

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

**STABILITY THEORY OF SET VALUED
DIFFERENTIAL EQUATIONS**

MUHARREM BARKIN
A THESIS SUBMITTED FOR THE DEGREE OF
MASTER OF SCIENCE
DEPARTMENT OF MATHEMATICS

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ASSOC. PROF. DR. COŞKUN YAKAR

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SUMMARY

In this thesis, we have given some stability concepts and boundedness criteria for the set valued differential equations. It is well known that the theory of stability in the sense of Lyapunov has been successfully investigated to understand qualitative properties of set valued differential equations [1]-[5]. In set valued differential equations, the Lyapunov's second method allows us to obtain sufficient conditions for the stability of a system without explicitly solving the set valued differential equations. The method generalizes the idea which shows that the system is stable if there are some Lyapunov function satisfy some good properties for the set valued differential equations. The investigation of initial value problems of set valued differential equations with a perturbation in the space variable is well known when the perturbation is restricted to the space variable. We have worked on stability and boundedness criteria that is the special case of the stability introduced the concept of stability in terms of two measures which unified the foregoing stability concepts. Then, the theories of the stability have been successfully developed and some results are obtained by means of various types of Lyapunov functions for several kinds of differential equations. These concepts of stability and boundedness criteria also could have investigated to ITD stability, boundedness and Lagrange stability in terms of two measures for perturbed set valued differential equations by using comparison method via Lyapunov-like function.

Key Words: Set Valued Differential Equations, Lyapunov Functions, Stability, Boundedness Criteria, Comparison Results.

ÖZET

Bu tezde, küme değerli diferansiyel denklemler için kararlılık kavramlarını ve sınırlılık kriterini ele aldık. Lyapunov açısından kararlılık teorisi, küme değerli diferansiyel denklemlerin [1]-[5] niteliksel özelliklerini anlamak için yapılan çalışmalarda başarılı olduğunu biliyoruz. Küme değerli diferansiyel denklemlerdeki, Lyapunov'un ikinci methodu, çözümleri açık olmayan küme değerli diferansiyel denklemler sisteminin kararlılığı için yeterli koşulları elde etmemize yardımcı olur. Bu method, eğer küme değerli diferansiyel denklemler için bazı iyi özellikleri sağlayan bazı Lyapunov fonksiyonları varsa bu sistemin kararlı olduğunu gösterme fikrinin genelleştirilmesidir. Değişken aralığındaki bir düzensizlik ile küme değerli diferansiyel denklemlerin başlangıç değer problemlerinin incelenmesi, düzensizlik sınırlı bir değişken aralığına sıkıştırılmışsa iyi bilinir. Biz burada kararlılığın özel bir durumu için sınırlılık ve kararlılık kriteri üzerinde çalıştık. Kararlılığın bu özel durumu önceki kararlılık kavramlarını birleştiren iki ölçünün açısından kararlılık kavramıdır. Daha sonra, kararlılık teorileri başarıyla geliştirilmiş ve diferansiyel denklemlerin birkaçı çeşidi için çeşitli tipteki Lyapunov fonksiyonlarının aracılığıyla bazı sonuçlar elde edilmiştir. Bu kararlılık ve sınırlılık kriteri kavramları ayrıca saptırılmış küme değerli diferansiyel denklemler için iki ölçü açısından başlangıç zaman farklı kararlılık, sınırlılık ve Lagrange kararlılık kriterleri için de incelenebilir.

Anahtar Kelimeler: Küme Değerli Diferansiyel Denklemler, Lyapunov Fonksiyonları, Kararlılık, Sınırlılık Kriteri, Karşılaştırma Sonuçları.

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TABLE of CONTENTS

	<u>Page</u>
SUMMARY	v
ÖZET	vi
ACKNOWLEDGEMENTS	vii
TABLE of CONTENTS	viii
LIST of ABBREVIATIONS and ACRONYMS	ix
1. INTRODUCTION	1
2. BASIC DEFINITIONS AND THEOREMS	2
3. BASIC THEORY	12
3.1. Comparison Principles	12
3.2. Local Existence and Uniqueness	19
3.3. Local Existence and Extremal Solutions	24
4. STABILITY THEORY	31
4.1. Lyapunov-like Functions	31
4.2. Global Existence	34
4.3. Fundamental Definitions of Stability of Set Differential Equations	36
4.4. Classical Notion of Stability of Set Differential Equation	41
4.5. Stability Criteria	42
4.6. Nonuniform Stability Criteria	49
4.7. Boundedness Criteria	56
5. CONCLUSION	63
REFERENCES	64
BIOGRAPHY	66

LIST of ABBREVIATIONS and ACRONYMS

<u>Abbreviations</u>	<u>Explanations</u>
<u>and Acronyms</u>	
co	: Convex hull
$\mathbb{R}$	: Real space
$\int$	: Integral
$d_H(B, A)$	: Hausdorff separation of B from A
$D(A, B)$	: Hausdorff distance between A and B
D_H	: Hukuhara differentiation
$\lim$	: Limit
$C(\mathbb{R}^n)$	: Nonempty closed subsets of n dimensional real space
$K(\mathbb{R}^n)$	: Nonempty compact subsets of n dimensional real space
$K_c(\mathbb{R}^n)$	: Nonempty compact convex subsets of n th dimensional real space
S_ϵ	: ϵ - neighborhood of set
GTÜ	: Gebze Teknik Üniversitesi
ITD	: Initial Time Difference
SDE	: Set Differential Equations

1. INTRODUCTION

In the qualitative theory of differential equations one of the main problems is stability criteria of the solutions. It is well known that the theory of stability with respect to position in the sense of Lyapunov has been successfully investigated to understand qualitative properties of dynamic systems for many years [1]-[5]. In nonlinear differential systems, Lyapunov's direct or Lyapunov's second method allows us to obtain sufficient conditions for the stability of a system without explicitly solving them. The method generalizes the idea which shows that the system is stable if there are some Lyapunov function such that satisfy some properties. The investigation of initial value problems of differential equations with a perturbation in the space variable is well known when the perturbation is restricted to the space variable [1]-[5].

In stability theory, there is another important problem that is to determine which stability properties of a special differential system are preserved under sufficiently small perturbations. This problem was investigated in several ways in [7]-[9]. In 1979, S.P.Gordon investigated the problem of determining the behavior of the solutions of a perturbed differential equation with respect to the solutions of the original unperturbed differential equation [17]. The principal mathematical technique employed is a modification of Lyapunov's Direct Method which is applied to the difference of the solutions of perturbed and unperturbed system where the initial positions are sufficiently close. In [18], the authors applied Variational Lyapunov Method, combines the method of variation of parameters and the method of Lyapunov, to connect the solutions of perturbed and unperturbed system.

There are many stability concepts are presented in the literature such as the partial stability, eventual stability, conditional stability, Lipschitz stability, relative stability and so on. Then, the theories of the stability in terms of two measures, Movchan in 1960, have been successfully developed [11] and some results are obtained by means of various types of Lyapunov functions for several kinds of differential equations in [12].

2. BASIC DEFINITIONS AND THEOREMS

Definition 2.1: Let A, B be two nonempty subsets of $\mathbb{R}^n$ and let $\lambda \in \mathbb{R}$. We define following Minkowski addition and scalar multiplication by

$$A + B = \{a + b : a \in A, b \in B\} \quad (2.1)$$

$$\lambda A = \{\lambda a : a \in A\} \quad (2.2)$$

Definition 2.2: Let A be nonempty subsets of $\mathbb{R}^n$. We call subset A of $\mathbb{R}^n$ is convex if for $a_1, a_2 \in A$ and $\forall \lambda \in [0,1]$ the point

$$a = \lambda a_1 + (1 - \lambda)a_2 \quad (2.3)$$

belongs to A .

Definition 2.3: We shall consider the following three spaces of nonempty compact convex subsets of $\mathbb{R}^n$, namely

- i) $C(\mathbb{R}^n) = \{A : A \text{ is nonempty closed subsets of } \mathbb{R}^n\}$*
- ii) $K(\mathbb{R}^n) = \{A : A \text{ is nonempty compact subset of } \mathbb{R}^n\}$*
- iii) $K_c(\mathbb{R}^n) = \{A : A \text{ is convex subset of } K(\mathbb{R}^n)\}$*

Definition 2.4: For any nonempty subsets of $\mathbb{R}^n$ we denote by coA its convex hull that is

$$coA = \{A' : A \subseteq A_\alpha \text{ for } A_\alpha \in K_c(\mathbb{R}^n), \alpha \in I, A' = \bigcap_{\alpha \in I} A_\alpha\} \quad (2.4)$$

Proposition 2.1: The spaces $C(\mathbb{R}^n)$, $K(\mathbb{R}^n)$ and $K_c(\mathbb{R}^n)$ are closed under the operations of addition and multiplication. And the following properties hold.

- i) $A + \theta = \theta + A = A$, θ is the zero element of $\mathbb{R}^n$ treated as singleton*

- ii) $(A + B) + C = A + (B + C)$
- iii) $A + B = B + A$
- iv) $A + C = B + C$ implies $A = B$
- v) $1.A = A$
- vi) $\lambda(A + B) = \lambda A + \lambda b$
- vii) $(\lambda + \mu)A = \lambda A + \mu A$

where $A, B, C \in K_c(\mathbb{R}^n)$ and $\lambda, \mu \in \mathbb{R}_+$.

Definition 2.5: Let A and B be element of $K_c(\mathbb{R}^n)$. If there exist an element $C \in K_c(\mathbb{R}^n)$ such that $A = B + C$, then we say that the Hukuhara difference of A and B exists and it is denoted by $A - B$.

Definition 2.6: Let $x \in \mathbb{R}^n$ and A is nonempty subset of $\mathbb{R}^n$. The distance $d(x, A)$ from x to A is defined by

$$d(x, A) = \inf\{\|x - a\| : a \in A\} \quad (2.5)$$

Remark 2.1: Let $x \in \mathbb{R}^n$ and A be nonempty subset of $\mathbb{R}^n$.

- i) $d(x, A) = d(x, \bar{A}) \geq 0$;
- ii) $d(x, A) = 0 \Leftrightarrow x \in \bar{A}$, where A is subset of $\mathbb{R}^n$.

Definition 2.7: We call the subset

$$S_\epsilon = \{x \in \mathbb{R}^n : d(x, A) < \epsilon\} \quad (2.6)$$

an ϵ – neighborhood of A .

Its closure is the subset

$$\bar{S}_\epsilon = \{x \in \mathbb{R}^n : d(x, A) \leq \epsilon\} \quad (2.7)$$

In particular, we shall denote by

$$\bar{S}_1^n = \bar{S}_1(\theta) \quad (2.8)$$

which is obviously compact subset of $\mathbb{R}^n$.

Remark 2.2: Let $x \in \mathbb{R}^n$ and A be nonempty subset of $\mathbb{R}^n$. $\bar{S}_\epsilon(A) = S(A, \epsilon) = A + \epsilon S_1^n, \forall \epsilon > 0$.

Definition 2.8: Let A, B be nonempty subsets of $\mathbb{R}^n$. We define the Hausdorff separation of B from A by

$$d_H(B, A) = \sup\{d(b, A) : b \in B\} \quad (2.9)$$

or

$$d_H(B, A) = \inf\{\epsilon > 0 : B \subseteq A + \epsilon \bar{S}_1^n\} \quad (2.10)$$

Remark 2.3: Let A, B and C be nonempty subsets of $\mathbb{R}^n$.

- i) $d_H(B, A) = 0 \Leftrightarrow B \subseteq \bar{A}$.
- ii) $d_H(B, A) \leq d_H(B, C) + d_H(C, A)$.
- iii) $d_H(B, A) \neq d_H(A, B)$.

Definition 2.9: Let A, B be nonempty subsets of $\mathbb{R}^n$. Hausdorff distance between A and B is defined

$$D(A, B) = \max\{d_H(B, A), d_H(A, B)\} \quad (2.11)$$

Remark 2.4: Let A, B , and C be nonempty subsets of $\mathbb{R}^n$.

- i) $D(B, A) = 0 \Leftrightarrow \bar{B} = \bar{A}$
- ii) $D(A, B) \leq D(A, C) + D(C, B)$
- iii) $D(B, A) = D(A, B)$

If we restrict our attention to $C(\mathbb{R}^n)$, then we find that the Hausdorff distance is a metric known as the Hausdorff metric. Thus $(C(\mathbb{R}^n), D)$ is a metric space.

Lemma 2.1: If $A, B \in K_c(\mathbb{R}^n)$, $C \in K(\mathbb{R}^n)$ and $A + C \subseteq B + C$ then $A \subseteq B$.

Proof 2.1: Let a be any element of A . We need to show that $a \in B$. Given any $c_1 \in C$, we have $a + c_1 \in B + C$, that is, there exist $b_1 \in B$ and $c_2 \in C$ with $a + c_1 = b_1 + c_2$. For the same reason $b_2 \in B$ and $c_3 \in C$ with $a + c_2 = b_2 + c_3$ must exist. Repeat the procedure indefinitely and sum the first n of the equations obtained. We get $a + c_1 = b_1 + c_2$, $a + c_2 = b_2 + c_3, \dots, a + c_n = b_n + c_{n+1}$

$$na + \sum_{i=2}^n c_i + c_1 = \sum_{i=1}^n b_i + \sum_{i=2}^n c_i + c_{n+1} \quad (2.12)$$

$$na = \sum_{i=1}^n b_i + c_{n+1} - c_1 \quad (2.13)$$

$$a = \frac{1}{n} \sum_{i=1}^n b_i + \frac{1}{n} (c_{n+1} - c_1) \quad (2.14)$$

$$a = \lim_{n \rightarrow \infty} a = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n b_i \quad (2.15)$$

where $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n b_i \in B$ and $\lim_{n \rightarrow \infty} \frac{1}{n} (c_{n+1} - c_1) = 0$ since B is convex and C is compact. Thus $\frac{1}{n} \sum_{i=1}^n b_i$ converges to a . Hence B is compact, $a \in B$. This completes the proof of Lemma 2.1.

Proposition 2.2: If $A, B \in K_c(\mathbb{R}^n)$ and $C \in K(\mathbb{R}^n)$ then

$$D(A + C, B + C) = D(A, B) \quad (2.16)$$

Proof 2.2: Let $\lambda \geq 0$ and S denote the closed unit sphere of the space. Consider the following inequalities

- i) $A + \lambda S \supset B$,*
- ii) $B + \lambda S \supset A$,*
- iii) $A + C + \lambda S \supset B + C$,*
- iv) $B + C + \lambda S \supset A + C$.*

Let $d_1 = D(A, B)$ and $d_2 = D(A + C, B + C)$. Then d_1 is the infimum of the positive numbers λ for which i) and ii) hold. Similarly, d_2 is the infimum of the positive numbers λ for which iii) and iv) hold. Since iii) and iv) follow from i) and ii) respectively, by adding C , we have $d_1 \geq d_2$. Conversely, since by Lemma 2.1, canceling C is allowed in iii) and iv), we obtain $d_1 \leq d_2$, which proves the proposition.

Proposition 2.3: If $A, B \in K(\mathbb{R}^n)$, then

$$D(\text{co}A, \text{co}B) \leq D(A, B) \quad (2.17)$$

If $A, A', B, B' \in K_c(\mathbb{R}^n)$, then

$$D(tA, tB) = tD(A, B) \text{ for all } t \geq 0 \quad (2.18)$$

$$D(A + A', B + B') \leq D(A, B) + D(A', B') \quad (2.19)$$

Further,

$$D(A - A', B - B') \leq D(A, B) + D(A', B') \quad (2.20)$$

if the differences $A - A'$ and $B - B'$ exist. Moreover, with $\beta = \max\{\lambda, \mu\}$, we have

$$D(\lambda A, \mu B) \leq \beta D(A, B) + |\lambda - \mu| [D(A, \theta) + D(B, \theta)] \quad (2.21)$$

and

$$D(\lambda A, \lambda B) = \lambda D(A - B, \theta) \quad (2.22)$$

whenever $A - B$ exists.

Proposition 2.4: A nonempty subset $\mathcal{A}$ of the metric space $(K(\mathbb{R}^n), D)$ or $(K_c(\mathbb{R}^n), D)$, is compact if and only if it is closed and uniformly bounded.

Set inclusion induces partial ordering on $K(\mathbb{R}^n)$. Write $A \leq B$ if and only if $A \subseteq B$, where $A, B \in K(\mathbb{R}^n)$. Then

$$\mathcal{L}(B) = \{A \in K(\mathbb{R}^n) : B \leq A\} \quad (2.23)$$

$$\mathcal{U}(B) = \{A \in K(\mathbb{R}^n) : A \leq B\} \quad (2.24)$$

are closed subsets of $K(\mathbb{R}^n)$ for any $B \in K(\mathbb{R}^n)$. In fact, Proposition 2.4, $\mathcal{U}(B)$ is compact subset of $K(\mathbb{R}^n)$.

Proposition 2.5: Let $\{A_j\} \subset K(\mathbb{R}^n)$ satisfy

$$\cdots \subseteq A_j \subseteq \cdots \subseteq A_2 \subseteq A_1 \quad (2.25)$$

Then $A = \bigcap_{j=1}^{\infty} A_j \in K(\mathbb{R}^n)$ and

$$D(A_n, A) \rightarrow 0 \text{ as } n \rightarrow \infty \quad (2.26)$$

on the other hand, if

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_j \subseteq \cdots \quad (2.27)$$

and $A = \bigcup_{j=1}^{\infty} A_j \in K(\mathbb{R}^n)$, then $D(A_n, A) \rightarrow 0$ as $n \rightarrow \infty$ holds.

Definition 2.10: Let A be nonempty subset of $\mathbb{R}^n$. The support function of A is defined for all $p \in \mathbb{R}^n$ by

$$s(p, A) = \sup\{\langle p, a \rangle; a \in A\} \quad (2.28)$$

which approaches to the value $+\infty$ when A is unbounded. However, when $A \in K(\mathbb{R}^n)$, the supremum is always attained and the support function $s(\cdot, A): \mathbb{R}^n \rightarrow \mathbb{R}$ is well defined. Indeed,

$$|s(p, A)| \leq \|A\| \|p\|, \forall p \in \mathbb{R}^n \quad (2.29)$$

and

$$|s(p, A) - s(q, A)| \leq \|A\| \|p - q\|, \forall p, q \in \mathbb{R}^n \quad (2.30)$$

Definition 2.11: We consider mappings F from a domain T in $\mathbb{R}^k$ into the metric space $(K_c(\mathbb{R}^n), D)$.

Thus, $F: T \rightarrow K_c(\mathbb{R}^n)$ or equivalently

$$F(t) \in K_c(\mathbb{R}^n), \text{ for all } t \in T \quad (2.31)$$

We shall call such a mapping F is a (compact convex) set valued mapping from T to $\mathbb{R}^n$.

Definition 2.12: We shall say that a set valued mapping F satisfying (2.31) is continuous at $t_0 \in T$ if $\forall \epsilon > 0$ there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that

$$D[F(t), F(t_0)] < \epsilon, \forall t \in T \text{ with } \|t - t_0\| < \delta \quad (2.32)$$

Alternatively, we can write (2.32) in terms of the convergence of sequences, that is

$$\lim_{t_n \rightarrow t_0} D[F(t_n), F(t_0)] = 0, \forall t_n \in T \text{ with } t_n \rightarrow t_0 \in T. \quad (2.33)$$

Using the Hausdorff separation d_H and neighborhoods, we see that (2.32) is combination of

$$d_H(F(t), F(t_0)) < \epsilon \quad (2.34)$$

that is

$$F(t) \subset S_\epsilon(F(t_0)) \equiv F(t_0) + \epsilon S_1^n \quad (2.35)$$

and

$$d_H(F(t_0), F(t)) < \epsilon \quad (2.36)$$

that is

$$F(t_0) \subset S_\epsilon(F(t)) \equiv F(t) + \epsilon S_1^n \quad (2.37)$$

for all $t \in T$ with $\|t - t_0\| < \delta$. As before, $S_1^n = \{x \in \mathbb{R}^n: \|x\| < 1\}$ is the open unit ball in $\mathbb{R}^n$. If the mapping F satisfies (2.34), (2.35), then we say that F is upper semicontinuous at t_0 , or that F is lower semicontinuous at t_0 , if it satisfies (2.36), (2.37). Thus F is continuous at t_0 if and only if it is both lower semicontinuous and upper semicontinuous at t_0 .

Theorem 2.1: $F: T \rightarrow K_c(\mathbb{R}^n)$ is measurable if and only if there exists a sequence $\{f_i\}$ of measurable selectors of F such that

$$F(t) = \overline{\bigcup \{f_i(t): i = 1, 2, \dots\}}, \text{ for each } t \in T \quad (2.38)$$

Definition 2.13: Let I be an interval of real numbers. Let a multifunction $U: I \rightarrow K_c(\mathbb{R}^n)$ be given. U is Hukuhara differentiable at a point $t_0 \in I$, if there exist $D_H U(t_0) \in K_c(\mathbb{R}^n)$ such that the limits

$$\lim_{\Delta t \rightarrow 0^+} \frac{U(t_0 + \Delta t) - U(t_0)}{\Delta t} \quad (2.39)$$

and

$$\lim_{\Delta t \rightarrow 0^+} \frac{U(t_0) - U(t_0 - \Delta t)}{\Delta t} \quad (2.40)$$

both exist and are equal to $D_H U(t_0)$.

Definition 2.14: Let $F: [0,1] \rightarrow K_c(\mathbb{R}^n)$ and let $S(F)$ denote the set of integrable selectors of F over $[0,1]$. Then the Aumann integral of F over $[0,1]$ is defined as

$$\int_0^1 F(t)dt = \overline{\left\{ \int_0^1 f(t)dt : f \in S(F) \right\}} \quad (2.41)$$

If $S(F) \neq \emptyset$, then the Aumann integral exist and F is said to be Aumann integrable. We shall say that F is integrally bounded on $[0,1]$ if there exists an integrable function $g: [0,1] \rightarrow \mathbb{R}$ such that

$$\|F(t)\| \leq g(t), \forall t \in [0,1] \quad (2.42)$$

If such an F has measurable selectors, then they are also integrable and $S(F)$ is nonempty.

Theorem 2.2: If $F: [0,1] \rightarrow K_c(\mathbb{R}^n)$ is measurable and integrally bounded, then it is Aumann integrable over each $[a,s] \subset [0,1]$ with $\int_a^s F(t)dt \in K_c(\mathbb{R}^n)$ for all $s \in [a,1]$. For a set valued mapping F as Theorem 2.1, the Castaing Representation Theorem 2.1 applies and provides a sequence $\{f_i\}$ of integrable selectors which are pointwise dense in F . Moreover,

$$\int_0^1 F(t)dt = \overline{\left\{ \int_0^1 f_i(t)dt : i = 1,2,\dots \right\}} \quad (2.43)$$

And also, we need only to consider these selectors to evaluate $\int_0^1 F(t)dt$.

Theorem 2.3: Let $F, F_i: [0,1] \rightarrow K_c(\mathbb{R}^n), i = 1,2,\dots$, be measurable and uniformly integrally bounded. If $F_i(t) \rightarrow F(t)$ for all $t \in [0,1]$ as $i \rightarrow \infty$, then

$$A_i = \int_0^1 F_i(t)dt \rightarrow A = \int_0^1 F(t)dt \text{ as } i \rightarrow \infty \quad (2.44)$$

Theorem 2.4: Let $F_i: [0,1] \rightarrow K_c(\mathbb{R}^n)$, $i = 1,2,\dots$, be measurable and uniformly integrally bounded, and suppose that $A_i = \int_0^1 F_i(t)dt \rightarrow A \in K_c(\mathbb{R}^n)$ as $i \rightarrow \infty$. Then there exists a measurable mapping $F: [0,1] \rightarrow K_c(\mathbb{R}^n)$ such that $A = \int_0^1 F(t)dt$.

Theorem 2.5: If $F, G: [0,1] \rightarrow K_c(\mathbb{R}^n)$ are integrable, then so also is

$$D[F(\cdot), G(\cdot)]: [0,1] \rightarrow \mathbb{R} \quad (2.45)$$

and

$$D \left[\int_0^1 F(t)dt, \int_0^1 G(t)dt \right] \leq \int_0^1 D[F(t), G(t)]dt \quad (2.46)$$

Theorem 2.6: Let $F: [0,1] \rightarrow K_c(\mathbb{R}^n)$ be measurable and integrally bounded. Then $A: [0,1] \rightarrow K_c(\mathbb{R}^n)$ defined by

$$A(t) = \int_0^t F(s)ds \quad (2.47)$$

for all $t \in [0,1]$ is Hukuhara differentiable for almost all $t_0 \in (0,1)$, with the Hukuhara derivative $D_H A(t_0) = F(t_0)$. A counterpart of the first fundamental theorem of calculus

$$F(t_1) = F(t_0) + \int_{t_0}^{t_1} D_H F(s)ds, \quad 0 \leq t_0 \leq t_1 \leq 1 \quad (2.48)$$

holds for a Hukuhara differentiable $F: [0,1] \rightarrow K_c(\mathbb{R}^n)$ with continuous Hukuhara derivative $D_H F$ on $[0,1]$.

3. BASIC THEORY

3.1. Comparison Principles

Let us consider the initial value problem (IVP) for the set differential equation

$$D_H U = F(t, U), U(t_0) = U_0 \in K_c(\mathbb{R}^n), t_0 \geq 0 \quad (3.1)$$

where $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$ and $D_H U$ is the Hukuhara derivative of U .

The mapping $U \in C^1[J, K_c(\mathbb{R}^n)]$ where $J = [t_0, t_0 + a]$, $a > 0$, is said to be a solution of (3.1) on J , if it satisfies (3.1) on J .

Since $U(t)$ is continuously differentiable, then we have

$$U(t) = U_0 + \int_{t_0}^t D_H U(s) ds, t \in J. \quad (3.2)$$

We also associate with the IVP (3.1) the following integral equation

$$U(t) = U_0 + \int_{t_0}^t F(s, U(s)) ds, t \in J \quad (3.3)$$

where the integral in (3.3) is the Hukuhara integral. Observe also that $U(t)$ is a solution of (3.1) if and only if it satisfies (3.3) on J .

Theorem 3.1: Let $F \in C[J \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$ and $t \in J, U, V \in K_c(\mathbb{R}^n)$

$$D[F(t, U), F(t, V)] \leq g(t, D[U, V]) \quad (3.4)$$

where $g \in C[J \times \mathbb{R}_+, \mathbb{R}_+]$ and $g(t, w)$ is monotone nondecreasing in $w \forall t \in J$.

Suppose further that the maximal solution $r(t, t_0, w_0)$ of the scalar differential equations

$$w' = g(t, w), w(t_0) = w_0 \geq 0 \quad (3.5)$$

exists on J . Then, if $U(t)$ and $V(t)$ are any two solutions through (t_0, U_0) and (t_0, V_0) respectively on J , it follows that

$$D[U(t), V(t)] \leq r(t, t_0, w_0), t \in J \quad (3.6)$$

provided $D[U_0, V_0] \leq w_0$.

Proof 3.1: Setting $D[U(t), V(t)]$, so that $m(t_0) = D[U_0, V_0] \leq w_0$. Then, in view of the properties of the metric D , then we get

$$\left\{ \begin{array}{l} m(t) = D \left[U_0 + \int_{t_0}^t F(s, U(s)) ds, V_0 + \int_{t_0}^t F(s, V(s)) ds \right] \\ \leq D \left[U_0 + \int_{t_0}^t F(s, U(s)) ds, U_0 + \int_{t_0}^t F(s, V(s)) ds \right] \\ + D \left[U_0 + \int_{t_0}^t F(s, V(s)) ds, V_0 + \int_{t_0}^t F(s, V(s)) ds \right] \\ = D \left[\int_{t_0}^t F(s, U(s)) ds, \int_{t_0}^t F(s, V(s)) ds \right] + D[U_0, V_0] \end{array} \right. \quad (3.7)$$

Now, by using the properties of the integrals and condition (3.4), we observe that

$$\left\{ \begin{array}{l} m(t) = m(t_0) + \int_{t_0}^t D[F(s, U(s)), F(s, V(s))] ds \\ \leq m(t_0) + \int_{t_0}^t g(s, D[U(s), V(s)]) ds \\ = m(t_0) + \int_{t_0}^t g(s, m(s)) ds, t \in J. \end{array} \right. \quad (3.8)$$

Now applying Theorem 1.9.2 given in Lakshmikantham and Leela [6], we conclude that

$$m(t) \leq r(t, t_0, w_0), t \in J \quad (3.9)$$

This establishes Theorem 3.1.

Remark 3.1: If we employ the theory of differential inequalities instead of integral inequalities, we can dispense with the monotone character of $g(t, w)$ assumed in Theorem 3.1. This is the content of the next comparison principle.

Theorem 3.2: Let the assumptions of Theorem 3.1 hold except the nondecreasing property of $g(t, w)$ in w . Then the conclusion (3.6) is valid.

Proof 3.2: For small $h > 0$, the Hukuhara differences $U(t + h) - U(t), V(t + h) - V(t)$ exist, and we have for $t \in J$

$$m(t + h) - m(t) = D[U(t + h), V(t + h)] - D[U(t), V(t)] \quad (3.10)$$

Using Remark 2.4

$$\left\{ \begin{array}{l} D[U(t + h), V(t + h)] \\ \leq D[U(t + h), V(t + h)] \\ \quad + D[U(t) + hF(t, U(t)), U(t) + hF(t, U(t))] \\ D[U(t + h), V(t + h)] \\ \leq D[U(t + h), U(t) + hF(t, U(t))] \\ \quad + D[U(t) + hF(t, U(t)), V(t + h)] \end{array} \right. \quad (3.11)$$

and

$$\left\{ \begin{array}{l} D[U(t) + hF(t, U(t)), V(t + h)] \\ \leq D[U(t) + hF(t, U(t)), V(t + h)] \\ \quad + D[V(t) + hF(t, V(t)), V(t) + hF(t, V(t))] \end{array} \right. \quad (3.12)$$

$$\left\{ \begin{array}{l} D[U(t) + hF(t, U(t)), V(t + h)] \\ \leq D[U(t) + hF(t, U(t)), V(t) + hF(t, V(t))] \\ \quad + D[V(t) + hF(t, V(t)), V(t + h)] \end{array} \right. \quad (3.13)$$

Also, we observe that

$$\left\{ \begin{array}{l} D[U(t) + hF(t, U(t)), V(t) + hF(t, V(t))] \\ \leq D[hF(t, U(t)), hF(t, V(t))] + D[U(t), V(t)] \end{array} \right. \quad (3.14)$$

Hence, it follows that

$$\left\{ \begin{array}{l} \frac{m(t + h) - m(t)}{h} \\ \leq \frac{1}{h} D[U(t + h), U(t) + hF(t, U(t))] \\ \quad + \frac{1}{h} D[V(t) + hF(t, V(t)), V(t + h)] \\ \quad + \frac{1}{h} D[hF(t, U(t)), hF(t, V(t))] \end{array} \right. \quad (3.15)$$

And consequently, in view of properties of D and the fact that $U(t), V(t)$ are solutions of (3.1), then we find that

$$\left\{ \begin{aligned} D^+m(t) &\leq \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [m(t+h) - m(t)] \\ &\leq \lim_{h \rightarrow 0^+} \sup D \left[\frac{U(t+h) - U(t)}{h}, F(t, U(t)) \right] \\ &\quad + \lim_{h \rightarrow 0^+} \sup D \left[F(t, V(t)), \frac{V(t+h) - V(t)}{h} \right] \\ &\quad + \lim_{h \rightarrow 0^+} \sup D [F(t, U(t)), F(t, V(t))] \end{aligned} \right. \quad (3.16)$$

Here, we have used the fact that

$$\left\{ \begin{aligned} &D[U(t+h), U(t) + hF(t, U(t))] \\ &= D[U(t) + Z(t, h), U(t) + hF(t, U(t))] \\ &= D[Z(t, h) + U(t), U(t) + hF(t, U(t))] \\ &= D[Z(t, h), h(F(t, U(t)))] \\ &= D[U(t+h) - U(t), hF(t, U(t))] \end{aligned} \right. \quad (3.17)$$

The conclusion (3.6) follows from Theorem 1.4.1 in Lakshmikantham and Leela[6].

The next comparison result provides an estimate under weaker assumptions.

Theorem 3.3: Assume that $F \in C[J \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$ and

$$\left\{ \begin{aligned} &\lim_{h \rightarrow 0} \sup \frac{1}{h} [D[U + hF(t, U), V + hF(t, V)] - D[U, V]] \\ &\leq g(t, D[U, V]), t \in J \end{aligned} \right. \quad (3.18)$$

where $U, V \in K_c(\mathbb{R}^n)$, $g \in C[J \times \mathbb{R}_+, \mathbb{R}]$. The maximal solution $r(t, t_0, w_0)$ of (3.5) exists on J . Then the conclusion of Theorem 3.2 is valid.

Proof 3.3: Proceeding as in the proof of Theorem 3.2, we see that

$$\left\{ \begin{aligned} m(t+h) - m(t) &= D[U(t+h), V(t+h)] - D[U(t), V(t)] \\ &\leq D[U(t+h), U(t) + hF(t, U(t))] \\ &\quad + D[V(t) + hF(t, V(t)), V(t+h)] \\ &\quad + D[U(t) + hF(t, U(t)), V(t) + hF(t, V(t))] \\ &\quad - D[U(t), V(t)] \end{aligned} \right. \quad (3.19)$$

$$\left\{ \begin{aligned} D^+ m(t) &\leq \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [m(t+h) - m(t)] \\ &\leq \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [D[U(t) + hF(t, U(t)), V(t) + hF(t, V(t))]] \\ &\quad - \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [D[U(t), V(t)]] \\ &\quad + \lim_{h \rightarrow 0^+} \sup D \left[\frac{U(t+h) - U(t)}{h}, F(t, U(t)) \right] \\ &\quad + \lim_{h \rightarrow 0^+} \sup D \left[F(t, V(t)), \frac{V(t+h) - V(t)}{h} \right] \\ &\leq g(t, D[U(t), V(t)]) \\ &= g(t, m(t)), t \in J \end{aligned} \right. \quad (3.20)$$

The conclusion follows as before by Theorem 1.4.1 in Lakshmikantham and Leela[6] and the proof is complete.

Corollary 3.1: Assume that $F \in C[J \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$ and either

- i) $D[F(t, U), \theta] \leq g(t, D[U, \theta])$ or
- ii) $\lim_{h \rightarrow 0} \sup \frac{1}{h} [D[U + hF(t, U), \theta] - D[U, \theta]] \leq g(t, D[U, \theta])$, where $g \in C[J \times \mathbb{R}_+, \mathbb{R}]$.

Then, if $D[U_0, \theta] \leq w_0$, we have

$$D[U(t), \theta] \leq R(t, t_0, w_0), t \in J \quad (3.21)$$

where $R(t, t_0, w_0)$ is the maximal solution of (3.5) on J .

Corollary 3.2: The function $g(t, w) = \lambda(t)w$, $\lambda(t) \geq 0$ and continuous is admissible in Theorem 3.2 to give

$$m(t) \leq m(t_0) + \int_{t_0}^t \lambda(s)m(s)ds, \quad t \in J \quad (3.22)$$

then the application of Gronwall Lemma implies that

$$m(t) \leq m(t_0) \exp \left[\int_{t_0}^t \lambda(s)ds \right], \quad t \in J \quad (3.23)$$

which shows that (3.6) reduces to

$$D[U(t), V(t)] \leq D[U_0, V_0] \exp \left[\int_{t_0}^t \lambda(s)ds \right], t \in J \quad (3.24)$$

Corollary 3.3: The function $g(t, w) = -\lambda(t)w$, with $\lambda(t)$ as in Corollary 3.2 is also admissible in Theorem 3.3, and then we get

$$D[U(t), V(t)] \leq D[U_0, V_0] \exp \left[- \int_{t_0}^t \lambda(s)ds \right], t \in J \quad (3.25)$$

If $\lambda(t) = \lambda > 0$ then, we find that

$$D[U(t), V(t)] \leq D[U_0, V_0] e^{-\lambda(t-t_0)}, t \in J \quad (3.26)$$

3.2. Local Existence and Uniqueness

In this section we will begin by proving the existence and uniqueness result under the assumptions more general than the Lipschitz type conditions, which exhibits the idea of the comparison principle.

Theorem 3.4: Assume that

- i) $F \in C[R_0, K_c(\mathbb{R}^n)]$ and $D[F(t, U), \theta] \leq M_0$ where $R_0 = J \times B(U_0, b)$, $B(U_0, b) = [U \in K_c(\mathbb{R}^n) : D[U, U_0] \leq b]$ on R_0 ;
- ii) $g \in C[J \times [0, 2b], \mathbb{R}_+]$, $g(t, w) \leq M_1$ on $J \times [0, 2b]$, $g(t, 0) \equiv 0$, $g(t, w)$ is nondecreasing in w for each $t \in J$ and $w(t) \equiv 0$ is the only solution of

$$w' = g(t, w), w(t_0) = 0 \quad (3.27)$$

on J ;

- iii) $D[F(t, U), F(t, V)] \leq g(t, D[U, V])$ on R_0 .

Then the successive approximations defined by

$$U_{n+1}(t) = U_0 + \int_{t_0}^t F(s, U_n(s)) ds, \quad n = 0, 1, 2, \dots \quad (3.28)$$

exists on $J_0 = [t_0, t_0 + \eta)$, where $\eta = \min\{a, \frac{b}{M}\}$, $M = \max\{M_0, M_1\}$, as continuous functions and converge uniformly to the unique solutions $U(t)$ of the IVP (3.2) on J_0 .

Proof 3.4: Using the properties of Hausdorff metric, we get by induction

$$\left\{ \begin{aligned} [U_{n+1}(t), U_0] &= D \left[U_0 + \int_{t_0}^t F(s, U_n(s)) ds, U_0 \right] \\ &= D \left[\int_{t_0}^t F(s, U_n(s)) ds, \theta \right] \\ &\leq \int_{t_0}^t D[F(s, U_n(s)), \theta] ds \\ &\leq M_0(t - t_0) \leq M_0 a \leq b \end{aligned} \right. \quad (3.29)$$

and consequently, the successive approximations $\{U_n(t)\}$ are well defined on J_0 .

We shall next define the successive approximations of (3.27) as follows:

$$w_0(t) = M(t - t_0), \quad (3.30)$$

$$w_{n+1}(t) = \int_{t_0}^t g(s, w_n(s)) ds, t \in J_0, n = 0, 1, 2, \dots \quad (3.31)$$

By using the Mathematical induction, in view of the monotone character of $g(t, w)$ in w , proves that $\{w_n(t)\}$ are well defined and

$$0 \leq w_{n+1}(t) \leq w_n(t), t \in J_0 \quad (3.32)$$

Since $|w'_n(t)| \leq g(t, w_{n-1}(t)) \leq M_1$, we conclude by the Ascoli-Arzelà Theorem and the monotonicity of the sequence $\{w_n(t)\}$ that

$$\lim_{n \rightarrow \infty} w_n(t) = w(t) \quad (3.33)$$

uniformly on J_0 . It is also clear that $w(t)$ satisfies (3.27) and therefore by condition ii), $w(t) \equiv 0$ on J_0 .

We observe that

$$D[U_1(t), U_0(t)] \leq \int_{t_0}^t D[F(s, U_0(s)), \theta] ds \leq M(t - t_0) \equiv w_0(t) \quad (3.34)$$

Assume that for some $k > 1$, we have

$$D[U_k(t), U_{k-1}(t)] \leq w_0(t), \text{ on } J_0 \quad (3.35)$$

Since

$$D[U_{k+1}(t), U_k(t)] \leq \int_{t_0}^t D[F(s, U_k(s)), F(s, U_{k-1}(s))] ds \quad (3.36)$$

using condition iii) and the monotone character of $g(t, w)$, we get

$$\begin{cases} D[U_k(t), U_{k-1}(t)] \leq \int_{t_0}^t g(s, D[U_k(s), U_{k-1}(s)]) ds \\ \leq \int_{t_0}^t g(s, w_{k-1}(s)) ds = w_k(t) \end{cases} \quad (3.37)$$

Thus by induction again, we get the estimate

$$D[U_{n+1}(t), U_n(t)] = w_n(t), t \in J_0 \quad (3.38)$$

That is true for all n .

Letting $u(t) = D[U_{n+1}(t), U_n(t)]$, $t \in J_0$. The proof of the Theorem of 3.2 shows that

$$D^+u(t) \leq g(t, D[U_n(t), U_{n-1}(t)]) \leq g(t, w_{n-1}(t)), t \in J_0 \quad (3.39)$$

Now let $n \leq m$. Setting $v(t) = D[U_n(t), U_m(t)]$, we obtain from (3.28)

$$\left\{ \begin{array}{l} D^+v(t) \leq D[D_H U_n(t), D_H U_m(t)] \\ \quad = D[F(t, U_{n-1}(t)), F(t, U_{m-1}(t))] \\ \quad \leq [F(t, U_n(t)), F(t, U_{n-1}(t))] + D[F(t, U_n(t)), F(t, U_m(t))] \\ \quad \quad + D[F(t, U_m(t)), F(t, U_{m-1}(t))] \\ \quad \leq g(t, w_{n-1}(t)) + g(t, w_{m-1}(t)) + g(t, D[U_n(t), U_m(t)]) \\ \quad \leq g(t, v(t)) + 2g(t, w_{n-1}(t)), t \in J_0 \end{array} \right. \quad (3.40)$$

Here we have used the arguments of the proof of Theorem 3.2, the monotone character of $g(t, w)$ and the fact that $w_{m-1} \leq w_{n-1}$, since $n \leq m$ and $w_n(t)$ is a decreasing sequence. The comparison Theorem 1.4.1 in Lakshmikantham and Leela [6] yields the estimate

$$v(t) \leq r_n(t), \quad t \in J_0 \quad (3.41)$$

where $r_n(t)$ is the maximal solution of

$$r'_n(t) = g(t, r_n) + 2g(t, w_{n-1}(t)), \quad r_n(t_0) = 0 \quad (3.42)$$

for each n . Since as $n \rightarrow \infty$, $2g(t, w_{n-1}(t)) \rightarrow 0$ uniformly on J_0 , it follows by Lemma 1.3.1 in Lakshmikantham and Leela [6] that $r_n(t) \rightarrow 0$ as $n \rightarrow \infty$ uniformly, on J_0 . This implies from (3.5) and definition of $v(t)$ that $U_n(t)$ converges uniformly to $U(t)$ is a solution of (3.1). To show uniqueness, let us assume that there exist another solution $V(t)$ of (3.1) on J_0 . Then setting $m(t) = D[U(t), V(t)]$ and noting that $m(t_0) = 0$, we get, as before

$$D^+m(t) \leq g(t, m(t)), \quad t \in J_0 \quad (3.43)$$

and $m(t) \leq r(t, t_0, 0)$, $t \in J_0$, by Theorem 3.4. By assumption $r(t, t_0, 0) \equiv 0$, we get $U(t) \equiv V(t)$ on J_0 , proving the theorem.

Lemma 3.1: Let $F \in C[J \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$ and

$$G(t, r) = \max[D[F(t, U), \theta]: D[U, U_0] \leq R] \quad (3.44)$$

Assume that $R^*(t, t_0, 0)$ is the maximal solution of

$$w' = G(t, w), \quad w(t_0) = 0, \text{ on } J \quad (3.45)$$

Let $U(t) = U(t, t_0, U_0)$ be the solution of (3.2). Then

$$D[U(t), U_0] \leq R^*(t, t_0, 0), \quad t \in J. \quad (3.46)$$

Theorem 3.5: Suppose the assumptions i), ii), iii) of Theorem 3.4 hold. Assume further that the solutions $w(t, t_0, w_0)$ of (3.5) through (t_0, w_0) are continuous with respect to (t_0, w_0) . Then the solutions $U(t) = U(t, t_0, U_0)$ of (3.1) are continuous relative to (t_0, U_0) .

Proof 3.5: Let $U(t) = U(t, t_0, U_0), V(t) = V(t, t_0, V_0), U_0, V_0 \in K_c(\mathbb{R}^n)$, be two solutions of (3.2). Then defining $m(t) = D[U(t), V(t)]$, we get from Theorem 3.2, the estimate

$$D[U(t), V(t)] \leq R(t, t_0, D[U_0, V_0]), \quad t \in J. \quad (3.47)$$

Since $\lim_{u_0 \rightarrow v_0} R(t, t_0, D[U_0, V_0]) = R(t, t_0, 0)$ uniformly on J and by hypothesis $R(t, t_0, 0) \equiv 0$, it follows that $\lim_{u_0 \rightarrow v_0} U(t, t_0, U_0) = V(t, t_0, V_0)$ uniformly and hence continuity of $U(t, t_0, U_0)$ relative to U_0 is valid.

To prove the continuity relative to t_0 , let $U(t, t_0, U_0), V(t) = V(t, \tau_0, V_0)$ be the two solutions of (3.1) and $\tau_0 > t_0$. As before, setting $m(t) = D[U(t), V(t)]$, noting that $m(\tau) = D[U(\tau_0), U_0]$, we obtain from Lemma 3.1,

$$m(\tau_0) \leq R^*(\tau_0, t_0, 0) \quad (3.48)$$

and consequently, by Theorem 3.1, we arrive at

$$m(t) \leq \tilde{R}(t), \quad t \geq \tau_0 \quad (3.49)$$

where $\tilde{R}(t) = \tilde{R}(t, \tau_0, R^*(\tau_0, t_0, 0))$ is the maximal solution of (3.5) through $(\tau_0, R^*(\tau_0, t_0, 0))$. Since $R(\tau_0, t_0, 0) = 0$, we have

$$\lim_{\tau_0 \rightarrow t_0} \tilde{R}(t, \tau_0, R(\tau_0, t_0, 0)) = \tilde{R}(t, t_0, 0) \quad (3.50)$$

uniformly on J . By the hypothesis $\tilde{R}(t, t_0, 0) \equiv 0$ which proves the continuity of $U(t, t_0, U_0)$, with respect to t_0 and the proof is complete.

3.3. Local Existence and Extremal Solutions

Theorem 3.6: If $\{U_n(t)\}$ is a sequence of equicontinuous and equibounded multimappings defined on an interval J , we can extract a subsequence that converges uniformly to a continuous multimapping $U(t)$ on J .

Theorem 3.7: Assume that $F \in C[R_0, K_c(\mathbb{R}^n)]$ where $R_0 = J \times B[U_0, b], B[U_0, b] = \{U \in K_c(\mathbb{R}^n) : D[U, U_0] \leq b\}$ and $D[F(t, U), \theta] \leq M$ on R_0 . Then there exists at least one solution $U(t)$ for the IVP (3.1) on $J_0 = [t_0, t_0 + \alpha]$, where $\alpha = \min\left\{a, \frac{b}{M}\right\}$, $\alpha > 0$.

Proof 3.7: Let $U_0 \in C^1[[t_0 - \delta, t_0], K_c(\mathbb{R}^n)]$, $\delta > 0$ such that $U_0 = U_0(t_0)$, $D[U_0(t), U_0] \leq b$ and $D[D_H U_0(t), \theta] \leq M$. Consider $0 < \epsilon < \delta$ and define

$$\begin{cases} U_\epsilon(t) = U_0(t), & t_0 - \delta \leq t \leq t_0 \\ U_\epsilon = U_0 + \int_{t_0}^t F(s, U_\epsilon(s - \epsilon)) ds, & t_0 \leq t \leq t_0 + \alpha_1 \end{cases} \quad (3.51)$$

where $\alpha_1 = \min\{\alpha, \epsilon\}$. Then we have, by using Remark 2.4 ii) and the property (2.46) of D

$$\left\{ \begin{aligned} D[U_\epsilon(t), U_0] &= D \left[U_0 + \int_{t_0}^t F(s, U_\epsilon(s - \epsilon)) ds, U_0 \right] \\ &= D \left[\int_{t_0}^t F(s, U_\epsilon(s - \epsilon)) ds, \theta \right] \\ &\leq \int_{t_0}^t D[F(s, U_\epsilon(s - \epsilon)), \theta] ds \\ &\leq M(t - t_0) \leq M\alpha_1 \leq b \end{aligned} \right. \quad (3.52)$$

Furthermore,

$$D_H U_\epsilon(t) \leq F(t, U_\epsilon(t - \epsilon)) \text{ on } [t_0, t_0 + \alpha_1] \quad (3.53)$$

and therefore

$$D[D_H U_\epsilon(t), \theta] \leq M \quad (3.54)$$

If $\alpha_1 \leq \alpha$, we can use (3.51) to extend $U_\epsilon(t)$ satisfying the relations (3.53) and (3.54) on $[t_0 - \delta, t_0 + \alpha_2]$ where $\alpha_2 = \min\{\alpha_1, 2\epsilon\}$. Continuing this process, we arrive at an a.e. differentiable function $U_\epsilon(t)$ such that (3.51), (3.53) and (3.54) hold on $[t_0 - \delta, t_0 + \alpha]$. Thus $U_\epsilon(t)$ forms a family of equicontinuous and uniformly bounded functions. By Theorem 3.6, we obtain a decreasing sequence $\{\epsilon_n\}$ such that $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and $U(t) = \lim_{n \rightarrow \infty} U_{\epsilon_n}(t)$ exists uniformly for $t_0 - \delta \leq t \leq t_0 + \alpha$. Since F is uniformly continuous, we show that $F(t, U_{\epsilon_n}(t - \epsilon_n)) \rightarrow F(t, U(t))$ uniformly, as $n \rightarrow \infty$ on J_0 . This allows for term by term integration in (3.51) with $\epsilon = \epsilon_n$ and $\alpha = \alpha_1$ which yields

$$U(t) = U_0 + \int_{t_0}^t F(s, U(s)) ds, \quad t \in J_0 \quad (3.55)$$

Hence $U(t)$ is a solution of (3.1) on J_0 and the proof is complete.

In order to discuss the existence of extremal solutions for the IVP (3.1), we require a comparison result which demands introducing a partial order in the metric space $(K_c(\mathbb{R}^n), D)$. We denote by $K(K^0)$ the subfamily of $K_c(\mathbb{R}^n)$ consisting of sets $U \in K_c(\mathbb{R}^n)$ such that any $u \in U$ is nonnegative vector of n -components satisfying $u_i \geq 0$ ($u_i > 0$) for $i = 1, 2, \dots, n$. Thus K is a cone in $K_c(\mathbb{R}^n)$ and K^0 is the nonempty interior of K . We can therefore induce a partial ordering in $K_c(\mathbb{R}^n)$ as follows.

Definition 3.1: For any U and $V \in K_c(\mathbb{R}^n)$, if there exists a $Z \in K_c(\mathbb{R}^n)$ such that $Z \in K(K^0)$ and

$$U = V + Z \quad (3.56)$$

then, we write $U \geq V$ ($U > V$). Similarly, one can define $U \leq V$ ($U < V$).

Definition 3.2: Let $R(t)$ be a solution of the set differential equation (3.1). Then we say that $R(t)$ is the maximal solution of (3.1) if, for every solution $U(t)$ of (3.1) existing on J_0 , we have

$$U(t) \leq R(t), t \in J_0 \quad (3.57)$$

Definition 3.3: Let $r(t)$ be a solution of the set differential equation (3.1). Then, we say that $r(t)$ is the minimal solution of (3.1) if, for every solution $U(t)$ of (3.1) existing on J_0 , we have

$$U(t) \geq r(t), t \in J_0 \quad (3.58)$$

Theorem 3.8: Assume that

- i) $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$, $F(t, U)$ is monotone nondecreasing in U for each $t \in \mathbb{R}_+$; that is, whenever $U \leq V$, we have $F(t, U) \leq F(t, V)$, $t \in \mathbb{R}_+$;
- ii) $V, W \in C^1[\mathbb{R}_+, K_c(\mathbb{R}^n)]$

$$D_H V < F(t, V) \text{ and } D_H W \geq F(t, W), t \in \mathbb{R}_+; \quad (3.59)$$

$$\text{iii) } V(t_0) < W(t_0).$$

Then,

$$V(t) < W(t), t \geq t_0. \quad (3.60)$$

Proof 3.8: Let $t_1 > 0$ be supremum of all positive numbers $\delta > 0$ such that $V(t_0) < W(t_0)$ implies that $V(t) < W(t)$ on $[t_0, \delta]$.

Clearly, $t_1 > t_0$ and $V(t_1) \leq W(t_1)$. Now, using the nondecreasing nature of $F(t, U)$ in U and assumption ii), we arrive at

$$D_H V(t_1) < F(t_1, V(t_1)) \leq F(t_1, W(t_1)) \leq D_H W(t_1) \quad (3.61)$$

If therefore follows that there exist a $\eta > 0$ satisfying

$$V(t) - W(t) > V(t_1) - W(t_1), \quad t_1 - \eta < t < t_1 \quad (3.62)$$

This implies that $t_1 - t_0$ cannot be the supremum due to the continuity of the functions involved, and hence relation (3.60) holds, completing the proof.

Remark 3.2: It is clear that the inequalities (3.56) can be replaced by

$$D_H V \leq F(t, V) \text{ and } D_H W > F(t, W), t \in \mathbb{R}_+ \quad (3.63)$$

respectively, to get the conclusion of Theorem 3.8.

Theorem 3.9: Let the assumptions of Theorem 3.7 hold and suppose further $F(t, U)$ is nondecreasing in U for each $t \in J$. Then the IVP (3.1) possesses the extremal solutions on $J_0 = [t_0, t_0 + \alpha_0]$, where $\alpha_0 = \min\{a, \frac{b}{2M+b}\}$.

Proof 3.2: Let $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_n) > 0$ be such that $\|\epsilon\| \leq \frac{b}{2}$. Then consider for each positive integer N , the following IVP for $t \in J$

$$D_H U = F(t, U) + \frac{\epsilon}{N}, \quad U(t_0) = U_0 + \frac{\epsilon}{N} \quad (3.64)$$

We observe that $F_N(t, U) = F(t, U) + \frac{\epsilon}{N}$ is defined and is continuous on $R_\epsilon = J \times B\left[U_0, \frac{b}{2}\right]$, where $B\left[U_0, \frac{b}{2}\right] = \left\{U \in K_c(\mathbb{R}^n) : D\left[U, U_0 + \frac{\epsilon}{N}\right] \leq \frac{b}{2}\right\}$. Also $D[F_N(t, U), \theta] \leq M + \frac{b}{2}$ on R_ϵ . Hence, we deduce from Theorem 3.7 that (3.64) has solution $U_N(t, \epsilon) \in K_c(\mathbb{R}^n)$ on J_0 . For $0 < \epsilon_2 < \epsilon_1 \leq \epsilon$, we see that

$$\left\{ \begin{array}{l} U_N(t_0, \epsilon_2) < U_N(t_0, \epsilon_1) \\ D_H U_N(t, \epsilon_2) \leq F(t, U_N(t, \epsilon_2)) + \frac{\epsilon_2}{N} \\ D_H U_N(t, \epsilon_1) > F(t, U_N(t, \epsilon_1)) + \frac{\epsilon_2}{N}, \text{ on } J_0 \end{array} \right. \quad (3.65)$$

We can apply the Theorem 3.8 to get

$$U_N(t, \epsilon_2) < U_N(t, \epsilon_1), \text{ on } J_0 \quad (3.66)$$

Since the family of functions $\{U_N(t, \epsilon)\}$ is equicontinuous and uniformly bounded on J_0 , it follows by Theorem 3.6 that there exists a decreasing sequence $\left\{\frac{\epsilon}{N_k}\right\}$ such that $\frac{\epsilon}{N_k} \rightarrow 0$ as $k \rightarrow \infty$ and the uniform limit

$$R(t) = \lim_{k \rightarrow \infty} U_{N_k}(t, \epsilon) \quad (3.67)$$

exists on J_0 . Obviously $R(t_0) = U_0$. The uniform continuity of F implies that $F(t, U_{N_k}(t, \epsilon))$ tends uniformly to $F(t, R(t))$ as $k \rightarrow \infty$ and thus term by term integration is applicable to

$$U_{N_K}(t, \epsilon) = U_0 + \frac{\epsilon}{N_K} + \int_{t_0}^t F(s, U_{N_K}(t, \epsilon)) ds \quad (3.68)$$

which in turn yields that limit $R(t)$ is required maximal solution of IVP (3.1) on J_0 .

We shall next show that $R(t)$ is the required of IVP (3.1) on J_0 . For this purpose, we observe that

$$U(t_0) = U_0 < U_0 + \frac{\epsilon}{N} = U_N(t_0, \epsilon), \quad (3.69)$$

$$D_H U(t) < F(t, U(t)) + \frac{\epsilon}{N} \quad (3.70)$$

$$D_H U_N(t, \epsilon) \geq F(t, U_N(t, \epsilon)) + \frac{\epsilon}{N}, \text{ on } J^0 \quad (3.71)$$

We then obtain from Theorem 3.8, that

$$U(t) < U_N(t, \epsilon), \text{ on } J_0 \quad (3.72)$$

The uniqueness of maximal solution $R(t)$ shows that $U_N(t, \epsilon)$ tends uniformly to $R(t)$ on J_0 as $N \rightarrow \infty$. This proves that $R(t)$ is the maximal solution of IVP (3.1). Similarly, once can prove the existence of the minimal solution $\rho(t)$ of IVP (3.1) by considering the IVP

$$D_H U = F(t, U) - \frac{\epsilon}{N}, U(t_0) = U_0 - \frac{\epsilon}{N} \quad (3.73)$$

Hence the proof is complete by proceeding with suitable change of arguments.

Theorem 3.10: Assume that the conditions of Theorem 3.9 are satisfied. Suppose that $M \in C^1[J, K_c(\mathbb{R}_+^n)]$ and

$$D_H M(t) \leq F(t, M(t)), M(t_0) \leq U_0 \quad (3.74)$$

Then $M(t) \leq R(t)$ on J_0 .

Proof 3.10: Let $U_N(t, \epsilon)$ be a solution of (3.64) on J_0 . Then

$$M(t_0) < U_0 + \frac{\epsilon}{N} \quad (3.75)$$

$$D_H U_N(t, \epsilon) > F(t, U_N(t, \epsilon)) \text{ on } J_0 \quad (3.76)$$

This together with (3.74), yields, by Theorem 3.8

$$M(t) < U_N(t, \epsilon) \text{ on } J_0 \quad (3.77)$$

The last inequality, in view of (3.67) proves the assertion of Theorem.

4. STABILITY THEORY

4.1. Lyapunov-like Functions

Let us consider the IVP for set differential equations

$$D_H U = F(t, U), U(t_0) = U_0 \in K_c(\mathbb{R}^n) \quad (4.1)$$

where $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$. To investigate the qualitative behavior of solutions of (4.1), the following comparison result in terms of a Lyapunov like function is very important and could be proved by means of the theory of ordinary differential inequalities. Here the Lyapunov-like function serves as a vehicle to transform the set differential equation into a scalar comparison differential equation. Thus, it is enough to consider the qualitative properties of the simpler comparison equation, under suitable conditions for the Lyapunov-like function.

We also require the IVP for scalar differential equation

$$\omega' = g(t, \omega), \omega(t_0) = \omega_0 \geq 0 \quad (4.2)$$

where $g \in C[\mathbb{R}_+^2, \mathbb{R}]$.

Theorem 4.1: Assume that

- i) $V \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ and $|V(t, A) - V(t, B)| \leq LD[A, B]$, where L is the local Lipschitz constant, for $A, B \in K_c(\mathbb{R}^n), t \in \mathbb{R}_+$;
- ii) $g \in C[\mathbb{R}_+^2, \mathbb{R}]$ and for $t \in \mathbb{R}_+, A \in K_c(\mathbb{R}^n)$

$$\begin{cases} D^+ V(t, A) = \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [V(t+h, A+hF(t, A)) - V(t, A)] \\ \leq g(t, V(t, A)) \end{cases} \quad (4.3)$$

Then if $U(t) = U(t, t_0, U_0)$ is any solution of (4.1) existing on $[t_0, T)$ such that $V(t_0, U_0) \leq \omega_0$, we have

$$V(t, U(t)) \leq R(t, t_0, \omega_0), \quad t \in [t_0, T) \quad (4.4)$$

where $R(t, t_0, \omega_0)$ is the maximal solution of (4.2) existing on $[t_0, T)$.

Proof 4.1: Let $U(t) = U(t, t_0, U_0)$ be any solution of (4.1) existing on $[t_0, T)$.

Define $m(t) = V(t, U(t))$ so that $m(t_0) = V(t_0, U_0) \leq \omega_0$. Now for small $h > 0$

$$\left\{ \begin{array}{l} m(t+h) - m(t) = V(t+h, U(t+h)) - V(t, U(t)) \\ \\ = V(t+h, U(t+h)) - V(t+h, U(t) + hF(t, U(t))) \\ \\ + V(t+h, U(t) + hF(t, U(t))) - V(t, U(t)) \\ \\ \leq LD[U(t+h), U(t) + hF(t, U(t))] \\ \\ + V(t+h, U(t) + hF(t, U(t))) - V(t, U(t)) \end{array} \right. \quad (4.5)$$

by using the Lipschitz condition given in i). Thus

$$\left\{ \begin{array}{l} D^+ m(t) = \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [m(t+h) - m(t)] \\ \\ \leq D^+ V(t, U(t)) \\ \\ + L \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [D[U(t+h), U(t) + hF(t, U(t))]] \end{array} \right. \quad (4.6)$$

Since

$$\left\{ \begin{array}{l} \frac{1}{h} [D[U(t+h), U(t) + hF(t, U(t))]] \\ \\ = D \left[\frac{U(t+h) - U(t)}{h}, F(t, U(t)) \right] \end{array} \right. \quad (4.7)$$

we find that

$$\left\{ \begin{aligned} & \lim_{h \rightarrow 0^+} \sup \frac{1}{h} \left[D[U(t+h), U(t) + hF(t, U(t))] \right] \\ &= \lim_{h \rightarrow 0^+} \sup D \left[\frac{U(t+h) - U(t)}{h}, F(t, U(t)) \right] \\ &= D[D_H U(t), F(t, U(t))] \equiv 0 \end{aligned} \right. \quad (4.8)$$

since $U(t)$ is a solution of (4.1). We therefore have the scalar differential inequality

$$D^+ m(t) \leq g(t, m(t)), m(t_0) \leq \omega_0 \quad (4.9)$$

This proves the claimed estimate of the Theorem.

Corollary 4.1: The function $g(t, \omega) = 0$ is admissible in Theorem 4.1 to yield the estimate

$$V(t, U(t)) \leq V(t_0, U_0), t \in [t_0, T) \quad (4.10)$$

Corollary 4.2: If, in Theorem 4.1, we assume that $g(t, \omega) = -\alpha\omega, \alpha > 0$, we get the relation

$$V(t, U(t)) \leq V(t_0, U_0) \exp(-\alpha(t - t_0)), t \in [t_0, T) \quad (4.11)$$

Corollary 4.3: If, in Theorem 4.1, we strengthen the assumption on $D^+V(t, U)$ to

$$D^+V(t, U) + c[\omega(t, U)] \leq g(t, V(t, U)) \quad (4.12)$$

where $\omega \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$, $c \in \mathcal{K}$ and $g(t, \omega)$ is nondecreasing in ω for each $t \in [t_0, T)$, then we get

$$V(t, U(t)) + \int_{t_0}^t c[\omega(s, U(s))] ds \leq r(t, t_0, V(t_0, U_0)), t \in [t_0, T) \quad (4.13)$$

whenever $\omega_0 = V(t_0, U_0)$.

Proof 4.3: Setting $L(t, U(t)) = V(t, U(t)) + \int_{t_0}^t c[\omega(s, U(s))]ds$ and note that

$$\begin{cases} D^+L(t, U(t)) \leq D^+V(t, U(t)) + c[\omega(t, U(t))] \\ \leq g(t, V(t, U(t))) \leq g(t, L(t, U(t))) \end{cases} \quad (4.14)$$

using the monotone character of $g(t, \omega)$. We then get immediately by Theorem 4.1 the desired estimate (4.4).

4.2. Global Existence

Theorem 4.2: Assume that

- i) $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$, F maps bounded sets onto bounded sets, and there exists a local solution of (4.1) for every (t_0, U_0) , $t_0 \geq 0$ and $U_0 \in K_c(\mathbb{R}^n)$;*
- ii) $V \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$;*

$$|V(t, A) - V(t, B)| \leq LD[A, B] \quad (4.15)$$

where L is the local Lipschitz constant, for $A, B \in K_c(\mathbb{R}^n)$, $t \in \mathbb{R}_+$, $V(t, A) \rightarrow \infty$ as $D[A, \theta] \rightarrow \infty$ uniformly for $[0, T]$ for every $T > 0$ and for $t \in \mathbb{R}_+$, $A \in K_c(\mathbb{R}^n)$

$$D^+V(t, A) \leq g(t, V(t, A)) \quad (4.16)$$

where $g \in C[\mathbb{R}_+^2, \mathbb{R}]$, $D^+V(t, A)$ is as defined in Theorem 4.1;

- iii) The maximal solution $R(t) = R(t, t_0, \omega_0)$ of (4.2) exists on $[t_0, \infty)$, and is positive whenever $\omega_0 > 0$.*

Then, for every $U_0 \in K_c(\mathbb{R}^n)$ such that $V(t_0, U_0) \leq \omega_0$, the IVP (4.2) has a solution $U(t)$ on $[t_0, \infty)$ which satisfies

$$V(t, U(t)) \leq R(t), t \geq t_0 \quad (4.17)$$

Proof 4.2: Let S denote the set of all functions U defined on $I_U = [t_0, c_U]$ with values in $K_c(\mathbb{R}^n)$ such that $U(t)$ is a solution of (3.27) on I_U and

$$V(t, U(t)) \leq R(t) \text{ on } I_U \quad (4.18)$$

Define a partial order $\leq$ on S as follows:

the relation $U \leq V$ implies $I_U \leq I_V$ and $U(t) = V(t)$ on I_U .

We shall first show that S is nonempty. By i) there exists a solution $U(t)$ of (3.1) defined on $I_U = [t_0, c_U]$. Setting $m(t) = V(t, U(t))$, $t \in I_U$ and using assumption ii), we get by Theorem 4.1, that

$$V(t, U(t)) \leq R(t), t \in I_U \quad (4.19)$$

where $R(t)$ is the maximal solution of (4.2). This proves that $U \in S$ and hence S is not empty.

If $(U_\beta)_\beta$ is a chain in $(S, \leq)$, then there is uniquely defined map V on $I_V = [t_0, \sup_\beta c_{U_\beta}]$ that coincides with U_β on I_{U_β} . Clearly, $V \in S$ and therefore V is an upperbound of $(U_\beta)_\beta$ in $(S, \leq)$. Then Zorn's lemma assures the existence of a maximal element Z in $(S, \leq)$. The proof of the Theorem is complete if we show that $c_Z = \infty$.

Suppose that it is not true, so that $c_Z < \infty$. Since $R(t)$ is assumed to exist on $[t_0, \infty)$, $R(t)$ is bounded on I_Z . Since $V(t, A) \rightarrow \infty$ as $D[A, \theta] \rightarrow \infty$ uniformly in t on $[t_0, c_Z]$, the relation $V(t, Z(t)) \leq R(t)$ on I_Z . By i), this shows that there is a constant $M > 0$ such that

$$D[F(t, Z(t)), \theta] \leq M \text{ on } I_Z \quad (4.20)$$

We then have, for all $t_1, t_2 \in I_Z$, $t_1 \leq t_2$,

$$D[Z(t_2), Z(t_1)] \leq \int_{t_1}^{t_2} D[F(s, Z(s)), \theta] ds \leq M(t_2 - t_1), \quad (4.21)$$

which proves that Z is Lipschitzian on I_Z and consequently has a continuous extension $Z_0(t)$ on $[t_0, c_Z]$.

By the continuity, we get

$$Z_0(c_Z) = U_0 + \int_{t_0}^{c_Z} F(s, Z_0(s)) ds \quad (4.22)$$

This implies that $Z_0(t)$ is a solution of (4.1) on $[t_0, c_Z]$ and obviously

$$V(t, Z_0(t)) \leq r(t), \quad t \in [t_0, c_Z] \quad (4.23)$$

Consider the IVP

$$D_H U(t) = F(t, U), \quad U(c_Z) = Z_0(c_Z) \quad (4.24)$$

By the assumed local existence there is a solution of $U_0(t)$ on $[c_Z, c_Z + \delta)$, $\delta > 0$.

Define

$$Z_1(t) = \begin{cases} Z_0(t) & \text{for } t_0 \leq t \leq c_Z, \\ U_0(t) & \text{for } c_Z \leq t \leq c_Z + \delta, \end{cases} \quad (4.25)$$

Clearly, $Z_1(t)$ is a solution of (4.1) on $[t_0, c_Z + \delta)$ and repeating the argument, we get

$$V(t, Z_1(t)) \leq r(t), \quad t \in [t_0, c_Z + \delta) \quad (4.26)$$

This contradicts the maximality of Z and hence $c_Z = \infty$. The proof is complete.

4.3. Fundamental Definitions of Stability of Set Differential Equations

Let us consider the initial value problem for the set differential equations

$$D_H U = f(t, U), \quad U(t_0) = U_0 \text{ for } U_0 \in K_c(\mathbb{R}^n) \text{ and } t \geq t_0 \geq 0 \quad (4.27)$$

$$D_H U = f(t, U), U(\tau_0) = V_0 \text{ for } V_0 \in K_c(\mathbb{R}^n) \text{ and } t \geq \tau_0 \geq t_0 \geq 0 \quad (4.28)$$

the perturbed systems of (4.28)

$$D_H V = F(t, V), V(\tau_0) = V_0 \text{ for } V_0 \in K_c(\mathbb{R}^n) \text{ and } t \geq \tau_0 \quad (4.29)$$

and

$$\begin{cases} D_H W = H(t, W), W(\tau_0) = V_0 - U_0 \text{ for} \\ W(\tau_0) = W_0 \in K_c(\mathbb{R}^n) \text{ and } t \geq \tau_0 \end{cases} \quad (4.30)$$

where $f, F, H \in \mathcal{C}[\mathbb{R}_+ \times S_\rho, K_c(\mathbb{R}^n)]$, $S_\rho = [U \in K_c(\mathbb{R}^n) : D[U, \tilde{0}] < \rho < \infty$, $f(t, \tilde{0}) \equiv \tilde{0}$ for $t \geq 0$. We have a special case of (4.29) that is perturbation equation of (4.28) if $F(t, V) = f(t, V) + R(t, V)$ where $R(t, V)$ is the perturbation term. The above assumptions imply the existence of trivial solutions of (4.27) and (4.28) through (t_0, U_0) and (τ_0, V_0) respectively.

The mappings $U \in \mathcal{C}^1[J_1, K_c(\mathbb{R}^n)]$ on $J_1 = [t_0, t_0 + T_1]$, $U, V, W \in \mathcal{C}^1[J_2, K_c(\mathbb{R}^n)]$ on $J_2 = [\tau_0, \tau_0 + T_2]$ are said to be solutions of (4.27) on J_1 if it satisfies (4.27) on J_1 and (4.28), (4.29), (4.30) on J_2 if it satisfy (4.28), (4.29), (4.30) on J_2 respectively. Since $U(t)$, $V(t)$ and $W(t)$ are continuously differentiable, we have

$$U(t) = U_0 + \int_{t_0}^t D_H U(s) ds, \quad U_0 \in K_c(\mathbb{R}^n) \text{ and } t \in J_1 \quad (4.31)$$

$$U(t) = U_0 + \int_{\tau_0}^t D_H U(s) ds, \quad U_0 \in K_c(\mathbb{R}^n) \text{ and } t \in J_2 \quad (4.32)$$

$$V(t) = V_0 + \int_{\tau_0}^t D_H V(s) ds, \quad V_0 \in K_c(\mathbb{R}^n) \text{ and } t \in J_2 \quad (4.33)$$

$$W(t) = W_0 + \int_{\tau_0}^t D_H W(s) ds, \quad W_0 \in K_c(\mathbb{R}^n) \text{ and } t \in J_2 \quad (4.34)$$

Thus the corresponding with the initial value problems (IVPs) of (4.27), (4.28), (4.29), (4.30) are followings

$$U(t) = U_0 + \int_{t_0}^t f(s, U(s)) ds, \quad U_0 \in K_c(\mathbb{R}^n) \text{ and } t \in J_1 \quad (4.35)$$

$$U(t) = U_0 + \int_{\tau_0}^t f(s, U(s)) ds, \quad U_0 \in K_c(\mathbb{R}^n) \text{ and } t \in J_2 \quad (4.36)$$

$$V(t) = V_0 + \int_{\tau_0}^t F(s, V(s)) ds, \quad V_0 \in K_c(\mathbb{R}^n) \text{ and } t \in J_2 \quad (4.37)$$

$$W(t) = W_0 + \int_{\tau_0}^t H(s, W(s)) ds, \quad W_0 \in K_c(\mathbb{R}^n) \text{ and } t \in J_2 \quad (4.38)$$

respectively where the integrals are the Hukuhara integrals. Note that $U(t)$ and $U(t), V(t), W(t)$ are also the solutions of (4.27) and (4.28), (4.29), (4.30) if and only if they satisfy (4.35) on J_1 and (4.36), (4.37), (4.38) on J_2 respectively.

Definition 4.1: The solution $V(t, \tau_0, V_0)$ of the system (4.29) through (τ_0, V_0) for $t \geq \tau_0$, $\tau_0 \in \mathbb{R}_+$ is said to be initial time difference

i) stable with respect to the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, where $U(t, t_0, U_0)$ is any solution of the system (4.27) for $t \geq t_0 \geq 0$, $t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0$ if and only if given any $\epsilon > 0$ there exist $\delta_1 = \delta_1(\epsilon, \tau_0) > 0$ and $\delta_2 = \delta_2(\epsilon, \tau_0) > 0$ such that

$$D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] < \epsilon \quad (4.39)$$

provided that $D[V_0 - U_0, \tilde{0}] < \delta_1$ and $d_s[\tau_0 - t_0, 0] < \delta_2$ for $t \geq \tau_0$;

ii) uniformly stable with respect to the solution $U(t - \eta, t_0, U_0)$ if δ_1, δ_2 in i) are independent of $\tau_0 \in \mathbb{R}_+$;

iii) attractive with respect to the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, where $U(t, t_0, U_0)$ is any solution of the system (4.27) for $t \geq t_0 \geq 0$, $t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0$ if and only if given any $\epsilon > 0$ and $\tau_0 \in \mathbb{R}_+$ there exist $\delta_1 = \delta_1(\tau_0) > 0$, $\delta_2 = \delta_2(\tau_0) > 0$ and a $T = T(\epsilon, \tau_0) > 0$ such that

$$D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] < \epsilon \quad (4.40)$$

provided that $D[V_0 - U_0, \tilde{0}] < \delta_1$ and $d_s[\tau_0 - t_0, 0] < \delta_2$ for $t \geq \tau_0 + T$;

iv) uniformly attractive with respect to the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, where $U(t, t_0, U_0)$ is any solution of the system (4.48) for $t \geq t_0 \geq 0$, $t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0$ if $\delta_1 > 0$, $\delta_2 > 0$ and a $T > 0$ in iii) are independent of $\tau_0 \in \mathbb{R}_+$;

v) asymptotically stable with respect to the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$ if i) and iii) hold simultaneously. Or, if i) holds and there exist $\gamma_1(\tau_0) > 0$ and $\gamma_2(\tau_0) > 0$ such that

$$\lim_{t \rightarrow \infty} D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] = 0 \quad (4.41)$$

for all $V(t, \tau_0, V_0)$ and $U(t - \eta, t_0, U_0)$ with $D[V_0 - U_0, \tilde{0}] < \gamma_1$ and $d_s[\tau_0 - t_0, 0] < \gamma_2$ for each $t \geq \tau_0$;

vi) uniformly asymptotically stable with respect to the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$ if ii) and iv) hold simultaneously. Or, if ii) holds and there exist $\gamma_1 = \gamma_1(\epsilon) > 0$ and $\gamma_2 = \gamma_2(\epsilon) > 0$ and a $T(\epsilon) > 0$ such that

$$D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] < \epsilon \quad (4.42)$$

provided that $D[V_0 - U_0, \tilde{0}] < \gamma_1$ and $d_s[\tau_0 - t_0, 0] < \gamma_2$ for each $t \geq \tau_0 + T(\epsilon)$;

vii) exponentially asymptotically stable with respect to the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$ if there exist an estimate such that

$$\begin{cases} D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] \\ \leq D[V_0 - U_0, \tilde{0}] \exp[-\alpha(t - \tau_0)] \end{cases} \quad (4.43)$$

for $\alpha > 0$ and $t \geq t_0 \in \mathbb{R}_+$.

Definition 4.2: If a function $\phi \in C[(0, \rho), \mathbb{R}_+]$, $\phi(0) = 0$, and $\phi(r)$ is strictly monotone increasing in r , then $\phi(r)$ is said to belong to the class $\mathcal{K}$. Moreover, if $\rho = \infty$ and $\phi(r) \rightarrow \infty$ as $r \rightarrow \infty$ then it is said to belong to class K_∞ .

Definition 4.3: If a function $a \in C[\mathbb{R}_+^2, \mathbb{R}_+]$, $a(t, u) \in \mathcal{K}$ for each $t \in \mathbb{R}_+$, then $a(t, u)$ is said to belong to the class $\mathcal{CK}$.

Definition 4.4: If a function $h \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$, $\inf_{(t,x)} h(t, x) = 0$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, then $h(t, x)$ is said to belong to the class Γ .

Definition 4.5: If a function $h \in \Gamma$ and $\sup_{t \in \mathbb{R}_+} h(t, x)$ exists for $x \in \mathbb{R}^n$, then $h(t, x)$ is said to belong to the class Γ_0 .

Definition 4.6: For a real valued function $\mathcal{V} \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ we define the Dini derivatives as follows for $(t, U) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n)$

$$D^+\mathcal{V}(t, U) \equiv \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [\mathcal{V}(t + h, U + hf(t, U)) - \mathcal{V}(t, U)] \quad (4.44)$$

$$D_-\mathcal{V}(t, U) \equiv \lim_{h \rightarrow 0^-} \inf \frac{1}{h} [\mathcal{V}(t + h, U + hf(t, U)) - \mathcal{V}(t, U)]. \quad (4.45)$$

Definition 4.7: For a real valued function $\mathcal{V} \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ we define the generalized derivatives (Dini-like derivatives) as follows

$$\left\{ \begin{array}{l} D_*^+ \mathcal{V}(t, V - \tilde{U}) \\ \equiv \lim_{h \rightarrow 0^+} \sup \frac{1}{h} \left[\mathcal{V} \left(t + h, V - \tilde{U} + h \left(F(t, V) - \tilde{f}(t, \tilde{U}) \right) \right) \right. \\ \left. - \mathcal{V}(t, V - \tilde{U}) \right] \end{array} \right. \quad (4.46)$$

$$\left\{ \begin{array}{l} D_{*-} \mathcal{V}(t, V - \tilde{U}) \\ \equiv \lim_{h \rightarrow 0^+} \inf \frac{1}{h} \left[\mathcal{V} \left(t + h, V - \tilde{U} + h \left(F(t, V) - \tilde{f}(t, \tilde{U}) \right) \right) \right. \\ \left. - \mathcal{V}(t, V - \tilde{U}) \right] \end{array} \right. \quad (4.47)$$

for $(t, U) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n)$ where $V(t, \tau_0, V_0)$ is the solution of the system (4.29) and $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, where $U(t, t_0, U_0)$ is any solution of the system (4.48) for $t \geq \tau_0 \geq 0$, $t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0$.

4.4. Classical Notion of Stability of Set Differential Equation

Let $V(t, \tau_0, V_0)$ be any solution of the set differential equation

$$D_H V = f(t, V), V(\tau_0) = V_0 \text{ for } V_0 \in K_c(\mathbb{R}^n) \text{ and } t \geq \tau_0, \tau_0 \in \mathbb{R}_+ \quad (4.48)$$

where $f \in C[\mathbb{R}_+ \times S_\rho, K_c(\mathbb{R}^n)]$ and S_ρ is the set $S_\rho = [V \in K_c(\mathbb{R}^n): D[V, \tilde{0}] < \rho < \infty]$.

Assume that $f(t, \tilde{0}) = \tilde{0}$, $t \in \mathbb{R}_+$ so that $V = 0$ is a null solution of (4.48) through $(t_0, 0)$. Now, we can state the well know definitions concerning the stability of the null solution of the set differential equation.

Definition 4.8: The null solution $V = 0$ is said to be stable if and only if for each $\epsilon > 0$ and for all $\tau_0 \in \mathbb{R}_+$, there exist two positive functions $\delta_1 = \delta_1(\epsilon, \tau_0) > 0$ and $\delta_2 = \delta_2(\epsilon, \tau_0) > 0$ that are continuous in τ_0 for each ϵ such that

$$\begin{cases} D[V(t, \tau_0, V_0), \tilde{0}] < \epsilon \text{ provided} \\ D[V_0, \tilde{0}] < \delta_1 \text{ and } d_s[\tau_0, 0] < \delta_2, t \geq \tau_0; \end{cases} \quad (4.49)$$

If δ_1 and δ_2 are independent of $\tau_0 \in \mathbb{R}_+$, then the null solution $V = 0$ of (4.48) is said to be uniformly stable.

Definition 4.9: The solution of $V(t, \tau_0, V_0)$ (4.48) through (τ_0, V_0) is said to be stable with respect to the solution $V(t, \tau_0, U_0)$ of (4.48) for $t \geq \tau_0 \in \mathbb{R}_+$ if and only if given any $\epsilon > 0$, there exist two positive functions $\delta_1 = \delta_1(\epsilon, \tau_0)$ and $\delta_2 = \delta_2(\epsilon, \tau_0)$ that are continuous in τ_0 for each ϵ such that

$$D[V(t, \tau_0, V_0) - V(t, \tau_0, U_0), \tilde{0}] < \epsilon \quad (4.50)$$

provided that $D[V_0 - U_0, \tilde{0}] < \delta_1$ and $d_s[\tau_0, 0] < \delta_2$ for $t \geq \tau_0$;

If δ_1 and δ_2 are independent of $\tau_0 \in \mathbb{R}_+$, then the solution of the system (4.48) is uniformly stable with respect to the solution $V(t, \tau_0, U_0)$.

4.5. Stability Criteria

Having necessary comparison result in terms of Lyapunov-like functions, to establish the stability results for the set differential equations (SDE) (4.1) analogous to the original Lyapunov result and their extensions. However, in order to investigate the stability criteria for the trivial solutions of (4.1), we need to employ, in a natural way, the measure $D[U(t), \theta] = \|U(t)\| = \text{diam } U(t)$ for $t \geq t_0$, where $U(t, t_0, U_0)$ is the solution of (4.1). This implies that the $\text{diam } U(t)$ is nondecreasing in t for $t \geq t_0$. This is due to the fact that, in the generation of the SDE from ordinary differential equations (ODEs), certain undesirable elements may enter the solution $U(t)$ and the measure to be employed, namely, $\|U(t)\|$ is therefore unsuitable to develop stability theory without some adjustment. Recall that SDE (4.1) reduces to ODE when $U(t)$ is single valued and SDE (4.1) can be generated from ODE as well. The latter is done as follows.

We let $F(t, A) = \overline{c\bar{o}} f(t, A)$, $A \in K_c(\mathbb{R}^n)$, where $f \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$ arising from the ODE

$$u' = f(t, u), u(t_0) = u_0 \in \mathbb{R}^n \quad (4.51)$$

Consequently, the solutions of $u(t)$ of ODE (4.51) are imbedded in the solution $U(t)$ of SDE (4.1).

Since the cause of the problem in SDE is the requirement of the existence of Hukuhara differences in formulating SDE, we need to incorporate the Hukuhara difference in the initial conditions also, to match the behavior of solutions of SDE with corresponding ODE. This is precisely what we plan to do.

Suppose that the Hukuhara difference exists for any given initial values $U_0, V_0 \in K_c(\mathbb{R}^n)$ so that $U_0 - V_0 = W_0$ is defined. Then, we consider the stability of the solution $U(t, t_0, U_0 - V_0) = U(t, t_0, W_0)$ of (4.1). Before presenting the stability results, in this set up, let us present some examples to illustrate our approach. Consider the ODE

$$u' = -u, \quad u(0) = u_0 \in \mathbb{R} \quad (4.52)$$

and the corresponding SDE

$$D_H U = -U, \quad U(0) = U_0 \in K_c(\mathbb{R}) \quad (4.53)$$

Since the values of the solution (4.53) are interval functions, the equation (4.53) can be written as

$$[u'_1, u'_2] = (-1)U = [-u_2, -u_1] \quad (4.54)$$

where $U(t) = [u_1(t), u_2(t)]$ and $U(0) = [u_{10}, u_{20}]$. The relation (4.54) is equivalent to the system of equations

$$\begin{cases} u'_1 = -u_2, & u_1(0) = u_{10} \\ u'_2 = -u_1, & u_2(0) = u_{20} \end{cases} \quad (4.55)$$

whose solutions as follow for $t \geq 0$

$$\begin{cases} u_1(t) = \frac{1}{2}[u_{10} + u_{20}]e^{-t} + \frac{1}{2}[u_{10} - u_{20}]e^t \\ u_2(t) = \frac{1}{2}[u_{20} + u_{10}]e^{-t} + \frac{1}{2}[u_{20} - u_{10}]e^t \end{cases} \quad (4.56)$$

Given $U_0 \in K_c(\mathbb{R})$, if there exists $V_0, W_0 \in K_c(\mathbb{R})$ such that $U_0 = V_0 + W_0$, then the Hukuhara difference $U_0 - V_0 = W_0$ exists. Let us choose

$$U_0 = [u_{10}, u_{20}], \quad V_0 = \frac{1}{2}[(u_{10} - u_{20}), (u_{20} - u_{10})] \quad (4.57)$$

so that

$$W_0 = \frac{1}{2}[(u_{10} + u_{20}), (u_{20} + u_{10})] \quad (4.58)$$

If $u_{10} \neq -u_{20}$, then we have for $t \geq 0$

$$\begin{cases} U(t, U_0) = \frac{1}{2}[-(u_{20} - u_{10}), (u_{20} - u_{10})]e^t \\ \quad + \frac{1}{2}[(u_{10} + u_{20}), (u_{10} + u_{20})]e^{-t} \\ U(t, V_0) = \frac{1}{2}[-(u_{10} - u_{20}), (u_{20} - u_{10})]e^t \\ U(t, W_0) = \frac{1}{2}[(u_{10} + u_{20}), (u_{10} + u_{20})]e^{-t} \end{cases} \quad (4.59)$$

On the other hand, if $u_{10} = -u_{20}$, implies that we choose $U_0 = [-d, d]$ with $d = u_{20}$. Then $U_0 = V_0$ and $W_0 = [0, 0]$. This choice eliminates the term with e^{-t} and we have only undesirable part of the solution. The other situation is to choose $U_0 = [c, c]$ for some c , which eliminates the term with e^t and retains only the desirable part of the solution compared with the ODE. Even when U_0 is chosen as $U_0 = [-d, d]$, we can find $V_0 = [c - d, c + d]$ for some c so that we have $V_0 = U_0 + W_0$ where $W_0 = [c, c]$.

We note that for any general initial value U_0 , the solution (4.52) contains both desired and undesired parts compared to the solution of the ODE (4.52). In order to isolate the desired part of the solution $U(t, U_0)$ of (4.52) that matches the solution of the ODE (4.52), we need to use the initial values satisfying the desired Hukuhara difference of the given two initial values.

If, on the other hand, we define the SDE as

$$D_H U = \lambda(t)U, \quad U(0) = U_0 \quad (4.60)$$

which is generated by

$$u' = \lambda(t)u, \quad u(0) = u_0 \quad (4.61)$$

where $\lambda(t) > 0$ is a real valued function from $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\lambda \in L^1(\mathbb{R}_+)$, then we see that, with similar computation,

$$U(t, U_0) = U_0 \exp \left[\int_0^t \lambda(s) ds \right], \quad t \geq 0 \quad (4.62)$$

for any $U_0 \in K_c(\mathbb{R}^n)$.

Hence, we get the stability of the trivial (4.60). In this case, we note that the solutions of both equations (4.60) and (4.61) match, providing the same stability results. There is no necessity, therefore, to choose the initial values as before, since the undesirable part of the solution does not exist among solutions of (4.60). Consequently, it does not matter, whether we use the Hukuhara difference or not, we get the same conclusion. To be consistent and to take care of all the situations, most of the results are formulated in terms of Hukuhara differences of initial values.

To consider the stability of the trivial solution of (4.1), let us assume that $F(t, \theta) = \theta$, the solutions are unique and exist for all $t \geq t_0$. Also, we assume, as a standard hypothesis, that the Hukuhara difference $W_0 = U_0 - V_0$ exists, since we suppose that $U_0 = V_0 + W_0$. Consequently, we utilize the solutions $U(t) = U(t, t_0, U_0 - V_0) = U(t, t_0, W_0)$, throughout (t_0, W_0) .

Theorem 4.3: Assume that the following hold

- i) $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$, $|V(t, U_1) - V(t, U_2)| \leq L D[U_1, U_2]$, $L > 0$ and for $(t, U) \in \mathbb{R}_+ \times S(\rho)$, where $S(\rho) = [U \in K_c(\mathbb{R}^n): D[U, \theta] < \rho]$;

$$D^+V(t, U) = \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [V(t + h, U + hF(t, U)) - V(t, U)] \leq 0 \quad (4.63)$$

- ii) $b(\|U\|) \leq V(t, U) \leq a(t, \|U\|)$, for $(t, U) \in \mathbb{R}_+ \times S(\rho)$ where $b, a(t, \cdot) \in \mathcal{K} = \{\sigma \in C[[0, \rho), \mathbb{R}_+]: \sigma(0) = 0 \text{ and } \sigma(w) \text{ is increasing in } w\}$.

Then, the trivial solution of (4.1) is equi-stable.

Proof 4.3: Let $0 < \epsilon < \rho$ and $t_0 \in \mathbb{R}_+$, be given. Choose a $\delta = \delta(t_0, \epsilon) > 0$ such that

$$a(t_0, \delta) < b(\epsilon) \quad (4.64)$$

We claim that with this δ , equi-stability holds. If this is not true then, there would exist a solution $U(t) = U(t, t_0, W_0)$ of (4.1) and $t_1 > t_0$ such that

$$\|U(t_1)\| = \epsilon \text{ and } \|U(t)\| \leq \epsilon < \rho, \quad t_0 \leq t \leq t_1 \quad (4.65)$$

whenever $\|W_0\| < \delta$. By Corollary 4.1, then we have

$$V(t, U(t)) \leq V(t_0, W_0), \quad t_0 \leq t \leq t_1 \quad (4.66)$$

Consequently, using ii), (4.64) and (4.65), we arrive at the following contradiction

$$\begin{cases} b(\epsilon) = b(\|U(t_1)\|) \leq V(t_1, U(t_1)) \leq V(t_0, W(t_0)) \\ \leq a(t_0, \|W_0\|) \leq a(t_0, \delta) < b(\epsilon) \end{cases} \quad (4.67)$$

Hence equi-stability holds, completing the proof.

Theorem 4.4: Let the assumptions of Theorem 4.3 hold, except that estimate (4.63) be strengthened to

$$D^+V(t, U) \leq -\beta V(t, U), \quad (t, U) \in \mathbb{R}_+ \times S(\rho) \quad (4.68)$$

Then the trivial solution of (4.1) is equi-asymptotically stable.

Proof 4.4: Clearly, the trivial solution of (4.1) is equi-stable. Hence, taking $\epsilon = \rho$ and designating $\delta_0 = \delta(t_0, \rho) > 0$, we have by Theorem 4.3

$$\|W_0\| < \delta_0 \text{ implies } \|U(t)\| < \rho \quad (4.69)$$

where $U(t) = U(t, t_0, W_0)$.

Consequently, we get from the assumption (4.68), the estimate

$$V(t, U(t)) \leq V(t_0, W_0) \exp[-\beta(t - t_0)], \quad t \geq t_0 \quad (4.70)$$

Given $\epsilon > 0$, we choose $T = T(t_0, \epsilon) = \frac{1}{\beta} \ln \frac{a(t_0, \delta_0)}{b(\epsilon)} + 1$. Then it is easy to see that

$$b(\|U(t)\|) \leq V(t, U(t)) \leq a(t_0, \delta) e^{-\beta(t-t_0)} < b(\epsilon), \quad t \geq t_0 + T \quad (4.71)$$

The proof is complete.

Theorem 4.5: Assume that for $(t, U) \in \mathbb{R}_+ \times S(\rho) \cap S^c(\eta)$ for each $0 < \eta < \rho$, $V \in \mathcal{C}[\mathbb{R}_+ \times S(\rho) \cap S^c(\eta), \mathbb{R}_+]$, we have

$$|V(t, U_1) - V(t, U_2)| \leq L D[U_1, U_2], \quad L > 0 \quad (4.72)$$

$$D^+V(t, U) \leq 0 \quad (4.73)$$

and

$$b(\|U\|) \leq V(t, U) \leq a(\|U\|), \quad a, b \in \mathcal{K} \quad (4.74)$$

Then the trivial solution of (4.1) is uniformly stable.

Proof 4.5: Let $0 < \epsilon < \rho$ and $t_0 \in \mathbb{R}_+$ be given. Choose $\delta = \delta(\epsilon) > 0$ such that $a(\delta) < b(\epsilon)$. Then we claim that with this δ , uniform stability follows. If not, there would exist a solution $U(t)$ of (4.1), and a $t_2 > t_1 > t_0$ to satisfying

$$\begin{cases} \|U(t_1)\| = \delta, \|U(t_2)\| = \epsilon \text{ and} \\ \delta \leq \|U(t)\| \leq \epsilon < \rho, \quad t_1 \leq t \leq t_2 \end{cases} \quad (4.75)$$

Taking $\eta = \delta$, we get from (4.73), the estimate

$$V(t_2, U(t_2)) \leq V(t_1, U(t_1)) \quad (4.76)$$

and therefore, (4.74) and (4.75), together with the definition of δ , yield

$$\begin{cases} b(\epsilon) = b(\|U(t_2)\|) \leq V(t_2, U(t_2)) \leq V(t_1, U(t_1)) \\ \leq a(\|U(t_1)\|) = a(\delta) < b(\epsilon) \end{cases} \quad (4.77)$$

This contradiction proves uniform stability, completing the proof.

Theorem 4.6: Let the assumptions of Theorem 4.5 hold, except that estimate (4.73) be strengthened to

$$D^+V(t, U) \leq -c(\|U\|), \quad c \in \mathcal{K} \quad (4.78)$$

Then the trivial solution of (4.1) is uniformly asymptotically stable.

Proof 4.6: By Theorem 4.5, uniform stability follows. Now, for $\epsilon = \rho$, we designate $\delta_0 = \delta(\rho)$. This means

$$\|W_0\| < \delta_0 \text{ implies that } \|U(t)\| < \rho, \quad t \geq t_0 \quad (4.79)$$

In view of the uniform stability, it is enough to show that there exists a t^* such that for $t_0 \leq t^* \leq t_0 + T$, where $T = 1 + \frac{a(\delta_0)}{\delta}$

$$\|U(t^*)\| < \delta \quad (4.80)$$

If this is not true, then $\delta \leq \|U(t)\|$ for $t_0 \leq t \leq t_0 + T$. Hence (4.78) gives

$$V(t, U(t)) \leq V(t_0, W_0) - \int_{t_0}^t c(\|U(s)\|) ds, \quad t_0 \leq t \leq t_0 + T \quad (4.81)$$

Thus, we have, in view of the choice of T

$$0 \leq V(t_0 + T, U(t_0 + T)) \leq a(\delta_0) - c(\delta)T < 0 \quad (4.82)$$

which is a contradiction. Therefore, there exists a t^* satisfying (4.80) and uniform stability then shows that and the proof is complete.

$$\|W_0\| < \delta_0 \text{ implies } \|U(t)\| < \epsilon, \quad t \geq t_0 + T \quad (4.83)$$

4.6. Nonuniform Stability Criteria

Theorem 4.7: Assume that

- i) $V_1 \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$,
- ii) $|V_1(t, U_1) - V_1(t, U_2)| \leq LD[U_1, U_2]$, $L_1 > 0$, $V_1(t, U) \leq a_0(t, \|U\|)$ where $a_0 \in C[\mathbb{R}_+ \times [0, \rho), \mathbb{R}_+]$ and $a_0(t, \cdot) \in \mathcal{K}$ for each $t \in \mathbb{R}_+$;
- iii) $D^+V_1(t, U) \leq g_1(t, V_1(t, U))$, $(t, U) \in \mathbb{R}_+ \times S(\rho)$ where $g_1 \in C[\mathbb{R}_+^2, \mathbb{R}]$ and $g_1(t, 0) \equiv 0$;
- iv) for every $\eta > 0$, there exists a $V_\eta \in C[\mathbb{R}_+ \times S(\rho) \cap S^c(\eta), \mathbb{R}_+]$,

$$|V_\eta(t, U_1) - V_\eta(t, U_2)| \leq L_\eta D[U_1, U_2] \quad (4.84)$$

$$b(\|U\|) \leq V_\eta(t, U) \leq a(\|U\|), a, b \in \mathcal{K}; \quad (4.85)$$

and

$$D^+V_1(t, U) + D^+V_\eta(t, U) \leq g_2\left(t, V_1(t, U) + V_\eta(t, U)\right) \quad (4.86)$$

for $(t, U) \in \mathbb{R}_+ \times S(\rho) \cap S^c(\eta)$, where $g_2 \in C[\mathbb{R}_+^2, \mathbb{R}]$ and $g_2(t, 0) \equiv 0$;

v) the trivial solution $w_1 \equiv 0$ of

$$w_1' = g_1(t, w_1), w_1(t_0) = w_{10} \geq 0 \quad (4.87)$$

is equistable;

vi) the trivial solution $w_2 \equiv 0$ of

$$w_2' = g_2(t, w_2), w_2(t_0) = w_{20} \geq 0 \quad (4.88)$$

is uniformly stable.

Then, the trivial solution of (4.1) is equi-stable.

Proof 4.7: Let $0 < \epsilon < \rho$ and $t_0 \in \mathbb{R}_+$, be given. Since the trivial solution of (4.88) is uniformly stable, given any $b(\epsilon) > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta_0 = \delta_0(\epsilon) > 0$ satisfying

$$0 < w_{20} < \delta_0 \text{ implying } w_2(t, t_0, w_{20}) < b(\epsilon), t \geq t_0 \quad (4.89)$$

where $w_2(t, t_0, w_{20})$ is any solution of (4.88). In view of the hypothesis on $a(w)$, there is a $\delta_2 = \delta_2(\epsilon) > 0$ such that

$$a(\delta_2) < \frac{\delta_0}{2} \quad (4.90)$$

Since the trivial solution of (4.87) is equi-stable, given $\frac{\delta_0}{2} > 0$ and $t_0 \in \mathbb{R}_+$, we can find a $\delta^* = \delta^*(t_0, \epsilon) > 0$ such that

$$0 < w_{10} < \delta^* \text{ implies } w_1(t, t_0, w_{10}) < \frac{\delta_0}{2}, \quad t \geq t_0 \quad (4.91)$$

where $w_1(t, t_0, w_{10})$ is any solution of (4.87). Choose $w_{10} = V_1(t_0, W_0)$. Since $V_1(t, U) \leq a_0(t, \|U\|)$, we see that there exists $\delta_1 = \delta_1(t_0, \epsilon) > 0$ satisfying

$$\|W_0\| < \delta_1 \text{ and } a_0(t_0, \|W_0\|) < \delta^* \quad (4.92)$$

simultaneously. Define $\delta = \min(\delta_1, \delta_2)$.

Then, we claim that

$$\|W_0\| < \delta \text{ implies } \|U(t)\| < \epsilon, \quad t \geq t_0 \quad (4.93)$$

for any solution $U(t) = U(t, t_0, W_0)$ of (4.1). If this is false, there would exist a solution $U(t)$ of (4.1) with $\|W_0\| < \delta$ and $t_1, t_2 > t_0$ such that

$$\|U(t_1)\| = \delta_2, \quad \|U(t_1)\| = \epsilon \quad (4.94)$$

and

$$\delta_2 \leq \|U(t)\| \leq \epsilon < \rho \quad (4.95)$$

for $t_1 \leq t \leq t_2$. We let $\eta = \delta_2$ so that the existence of a V_η satisfying hypothesis iii) is assured. Hence, setting

$$m(t) = V_1(t, U(t)) + V_\eta(t, U(t)), \quad t \in [t_1, t_2] \quad (4.96)$$

we obtain the differential inequality

$$D^+m(t) \leq g_2(t, m(t)), \quad t_1 \leq t \leq t_2 \quad (4.97)$$

which yields

$$V_1(t_2, U(t_2)) + V_\eta(t_2, U(t_2)) \leq R_2(t_2, t_1, w_{20}) \quad (4.98)$$

where $w_{20} = V_1(t_1, U(t_1)) + V_\eta(t_1, U(t_1))$, and $R_2(t, t_1, w_{20})$ is the maximal solution of (4.88). Because of the assumptions i) and ii), we also have

$$V_1(t_1, U(t_1)) \leq R_1(t_1, t_0, w_{10}) \quad (4.99)$$

with $w_{10} = V_1(t_0, W_0)$ where $R_1(t, t_0, w_{10})$ is the maximal solution of (4.87).

By (4.91) and (4.92), we get

$$V_1(t_1, U(t_1)) < \frac{\delta_0}{2} \quad (4.100)$$

Also, by (3.6.4), (3.6.8) and assumption iii), we arrive at

$$V_\eta(t_1, U(t_1)) \leq a(\delta_2) < \frac{\delta_0}{2} \quad (4.101)$$

Thus, (3.6.10) and (3.6.11) and the definition of w_{20} shows that $w_{20} < \delta_0$ which, in view of (3.6.3), shows that $w_2(t_2, t_1, w_{20}) < b(\epsilon)$. It then follows from (3.6.9), $V_1(t, U) \geq 0$ and assumption iii)

$$b(\epsilon) = b(\|U(t_2)\|) \leq V_1(t_2, U(t_2)) \leq R_2(t_2, t_1, w_{20}) < b(\epsilon) \quad (4.102)$$

This contradiction proves equi-stability of the trivial solution of (4.1) since (4.93) is then true. The proof is complete.

Theorem 4.8: Let the assumptions of Theorem 4.7 hold except that condition ii) is strengthened to

$$D^+V_1(t, U) \leq -c(w(t, U)) + g_1(t, V_1(t, U)), \quad c \in \mathcal{K} \quad (4.103)$$

where $w \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$

$$|w(t, U_1) - w(t, U_2)| \leq ND[U_1, U_2], \quad N > 0 \quad (4.104)$$

and $D^+w(t, U)$ is bounded above or below. Then, the trivial solution of (4.1) is equi-asymptotically stable, if $g_1(t, w)$ is monotone nondecreasing in w and

$$w(t, U) \geq b_0(\|U\|), \quad b_0 \in \mathcal{K} \quad (4.105)$$

Proof 4.8: By Theorem 4.7, the trivial solution of (4.1) is equi-stable. Hence letting $\epsilon = \rho$ so that $\delta_0 = \delta(\rho, t_0) > 0$, we get, by equi-stability

$$\|W_0\| < \delta_0 \text{ implies that } \|U(t_0)\| < \rho, \quad t \geq t_0 \quad (4.106)$$

We shall show that, for any solution $U(t)$ of (4.1) with $\|W_0\| < \delta_0$, we have $\lim_{t \rightarrow \infty} w(t, U(t)) = 0$. This and (4.105) imply $\lim_{t \rightarrow \infty} \|U(t)\| = 0$, completing the proof.

Suppose that $\limsup_{t \rightarrow \infty} w(t, U(t)) \neq 0$. Then there would exist two divergent sequences $\{t'_i\}$, $\{t''_i\}$ and a $\sigma > 0$ satisfying

$$\begin{aligned} \text{i) } & w(t'_i, U(t'_i)) = \frac{\sigma}{2}, \quad w(t''_i, U(t''_i)) = \sigma \text{ and } w(t, U(t)) > \frac{\sigma}{2}, \quad t \in (t'_i, t''_i) \text{ or} \\ \text{ii) } & w(t'_i, U(t'_i)) = \sigma, \quad w(t''_i, U(t''_i)) = \frac{\sigma}{2} \text{ and } w(t, U(t)) > \frac{\sigma}{2}, \quad t \in (t'_i, t''_i). \end{aligned}$$

Suppose that $D^+w(t, U(t)) \leq M$. Then using i), we obtain

$$\frac{\sigma}{2} = \sigma - \frac{\sigma}{2} = w(t''_i, U(t''_i)) - w(t'_i, U(t'_i)) \leq M(t''_i - t'_i) \quad (4.107)$$

which shows that $t''_i - t'_i \geq \frac{\sigma}{2M}$ for each i . Hence, by (4.103) and Corollary 4.3, we have

$$V_1(t, U(t)) \leq r_1(t, t_0, w_{10}) - \sum_{i=1}^n \int_{t'_i}^{t''_i} c[w(s, U(s))] ds, \quad t \geq t_0 \quad (4.108)$$

Since $w_{10} = V_1(t_0, W_0) \leq a_0(w_0, \|W_0\|) \leq a_0(t_0, \delta_0) < \delta^*$, we get from (4.91)

$$w_1(t, t_0, w_{10}) < \frac{\delta_0(\rho)}{2}, \quad t \geq t_0 \quad (4.109)$$

We thus obtain

$$0 \leq V_1(t, U(t)) \leq \frac{\delta_0(\rho)}{2} - c\left(\frac{\sigma}{2}\right) \frac{\sigma}{2M} n \quad (4.110)$$

For sufficiently large n , we get a contradiction and therefore $\limsup_{t \rightarrow \infty} w(t, U(t)) = 0$. Since $w(t, U) \geq b_0(\|U(t)\|)$ by assumption, it follows that $\lim_{t \rightarrow \infty} \|U(t)\| = 0$ and the proof is complete.

Note 4.1: The functions $g_1(t, w) = g_2(t, w) \equiv 0$ are admissible in Theorem 4.7, and so the same conclusion can be reached. If $V_1(t, U) \equiv 0$ and $g_1(t, w) \equiv 0$, then we get uniform stability from Theorem 4.7. If, on the other hand, $V_\eta(t, U) \equiv 0$, $g_2(t, w) \equiv 0$ and $V_1(t, U) \geq b(\|U\|)$, $b \in \mathcal{K}$, then Theorem 4.7 yields equi-stability. We note that known results on equi-stability require the assumption to hold everywhere in $S(\rho)$ and Theorem 4.7 relaxes such a requirement considerably by method of perturbing Lyapunov functions.

Note 4.2: The functions $g_1(t, w) = g_2(t, w) \equiv 0$ are admissible in Theorem 4.8 to yield equi-asymptotic stability. Similarly, if $V_\eta(t, U) \equiv 0$, $g_2(t, w) \equiv 0$ with $V_1(t, U) \geq b(\|U\|)$, $b \in \mathcal{K}$, implies the same conclusion. If $V_1(t, U) \equiv 0$ and $g_1(t, w) \equiv 0$ in Theorem 4.7, to get uniformly asymptotic stability, one needs to strengthen the estimate on $D^+ V_\eta(t, U)$. This we state as a corollary.

Corollary 4.4: Suppose that the assumptions of Theorem 4.7 hold with $V_1(t, U) \equiv 0$, $g(t, w) \equiv 0$. Suppose further that

$$\begin{cases} D^+V_\eta(t, U) \leq -c[w(t, U)] + g_2(t, V_\eta(t, U)), \\ (t, U) \in \mathbb{R}_+ \times S(\rho) \cap S^c(\eta) \end{cases} \quad (4.111)$$

where $w \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$, $w(t, U) \geq b(\|U\|)$, $c, b \in \mathcal{K}$ and $g_2(t, w)$ is nondecreasing in w . Then, the trivial solution of (4.1) is uniformly asymptotically stable.

Proof 4.4: The trivial solution of (4.1) is uniformly stable by Note 4.1 in the present case. Hence taking $\epsilon = \rho$ and designating $\delta_0 = \delta(\rho)$, we have

$$\|W_0\| < \delta_0 \text{ implies } \|U(t)\| < \rho, \quad t \geq t_0 \quad (4.112)$$

To prove uniform attractivity, let $0 < \epsilon < \rho$ be given. Let $\delta = \delta(\epsilon) > 0$ be the number relative to ϵ in uniform stability. Choose $T = \frac{b(\rho)}{c(\delta)} + 1$. Then we shall show that there exist a $t^* \in [t_0, t_0 + T]$ such that $w(t^*, U(t^*)) < b(\delta)$ for any solution $U(t)$ of (4.1) with $\|W_0\| < \delta_0$.

If this is not true, $w(t, U(t)) \geq b(\delta)$, $t \in [t_0, t_0 + T]$. Now using the assumption (4.111) and arguing as in Corollary 4.3, we get

$$\begin{cases} 0 \leq V_\eta(t_0 + T, U(t_0 + T)) \\ \leq r_2(t_0 + T, t_0, w_{20}) - \int_{t_0}^{t_0+T} w(s, U(s)) ds \end{cases} \quad (4.113)$$

This yields, since $r_2(t, t_0, w_{20}) < b(\rho)$ and the choice of T

$$0 \leq V_\eta(t_0 + T, U(t_0 + T)) \leq b(\rho) - c(\delta)T < 0 \quad (4.114)$$

which is a contradiction. Hence there exists a $t^* \in [t_0, t_0 + T]$ satisfying $w(t^*, U(t^*)) < b(\delta)$, which implies $\|U(t)\| < \delta$. Consequently, it follows, by uniform stability that

$$\|W_0\| < \delta_0 \text{ implies that } \|U(t)\| < \epsilon \quad (4.115)$$

$t \geq t_0 + T$ and the proof is complete.

4.7. Boundedness Criteria

In this section, we have investigated that the boundedness of solutions of the set differential equation

$$D_H U = F(t, U), U(t_0) = U_0 \in K_c(\mathbb{R}^n) \quad (4.116)$$

where $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$.

Definition 4.10: The solution of (4.116) is said to be

i) equi-bounded, if for any $\alpha > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\beta = \beta(t_0, \alpha) > 0$ such that

$$\|W_0\| < \alpha \text{ implies } \|U(t)\| < \beta, \quad t \geq t_0 \quad (4.117)$$

ii) uniform equi-bounded, if β in i) does not depend on t_0 .

iii) quasi-equi-ultimately bounded if for each $\alpha \geq 0$, $t_0 \in \mathbb{R}_+$, there exists a $B > 0$ and a $T = T(t_0, \alpha) > 0$ such that

$$\|W_0\| < \alpha \text{ implies } \|U(t)\| < B, \quad t \geq t_0 + T \quad (4.118)$$

iv) quasi-uniform ultimately bounded if T in iii) is independent of t_0 .

v) equi-ultimately bounded, if i) and iii) hold simultaneously.

vi) uniform ultimately bounded if ii) and iv) hold simultaneously.

vii) equi-Lagrange stable if i) and Definition 4.1 iii) hold.

viii) uniformly Lagrange stable if ii) and Definition 4.1 iv) hold.

Theorem 4.9: Assume that

- i) $V \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$, $|V(t, U_1) - V(t, U_2)| \leq LD[U_1, U_2]$, $L > 0$ and for $(t, U) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n)$, $D^+V(t, U) \leq 0$;
- ii) $b(\|U\|) \leq V(t, U) \leq a(t, \|U\|)$, for $(t, U) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n)$ where $b, a(t, \cdot) \in \mathcal{K} = [\sigma \in C[\mathbb{R}_+, \mathbb{R}_+]: \sigma(w) \text{ is incereasing in } w \text{ and } \sigma(w) \rightarrow \infty \text{ as } w \rightarrow \infty]$.

Proof 4.9: Let $\alpha > 0$ and $t_0 \in \mathbb{R}_+$, be given. Choose $\beta = \beta(t_0, \alpha) > 0$ such that

$$a(t_0, \alpha) < b(\beta) \quad (4.119)$$

with this β , i) holds. If this is not true, there would exist a solution $U(t) = U(t, t_0, W_0)$ of (4.116) an a $t_0 < t_1$ such that

$$\|U(t_1)\| = \beta \text{ and } \|U(t)\| \leq \beta, \quad t_0 \leq t \leq t_1 \quad (4.120)$$

Assumption i) and Corollary 4.1 show that

$$V(t, U(t)) \leq V(t_0, W_0), \quad t_0 \leq t \leq t_1 \quad (4.121)$$

Thus, condition ii) and (4.119) yield

$$\begin{cases} b(\beta) = b(\|U(t_1)\|) \leq V(t_1, U(t_1)) \\ \leq V(t_0, W_0) \leq a(t_0, \|W_0\|) < a(t_0, \alpha) < b(\beta) \end{cases} \quad (4.122)$$

This contradiction proves i) and we have completed the proof.

For uniform boundedness, the following result is obtained under weaker assumptions.

Theorem 4.10: Assume that

- i) $V \in C[\mathbb{R}_+ \times S^c(\rho), \mathbb{R}_+]$, where ρ may be large; $|V(t, U_1) - V(t, U_2)| \leq LD[U_1, U_2]$, and for $(t, U) \in \mathbb{R}_+ \times S^c(\rho)$, $D^+V(t, U) \leq 0$;

ii) $b(\|U\|) \leq V(t, U) \leq a(\|U\|)$, for $(t, U) \in \mathbb{R}_+ \times S^c(\rho)$ where $a, b \in \mathcal{K}$,
which are defined only on $[\rho, \infty)$

Then, Definition 4.10 ii) holds

Proof 4.10: The proof is similar to the proof of Theorem 4.9 except that the choice of β is now made so that $a(\alpha) < b(\beta)$ and consequently β is independent of t_0 . Also, $\alpha > \rho$ for the proof since the assumptions are only for $S^c(\rho)$. However, if $0 < \alpha \leq \rho$, we can take $\beta = \beta(\rho)$ and again the proof follows.

We shall give a typical result that offers conditions for equi-ultimate boundedness, that is for Definition 4.10 v).

Theorem 4.11: Let the assumptions of Theorem 4.9 hold except that we strengthen the estimate on $D^+V(t, U)$ as

$$D^+V(t, U) \leq -\eta V(t, U), \quad \eta > 0, \quad (t, U) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n) \quad (4.123)$$

and suppose that condition ii) holds for $\|U\| \geq B$. Then (v) holds.

Proof 4.11: Clearly Definition 4.10 i) is obtained from Theorem 4.9. Hence

$$\|W_0\| < \alpha \text{ implies } \|U(t)\| < \beta, \quad t \geq t_0 \quad (4.124)$$

Now (4.123) yields the estimate

$$V(t, U(t)) \leq V(t_0, W_0)e^{-\eta(t-t_0)}, \quad t \geq t_0 \quad (4.125)$$

Let $T = \frac{1}{\eta} \ln \frac{a(t_0, \alpha)}{b(B)}$ and suppose that for $t \geq t_0 + T$, $\|U(t)\| \geq B$. Then, we get from (4.125)

$$b(B) \leq b(\|U(t)\|) \leq V(t, U(t)) < a(t_0, \alpha)e^{-\eta T} = b(B) \quad (4.126)$$

The contradiction proves Definition 4.10 v) and the proof is complete.

Finally, we shall provide a result on nonuniform boundedness property using the method of perturbing Lyapunov functions.

Theorem 4.12: Assume that

i) $\rho > 0$, $V_1 \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$, V_1 is bounded for $(t, U) \in \mathbb{R}_+ \times \partial S(\rho)$, and

$$|V_1(t, U_1) - V_1(t, U_2)| \leq L_1 D[U_1, U_2], \quad L_1 > 0 \quad (4.127)$$

$$\begin{cases} D^+ V_1(t, U) \\ = \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [V_1(t+h, U+hF(t, U)) - V_1(t, U)] \\ \leq g_1(t, V_1) \end{cases} \quad (4.128)$$

$(t, U) \in \mathbb{R}_+ \times S^c(\rho)$, where $g_1 \in C[\mathbb{R}_+^2, \mathbb{R}]$;

ii) $V_2 \in C[\mathbb{R}_+ \times S^c(\rho), \mathbb{R}_+]$,

$$b(\|U\|) \leq V_2(t, U) \leq a(\|U\|), \quad a, b \in \mathcal{K} \quad (4.129)$$

$$\begin{cases} D^+ V_1(t, U) + D^+ V_2(t, U) \leq g_2(t, V_1(t, U) + V_2(t, U)), \\ g_2 \in C[\mathbb{R}_+^2, \mathbb{R}]; \end{cases} \quad (4.130)$$

iii) the scalar differential equations

$$w_1' = g_1(t, w_1), \quad w_1(t_0) = w_{10} \geq 0 \quad (4.131)$$

and

$$w_2' = g_2(t, w_2), \quad w_2(t_0) = w_{20} \geq 0 \quad (4.132)$$

are equi-bounded and uniformly bounded respectively.

Then the system (4.116) is equi-bounded.

Proof 4.12: Let $B_1 > \rho$ and $t_0 \in \mathbb{R}_+$, be given. Let

$$\alpha_1 = \alpha_1(t_0, B_1) = \max\{\alpha_0, \alpha^*\} \quad (4.133)$$

where $\alpha_0 = \max\{V_1(t_0, W_0) : W_0 \in cl\{S(B_1) \cap S^c(\rho)\}\}$ and $\alpha^* \geq V_1(t, U)$ for $(t, U) \in \mathbb{R}_+ \times \partial S(\rho)$.

Since equation (4.131) is equi-bounded, given $\alpha_1 > 0$, and $t_0 \in \mathbb{R}_+$, there exist a $\beta_0 = \beta(t_0, \alpha_1) > 0$ such that

$$w_1(t, t_0, w_{10}) < \beta_0, \quad t \geq t_0 \quad (4.134)$$

provided $w_{10} < \alpha_1$, where $w_1(t, t_0, w_{10})$ is any solution (4.131). Let $\alpha_2 = a(B_1) + \beta_0$, the uniform boundedness of equation (4.132) yields that

$$w_2(t, t_0, w_{20}) < \beta_1(\alpha_2), \quad t \geq t_0 \quad (4.135)$$

provided $w_{20} < \alpha_2$, where $w_2(t, t_0, w_{20})$ is any solution of (4.134). Choose B_2 satisfying

$$b(B_2) < \beta_1(\alpha_2) \quad (4.136)$$

We now claim that $W_0 \in S(B_1)$ implies that

$$U(t, t_0, W_0) \in S(B_2) \quad (4.137)$$

for $t \geq t_0$, where $U(t, t_0, W_0)$ is any solution of (4.116).

If it is not true, then there exists a solution $U(t, t_0, W_0)$ of (4.134) with $W_0 \in S(B_1)$, such that for some $t^* > t_0$, $B_2 = \|U(t^*, t_0, W_0)\|$. Since $B_1 > \rho$, there are two possibilities to consider:

- i) $U(t, t_0, W_0) \in S^c(\rho)$ for $t \in [t_0, t^*]$;
- ii) there exists a $\bar{t} \geq t_0$ such that $U(\bar{t}, t_0, W_0) \in \partial S(\rho)$ and $U(t, t_0, W_0) \in S^c(\rho)$ for $t \in [\bar{t}, t^*]$.

If i holds, we can find $t_1 > t_0$ such that

$$U(t_1, t_0, W_0) \in \partial S(B_1) \quad (4.138)$$

$$\begin{cases} U(t^*, t_0, W_0) \in \partial S(B_2) \text{ and} \\ (t, t_0, W_0) \in S^c(B_1), \quad t \in [t_1, t^*] \end{cases} \quad (4.139)$$

Setting $m(t) = V_1(t, U(t, t_0, W_0)) + V_2(t, U(t, t_0, W_0))$ for $t \in [t_1, t^*]$, and then using Theorem 4.1, we can obtain differential inequality

$$D^+m(t) \leq g_2(t, m(t)), \quad t \in [t_1, t^*] \quad (4.140)$$

and so

$$m(t) \leq \gamma_2(t, t_1, m(t_1)), \quad t \in [t_1, t^*] \quad (4.141)$$

where $\gamma_2(t, t_1, v_0)$ is the maximal solution of (4.132) with $\gamma_2(t_1, t_1, v_0) = v_0$.

Thus,

$$\begin{cases} V_1(t^*, U(t^*, t_0, W_0)) + V_2(t^*, U(t^*, t_0, W_0)) \\ \leq \gamma_2(t^*, t_1, V_1(t_1, U(t_1, t_0, W_0)) + V_2(t_1, U(t_1, t_0, W_0))) \end{cases} \quad (4.142)$$

Similarly, we also have

$$V_1(t_1, U(t_1, t_0, W_0)) \leq \gamma_1(t_1, t_0, V_1(t_0, W_0)) \quad (4.143)$$

where $\gamma_1(t_1, t_0, u_0)$ is the maximal solution of (4.131). Setting $w_{10} = V_1(t_0, W_0) < \alpha_1$. Then

$$V_1(t_1, U(t_1, t_0, W_0)) \leq \gamma_1(t_1, t_0, V_1(t_0, W_0)) \leq \beta_0 \quad (4.144)$$

since (4.134) holds.

Furthermore, $V_2(t_1, U(t_1, t_0, W_0)) \leq a(B_1)$. Consequently, we have

$$\begin{cases} w_{20} = V_1(t_1, U(t_1, t_0, W_0)) + V_2(t_1, U(t_1, t_0, W_0)) \\ \leq \beta_0 + a(B_1) = \alpha_2 \end{cases} \quad (4.145)$$

Combining (4.135), (4.136), (4.139) and (4.145), we obtain

$$b(B_2) \leq m(t^*) \leq \gamma(t^*) \leq \beta_1(\alpha_2) < b(B_2) \quad (4.146)$$

which is a contradiction.

If case ii) holds, we also arrive at the inequality (4.142), where $t_1 > \bar{t}$ satisfies (4.139). We then have, in place of (4.143), the relation

$$V_1(t_1, U(t_1, t_0, W_0)) \leq \gamma_1(t_1, \bar{t}, V_1(\bar{t}, U(\bar{t}, t_0, W_0))) \quad (4.147)$$

Since $U(\bar{t}, t_0, W_0) \in \partial S(\rho)$ and $V_1(\bar{t}, U(\bar{t}, t_0, W_0)) \leq \alpha^* \leq \alpha_1$, arguing as before, we get the contradiction (4.146). This proves that for any given $B_1 > \rho$, $t_0 > 0$, there exists a B_2 such that

$$W_0 \in S(B_1) \text{ implies } U(t, t_0, W_0) \in S(B_2), \quad t \geq t_0. \quad (4.148)$$

For $B_1 < \rho$, we set $B_2(t_0, B_2) = B_2(t_0, \rho)$ and hence the proof is complete.

5. CONCLUSION

In this work the set valued differential equations have been worked on in terms of stability and boundedness criteria by using comparison theorems. It can be generalized to the concept of stability in terms of two measures with initial time difference. The theory of stability has been successfully studied. Some results are used by means of various types of Lyapunov functions for set valued differential equations. These concepts of stability theory and boundedness criteria also could have been applied to initial time difference qualitative behavior for perturbed set valued differential equations with respect to unperturbed set valued differential equations by using comparison method via Lyapunov functions, Lyapunov functionals and Lyapunov-like functions.

REFERENCES

- [1] Banks H. T., Jacobs M. G., (1970), "A differential Calculus for multifunctions", *Journal of Mathematical Analysis and Applications*, 29 (2), 246 - 272.
- [2] Pinto Brandao Lopes A. J., De Blasi F. S., Iervolino F., (1970), "Uniqueness and existence theorems for differential equations with compact convex-valued solutions", *Bollettino dell'Unione Matematica Italy*, 3 (1), 47-54.
- [3] Bhaskar T. Gnana, Lakshmikantham V., (2003), "Set Differential Equations and Flow Invariance", *Applicable Analysis*, 82, 357-368.
- [4] Hukuhara M., (1967), "Sur l'application Semicontinue Dont la Valeur est un Compact Convexe", *Funkcialaj Ekvacioj*, 10, 43-68.
- [5] Hukuhara M., (1967), "Int'egration des Applications Mesurables Dont la Valeur est un Compact Convex", *Funkcialaj Ekvacioj*, 10, 205-229.
- [6] Laksmikantham V., Leela S., (1969), "Differential and Integral Inequalities, Vol. I", 1st Edition, Academic Press.
- [7] Laksmikantham V., Leela S., Vatsala A. S., (2002), "Set valued Hybrid Differential Equations and Stability in Terms of Two Measures", *Journal of Hybrid Systems*, 2 (11), 169-187.
- [8] Laksmikantham V., Leela S., Vatsala A. S., (2003), "Interconnection between Set and Fuzzy Differential Equations" (2), *Nonlinear Analysis*, 54, 351-360.
- [9] Tolstonogov A., (2000), "Differential Inclusions in a Banach Space", 1th Edition, Kluwer Academic Publishers.
- [10] Yakar C., (2015), "An Application of Dynamic Systems on Time Scales with Initial Time Difference Involving a Comparison Result and Lyapunov Stability", *Dynamics of Continuous, Discrete and Impulsive Systems. Series A: Mathematical Analysis*, 22 (1), 143-158
- [11] Yakar C., Oğur B., (2015), "Stability of Perturbed Dynamic System on Time Scales with Initial Time Difference", *Turkish Journal of Mathematics*, 39 (1), 1-15.
- [12] Yakar C., Gücen M. B., (2014), "Initial Time Difference Stability of Causal Differential Systems in terms of Lyapunov Functions and Lyapunov Functionals", *Journal of Applied Mathematics*, vol. 2014 Article ID 832015, 1-7.
- [13] Lakshmikantham V., Bhaskar T.G., Devi J. V., (2006), "Theory of Set Differential Equations in Metric Spaces", 6th Edition Cambridge Scientific Publishers.

- [14] Yakar C., Çiçek M., Gücen M. B., (2014), “Practical stability in terms of two measures for fractional order dynamic systems in Caputo's sense with initial time difference.”, *Journal of the Franklin Institute*, 351 (2), 732-742.
- [15] Yakar C., Çiçek M., Gücen M. B., (2012), “Practical stability, boundedness criteria and Lagrange stability of Fuzzy differential systems”, *Journal of Computers and Mathematics with Applications*, 64 (2), 2118-2127.
- [16] Yakar C., Shaw M.D., (2005), “A comparison result and Lyapunov stability criteria with initial time difference”, *Dynamics of Continuous Discrete and Impulsive Systems, Ser. A: Mathematical Analysis*, 12 (6), 731–741.
- [17] Sheldon P., Gordon, (1979), “A stability theory for perturbed differential equations”, *International Journal of Mathematics and Mathematical Sciences*, 2 (2), Pages 283-297
- [18] Lakshmikantham V., Liu X., Leela S. Leela, (1998), “Variational Lyapunov method and stability theory”, *Mathematical Problems in Engineering*, 3 (6), Pages 555-571
- [19] Knops, R. J., Wilkes, E. W., (1966), “On Movchan's theorems for stability of continuous systems”, *International Journal of Engineering*, 4 (2), 303–329.

BIOGRAPHY

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