T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

THEORETICAL AND NUMERICAL INVESTIGATION OF INVERSE PROBLEMS OF FINDING THE LOWEST TERM IN 1D AND 2D HEAT EQUATIONS

SAİT ERKOVAN
A THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
DEPARTMENT OF MATHEMATICS

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GEBZE TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

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SUMMARY

In this thesis, we consider the inverse problems of finding time-dependent coefficient of the lowest term in 1D heat equation with Wentzell-Neumann boundary condition and 2D heat equation with Ionkin-type non-local boundary condition. The Fourier series analysis of these inverse problems in the heat equations with boundary and integral over-determination conditions is presented. The existence and uniqueness of the classical solutions are shown by using the generalized Fourier method combined with the unique solvability of the second kind Volterra integral equation under some regularity, consistency and orthogonality conditions on the data and additional conditions on the sign of the Fourier coefficients of the initial data and the known part of source term. The numerical solutions of the inverse problems are introduced by using uniform finite difference method combined with the composite trapezoidal rule in 1D problem and the composite product Simpson's rule in 2D problem on a uniform grid; and on a non-uniform grid, using non-uniform finite difference method combined with Gauss-Lobatto quadrature in 1D problem and product Gauss-Lobatto cubature in 2D problem. Numerical examples illustrate how to implement the methods.

Keywords: 1D (2D) Heat Equation, Inverse Coefficient Problem, Wentzell-Neumann Boundary Condition, Ionkin-type Boundary Condition, (Non-)Uniform Finite Difference Method, Numerical Integration.

ÖZET

Bu tezde, Wentzell-Neumann sınır koşulna sahip 1 boyutlu ısı denkleminde ve yerel olmayan Ionkin-tipli sınır koşuluna sahip 2 boyutlu ısı denkleminde zamana-bağlı en küçük katsayıyı bulma ters problemleri ele alınmıştır. Bu ters problemlerin, sınır ve ek integraller koşulları altında Fourier seri analizleri verilmiştir. Bu problemlerin çözümlerinin varlığı - tekliği ve kararlı çözümleri, ikinci tip Volterra integral denklemleri kullanılarak ve başlangıç koşulu, kaynak terimin bilinen kısmı ve Fourier katsayıları üzerine bazı düzenlilik, uyumluluk ve diklik koşulları eklenerek genelleştirilmiş Fourier metoduyla elde edilmiştir. Ters problemlerin sayısal çözümleri, düzgün bölmelenmiş bölge üzerinde, 1 boyutta bileşik yamuk kuralı ve 2 boyutta bileşik Simpson çarpım kuralıyla düzgün sonlu farklar metodu kullanılarak; ve düzgün bölmelenmemiş bölge üzerinde, 1 boyutta Gauss-Lobatto kuadratür formülü ve 2 boyutta Gauss-Lobatto kübatür formülü ile düzgün olmayan sonlu farklar metodu kullanılarak gösterilmiştir. Sayısal yöntemlerin uygulamalarını karşılaştırmak için bazı örnekler verilmiştir.

Anahtar Kelimeler: 1 ve 2 Boyutlu Isı Denklemleri, Ters Katsayı Problemi, Wentzell-Neumann Sınır Koşulu, Ionkin-tipli Sınır Koşulu, Düzgün (Olmayan) Sonlu Farklar Yöntemi, Sayısal İntegralleme

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TABLE OF CONTENTS

	Page
SUMMARY	v
ÖZET	vi
ACKNOWLEDGEMENTS	vii
TABLE of CONTENTS	viii
LIST of ABBREVIATIONS and ACRONYMS	x
1. INTRODUCTION	1
2. BASIC DEFINITIONS AND PRELIMINARY RESULTS	9
3. 1D HEAT EQUATION WITH WENTZELL - NEUMANN BOUNDARY CONDITION	13
3.1. Well posedness of the inverse problem	13
3.1.1. Spectral problem	13
3.1.2. Existence-uniqueness and continuous dependence upon the data of the solution	15
3.2. Numerical methods for the inverse problem	24
3.2.1. Non-uniform finite difference method	25
3.2.2. Uniform finite difference method	28
4. 2D HEAT EQUATION WITH IONKIN-TYPE BOUNDARY CONDITION	42
4.1. Well-posedness of the inverse problem	42
4.1.1. Spectral problem	42
4.1.2. Existence-uniqueness and continuous dependence upon the data of the solution	44
4.2. Numerical methods for the inverse problem	51
4.2.1. Uniform finite difference method	52
4.2.2. Non-uniform finite difference method	55

5. CONCLUDING REMARKS 65

REFERENCES 68

BIOGRAPHY 74

APPENDICES 75

LIST of ABBREVIATIONS and ACRONYMS

Abbreviations and Acronyms	**Explanations**
$C[a,b]$	: The space of the continuous functions on $[a,b]$
$C([a,b]\times[a,b])$	: The space of continuous function over the rectangular region $[a,b]\times[a,b]$
$C^n[a,b]$	: The space of functions which n-th derivatives are continuous on $[a,b]$
$C^{k,l,m}(\Omega)$	: The space of three variable continuous functions which k-th partial derivative with respect to first variable, l-th partial derivative with respect to second variable and m-th partial derivative with respect to third variable are continuous over the region $\Omega\in\mathbb{R}^3$
$L_2(a,b)$	: The space of measurable and square integrable functions on (a,b)

1. INTRODUCTION

In theory of differential equations, the boundary conditions associated with a second order partial differential operator are usually classical, since they involve the function and its first derivative (including Dirichlet, Neumann and Robin conditions). The Fourier series analysis of the problems with classical boundary conditions are applicable uneventfully since the expansion in terms of eigenfuntions of auxiliary spectral problem is perfectly studied in various literature (see, for example [1]).

In recent years, many authors have focused on non-classical boundary conditions. In Markov process theory, it was recognized that it is also natural to include boundary conditions involving the operator itself (Wentzell boundary conditions) [2]. One may think of the Wentzell boundary condition as describing a situation like the Robin boundary condition, with the additional feature that the boundary has the capacity for storing (but not conducting) heat. For the classical results along these lines we refer to [3, 4, 5, 6] and more recently, [7, 8]. We refer to [9] and [10], and to the references contained therein for the motivations and the study of the general Wentzell boundary conditions in the theory of parabolic partial differential equations.

The aim of the first part of this thesis is to consider the heat equation

$$u_t = u_{xx} + p(t)u + f(x,t),\ (x,t) \in \Omega_T, \tag{1.1}$$

with the initial condition

$$u(x,0) = \varphi(x),\ x \in [0,1], \tag{1.2}$$

Dirichlet boundary condition

$$u(0,t) = 0,\ t \in [0,T] \tag{1.3}$$

and a condition with non-local and/or non-classical type:

- a condition in which $u(0,t)$ and/or $u_x(0,t)$ is linearly related with $u(1,t)$ and/or $u_x(1,t)$ (non-local);
- a condition which contains the term of maximal order $u_{xx}(0,t)$ or $u_{xx}(1,t)$ (non-classical or Wentzell);
- a condition in which $u(0,t)$ and/or $u_x(0,t)$ is linearly related with $u_{xx}(1,t)$ (non-local and non-classical),

where $\Omega_T = \{(x,t) : 0 < x < 1,\ 0 < t \leq T\}$ with fixed $T > 0$. The functions $f,\ \varphi$ are given functions on $\bar{\Omega}_T$, which is the closure of Ω_T, and $0 \leq x \leq 1$, respectively.

The Fourier series analysis of direct and inverse initial boundary value problems (IBVPs) for (1.1) with first (non-local) and second kind (non-classical) boundary conditions are considered in [11, 12]. In this study we adress the IBVP with a third kind (non-local and non-classical) boundary conditions. More precisely, we consider the problem (1.1)–(1.3) with the non-local Wentzell-Neumann boundary condition

$$u_x(0,t) + \alpha u_{xx}(1,t) = 0,\ t \in [0,T] \tag{1.4}$$

for $\alpha > 0$. Note that (1.4) is a dynamic boundary condition involving motion on the boundary. In the case that (1.1) is valid at $x = 1$, we plug (1.1) into (1.4), the boundary condition becomes

$$u_t(1,t) + \frac{1}{\alpha} u_x(0,t) - p(t)u(1,t) = f(1,t). \tag{1.5}$$

Such a boundary condition is mentioned in [13] as modelling a boundary reaction in diffusion of a chemical, where the term $\frac{1}{\alpha}u_x(0,t)$ represents the diffusion transport of materials to the boundary.

The dynamic boundary conditions are not very common in the mathematical literature. Nevertheless, they appear in many mathematical models including heat transfer in a solid in contact with a moving fluid, thermoelectricity, diffusion phenomena, problems in fluid dynamics. Such conditions arise in the applications and

have been studied in [14, 15, 16] and more recently in [17, 18]. The time evolution presented above describes particles which diffuse in and which may stick (for infinitesimally short times) to the boundary.

Numerical methods for the problem of finding the lowest coefficient in parabolic equations are satisfactorily considered in the cases of classical boundary conditions. We give the references [20, 21, 22, 23, 24] for the numerical solution of such type of inverse parabolic problems. In these references, different kinds of uniform finite difference methods are applied to the solution of inverse problems. In the present work, we use non-uniform finite difference discretization combined with Gauss-Lobatto quadrature rule, and we compare the efficiency of this method with uniform finite difference discretization combined with composite trapezoidal rule.

When the coefficient $p(t),\ 0 \leq t \leq T$ is also given, the problem of finding $u(x,t)$ from the equation (1.1), initial condition (1.2), boundary conditions (1.3) and (1.4) is termed as the direct (or forward) problem.

When the function $p(t),\ 0 \leq t \leq T$ is unknown, the inverse problem formulates as a problem of finding a pair of functions $\{p(t),\ u(x,t)\}$ which satisfy the equation (1.1), initial condition (1.2), boundary conditions (1.3) and (1.4) and the overdetermination condition

$$\int_0^1 u(x,t)dx = E(t),\ 0 \leq t \leq T, \tag{1.6}$$

where $E(t)$ is a given function which is the prescription of the total energy, or mass.

In contrast to the direct problems, the inverse parabolic problems with Wentzell boundary conditions are scarce ([11, 12, 19]), and need additional consideration.

To study the initial boundary value problem (IBVP) (1.1)–(1.3) with the general form of non-local and/or non-classical boundary condition is difficult in technical point of view. The allowed form of non-local boundary conditions are considered in [25, 26] where the Green's function representation of the solution is obtained by the

techniques of fundamental solution of heat equation. For some of the cases of non-local boundary conditions, when the fundamental solution of the heat equation does not work, we used the expansion in terms of eigenfunctions for the auxiliary spectral problem corresponding to the considered initial boundary value problem [11]. Then this method is extended to IBVPs with non-classical boundary conditions [12].

The important works [5, 7, 8, 9, 10, 16, 17] on evolution equations which subject to Wentzell/dynamic boundary conditions are generally analysed from the point of view of semi-group theory. The same method is used in an earlier work [34] which focuses on a heat transfer model with dynamic boundary condition. Here, we consider the heat equation with Wentzell boundary conditions from the point of view of spectral theory. Under some regularity, consistency and orthogonality conditions on the initial data and the given source term, the well posedness of the classical solution is shown by using the generalized Fourier method. Some of the classes of the problems with Wentzell boundary condition can not be solved by the other approaches, for example, semi-group approach.

The inverse coefficient problems for 2D heat equation are of importance in some engineering, industrial and medical point of views. Among these inverse problems, much attention is given to the determination of the lowest order coefficient in multidimensional heat equation, in particular, when this coefficient depends solely on time [38, 39, 40, 41, 42, 43, 44, 45], although many researchers have reported its difficulties. The bioheat model is composed of a partial differential equation [46]:

$$\rho c U_\tau - \kappa \Delta U + w_b c_b (U - U_a) = h_m + h_e, \tag{1.7}$$

where κ is the thermal conductivity of the tissue, U is the temperature of the tissue, w_b is the mass flow rate of blood, c_b is specific heat of the blood, U_a is the temperature of the arterial blood, h_m is a volumetric rate of metabolic heating generation, h_e volumetric rate of external heat, ρ is the density, and c is specific heat of the tissue.

In two-dimensional (2D) case, (1.7) takes the following dimensionless form

$$u_t = u_{xx} + u_{yy} - p(t)u(x, y, t) + f(x, y, t), \quad (x, y, t) \in D_T, \tag{1.8}$$

by using the change of variables, letting g_0 as a reference source of heating generation, we have

$$u = \frac{\kappa(U - U_a)}{g_0}, \quad t = \frac{\kappa\tau}{\rho c}, \quad p = \frac{w_b c_b}{\kappa}, \quad f = \frac{h_e + h_m}{g_0}, \tag{1.9}$$

where the domain is $D_T = D_{xy} \times (0 < t \leq T)$ with $D_{xy} = \{(x, y) : 0 < x, y < 1\}$, the parameter $p(t)$ is the blood perfusion coefficient, and f is the dimensionless source of heating generation.

Inverse blood perfusion determination techniques for one dimensional (1D) case based on known initial and classicial boundary conditions (Dirichlet, Neumann, Robin) and additional classical measurements have been described in [45, 47, 48, 49, 20]. Moreover, these inverse problems in multidimensional cases have been studied in [50, 51]. The determination techniques for the inverse blood perfusion on 1D case based on nonlocal or nonclassical and additional integral boundary conditions have been described in [11, 12, 52].

The determination of a time-dependent blood perfusion coefficient in a 2D bioheat equation is formulated as an inverse problem of finding the coefficient of the lowest term in a 2D heat equation with non-local boundary conditions (Ionkin-type boundary conditions) will be considered in the present study.

Numerical methods for solving the problem of the identification of the lowest coefficient of 1D parabolic equations are considered in many works. The works [20, 21, 22, 23, 24] deal the numerical solution of inverse problems for parabolic equations from an application point of view. We highlight separately studies dealing with numerical solution of inverse problems for finding time-dependent lowest coefficient in multidimensional parabolic equations [53, 54, 55]. Discretization in our problem is performed using uniform and non-uniform finite difference methods combined with

suitable nodes for numerical integration on uniform and non-uniform grids.

We consider (1.8) with the initial condition

$$u(x, y, 0) = \varphi(x, y), \quad 0 \leq x, y \leq 1, \tag{1.10}$$

nonlocal boundary conditions

$$\begin{aligned} &u(0, y, t) = u(1, y, t), \quad u_x(1, y, t) = 0, \quad 0 \leq y \leq 1, \quad 0 \leq t \leq T, \\ &u(x, 0, t) = u(x, 1, t) = 0, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T. \end{aligned} \tag{1.11}$$

In the mathematical model of this problem, we consider a rectangular perfused tissue with both thickness and length are 1 (see Figure 1.1). We show the temperature at position (x, y) and time t with $u(x, y, t)$. The boundary conditions (1.11) include the same temperature at $x = 0$ and $x = 1$ which is denoted by u^*, 0° temperature on the upper skin surface at $y = 1$, and on a wall between the tissue and an adjoint large blood vessel at $y = 0$, and adiabatic condition at $x = 1$ in Figure 1.1.

The model which is considered in a rectangular perfused tissue with length and thickness equal to some value L and 1, respectively, including (some) given temperature on the upper skin $y = 1$, adiabatic conditions at $x = 0$ and $x = L$, and convective heat transfer between the tissue and an adjoint large blood vessel at $y = 0$ as a space-dependent bioheat transfer problem presented in [61, 62].

The problem of finding a pair $\{p(t), u(x, y, t)\} \in C[0, T] \times C^{2,2,1}(\overline{D_T})$ satisfying the equation (1.8), initial condition (1.10), boundary conditions (1.11) and the overdetermination condition

$$\int_0^1 \int_0^1 u(x, y, t) dx dy = E(t). \tag{1.12}$$

will be called an inverse problem.

The thesis is organized as follows. In Chapter 2, we recall some definitions,

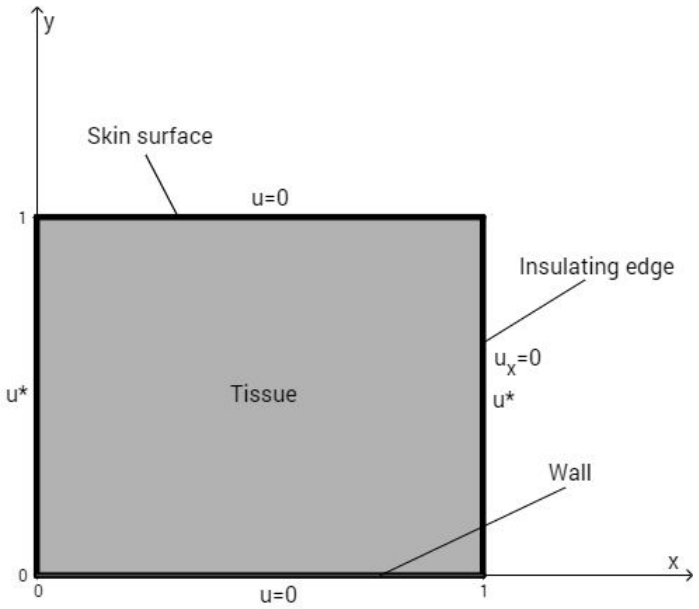


Figure 1.1: Mathematical model for perfusion determination in D_T.

some necessary and preliminary results. In Chapter 3, we consider a 1D heat equation with Wentzell-Neumann boundary condition. In subsections, we recall the eigenvalues and their corresponding eigenfunctions of the spectral problem and some of their properties. The existence and uniqueness of the solution of inverse IBVP (1.1)-(1.6) is shown by the generalized Fourier method combined with the unique solvability of a second kind Volterra integral equation. We give the numerical methods of the inverse problem both on a uniform and on a non-uniform grid of the interval $[0, 1]$. Two numerical examples will be provided to investigate the efficiency and the stability of the methods with respect to the coefficient of diffusive transport to the boundary. We also compare the performance of the numerical methods on these examples. In Chapter 4, we consider the perfusion coefficient determination in a 2D bioheat transfer process. In subsections, we recall some necessary results on basisness of root functions concerning the two-dimensional spectral problem with Ionkin-type boundary condition, and the well-posedness of inverse problem (1.8)-(1.12) for global T is shown by using the generalized Fourier method combined with the unique solvability of the second kind of Volterra integral equation. The numerical methods for solving the

inverse problem (1.8)-(1.12) by applying uniform and non-uniform finite difference with suitable numerical integration rules are given. We also present several numerical examples intended to illustrate the behaviour of the proposed methods. The simulations are performed by a code written by us using MATLAB. Finally in Chapter 5, some concluding remarks with the comparison of two different numerical methods on uniform and non-uniform grids in 1D and in 2D are presented, respectively.

2. BASIC DEFINITIONS AND PRELIMINARY RESULTS

In this chapter, we give some basic definitions and preliminary results that we will use for the following chapters. At first, we give some inequalities, a theorem and a definition which we use for the theoretical part of this thesis.

Definition 2.1: [56] Let H be a Hilbert space. A collection of vectors x_k is a Riesz basis in H if it is the image of an orthonormal basis under an invertible linear transformation T, i.e., $Te_k = x_k$ for all k.

Lemma 2.1: [57] Any two elements x, y of an inner product space satisfy the Schwarz inequality

$$|\langle x, y\rangle| \leq \|x\|\|y\|, \tag{2.1}$$

where

$$\|x\| = \sqrt{\langle x, x\rangle}, \qquad \|y\| = \sqrt{\langle y, y\rangle}. \tag{2.2}$$

Lemma 2.2 (Bessel inequality, [58]). Let (e_k) be an orthonormal sequence in an inner product space X. Then for every $x \in X$

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2. \tag{2.3}$$

We now get the following existence and uniqueness theorem of the Volterra integral equation given by $a \leq \tau \leq t$ and $a \leq t \leq b$

$$x(t) - \mu \int_a^t k(t, \tau)x(\tau)d\tau = v(t), \tag{2.4}$$

without any restriction on μ. Here, $[a, b]$ is a given interval. x is an unknown function on $[a, b]$. μ is a parameter. The kernel k of the equation is a given function on the square $[a, b] \times [a, b]$, and v is a given function on $[a, b]$. The proof of the following theorem about the uniqueness of the solution of (2.4) is given in [58].

Theorem 2.3: Suppose that v *in* (2.4) *is continuous on* $[a,b]$ *and the kernel* k *is continuous on the triangular region in the* $t\tau-$*plane given by* $a \leq \tau \leq t$, $a \leq t \leq b$. *Then* (2.4) *has a unique continuous solution* x *on* $[a,b]$ *for every* μ.

Now, we give some formulations for numerical part of this thesis.

Definition 2.2: [59] Let $f \in C^2[a,b]$, $h = (b-a)/n$, *and* $x_j = a + jh$, *for each* $j = 0, 1, \ldots, n$. *There exists a* $\xi \in (a,b)$ *for which the composite trapezoidal rule for* n *subintervals can be written with its error term as*

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12}h^2 f''(\xi). \tag{2.5}$$

Definition 2.3: [59] Let $f \in C^4[a,b]$, n *be even,* $h = (b-a)/n$, *and* $x_j = a + jh$, *for each* $j = 0, 1, \ldots, n$. *There exists a* $\xi \in (a,b)$ *for which the composite Simpson's rule for* n *subintervals can be written with its error term as*

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right] - \frac{b-a}{180}h^4 f^{(4)}(\xi). \tag{2.6}$$

Now, let $R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$ be a rectangular domain for some constants a, b, c, d and consider the double integral

$$\iint_R f(x,y)dydx. \tag{2.7}$$

If we apply the composite Simpson's rule using two subintervals in each coordinate direction to this integral, we get the following *product composite Simpson's rule* [59]:

$$
\begin{aligned}
\iint_R f(x,y)dydx \approx \frac{hk}{9}\Bigg\{ & \Bigg[f(x_0,y_0) + 2\sum_{i=1}^{n/2-1} f(x_{2i},y_0) \\
& \quad + 4\sum_{i=1}^{n/2} f(x_{2i-1},y_0) + f(x_n,y_0) \Bigg] \\
& + 2\Bigg[\sum_{j=1}^{m/2-1} f(x_0,y_{2j}) + 2\sum_{j=1}^{m/2-1}\sum_{i=1}^{n/2-1} f(x_{2i},y_{2j}) \\
& \quad + 4\sum_{j=1}^{m/2-1}\sum_{i=1}^{n/2} f(x_{2i-1},y_{2j}) + \sum_{j=1}^{m/2-1} f(x_n,y_{2j}) \Bigg] \\
& + 4\Bigg[\sum_{j=1}^{m/2} f(x_0,y_{2j-1}) + 2\sum_{j=1}^{m/2}\sum_{i=1}^{n/2-1} f(x_{2i},y_{2j-1}) \\
& \quad + 4\sum_{j=1}^{m/2}\sum_{i=1}^{n/2} f(x_{2i-1},y_{2j-1}) + \sum_{j=1}^{m/2} f(x_n,y_{2j-1}) \Bigg] \\
& + \Bigg[f(x_0,y_m) + 2\sum_{i=1}^{n/2-1} f(x_{2i},y_m) \\
& \quad + 4\sum_{i=1}^{n/2} f(x_{2i-1},y_m) + f(x_n,y_m) \Bigg] \Bigg\},
\end{aligned}
\tag{2.8}
$$

by letting $x_i = a + ih$ and $y_j = b + jk$ with $h = (b-a)/n$ and $k = (d-c)/m$ for each $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, m$, respectively. And the error term is given by

$$
E = -\frac{(d-c)(b-a)}{180}\left[h^4 \frac{\partial^4 f}{\partial x^4}(\overline{\eta},\overline{\xi}) + k^4 \frac{\partial^4 f}{\partial y^4}(\hat{\eta},\hat{\xi}) \right], \tag{2.9}
$$

for some $(\overline{\eta},\overline{\xi})$ and $(\hat{\eta},\hat{\xi})$ in R.

Up to now, we give some formulas about what we use on a uniform grid for numerical integration. Now, we give the formula of Gauss-Lobatto rule for numerical integration on a non-uniform grid.

Definition 2.4: [30] Let $f \in C^{2n-2}[-1,1]$ *and* $-1 = x_1 < x_2 < \cdots < x_n = 1$. *There exists a* $\nu \in (-1,1)$ *for which the Gauss-Lobatto integration formula can be written with its error term as*

$$\int_{-1}^{1} f(x)dx = \frac{2}{n(n-1)}[f(-1)+f(1)] + \sum_{i=2}^{n-1} w_i f(x_i) - \frac{n(n-1)^3 2^{2n-1}[(n-2)!]^4}{(2n-1)[(2n-2)!]^3} f^{(2n-2)}(\nu), \tag{2.10}$$

where x_i *are the roots of the polynomial* $P'_{n-1}(x)$, *(*$P_{n-1}(x)$ *is Legendre polynomial of degree* $n-1$*), and the weights*

$$w_i = \frac{2}{n(n-1)[P_{n-1}(x_i)]^2}, \tag{2.11}$$

for $i = 2, \ldots, n-1$.

We use Gauss-Lobatto rule instead of the other Gauss quadrature rules because of the advantage of using the boundaries of the spatial axis as nodes. Therefore this rule has the higher degree of precision.

3. 1D HEAT EQUATION WITH WENTZELL - NEUMANN BOUNDARY CONDITION

In this section, the Fourier series analysis of the inverse problem of finding the coefficient of lowest term in the heat equation with a nonlocal Wentzell-Neumann boundary and integral overdetermination conditions is presented [60]. the existence and uniqueness of the classical solution is shown by using the generalized Fourier method under some regularity, consistency and orthogonality conditions on the data, and additional conditions on the sign of the Fourier coefficients of the initial data, and known part of source term. Numerical solutions of the inverse problem are given both on a uniform grid and on a non-uniform grid with uniform finite difference method combined with the composite trapezoidal rule and with non-uniform finite difference method combined with Gauss-Lobatto quadrature, respectively. Two numerical examples (one is smooth while the other is non-smooth) will be provided to investigate the efficiency and the stability of the methods with respect to the coefficient of diffusive transport to the boundary.

3.1. Well posedness of the inverse problem

At first, we give the spectral problem of the inverse problem in 1D. Then, we give the existence and uniqueness of the solution and its continuous dependence upon the data.

3.1.1. Spectral problem

The auxiliary spectral problem of the IBVP considered in the present chapter is

$$\begin{cases} -y''(x) = \lambda y(x),\ 0 \le x \le 1, \\ y(0) = 0,\ y'(0) - \alpha\lambda y(1) = 0. \end{cases} \tag{3.1}$$

We consider the spectral problem given above with $\alpha \neq \frac{1}{x_i \sin x_i}$ where x_i on $(0, +\infty)$ are the roots of $\sin x + x \cos x = 0$. We know that this problem has at most finitely many complex eigenvalues, and their numbers are dependent on α [27, 28].

Here, the problem (3.1) has only the eigenfunctions

$$X_n(x) = \sqrt{2}\sin\left(\sqrt{\lambda_n}x\right),\ n = 0, 1, 2, ... \tag{3.2}$$

where the eigenvalues $\lambda_n, n = 0, 1, 2, ...$ satisfy the equation $\alpha\sqrt{\lambda}\sin\sqrt{\lambda} = 1,$ with $\mathrm{Re}\,\sqrt{\lambda} > 0.$

Let $\lambda_n, n = 0, 1, 2, ..., n_\alpha$ be complex eigenvalues, $X_n(x), n = 0, 1, 2, ..., n_\alpha$ be the corresponding eigenfunctions.

The asymptotic estimate for the eigenvalues

$$\sqrt{\lambda_n} = \pi n + \frac{(-1)^n}{\pi\alpha n} + O(\frac{1}{n^3}) \tag{3.3}$$

is valid for large $n.$

By [27], the system of eigenfunctions $X_n(x),\ n = 1, 2, \ldots,$ with anyone of them is deleted, is a Riesz basis in $L_2[0, 1]$ and the system

$$Y_n(x) = \sqrt{2}\frac{\overline{\sqrt{\lambda_0}}\sin\overline{\sqrt{\lambda_0}}(1-x) - \overline{\sqrt{\lambda_n}}\sin\overline{\sqrt{\lambda_n}}(1-x)}{\overline{\sqrt{\lambda_n}}\cos\overline{\sqrt{\lambda_n}} + \sin\overline{\sqrt{\lambda_n}}},\ n = 1, 2, ... \tag{3.4}$$

is a bi-orthogonal to the system $X_n(x),\ n = 1, 2, ...,$ i.e.

$$(X_n, Y_m) = \int_0^1 X_n(x)Y_m(x)dx = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases} \tag{3.5}$$

Definition 3.1: The pair $\{p(t), u(x, t)\}$ *from the class* $C[0, T]\times(C^{2,1}(D_T)\cap C^{2,0}(\bar{D}_T))$ *for which the conditions* (1.1)–(1.4) *are satisfied, is called a classical solution of the inverse problem* (1.1)–(1.6).

The uniform convergence of the Fourier series expansion in the system $X_n(x),$ $n = 1, 2, ...$ is important, since we are considering the classical solution of the inverse problem (1.1)–(1.6).

Lemma 3.1: (Theorem 1, [27]) Suppose that a function $\varphi \in C[0, 1]$ *has a uni-*

formly convergent Fourier series expansion in the system $\sin(\pi n x),\ n = 1, 2, ...$ *on* $[0, 1]$. *Then, this function can be expanded in a Fourier series in the system* $X_n(x),\ n = 0, 1, 2, ...$ *and this expansion is uniformly convergent on the interval [0,1] if* $\left(\varphi, \sin\sqrt{\lambda_0}(1-x)\right) = 0$.

We choose the method of annihilation of the these terms of Fourier series expansions by the conditions $\int_0^1 \varphi(x)\sin(\sqrt{\lambda_n}(1-x))dx = 0,\ n = 0, 1, 2, ..., n_\alpha$, since the temperature distribution $u(x,t)$ is real valued, and only a finite number of the eigenvalues are complex. The class of functions that also contains the conditions of Lemma 1 will be denoted by

$$C^3_{n_\alpha}[0,1] \equiv \left\{ \begin{array}{c} \varphi(x) \in C^3[0,1] :\ \varphi(0) = \varphi'(0) = \varphi''(0) = 0, \\ \varphi(1) = \varphi''(1) = 0, \\ \int_0^1 \varphi(x)\sin(\sqrt{\lambda_n}(1-x))dx = 0,\ n = 0, 1, 2, ..., n_\alpha. \end{array} \right\} \tag{3.6}$$

3.1.2. Existence-uniqueness and continuous dependence upon the data of the solution

Let $p(t),\ t \in [0, T]$ be an unknown function. The smoothness conditions $f(x,t) \in C\left(\overline{D}_T\right),\ \varphi(x) \in C^2[0,1],\ E(t) \in C[0,T]$ and the consistency conditions

$$\varphi(0) = 0,\ \varphi'(0) + \alpha\varphi''(1) = 0,\ E(0) = \int_0^1 \varphi(x)dx \tag{3.7}$$

are necessary for the existence of a classical solution of (1.1)–(1.6).

For arbitrary $p(t) \in C[0, T]$, we use the generalized Fourier method to construct the formal solution of the problem (1.1)–(1.4). Similarly to this method, the solution $u(x,t)$ is aimed to be obtained in a Fourier series in terms of the eigenfunctions and associated functions $\{X_n(x)\}_{n=1}^{\infty}$ of the auxiliary spectral problem:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)X_n(x). \tag{3.8}$$

We obtain the Cauchy problems for the functions $u_n(t), n = 1, 2, ...,$

$$u_n'(t) = -\lambda_n u_n(t) + p(t)u_n(t) + f_n(t),\ u_n(0) = \varphi_n,\ n = 1, 2, ..., \tag{3.9}$$

where $f_n(t) = (f, Y_n) = \int\limits_0^1 f(x,t)Y_n(x)dx,\ \varphi_n = (\varphi, Y_n) = \int\limits_0^1 \varphi(x)Y_n(x)dx,\ n = 1, 2, ...$

We get the following by solving these Cauchy problems,

$$u_n(t) = \varphi_n e^{-\lambda_n t + \int\limits_0^t p(s)ds} + \int\limits_0^t f_n(\tau) e^{-\lambda_n (t-\tau) + \int\limits_\tau^t p(s)ds} d\tau. \tag{3.10}$$

Under the conditions that $\varphi(x) \in C_{n_\alpha}^3$ and $f(x,t) \in C_{n_\alpha}^3, \forall t \in [0,T]$, in particular,

$$\int_0^1 \varphi(x)\sin(\sqrt{\lambda_n}(1-x))dx = 0, \quad \int_0^1 f(x,t)\sin(\sqrt{\lambda_n}(1-x))dx = 0, \tag{3.11}$$

the Fourier coefficients $\varphi_n = 0,\ f_n(t) = 0,\ n = 0, 1, 2, ..., n_\alpha$ and for $n > n_\alpha$,

$$\varphi_n = \sqrt{2}\frac{\sqrt{\lambda_n}}{\sqrt{\lambda_n}\cos\sqrt{\lambda_n} + \sin\sqrt{\lambda_n}} \int_0^1 \varphi(x)\sin\left(\sqrt{\lambda_n}(x-1)\right)dx, \tag{3.12}$$

are real numbers and

$$f_n(t) = \sqrt{2}\frac{\sqrt{\lambda_n}}{\sqrt{\lambda_n}\cos\sqrt{\lambda_n} + \sin\sqrt{\lambda_n}} \int_0^1 f(x,t)\sin\left(\sqrt{\lambda_n}(x-1)\right)dx, \tag{3.13}$$

are real valued functions.

The following series is the formal solution of the problem (1.1)–(1.4)

$$u(x,t) = \sum_{n=n_\alpha+1}^{\infty} \left[\varphi_n e^{-\lambda_n t + \int\limits_0^t p(s)ds} + \int\limits_0^t f_n(\tau) e^{-\lambda_n (t-\tau) + \int\limits_\tau^t p(s)ds} d\tau\right] X_n(x). \tag{3.14}$$

Lemma 3.2: If $\varphi(x) \in C_{n_\alpha}^3[0,1]$ *then the inequality*

$$\sum_{n=n_\alpha+1}^{\infty} |\lambda_n \varphi_n| \le c \left\|\varphi'''\right\|_{L_2[0,1]}^2,\ c = const > 0, \tag{3.15}$$

holds, where $\varphi_n = (\varphi, Y_n)$.

Proof The following equality holds

$$\varphi_n = -\frac{\sqrt{2}}{\lambda_n\sqrt{\lambda_n}}\frac{\sqrt{\lambda_n}}{\sqrt{\lambda_n}\cos\sqrt{\lambda_n}+\sin\sqrt{\lambda_n}}\int_0^1 \varphi'''(1-s)\cos\left(\sqrt{\lambda_n}s\right)ds \qquad (3.16)$$

by three times integrating by parts since $\varphi(0)=\varphi'(0)=\varphi''(0)=0$, $\varphi(1)=\varphi''(1)=0$,. *The eigenfunctions of the differential operator generated by* $l(y)=-y''$ *and regular boundary conditions*

$$y'(0)=0,\ y(0)+\alpha y(1)=0, \qquad (3.17)$$

are $\sqrt{2}\cos\left(\sqrt{\lambda_n}x\right)$, $n=1,2,....$ *Here, the set of these eigenfunctions and associated functions is a Riesz basis in the space* $L_2[0,1]$ *[1]. So this implies the following Bessel type inequality*

$$\sum_{n=n_\alpha+1}^{\infty}\left|(h,\sqrt{2}\cos\left(\sqrt{\lambda_n}x\right))\right|^2 \le c\left\|h\right\|^2_{L_2[0,1]} \qquad (3.18)$$

for each $h(x)\in L_2[0,1]$.

We obtain from the earlier discussion by using the Schwarz and Bessel inequalities

$$\begin{aligned}\sum_{n=n_\alpha+1}^{\infty}|\lambda_n\varphi_n| &\le \sum_{n=n_\alpha+1}^{\infty}\left|\mu_n\int_0^1\varphi'''(1-s)\sqrt{2}\cos\left(\sqrt{\lambda_n}s\right)ds\right| \\ &\le \left[\sum_{n=n_\alpha+1}^{\infty}|\mu_n|^2\right]^{\frac{1}{2}}\left[\sum_{n=n_\alpha+1}^{\infty}\left|\int_0^1\varphi'''(1-s)\sqrt{2}\cos\left(\sqrt{\lambda_n}s\right)ds\right|^2\right]^{\frac{1}{2}} \qquad (3.19)\\ &\le C_1\left\|\varphi'''\right\|_{L_2(0,1)} \le C_1\left\|\varphi\right\|_{C^3[0,1]},\end{aligned}$$

for some constant C_1, *where* $\mu_n=\frac{1}{\sqrt{\lambda_n}\cos\sqrt{\lambda_n}+\sin\sqrt{\lambda_n}}$ *and* $\sum_{n=1}^{\infty}|\mu_n|^2<+\infty$, *since* $\mu_n\sim\frac{1}{\pi n}$, $n\to+\infty$. ■

The followings are the assumptions for the main result.

On φ, E and f, we have the following assumptions.

(A_1) $(\mathrm{A}_1)_1$ $\varphi(x)\in C^3_{n_\alpha}[0,1]$;
$(\mathrm{A}_1)_2$ $\varphi_{n_\alpha+1}>0,\ \varphi_{n_\alpha+k}\ge 0,\ k=2,3,\ldots,$

$$
\begin{array}{ll}
(\mathrm{A}_2) & (\mathrm{A}_2)_1\ E(t)\in C^1[0,T]\,;\ E(0)=\int\limits_0^1\varphi(x)dx;\\
& (\mathrm{A}_2)_2\ E(t)>0,\forall t\in[0,T]\,;\\
(\mathrm{A}_3) & (\mathrm{A}_3)_1\ f(x,t)\in C\left(\overline{D}_T\right);\ f(x,t)\in C^3_{n_\alpha}[0,1],\ \forall t\in[0,T]\,;\\
& (\mathrm{A}_3)_2\ f_{n_\alpha+k}(t)\geq 0,\ k=1,2,\ldots,
\end{array}
$$

where $\varphi_{n_\alpha+k}=\int\limits_0^1\varphi(x)Y_{n_\alpha+k}(x)dx,\ f_{n_\alpha+k}(t)=\int\limits_0^1 f(x,t)Y_{n_\alpha+k}(x)dx,\ k=1,2,....$

The main results are presented as follows.

Theorem 3.3: (Existence and uniqueness) Let $(A_1)-(A_3)$ *be satisfied. Then the inverse problem* (1.1)–(1.6) *has a unique classical solution pair* $\{p(t),\ u(x,t)\}$ *as* $u(x,t)\in C^{2,1}\left(\overline{\Omega}_T\right)$.

Proof For arbitrary $p(t)\in C[0,T]$*, to be constructed the formal solution of the problem* (1.1)–(1.4) *we will use generalized Fourier method as* (3.14)*:*

$$u(x,t)=\sum_{n=n_\alpha+1}^{\infty}u_n(t)X_n(x). \tag{3.20}$$

where

$$u_n(t)=\varphi_n e^{-\lambda_n t+\int\limits_0^t p(s)ds}+\int\limits_0^t f_n(\tau)e^{-\lambda_n(t-\tau)+\int\limits_\tau^t p(s)ds}d\tau. \tag{3.21}$$

Under the conditions $(A_1)_1$ *and* $(A_3)_1$*, since their majorizing sums are absolutely convergent by using Lemma 2, the series* (3.20)*, its* t*-partial derivative and the* xx*-second order partial derivative series are uniformly convergent in* $\overline{\Omega}_T$*. Therefore, their sums* $u(x,t),\ u_t(x,t)$ *and* $u_{xx}(x,t)$ *are continuous in* $\overline{\Omega}_T$*. Thus,* $u(x,t)\in C^{2,1}\left(\overline{\Omega}_T\right)$ *and satisfies the condition* (1.1)–(1.4) *for arbitrary* $p(t)\in C[0,T]$*. Besides, the series* (3.20) *is termwise integrable. Applying the overdetermination condition* (1.6)*, we obtain the following Volterra integral equation of the second kind with respect to* $q(t)=e^{-\int\limits_0^t p(s)ds}$*:*

$$q(t)=F(t)+\int\limits_0^t K(t,\tau)q(\tau)d\tau, \tag{3.22}$$

where

$$F(t)=\frac{1}{E(t)}\left(\sum_{n=n_\alpha+1}^{\infty}\left[\frac{\sqrt{2}}{\sqrt{\lambda_n}}(1-\cos\sqrt{\lambda_n})\varphi_n\right]e^{-\lambda_n t}\right),$$
$$K(t,\tau)=\frac{1}{E(t)}\left(\sum_{n=n_\alpha+1}^{\infty}\left[\frac{\sqrt{2}}{\sqrt{\lambda_n}}(1-\cos\sqrt{\lambda_n})f_n(\tau)\right]e^{-\lambda_n (t-\tau)}\right). \tag{3.23}$$

Thus

$$\int_0^1 X_n(x)dx=\frac{\sqrt{2}}{\sqrt{\lambda_n}}(1-\cos\sqrt{\lambda_n}). \tag{3.24}$$

In the case of the existence of the positive solution of (3.22) *in class* $C^1[0,T]$*, the function* $p(t)$ *can be determined from* $q(t)=e^{-\int_0^t p(s)ds}$ *as*

$$p(t)=-\frac{q'(t)}{q(t)}. \tag{3.25}$$

By using Lemma 1, the right-hand side of $F(t)$ *and the kernel* $K(t,\tau)$ *are continuously differentiable functions under the assumptions* $(A_1)_1$ *and* $(A_3)_1$ *in* $[0,T]$ *and* $[0,T]\times[0,T]$*, respectively. Additionally, according to the assumptions* $(A_1)_2-(A_3)_2$ *the conditions* $F(t)>0$ *and* $K(t,\tau)\geq 0$ *are satisfied in* $[0,T]$ *and* $[0,T]\times[0,T]$*, respectively.*

Therefore, we obtain a unique, positive and continuously differentiable function $q(t)$ *in* $[0,T]$*, which the function* (3.25) *together with the solution of the problem* (1.1)–(1.4) *given by the Fourier series* (3.14)*, form the unique solution of the inverse problem* (1.1)–(1.6). ■

Theorem 3.4: Let $\Im$ *be the class of triples in the form of* $\{f,\ \varphi,\ E\}$ *which satisfy the assumptions* $(A_1)-(A_3)$ *and*

$$\|f\|_{C^{3,0}(\overline{D}_T)}\leq N_0,\ \|\varphi\|_{C^3[0,1]}\leq N_1,\ \|E\|_{C^1[0,T]}\leq N_2,$$
$$0<N_3\leq\min_{t\in[0,T]}|E(t)|, \tag{3.26}$$

for some positive constants $N_i,\ i = 0,\ 1,\ 2,\ 3$. *Then the solution pair* (p, u) *of the inverse problem* (1.1)–(1.6) *continuously depends upon the data in* $\Im$.

Proof Take the norm $\|\Phi\| = \|f\|_{C^{3,0}(\overline{D}_T)} + \|\varphi\|_{C^3[0,1]} + \|E\|_{C^1[0,T]}$. *Let* $\Phi = \{f, \varphi, E\}$, $\overline{\Phi} = \{\overline{f}, \overline{\varphi}, \overline{E}\} \in \Im$ *be two sets of data and* (p, u) *and* $(\overline{p}, \overline{u})$ *be the solutions of the inverse problem* (1.1)-(1.6) *corresponding the data* Φ *and* $\overline{\Phi}$, *respectively. According to* (3.20) *and* (3.22), *we have*

$$q(t) = F(t) + \int_0^t K(t,\tau)q(\tau)d\tau, \quad \textit{and} \quad \overline{q}(t) = \overline{F}(t) + \int_0^t \overline{K}(t,\tau)\overline{q}(\tau)d\tau, \tag{3.27}$$

where

$$q(t) = e^{-\int_0^t p(s)ds}, \qquad \overline{q}(t) = e^{-\int_0^t \overline{p}(s)ds}, \tag{3.28}$$

$$\begin{aligned}
F(t) &= \frac{1}{E(t)}\left(\sum_{n=n_\alpha+1}^{\infty}\left[\frac{\sqrt{2}}{\sqrt{\lambda_n}}(1-\cos\sqrt{\lambda_n})\varphi_n\right]e^{-\lambda_n t}\right), \\
K(t,\tau) &= \frac{1}{E(t)}\left(\sum_{n=n_\alpha+1}^{\infty}\left[\frac{\sqrt{2}}{\sqrt{\lambda_n}}(1-\cos\sqrt{\lambda_n})f_n(\tau)\right]e^{-\lambda_n (t-\tau)}\right),
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
\overline{F}(t) &= \frac{1}{\overline{E}(t)}\left(\sum_{n=n_\alpha+1}^{\infty}\left[\frac{\sqrt{2}}{\sqrt{\lambda_n}}(1-\cos\sqrt{\lambda_n})\overline{\varphi}_n\right]e^{-\lambda_n t}\right), \\
\overline{K}(t,\tau) &= \frac{1}{\overline{E}(t)}\left(\sum_{n=n_\alpha+1}^{\infty}\left[\frac{\sqrt{2}}{\sqrt{\lambda_n}}(1-\cos\sqrt{\lambda_n})\overline{f}_n(\tau)\right]e^{-\lambda_n (t-\tau)}\right).
\end{aligned} \tag{3.30}$$

The following inequalities are true by using the Schwarz and Bessel inequalities:

$$\begin{aligned}
|F(t)| &\le \frac{4}{|E(t)|}\sum_{n=n_\alpha+1}^{\infty}\left|\mu_n\int_0^1 \varphi(x)\sin\left(\sqrt{\lambda_n}(x-1)\right)dx\right| \\
&\le \frac{4C_1}{N_3}\|\varphi\|_{C^3[0,1]} \le \frac{4C_1}{N_3}N_1,
\end{aligned} \tag{3.31}$$

$$
\begin{aligned}
|\overline{K}(t,\tau)| &\leq \frac{4}{|E(t)|}\sum_{n=n_\alpha+1}^{\infty}\left|\mu_n\int_0^1 f(x,\tau)\sin\left(\sqrt{\lambda_n}(x-1)\right)dx\right| \\
&\leq \frac{4C_1}{N_3}\|f\|_{C^{3,0}(\overline{D}_T)} \leq \frac{4C_1}{N_3}N_0,
\end{aligned}
\tag{3.32}
$$

where C_1 is the constant in Lemma 3.2.

Let us estimate the difference $q-\overline{q}$, firstly. From (3.27)*, we have the following*

$$
\begin{aligned}
q(t)-\overline{q}(t) = F(t)-\overline{F}(t) &+ \int_0^t [K(t,\tau)-\overline{K}(t,\tau)]q(\tau)d\tau \\
&+ \int_0^t \overline{K}(t,\tau)[q(\tau)-\overline{q}(\tau)]d\tau.
\end{aligned}
\tag{3.33}
$$

Let $\beta = \|F-\overline{F}\|_{C[0,T]} + T\|K-\overline{K}\|_{C([0,T]\times[0,T])}\|q\|_{C[0,T]}$ and $Q(t)=|q(t)-\overline{q}(t)|$. Then (3.33) *implies*

$$
Q(t) \leq \beta + \int_0^t |\overline{K}(t,\tau)|Q(\tau)d\tau. \tag{3.34}
$$

Applying Gronwall inequality in [29] to (3.34)*, we get*

$$
Q(t) \leq \beta e^{\int_0^t \sup_{s\in[\tau,t]}\overline{K}(s,\tau)d\tau}. \tag{3.35}
$$

Therefore, using (3.31) *and* (3.32)*, we obtain the following*

$$
\|q-\overline{q}\|_{C[0,T]} \leq C_2\left(\|F-\overline{F}\|_{C[0,T]} + C_2\|K-\overline{K}\|_{C([0,T]\times[0,T])}\right), \tag{3.36}
$$

where $C_2 = e^{(4C_1/N_3)N_0T}$ and $C_3 = (4C_1T/N_3)N_1C_2$, since (from Gronwall inequality) $\|q\|_{C[0,T]} \leq \|F\|_{C[0,T]}e^{T\|K\|_{C([0,T]\times[0,T])}}$. Thus, q depends continuously upon F and K.

Now, let us show that F and K depend continuously upon the data. Taking into account the inequality in Lemma 3.2, we get

$$
\begin{aligned}
|F(t)-\overline{F}(t)| &= \left| \sum_{n=n_\alpha+1}^{\infty} \left[\frac{\sqrt{2}}{\sqrt{\lambda_n}}(1-\cos\sqrt{\lambda_n}) \left(\frac{\varphi_n}{E(t)} - \frac{\overline{\varphi}_n}{\overline{E}(t)} \right) \right] e^{-\lambda_n t} \right| \\
&\leq 4 \sum_{n=n_\alpha+1}^{\infty} \left| \mu_n \int_0^1 \left(\frac{\varphi}{E(t)} - \frac{\overline{\varphi}}{\overline{E}(t)} \right) \sin\left(\sqrt{\lambda_n}(x-1) \right) dx \right| \qquad (3.37) \\
&\leq \frac{4C_1}{N_3} \|\varphi - \overline{\varphi}\|_{C^3[0,1]} + \frac{4C_1 N_1}{N_3^2} \|E - \overline{E}\|_{C^1[0,1]},
\end{aligned}
$$

$$
\begin{aligned}
&|K(t,\tau)-\overline{K}(t,\tau)| \\
&\leq 4 \sum_{n=n_\alpha+1}^{\infty} \left| \mu_n \int_0^1 \left(\frac{f(x,\tau)}{E(t)} - \frac{\overline{f}(x,\tau)}{\overline{E}(t)} \right) \sin\left(\sqrt{\lambda_n}(x-1) \right) dx \right| \qquad (3.38) \\
&\leq \frac{4C_1}{N_3} \|f - \overline{f}\|_{C^{3,0}(\overline{D}_T)} + \frac{4C_1 N_0}{N_3^2} \|E - \overline{E}\|_{C^1[0,1]}.
\end{aligned}
$$

These inequalities imply that

$$
\begin{aligned}
\|F - \overline{F}\|_{C[0,T]} &\leq M_1 (\|f - \overline{f}\|_{C^{3,0}(\overline{D}_T)} + \|\varphi - \overline{\varphi}\|_{C^3[0,1]} + \|E - \overline{E}\|_{C^1[0,T]}) \qquad (3.39) \\
&\leq M_1 \|\Phi - \overline{\Phi}\|,
\end{aligned}
$$

$$
\|K - \overline{K}\|_{C([0,T]\times[0,T])} \leq M_2 \|\Phi - \overline{\Phi}\|, \qquad (3.40)
$$

where M_1 *and* M_2 *are constants obtained from the constants* C_1 *and* N_k, $k = 0,1,2,3$. *Hence,* F *and* K *depend continuously upon the data, and so* q.

Finally, let us show that q' *also continuously depends upon the data. By differentiating* (3.27) *with respect to* t, *we get the followings:*

$$
\begin{aligned}
q'(t) &= F'(t) + K(t,t)q(t) + \int_0^t K_t(t,\tau)q(\tau)d\tau, \\
\overline{q}'(t) &= \overline{F}'(t) + \overline{K}(t,t)\overline{q}(t) + \int_0^t \overline{K}_t(t,\tau)\overline{q}(\tau)d\tau.
\end{aligned} \qquad (3.41)
$$

The following estimation holds:

$$\begin{aligned}\|q'-\overline{q}'\|_{C[0,T]} &\le \|F'-\overline{F}'\|_{C[0,T]}\\ &+\left(\|K-\overline{K}\|_{C([0,T]\times[0,T])}+T\|K_t-\overline{K}_t\|_{C([0,T]\times[0,T])}\right)\|q\|_{C[0,T]}\\ &+\left(\|\overline{K}\|_{C([0,T]\times[0,T])}+T\|\overline{K}_t\|_{C([0,T]\times[0,T])}\right)\|q-\overline{q}\|_{C[0,T]}.\end{aligned} \tag{3.42}$$

Since q, F and K depend continuously upon the data, using the following inequality

$$\begin{aligned}&|\overline{K}_t(t,\tau)|\\ &=\left|\frac{-1}{\overline{E}^2(t)}\left(\sum_{n=n_\alpha+1}^{\infty}\left[\frac{\sqrt{2}}{\sqrt{\lambda_n}}(1-\cos\sqrt{\lambda_n})\overline{f}_n(\tau)\right.\right.\right.\\ &\qquad\left.\left.\left.\left(\lambda_n\overline{E}(t)+\overline{E}'(t)\right)\right]e^{-\lambda_n(t-\tau)}\right)\right|\\ &\le\frac{4C_1}{N_3}\|\overline{f}\|_{C^{3,0}(\overline{D}_T)}+\frac{4C_1}{N_3^2}N_2\|\overline{f}\|_{C^{3,0}(\overline{D}_T)}\le 4C_1N_0\left(\frac{1}{N_3}+\frac{N_2}{N_3^2}\right).\end{aligned} \tag{3.43}$$

Therefore, q' continuously depends upon F' and K_t. We can obtain the following estimates by using similar operations:

$$\begin{aligned}&\|F'-\overline{F}'\|_{C[0,T]}\\ &\le 4C_1\left(\frac{N_2+N_3}{N_3^2}\right)\|\varphi-\overline{\varphi}\|_{C^3[0,1]}+8C_1N_1\left(\frac{N_2^2+N_3^2}{N_3^4}\right)\|E-\overline{E}\|_{C^1[0,1]},\end{aligned} \tag{3.44}$$

$$\begin{aligned}&\|K_t-\overline{K}_t\|_{C([0,T]\times[0,T])}\\ &\le 4C_1\left(\frac{N_2+N_3}{N_3^2}\right)\|f-\overline{f}\|_{C^{3,0}(\overline{D}_T)}\\ &\qquad+8C_1N_0\left(\frac{N_2^2+N_3^2}{N_3^4}\right)\|E-\overline{E}\|_{C^1[0,1]}.\end{aligned} \tag{3.45}$$

Thus, using the last inequalities, from (3.42)*, we get*

$$\|q'-\overline{q}'\|_{C[0,T]}\le M_3\|\Phi-\overline{\Phi}\|, \tag{3.46}$$

where M_3 is a constant in terms of C_1 and N_k, $k=0,1,2,3$. This means q' also

continuously depends upon the data.

From (3.25)*, we see that* p *continuously depends upon the data. The continuously dependence upon the data of* u *can be showed similarly.* ■

Remark 3.5: The case in which $\alpha = \frac{1}{x_0 \sin x_0}$ *where* x_0 *is one of the positive root of the equation* $\sin x + x \cos x = 0$*, the spectral problem has the eigenfunctions as well as one associated function corresponding to eigenvalue* $\lambda = x_0$*. The entire system of eigenfunctions and corresponding functions, unlike to the previous case, is a Riesz basis in* $L_2[0, 1]$*, [27].*

3.2. Numerical methods for the inverse problem

Theorem 1 and Remark 2 establish that the inverse problem under investigation given by Eqs. (1.1)–(1.6) is well-posed in appropriate spaces of regular functions. However, in practice the input data, especially the measured one, such as the energy (1.6), is non-smooth and hence, the solution of the inverse problem needs some alternative investigation.

We consider the numerical methods of the inverse problem (1.1)–(1.6) for not only uniform but also non-uniform grids on spatial axis. For the convenience of the numerical discussion of the problem, we rewrite the equations as follows:

$$v_t = v_{xx} + r(t)f(x,t), \quad (x,t) \in \Omega_T, \tag{3.47}$$

$$v(x,0) = \varphi(x), \quad x \in [0,1], \tag{3.48}$$

$$v(0,t) = 0, \quad t \in [0,T], \tag{3.49}$$

$$v_x(0,t) + \alpha v_{xx}(1,t) = 0, \quad t \in [0,T], \tag{3.50}$$

$$\int_0^1 v(x,t)dx = E(t)r(t), \quad t \in [0,T], \tag{3.51}$$

by using transformations

$$r(t) = e^{-\int_0^t p(\tau)d\tau} \tag{3.52}$$

$$v(x,t) = u(x,t)r(t). \tag{3.53}$$

At first, we give the numerical method of the problem (3.47)–(3.51) with Crank-Nicholson implicit method and the numerical integration of (3.51) by using the transformed Gauss-Lobatto nodes [30] from the unit square on non-uniform grids [31]. Moreover, we give Crank-Nicholson implicit finite difference discretization for the numerical method of the same problem with composite trapezoidal rule for numerical integration of (3.51) [32].

3.2.1. Non-uniform finite difference method

We use Crank-Nicholson method for the problem (3.47)–(3.51) on a non-uniform grid of the interval $[0,1]$. Here, we choose the non-uniform grid considering the transformed Gauss-Lobatto nodes from the unit square which is also used for numerical integration of (3.51). We use a fictitious point x_{-1} in order to deal with the boundary condition (3.50).

$$\begin{aligned}\frac{v_i^{j+1}-v_i^j}{k} =& \frac{v_{i-1}^j + v_{i-1}^{j+1}}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})} + \frac{v_i^j + v_i^{j+1}}{(x_i - x_{i-1})(x_i - x_{i+1})} \\ &+ \frac{v_{i+1}^j + v_{i+1}^{j+1}}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)} + \quad \frac{1}{4}(r^j + r^{j+1})(f_i^j + f_i^{j+1}),\end{aligned} \tag{3.54}$$

(if we denote $h_i = x_i - x_{i-1}$,)

$$\begin{aligned}&-\frac{k}{h_i(h_i+h_{i+1})}v_{i-1}^{j+1} + \left(1 + \frac{k}{h_i h_{i+1}}\right)v_i^{j+1} - \frac{k}{h_{i+1}(h_i+h_{i+1})}v_{i+1}^{j+1} \\ &\qquad - \frac{k}{4}(f_i^j + f_i^{j+1})r^{j+1} \\ &= \frac{k}{h_i(h_i+h_{i+1})}v_{i-1}^{j} + \left(1 - \frac{k}{h_i h_{i+1}}\right)v_i^{j} + \frac{k}{h_{i+1}(h_i+h_{i+1})}v_{i+1}^{j} \\ &\qquad + \frac{k}{4}(f_i^j + f_i^{j+1})r^{j},\end{aligned} \tag{3.55}$$

$$v_i^0 = \varphi_i, \tag{3.56}$$

$$v_0^j = 0, \tag{3.57}$$

$$\frac{(v_1^j - v_{-1}^j) + (v_1^{j+1} - v_{-1}^{j+1})}{2(x_1 - x_{-1})} + \alpha \frac{v_M^{j+1} - v_M^j}{k} = \frac{\alpha}{4}(r^j + r^{j+1})(f_M^j + f_M^{j+1}), \tag{3.58}$$

(we eliminate v_{-1}^j and v_{-1}^{j+1} by taking $i = 0$ in (3.55), then we get)

$$\begin{aligned} &\frac{v_1^j + v_1^{j+1}}{2x_1} + \alpha \frac{v_M^{j+1} - v_M^j}{k} \\ &\qquad = \frac{1}{8}(r^j + r^{j+1}) \left[2\alpha(f_M^j + f_M^{j+1}) - x_1(f_0^j + f_0^{j+1}) \right], \end{aligned} \tag{3.59}$$

$$\begin{aligned} &\frac{v_1^{j+1}}{2} + \frac{\alpha x_1}{k} v_M^{j+1} - \frac{2\alpha x_1(f_M^j + f_M^{j+1}) - x_1^2(f_0^j + f_0^{j+1})}{8} r^{j+1} \\ &= -\frac{v_1^j}{2} + \frac{\alpha x_1}{k} v_M^j + \frac{2\alpha x_1(f_M^j + f_M^{j+1}) - x_1^2(f_0^j + f_0^{j+1})}{8} r^j, \end{aligned} \tag{3.60}$$

$$r^j = \frac{1}{E^j} \int_0^1 v(x, t_j) dx, \tag{3.61}$$

where $0 = x_0 < x_1 < \cdots < x_M = 1$, and $t_j = j \cdot k$ for $j = 0, 1, \ldots, N$ with $k = 1/N$,

$$\begin{aligned} &v_i^j = v(x_i, t_j), \quad r^j = r(t_j), \quad f_i^j = f(x_i, t_j), \quad E^j = E(t_j), \\ &\qquad\qquad i = 1, 2, \ldots, M-1, \;\; j = 0, 1, \ldots, N-1. \end{aligned} \tag{3.62}$$

Here, we approximate the integral in (3.61) by numerical integration of using $M + 1$ transformed Gauss-Lobatto nodes from the unit square such that

$$\int_0^1 v(x, t_j) dx \approx \frac{1}{2} \sum_{i=0}^{M} w_i v_i^j, \tag{3.63}$$

where w_i are the weights of Gauss-Lobatto quadrature. When we substitute (3.63) into (3.61), we get $M \times M$ linear system of equations for each $j = 0, 1, \ldots, N-1$ from (3.55)–(3.61) such that

$$\left(A+\frac{1}{E^{j+1}}\widetilde{A}\right)V^{j+1}=\left(B+\frac{1}{E^{j}}\widetilde{B}\right)V^{j}, \tag{3.64}$$

where

$$V^{j}=(v_1^j,v_2^j,\ldots,v_M^j)^t, \tag{3.65}$$

$$A=\begin{bmatrix} 1+\frac{k}{h_1h_2} & -\frac{k}{h_2(h_1+h_2)} & 0 & \cdots & 0 \\ -\frac{k}{h_2(h_2+h_3)} & 1+\frac{k}{h_2h_3} & -\frac{k}{h_3(h_2+h_3)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\frac{k}{h_{M-1}(h_{M-1}+h_M)} & 1+\frac{k}{h_{M-1}h_M} & -\frac{k}{h_M(h_{M-1}+h_M)} \\ \frac{1}{2} & 0 & \cdots & 0 & \frac{\alpha x_1}{k} \end{bmatrix},$$

$$\widetilde{A}=\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,M} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,M} \\ \vdots & \ddots & \ddots & \vdots \\ a_{M-1,1} & a_{M-1,2} & \cdots & a_{M-1,M} \\ a_{M,1} & a_{M,2} & \cdots & a_{M,M} \end{bmatrix}, \tag{3.66}$$

$$a_{i,l}=-\frac{k}{8}w_l(f_i^j+f_i^{j+1}),\ \ i=1,2,\ldots,M-1, \tag{3.67}$$

$$a_{M,l}=-\frac{x_1}{16}w_l\left[2\alpha(f_M^j+f_M^{j+1})-x_1(f_0^j+f_0^{j+1})\right],\ l=1,2,\ldots,M, \tag{3.68}$$

$$B=\begin{bmatrix} 1-\frac{k}{h_1h_2} & \frac{k}{h_2(h_1+h_2)} & 0 & \cdots & 0 \\ \frac{k}{h_2(h_2+h_3)} & 1-\frac{k}{h_2h_3} & \frac{k}{h_3(h_2+h_3)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{k}{h_{M-1}(h_{M-1}+h_M)} & 1-\frac{k}{h_{M-1}h_M} & \frac{k}{h_M(h_{M-1}+h_M)} \\ -\frac{1}{2} & 0 & \cdots & 0 & \frac{\alpha x_1}{k} \end{bmatrix},$$

$$\widetilde{B} = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,M} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,M} \\ \vdots & \ddots & \ddots & \vdots \\ b_{M-1,1} & b_{M-1,2} & \cdots & b_{M-1,M} \\ b_{M,1} & b_{M,2} & \cdots & b_{M,M} \end{bmatrix}, \tag{3.69}$$

$$b_{i,l} = \frac{k}{8} w_l (f_i^j + f_i^{j+1}), \;\; i = 1, 2, \ldots, M-1, \tag{3.70}$$

$$b_{M,l} = \frac{x_1}{16} w_l \left[2\alpha (f_M^j + f_M^{j+1}) - x_1 (f_0^j + f_0^{j+1}) \right], \; l = 1, 2, \ldots, M. \tag{3.71}$$

We see that the truncation errors $\mathcal{T}_{i,j}(v)$ of the equations (3.55) for $i = 1, \ldots, M-1$ and $j = 0, 1, \ldots, N$ are of the form $\mathcal{O}(h_{i+1} - h_i) + \mathcal{O}(k^2)$ by expanding the functions v, f and r with their Taylor series about the point (x_i, t_j) and t_j, respectively [31]. Moreover, if we expand v_1^j, v_1^{j+1}, f_0^j and f_0^{j+1} at the point (x_0, t_j); v_M^j, v_M^{j+1}, f_M^j and f_M^{j+1} at the point (x_M, t_j) and the function r at the point t_j in the equation (3.60), we see that the truncation error of the equation is $\mathcal{O}(k) + \mathcal{O}(h_1 k) + \mathcal{O}(h_1^2)$. Hence, the system (3.64) is consistent. We solve this system with Gauss elimination method to obtain V^{j+1}, and then we find r^{j+1}. Since $v(x,t) = u(x,t)r(t)$ and $p(t) = -\frac{r'(t)}{r(t)}$, after finding V^j and r^j, we get U^j and p^j for all $j = 0, 1, 2, \ldots, N$ from the following equations

$$U^j = \frac{V^j}{r^j}, \tag{3.72}$$

$$\begin{gathered} p^0 = -\frac{r^1 - r^0}{r^0 k}, \; p^l = -\frac{r^{l+1} - r^{l-1}}{2 r^l k}, \; l = 1, 2, \ldots, N-1, \\ p^N = -\frac{r^n - r^{n-1}}{r^n k}. \end{gathered} \tag{3.73}$$

3.2.2. Uniform finite difference method

We also consider the numerical method of the same problem on a uniform grid with composite trapezoidal rule for the integral in (3.51) by taking equally spaced nodes, say $x_i = i \cdot h$ and $i = 0, 1, \ldots, M$ with $h = 1/M$ and $w_0 = w_M = h$, $w_1 = w_2 = \cdots = w_{M-1} = 2h$ in (3.63). Even if the uniform finite difference method

is a special case of the non-uniform finite difference method, the truncation errors $\mathcal{T}_{i,j}(v)$ are of the form $\mathcal{O}(h^2) + \mathcal{O}(k^2)$. Then the linear system (3.64) returns

$$\left(C + \frac{1}{E^{j+1}}\widetilde{C}\right) V^{j+1} = \left(D + \frac{1}{E^{j}}\widetilde{D}\right) V^{j}, \tag{3.74}$$

where

$$V^j = (v_1^j, v_2^j, \ldots, v_M^j)^t, \tag{3.75}$$

$$C = \begin{bmatrix} 1+2\gamma & -\gamma & 0 & \cdots & 0 \\ -\gamma & 1+2\gamma & -\gamma & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & -\gamma & 1+2\gamma & -\gamma \\ 1/2 & 0 & \cdots & 0 & \alpha h/k \end{bmatrix}, \tag{3.76}$$

$$\widetilde{C} = \begin{bmatrix} 2c_1 & 2c_1 & 2c_1 & \cdots & c_1 \\ 2c_2 & 2c_2 & 2c_2 & \cdots & c_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 2c_{M-1} & 2c_{M-1} & 2c_{M-1} & \cdots & c_{M-1} \\ 2c_M & 2c_M & 2c_M & \cdots & c_M \end{bmatrix} \tag{3.77}$$

$$\gamma = \frac{k}{2h^2}, \;\; c_i = -\frac{hk}{8}(f_i^j + f_i^{j+1}), \;\; i = 1, 2, \ldots, M-1, \tag{3.78}$$

$$c_M = -\frac{h^2}{16}\left[2\alpha(f_M^j + f_M^{j+1}) - h(f_0^j + f_0^{j+1})\right], \tag{3.79}$$

$$D = \begin{bmatrix} 1-2\gamma & \gamma & 0 & \cdots & 0 \\ \gamma & 1-2\gamma & \gamma & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \gamma & 1-2\gamma & \gamma \\ -1/2 & 0 & \cdots & 0 & \alpha/k \end{bmatrix}, \tag{3.80}$$

$$\widetilde{D} = \begin{bmatrix} 2d_1 & 2d_1 & 2d_1 & \cdots & d_1 \\ 2d_2 & 2d_2 & 2d_2 & \cdots & d_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 2d_{M-1} & 2d_{M-1} & 2d_{M-1} & \cdots & d_{M-1} \\ 2d_M & 2d_M & 2d_M & \cdots & d_M \\ \end{bmatrix}, \tag{3.81}$$

$$d_i = \frac{hk}{8}(f_i^j + f_i^{j+1}), \;\; i = 1, 2, \ldots, M-1, \tag{3.82}$$

$$d_M = \frac{h^2}{16}\left[2\alpha(f_M^j + f_M^{j+1}) - h(f_0^j + f_0^{j+1})\right]. \tag{3.83}$$

Now, let us consider the following two examples (the first one is smooth, second one is non-smooth on the input data) on a uniform and a non-uniform grid of the interval $[0, 1]$, and compare the numerical results of them for different values of α. Moreover, we give some noises on the input data and then employ the mollification method.

Example 3.6: Consider the inverse problem (3.47)–(3.51)*, with given*

$$f(x,t) = \sin(\pi x)e^t\left[\sin^2(\pi x)(1 + 3\pi^2 - e^{-2t}) - 6\pi^2\cos^2(\pi x)\right], \tag{3.84}$$

$$\varphi(x) = \sin^3(\pi x), \quad E(t) = \frac{4}{3\pi}e^t, \tag{3.85}$$

for $T = 1$*. It is easy to check that*

$$\{p(t), u(x,t)\} = \{e^{-2t}, \sin^3(\pi x)e^t\}. \tag{3.86}$$

If we look at the problem (3.47)–(3.51)*, we get*

$$v_t = v_{xx} + r(t)\sin(\pi x)e^t\left[\sin^2(\pi x)(1 + 3\pi^2 - e^{-2t}) - 6\pi^2\cos^2(\pi x)\right], \quad 0 < x < 1, \;\; 0 < t < 1, \tag{3.87}$$

$$v(x,0)=\varphi(x)=\sin^3(\pi x),\quad 0\le x\le 1,$$

$$v(0,t)=0,\quad 0\le t\le 1, \tag{3.88}$$

$$v_x(0,t)+\alpha v_{xx}(1,t)=0,\quad 0\le t\le 1, \tag{3.89}$$

$$\int_0^1 v(x,t)dx=\frac{4}{3\pi}e^t r(t),\quad 0\le t\le T, \tag{3.90}$$

where the exact solution of $r(t)$ *is*

$$r(t)=e^{-1/2+(1/2)e^{-2t}}. \tag{3.91}$$

We give the numerical results of this problem by using the system (3.74) *on the uniform grid with* $M=40$ *and the system* (3.64) *on the non-uniform grid by considering the* 41 *Gauss-Lobatto nodes (with* $M=40$*) for different values of* α *with* $N=50$*. We give the results with relative errors in* ∞*-norm on Table 3.1 and illustrate the exact and approximate functions in Figure 3.1-3.6 for different values of* α*:*

Table 3.1: Errors of $u(x,t)$ for $t=1$ and $p(t)$.

	Error ($\alpha=1/3$)		Error ($\alpha=1$)		Error ($\alpha=7$)	
	Uniform	Non-uniform	Uniform	Non-uniform	Uniform	Non-uniform
$u(x,t)$	0.0093	0.0039	0.0031	0.0028	0.0019	0.0027
$p(t)$	0.1040	0.0456	0.0347	0.0168	0.0147	0.0156

For the perfect stability analysis of the method, the noises can be entered to the input data. This is a basic process in solving the inverse problems which is focused in the reference [35] for integral overdetermined data and in the reference [36] for the initial data.

Now, the stability of the numerical solution with respect to the noisy over-determination data (1.6)*, defined by*

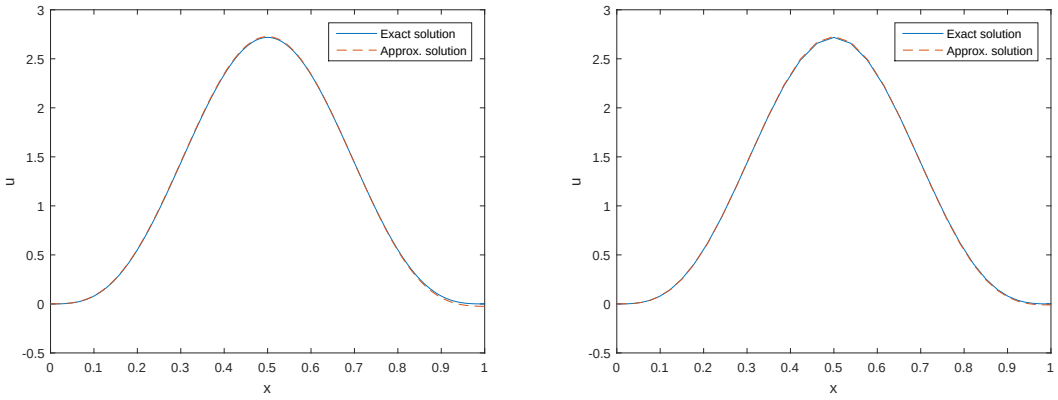


Figure 3.1: Uniform and non-uniform (from left to right) finite difference solutions of $u(x,t)$ for $\alpha = 1/3$.

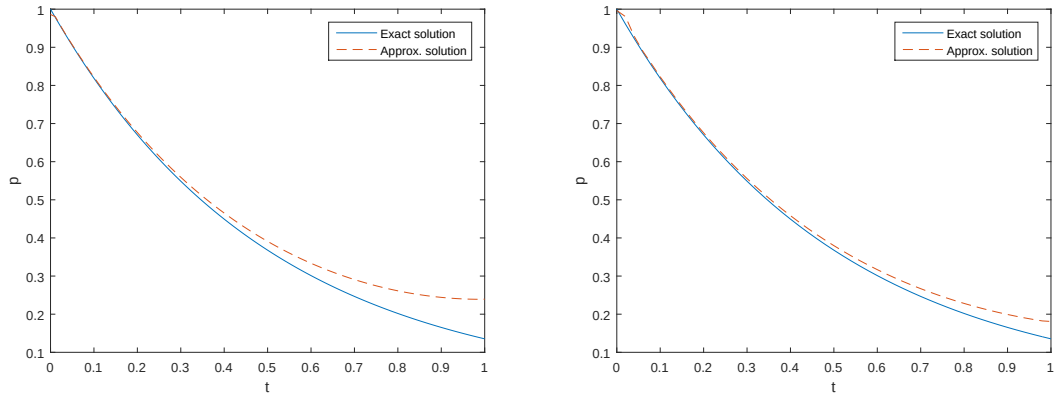


Figure 3.2: Uniform and non-uniform (from left to right) finite difference solutions of $p(t)$ for $\alpha = 1/3$.

$$E_\gamma(t) = E(t)(1 + \gamma\theta), \tag{3.92}$$

is investigated. Here, γ is the percentage of noise and θ are random variables (which are generated using rand command in MATLAB) generated from a uniform distribution in $[-1, 1]$. From these figures, we can see the stability of numerical solutions.

Table 3.2, 3.3 and Figure 3.7-3.10 show the exact and numerical solutions of $u(x,t)$ and $p(t)$ when the input data (1.6) *are contaminated by $\gamma = 2\%$, and 20%*

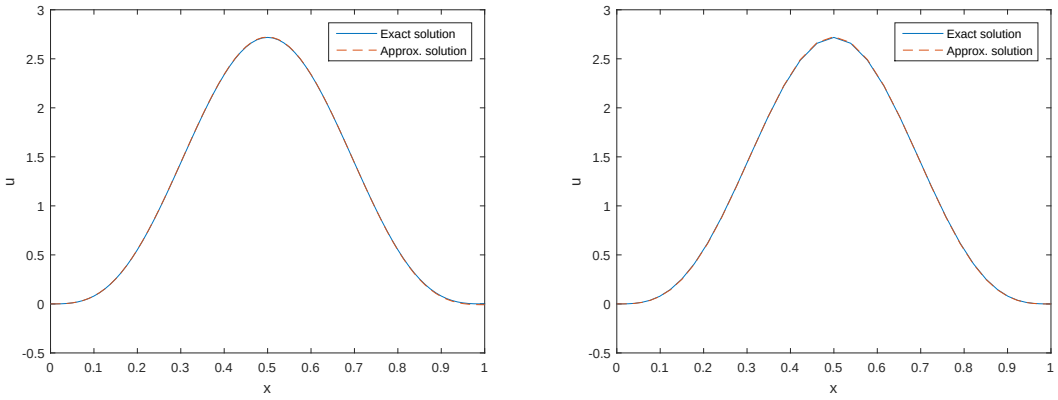


Figure 3.3: Uniform and non-uniform (from left to right) finite difference solutions of $u(x,t)$ for $\alpha = 1$.

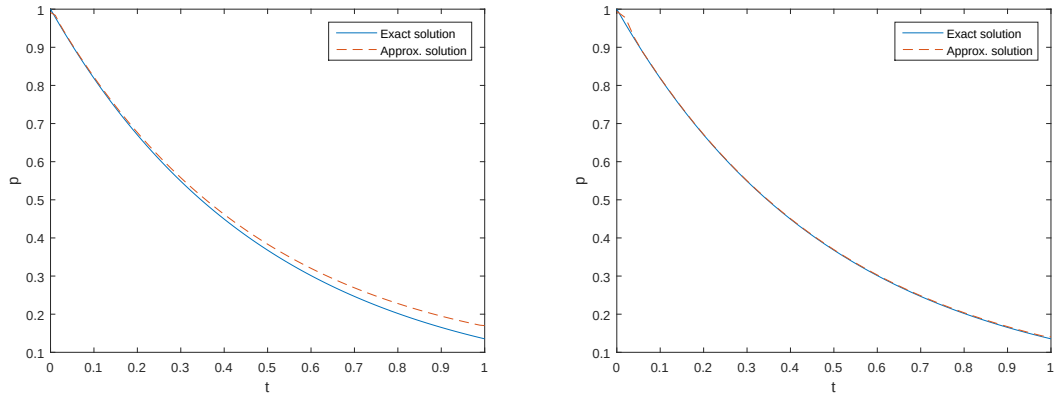


Figure 3.4: Uniform and non-uniform (from left to right) finite difference solutions of $p(t)$ for $\alpha = 1$.

noise for different values of α.

Table 3.2: Errors of $u(x,t)$ for $t = 1$ and $p(t)$ with noise $\gamma = 2\%$.

	Error ($\alpha = 1/3$, $\gamma = 2\%$)		Error ($\alpha = 1$, $\gamma = 2\%$)		Error ($\alpha = 7$, $\gamma = 2\%$)	
	Uniform	Non-uniform	Uniform	Non-uniform	Uniform	Non-uniform
$u(x,t)$	0.0026	0.0048	0.0033	0.0022	0.0016	0.0019
$p(t)$	0.1951	0.1383	0.1762	0.0903	0.0801	0.0833

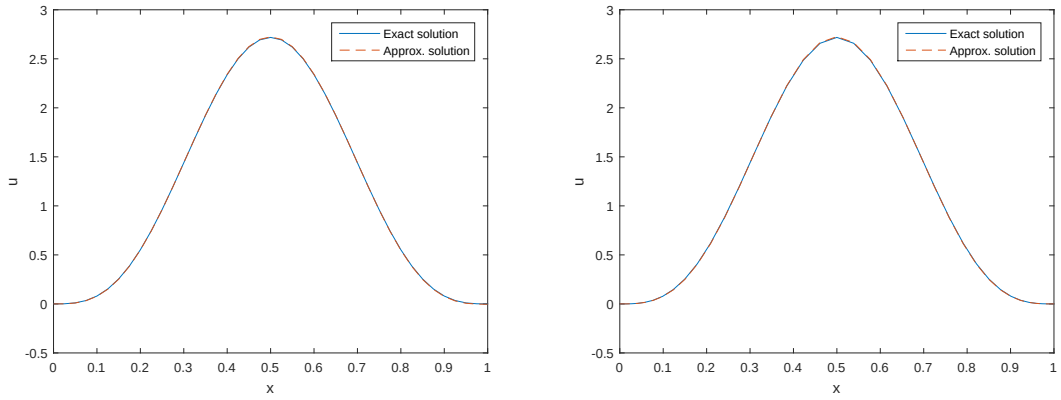


Figure 3.5: Uniform and non-uniform (from left to right) finite difference solutions of $u(x,t)$ for $\alpha = 7$.

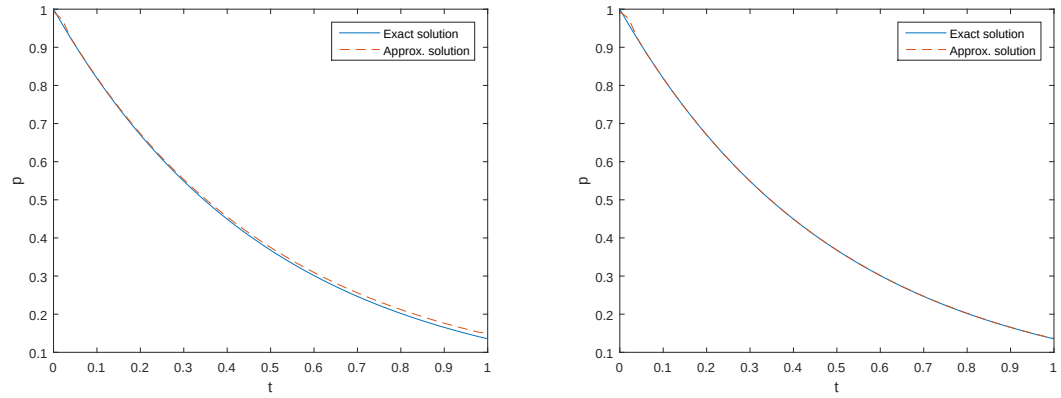


Figure 3.6: Uniform and non-uniform (from left to right) finite difference solutions of $p(t)$ for $\alpha = 7$.

Table 3.3: Errors of $u(x,t)$ for $t = 1$ and $p(t)$ with noise $\gamma = 20\%$.

	Error($\alpha = 1/3$, $\gamma = 20\%$)		Error($\alpha = 1$, $\gamma = 20\%$)		Error($\alpha = 7$, $\gamma = 20\%$)	
	Uniform)	Non-uniform)	Uniform	Non-uniform	Uniform	Non-uniform
$u(x,t)$	0.0101	0.0219	0.0064	0.0070	0.0108	0.0021
$p(t)$	1.7322	0.9904	1.5684	0.7510	0.7933	0.6223

Now, we employ the mollification method with a Gaussian mollifier [33], given by

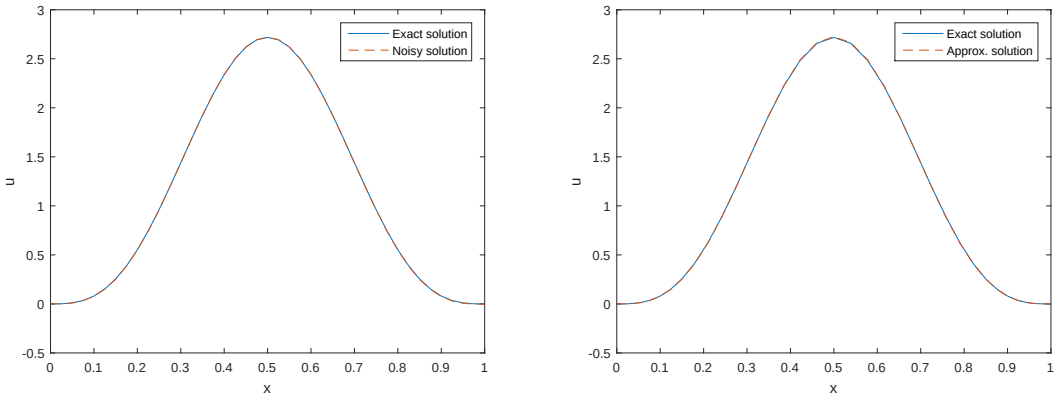


Figure 3.7: Uniform and non-uniform (from left to right) finite difference solutions of $u(x,t)$ for $\alpha = 7$ with noise $\gamma = 2\%$.

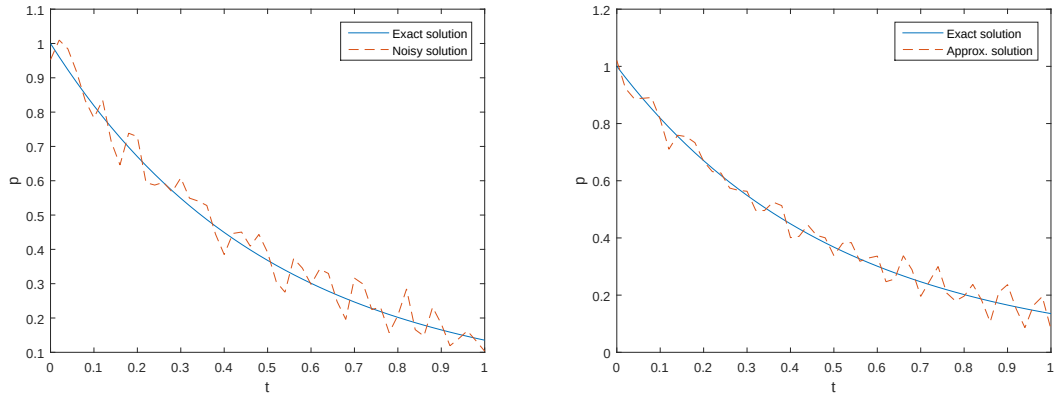


Figure 3.8: Uniform and non-uniform (from left to right) finite difference solutions of $p(t)$ for $\alpha = 7$ with noise $\gamma = 2\%$.

$$J_\delta = \frac{1}{\delta\sqrt{\pi}} e^{-t^2/\delta^2}, \tag{3.93}$$

where $\delta > 0$ is the radius of mollification. Its choice depends on the standard methods for choosing regularization parameter in ill-posed problems. The mollification of the noisy data (3.92) *are performed through the following convolution*

$$(J_\delta * E_\gamma)(t) = \int_{-\infty}^{+\infty} J_\delta(\tau) E_\gamma(t-\tau) d\tau. \tag{3.94}$$

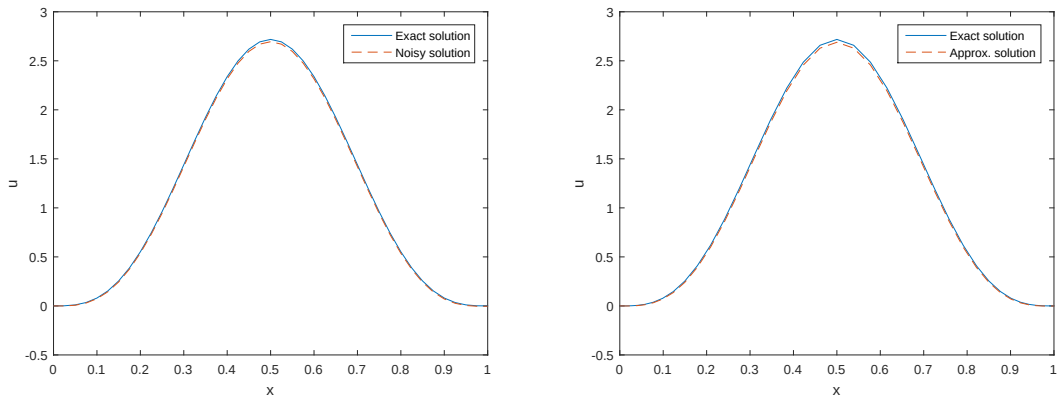


Figure 3.9: Uniform and non-uniform (from left to right) finite difference solutions of $u(x,t)$ for $\alpha = 7$ with noise $\gamma = 20\%$.

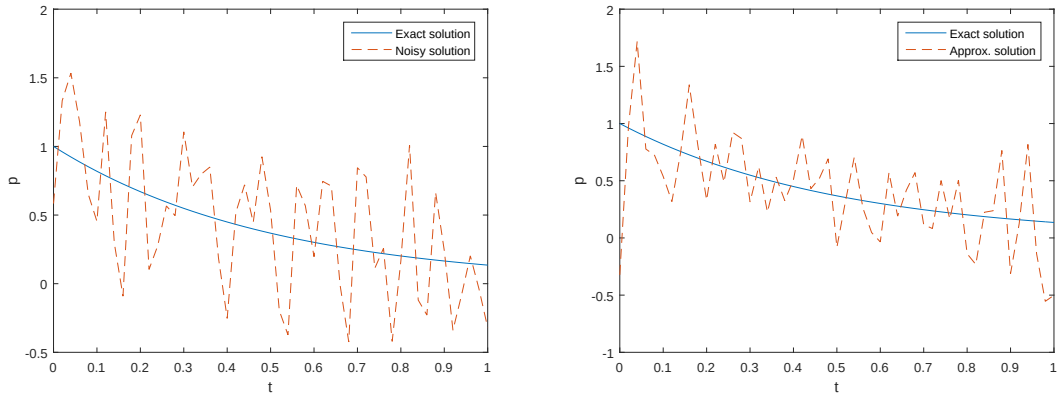


Figure 3.10: Uniform and non-uniform (from left to right) finite difference solutions of $p(t)$ for $\alpha = 7$ with noise $\gamma = 20\%$.

Notice that although $E_\gamma(t)$ is non-smooth, its mollification $(J_\delta * E_\gamma)(t)$ is C^∞ function, hence differentiable.

Table 3.4, 3.5 and Figure 3.11-3.13 show the exact and numerical solutions of $u(x,t)$ and $p(t)$ obtained after mollification, when the input data (1.6) *is contaminated by $\gamma = 2\%$ and 20% noise for different values of α. We can see from these figures that the application of the mollification to stabilize the derivative of the noisy*

function $E_\gamma(t)$ *to produce a stable numerical solutions for* $u(x,t)$ *and* $p(t)$*, respectively.*

Table 3.4: Errors of $u(x,t)$ for $t = 1$ and $p(t)$ with mollification $\gamma = 2\%$.

	Error ($\alpha = 1/3$, $\gamma = 2\%$)		Error ($\alpha = 1$, $\gamma = 2\%$)		Error ($\alpha = 7$, $\gamma = 2\%$)	
	Uniform	Non-uniform	Uniform	Non-uniform	Uniform	Non-uniform
$u(x,t)$	0.0165	0.0056	0.0040	0.0022	0.0017	0.0021
$p(t)$	0.1892	0.0759	0.0784	0.0586	0.0596	0.0487

Table 3.5: Errors of $u(x,t)$ for $t = 1$ and $p(t)$ with mollification $\gamma = 20\%$.

	Error($\alpha = 1/3$, $\gamma = 20\%$)		Error($\alpha = 1$, $\gamma = 20\%$)		Error($\alpha = 7$, $\gamma = 20\%$)	
	Uniform	Non-uniform	Uniform	Non-uniform	Uniform	Non-uniform
$u(x,t)$	0.0052	0.0164	0.0092	0.0051	0.0044	0.0043
$p(t)$	0.2331	0.1661	0.1822	0.1467	0.1269	0.1128

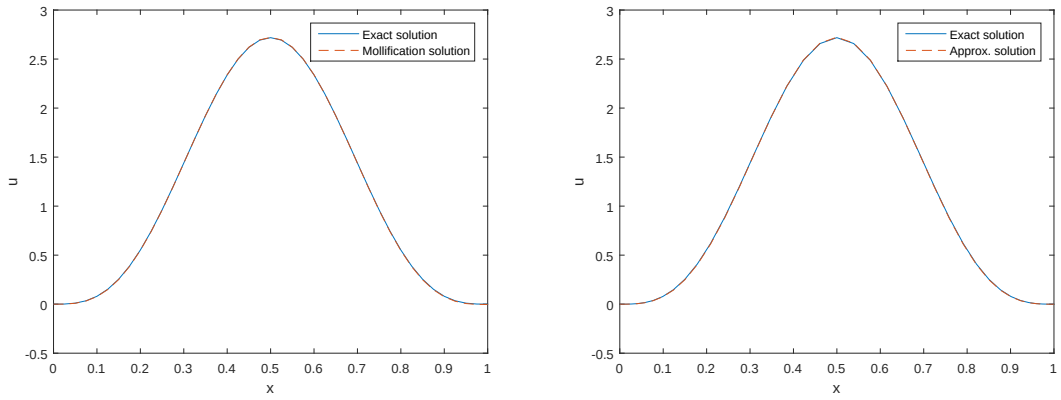


Figure 3.11: Uniform and non-uniform (from left to right) finite difference solutions of $u(x,t)$ for $\alpha = 7$ with mollification $\gamma = 2\%$.

Now, we consider an example with non-smooth input data.

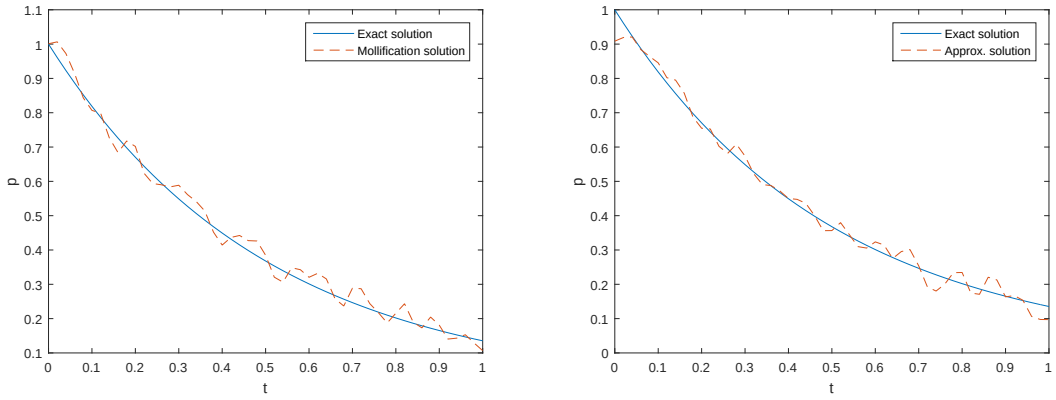


Figure 3.12: Uniform and non-uniform (from left to right) finite difference solutions of $p(t)$ for $\alpha = 7$ with mollification $\gamma = 2\%$.

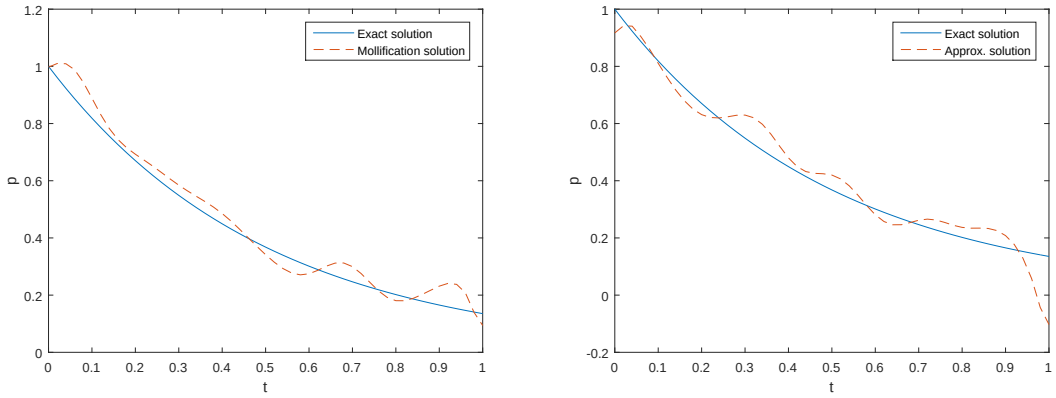


Figure 3.13: Uniform and non-uniform (from left to right) finite difference solutions of $p(t)$ for $\alpha = 7$ with mollification $\gamma = 20\%$.

Example 3.7: Let us consider an example for which an explicit analytical solution for $u(x,t)$ *does not exist. Take the initial condition* (1.2) *given by*

$$u(x,0) = \varphi(x) = \begin{cases} \frac{1}{4}, & 0 \le x \le 1/2, \\ x - \frac{1}{4}, & 1/2 < x \le 1. \end{cases} \tag{3.95}$$

Clearly, for this example, some of the conditions of Theorem 1 is not applicable. Nevertheless, the inverse problem can be made solvable numerically by considering the direct problem (1.1)-(1.4) *with* $\varphi(x)$ *and* $p(t)$ *given by*

$$p(t) = \frac{1}{t+1}, \qquad t \in [0,1], \tag{3.96}$$

in order to get the data (1.6).

We test some various mesh sizes as in [37], and we see that it's suitable taking as $M = 10$ *and* $N = 100$ *for getting a (close) better values of* $E(t)$ *(Figure 3.14), numerically.*

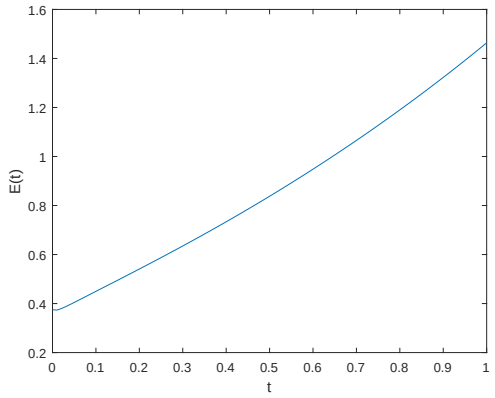


Figure 3.14: $E(t)$ with mesh sizes $M = 10$ and $N = 100$.

After we get the values of E^j *for* $j = 0, 1, \ldots, N$*, we can find a solution of the inverse problem for giving* $\varphi(x)$*,* E^j *and*

$$f(x,t) = \left(x - \frac{1}{4}\right)\left(1 + \frac{1}{t+1}\right)e^{t-1/2}, \tag{3.97}$$

by using the system (3.74)*. We give some numerical results with* $\alpha = 7$ *on Table 3.6, 3.7 and Figure 3.15-3.17.*

Table 3.6: Errors of $p(t)$ with different noises for $\alpha = 7$.

	$\alpha = 7, \gamma = 0\%$	$\alpha = 7, \gamma = 2\%$	$\alpha = 7, \gamma = 20\%$
	Error	Error	Error
$p(t)$	0.0083	0.2500	2.5679

Table 3.7: Errors of $p(t)$ with different mollifications for $\alpha = 7$.

	$\alpha = 7, \gamma = 0\%$	$\alpha = 7, \gamma = 2\%$	$\alpha = 7, \gamma = 20\%$
	Error	Error	Error
$p(t)$	0.0083	0.3766	2.2430

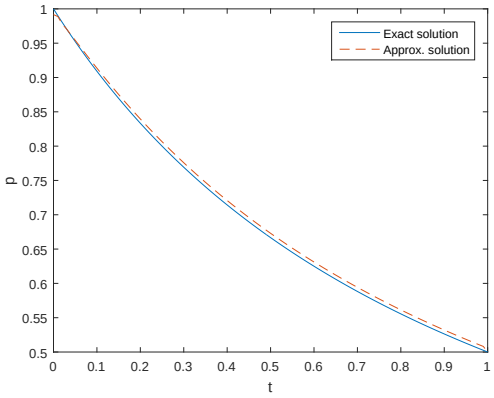


Figure 3.15: Finite difference solutions of $p(t)$ for $\alpha = 7$.

Remark 3.8: It is natural that α *must be non-zero. Unlike to the theoretical case, the numerical procedure of the mentioned inverse problem can be applied in the case* $\alpha < 0$. *By the way, the inverse problem* (1.1)-(1.6) *in the case* $\alpha = -1$ *has the following numerical performance:*

The errors on $u(x,t)$ *are* 0.0043 *and* 0.0027 *in uniform and non-uniform finite difference methods, respectively. The errors on* $p(t)$ *are* 0.0204 *and* 0.0139 *in uniform and non-uniform finite difference methods, respectively. By given* 2% *(*20%*) noises to the input data* (1.6) *in both uniform and non-uniform finite difference methods we get the errors on* $u(x,t)$ *as* 0.0013 *(*0.0075*)and* 0.0041 *(*0.0157*); and the errors on* $p(t)$ *as* 0.1632 *(*1.6616*) and* 0.1132 *(*1.1501*), respectively. After the application of the mollification to* 2% *(*20%*) noisy input data, we get the errors on* $u(x,t)$ *and* $p(t)$ *as* 0.0012 *(*0.0071*) and* 0.1359 *(*0.3440*) for uniform finite difference method; and*

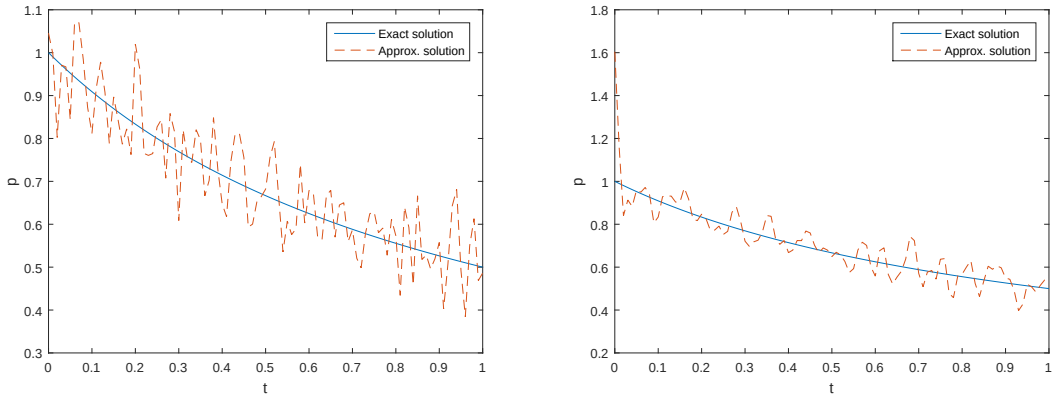


Figure 3.16: Finite difference solutions of $p(t)$ for $\alpha = 7$ (from left to right) with noise and mollification $\gamma = 2\%$.

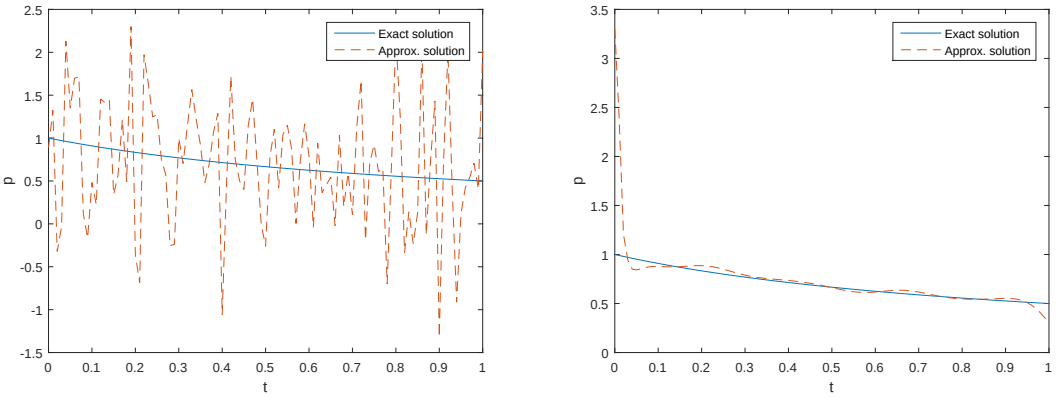


Figure 3.17: Finite difference solutions of $p(t)$ for $\alpha = 7$ (from left to right) with noise and mollification $\gamma = 20\%$.

as 0.0036 (0.0073) *and* 0.0514 (0.2780) *for non-uniform finite difference method, respectively.*

It is obviously observed that the numerical results in the case $\alpha < 0$ *is so alike to the case* $\alpha > 0$.

4. 2D HEAT EQUATION WITH IONKIN-TYPE BOUNDARY CONDITION

In this section, an inverse problem of determining the time-dependent perfusion coefficient of two-dimensional (2D) heat equation with a nonlocal and total energy integral overdetermination condition is considered [67]. The global well-posedness of the problem is obtained by generalized Fourier method combined with the unique solvability of the second kind Volterra integral equation. The numerical solutions of the inverse problem are proposed by traditional method, uniform finite difference combined with numerical integration on a uniform grid using the product Simpson's rule and by the method of non-uniform finite difference which is combined by a numerical integration on a non-uniform grid constructed with transformed Gauss-Lobatto nodes. Numerical examples illustrate how to implement the method.

4.1. Well-posedness of the inverse problem

At first, let us recall some necessary results on basisness of root functions concerning the two-dimensional spectral problem with Ionkin-type boundary conditions (see [63]).

4.1.1. Spectral problem

Consider the following spectral problem:

$$\frac{\partial^2 Z}{\partial x^2}+\frac{\partial^2 Z}{\partial y^2}+\mu Z=0, \quad 0<x,y<1, \tag{4.1}$$

$$Z(0,y)=Z(1,y), \ \ \frac{\partial Z(1,y)}{\partial x}=0, \ \ Z(x,0)=Z(x,1)=0, \ \ 0\leq x,y\leq 1, \tag{4.2}$$

where μ is the separation parameter. We present the solution as follows

$$Z(x,y)=X(x)V(y). \tag{4.3}$$

When we substitute this expression into (4.1) and (4.2), we obtain the following problems

$$V''(y) + \lambda V(y) = 0, \quad 0 < y < 1, \quad V(0) = V(1) = 0, \tag{4.4}$$

$$X''(x) + \gamma X(x) = 0, \quad 0 < x < 1, \quad X(0) = X(1), \quad X'(1) = 0, \tag{4.5}$$

where $\gamma = \mu - \lambda$. It is known that the solutions of (4.4) are

$$\lambda_k = (\pi k)^2, \quad V_k(y) = \sqrt{2}\sin(\pi k y), \tag{4.6}$$

for $k = 1, 2, \ldots$. We give the constants multiplying the eigenfunctions and associated functions, here and in the following, suitable for normalization conditions.

The eigenvalues the eigenfunctions of the problem (4.5) are (see [11])

$$\gamma_m = (2\pi m)^2, \quad m = 0, 1, 2, \ldots, \tag{4.7}$$

$$X_0 = 2, \quad X_m(x) = 4\cos(2\pi m x), \quad m = 1, 2, \ldots. \tag{4.8}$$

Thus, the eigenvalues and eigenfunctions of the problem (4.1), (4.2), with the representation (4.3), are

$$\mu_{m,k} = \gamma_m + \lambda_k = (2\pi m)^2 + (\pi k)^2, \tag{4.9}$$

$$Z_{m,k}(x, y) = X_m(x) V_k(y), \quad m = 0, 1, 2, \ldots, \quad k = 1, 2, \ldots. \tag{4.10}$$

Here, $Z_{m,k}(x, y)$ is not complete in $L_2(D_{xy})$; therefore, it is supplemented the set of eigenfunctions by the following associated functions $\widetilde{Z}_{m,k}(x, y)$, $m, k = 1, 2 \ldots$ in [63] by following the lines of [64]

$$\widetilde{Z}_{m,k} = 4(1 - x)\sin(2\pi m x)\sqrt{2}\sin(\pi k y). \tag{4.11}$$

Let the system of root functions of problem (4.1), (4.2) has the form

$$Z_{0,k}(x,y) = X_0(x)V_k(y), \quad Z_{2m-1,k}(x,y) = X_m(x)V_k(y),$$
$$Z_{2m,k}(x,y) = \widetilde{Z}_{m,k}, \quad k,m = 1,2,\ldots \tag{4.12}$$

The adjoint problem of (4.1), (4.2) is

$$\partial^2 W/\partial x^2 + \partial^2 W/\partial y^2 + \mu W = 0, \quad 0 < x,y < 1,$$
$$W(0,y) = 0, \quad \partial W(0,y)/\partial x = \partial W(1,y)/\partial x, \tag{4.13}$$
$$W(x,0) = W(x,1) = 0, \quad 0 \le x,y \le 1.$$

The root functions of the adjoint problem (4.13) have the form

$$W_{0,k}(x,y) = xV_k(y), \quad W_{2m,k}(x,y) = \sin(2\pi mx)V_k(y),$$
$$W_{2m-1,k}(x,y) = x\cos(2\pi mx)V_k(y), \quad m,k = 1,2,\ldots. \tag{4.14}$$

Then, the sequences (4.12), (4.14) form a bi-orthonormal system of functions on $\overline{D}_{xy}$; i.e., for any indices $m,k,l,$ and p, we have $(Z_{m,k},W_{l,p}) = 1$ if $m = l$ and $k = p$ and $(Z_{m,k},W_{l,p}) = 0$ otherwise. And each of these sequence is a basis in the space $L_2(D_{xy})$ [63]. Here, the inner product is given by

$$(\psi,\xi) = \int_0^1 \int_0^1 \psi(x,y)\xi(x,y)dxdy. \tag{4.15}$$

4.1.2. Existence-uniqueness and continuous dependence upon the data of the solution

We have the following assumptions on $\varphi(x,y)$, $f(x,y,t)$ and $E(t)$.

$(A_1)_1$ $\varphi(x,y) \in C^{2,2}(\overline{D}_{xy})$,

$(A_1)_2$ $\varphi(0,y) = \varphi(1,y), \quad \varphi(x,0) = \varphi(x,1) = 0, \quad \varphi_x(1,y) = 0,$

$(A_1)_3$ $\varphi_{0,2k-1} > 0, \quad \varphi_{2m,2k-1} \ge 0, \quad m,k = 1,2\ldots,$

$(A_2)_1$ $f(x,y,t) \in C(\overline{D_T}),\ \ f(x,y,t) \in C^{2,2}(\overline{D}_{xy}),\ \ \forall t \in [0,T]$

$(A_2)_2$ $f(0,y,t) = f(1,y,t), \quad f(x,0,t) = f(x,1,t) = 0,$

$(A_2)_3$ $f_{2m,2k-1}(t) \geq 0, \quad m = 0,1,\ldots,\ k = 1,2,\ldots,$

$(A_3)_1$ $E(t) \in C^1[0,T],$

$(A_3)_2$ $E(0) = \int_0^1 \int_0^1 \varphi(x,y)dxdy,$

$(A_3)_3$ $E(t) > 0, \quad \forall t \in [0,T].$

The presentation of the main result on existence and uniqueness of the solution of the inverse problem (1.8)-(1.12) follows.

Theorem 4.1 (Existence and uniqueness). Under the conditions (A_1)-(A_3) the inverse problem (1.8)-(1.12) *has a unique solution.*

Proof Since (4.12) *is a basis in* $L_2(\overline{D}_{xy})$*, we present the solution of* (1.8)-(1.11) *in the following form for arbitrary* $p(t) \in C[0,T]$*:*

$$u(x,y,t) = \sum_{k=1}^{\infty} \left[\alpha_{0,k}(t) Z_{0,k}(x,y) + \sum_{m=1}^{\infty} \alpha_{2m-1,k}(t) Z_{2m-1,k}(x,y) + \sum_{m=1}^{\infty} \alpha_{2m,k}(t) Z_{2m,k}(x,y) \right], \tag{4.16}$$

where

$$\alpha_{0,k}(t) = \varphi_{0,k} \cdot e^{-(\pi k)^2 t - \int_0^t p(s)ds} + \int_0^t f_{0,k}(\tau) e^{-(\pi k)^2 (t-\tau) - \int_\tau^t p(s)ds} d\tau, \tag{4.17}$$

$$\alpha_{2m,k}(t) = \varphi_{2m,k} e^{-(2\pi m)^2 t - (\pi k)^2 t - \int_0^t p(s)ds} + \int_0^t f_{2m,k}(\tau) e^{-(2\pi m)^2 (t-\tau) - (\pi k)^2 (t-\tau) - \int_\tau^t p(s)ds} d\tau, \tag{4.18}$$

$$\alpha_{2m-1,k}(t) = [\varphi_{2m-1,k} - 4\pi m \cdot \varphi_{2m,k} t]\, e^{-(2\pi m)^2 t - (\pi k)^2 t - \int_0^t p(s)ds}$$

$$+ \int_0^t [f_{2m-1,k}(\tau) - 4\pi m f_{2m,k}(\tau)(t-\tau)] \tag{4.19}$$

$$e^{-(2\pi m)^2 (t-\tau) - (\pi k)^2 (t-\tau) - \int_\tau^t p(s)ds} d\tau.$$

Here

$$\varphi_{m,k} = \iint_{D_{xy}} \varphi(x,y) W_{m,k}(x,y) dxdy,$$

$$f_{m,k}(t) = \iint_{D_{xy}} f(x,y,t) W_{m,k}(x,y) dxdy, \tag{4.20}$$

$$m = 0, 1, 2, \ldots,\ k = 1, 2, \ldots.$$

Under conditions (A_1) *and* (A_2)*, the series* (4.16)*, its* $t-$*partial derivative, the* $xx-$*second order and* $yy-$*second order partial derivatives converge uniformly in* $\overline{D_T}$ *that their majorizing sums absolutely convergence. Thus* $u(x,y,t) \in C^{2,2,1}(\overline{D_T})$*. Since the conditions that the series* (4.16) *can be term-wise differentiable by* t*, applying the overdetermination condition* (1.12)*, we obtain the following Volterra integral equation of the second kind with respect to* $r(t) = e^{\int_0^t p(s)ds}$*:*

$$r(t) = F(t) + \int_0^t K(t,\tau) r(\tau) d\tau, \tag{4.21}$$

where

$$F(t) = \frac{4\sqrt{2}}{\pi E(t)} \sum_{k=1}^{\infty} \left[\frac{\varphi_{0,2k-1}}{2k-1} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\varphi_{2m,2k-1}}{(2k-1)m} e^{-(2\pi m)^2 t} \right] e^{-(\pi(2k-1))^2 t},$$

$$K(t,\tau) = \frac{4\sqrt{2}}{\pi E(t)} \sum_{k=1}^{\infty} \left[\frac{f_{0,2k-1}(\tau)}{2k-1} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{f_{2m,2k-1}(\tau)}{(2k-1)m} e^{-(2\pi m)^2 (t-\tau)} \right] \tag{4.22}$$

$$e^{-(\pi(2k-1))^2 (t-\tau)}.$$

In the fact that (4.21) *has a positive solution in* $C^1[0,T]$*, the function* $p(t)$ *can be determined from the equation*

$$p(t) = \frac{r'(t)}{r(t)}. \tag{4.23}$$

The right-hand side of $F(t)$ *and the kernel* $K(t,\tau)$ *are continuously differentiable functions in* $[0,T]$ *and* $[0,T]\times[0,T]$*, respectively, under the assumptions* $(A_1)_{1,2}-(A_3)_{1,2}$*. Moreover, under the conditions* $(A_1)_3-(A_3)_3$*,* $F(t)>0$ *in* $[0,T]$ *and* $K(t,\tau)\geq 0$ *in* $[0,T]\times[0,T]$*. In addition, as in [12], the following inequality holds:*

$$\|r\|_{C[0,T]} \leq \|F\|_{C[0,T]} e^{T\|K\|_{C([0,T]\times[0,T])}}. \tag{4.24}$$

Then it is obtained a unique positive function $r(t)\in C^1[0,T]$*, which the function* (4.23) *together with the solution of the problem* (1.8)-(1.11) *given by the Fourier series* (4.16)*, form the unique solution of the inverse problem* (1.8)-(1.12)*.*

■

The following result about the dependence continuously upon the data of the solution of the inverse problem (1.8)-(1.12) holds.

Theorem 4.2 (Stability). Under assumptions $(A_1)-(A_3)$*, the solution* (p,u) *depends continuously upon the data.*

Proof Let $\Phi=\{\varphi,E,f\}$ *and* $\overline{\Phi}=\{\overline{\varphi},\overline{E},\overline{f}\}$ *be two sets of data, which satisfy the conditions* $(A_1)-(A_3)$*. Denote* $\|\Phi\| = \|\varphi\|_{C^{2,2}(\overline{D}_{xy})} + \|E\|_{C^1[0,T]} + \|f\|_{C^{2,2,0}(\overline{D_T})}$*. Suppose that there exist positive constants* M_1 *and* M_2 *such that*

$$0<M_1\leq|E|, \quad 0<M_1\leq|\overline{E}|, \qquad \|\Phi\|\leq M_2, \quad \|\overline{\Phi}\|\leq M_2. \tag{4.25}$$

Let (p,u) *and* $(\overline{p},\overline{u})$ *be the solutions of inverse problem* (1.8)-(1.12) *corresponding the data* Φ *and* $\overline{\Phi}$*, respectively. According to* (4.16) *and* (4.21)*, we get*

$$r(t) = F(t) + \int_0^t K(t,\tau)r(\tau)d\tau, \text{ and } \overline{r}(t) = \overline{F}(t) + \int_0^t \overline{K}(t,\tau)\overline{r}(\tau)d\tau, \tag{4.26}$$

where

$$F(t) = \frac{4\sqrt{2}}{\pi E(t)} \sum_{k=1}^{\infty} \left[\frac{\varphi_{0,2k-1}}{2k-1} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\varphi_{2m,2k-1}}{(2k-1)m} e^{-(2\pi m)^2 t} \right] e^{-(\pi(2k-1))^2 t},$$

$$K(t,\tau) = \frac{4\sqrt{2}}{\pi E(t)} \sum_{k=1}^{\infty} \left[\frac{f_{0,2k-1}(\tau)}{2k-1} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{f_{2m,2k-1}(\tau)}{(2k-1)m} e^{-(2\pi m)^2 (t-\tau)} \right] \tag{4.27}$$

$$e^{-(\pi(2k-1))^2 (t-\tau)},$$

and

$$\overline{F}(t) = \frac{4\sqrt{2}}{\pi \overline{E}(t)} \sum_{k=1}^{\infty} \left[\frac{\overline{\varphi}_{0,2k-1}}{2k-1} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\overline{\varphi}_{2m,2k-1}}{(2k-1)m} e^{-(2\pi m)^2 t} \right] e^{-(\pi(2k-1))^2 t},$$

$$\overline{K}(t,\tau) = \frac{4\sqrt{2}}{\pi \overline{E}(t)} \sum_{k=1}^{\infty} \left[\frac{\overline{f}_{0,2k-1}(\tau)}{2k-1} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\overline{f}_{2m,2k-1}(\tau)}{(2k-1)m} e^{-(2\pi m)^2 (t-\tau)} \right] \tag{4.28}$$

$$e^{-(\pi(2k-1))^2 (t-\tau)}.$$

Firstly, let us estimate the difference $r - \overline{r}$. *We obtain from* (4.21)

$$r(t) - \overline{r}(t) = F(t) - \overline{F}(t) + \int_0^t \left[K(t,\tau) - \overline{K}(t,\tau) \right] r(\tau) d\tau + \int_0^t \overline{K}(t,\tau) \left[r(\tau) - \overline{r}(\tau) \right] d\tau. \tag{4.29}$$

Let $\gamma = \|F - \overline{F}\|_{C[0,T]} + T \|K - \overline{K}\|_{C([0,T]\times[0,T])} \|r\|_{C[0,T]}$ *and denote* $R(t) = |r(t) - \overline{r}(t)|$. *Then* (4.29) *implies the inequality*

$$R(t) \leq \gamma + \int_0^t |\overline{K}(t,\tau)| R(\tau) d\tau \tag{4.30}$$

By applying the Gronwall inequality [29], we obtain from (4.30)

$$R(t) \leq \gamma \cdot e^{\int_0^t \sup_{s\in[\tau,t]} |\overline{K}(s,\tau)| d\tau}. \tag{4.31}$$

Moreover,

$$|F(t)| \leq \frac{4\sqrt{2}}{\pi}\left|\sum_{k=1}^{\infty}\frac{1}{2k-1}\frac{\varphi_{0,2k-1}}{E(t)}\right| + \frac{4\sqrt{2}}{\pi^2}\left|\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(2k-1)m}\frac{\varphi_{2m,2k-1}}{E(t)}\right| \leq M_3, \tag{4.32}$$

and similarly

$$|K(t,\tau)| \leq M_3, \tag{4.33}$$

where

$$M_3 = \frac{4\sqrt{2}}{\pi}C\frac{M_2}{M_1} + \frac{4\sqrt{2}}{\pi^2}C\overline{C}\frac{M_2}{M_1}, \tag{4.34}$$

and

$$C = \sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\right)^2, \quad \overline{C} = \sum_{m=1}^{\infty}\left(\frac{1}{m}\right)^2. \tag{4.35}$$

Finally, we have from (4.29)*,* (4.32) *and* (4.33)

$$\|r-\overline{r}\|_{C[0,T]} \leq C_1(\|F-\overline{F}\|_{C[0,T]} + C_2\|K-\overline{K}\|_{C([0,T]\times[0,T])}), \tag{4.36}$$

where $C_1 = e^{TM_3}$ *and* $C_2 = TM_3C_1$*. Therefore,* r *continuously depend on* F *and* K*.*

Now let us see that F *and* K *continuously depend on the data. It is easy to derive the following inequalities:*

$$\begin{aligned} |F(t)-\overline{F}(t)| &\leq M_4\|\varphi-\overline{\varphi}\|_{C^{2,2}(\overline{D}_{xy})} + M_5\|E-\overline{E}\|_{C^1[0,T]}, \\ |K(t,\tau)-\overline{K}(t,\tau)| &\leq M_6\|f-\overline{f}\|_{C^{2,2,0}(\overline{D_T})} + M_7\|E-\overline{E}\|_{C^1[0,T]}, \end{aligned} \tag{4.37}$$

where M_4*,* M_5*,* M_6*, and* M_7 *constants that are determined from* M_1*,* M_2*,* C*, and* $\overline{C}$*. Thus,* F *and* K *continuously depend on the data. So,* r *also continuously depends on the data.*

We also need to show that r' *depends continuously on the data in order to show that* p *depends continuously on the data. Differentiating both sides of* (4.21) *with respect to* t*, we get the followings:*

$$
\begin{aligned}
r'(t) &= F'(t) + K(t,t)r(t) + \int_0^t K_t(t,\tau)r(\tau)d\tau, \\
\overline{r}'(t) &= \overline{F}'(t) + \overline{K}(t,t)\overline{r}(t) + \int_0^t \overline{K}_t(t,\tau)\overline{r}(\tau)d\tau.
\end{aligned}
\tag{4.38}
$$

Then, the following estimation holds:

$$
\begin{aligned}
\|r' - \overline{r}'\|_{C[0,T]} \leq & \|F' - \overline{F}'\|_{C[0,T]} + \|r\|_{C[0,T]}\|K - \overline{K}\|_{C([0,T]\times[0,T])} \\
& + \|\overline{K}\|_{C([0,T]\times[0,T])}\|r - \overline{r}\|_{C[0,T]} \\
& + T\|r\|_{C[0,T]}\|K_t - \overline{K}_t\|_{C([0,T]\times[0,T])} \\
& + T\|\overline{K}_t\|_{C([0,T]\times[0,T])}\|r - \overline{r}\|_{C[0,T]}.
\end{aligned}
\tag{4.39}
$$

Since we know that r, F, and K are dependent continuously on the data we need to show continuously dependence of the remaining statements. Since

$$
|K_t(t,\tau)| \leq M_8, \tag{4.40}
$$

where M_8 is determined from M_1, M_2, C, and $\overline{C}$ and r' depends continuously on F' and K_t (it is easy to be seen), we can easily obtain the following inequalities similarly:

$$
\begin{aligned}
|F'(t) - \overline{F}'(t)| &\leq M_9\|\varphi - \overline{\varphi}\|_{C^{2,2}(\overline{D}_{xy})} + M_{10}\|E - \overline{E}\|_{C^1[0,T]}, \\
|K_t(t,\tau) - \overline{K}_t(t,\tau)| &\leq M_{11}\|f - \overline{f}\|_{C^{2,2,0}(\overline{D_T})} + M_{12}\|E - \overline{E}\|_{C^1[0,T]},
\end{aligned}
\tag{4.41}
$$

where M_9, M_{10}, M_{11}, and M_{12} constants that are determined from M_1, M_2, C, and $\overline{C}$. This means that r' continuously depends on the data as well. Thus, from (4.23) *p also depends continuously on the data.*

Similarly, we obtain the estimate the difference $u - \overline{u}$ from (4.16)*:*

$$
\|u - \overline{u}\|_{C(\overline{D_T})} \leq M_{13}\|\Phi - \overline{\Phi}\|. \tag{4.42}
$$

■

Now, let's see how we implement the numerical solution of the inverse problem in two different methods.

4.2. Numerical methods for the inverse problem

We give the numerical methods of the inverse problem (1.8)-(1.12) both on equidistant and non-equidistant points of the interval $[0,1]$. We will rewrite the equations for the convenience of discussion of the numerical method as follows:

$$\partial v/\partial t = \partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 + r(t)f(x,y,t), \tag{4.43}$$

$$v(x,y,0) = \varphi(x,y), \quad 0 \le x,y \le 1, \tag{4.44}$$

$$\begin{aligned} &v(0,y,t) = v(1,y,t), \quad v_x(1,y,t) = 0, \quad 0 \le y \le 1,\ 0 \le t \le T, \\ &v(x,0,t) = v(x,1,t) = 0, \quad 0 \le x \le 1, \quad 0 \le t \le T, \end{aligned} \tag{4.45}$$

$$\int_0^1 \int_0^1 v(x,y,t)dxdy = E(t)r(t), \tag{4.46}$$

by using the transformations

$$r(t) = e^{\int_0^t p(s)ds}, \tag{4.47}$$

$$v(x,y,t) = u(x,y,t)r(t). \tag{4.48}$$

At first, we give the implicit finite difference discretization of the numerical method on a uniform grid and then on a non-uniform grid (see [31, 65]). Moreover, we use the product Simpson's rule on a uniform grid and the product rule with transformed Gauss-Lobatto nodes from the unit square and weights for the numerical integration on a non-uniform grid [30].

4.2.1. Uniform finite difference method

Set

$$\begin{aligned} x_i &= i \cdot hx, \quad i = 0, 1, \ldots, Nx, \quad hx \cdot Nx = 1, \\ y_j &= j \cdot hy, \quad j = 0, 1, \ldots, Ny, \quad hy \cdot Ny = 1, \\ t_n &= n \cdot ht, \quad n = 0, 1, \ldots, Nt, \quad ht \cdot Nt = T. \end{aligned} \tag{4.49}$$

We consider Crank-Nicholson implicit finite difference discretization of the problem (4.43)-(4.46) on a uniform grid. In order to deal with the boundary conditions (3.49), we add the fictitious point x_{Nx+1}.

$$\begin{aligned} \frac{v_{ij}^{n+1} - v_{ij}^{n}}{ht} =& \frac{1}{2hx^2}\left[(v_{i+1j}^{n} - 2v_{ij}^{n} + v_{i-1j}^{n}) + (v_{i+1j}^{n+1} - 2v_{ij}^{n+1} + v_{i-1j}^{n+1})\right] \\ &+ \frac{1}{2hy^2}\left[(v_{ij+1}^{n} - 2v_{ij}^{n} + v_{ij-1}^{n}) + (v_{ij+1}^{n+1} - 2v_{ij}^{n+1} + v_{ij-1}^{n+1})\right] \\ &+ \frac{1}{2}\left((rf)_{ij}^{n} + (rf)_{ij}^{n+1}\right), \end{aligned} \tag{4.50}$$

$$\begin{aligned} \Rightarrow & - R_1 v_{i-1j}^{n+1} - R_2 v_{ij-1}^{n+1} + (1 + 2R_1 + 2R_2) v_{ij}^{n+1} \\ & \qquad - R_1 v_{i+1j}^{n+1} - R_2 v_{ij+1}^{n+1} - \frac{ht}{2}(rf)_{ij}^{n+1} \\ & = R_1 v_{i-1j}^{n} + R_2 v_{ij-1}^{n} + (1 - 2R_1 - 2R_2) v_{ij}^{n} \\ & \qquad + R_1 v_{i+1j}^{n} + R_2 v_{ij+1}^{n} + \frac{ht}{2}(rf)_{ij}^{n}, \end{aligned} \tag{4.51}$$

$$\varphi_{ij} = v_{ij}^{0}, \tag{4.52}$$

$$v_{0j}^{n} = v_{Nxj}^{n}, \quad v_{Nx+1j}^{n} = v_{Nx-1j}^{n}, \quad v_{i0}^{n} = v_{iNy}^{n} = 0, \tag{4.53}$$

$$r^n = \frac{1}{E^n}\int_0^1 \int_0^1 v(x, y, t_n)\, dx dy, \tag{4.54}$$

where

$$v_{ij}^n = v(x_i, y_j, t_n), \quad r^n = r(t_n), \quad E^n = E(t_n), \quad f_{ij}^n = f(x_i, y_j, t_n),$$

$$R_1 = \frac{ht}{2hx^2}, \quad R_2 = \frac{ht}{2hy^2}, \tag{4.55}$$

$$n = 0, \ldots, Nt,\ i = 1, \ldots, Nx,\ j = 1, \ldots, Ny - 1.$$

Now, we approximate the double integral in (4.54) by the product Simpson's rule to write r^n discretely as follows:

$$\begin{aligned}\int_0^1 \int_0^1 v(x, y, t_n)dxdy \approx & \frac{hx \cdot hy}{9} \sum_{i=1}^{Nx/2} \sum_{j=1}^{Ny/2} \left[v_{2i-2,2j-2}^n + 4v_{2i-1,2j-2}^n \right. \\ & + v_{2i,2j-2}^n + 4\left(v_{2i-2,2j-1}^n + 4v_{2i-1,2j-1}^n + v_{2i,2j-1}^n\right) \\ & \left. + v_{2i-2,2j}^n + 4v_{2i-1,2j}^n + v_{2i,2j}^n \right],\end{aligned} \tag{4.56}$$

where Nx and Ny must be even. Therefore, substituting (4.56) into (4.54), we get $Nx(Ny-1) \times Nx(Ny-1)$ linear system of equations, for each $n = 0, \ldots, Nt$, from the equations (4.51)-(4.54) such that

$$\left(\mathcal{A} - \frac{1}{E^{n+1}}\widetilde{\mathcal{A}}\right) V^{n+1} = \left(\mathcal{B} + \frac{1}{E^n}\widetilde{\mathcal{B}}\right) V^n, \tag{4.57}$$

where

$$\begin{aligned}V^n = (& v_{1,1}^n, v_{1,2}^n, \ldots, v_{1,Ny-1}^n, v_{2,1}^n, v_{2,2}^n, \ldots, v_{2,Ny-1}^n, \\ & \ldots, v_{Nx,1}^n, v_{Nx,2}^n, \ldots, v_{Nx,Ny-1}^n)^t,\end{aligned} \tag{4.58}$$

$$\mathcal{A} = \begin{bmatrix} R_{D^-} & -R_F & \mathbf{0} & \cdots & -R_F \\ -R_F & R_{D^-} & -R_F & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & -R_F & R_{D^-} & -R_F \\ \mathbf{0} & \mathbf{0} & \cdots & -2 \cdot R_F & R_{D^-} \end{bmatrix}, \tag{4.59}$$

$$\mathcal{B} = \begin{bmatrix} R_{D^+} & R_F & \mathbf{0} & \cdots & R_F \\ R_F & R_{D^+} & R_F & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & R_F & R_{D^+} & R_F \\ \mathbf{0} & \mathbf{0} & \cdots & 2 \cdot R_F & R_{D^+} \end{bmatrix}, \tag{4.60}$$

$$R_{D^-} = \begin{bmatrix} 1+2R_1+2R_2 & -R_2 & 0 & \cdots & 0 \\ -R_2 & 1+2R_1+2R_2 & \ddots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & 1+2R_1+2R_2 & -R_2 \\ 0 & \cdots & 0 & -R_2 & 1+2R_1+2R_2 \end{bmatrix},$$

$$R_{D^+} = \begin{bmatrix} 1-2R_1-2R_2 & R_2 & 0 & \cdots & 0 \\ R_2 & 1-2R_1-2R_2 & R_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & R_2 & 1-2R_1-2R_2 & R_2 \\ 0 & \cdots & 0 & R_2 & 1-2R_1-2R_2 \end{bmatrix},$$

$$R_F = R_1 \cdot I, \tag{4.61}$$

$$\widetilde{\mathcal{A}} = \begin{bmatrix} 2B_{1,1}^{n+1} & B_{1,1}^{n+1} & 2B_{1,1}^{n+1} & \cdots & 2B_{1,1}^{n+1} & B_{1,1}^{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2B_{1,Ny-1}^{n+1} & B_{1,Ny-1}^{n+1} & 2B_{1,Ny-1}^{n+1} & \cdots & 2B_{1,Ny-1}^{n+1} & B_{1,Ny-1}^{n+1} \\ 2B_{2,1}^{n+1} & B_{2,1}^{n+1} & 2B_{2,1}^{n+1} & \cdots & 2B_{2,1}^{n+1} & B_{2,1}^{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2B_{2,Ny-1}^{n+1} & B_{2,Ny-1}^{n+1} & 2B_{2,Ny-1}^{n+1} & \cdots & 2B_{2,Ny-1}^{n+1} & B_{2,Ny-1}^{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2B_{Nx,1}^{n+1} & B_{Nx,1}^{n+1} & 2B_{Nx,1}^{n+1} & \cdots & 2B_{Nx,1}^{n+1} & B_{Nx,1}^{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2B_{Nx,Ny-1}^{n+1} & B_{Nx,Ny-1}^{n+1} & 2B_{Nx,Ny-1}^{n+1} & \cdots & 2B_{Nx,Ny-1}^{n+1} & B_{Nx,Ny-1}^{n+1} \end{bmatrix},$$

$$\widetilde{\mathcal{B}} = \begin{bmatrix} 2B^n_{1,1} & B^n_{1,1} & 2B^n_{1,1} & \cdots & 2B^n_{1,1} & B^n_{1,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2B^n_{1,Ny-1} & B^n_{1,Ny-1} & 2B^n_{1,Ny-1} & \cdots & 2B^n_{1,Ny-1} & B^n_{1,Ny-1} \\ 2B^n_{2,1} & B^n_{2,1} & 2B^n_{2,1} & \cdots & 2B^n_{2,1} & B^n_{2,1}2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2B^n_{2,Ny-1} & B^n_{2,Ny-1} & 2B^n_{2,Ny-1} & \cdots & 2B^n_{2,Ny-1} & B^n_{2,Ny-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2B^n_{Nx,1} & B^n_{Nx,1} & 2B^n_{Nx,1} & \cdots & 2B^n_{Nx,1} & B^n_{Nx,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2B^n_{Nx,Ny-1} & B^n_{Nx,Ny-1} & 2B^n_{Nx,Ny-1} & \cdots & 2B^n_{Nx,Ny-1} & B^n_{Nx,Ny-1} \end{bmatrix}.$$

Here $\mathbf{0}$ and I are $(Ny-1)\times(Ny-1)$ matrices which are zero and identity matrices, respectively and

$$B^n_{i,j} = \frac{ht}{2}\frac{hxhy}{9} f^n_{i,j}[8\ 4\ 8\ \cdots\ 4\ 8], \tag{4.62}$$

are $1 \times (Ny-1)$ row vectors for $i = 1, \ldots, Nx$, and $j = 1, \ldots, Ny-1$. Thus, R_{D^-}, R_{D^+}, are $(Ny-1)\times(Ny-1)$ tridiagonal matrices.

We solve the system (4.57) by the Gauss elimination method to obtain V^{n+1} and then we evaluate r^{n+1}.

Now, let us consider the same problem on a non-uniform grid.

4.2.2. Non-uniform finite difference method

We use Crank-Nicholson method for the problem (4.43)-(4.46) on a non-uniform grid and again we add the fictitious point x_{Nx+1} for dealing with the boundary conditions (3.49). Here, we choose the non-uniform grid considering transformed Gauss-Lobatto nodes from the unit square which we use for numerical integration.

$$\frac{v_{i,j}^{n+1}-v_{i,j}^{n}}{ht}=\left(\frac{v_{i-1,j}^{n}+v_{i-1,j}^{n+1}}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})}+\frac{v_{i,j}^{n}+v_{i,j}^{n+1}}{(x_i-x_{i-1})(x_i-x_{i+1})}\right.$$
$$\left.+\frac{v_{i+1,j}^{n}+v_{i+1,j}^{n+1}}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)}\right)$$
$$+\left(\frac{v_{i,j-1}^{n}+v_{i,j-1}^{n+1}}{(y_{j-1}-y_j)(y_{j-1}-y_{j+1})}+\frac{v_{i,j}^{n}+v_{i,j}^{n+1}}{(y_j-y_{j-1})(y_j-y_{j+1})}\right.$$
$$\left.+\frac{v_{i,j+1}^{n}+v_{i,j+1}^{n+1}}{(y_{j+1}-y_{j-1})(y_{j+1}-y_j)}\right)+\frac{1}{2}\left((rf)_{i,j}^{n}+(rf)_{i,j}^{n+1}\right), \tag{4.63}$$

$$\frac{v_{i,j}^{n+1}-v_{i,j}^{n}}{ht}=\left(\frac{v_{i-1,j}^{n}+v_{i-1,j}^{n+1}}{hx_i(hx_i+hx_{i+1})}-\frac{v_{i,j}^{n}+v_{i,j}^{n+1}}{hx_ihx_{i+1}}+\frac{v_{i+1,j}^{n}+v_{i+1,j}^{n+1}}{hx_{i+1}(hx_i+hx_{i+1})}\right)$$
$$+\left(\frac{v_{i,j-1}^{n}+v_{i,j-1}^{n+1}}{hy_j(hy_j+hy_{j+1})}-\frac{v_{i,j}^{n}+v_{i,j}^{n+1}}{hy_jhy_{j+1}}+\frac{v_{i,j+1}^{n}+v_{i,j+1}^{n+1}}{hy_{j+1}(hy_j+hy_{j+1})}\right) \tag{4.64}$$
$$+\frac{1}{2}\left((rf)_{i,j}^{n}+(rf)_{i,j}^{n+1}\right),$$

with the initial condition

$$\varphi_{i,j}=v_{i,j}^{0}, \tag{4.65}$$

nonlocal boundary conditions

$$v_{0,j}^{n}=v_{Nx,j}^{n},\quad v_{Nx+1,j}^{n}=v_{Nx-1,j}^{n},\quad v_{i,0}^{n}=v_{i,Ny}^{n}=0, \tag{4.66}$$

and overdetermination condition

$$r^n=\frac{1}{E^n}\int_0^1\int_0^1 v(x,y,t_n)dxdy, \tag{4.67}$$

where $0=x_0<x_1<\cdots<x_{Nx}=1$, $0=y_0<y_1<\cdots<y_{Ny}=1$, and $t_n=n\cdot ht$ with $ht\cdot Nt=T$,

$$v_{i,j}^{n}=v(x_i,y_j,t_n),\quad r^n=r(t_n),\quad E^n=E(t_n),\quad f_{i,j}^{n}=f(x_i,y_j,t_n), \tag{4.68}$$

$$hx_i=x_i-x_{i-1},\quad h_{Nx+1}=h_{Nx},\quad hy_j=y_j-y_j-1, \tag{4.69}$$

for $i = 1, \ldots, Nx$, $j = 1, \ldots, Ny-1$, $n = 0, \ldots, Nt$. We approximate $\int_0^1 \int_0^1 v(x, y, t)dxdy$ by the product rules on the square using Gauss-Lobatto nodes which are transformed from the unit square and the weights,

$$\int_0^1 \int_0^1 v(x, y, t)dxdy \approx \frac{1}{4} \sum_{i=0}^{Nx} \sum_{j=0}^{Ny} A_{i,j} \cdot v(x_i, y_j, t), \tag{4.70}$$

where $A_{i,j} = A_i A_j$ are products of the weights A_i and A_j. Then, substituting (4.70) into (4.67), we get $Nx(Ny-1) \times Nx(Ny-1)$ linear system of equations, for each $n = 0, \ldots, Nt$, from the equations (4.64)-(4.67) such that

$$\left(\mathcal{C} - \frac{1}{E^{n+1}}\widetilde{\mathcal{C}}\right) V^{n+1} = \left(\mathcal{D} + \frac{1}{E^n}\widetilde{\mathcal{D}}\right) V^n, \tag{4.71}$$

where

$$\begin{aligned} V^n = (&v^n_{1,1}, v^n_{1,2}, \ldots, v^n_{1,Ny-1}, v^n_{2,1}, v^n_{2,2}, \ldots, v^n_{2,Ny-1}, \\ &\ldots, v^n_{Nx,1}, v^n_{Nx,2}, \ldots, v^n_{Nx,Ny-1})^t, \end{aligned} \tag{4.72}$$

$$\mathcal{C} = \begin{bmatrix} R_{D_1^-} & -R_{F_{2,1}} & \mathbf{0} & \cdots & -R_{F_{1,1}} \\ -R_{F_{1,2}} & R_{D_2^-} & -R_{F_{2,2}} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & -R_{F_{1,Nx-1}} & R_{D^-_{Nx-1}} & -R_{F_{2,Nx-1}} \\ \mathbf{0} & \mathbf{0} & \cdots & -R_{F_{1,Nx}} - R_{F_{2,Nx}} & R_{D^-_{Nx}} \end{bmatrix}, \tag{4.73}$$

$$\mathcal{D} = \begin{bmatrix} R_{D_1^+} & R_{F_{2,1}} & \mathbf{0} & \cdots & R_{F_{1,1}} \\ R_{F_{1,2}} & R_{D_2^+} & R_{F_{2,2}} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & R_{F_{1,Nx-1}} & R_{D^+_{Nx-1}} & R_{F_{2,Nx-1}} \\ \mathbf{0} & \mathbf{0} & \cdots & R_{F_{1,Nx}} + R_{F_{2,Nx}} & R_{D^+_{Nx}} \end{bmatrix}, \tag{4.74}$$

$$R_{D_i^-} = \begin{bmatrix} 1+rx_i+ry_1 & -sy_{2,1} & \cdots & 0 \\ -sy_{1,2} & 1+rx_i+ry_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1+rx_i+ry_{Ny-2} & -sy_{2,Ny-2} \\ 0 & \cdots & -sy_{1,Ny-1} & 1+rx_i+ry_{Ny-1} \end{bmatrix},$$

$$R_{D_i^+} = \begin{bmatrix} 1-rx_i-ry_1 & sy_{2,1} & \cdots & 0 \\ sy_{1,2} & 1-rx_i-ry_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1-rx_i-ry_{Ny-2} & sy_{2,Ny-2} \\ 0 & \cdots & sy_{1,Ny-1} & 1-rx_i-ry_{Ny-1} \end{bmatrix},$$

$$R_{F_{1,i}} = sx_{1,i}I, \qquad R_{F_{2,i}} = sx_{2,i}I, \tag{4.75}$$

$$rx_i = \frac{ht}{hx_i hx_{i+1}}, \quad ry_j = \frac{ht}{hy_j hy_{j+1}}, \tag{4.76}$$

$$sx_{1,i} = \frac{ht}{hx_i(hx_i+hx_{i+1})}, \quad sx_{2,i} = \frac{ht}{hx_{i+1}(hx_i+hx_{i+1})}, \tag{4.77}$$

$$sy_{1,j} = \frac{ht}{hy_j(hy_j+hy_{j+1})}, \quad sy_{2,j} = \frac{ht}{hy_{j+1}(hy_j+hy_{j+1})}, \tag{4.78}$$

$$\widetilde{\mathcal{C}} = \begin{bmatrix} D_{1,1}^{n+1} \\ \vdots \\ D_{1,Ny-1}^{n+1} \\ D_{2,1}^{n+1} \\ \vdots \\ D_{2,Ny-1}^{n+1} \\ \vdots \\ D_{Nx,1}^{n+1} \\ \vdots \\ D_{Nx,Ny-1}^{n+1} \end{bmatrix}, \qquad \widetilde{\mathcal{D}} = \begin{bmatrix} D_{1,1}^{n} \\ \vdots \\ D_{1,Ny-1}^{n} \\ D_{2,1}^{n} \\ \vdots \\ D_{2,Ny-1}^{n} \\ \vdots \\ D_{Nx,1}^{n} \\ \vdots \\ D_{Nx,Ny-1}^{n} \end{bmatrix}. \tag{4.79}$$

Here $\mathbf{0}$ and I are $(Ny-1)\times(Ny-1)$ matrices which are zero and identity matrices, respectively and

$$D^n_{i,j} = \frac{ht}{2}\frac{1}{4} f^n_{i,j}[A_{1,1} \ \cdots \ A_{1,Ny-1} \ A_{2,1} \ \cdots \ A_{2,Ny-1} \ \cdots \ A_{Nx,Ny-1} \ \cdots \ A_{Nx,Ny-1}], \tag{4.80}$$

are $1 \times Nx(Ny-1)$ row vectors, $R_{D_i^-}$ and $R_{D_i^+}$ are $(Ny-1)\times(Ny-1)$ tridiagonal matrices for $i = 1, \ldots, Nx$, and $j = 1, \ldots, Ny-1$.

We solve the system (4.71) by the Gauss elimination method to obtain V^{n+1} and then we evaluate r^{n+1}.

Now we consider two examples and see the results of using the product rules on uniform and non-uniform grids. At first let consider the example which does not satisfy the conditions $(A_1) - (A_3)$.

Example 4.3: Consider the inverse problem (1.8)-(1.12)*, with*

$$\begin{aligned} f(x,y,t) = e^{tx(x-1)^2} \big(&9y^2t^2x^4 - 24y^2t^2x^3 + 22y^2t^2x^2 - 8y^2t^2x - 6ytx \\ &+ 6y^2tx - 9yt^2x^4 + 24yt^2x^3 - 22yt^2x^2 + 8yt^2x + yx^3 \\ &-2yx^2 - y^2x^3 + 2y^2x^2 - y^2x + 2 + yx\big), \end{aligned} \tag{4.81}$$

$$\varphi(x,y) = y(1-y), \qquad E(t) = \frac{1}{6}\int_0^1 e^{tx^3-2tx^2+tx}dx, \qquad T = 1. \tag{4.82}$$

It is easy to check

$$\{p(t), u(x,y,t)\} = \{t^2 - 4t, \ y(1-y)e^{tx(x-1)^2}\}. \tag{4.83}$$

Then the problem (4.43)-(4.46) *will be*

$$\begin{aligned} v_t = v_{xx} + v_{yy} + r(t)e^{tx(x-1)^2} \big(&9y^2t^2x^4 - 24y^2t^2x^3 + 22y^2t^2x^2 - 8y^2t^2x \\ &- 6ytx + 6y^2tx - 9yt^2x^4 + 24yt^2x^3 - 22yt^2x^2 + 8yt^2x + yx^3 - 2yx^2 \\ &-y^2x^3 + 2y^2x^2 - y^2x + 2 + yx\big), \quad 0 < x, y < 1, \quad 0 < t < 1, \end{aligned} \tag{4.84}$$

$$v(x,y,0) = y(1-y), \quad 0 \le y \le 1, \tag{4.85}$$

$$v(0,y,t) = v(1,y,t), \quad 0 \le y,t \le 1, \tag{4.86}$$

$$v(x,0,t) = v(x,1,t) = 0, \quad 0 \le x,t \le 1, \tag{4.87}$$

$$v_x(1,y,t) = 0, \quad 0 \le y,t \le 1, \tag{4.88}$$

$$\int_0^1 \int_0^1 v(x,y,t)dxdy = r(t)\frac{1}{6}\int_0^1 e^{tx^3-2tx^2+tx}dx, \tag{4.89}$$

where the exact solution of $r(t)$ *is* $\exp\left(\frac{t^3}{3} - 2t^2\right)$. *We use the Crank-Nicholson implicit finite difference method to solve the problem for finding the values of* u, *and then use* (4.54) *and product Simpson's rule on uniform grid;* (4.67) *and product rule with transformed Gauss-Lobatto nodes and weights to find* $p(t)$ *approximately. We get the following results on Table 4.1 and give the figures (Figure 4.1 - Figure 4.4) for* $Nx = Ny = 26$ *and* $Nt = 750$ *on uniform grid,* $Nt = 1500$ *on non-uniform grid:*

Table 4.1: Errors of $u(x,y,t)$ for $t = 1$ and $p(t)$.

	Error (Uniform)	Error (Non-uniform)
$u(x,y,t)$	6.4009e-05	7.9942e-05
$p(t)$	0.0071	0.0062

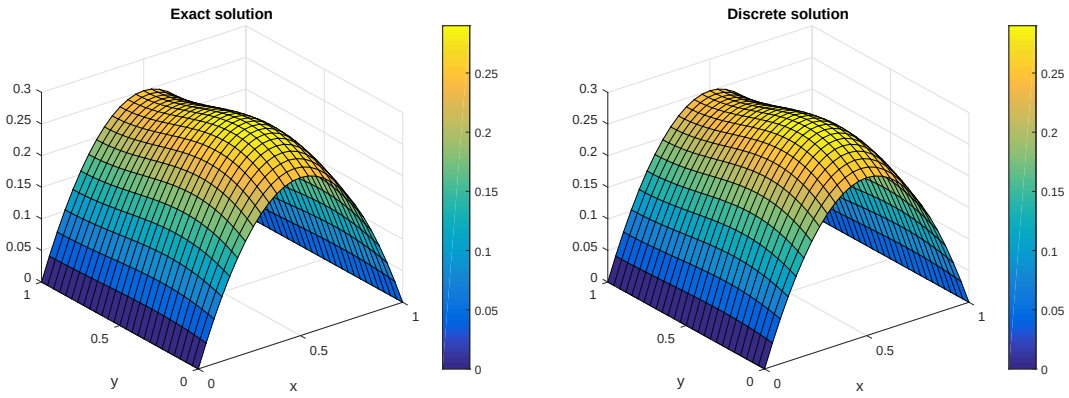


Figure 4.1: Uniform finite difference, exact and discrete solution of $u(x, y, t)$ for $t = 1$.

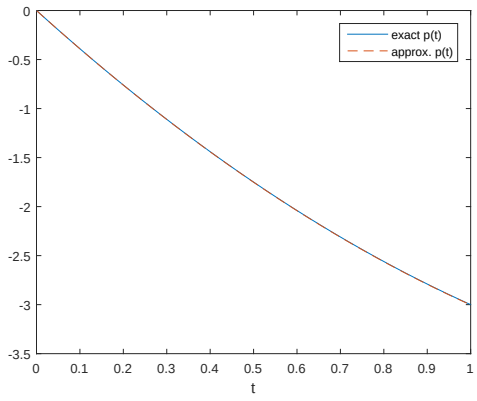


Figure 4.2: Uniform finite difference, exact and approximate solution of $p(t)$.

Now let's consider an example which satisfies the conditions $(A_1) - (A_3)$.

Example 4.4: Consider the inverse problem (1.8)-(1.12)*, with*

$$f(x, y, t) = -\sin(\pi y)e^{-\pi^2 t}\sin(\pi x) \left(9\pi^2\cos^2(\pi x) - 3\pi^2 - e^{\pi^2 t} + e^{\pi^2 t}\cos^2(\pi x)\right), \tag{4.90}$$

$$\varphi(x, y) = \sin(\pi y)\sin^3(\pi x), \quad E(t) = \frac{8}{3\pi^2}e^{-\pi^2 t}, \quad T = \frac{1}{3}. \tag{4.91}$$

It is easy to check that

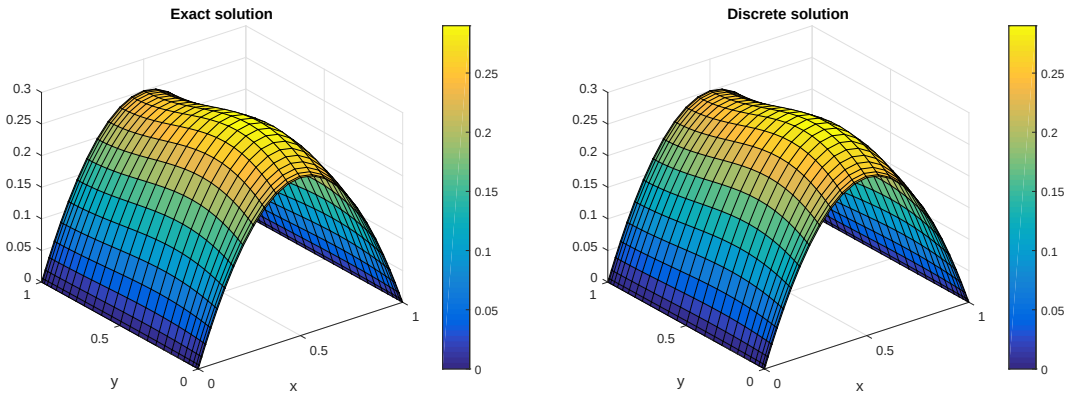


Figure 4.3: Non-uniform finite difference, exact and discrete solution of $u(x, y, t)$ for $t = 1$.

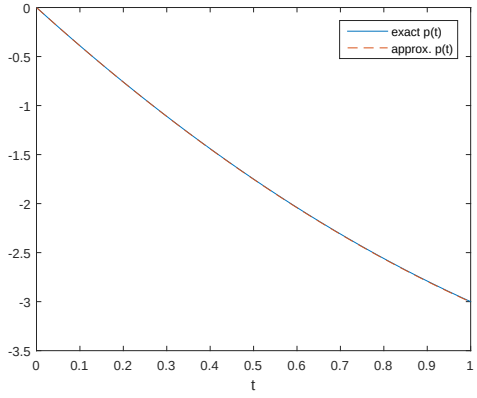


Figure 4.4: Non-uniform finite difference, exact and approximate solution of $p(t)$.

$$\{p(t), u(x, y, t)\} = \{e^{\pi^2 t},\ \sin(\pi y)e^{-\pi^2 t}\sin^3(\pi x)\}. \tag{4.92}$$

Then the problem (4.43)-(4.46) *will be*

$$\begin{aligned} v_t = v_{xx} + v_{yy} + r(t)\Big(-\sin(\pi y)e^{-\pi^2 t}\sin(\pi x)\Big(9\pi^2\cos^2(\pi x) - 3\pi^2 - e^{\pi^2 t} \\ + e^{\pi^2 t}\cos^2(\pi x)\Big)\Big), \quad 0 < x, y < 1, \quad 0 < t < \frac{1}{4}, \end{aligned} \tag{4.93}$$

$$v(x, y, 0) = \sin(\pi y)\sin^3(\pi x), \quad 0 \le x, y \le 1, \tag{4.94}$$

$$v(0,y,t) = v(1,y,t), \quad 0 \le y \le 1,\ 0 \le t \le \frac{1}{4}, \tag{4.95}$$

$$v(x,0,t) = v(x,1,t) = 0, \quad 0 \le x \le 1,\ 0 \le t \le \frac{1}{4}, \tag{4.96}$$

$$v_x(1,y,t) = 0, \quad 0 \le y \le 1,\ 0 \le t \le \frac{1}{4}, \tag{4.97}$$

$$\int_0^1 \int_0^1 v(x,y,t)dxdy = r(t)\frac{8}{3\pi^2}e^{-\pi^2 t}, \tag{4.98}$$

where the exact solution of $r(t)$ *is* $\exp\left(\frac{-1+e^{\pi^2 t}}{\pi^2}\right)$. *We again use Crank-Nicholson implicit finite difference method. Similarly, we give the following results on Table 4.2 and the figures (Figure 4.5 - Figure 4.8) again for* $Nx = Ny = 26$ *and* $Nt = 750$ *on uniform grid,* $Nt = 1500$ *on non-uniform grid:*

Table 4.2: Errors of $u(x,y,t)$ for $t = \frac{1}{3}$ and $p(t)$.

	Error (Uniform)	Error (Non-uniform)
$u(x,y,t)$	1.6231e-04	1.9031e-04
$p(t)$	0.1211	0.1110

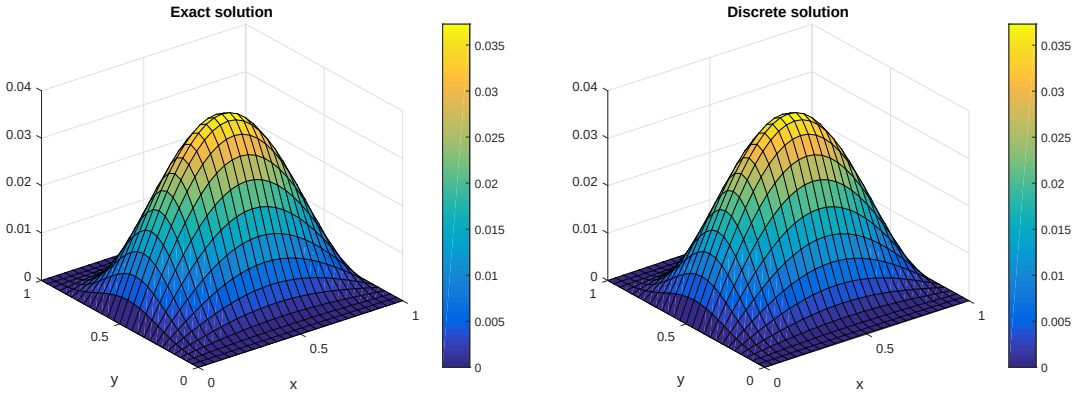


Figure 4.5: Uniform finite difference, exact and discrete solution of $u(x,y,t)$ for $t = \frac{1}{3}$.

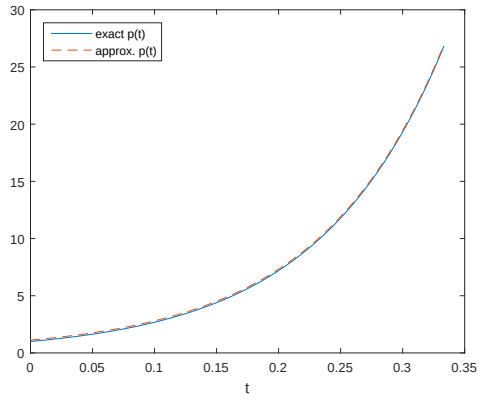


Figure 4.6: Uniform finite difference, exact and approximate solution of $p(t)$.

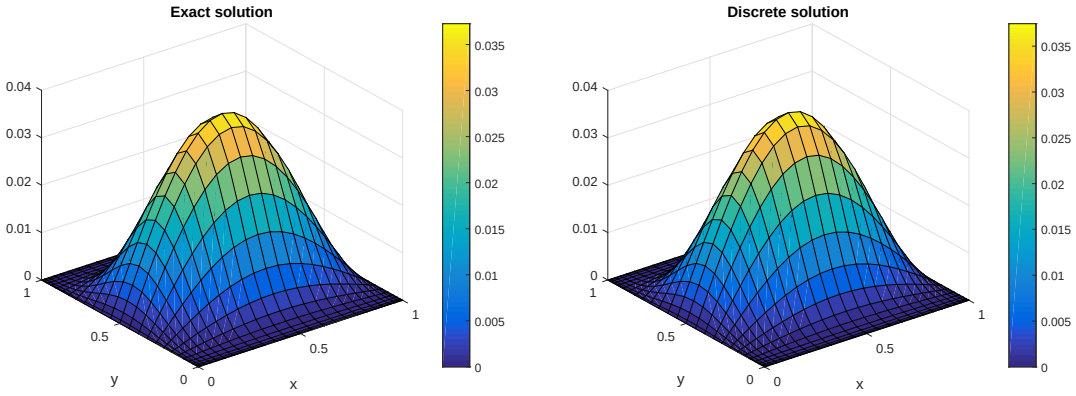


Figure 4.7: Non-uniform finite difference, exact and discrete solution of $u(x, y, t)$ for $t = \frac{1}{3}$.

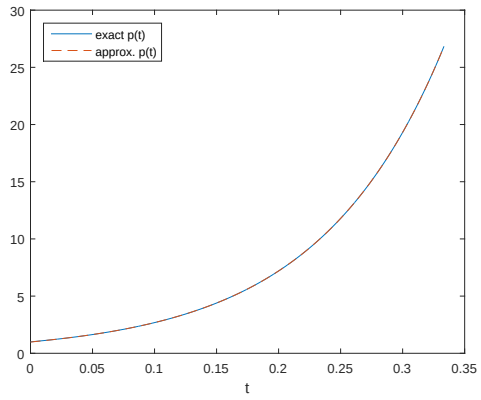


Figure 4.8: Non-uniform finite difference, exact and approximate solution of $p(t)$.

5. CONCLUDING REMARKS

The inverse source problems are investigated for the heat equation with a non-local Wentzell-Neumann boundary condition and integral overdetermination condition in Chapter 3. With some consistency, regularity and orthogonality conditions on the initial data and known part of the source term, the well-posedness of the classical solution is shown by using the generalized Fourier method.

The equation (1.1) turns to (3.47), under the transformation $v(x,t) = r(t)u(x,t)$ and $r(t) = \exp\left(-\int_0^t p(\tau)d\tau\right)$. The introduced inverse problem of such a model gives an idea of how the amount of the total energy might be externally controlled. The external energy is given to a target at a controlled level by the microwave generating equipment. However, the dielectric constant of the target material varies in time and space, arise from spatially heterogeneous conversion of electromagnetic energy to heat. This can correspond to the source term $r(t)f(x,t)$, where $f(x,t)$ is the local conversion rate of microwave energy and $r(t)$ is proportional to the power of the external energy source. If $v(x,t)$ denotes the concentration of absorbed energy, therefore its integral over all volume of the material determining the time dependence absorbed energy.

The numerical methods of the inverse problem are considered as a suitable combination of finite difference discretization and quadrature rules for the numerical integration with different values of α. It is seen from the Example 1 that the numerical methods on non-uniform finite difference discretization with Gauss-Lobatto quadrature give better results for u and p for smaller values of $\alpha > 0$. Otherwise, the numerical results on uniform finite difference discretization with using composite trapezoidal rule for the numerical integration (even though there are better quadrature rules "which means having higher degree of precision" than the composite trapezoidal rule) give better results. Moreover, some tables and some figures are given to illustrate how to implement the method for different value of α in this example and to see the

effect of the given noises to the integral overdetermination condition and their mollification. On the other hand, it is seen that although the initial condition in Example 2 is non-smooth, the numerical results are pretty good. It is also given some noises to the integral overdetermination condition and employ the mollification of them. Also some tables and figures are given in this example.

In Chapter 4, it is considered the problem of determining the lowest coefficient that depends on time only, for a two-dimensional parabolic equation with Ionkin type non-local boundary condition and the total energy measurement. The existence and uniqueness of the solution of such an inverse problem and global well-posedness of this problem are examined by using the method of series expansion in terms of eigenfunctions and associated functions of corresponding spatial differential operator which is nonself adjoint and hence the system of eigenfunctions must be supplemented by associated functions since it is not complete. The numerical method can be considered as a suitable combination of finite difference scheme and numerical integration. The traditional approach is the method of uniform finite difference of the equation, initial condition and boundary conditions combined with numerical integration on a uniform grid of the integral overdetermination condition using the product Simpson's rule. Consequently, the method of non-uniform finite difference is combined by the transformed Gauss-Lobatto nodes from the unit square and weights for the integration. If we use explicit finite difference method we would get better results for both u and p on the non-uniform grid in comparison to the uniform grid [67]. However, we would have to take Nt much greater (for example, 147000 and 2700 for non-uniform and uniform finite difference method, respectively in Example 1) because of satisfying the convergence of the numerical methods. Since Crank-Nicholson method is unconditionally stable [66], the numerical experiments show that it is enough to take Nt, one time greater than what we take for uniform grid, to get better results for only p. Moreover, the uniform finite difference method has a better numerical result for

u than non-uniform finite difference method for any different choice of Nt since it has higher degree of precision on the spatial axes. Two numerical examples illustrate how to implement the method. Some tables and figures demonstrate that the method is effective for the 2D heat equation.

REFERENCES

[1] Naimark M. A., (1967), "Linear Differential Operators: Elementary Theory of Linear Differential Operators", Frederick Ungar Publishing Company, New York, USA.

[2] Wentzell A. D., (1959), "On boundary conditions for multidimensional diffusion processes", Theory of Probability and Its Applications, 4, 164-177.

[3] Luo Y., Trudinger N. S., (1991), "Linear second order elliptic equations with Venttsel boundary conditions", Proceedings of the Royal Society of Edinburgh, 118A , 193-207.

[4] Campiti M., Metafune G., (1998), "Ventcel's boundary conditions and analytic semigroups", Archiv der Mathematik, 70 (5), 377-390.

[5] Favini A., Goldstein G. R., Goldstein J. A., Romanelli S., (2000), "Cosemigroups generated by second order differential operators with general Wentzell boundary conditions", Proceedings of the American Mathematical Society, 128, 1981-1989.

[6] Vogt H., Voigt J., (2003), "Wentzell boundary conditions in the context of Dirichlet forms", Advances in Differential Equations 8 (7), 821-842.

[7] Favini A., Goldstein G. R., Goldstein J. A., Romanelli S., (2006), "The heat equation with nonlinear general Wentzell boundary condition", Advances in Differential Equations 11 (5), 481-510.

[8] Coclite G., Goldstein G. and Goldstein J., (2009), "Stability of parabolic problems with nonlinear Wentzell boundary conditions", Journal of Differential Equations, 246, 2434-2447.

[9] Favini A., Goldstein G. R., Goldstein J. A., Romanelli S., (2002), "The heat equation with generalized Wentzell boundary condition", Journal of Evolution Equations, 2, 1-19.

[10] Vázquez J. L., Vitillaro E., (2011), "Heat equation with dynamical boundary conditions of reactive-diffusive type", Journal of Differential Equations, 250 (4), 2143-2161.

[11] Ismailov M. I., Kanca F., (2011), "An inverse coefficient problem for a parabolic equation in the case of nonlocal boundary and overdetermination conditions", Mathematical Methods in the Applied Sciences, 34, 692-702.

[12] Kerimov N. B., Ismailov M. I., (2015), "Direct and inverse problems for the heat equation with a dynamic-type boundary condition", IMA Journal of Applied Mathematics, 80 (5), 1519-1533.

[13] Cannon J. R., "The one-dimensional heat equation", (1984), Encyclopedia of Mathematics and its Applications (Book 23), Cambridge University Press.

[14] Hintermann T., (1989), "Evolution equations with dynamic boundary conditions", Proceedings of the Royal Society of Edinburgh, 113A, 43-60.

[15] Sauer N., (1989), "Dynamical processes associated with dynamic boundary conditions for partial differential equations", Differential Equations and Applications Ohio University Press, 374-378.

[16] Escher J., (1993), "Quasilinear parabolic systems with dynamical boundary conditions", Communications in Partial Differential Equations, 18 (7-8), 1309-1364.

[17] Igbida N., Kirane M., (2002), "A degenerate diffusion problem with dynamical boundary conditions diffusion problem with dynamical boundary conditions", Mathematische Annalen, 323 (2), 377-396.

[18] Vrabel V., Slodicka M., (2013), "Nonlinear parabolic equation with a dynamical boundary condition of diffusive type", Applied Mathematics and Computation, 222, 372-380.

[19] Slodicka M., (2015), "A parabolic inverse source problem with a dynamical boundary condition", Applied Mathematics and Computation, 256, 529-539.

[20] Grabski J. K., Lesnic D., Johansson B. T., (2016), "Identification of a time-dependent bio-heat blood perfusion coefficient", International Communications in Heat and Mass Transfer, 75, 218-222.

[21] Alifanov O. M., (2011), "Inverse Heat Transfer Problems", Springer-Verlag, Berlin, Germany.

[22] Samarskii A. A., Vabishchevich P. N., (2007),"Numerical Methods for Solving Inverse Problems of Mathematical Physics", De Gruyter, Germany.

[23] Tarantola A., (1987), "Inverse Problem Theory: Methods for Data Fitting and Model Parameter Estimation", Elsevier, Netherlands.

[24] Vogel C. R., (2002), "Computational Methods for Inverse Problems", Society for Industrial and Applied Mathematics, Philadelphia, PA, USA.

[25] Ivanchov M. I., Pabyrivska N. V., (2001), "Simultaneous determination of two coefficients of a parabolic equation in the case of nonlocal and integral condi-

tions", Ukrainian Mathematical Journal, 53 (5), 674-684.

[26] Cannon J. R., Lin Y., Wang S., (1991), "Determination of a control parameter in a parabolic partial differential equation", Journal of the Australian Mathematical Society: Series B, 33, 149-163.

[27] Marchenkov D. B., (2006), "The basis property in the space Lp(0,1) of a system of eigenfunctions corresponding to a problem with a spectral parameter in the boundary condition", Differential Equations, 42 (6), 847-849.

[28] Marchenkov D. B., (2005), "On the convergence of spectral expansions of functions for a problem with a spectral parameter in the boundary condition", Differential Equations, 41 (10), 1496-1500.

[29] Dragomir S. S., (2002), "Some Gronwall Type Inequalities and Applications", RGMIA Monographs, Victoria University, Australia.

[30] Abramowitz M., Stegun A., (1965), "Handbook of Mathematical Functions", Dover, Mineola, NY, USA.

[31] Bowen M. K., Smith R., (2005), "Derivative Formulae and Errors for Non-Uniformly Spaced Points", Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 461 (2059), 1975-1997.

[32] Thomas J. W., (1995), "Numerical Partial Differential Equations: Finite Difference Methods", Texts in Applied Mathematics, vol.22, Springer-Verlag, New York, USA.

[33] Murio D. A., (1993), "The Mollification Method and the Numerical Solution of Ill-Posed Problems", Wiley Interscience, New York, USA.

[34] Kerbal S., (1998), Optimal control of systems governed by B-evolutions". Thesis (Ph.D.) University of Ottawa, 71 pp., Canada.

[35] Shivanian E., ,Jafarabadi A., (2017), "Numerical solution of two-dimensional inverse force function in the wave equation with nonlocal boundary conditions", Inverse Problems in Science and Engineering, 25 (12), 1743-1767.

[36] Shivanian E., Jafarabadi A., (2017), "Inverse Cauchy problem of annulus domains in the framework of spectral meshless radial point interpolation", Engineering with Computers, 33, 431-442.

[37] Hussein M. S., Lesnic D., Ismailov M., (2016), "An inverse problem of finding the time-dependent diffusion coefficient from an integral condition", Mathematical Methods in the Applied Sciences, 39 (5), 963–980.

[38] Isakov V., (1998), “Inverse Problems for Partial Differential Equations”, Springer, Switzerland.

[39] Lavrent’ev M. M., Romanov V. G., Shishatskii S. P., (1986), “Ill-posed Problems of Mathematical Physics and Analysis”, American Mathematical Society.

[40] Choulli M., (1994), “An Inverse Problem for a Semilinear Parabolic Equation”, Inverse Problems, 10, 1123-1132.

[41] Kamynin V. L., (2013), “The Inverse Problem of Determining the Lower-Order Coefficient in Parabolic Equations with Integral Observation”, Mathematical Notes, 94 (1), 205-213.

[42] Kostin A. B., (2015), “Inverse Problem of Finding the Coefficient of u in a Parabolic Equation on the Basis of a Nonlocal Observation Condition”, Differential Equations, 51 (5), 605-619.

[43] Kozhanov A. I., (2006), “Parabolic Equations with an Unknown Absorption Coefficient”, Doklady Mathematics, 74 (1), 573-576.

[44] Kozhanov A. I., (2002), “On Solvability of an Inverse Problem with an Unknown Coefficient and Right-Hand Side for a Parabolic Equation”, Journal of Inverse and Ill-Posed Problems, 10 (6), 611-627.

[45] Prilepko A. I., Solov’ev V. V., (1987), “On the solvability of inverse boundary value problems for the determination of the coefficient preceding the lower derivative in a parabolic equation”, (Russian) Differentsial’nye Uravneniya, 23 (1), 136-143.

[46] Pennes H. H., (1948), “Analysis of tissue and arterial blood temperatures in the resting human forearm”, Journal of Applied Physiology, 1 (2), 93-122.

[47] Lin Y., (1991), “An inverse problem for a class of quasi-linear parabolic equations”, SIAM Journal on Mathematical Analysis, 22, 146-156.

[48] Lesnic D., (2009), “Identification of the time-dependent perfusion coefficient in bio-heat conduction equation”, Journal of Inverse and Ill-posed Problems, 17, 1-12.

[49] Lesnic D., Ivanchov M., (2015), “Determination of the time-dependent perfusion coefficient in the bio-heat equation”, Applied Mathematics Letters, 39, 96-100.

[50] Cannon J. R., Lin Y., Wang S., (1992), “Determination of source parameter in parabolic equations”, Meccanica, 27 (2), 85-94.

[51] Pyatkov S. G., (2004), "Solvability of Some Inverse Problems for Parabolic Equations", Journal of Inverse and Ill-Posed Problems, 12 (4), 397-412.

[52] Kerimov N. B., Ismailov M. I., (2012), "An inverse coefficient problem for the heat equation in the case of nonlocal boundary conditions", Journal of Mathematical Analysis and Applications, 396 (2), 546-554.

[53] Daoud D. S., Subasi D., (2005), "A Splitting up Algorithm for the Determination of the Control Parameter in Multi Dimensional Parabolic Problem", Applied Mathematics and Computation, 166, 584-595.

[54] Dehghan M., (2000), "Numerical Methods for Two-Dimensional Parabolic Inverse Problem with Energy Overspecification", International Journal of Computer Mathematics, 77 (3), 441-455.

[55] Vabishchevich P. N., Vasil'ev V. I., (2014), "Numerical Identification of the Lowest Coefficient of a Parabolic Equation", (Russian) Doklady Mathematics, 89 (2), 179–181, translated from Doklady Akademii Nauk, 455 (3), 258-260.

[56] Bari N. K., (1951), "Biorthogonal systems and bases in Hilbert space", Uchenye Zapiski Moskovskogo Gosudarstvennogo Universiteta, 148, 69-107.

[57] Kolmogorov A. N., Fomin S. V., (1975), "Introductory Real Analysis" (Dover Publications, 1st edition.

[58] Kreyszig E., (1989), "Introductory Functional Analysis with Applications", Wiley, 1st edition, USA.

[59] Burden R. L., Faires J. D., (2010), "Numerical Analysis", Brooks Cole, 9th edition, USA.

[60] Ismailov M. I., Tekin I., Erkovan S., (2018), "An inverse problem for finding the lowest term of a heat equation with Wentzell–Neumann boundary condition", Inverse Problems in Science and Engineering,
https://doi.org/10.1080/17415977.2018.1553968.

[61] Bazan F. S. V., Bedin L., Borges L. S., (2017), "Space-dependent perfusion coefficient estimation in a 2D bioheat transfer problem", Computer Physics Communications, 214, 18-30.

[62] Bedin L., Bazan F. S. V., (2014), "On the 2D bioheat equation with convective boundary conditions and its numerical realization via a highly accurate approach", Applied Mathematics and Computation, 236, 422-436.

[63] Ionkin N. I., Morozova V. A., (2000), "The Two-Dimensional Heat Equation with Nonlocal Boundary Conditions", (Russian) Differentsial'nye Uravneniya,

36 (7), 884-888, translation in Differential Equations, 36 (7), 982-987.

[64] Il'in V. A., (1984), "On the absolute and uniform convergence of the expansions in eigen- and associated functions of a nonselfadjoint elliptic operator", (Russian) Doklady Akademii Nauk SSSR, 274 (1), 19–22.

[65] Gulin A. V., Ionkin N. I., Morozova V. A., (2001), "Stability of a Nonlocal Two-Dimensional Finite-Difference Problem", (Russian) Differentsial'nye Uravneniya, 37 (7), 926-932, translation in Differential Equations, 37 (7), 970-978.

[66] Samarskii A. A., (2001), "The Theory of Difference Schemes", Marcel Dekker Inc., New York, USA.

[67] Ismailov M. I., Erkovan S., (2019), "Inverse Problem of Finding the Coefficient of the Lowest Term in Two-dimensional Heat Equation with Ionkin-type Boundary Condition", Computational Mathematics and Mathematical Physics, 49 (5).

BIOGRAPHY

Sait Erkovan, was born in Istanbul in 1987. Until his high school graduation, he stayed in Istanbul. He went to Atatürk University for his bachelor's degree. He had his Master's degree from Gebze Technical University in 2012. He also worked as a research assistant from 2010 to 2018 in Gebze Technical University. He is married and the father of a daughter.

APPENDICES

Appendix A: Publications Based on the Thesis

Ismailov M. I., Tekin I., Erkovan S., (2018), "An Inverse Problem for Finding the Lowest Term of a Heat Equation with Wentzell–Neumann Boundary Condition ", Inverse Problems in Science and Engineering,
https://doi.org/10.1080/17415977.2018.1553968.

Ismailov M. I., Erkovan S., (2019), "Inverse Problem of Finding the Coefficient of the Lowest Term in Two-dimensional Heat Equation with Ionkin-type Boundary Condition ", Computational Mathematics and Mathematical Physics, accepted, 49 (5).

Appendix B: Conference Presentations Based on the Thesis

Erkovan S., Ismailov M. I., (2017), "Time-dependent Perfusion Coefficient Determination in a 2D Bioheat Transfer Process", International Conference on Applied Analyis and Mathematical Modelling (ICAAMM 2017), 03 – 07 July 2017.

Erkovan S., Ismailov M. I., (2017), "Inverse Problem of Finding Blood Perfusion Coefficient in a 2D Bioheat Equation with Nonlocal Boundary Conditions", GTÜ Lisansüstü Araştırmalar Sempozyumu ve Tanıtım Günleri 2017, 17 – 18 Mayıs 2017.

Erkovan S., Ismailov M. I., (2018), "Numerical Solution of an Inverse Problem of Heat Equation with Wentzell-Neumann Boundary Condition", GTÜ Lisansüstü Araştırmalar Sempozyumu ve Tanıtım Günleri 2018, 14 – 16 Mayıs 2018.