathrm{r})\times\mathrm{z}]] \\ \psi(\mathrm{r})\partial_z A(z,t) = -I(z,t)\nabla_\perp.\nabla_\perp\psi(\mathrm{r}) \end{cases} \tag{2.22}$$

are obtained. After applying the triple product expansion to the right-hand side of the first line, and making use of $\nabla_\perp.\nabla_\perp\psi(\mathrm{r}) = -\nabla^2{}_\perp\psi(\mathrm{r})$ we obtaine

$$\begin{cases} -\mu_0\psi(\mathrm{r})\partial_t A(z,t) = [-\nabla^2{}_\perp\psi(\mathrm{r})]V(z,t) \\ \psi(\mathrm{r})\partial_z A(z,t) = [-\nabla^2{}_\perp\psi(\mathrm{r})]I(z,t) \end{cases} \tag{2.23}$$

Which indicates that $\psi(\mathrm{r})$ is twice differentiable with respect to the transverse coordinates, (r). Lastly substituting (2.21) and $H_z = A(z,t)\psi(\mathrm{r})$ in (2.16a) and using $[\mathrm{z}\times[\nabla_\perp\psi(\mathrm{r})\times\mathrm{z}]] = \nabla_\perp\psi(\mathrm{r})$ and eliminating the term $\nabla_\perp\psi(\mathrm{r})$ from both sides of the equation,

$$\nabla_{\perp} H_z = \varepsilon_0 \partial_t [\mathrm{z} \times \mathrm{E}] + \partial_z \mathrm{H} \tag{2.24}$$

$$\nabla_{\perp}[A(z,t)\psi(\mathrm{r})] = \varepsilon_0 \partial_t \left[\mathrm{z} \times \left[V(z,t)[\nabla_{\perp}\psi(\mathrm{r}) \times \mathrm{z}]\right]\right] + \partial_z [I(z,t)\nabla_{\perp}\psi(\mathrm{r})] \tag{2.25}$$

$$A(z,t) = \varepsilon_0 \partial_t V(z,t) + \partial_z I(z,t) \tag{2.26}$$

are found. In (2.17), every boundary condition for the transverse componenets of electric and magnetic field vectors E and H gives the same boundary condition for the function $\psi(\mathbf{r})$ as follows; Substitution of (2.21) into (2.17) yields the following conditions;

$$\begin{cases} I(z,t)[\mathrm{n}.\nabla_{\perp}\psi(\mathrm{r})]\big|_L = 0 \\ V(z,t)[l.\nabla_{\perp}\psi(\mathrm{r}) \times \mathrm{z}]\big|_L = 0 \end{cases} \tag{2.27}$$

Notice that $I(z,t)$ and $V(z,t)$ must not be zero, otherwise, the transverse components of electric and magnetic field vectors in (2.21) will be zero. Since nontrivial solutions are searched for, above equations should be interpreted as follows;

$$\begin{cases} [\mathrm{n}.\nabla_{\perp}\psi(\mathrm{r})]\big|_L = 0 \\ [l.\nabla_{\perp}\psi(\mathrm{r}) \times \mathrm{z}]\big|_L = 0 \end{cases} \tag{2.28}$$

Hence, these boundary conditions yield the same condition for the potential $\psi(\mathrm{r})$ as follows;

$$\partial_n \psi(\mathrm{r})\big|_L = 0 \tag{2.29}$$

Stating that the normal derivative of $\psi(\mathrm{r})$ on the contour L equals to zero. Moreover, the definitions in (2.21) satisfy the boundary conditions in (2.19). It is seen from (2.23) that the potential $\psi(\mathrm{r})$ must be twice differentiable. This situation and the boundary condition (2.29) propose to use Neumann boundary-eigenvalue problem to obtain the TE modal fields. $\mathrm{r} \in S$ and $\upsilon_n^2 \geq 0$ $(n = 0,1,2..)$ are the real eigenvalues. For the TE modes, the following problem which is a Neumann boundary-eigenvalue problem is solved.

$$\begin{cases} (\nabla_\perp^2 + \upsilon_n^2)\psi_m(\mathrm{r}) = 0 \\ \dfrac{\partial \psi_n(\mathrm{r})}{\partial n}_{|L} = 0 \\ \dfrac{\upsilon_n^2}{S}\int_S |\psi_n(\mathrm{r})|^2 \, ds = 1N \end{cases} \tag{2.30}$$

$\partial_n = \mathrm{n} \cdot \nabla_\perp, L$ is normal derivative on the contour L, $\upsilon_n^2 > 0$ $m = 1,2..$ are eigenvectors, the index n is numerical values listed in ascending order on the real axis, $\psi_n(\mathrm{r})$ are eigenvectors corresponding to these eigenvalues. Force size of (2.30) equation should be N (Newton) so that Vm^{-1} and Am^{-1} physical dimensions are respectively provided for E_n and H_n field vector extensions.

In (2.30) equations set $\{\psi_n(\mathrm{r})\}_{n=0}^{\infty}$ are the eigenfunctions according to the eigenvalues $\{\upsilon_n^2\}_{n=0}^{\infty}$ in problem (2.30) and these functions are complete in Hilbert space $L_2(S)$ space. Thus, any potential $\psi(\mathrm{r})$ that satisfy the condition (2.29) can be written in terms of functions $\psi_n(\mathrm{r})$. To this aim the functions $\psi_n(\mathrm{r})$ should be normalized. The complete set of functions $\{\psi_n(\mathrm{r})\}_{n=0}^{\infty}$, generates a complete set of TE modes in the time domain.

When $[-\nabla_\perp^2 \psi_n(\mathrm{r})] = \upsilon_n^2 \psi_n(\mathrm{r})\sqrt{2}$ is substituted in (2.23) and after introducing $A(z,t)$ as $A(z,t) = \upsilon_n^2 h_n(z,t)$ it is alsıo used in (2.23). Then,

$$V_n(z,t) = -\mu_0 \partial_t h_n(z,t) \quad , \quad I_n(z,t) = \partial_z h_n(z,t) \tag{2.31}$$

are found. These equalities and the expression $A(z,t) = \upsilon_n^2 h_n(z,t)$ are substituted in (2.26).

$$\begin{cases} E_{zn}^{h} = 0 \\ \upsilon_n^{-1} E_n^{h} = \left\langle -\partial_{(\upsilon_n \mathrm{ct})} h_n(\mathrm{z,t}) \right\rangle \left[\sqrt[-2]{\varepsilon_0} \nabla_{\perp} \psi_n(\mathrm{r}) \times \mathrm{z} \right] \\ \upsilon_n^{-1} H_n^{h} = \left\langle \partial_{(\upsilon_n \mathrm{z})} h_n(\mathrm{z,t}) \right\rangle \left[\sqrt[-2]{\mu_0} \nabla_{\perp} \psi_n(\mathrm{r}) \right] \\ \upsilon_n^{-1} H_{zn}^{h} = \left\langle h_n(\mathrm{z,t}) \right\rangle \left[\upsilon_n \sqrt[-2]{\mu_0} \psi_n(\mathrm{r}) \right] \end{cases} \tag{2.32}$$

$\partial_{(\upsilon_n ct)} = (1/c\upsilon_n)\partial_t, \partial_{(\upsilon_n \mathrm{z})} = (1/\upsilon_n)\partial_z$ *ve* $c = \sqrt[-2]{\varepsilon_0 \mu_0}$ is the speed of elegant space.

Modal field pairs is shown with brackets $[*]$ in waveguide section-area. All of the potential here is obtained from the Laplacian. Factors expressed in broken bracket $\langle * \rangle$ has physical meaning which is modal amplitudes of the appropriate extension of modal fields. All is determined by potential $h_n(\mathrm{z,t})$ derived from Klein-Gordon equation.

$$(\partial_{\upsilon_n ct}^2 - \partial_{\upsilon_n z}^2 + 1)\,\mathrm{h}_n(\mathrm{z,t}) = 0 \tag{2.33}$$

(2.30) problem has a more zero obvious solution. This solution corresponds to $\upsilon_0^2 = 0$ eigenvalues and generates $\nabla_{\perp}^2 \Psi_o(\mathrm{r}) = 0,\ \partial_n \Psi_0(\mathrm{r})_{|L} = 0$ problem for harmonic functions $\Psi_0(\mathrm{r})$. $\mathrm{r} \in L + S$ and c_1 is a constant, including harmonic functions as $\Psi_0(\mathrm{r}) = c_1$ is obtained from the maximum-minimum theorem. $\Psi_0(\mathrm{r})$ potetial produces more TE mode.

$$E_0^h(\mathrm{r,z,t}) = 0, \quad H_0^h(\mathrm{r,z,t}) = \mathrm{z}c_1 \tag{2.34}$$

Modal amplitude of the H_0^h field is a constant.

Weyl Theorem refers that modes of TE time-domain is complete in L_2 Hilbert space [Weyl, 1940].

Example 2.1: Suppose that L contour which is waveguide as rectangular with determined the boundaries of $\nabla_{\perp}^{2}\Psi_{o}(\mathrm{r})=0,\ \partial_{n}\Psi_{0}(\mathrm{r})_{|L}=0$. *The normalization constant,* A_{n}^{h} *is potential, let take* $\Psi_{n}(\mathrm{r})$ *as* $\Psi_{n}(\mathrm{r})=A_{n}^{h}\psi_{n}(\mathrm{r})$. *The equation is* $\psi_{n}(\mathrm{r})=\cos(p\pi x/a)\cos(q\pi y/b)$. *Parameters* p *and* q *are integers provide the requirement which is* $p,q=0,1,2..,\ p+q\neq 0$. *The following equation is obtained by solving (2.30) equation.*

$$\begin{cases} \upsilon_{n}^{2}=\pi^{2}\left(\dfrac{p^{2}}{a^{2}}+\dfrac{q^{2}}{b^{2}}\right)\equiv \upsilon_{p,q}^{2} \\ A_{n}^{h}=\dfrac{\sqrt{(2-\delta_{p,0})(2-\delta_{\mathrm{q},0})}}{\upsilon_{p,q}}\equiv A_{p,q}^{h} \end{cases} \tag{2.35}$$

$\delta_{p,0}$ *and* $\delta_{\mathrm{q},0}$ *are Kronecker deltas. (Remember;* $\delta_{m,n}=\begin{cases}0,\ if\ \ m\neq n\\ 1,\ if\ \ m=n\end{cases}$ *)* m *is index that position* υ_{n}^{2} *eigenvalues on the real axis.* $n\rightarrow(\mathrm{p,q}).\Psi_{n}(\mathrm{r})$ *is part of the definition of the potential of time-domain* TE *mode.*

2.3. TM Time-Domain Modes

Firstly, Dirichlet problem should be solved.

$$\begin{cases} (\nabla_{\perp}^{2}+\kappa_{m}^{2})\phi_{m}(\mathrm{r})=0 \\ \phi_{m}(\mathrm{r})_{|L}=0 \\ \dfrac{\kappa_{m}^{2}}{S}\displaystyle\int_{S}\left|\phi_{m}(\mathrm{r})\right|^{2}ds=1N \end{cases} \tag{2.36}$$

In here, $\kappa_{m}^{2}>0,\ m=1,2..$ are eigenvalues, m indexes are numerical values that listed by ascending order in the real axis and $\phi_{m}(\mathrm{r})$ are corresponding to these eigenvalues eigenvectors. The physical dimensions of extension of E_m and H_m field vector are respectively Vm^{-1} and Am^{-1}. $\phi_0(\mathrm{r})$ solution which is corresponding to $\kappa_0^2 = 0$ eigenvalues is zero.

Example 2.2: Let's consider an example of a more rectangular waveguide. A_m^e *normalization including the constant,* $\Phi_m(\mathrm{r})$ *potential received as* $\Phi_m(\mathrm{r}) = A_m^e \phi_m(\mathrm{r})$. $\phi_m(\mathrm{r}) = \sin(p\pi x/a)\sin(q\pi y/\mathrm{b})$ *provided* $p+q \neq 0$, p *and* q *are parameters,* $p,q = 0,1,2..$ *are integers. By (2.36) problem is solving,*

$$\begin{cases} \kappa_m^2 = \pi^2\left(\dfrac{p^2}{a^2}+\dfrac{q^2}{b^2}\right) \equiv \kappa_{p,q}^2 \\ A_m^e = \dfrac{2}{\kappa_m} \equiv A_{p,q}^e \end{cases} \tag{2.37}$$

In TE *modes like* $\Phi_m(\mathrm{r})$ *potential,* $\Psi_m(\mathrm{r})$ *potential is part of definition of time-domain* TM *modes too.*

$$\begin{cases} \mathrm{H}_{zm}^e = 0 \\ \kappa_m^{-1}\mathrm{H}_m^e = \left\langle -\partial_{(\kappa_m \mathrm{ct})} e_m(\mathrm{z},\mathrm{t})\right\rangle \left[z \times \sqrt[-2]{\mu_0}\nabla_\perp \phi_m(\mathrm{r})\right] \\ \kappa_m^{-1}\mathrm{E}_m^e = \left\langle \partial_{(\kappa_m \mathrm{z})} e_m(\mathrm{z},\mathrm{t})\right\rangle \left[\sqrt[-2]{\varepsilon_0}\nabla_\perp \phi_m(\mathrm{r})\right] \\ \kappa_m^{-1}\mathrm{E}_{zm}^e = \left\langle e_m(\mathrm{z},\mathrm{t})\right\rangle \left[\kappa_m \sqrt[-2]{\varepsilon_0}\phi_m(\mathrm{r})\right] \end{cases} \tag{2.38}$$

$\partial_{(\kappa_m ct)} = (1/c\kappa_m)\partial_t, \partial_{(\kappa_m \mathrm{z})} = (1/\kappa_m)\partial_z$ *ve* $c = \sqrt[-2]{\varepsilon_0\mu_0}$ *is the speed of light in free space.*

3. GENERAL CHARACTERISTICS OF TIME-DOMAIN MODES

3.1. Completeness of Time-Domain Modes

Both of (2.32) and (2.23) time-domain modes defines arbitrary X_1, X_2 vector pair determined by following inner product in the real-valued functional vectors space with 6 components.

$$(\mathrm{X}_1, \mathrm{X}_2) = \frac{1}{S}\int_S (\varepsilon_0 \mathrm{E}_1.\mathrm{E}_2 + \mu_0 \mathrm{H}_1.\mathrm{H}_2)\,\mathrm{ds} < \infty \tag{3.1}$$

$X_1 = col(\mathrm{E}_1, \mathrm{H}_1)$ ve $X_2 = col(\mathrm{E}_2, \mathrm{H}_2)$. "Col" means "column" and is the scalar product for vectors with 3 components.

Lets show respectively (2.32) and (2.23) set of TE and TM modes as following,

$$\mathrm{G} = \left\{X_m^h\right\}_{m=0}^{\infty}, \; \Im = \left\{X_m^e\right\}_{m=1}^{\infty} \tag{3.2}$$

$X_m^h = col(\mathrm{E}_m^h, \mathrm{H}_m^h)$ ve $X_m^e = col(\mathrm{E}_m^e, \mathrm{H}_m^e)$. $m = m'$ including, we can found $(X_m^h, X_{m'}^h) = 0$ results if we take $X_m^h \, ve \, X_{m'}^h$ for the first couple of different elements and put these instead of X_1, X_2 in the (3.1) equation. This show us that in the G set, all of elements are *mutually orthogonal*. The same situation applies for element of $\Im$ set. Now, lets take X_m^h element from G set and $X_{m'}^h$ element from $\Im$ set and put in the (3.1) equation. Again m, m' including arbitrary, we get $(X_m^h, X_{m'}^e) = 0$ results.

The completeness of each of G and $\Im$ has been proven in studies [Tretyakov 1990] and [Tretyakov, 1993]. In addition, G and $\Im$ form a subspace in the space of solutions. The direct sum of them eventually: $\mathrm{G} \oplus \Im$. Therefore, a subspace is orthogonal to the other.

3.2. Relativistic invariance of Time-Domain Modes

Each of time-domain $(\mathrm{E}_m, \mathrm{H}_m)$ modal field of (2.32) and (2.23) equations is a particular solution of Maxwell's equations system which protect ∂_t time derivative.

$$\nabla \times \mathrm{E}_m = -\mu_0 \partial_t \mathrm{H}_m \quad \text{and} \quad \nabla \times \mathrm{H}_m = -\varepsilon_0 \partial_t \mathrm{E}_m \tag{3.3}$$

The time-domain modal fields as solution of Maxwell's equations system which protect ∂_t time derivative should be calculated according to the special theory of relativity. Lets take stable framework of (2.32) and (2.23) the time-domain modes with $(\mathrm{r}, \mathrm{z}, \mathrm{t})$ coordinates and t is F. Let the new F' reference framework. Let's assume that there is ongoing with a constant speed ν along the movement of the Oz axis. Lets show coordinates and time with $(\mathrm{r'}, \mathrm{z'}, \mathrm{t'})$ in F'. The corresponding (relationhip) between $(\mathrm{r}, \mathrm{z}, \mathrm{t})$ in F and $(\mathrm{r'}, \mathrm{z'}, \mathrm{t'})$ in F' is determined by direct Lorentz transformation:

$$\mathrm{r} = \mathrm{r'},\ z' = (\mathrm{z'} - \nu\, \mathrm{t'})\gamma,\ t' = (t' - \nu z'/c^2)\gamma \tag{3.4}$$

$t' = (t' - \nu z'/c^2)\gamma$ Inverse Lorentz transformation located symmetrically as following:

$$\mathrm{r} = \mathrm{r'}\ ,\ \ z' = (\mathrm{z} - \nu\, \mathrm{t})\gamma\ ,\ \ t' = (t - \nu z/c^2)\gamma \tag{3.5}$$

$\mathrm{r} = \mathrm{r'}$ is received, the solution of the problems (2.30) and (2.36) are overlap each other in F and F' reference framework. This means; physically, waves quide is not the same as the modal base in the F and F'. This means; physically, wavesguide of the modal bases is same in F and F'. So the waveguide modal field is invariant in F and F'. Evidently, υ_m^2 *and* κ_m^2 eigenvalues is same in the inertial reference framework.[Tretyakov and Akgun, 2010].

Last actually, TE and TM time-domain modes is uniform in the framework too [Tretyakov and Akgun, 2010]. Lets choose "Dimensionless" time τ and axial coordinates ξ.

$$\tau = \nu_m ct \quad ve \quad \xi = \nu_m z \quad \text{for } TE \tag{3.6a}$$

$$\tau = \kappa_m ct \quad ve \quad \xi = \kappa_m z \quad \text{for } TM \tag{3.6b}$$

So, (2.33) and (3.1) are the same equations.

The invariance of $h_m(\xi,\tau)$ and $e_m(\xi,\tau)$ amplitude in Maxwell's equations presented with (3.7) Klein-Gordon equation.

$$(\partial_\tau^2 - \partial_\xi^2 + 1) f(\xi,\tau) = 0 \tag{3.7}$$

Provided that $\tau = \upsilon_m ct,\ \xi = \upsilon_m z$, $f(\xi,\tau), \mathrm{h}_m(\xi,\tau)$ is instead of $f(\xi,\tau), \mathrm{e}_m(\xi,\tau)$ provided that $\tau = \kappa_m ct,\ \xi = \kappa_m z$.

It can be shown that (3.7) equation have the same form in both of the reference frame of the F and F'. We should take (3.4) and (3.5) Lorentz transformation and adapt to the variable (ξ,τ) and (ξ',τ').

$$\xi = (\xi' - \beta\tau')\gamma,\ \tau = (\tau' - \beta\xi')\gamma \tag{3.8a}$$

$$\xi' = (\xi - \beta\tau)\gamma,\ \tau' = (\tau - \beta\xi)\gamma \tag{3.8b}$$

Lets take (ξ',τ') instead of (ξ,τ) is solution of (3.7) equation and determine $f(\xi,\tau)$ founction with new variable as $f[\xi'(\xi,\tau),\tau'(\xi,\tau)]$ with the aid of inverse Lorentz transformation:

Taking the partial derivative of $\partial_{\xi'}, \partial_{\tau'}$ instead of $\partial_\xi, \partial_\tau$;

$$\partial_{\tau} f(\xi', \tau') = (-\beta\gamma\partial_{\xi'} + \gamma\partial_{\tau'}) f(\xi', \tau') \tag{3.9a}$$

$$\partial_{\xi} f(\xi', \tau') = (\gamma\partial_{\xi'} - \beta\gamma\partial_{\tau'}) f(\xi', \tau') \tag{3.9b}$$

In (3.7) equation, the following equation is obtained by repeated twice.

$$(\partial_{\tau'}^2 - \partial_{\xi'}^2 + 1) f(\xi', \tau') = 0 \tag{3.10}$$

(3.10) equation is valid for the reference framework F'. (3.8) equation in the reference framework F has the same form.[Tretyakov and Akgun, 2010].

3.3. Initial Conditions for the Klein-Gordon Equation

Klein-Gordon equation like other second degree partial differential equations (PDE) must be supported by "initial conditions" pairs. Physically, these are involved in excitation of suitable signal source. Assume that these source is not induction before $t = 0$ (which is stationary) but it started to excitation at $t = 0$.
If so, the initial condition is as follows.

$$f(\xi, \tau)_{|\xi=0} = \begin{cases} \varphi(\tau), \tau \geq 0 \Rightarrow \; for \; t \geq 0 \\ 0, \tau < 0 \Rightarrow \; for \; \tau < 0 \end{cases}$$
$$\frac{\partial}{\partial \tau} f(\xi, \tau)_{|\xi=0} = \begin{cases} \hat{\varphi}(\tau), \tau \geq 0 \Rightarrow \; for \; t \geq 0 \\ 0, \tau < 0 \Rightarrow \; for \; \tau < 0 \end{cases} \tag{3.11}$$

$\varphi(\tau), \hat{\varphi}(\tau)$ must be given. $\xi = 0 \Rightarrow z = 0$ should be.

3.4. The Principle of Causality

The solution of the Klein-Gordon equation must be adhered to requirements of principle of causality. Here are the latest two comments and one supports the other.

If resources are zero at this initial time, there is weak causality that whole fields are zero. In our problem, this corresponds to $\tau < 0$.Strong causality condition follow the claim that magnetic field emitted signal with speed of light, c in the space, is suggested by Einstein. Our problem is the source is taking place at $\xi = 0$, inside sectional area of the waveguide. This define that solution of Klein-Gordon must be zero beyond $\xi = 0$ source point, after $\xi = \tau$ (i.e. $z = ct$). Therefore, if the signal is spread along the axis Oz, the solution of Klein-Gordon equation must be physically regarded as follows.

$$f(\tau,\xi) = \begin{cases} f(\tau,\xi) = 0 \ if \ \ \tau < 0 \\ f(\tau,\xi) \neq 0 \ if \ \ 0 \leq \xi \leq \tau \\ f(\tau,\xi) = 0 \ if \ \ \xi > \tau \end{cases} \tag{3.12}$$

4.MODAL AMPLITUDE PROBLEMS

The Klein-Gordon equation invariant under the Lorentz transformation was discussed. In addition, the equation retains its properties when it is worked under the Poincaregroup in the skeleton framework of group theory. W. Jr. Miller, who worked with Klein-Gordon equation and has been created "orbit symmetry" which are orthogonal to each other in the solution of this equation. A result of studies have found wide application area in the development of electromagnetic field theory in the time-domain. You will find an example in this regard.

Its remembered that $f(\xi,\tau)$ solution of (3.7) the Klein-Gordon equation for determine the potential for modal amplitude. Its considered that given $f\left[u(\xi,\tau),v(\xi,\tau)\right]$ function has yet unknown "new" (u,v) variable and (u,v) is twice differentiable function which has "old" (ξ,τ) variable. The replacement of $f\left[u(\xi,\tau),v(\xi,\tau)\right]$ in the solution of the equation (3.7) gives a new form of equation after the following manipulations.

$$f \equiv f(\xi,\tau) = f\left[u(\xi,\tau),v(\xi,\tau)\right] \equiv f(u,v) = U(u)V(v) \tag{4.1}$$

And then,

$$\frac{\partial}{\partial\tau} f(u,v) = \frac{\partial f}{\partial u}\frac{\partial u}{\partial\tau} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial\tau} \tag{4.2}$$

$$\frac{\partial}{\partial\xi} f(u,v) = \frac{\partial f}{\partial u}\frac{\partial u}{\partial\xi} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial\xi} \tag{4.3}$$

$$\frac{\partial^2}{\partial\tau^2} f(u,v) = \frac{\partial^2 f}{\partial u^2}\left(\frac{\partial u}{\partial\tau}\right)^2 + \frac{\partial f}{\partial u}\frac{\partial^2 u}{\partial\tau^2} + \frac{\partial^2 f}{\partial v^2}\left(\frac{\partial v}{\partial\tau}\right)^2 + \frac{\partial f}{\partial v}\frac{\partial^2 v}{\partial\tau^2} + \frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial\tau}\frac{\partial v}{\partial\tau} + \frac{\partial^2 f}{\partial v\partial u}\frac{\partial v}{\partial\tau}\frac{\partial u}{\partial\tau} \tag{4.4}$$

$$\frac{\partial^2}{\partial\tau^2} f(u,v) = \frac{\partial^2 f}{\partial u^2}\left(\frac{\partial u}{\partial\tau}\right)^2 + \frac{\partial f}{\partial u}\frac{\partial^2 u}{\partial\tau^2} + \frac{\partial^2 f}{\partial v^2}\left(\frac{\partial v}{\partial\tau}\right)^2 + \frac{\partial f}{\partial v}\frac{\partial^2 v}{\partial\tau^2} + 2\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial\tau}\frac{\partial v}{\partial\tau} \qquad (4.5)$$

$$\frac{\partial^2}{\partial\xi^2} f(u,v) = \frac{\partial^2 f}{\partial u^2}\left(\frac{\partial u}{\partial\xi}\right)^2 + \frac{\partial f}{\partial u}\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 f}{\partial v^2}\left(\frac{\partial v}{\partial\xi}\right)^2 + \frac{\partial f}{\partial v}\frac{\partial^2 v}{\partial\xi^2} + 2\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial\xi}\frac{\partial v}{\partial\xi} \qquad (4.6)$$

İt's written (4.2) - (4.6) equations instead of (3.7)

$$\left[\left(\frac{\partial u}{\partial\tau}\right)^2 - \left(\frac{\partial u}{\partial\xi}\right)^2\right]\frac{\partial^2 f}{\partial u^2} + \left[\left(\frac{\partial v}{\partial\tau}\right)^2 - \left(\frac{\partial v}{\partial\xi}\right)^2\right]\frac{\partial^2 f}{\partial v^2} + \left[\frac{\partial^2 u}{\partial\tau^2} - \frac{\partial^2 u}{\partial\xi^2}\right]\frac{\partial f}{\partial u}$$
$$+\left[\frac{\partial^2 v}{\partial\tau^2} - \frac{\partial^2 v}{\partial\xi^2}\right]\frac{\partial f}{\partial v} + 2\left[\frac{\partial u}{\partial\tau}\frac{\partial v}{\partial\tau} - \frac{\partial u}{\partial\xi}\frac{\partial v}{\partial\xi}\right]\frac{\partial^2 f}{\partial u\partial v} + f = 0 \qquad (4.7)$$

Here the f function is a source function which is dependent on (u,v) independent values. Next goal is the elimination of old (ξ,τ) arguments from (4.7) equation. In the study of orbits of symmetry, a set of pairs of $\xi(u,v)$ and $\tau(u,v)$ inverse function of Miller is shown.[Appendix]. It is a simple case to achieve the dual function $u(\xi,\tau)$ and $v(\xi,\tau)$.

Time-harmonic modal amplitude is a particular case among others. In Miller's list, it corresponds case1. Really, when $u=\tau, v=\xi$ writing in the equation (4.7), the equation reverts to the original (3.7) equation. It is obvious that (ξ,τ) variable lead to the separation of time-harmonic mode. The physical sense of υ_m^2 and κ_m^2 eigenvalues is established.

4.1. An Overview For Bessel Functions

4.1.1.Series Solutions Near a Regular Singular Point

n-th order linear homogeneous ordinary differential equations be given as follows.

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + ... + p_0(x)y = 0 \qquad (4.8)$$

i) If all of $p_0(x), p_1(x),..., p_{n-1}(x)$ coefficients are not analytical at $x = x_0$, x_0 is called a singular point of the given differential equation.

ii) If all of $p_k(x)$ coefficients are not analytical but all of $(x - x_o)^{n-k} p_k(x)$ are analytical for $k = 0,1,..,(n-1)$, x_0 is called a regular singular point.

iii) If x_0 is the neither ordinary point nor the regular singular point of given differential equations in this case x_0 is irregular singular point.

$y'' + P(x)y' + Q(x)y = 0$ second order linear homogeneous ordinary differential equations been given. If $x = 0$ is a regular singular point, $xP(x)$ and $x^2Q(x)$ functions can expand into power series and can write as follows.

$$\begin{cases} xP(x) = \sum_{n=0}^{\infty} P_n x^n \,, |x| < r \\ x^2Q(x) = \sum_{n=0}^{\infty} Q_n x^n \,, |x| < r \end{cases} \quad \text{or} \quad \begin{cases} P(x) = \sum_{n=0}^{\infty} P_n x^{n-1} \,, |x| < r \,, x \neq 0 \\ Q(x) = \sum_{n=0}^{\infty} Q_n x^{n-2} \,, |x| < r \,, x \neq 0 \end{cases} \tag{4.9}$$

Frobenius solution of the given differential equation is suggested as followss form.

$$y(x) = x^a \sum_{n=0}^{\infty} a_n x^{n+a} = \sum_{n=0}^{\infty} a_n x^{n+a} \,, 0 < x < r, \tag{4.10}$$

When we take derivative of y according to x and write into equation, we can result as follows.

$$y'(x) = \sum_{n=0}^{\infty} (n+a) a_n x^{n+a-1} \,, \qquad y''(x) = \sum_{n=0}^{\infty} (n+a)(n+a-1) a_n x^{n+a-2} \tag{4.11}$$

$$\sum_{n=0}^{\infty}(n+a)(n+a-1)a_n x^{n+a-2} +$$

$$+\sum_{n=0}^{\infty}P_n x^{n-1}\sum_{n=0}^{\infty}(n+a)a_n x^{n+a-1}+\sum_{n=0}^{\infty}Q_n x^{n-2}\sum_{n=0}^{\infty}a_n x^{n+a} = 0 \qquad (4.12)$$

$$\sum_{n=0}^{\infty}Q_n x^{n-2}\sum_{n=0}^{\infty}a_n x^{n+a} = \sum_{n=0}^{\infty}(\sum_{m=0}^{n}Q_{n-m}a_m)x^{n+a-2}$$

$$=\sum_{n=0}^{\infty}\{(n+a)(n+a-1)a_n+\sum_{m=0}^{n}[(m+a)P_{n-m}+Q_{n-m}]a_m\}x^{n+a-2} = 0 \qquad (4.13)$$

$x^{n+a-2}, n=0,1,...$ coefficient must be zero in order to ensure the equality. For $n=0$; $[a(a-1)+aP_0+Q_0]a_0=0$ equation is obtained.

So, $a_0 = 0$ *or* $a(a-1)+aP_0+Q_0=0$ should be.

For $n=1$; If $(n+a)(n+a-1)a_n+\sum_{m=0}^{n}[(m+a)P_{n-m}+Q_{n-m}]a_m = 0$, the following equation can be written.

$$a_n = -\frac{1}{(n+a)(n+a-1)(n+a)P_0+Q_0}\sum_{m=0}^{n-1}[(m+a)\,\mathrm{P}_{n-m}+Q_{n-m}]a_m \qquad (4.14)$$

i) If $a_0 = 0$, $a_1 = a_2 = ... = 0$ and so, $a_0 = 0$ solution can be obtained.
ii) If $a_0 \neq 0$, $a(a-1)+aP_0+Q_0=0$ index equation can be obtained.

a_1 *and* a_2 roots can be obtain by solving that index equation for a. Because of that $a_0 \neq 0$ should be for not zero solution and a is root of index equation.

4.1.2. Series Solutions Near a Regular Singular Point

$$y''(x)+P(x)y'(x)+Q(x)y(x)=0 \qquad (4.15)$$

If $x=0$ is a regular singular point for second order linear homogeneous ordinary differential equation, the followsing equation can be write.

$$xP(x) = \sum_{n=0}^{\infty} P_n x^n \text{ , } x^2 Q(x) = \sum_{n=0}^{\infty} Q_n x^n \text{ , } |x| < r \tag{4.16}$$

Lets accept that $a(a-1) + aP_0 + Q_0 = 0$ index equation has two real root a_1 *and* a_2 that $a_1 \geq a_2$. So, giving differential equation has series solution as follows.

$$y_1(x) = x^{a_1} \sum_{n=0}^{\infty} a_n x^n \text{ , } a_o \neq 0, \quad 0 < x < r \tag{4.17}$$

a_n coefficients is found by replacing in $y_1(x)$ differantial equation. The second linear independent solution is found as follows. [Xie, 2010]

i) If $a_1 - a_2$ is not integer, second Frobenius series solution is as follows.

$$y_2(x) = x^{a_2} \sum_{n=0}^{\infty} b_n x^n \text{ , } \quad 0 < x < r \tag{4.18}$$

b_n coefficients is found by replacing in $y_2(x)$ differantial equation.

ii) If $a_1 = a_2 = a$,

$$y_2(x) = y_1(x) \ln x + x^a \sum_{n=0}^{\infty} b_n x^n \text{ , } \quad 0 < x < r \tag{4.19}$$

b_n coefficients is found by replacing in $y_2(x)$ differantial equation. So, $y_2(x)$ second solution is not a Frobenius series solution.

iii) If $a_1 - a_2$ is integer,

$$y_2(x) = a y_1(x) \ln x + x^{a_2} \sum_{n=0}^{\infty} b_n x^n \text{ , } \quad 0 < x < r \tag{4.20}$$

b_n and a coefficients is found by replacing in $y_2(x)$ differantial equation. a parameter can be zero and $y_2(x)$ second solution can be a Frobenius series solution.

As a result general solution of differantial solution can be given as follows.

$$y(x) = C_1 y_1(x) + C_2 y_2(x) \tag{4.21}$$

4.1.3.Bessel Differantial Equation and Solution

Bessel differantial equation is form as follows.

$$x^2 y'' + xy' + (x^2 - p^2)y = 0,\ x > 0 \tag{4.22}$$

$p \geq 0$ is a constant. The series solutions method near a regular singular point is used for solution of bessel differantial equation.

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0,\ P(x) = \frac{1}{x},\ Q(x) = \frac{x^2 - p^2}{x^2} \tag{4.23}$$

It is clear that $x = 0$ is a regular singular point. Because;

$$\begin{cases} xP(x) = 1 = 1 + 0.x + 0.x^2 + \ldots \Rightarrow P_0 = 1, \\ x^2 Q(x) = x^2 - p^2 = -p^2 + 0.x + 0.x^2 + \ldots \Rightarrow Q_0 = -p^2, \end{cases} \tag{4.24}$$

Both of $x^2 Q(x)$ and $xP(x)$ are analytical at $x = 0$ point and they can be expanded to series, these series is convergent for $|x| < \infty$. So, $x = 0$ is a regular singular point.

Index equation is as $a(a-1) + aP_0 + Q_0 = 0$

$$a(a-1) + a.1 - p^2 = 0 \Rightarrow a - p^2 = 0 \Rightarrow a_1 = p,\ a_2 = -p \tag{4.25}$$

Bessel differantial equation has series solution as follows.

$$y_1(x) = x^p \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+p},\ a_0 \neq 0,\ 0 < x < \infty \tag{4.26}$$

If we take derivatives by x,

$$\begin{cases} y_1'(x) = \sum_{n=0}^{\infty} (n+p) a_n x^{n+p-1}, \\ y_1''(x) = \sum_{n=0}^{\infty} (n+p)(n+p-1) a_n x^{n+p-2} \end{cases} \tag{4.27}$$

y_1, y_1', y_1'' values is replace in the equation,

$$x^2 \sum_{n=0}^{\infty} (n+p)(n+p-1) a_n x^{n+p-2} + x \sum_{n=0}^{\infty} (n+p) a_n x^{n+p-1} + \\ + (x^2 - p^2) \sum_{n=0}^{\infty} a_n x^{n+p} = 0 \tag{4.28}$$

By changing index of sum sign,

$$\sum_{n=0}^{\infty} a_n x^{n+p+2} \overset{n+2=m}{\Rightarrow} \sum_{n=2}^{\infty} a_{m-2} x^{m+p} = \sum_{n=2}^{\infty} a_{n-2} x^{n+p} \tag{4.29}$$

$$\begin{cases} x^p \{ \sum_{n=0}^{\infty} [(n+p)(n+p-1) + (n+p) - p^2] a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n \} = 0, \\ x^p \neq 0 \Rightarrow \sum_{n=0}^{\infty} n(n+2p) a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0 \end{cases} \tag{4.30}$$

$x^n, n = 0,1...$ should be zero to achieve this equation.

$x^0 : 0.(0+2p)a_0 = 0 \Rightarrow a_0 \neq 0$ is arbitrary constant.

$$x^1 : 1.(1+2p)a_1 = 0 \Rightarrow a_1 = 0 \tag{4.31}$$

for $n \geq 2$;

$$x^n : n(n+2p)a_n + a_{n-2} = 0 \Rightarrow a_n = -\frac{a_{n-2}}{n(n+2p)} \tag{4.32}$$

So, $a_{2n+1} = 0,\ n = 0,1...$ and

$$\begin{cases} a_2 = \dfrac{a_0}{2(2+2p)} = -\dfrac{a_0}{2^2.1(1+p)}, \\ a_4 = -\dfrac{a_2}{4(4+2p)} = -\dfrac{a_2}{2^2.2(2+p)} = (-1)^2 \dfrac{a_0}{2^4.2!(1+p)(2+p)}, \\ a_{2n} = (-1)^n \dfrac{a_0}{2^{2n}.\mathrm{n}!(1+p)(2+p)...(n+p)}, \end{cases} \tag{4.33}$$

$$y_1(x) = a_0 x^p \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(1+p)(2+p)...(n+p)} \left(\frac{x}{2}\right)^{2n}, 0 < x < \infty \tag{4.34}$$

For writing solution more simply,

$$\Gamma(p+1) = \int_0^{\infty} t^p e^{-t} dt,\ p > 0 \tag{4.35}$$

as defined the Gamma function is used.[Xie,2010].

$$\begin{aligned} \Gamma(p+1) &= -\int_0^{\infty} t^p d(e^{-t}) = \\ &= -t^p e^{-t} \Big|_{t=0}^{\infty} + \int_0^{\infty} e^{-t} p t^{p-1} dt = p\int_0^{\infty} t^{p-1} e^{-t} dt = p\Gamma(p) \end{aligned} \tag{4.36}$$

So,

$$\begin{aligned} \Gamma(n+p+1) &= (n+p)\Gamma(n+p) = (n+p)(n+p-1)\Gamma(n+p-1) \\ &= (n+p)(n+p-1)..(1+p)\Gamma(1+p) \end{aligned} \tag{4.37}$$

If $p = k$ is a integer,

$$\begin{cases} \Gamma(1) = \int_0^\infty e^{-t} dt = -e^{-t} \Big|_{t=0}^{\infty} = 1, \\ \quad \Gamma(2) = 1.\Gamma(1) = 1, \\ \Gamma(3) = 2.\Gamma(2) = 2.1 = 2!, \\ \quad \ldots \\ \Gamma(k+1) = k\Gamma(k) = k! \end{cases} \tag{4.38}$$

So,

$$a_0 = [2^p \Gamma(1+p)]^{-1} \tag{4.39}$$

Choosing that first Frobenious series solution is found,

$$y_1(x) = J_p(x) \tag{4.40}$$

$J_p(x)$ founction is p order first type Bessel Founction.

$$J_p(x) = \frac{1}{2^p \Gamma(1+p)} x^p \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(1+p)(2+p)...(n+p)} \left(\frac{x}{2}\right)^{2n} \tag{4.41}$$

$$J_p(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p} \ , \ 0 < x < \infty \tag{4.42}$$

According to Funch Theory, liear independent second solution can be changed according to difference of root of index equation. It is changed according to that $a_1 - a_2 = 2p$ is not integer, is zero or positive integer.

4.1.4. Relations Among Various Orders Bessel Functions

There are different recurrence formulas between Bessel functions. These may find briefly as follows.

$$J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(p+m+1)} \left(\frac{x}{2}\right)^{2m+p} \tag{4.43}$$

that founction is convergent for every x values, if we take derivative,

$$J'_p(x) = \frac{1}{2} \sum_{m=0}^{\infty} \frac{(-1)^m (2m+p)}{m!\Gamma(p+m+1)} \left(\frac{x}{2}\right)^{2m+p-1} \tag{4.44}$$

$$xJ'_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (2m+p)}{m!\Gamma(p+m+1)} \left(\frac{x}{2}\right)^{2m+p} \tag{4.45}$$

$$\begin{aligned} xJ'_p(x) = p \sum_{m=0}^{\infty} \frac{(-1)^m (2m+p)}{m!\Gamma(p+m+1)} \left(\frac{x}{2}\right)^{2m+p} \\ + 2m \sum_{m=1}^{\infty} \frac{(-1)^m (2m+p)}{m!\Gamma(p+m+1)} \left(\frac{x}{2}\right)^{2m+p} \\ = pJ_p(x) + x \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m!\Gamma(p+m+1)} \left(\frac{x}{2}\right)^{2m+p+1} \end{aligned} \tag{4.46}$$

We can obtained as follows.

$$xJ'_p(x) = pJ_p(x) - xJ_{p+1}(x) \tag{4.47}$$

Alike we can obtained as follows.

$$\begin{cases} xJ'_p(x) = -pJ_p(x) - xJ_{p-1}(x) \\ J_{p+1}(x) = \dfrac{2p}{x} J_p(x) - J_{p-1}(x) \\ J'_p = \dfrac{1}{2}\left(J_{p-1} - J_{p+1}\right) \end{cases} \tag{4.48}$$

This formulas is correct for $Y_p(x)$.

For example for $J_p(x)$,

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x) \tag{4.49}$$

If J_0 and J_1 is known, alike J_2 and $J_n(x)$ can be found, n is a integer.

With the help of the above recurrence formula, non-positive integers Bessel functions which correspond to the different p-values is easily writable in the same way.

For example; For $p = \frac{1}{2}$,

$$J_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma\left(m+\frac{3}{2}\right)} \left(\frac{x}{2}\right)^{2m+\frac{1}{2}} = \sqrt{\frac{x}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}\Gamma(m)\Gamma\left(m+\frac{3}{2}\right)} x^{2m} \tag{4.50}$$

On the other hand,

$$\Gamma\left(\frac{3}{2}+m\right) = \Gamma\left(\frac{3}{2}\right)\left[\frac{1.3.5...(2m+1)}{2^m}\right] = \frac{\sqrt{\pi}}{2}\left[\frac{1.3.5...(2m+1)}{2^m}\right] \tag{4.51}$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{x}{2}}\frac{2}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{(-1)^m}{2^m.3.5...(2m+1)}x^{2m} = \sqrt{\frac{2.x}{\pi}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{(2m+1)!}$$

$$= \sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}x^{2m+1} = \sqrt{\frac{2}{\pi x}}\sin(x) \tag{4.52}$$

With the same way,

$$\begin{cases} J_{\frac{-1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos(x) \\ J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin(x)}{x} - \cos(x)\right) \end{cases} \tag{4.53}$$

Let's take a look at the graph of the spherical Bessel function.

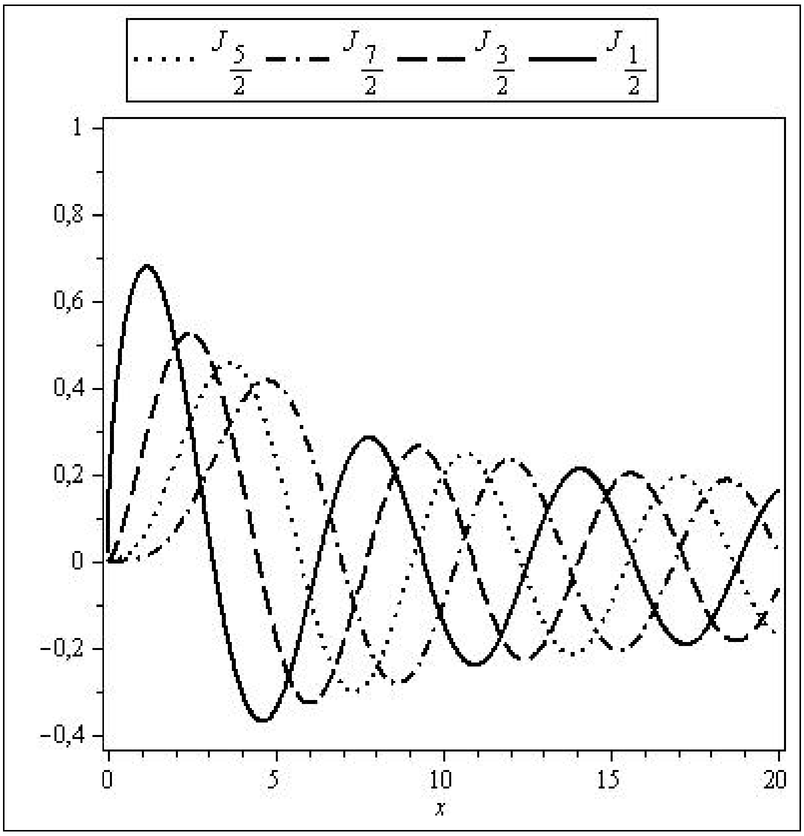

Figure 4.1: The graph of founctions $J_{1/2}(x), J_{3/2}(x), J_{5/2}(x), J_{7/2}(x)$

4.2.Expressions of Modal Amplitude with Bessel Functions

Lets examine **case-2** of list of Miller in detail.

$$\tau = u\cosh v,\ \ \xi = u\sinh v,\ \ 0 \le u \le \infty,\ -\infty \le v \le \infty \qquad (4.54)$$

It must be transform to $u(\xi,\tau)$ an $v(\xi,\tau)$. Therefore,

$$\begin{cases} \tau^2 = u^2\cosh^2 v \text{ and } \xi^2 = u^2\sinh^2 v, \\ \tau^2 - \xi^2 = u^2 \underbrace{(\cosh^2 v - \sinh^2 v)}_{1} = u^2 \\ u^2 = \tau^2 - \xi^2 \Rightarrow u = \sqrt{\tau^2 - \xi^2},\ \ 0 \le u < \infty \end{cases} \qquad (4.55)$$

Additionally,

$$\frac{\xi}{\tau}=\frac{u\sinh v}{u\cosh v}=\tanh v,\quad \operatorname{arc}\tanh\frac{\xi}{\tau}=v,\ -\infty<v<\infty \tag{4.56}$$

The equation can be made more useful as follows.

$$v=\operatorname{arc}\tanh\frac{\xi}{\tau}\equiv\frac{1}{2}\ln\frac{\tau+\xi}{\tau-\xi} \tag{4.57}$$

This expression has become utilizing from the following features.

$$\operatorname{arc}\tanh x\equiv\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ |x|<1 \tag{4.58}$$

Now, partial differential of $u(\xi,\tau)$ and $v(\xi,\tau)$ is calculated.

$$\frac{\partial u}{\partial\tau}=\frac{\partial}{\partial\tau}\sqrt{\tau^2-\xi^2}=\frac{\tau}{\sqrt{\tau^2-\xi^2}}=\frac{u\cosh v}{u}=\cosh v \tag{4.59}$$

$$\frac{\partial v}{\partial\tau}=\frac{\partial}{\partial\tau}\left(\frac{1}{2}\ln\frac{\tau+\xi}{\tau-\xi}\right)=-\frac{\xi}{\tau^2-\xi^2}=-\frac{u\sinh v}{u^2}=-\frac{\sinh v}{u} \tag{4.60}$$

$$\frac{\partial^2 u}{\partial\tau^2}=\frac{\partial}{\partial\tau}\left(\frac{\partial u}{\partial\tau}\right)=\frac{\partial}{\partial\tau}\cosh v=\frac{\partial v}{\partial\tau}\sinh v=-\frac{\sinh^2 v}{u} \tag{4.61}$$

$$\frac{\partial^2 v}{\partial\tau^2}=\frac{\partial}{\partial\tau}\left(\frac{\partial v}{\partial\tau}\right)=\frac{\partial}{\partial\tau}\left(-\frac{\sinh v}{u}\right)=\frac{\sinh 2v}{u^2} \tag{4.62}$$

$$\frac{\partial u}{\partial\xi}=\frac{\partial}{\partial\xi}\sqrt{\tau^2-\xi^2}=-\frac{\xi}{\sqrt{\tau^2-\xi^2}}=-\frac{u\sinh v}{u}=-\sinh v \tag{4.63}$$

$$\frac{\partial v}{\partial \xi} = \frac{\partial}{\partial \xi}\left(\frac{1}{2}\ln\frac{\tau+\xi}{\tau-\xi}\right) = \frac{\tau}{\sqrt{\tau^2-\xi^2}} = \frac{u\cosh v}{u^2} = \frac{\cosh v}{u} \tag{4.64}$$

$$\frac{\partial^2 u}{\partial \xi^2} = -\frac{\cosh^2 v}{u} \tag{4.65}$$

$$\frac{\partial^2 v}{\partial \xi^2} = -\frac{\sinh 2v}{u^2} \tag{4.66}$$

All this results are written in partial differential Klein-Gordon differential equation (4.35).

$$[\cosh^2 v - \sinh^2 v]\frac{\partial^2 f}{\partial u^2} + \left[\frac{\sinh^2 v}{u^2} - \frac{\cosh^2 v}{u^2}\right]\frac{\partial^2 f}{\partial v^2}$$

$$+\left[\frac{\sinh^2 v}{u} - \frac{\cosh^2 v}{u}\right]\frac{\partial f}{\partial u} + \left[\frac{\sinh 2v}{u^2} - \frac{\sinh 2v}{u^2}\right]\frac{\partial f}{\partial v} + \tag{4.67}$$

$$+\left[-\frac{\sinh v\cosh v}{u} + \frac{\sinh v\cosh v}{u}\right]\frac{\partial^2 f}{\partial u \partial v} + f = 0$$

Following the simplification, the following partial differential equations obtained.

$$\left(\frac{\partial^2}{\partial u^2} + \frac{1}{u}\frac{\partial}{\partial u} + 1 - \frac{1}{u^2}\frac{\partial^2}{\partial v^2}\right) f(u,v) = 0 \tag{4.68}$$

Bernoulli's multiplication method applied to the variables u and v,

$$f \equiv f(\xi,\tau) = f[u(\xi,\tau), v(\xi,\tau)] \equiv f(u,v) = U(u)V(v) \tag{4.69}$$

The devoted product to this last variable is writed instead of equation (4.68)

$$\left(\frac{\partial^2}{\partial u^2} + \frac{1}{u}\frac{\partial}{\partial u} + 1 - \frac{1}{u^2}\underbrace{\frac{V''(v)}{V(v)}}_{\alpha^2}\right) U(u) = 0 \tag{4.70}$$

$$\begin{cases} U''_\alpha(u) + \dfrac{1}{u}U'_\alpha(u) + \left(1 - \dfrac{\alpha^2}{u^2}\right)U_\alpha(u) = 0 \\ \\ V''_\alpha(v) - \alpha^2 V_\alpha(v) = 0 \end{cases} \tag{4.71}$$

Here, α is a constant of separation of variables method. Bessel function is producted from Bessel differential equation in part of the ordinary differential equation of U function of (4.71) equation and exponential function is producted from Bessel differential equation in part of the ordinary differential equation of V function of (4.71) equation. Both equations also has two linearly independent solution.

İncluding α arbitrary constant parameter, its writing $V''(v) = \alpha^2 V(v)$,

$$V(v) = a_\alpha e^{\alpha v} + b_\alpha e^{-\alpha v} \tag{4.72}$$

Here $a_\alpha, b_\alpha \in \Re$ are arbitrary real parameters. But for now, this arbitrary parameters are ignored for producing f,

$$V(v) = e^{\pm \alpha v} \tag{4.73}$$

Then, $U(u)$ function gives the Bessel differential equation.

$$\left(\frac{\partial^2}{\partial u^2} + \frac{1}{u}\frac{\partial}{\partial u} + 1 - \frac{\alpha^2}{u^2}\right)U(u) = 0 \tag{4.74}$$

This equations has two independent linear solution.

$$U(u) = A_\alpha J_\alpha(u) + B_\alpha Y_\alpha(u) \tag{4.75}$$

A_α and B_α are arbitrary constants here. In the equation, J is called first kind of Bessel functions and Y is called second kind of Bessel functions. However, the second kind of Bessel functions, $x \to 0 \Rightarrow Y(x) \to -\infty$. That is divergent around the origin. This divergence is an undesirable situation. The function must be convergent to find series expansion of solution. Therefore, arbitary coefficient must be selected $B_\alpha = 0$ to get rid of second kind of Bessel functions $Y(x)$. Finally, including A_α is arbitary constant,

$$U(u) = A_\alpha J_\alpha(u) \tag{4.76}$$

Now, f called as the product of the exponential function and Bessel functions is written. $f \equiv f(\xi,\tau) = f[u(\xi,\tau), v(\xi,\tau)] \equiv f(u,v) = U(u)V(v)$.
Here, combinations of (ξ,τ) is physically remarkable in terms of the independent variables.

$$f_\alpha(\xi,\tau) = \left[C_\alpha \left(\frac{\tau-\xi}{\tau+\xi} \right)^{\frac{\alpha}{2}} J_\alpha(\sqrt{\tau^2-\xi^2}) + \mathrm{D}_\alpha \left(\frac{\tau-\xi}{\tau+\xi} \right)^{\frac{\alpha}{2}} J_\alpha(\sqrt{\tau^2-\xi^2}) \right] \tag{4.77}$$

Its easy to see that solution is symmetrical to $\xi = 0$. After that, it will continue with the following equation in which $\xi \geq 0$ space is possible and α is ignored arbitrary constant.

$$f_\alpha(\xi,\tau) = \left(\frac{\tau-\xi}{\tau+\xi} \right)^{\frac{\alpha}{2}} J_\alpha(\sqrt{\tau^2-\xi^2}) \tag{4.78}$$

Physically, H_{zm} for TE mode and E_{zm} for TM mode are modal amplitude of longitudinal field components. Recognizing (2.32) and (2.23) equations, modal amplitudes of the transverse field components are shown as follows:

$$A_\alpha(\xi,\tau) = -\frac{\partial}{\partial\tau} f_\alpha \quad \text{and} \quad B_\alpha(\xi,\tau) = \frac{\partial}{\partial\xi} f_\alpha \tag{4.79}$$

In these derivatives by simple manipulation obtained partial derivatives solutions.

$$\begin{cases} A_\alpha(\xi,\tau) = -\dfrac{\partial}{\partial\tau} f_\alpha = \left(-\dfrac{1}{2}\right)(f_{\alpha-1} - f_{\alpha+1}) \\ B_\alpha(\xi,\tau) = \dfrac{\partial}{\partial\xi} f_\alpha = \left(-\dfrac{1}{2}\right)(f_{\alpha-1} + f_{\alpha+1}) \end{cases} \qquad (4.80)$$

$(\mathrm{E}_m^h, \mathrm{H}_m^h)$ which is 5-components is the electromagnetic field of the TE mode, $(\mathrm{E}_m^e, \mathrm{H}_m^e)$ which is 5-components is the electromagnetic field of the TM mode.

Remark: (2.32) and (2.23) formulas. $(\mathrm{E}_m^h, \mathrm{H}_m^h)$ has E_m^h field vector which is two-component while having H_m^h field vector which is 3-component. $(\mathrm{E}_m^e, \mathrm{H}_m^e)$ fields is the exactly opposite.

4.3. Cylindrical and Spherical Bessel Functions

Separation of variables constant, a is a free parameter. a is a non-negative integer, this form infinite set of functions as below.

$$f_n(\xi,\tau) = \left(\frac{\tau-\xi}{\tau+\xi}\right)^{\frac{n}{2}} J_n\left(\sqrt{\tau^2-\xi^2}\right), \quad n = 0,1,2... \qquad (4.81)$$

Each of $f_n(\xi,\tau)$ under (4.71) the initial conditions is particular solution of (3.7) Klein-Gordon equation.(Separately for each n). Here, $\varphi(\tau) = J_n(\tau)$ and $\widehat{\varphi}(\tau) = (1/2)(J_{n-1}(\tau) - J_{n+1}(\tau)), \ \tau \geq 0$. In addition, $J_n(\#)$ *and* $J_{n\pm1}(\#)$ notations show the cylindrical Bessel functions for integer n values. The following two functions family are considered:

$$\mathrm{J} = \left\{J_n(\tau)\right\}_{n=0}^{\infty} \quad \text{and} \quad \mathrm{F} = \left\{f_n(\xi,\tau)\right\}_{n=0}^{\infty} \qquad (4.82)$$

Each element of the set J is equal to $\xi = 0$, it equal to suitable element of set F. Bessel functions are full in J and his family members form a base. Hence, $f_n(\xi,\tau)$

elements can be interpreted as an expansion of the base element of $J_n(\tau) = f_n(\xi,\tau)_{\xi=0}$ towards right side of Oz axis.

In a base element of F, $f_n(\xi,\tau)$ is unique in between possible amplitude of the axial field component of the modal field (2.32) or (2.23). Variations in $\tau > \xi > 0$ space, physically, showed that this field components how to spread in the time through a waveguide. Therefore, it is very natural to call the set F in evolutionary base. Its elements, automatically is generated by the modal amplitude of transversal field components (4.80) formula with the help of $a = n$ or $a = n + 1/2$.

In addition, $J_n(\#)$ and $J_{n\pm1}(\#)$ notation shows the Spherical Bessel function for half-integer values of n.

We have to distinguish between the terms of "Modal base" and "Evolutionary base". First term generated by the solutions of the (2.30) ve (2.36) problem. The dual of modal area of the waveguide-sectional area is determined from (2.32) and (2.23) equation. The latter aims to identify the dynamic events in the modal amplitude while its spreading in the time-space. In fact, F set in time-domain plays the same role like $\{e^{i\omega t}\}_{\omega=-\infty}^{+\infty}$ set in the frequency domain.

A practical example of evolutionary base is considered. An input $z = 0(\xi = 0)$ which can spread with E_{zm} -component of mode of (2.23) TM equation in the waveguide section-area is considered. This entry, signal can be shown by the Heaviside function of time-domain are determined as follows:

$$H(\tau) = \begin{cases} 1 & ;\tau \geq 0 \\ 0 & ;\tau < 0 \end{cases} \tag{4.83}$$

The modal amplitudes of E_m^e *and* H_m^e fields which spread in the range of $\tau \geq \xi \geq 0$ must be present. In fact, under the (4.71) initial conditions by providing $f(\xi,\tau)_{|\xi=0} = H(\tau)$ and $\hat{\varphi}(\tau) = 0$ essential (3.7) Klein-Gordon equation has to be solved.

E_{zm} amplitude are shown $\hbar(\xi,\tau)$ in the range $\tau \geq \xi \geq 0$ .The lattest can not be obtained, for E_m^e and H_m^e transversal fields, amplitudes may be obtained respectively $\beta(\xi,\tau) = \partial_\xi \hbar$ and $A(\xi,\tau) = -\partial_\tau \hbar$.

$H(\tau)$ function expansion in $\tau \geq 0$ of Neumann as a Bessel function series may be found as follows.

$$H(\tau) = J_0(\tau) + 2\sum_{n=1}^{\infty} J_n(\tau) \tag{4.84}$$

Expansion of $H(\tau)$ (series expansion) is provided by the applying evolutionary base in the range of $\tau \geq \xi \geq 0$. Obtaining the results of all modal amplitudes are as follows.

$$\begin{cases} A(\xi,\tau) = \left(\xi \big/ \sqrt{\tau^2 - \xi^2}\right) J_1\left(\sqrt{\tau^2 - \xi^2}\right) \\ \hbar(\xi,\tau) = J_0\left(\sqrt{\tau^2 - \xi^2}\right) + \Lambda_{II}(\xi,\tau) \\ \hbar_\perp(\xi,\tau) = \left((2\xi - \tau) \big/ \sqrt{\tau^2 - \xi^2}\right) J_1\left(\sqrt{\tau^2 - \xi^2}\right) - \Lambda_\perp(\xi,\tau) \end{cases} \tag{4.85}$$

$\Lambda_{II}(\xi,\tau)$ and $\Lambda_\perp(\xi,\tau)$ are as follows infinite series.

$$\Lambda_{II} = 2\sum_{n=1}^{\infty} f_{2n} \qquad \text{and} \qquad \Lambda_\perp = 2\sum_{n=1}^{\infty} f_{2n-1} \tag{4.86}$$

Modal amplitude of E_{zm} -component have also been obtained previously in [Aksoy S., 2004]. $\Lambda_{II}(\xi,\tau)$ and $\Lambda_\perp(\xi,\tau)$ respectively modal amplitudes of the transverse components of the H_m^e *and* E_m^e field will be obtained for the first time. (4.79) series can be obtained in closed area in the front of the signal with (4.78) amplitudes as explicit. The results of obtaining a series provided $\xi \to \tau$ are as follows.

$$\begin{cases} \Lambda_{II} = \delta^2 + (1/12)\delta^4 + \mathrm{O}(\delta^5) \\ \Lambda_{\perp} = 2\delta + (1/3)\delta^3 + \mathrm{O}(\delta^5) \end{cases} \tag{4.87}$$

Here, $\delta = (\tau - \xi)/2 \ll 1$

Lets display $F_n(\xi, \omega)$ "frequency portrait" which is element of $f_n(\xi, \tau)$. This function is obtained by applying a Inverse Fourier transformation in accordance with the causality principles on (4.87) equation. The results are presented using the following z real coordinates:

$$F_n(\mathrm{z}, \omega) = \frac{e^{i\frac{z}{c}\sqrt{\omega^2 - \omega_m^2}}}{-i\frac{z}{c}\sqrt{\omega^2 - \omega_m^2}} \left(\frac{i\omega_m}{\omega + \sqrt{\omega^2 - \omega_m^2}} \right)^n \tag{4.88}$$

Here, $z > 0,\ \omega > 0,\ i = \sqrt{-1},\ \omega = \kappa_m^e c$ is cut-off frequency.

4.4. Expressions of Modal Amplitude with Airy Function

$$w''(z) - zw(z) = 0 \tag{4.89}$$

This is quite similar to the differential equation $w''(z) - zw(z) = 0$ for the hyperbolic sine and hyperbolic cosine functions, which has the general solution $w(z) = c_1 \sinh(\mathrm{z}) + \mathrm{c}_2 \cosh(\mathrm{z})$. Airy built two partial solutions $w_1(z)$ and $w_2(z)$ for the first equation in the form of a power series $w(z) = \sum_{j=0}^{\infty} a_j z^j$. These solutions were named the Airy functions. The Airy functions $Ai(z)$ and $Bi(z)$ are the special solutions of the differential equation:

$$\begin{cases} Ai(z) = \dfrac{1}{\pi}\int_0^\infty \cos\left(\dfrac{t^3}{3} + zt\right)dt \ ; \mathrm{Im(z)} = 0 \\ Bi(\mathrm{z}) = \dfrac{1}{\pi}\int_0^\infty \left(\sin\left(\dfrac{t^3}{3} + zt\right) + e^{zt - \frac{t^3}{3}}\right)dt \ ; z < 0 \end{cases} \tag{4.90}$$

By choosing u and v appropriate function, method of separation of variables are used to solve the equation (4.7). Energy waves that accompany the electromagnetic waves were investigated by helping the amplitude is obtained from Airy functions by considering the fifth of 11 dual function located in appendix and is described by Miller. This function, $-\infty < u, v < \infty$

$$\tau + \xi = 2(u + v) \tag{4.91}$$

$$\tau - \xi = (u - v)^2 \tag{4.92}$$

It is obtained as follows.

$$u = \frac{\tau + \xi}{4} - \frac{\sqrt{\tau - \xi}}{2}, \ v = \frac{\tau + \xi}{4} + \frac{\sqrt{\tau - \xi}}{2} \tag{4.93}$$

The following equation is obtained by writing this variables instead of (4.7) equation. [Tretyakov and Akgün, 2010], [Akgün, 2011]

$$\frac{1}{4(u - v)}\frac{\partial^2 f}{\partial u^2} - \frac{1}{4(u - v)}\frac{\partial^2 f}{\partial v^2} + f(u, v) = 0 \tag{4.94}$$

This equation is organized by the following process and the variable separation method is used (Tretyakov and Akgun, 2010; Akgun, 2011).

$$\frac{\partial^2 f(u, v)}{\partial u^2} + 4uf(u, v) = \frac{\partial^2 f(u, v)}{\partial v^2} + 4vf(u, v) \tag{4.95}$$

Including $f(u,v)=U(u)V(v)$, variable separation method is used.

$$\frac{1}{U(u)}\frac{d^2U(u)}{du^2}+4u=\frac{1}{V(v)}\frac{d^2V(v)}{\partial v^2}+4v=4\alpha \qquad (4.96)$$

Including α is a constant of separation of variables, u and v are defined as follows.

$$\bar{u}=\sqrt[3]{4}(\alpha-u) \qquad (4.97)$$

$$\bar{v}=\sqrt[3]{4}(\alpha-v) \qquad (4.98)$$

(Tretyakov and Akgun, 2010; Akgun, 2011). After changing of variable (4.96) equation is written in the form of Airy differential equations.

$$\frac{d^2U(\bar{u})}{d\bar{u}^2}-\bar{u}U(\bar{u})=0 \qquad (4.99)$$

$$\frac{d^2V(\bar{v})}{d\bar{v}^2}-\bar{v}V(\bar{v})=0 \qquad (4.100)$$

The solution of the Klein-Gordon equation is expressed as following partial function accordance with the principle of causality.

$$f(\xi,\tau)=\begin{cases} 0 & ,\tau<0 \\ U(\bar{u})V(\bar{v}) & ,0\le\xi\le\tau \\ 0 & ,\xi>\tau \end{cases} \qquad (4.101)$$

Physically this case, the field sources means that it is not active before $t=0$. The first line of (4.101) function is called a weak causality principle. When field source is zero for $t<0$, ıt is remarked that whole fields are zero. The bottom line is the powerfull causality condition. According to special relativity theory, any electromagnetic field transmits signal whose speed is c. These last two conditions are express that modal fields caused by source which becomes active at $t=0$ is zero beyond $z=ct$. (Akgün 2011)

Both of (4.92) and (4.93) equations have linearly independent solution.

$$U\left(\bar{u}\right)=a_1 Ai\left(\bar{u}\right)+b_1 Bi\left(\bar{u}\right) \tag{4.102}$$

$$V\left(\bar{v}\right)=a_2 Ai\left(\bar{v}\right)+b_2 Bi\left(\bar{v}\right) \tag{4.103}$$

a_1, a_2, b_1, b_2 are arbitrary constants and Ai (*) The first type of Airy function, Bi (*) is the the second type of Airy function. The amplitudes of modes of waveguide is expressed by the product of the Airy function (Tretyakov and Akgun, 2010; Akgun, 2011).

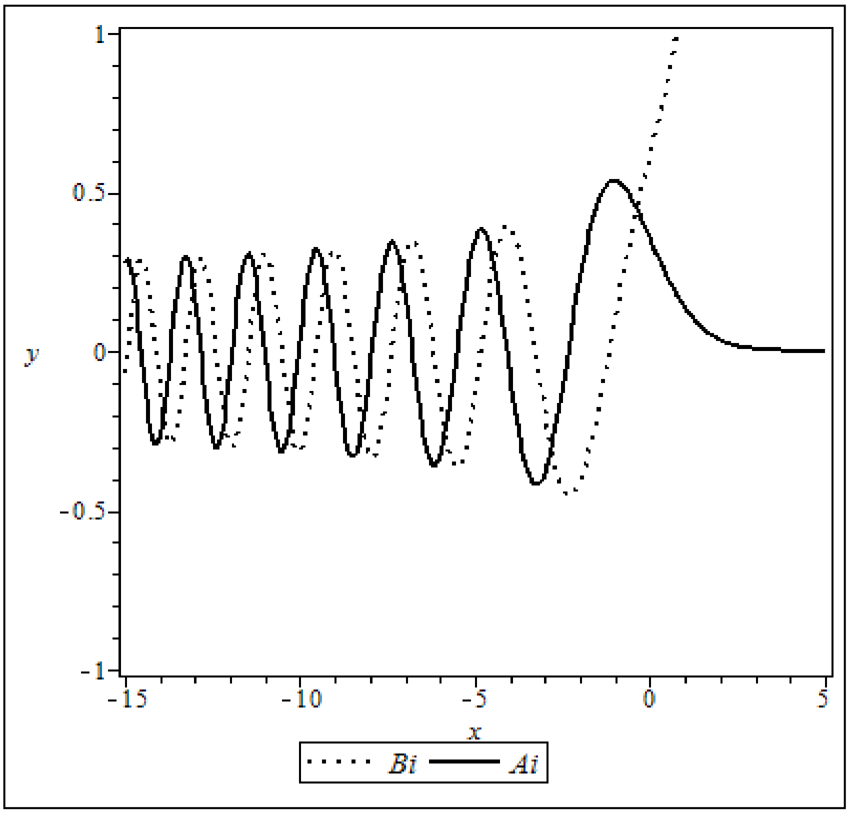

Figure 4.2: First and second type Airy function

While $x<0$, $Ai(x)$ and $Bi(x)$ is released with continuously increasing frequency and continuously decreasing amplitude. As a result, negative variable is providing the oscillations with decreasing amplitude of the Airy function. So, $\bar{u}$ and $\bar{v}$ variable should be negative to obtain finite value modal amplitude of z component.

$$\bar{u}<0 \Rightarrow \alpha<\frac{(\tau+\xi)}{4}-\frac{\sqrt{\tau-\xi}}{2} \tag{4.104}$$

$$\bar{v}<0 \Rightarrow \alpha<\frac{(\tau+\xi)}{4}+\frac{\sqrt{\tau-\xi}}{2} \tag{4.105}$$

u and v become negative when α value is choosen according to (4.104) and (4.105) criteria, namely A and B is taken pozitive value. This allows the release of the Airy function with diminishing amplitudes. For all possible multiplication of the Airy function is obtained as follows by taking as $\alpha=-1$ and writing solution of (4.95) and (4.96) instead of $f(\xi,\tau)=U(\bar{u})V(\bar{v})$ for $0\le\xi\le\tau<\infty$.

$$f_1(\xi,\tau) = Ai(\bar{u})Ai(\bar{v}) \tag{4.106}$$

$$f_2(\xi,\tau) = Ai(\bar{u})Bi(\bar{v}) \tag{4.107}$$

$$f_3(\xi,\tau) = Bi(\bar{u})Ai(\bar{v}) \tag{4.108}$$

$$f_4(\xi,\tau) = Bi(\bar{u})Bi(\bar{v}) \tag{4.109}$$

5. CONCLUSION

It can be highlighted as follows to compile obtained results and applied in this study.

In this study, the problem of power of the electromagnetic field produced by a source on surface of the time-dependent waveguide is excellent electrical conductor is discussed adhering to universal principles of analytical time-domain method called Evolutionary Approaches to Electromagnetic Theory. The study is designed to present the waveguide theory in time-domain in accordance with the scientific opinions and theories of theoreticians and engineers. Problem shaped on two autonomous blocks. One is modal base module and the other is the modal amplitude module. Modal base is common for some time-harmonic and time-domain modes. Therefore, the results obtained for modal bases in time-harmonic field theory freely is available in the time-domain study. Meanwhile, the time-domain amplitudes studies need to solve the Klein-Gordon equation. In this study, the symmetry properties of these equations was used in terms of group theory. This way seems to be a very promising way as it can fullfill requirements of special functions of mathematical physics and time-dependent arguments.

The basic principles of EAE is reached result from Maxwell's equations of the electromagnetic field while maintaining ∂t time derivative and by obtaining amplitude from Klein-Gordon equation and potential from Helmholtz equation. Eigenvalues and eigenfunctions depend on these eigenvalues is obtained from Helmholtz equation.(eigenvalues is real due to operator is self-adjoint.) Constant coefficients of these eigenfunctions are normalized by calculating under the boundary conditions. The solution of eigenfunctions is given as a series solution. A series solution of the time-dependent modal amptitudes produced from Klein-Gordon equation. Thus, the field is obtained by drawn the amptitude and potential embedded in the field from the above-mentioned equation and by replacing in constitutive equations.

Klein-Gordon equation and the Helmholtz equation are independent from each other in terms of modal amplitude and the modal base. However, these two independent equations provide to produce evolution equation.

In the above discussion, Klein-Gordon equation solved under the principle of causality, and provide the initial conditions, Helmholtz equation provide boundary conditions. In both equations remain invariant under a Lorentz transformation.

6. RESULTS

In this section, ıt has tried to display that f function in the cylindrical and spherical forms, the oscillations of A and B functions in the value $\alpha = 0,1,2...$ and images in different value of t namely τ and z namely ξ of this oscillations. Graphics was drawn by computer program Maple. ξ is hold steady while oscillation along the axis of τ, τ is hold steady while oscillation along the axis of ξ.

The first release of oscillation graphics are in cylindrical form.

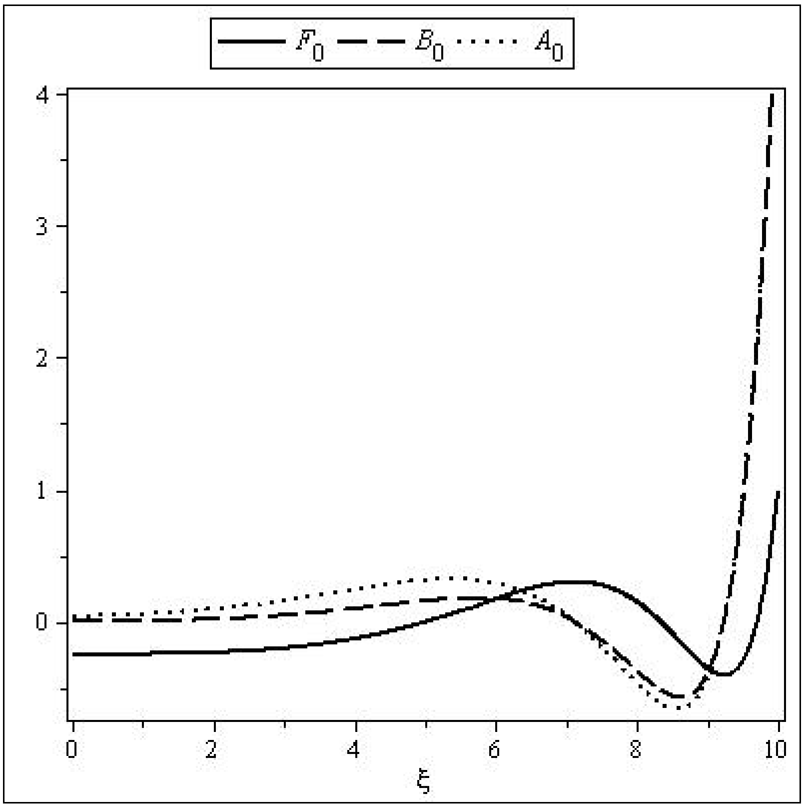

Figure 6.1: The change of modal amplitudes in the range of $0 \le \xi \le \tau$. $\tau = 10$ is constant, ξ is dimensionless axial coordinate, $\alpha = 0$

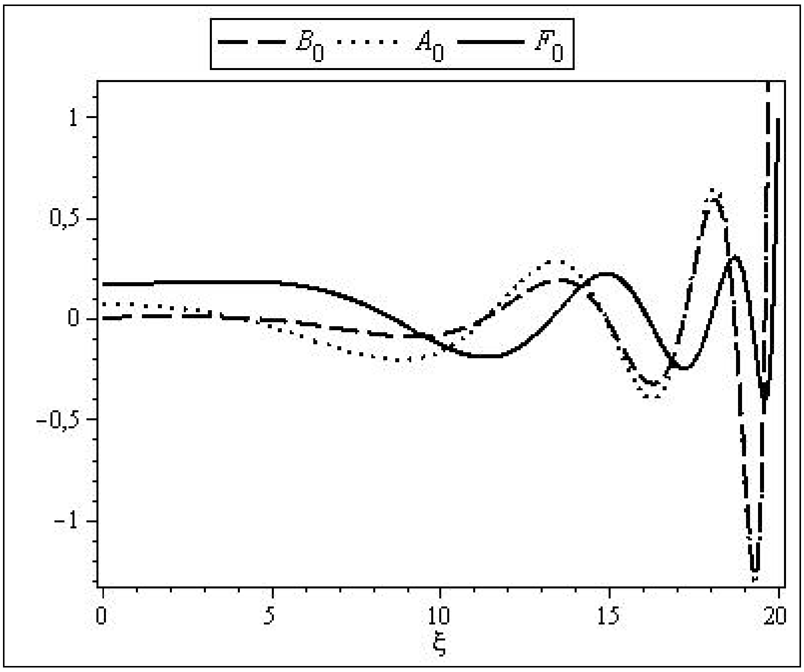

Figure 6.2: The change of modal amplitudes in the range of $0 \le \xi \le \tau$. $\tau = 20$ is constant, ξ is dimensionless axial coordinate, $\alpha = 0$

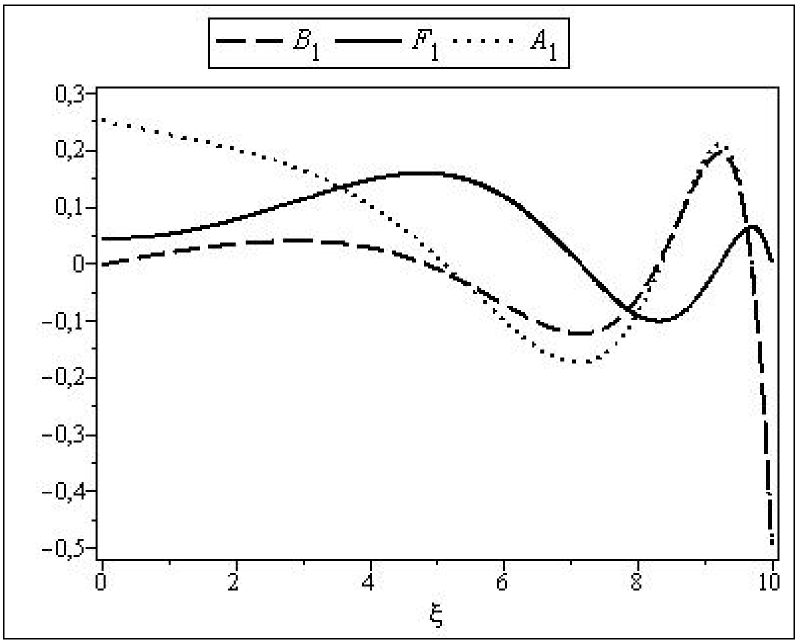

Figure 6.3: The change of modal amplitudes in the range of $0 \le \xi \le \tau$. $\tau = 10$ is constant, ξ is dimensionless axial coordinate, $\alpha = 1$

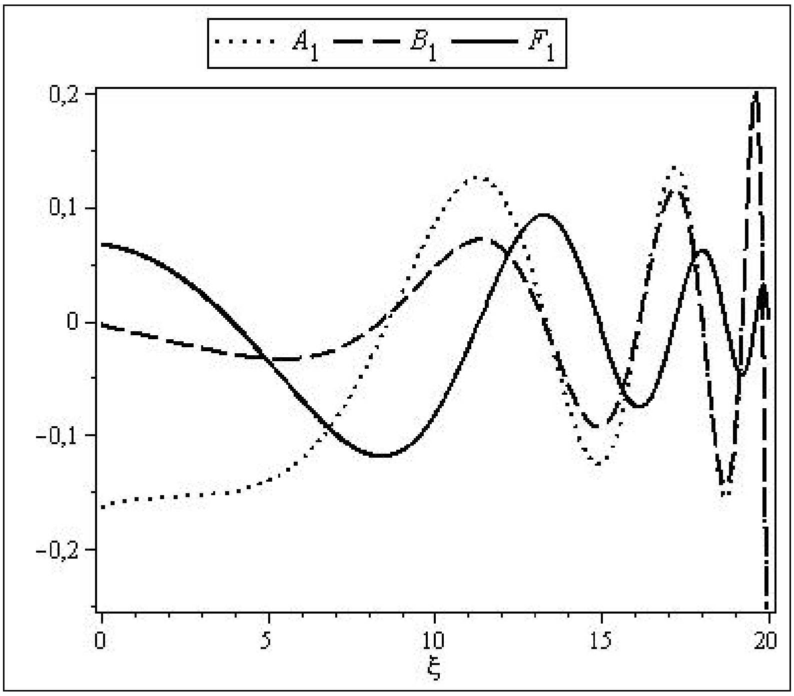

Figure 6.4: The change of modal amplitudes in the range of $0 \le \xi \le \tau$. $\tau = 20$ is constant, ξ is dimensionless axial coordinate, $\alpha = 1$

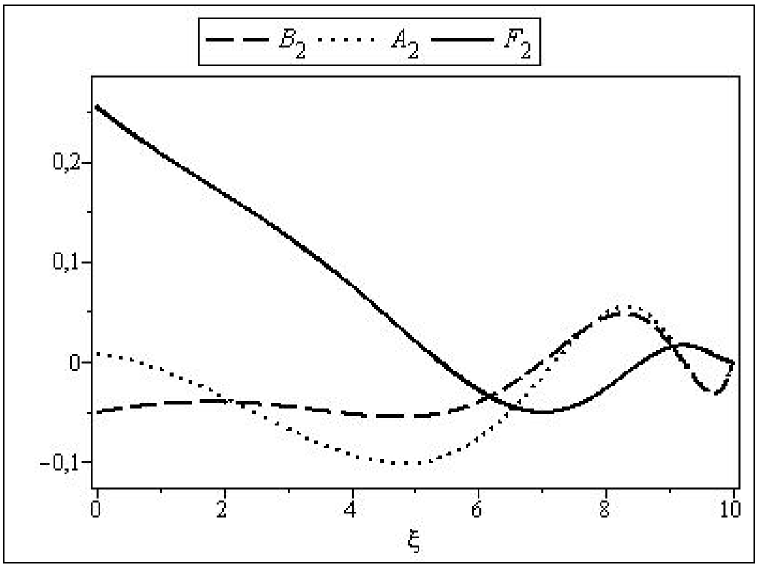

Figure 6.5: The change of modal amplitudes in the range of $0 \le \xi \le \tau$. $\tau = 10$ is constant, ξ is dimensionless axial coordinate, $\alpha = 2$

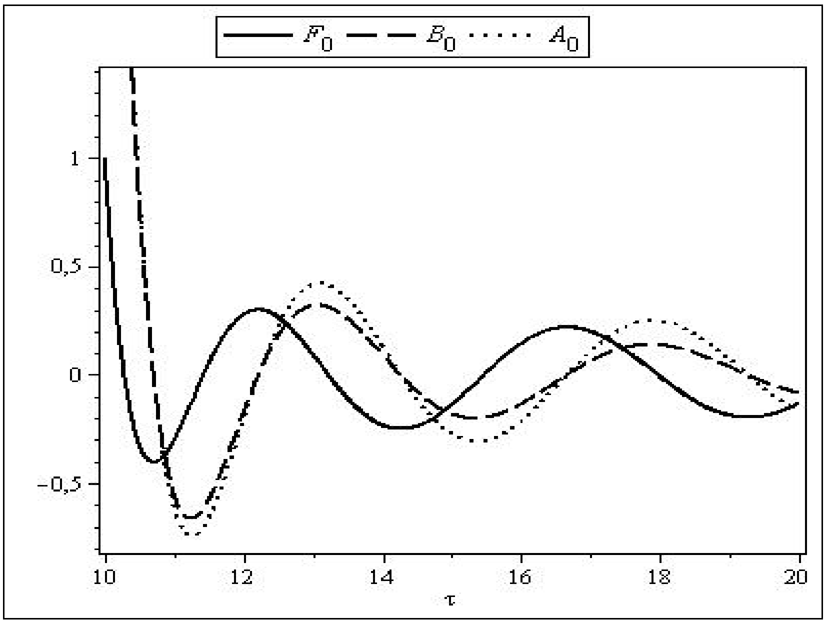

Figure 6.6: The change of modal amplitudes in the range of $10 \le \tau \le 20$. $\xi = 10$ is constant, $\alpha = 0$

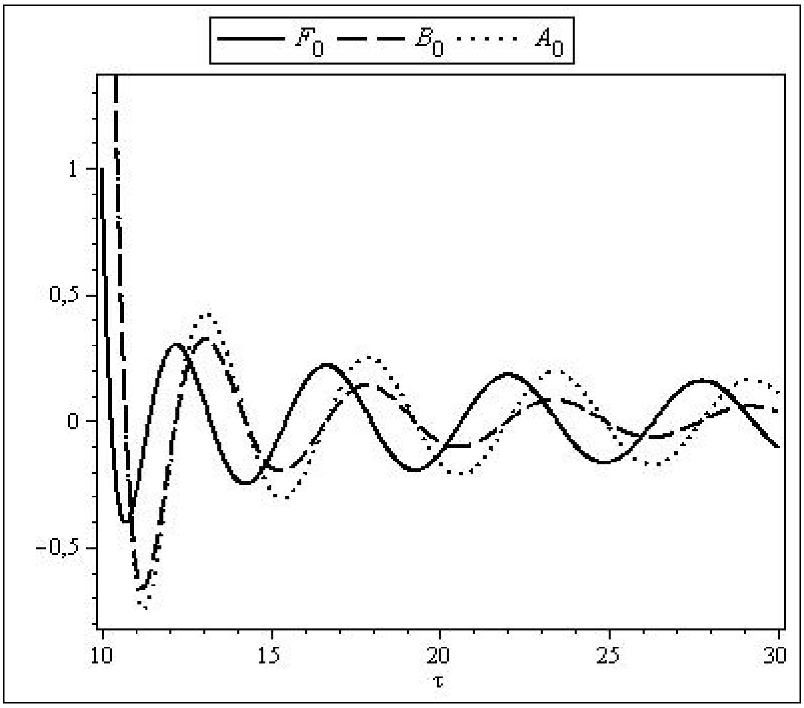

Figure 6.7: The change of modal amplitudes in the range of $10 \le \tau \le 30$. $\xi = 10$ is constant, $\alpha = 0$

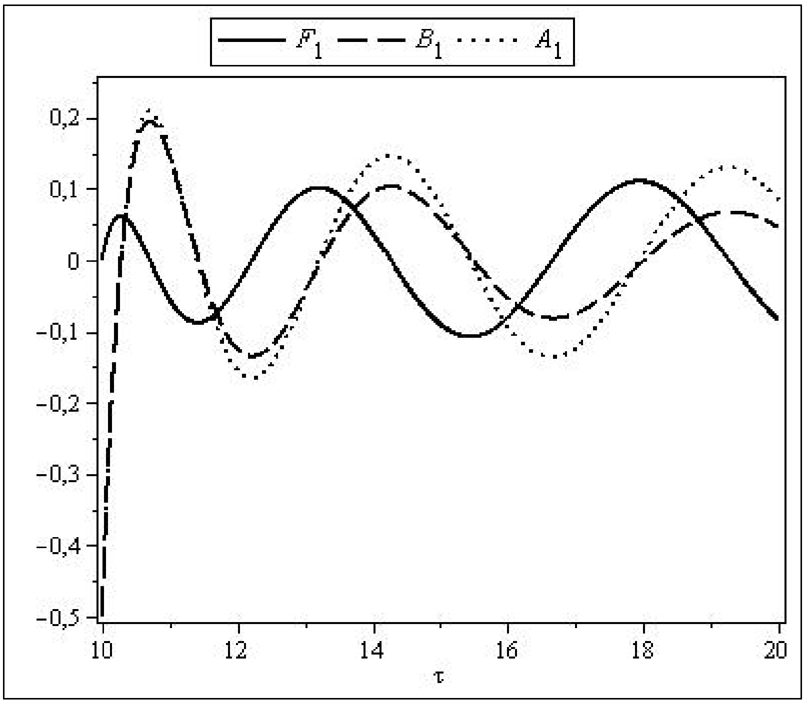

Figure 6.8: The change of modal amplitudes in the range of $10 \le \tau \le 20$. $\xi = 10$ is constant, $\alpha = 1$

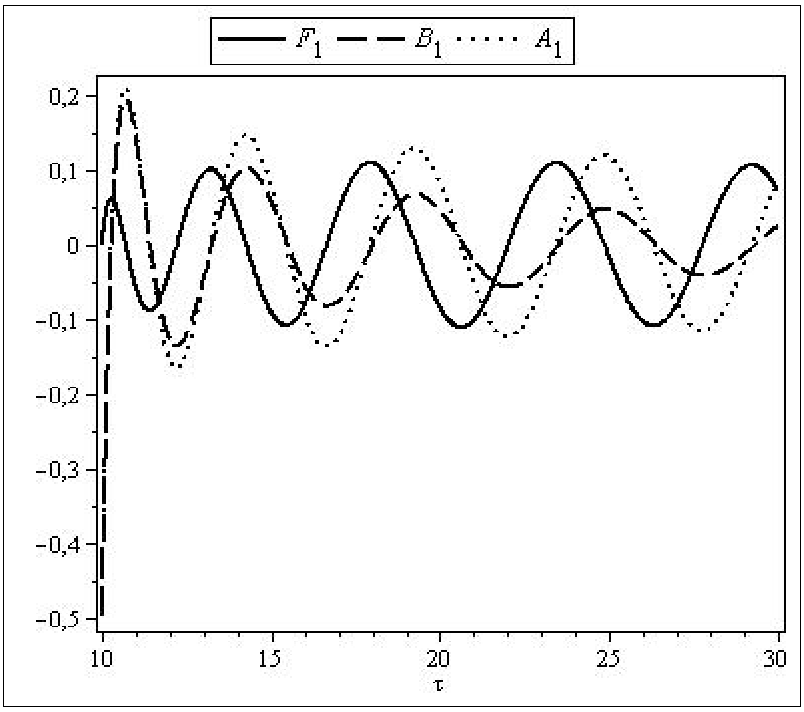

Figure 6.9: The change of modal amplitudes in the range of $10 \le \tau \le 30$. $\xi = 10$ is constant, $\alpha = 1$

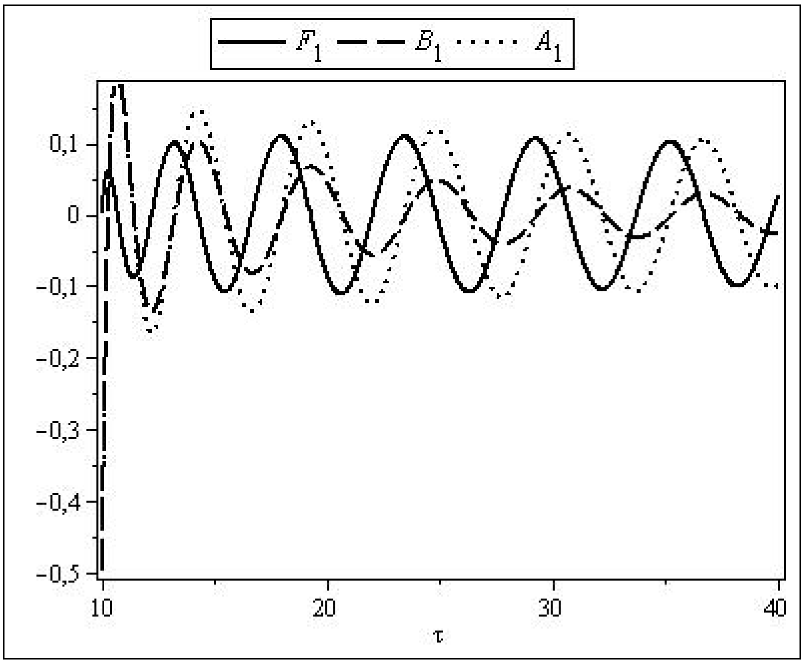

Figure 6.10: The change of modal amplitudes in the range of $10 \le \tau \le 40$. $\xi = 10$ is constant, $\alpha = 1$

In this release of oscillation graphics are in spherical form.

$(\alpha = n + 1/2)$, $n = 0,1,2,3...$

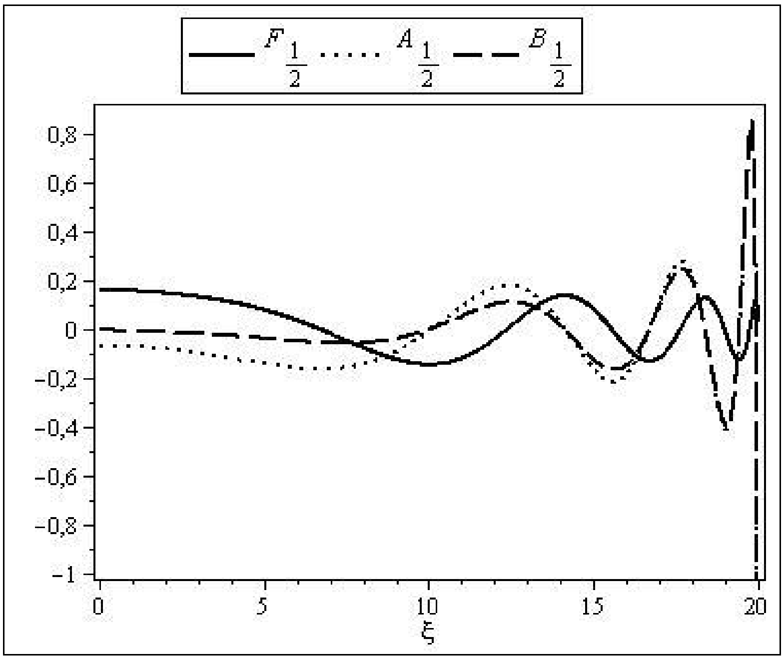

Figure 6.11: The change of modal amplitudes in the range of $0 \le \xi \le \tau$. $\tau = 20$ is constant, ξ is dimensionless axial coordinate, $n = 0$

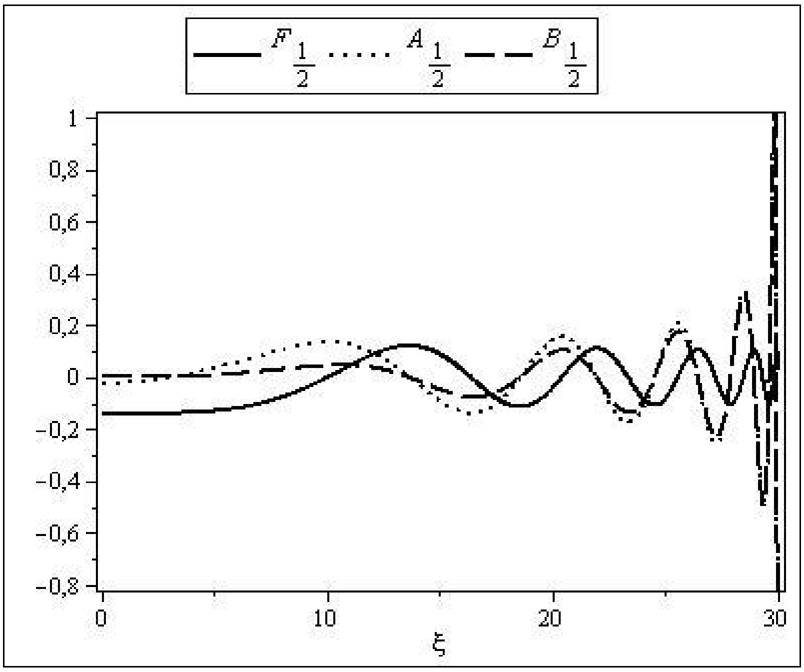

Figure 6.12: The change of modal amplitudes in the range of $0 \le \xi \le \tau$. $\tau = 30$ is constant, ξ is dimensionless axial coordinate, $n = 0$

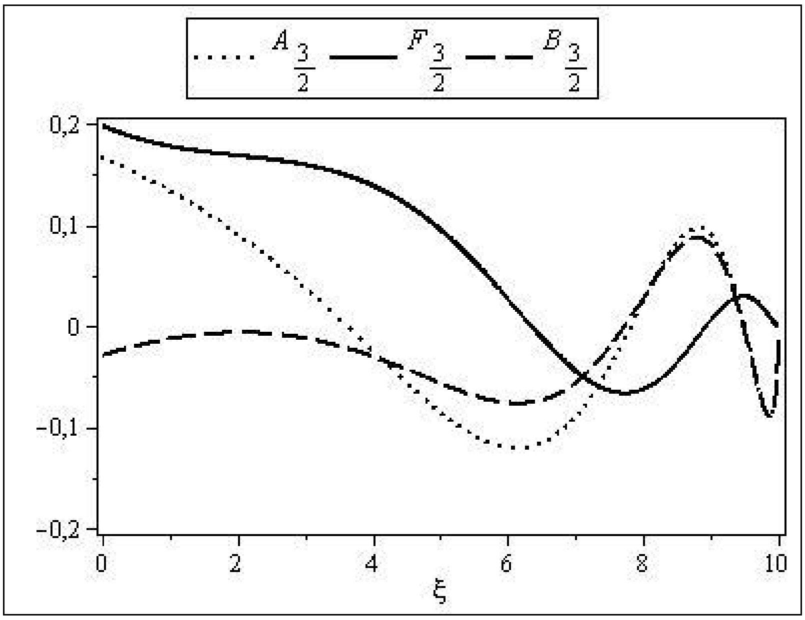

Figure 6.13: The change of modal amplitudes in the range of $0 \le \xi \le \tau$. $\tau = 10$ is constant, ξ is dimensionless axial coordinate, $n = 1$

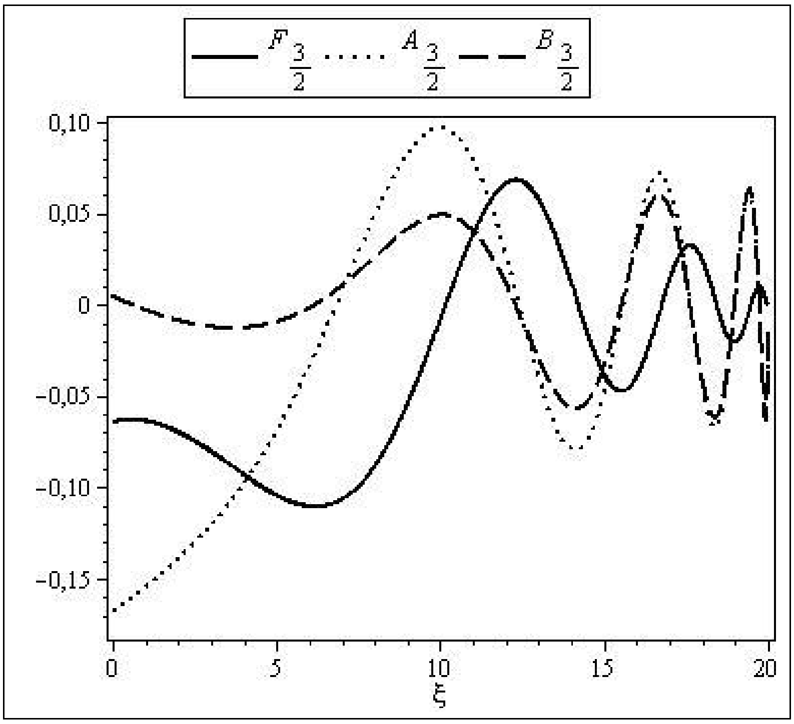

Figure 6.14: The change of modal amplitudes in the range of $0 \le \xi \le \tau$. $\tau = 20$ is constant, ξ is dimensionless axial coordinate, $n = 1$

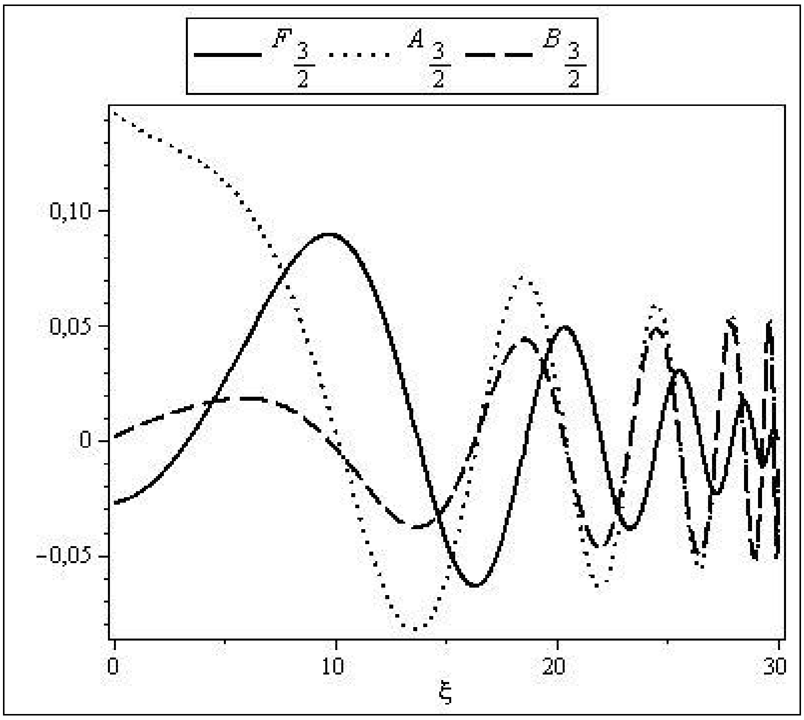

Figure 6.15: The change of modal amplitudes in the range of $0 \le \xi \le \tau$. $\tau = 30$ is constant, ξ is dimensionless axial coordinate, $n = 1$

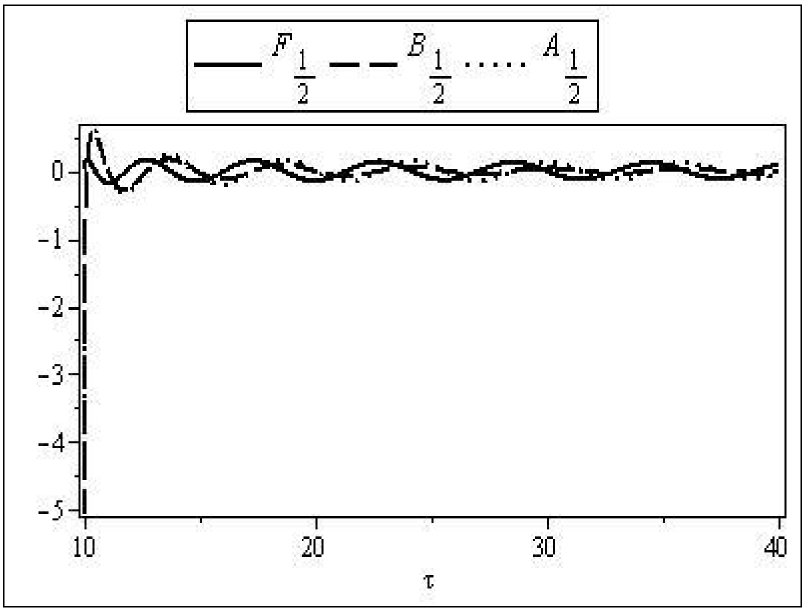

Figure 6.16: The change of modal amplitudes in the range of $10 \le \tau \le 40$. $\xi = 10$ is constant, $n = 0$

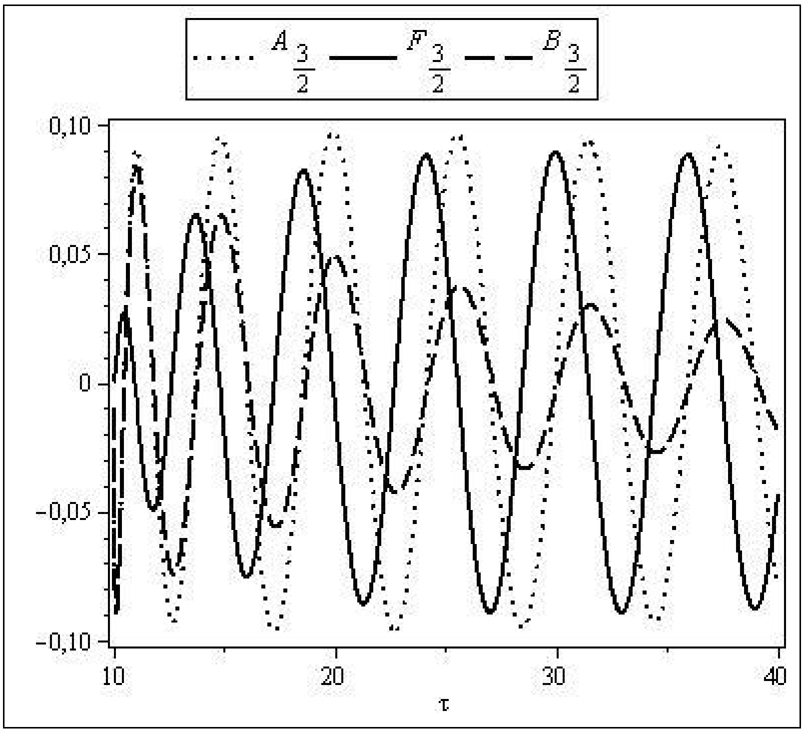

Figure 6.17: The change of modal amplitudes in the range of $10 \le \tau \le 40$. $\xi = 10$ is constant, $n = 1$

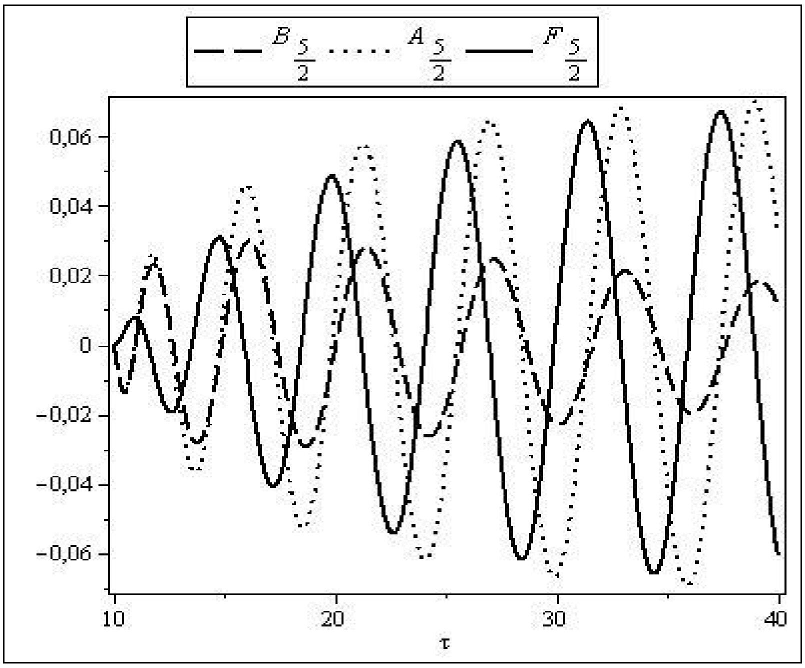

Figure 6.18: The change of modal amplitudes in the range of $10 \le \tau \le 40$. $\xi = 10$ is constant, $n = 2$

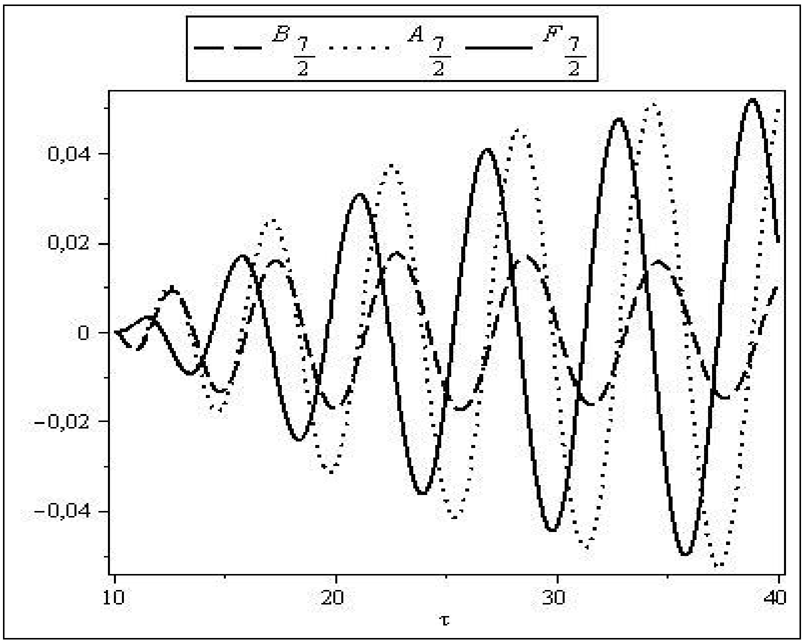

Figure 6.19: The change of modal amplitudes in the range of $10 \le \tau \le 40$. $\xi = 10$ is constant, $n = 3$

REFERENCES

Aksoy S., Antyufeyeva M., Basaran E., Ergin A. A., Tretyakov O. A., (2005), “Time-Domain Cavity Oscillations Supported by a Temporally Dispersive Dielectric”, IEEE Transactions on Microwave Theory ve Techniques, 53 (8), 2465-2471.

Aksoy S., Tretyakov O. A., (2002), “Study of a Time Variant Cavity System”, Journal of Electromagnetic Waves ve Applications, 16 (11), 1535-1553.

Aksoy S., Tretyakov O. A., (2003), “Evolution Equations for Analytical Study Digital Signals in Waveguides”, Journal of Electromagnetic Waves ve Applications, 17 (12), 1665-1682.

Aksoy S., Tretyakov O. A., (2004) ,“The Evolution Equations in Study of the Cavity Oscillations Excited by a Digital Signal”, IEEE Transactions on Antennas ve Propagation, 52 (1), 263-270.

Aksoy S., (2003), “Dalga Kılavuzlarında Elektromagnetik Zaman Domeni Analizlerinin Yeni Bir Analitik Metotla İncelenmesi”, Doctor of Philosophy Dissertation, Gebze Institute of Technology.

Baum C. E., Carin L., Stone A. P., (1997), “Ultra-Widebve Short-Pulse Electromagnetics”, 3rd Edition, Plenum Press.

Bladel J. V., (1964), “Electromagnetic Fields”, 1st Edition, Hemisphere.

Bennett C. L., Ross G. F., (1978), “Time-domain Electromagnetics ve Its Applications”, Proceedings of the IEEE, 66 (3), 299-318.

Camp M., Garbe H., (2004), “Susceptibility of personal Computer Systems to Electromagnetic Pulses with Double Exponential Character”, Advances in Radio Science, 2 (1), 63–69.

Cramer J. M., Scholtz R. A., Win M. Z., (1999), “On the Analysis of UWB Communication Channels”, Proceedings of the IEEE Military Communications Conference IEEE, 2 (1), 1191–1195.

Diaz R. E., Alexopoulos N. G., (1997), “An Analytic Continuation Method for the Analysis ve Design of Dispersive Materials”, IEEE Transactions on Antennas ve Propagation, 45 (11), 1602-1610.

Dvorak S. L., Dudley D. D., (1995), “Propagation of Ultra-Wide-Bve Electromagnetic Pulses Through Dispersive Media”, IEEE Transactions on Eletromagnetic Compatibility, 37 (2), 192-200.

Erden F., Tretyakov O. A., (2008), “Excitation of the Real-Valued Electromagnetic Fields in a Cavity by a Given Transient Signal”, Physical Review Series, 77 (1), 56-65.

Erden F., Tretyakov O. A., (2004), “Solenoidal Eşlenmiş Dejenere Kavite Modlarının Zaman Domeni Salınımları”, İkinci Ulusal Kongresi, 204-206, Bilkent Üniversitesi, Ankara, Türkiye, 15-18 September.

Hillion P., (1993), “Some Comments on Electromagnetic Signals”, 3rd Edition, World Scientific Publishing Company.

Harmuth H. F., (1993), “Electromagnetic Transients not Explained by Maxwell’s Equations”, 3rd Edition, World Scientific Publishing Company.

Kisun’ko G.V., (1949), “Electrodynamics of Hollow Systems”, 2nd Edition, VKAS Press, Leningrad.

Kurokawa K., (1958), “The Expansion of Electromagnetic Fields in Cavities”, IRE Transactions on Microwave Theory ve Techniques, 6 (1), 178-187.

Miller W. Jr., (1977), “Symmetry ve Separation of Variables”, 2nd. Edition, MA: Addison-Wesley Publication Company.

Müller R., (1961), “Theory of Cavity Resonators in Electromagnetic waveguides and cavities”, 2nd Edition, Pergamon Press.

Savic M. S., (2005), “Estimation of the Surge Arrester Outage Rate Caused by Lightning Overvoltages,” IEEE Transition on Power Delivery, 20 (1), 116–122.

Slater J. C., (1946), “Microwave Electronics”, Review of Modern Physics, 18 (4), 441-452.

Taflove A., Hagness S., (2005), “Computational Electrodynamics: The Finite-Difference Time-Domain Method”, 3rd Edition, Artech House.

Tretyakov O. A., (1993), “Essentials of Nonstationary ve Nonlinear Electromagnetic Field Theory”, 2nd Edition, Science House Company.

Tretyakov O. A., (1994), “Evolutionary Equations for the Theory of Waveguides”, Processing IEEE AP-S, 2465–2471, Seattle, USA, 12-15 June.

Tretyakov O. A. ve Akgün O., (2010), “Derivation of Klein-Gordon Equation from Maxwell’s Equations and Study of Relativistic Time-Domain Waveguide Modes”, Progress In Electromagnetics Research, 105 (1), 171-191.

Tretyakov O. A., Erden F., (2008), “Temporal Cavity Oscillations Caused by a Wide-Bve Waveform”, Progress in Electromagnetics Research B, 6 (1),183-204.

Tretyakov O. A., Erden F., (2006), “Temporal Cavity Oscillations Caused by a Wide-Bve Double-Exponential Waveform”, 4th International Workshop on Electromagnetic Wave Scattering , Gebze, Kocaeli, Turkey, 18-22 September.

Tretyakov O. A., Erden F., (2007), “Separation of the Instantaneous ve Dynamic Polarizations in Studies of Dispersive Dielectrics”, The Sixth International Kharkov

Symposium on Physics ve Engineering of Microwaves, Millimeter ve Submillimeter Waves, Kharkov, Ukraine, 42-48, 25-30 June.

Weyl H., (1940), “The Method of Orthogonal Projection in Potential Theory”, Duke Mathematics Journal, 7 (2), 411-44.

Xie W. C., (2010), “Differential Equations For Engineers”, 1st Edition, Cambridge University Press The Edinburgh Building.

Zhen J., Hagness S. C., Booske J. H., Mathur S., Meltz M. L., (2006), “FDTD Analysis of a Gigahertz TEM Cell for Ultra-widebve Pulse Exposure Studies of Biological Specimens”, IEEE Trans. on Biomedical Engineering, 53 (5), 780–789.

Eren N. (2015), “Special Functions in Transferring of Energy and Surplus of Energy”, Doctor of Philosophy Dissertation, Yildiz Technical University.

Işık Ö. (2014), ”Özel Fonksiyonlar Yardımıyla Sinyal Analizi”, Doctor of Philosophy Dissertation, Muğla Sıtkı Koçman University.

Eroğlu E. (2012) “Dalga Kılavuzları Boyunca Geçici Sinyallerin Transferi”, Doctor of Philosophy Dissertation, Gebze Institute of Technology.

BIOGRAPHY

I was born in 10 April 1990, Kırklareli. I have studied at Kırklareli Science High School. I graduated from Mathematical Engineering, Yıldız Technical University, in 2012. I also have worked as a systems analysis in Aegon Emeklilik ve Hayat A.Ş. for a year. I am studying at Gebze Technical University in the mathematics department.

APPENDICES

Appendix A: List of Miller

Including $f(u,v)=U(u)V(v)$,

- Including $\tau = u$ ve $\xi = v$ $-\infty < u < \infty, -\infty < v < \infty$ $f(u,v)$ is the product of the exponential function.
- Including $\tau = u\cosh v$ ve $\xi = u\sinh v$ $-\infty \le u < \infty, -\infty < v < \infty$ $f(u,v)$ is product by an exponential function and Bessel functions.
- Including $\tau = (u^2+v^2)/2$ ve $\xi = uv$ $-\infty \le u < \infty, -\infty < v < \infty$ $f(u,v)$ is parabolic cylinder functions.
- Including $\tau = uv$ ve $\xi = (u^2+v^2)/2$ $-\infty \le u < \infty, -\infty < v < \infty$ $f(u,v)$ is parabolic cylinder functions.
- Including $\tau + \xi = 2(u+v)$ ve $\tau - \xi = (u-v)^2/2$ $-\infty < u,\ v < \infty$ $f(u,v)$ is Airy functions.
- Including $\tau + \xi = \cosh\left[(u-v)/2\right]$ ve $\tau - \xi = \sinh\left[(u+v)/2\right]$ $-\infty < u,\ v < \infty$ $f(u,v)$ is Mathieu functions.
- Including $\tau + \xi = 2\sinh(u-v)$ ve $\tau - \xi = \exp(u-v)$ $-\infty < u,\ v < \infty$ $f(u,v)$ is Bessel functions.
- Including $\tau + \xi = 2\sinh(u-v)$ ve $\tau - \xi = \exp(u-v)$ $-\infty < u,\ v < \infty$ $f(u,v)$ is Bessel functions.
- Including $\tau = \sinh u \cosh v$ ve $\xi = \cosh u \sinh v$ $-\infty < u,\ v < \infty$ $f(u,v)$ is Mathieu functions.
- Including $\tau = \cosh u \cosh v$ ve $\xi = \sinh u \sinh v$ $-\infty < u < \infty, -\infty \le v < \infty$ $f(u,v)$ is Mathieu functions.
- Including $\tau = \cos u \cos v$ ve $\xi = \sin u \sin v$ $0 < u < 2\pi,\ 0 \le v < \pi$ $f(u,v)$ is Mathieu functions.