

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

**THE METHOD OF MONOTONE ITERATIVE TECHNIQUES
AND QUASILINEARIZATION IN TIME SCALE**

NOUR H. ALSHARIF
A THESIS SUBMITTED FOR THE DEGREE OF
MASTER OF SCIENCE
DEPARTMENT OF MATHEMATICS

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PROF. DR. COŞKUN YAKAR

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T.C.
GEBZE TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

ZAMAN SKALASINDA MONOTON İTERATİF
TEKNİK VE KUASİLİNEERİZASYON METODU

NOUR H. ALSHARIF
YÜKSEK LİSANS TEZİ
MATEMATİK ANABİLİM DALI

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SUMMARY

In this thesis, we have offered the fundamental definitions and results on time scales. Then we have proved the basic theorems of dynamic systems with IVP on time scale as: existence and uniqueness theorems, dynamic inequalities, existence of extremal solutions, comparison result, linear variation of parameters formula.

It is well known that the method of upper and lower solutions coupled with the monotone iterative technique gives an effective and flexible mechanism for showing constructive existence results in a segment. Therefore, we have discussed the method of upper and lower solutions of dynamic systems on time scale so that we can show the existence of a solution in the closed set. Furthermore, by using upper and lower solutions we have constructed monotone iterates which are solutions of dynamic systems with IVP on time scale such that these sequences converge uniformly and monotonically to extremal solutions of the problem considered on time scale. Moreover, we have generalized monotone iterative technique by using coupled lower and upper solutions for dynamic system with IVP on time scale. Then we have studied the method of mixed monotony such that construct monotone sequences converge uniformly to unique solution of problem considered on time scale when the functions involved do not possess any monotone properties. Finally, we have discussed the method of quasilinearization of dynamic system on time scale by using the method of upper and lower solutions such that it offers monotone sequences which converge to a unique solution but this convergence is superlinear to the solution of the considered problem.

Keywords: Time Scale, Dynamic Systems, Extremal Solutions, Comparison Result, Upper and Lower Solutions, Coupled Upper and Lower Solutions, Monotone Iterative Technique, Mixed Monotony, Quasilinearization Method.

ÖZET

Bu tezde zaman skalalarında temel tanımlar ve sonuçları sunulmuştur. Daha sonra varlık ve teklik teoremi, dinamik eşitsizlikler, ekstrem çözümlerin varlığı, karşılaştırma sonucu ve parametrelerin lineer varyasyon formülleri gibi zaman skalalarında başlangıç değer problemi ile dinamik sistemlerin temel teoremleri ispatlamışızdır. Bu yüzden, monoton iteratif teknik ile eşleşmiş alt ve üst çözümler yönteminin, yapıcı varlık sonuçlarını bir parçada gösteren etkili ve esnek bir aşamada mekanizma sağladığı iyi bilindiğinden dolayı, kapalı kümede bir çözümün varlığını gösterebilmek için zaman skalasında dinamik sistemlerin alt ve üst çözümler yöntemi tartışılmıştır. Ayrıca, alt ve üst çözümleri kullanarak zaman skalasında başlangıç değer problemiyle dinamik sistemlerin çözümü olan monoton iterasyon yapılmış, böylece bu diziler zaman skalasında göz önünde bulundurulanan problemin ekstremal çözümlerine düzgün ve monoton olarak yakınsadıkları gösterilmiştir. Buna ek olarak, zaman skalasında başlangıç değer problemiyle dinamik sisteme eşleşmiş alt ve üst çözümleri kullanarak monoton iterasyon tekniği geliştirilmiştir. Daha sonra ilgili fonksiyonlar herhangi bir monoton özelliğe sahip olmadığında zaman skalasında ele alınan benzersiz bir problem çözümüne düzgün yakınsak monoton dizileri yapılarak karışık monotoni yöntemi araştırılmıştır. Son olarak, bu yakınsaklık ele alınmış bir problemin çözümüne ancak süperlineer olarak benzersiz bir çözüme yakınsak monoton diziler sunduğundan dolayı alt ve üst çözümler yöntemi kullanılarak zaman skalasında dinamik sistemi kuasilineerizasyon tekniği tartışılmıştır.

Anahtar Kelimeler: Zaman Scale, Dinamik Sistemler, Ekstremel Çözümler, Alt ve Üst Çözümler, Eşleşmiş Alt ve Üst Çözümler, Monoton Iterative Teknik, Karışık Monotoni, Kuasilineerizasyon Yöntemi.

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LIST of ABBREVIATIONS and ACRONYMS

<u>Abbreviations</u>	<u>Explanations</u>
<u>and Acronyms</u>	
Δ	: Delta derivative.
rd	: Right dense.
rs	: Right scattered.
ld	: Left dense.
ls	: Left scattered.
σ	: Forward jump operator.
ρ	: Backward jump operator.
T	: Time scale.
T^k	: The set of all nondegenerate points of time scale T .
μ^*	: Graininess map.
$\mathbb{Z}$	: The set of integer numbers .
$\mathbb{R}$	: The set of real numbers.
$\mathbb{N}$	: The set of natural numbers.
C_{rd}	: Right dense continuous.
IVP	: Initial value problem.
$\mathbb{C}$	: The set of complex numbers.
$\mathbb{R} \setminus \mathbb{Q}$	: The set of irrational numbers
$\mathbb{Q}$	: The set of rational numbers.

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1. INTRODUCTION

Hilger introduced time scale when he gives the concept and calculus of measure chains with a view to combining continuous and discrete analysis [1].

The study of dynamic equations on time scale provides an important bridge between the fields of difference and differential equations. This means that the dynamic equation serves as difference equation when choosing the time scale to be the set of integers, but when choosing the time scale to be real numbers, the dynamic equation serves as a differential equation. The main thought is to show a result for a dynamic equation when the domain of the unknown function is so-called time scale $\mathbb{T}$ which is any closed nonempty subset of real numbers $\mathbb{R}$. We start this study by giving some preliminaries of time scale that can be found in Bohner and Peterson [2], [3]. Then we present some definitions and results that will be employed throughout the thesis.

In chapter 3, we mention the necessary theories of dynamic systems on time scale and the basic properties of the solutions of dynamic systems, such that the theory advanced includes the corresponding theories of difference and differential equations at the same time. In Sections, we give the local existence and uniqueness theorems for the dynamic equation with initial value problem (IVP) on time scale of existence condition and uniqueness respectively. We discuss some dynamic inequalities which will be used throughout the thesis. Moreover, we show the existence of extremal solutions and then we show comparison result on a time scale by using dynamic inequality and extremal solutions. We consider linear dynamic equation with initial value problem (IVP) to obtain the variation of parameters formula and give the basic properties of exponential functions on time scale. We refer in this chapter to [2] - [8].

It is well known that the method of monotone iterative technique coupled with the method of lower and upper solutions is effective and a flexible method to prove existence results for nonlinear differential equations [9]- [12]. This method was developed by Bhaskar and West in 2002 and 2004, was considered the generalized monotone iterative technique method with initial value problem (IVP), acquired the existence of minimal and maximal solutions for differential equations when the function consists two monotone functions including the sum of monotone non-decreasing and the other is non-increasing [13], [14] . At this stage, Ramirez and Vatsala extended this work

for Caputo fractional differential equations with periodic boundary condition via initial value problem [15]. Wang and Tian offered a different approach to get a unique solution for the boundary value problem. In the last decades, when using the monotone iterative technique coupled with the method of lower and upper solutions has played a key role in combining an assortment of nonlinear problems and have also shown to be of massive value [16], [17], [18].

In chapter 4, concern with the theoretical and constructive existence theorems employing the method of lower and upper solutions that generate a closed set, we get the existence of solutions in a closed set. Next, we construct monotone sequences from corresponding linear initial value problem of dynamic equations by using lower and upper solutions as an initial iteration, and these sequences that converge uniformly and monotonically to the minimal and maximal solutions with initial value problem (IVP) for nonlinear dynamic equations. Moreover, we have generalized the method of coupled lower and upper solutions for dynamic equations with initial value problem (IVP) on a time scale when f is rd-continuous function and consists of two functions including the difference of two monotones non-decreasing to obtain monotone iterative sequences that converge uniformly and monotonically to extremal solution of dynamic equation on $\mathbb{T}$. Finally, we discuss the method of mixed monotony such that monotone sequences converge to the unique solutions when f does not possess any monotone properties is natural and interesting. We refer in this Chapter to papers [19], [20].

The method of quasilinearization is creating a fruitful history starting with the works by Bellman, Bellman and Kalaba [21], [22]. Lakshmikantham, Leela, Vatsala and many authors have widely developed this method. The most important applications of the quasilinearization method have been to get a sequence of lower bounds that are the solutions of the linear differential equation which converge quadratically. When using the method of quasilinearization coupled with the method of lower and upper solutions and by using the idea Newton-Fourier, we can construct simultaneously lower and upper monotone sequences that converge to the unique solution. This method can also be applied to dynamic equations on time scale. Nevertheless, the convexity assumption that is desired by the quasilinearization method has been a stumbling block for further development.

In chapter 5, we study the method of quasilinearization of dynamic systems with

initial value problem (IVP) on time scale. When we use the method of upper and lower solutions coupled with this method of quasilinearization, it does not only give monotone sequences that converge to a unique solution but also this convergence is super-linear of the original problem. We in this chapter refer to [23], [24] .

2. BASIC DEFINITIONS AND THEOREMS

We start this chapter with basic definitions and results of time scale. We introduce the notion of differentiation and the necessary properties of the derivative on time scale. Next, we give the definition of the right-dense (rd) continuous, regulated functions, the concept of anti-derivative and give some result of the anti-derivative of functions which have important roles in the analysis of continuous and discrete scales in an integrated manner. At the end of this chapter, we introduce an induction principle that is a fundamental tool which is used to some theorems.

2.1. Preliminary Definitions and Results on Time Scales

A time scale denoted by $\mathbb{T}$ is any closed nonempty subset of the real number $\mathbb{R}$, for example

$$\mathbb{R}, \mathbb{Z}, \mathbb{N}, [1, 2] \cup [3, 4], [0, 2] \cup \mathbb{N} \quad \text{and the cantor set}$$

are a time scale, but

$$\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}, \mathbb{C} \quad \text{and} \quad (1, 2)$$

are not a time scale. A time scale $\mathbb{T}$ may or may not be connected, we define the concept of forward and backward jump operators on $\mathbb{T}$ as follows.

Definition 2.1: The forward jump operator $\sigma : \mathbb{T} \longrightarrow \mathbb{T}$ given by

$$\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}. \quad (2.1)$$

Definition 2.2: The backward jump operator $\rho : \mathbb{T} \longrightarrow \mathbb{T}$ given by

$$\rho(t) = \sup\{s \in \mathbb{T}, s < t\}. \quad (2.2)$$

Remark 2.3:

- If $\mathbb{T}$ is bounded above, then $\sigma(\max \mathbb{T}) = \max \mathbb{T}$, and if $\mathbb{T}$ is bounded below, then $\rho(\min \mathbb{T}) = \min \mathbb{T}$.

- If $\emptyset$ is the empty set, then $\sup \mathbb{T} = \inf \emptyset = \sigma(\emptyset)$, and also $\sup \emptyset = \inf \mathbb{T} = \rho(\emptyset)$.

The jump operators enable us to classify the points of $\mathbb{T}$ as follows :

- i) t is right-dense (rd), if $\sigma(t) = t$;
- ii) t is right-scattered (rs), if $\sigma(t) > t$;
- iii) t is left-dense (ld), if $\rho(t) = t$;
- iv) t is left-scattered (ls), if $\rho(t) < t$.

Remark 2.4: The points that are left-scattered and right-scattered at the same time are called isolated; that is, $\rho(t) < t < \sigma(t)$, but the points that are right-dense and left-dense at the same time are called dense; that is, $\rho(t) = t = \sigma(t)$.

Definition 2.5: A point $t \in \mathbb{T}$ is said to be degenerate point if t is maximal and left-scattered.

Definition 2.6: The mapping $\mu^* : \mathbb{T} \longrightarrow [0, \infty)$ is called graininess such that $\mu^*(t) = \mu(\sigma(t), t) = \sigma(t) - t$.

Remark 2.7:

- If $\mathbb{T} \equiv \mathbb{R}$, then $\mu^*(t) = 0$ if and only if $\sigma(t) = t$.
- If $\mathbb{T} \equiv \mathbb{Z}$, then $\mu^*(t) = 1$ if and only if $\sigma(t) = 1 + t$.

Remark 2.8: A triple $(\mathbb{T}, \mu, Y)$ is a dynamical triple if $(\mathbb{T}, \mu)$ is a measure chain and Y is Banach space.

Let $\mathbb{T}^k$ represents the set of all non-degenerate points of the time scale $\mathbb{T}$. In other words, let

$$\mathbb{T}^k := \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty; \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases} \quad (2.3)$$

Definition 2.9: The mapping $f : \mathbb{T} \longrightarrow Y$. If for any $\epsilon > 0$, there exists a neighborhood N of t such that f satisfies

$$| f(\sigma(t)) - f(r) - f_t^\Delta(\sigma(t) - r) | \leq \epsilon | \sigma(t) - r | \text{ for all } r \in N. \quad (2.4)$$

Then f_t^Δ is called delta derivative of f at $t \in \mathbb{T}$.

Definition 2.10: If f has exactly one delta derivative in t then f is differentiable at $t \in \mathbb{T}$.

Definition 2.11: The mapping $f : \mathbb{T} \longrightarrow \mathbb{R}$ is said to be pre-differentiable with region of differentiation D , if f satisfied the following conditions:

- i) if f is continuous;
- ii) $D \subset \mathbb{T}^k$;
- iii) $\mathbb{T}^k \setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$
- iv) f is differentiable at each $t \in D$.

Definition 2.12: Let $f : \mathbb{T} \longrightarrow Y$, if there exists (finite) left sided limit in each left-dense $t \in \mathbb{T}$ and there exists (finite) right sided limit in each right-dense $t \in \mathbb{T}$. Then f is called regulated.

Definition 2.13: Let $f : \mathbb{T} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be right-dense (rd) continuous if it satisfies the following conditions:

- i) It is continuous at each right-dense or maximal point in $\mathbb{T}$;
- ii) The left sided limit $f(t^-, x)$ exists in each left-dense point in $\mathbb{T}$.

We denote it by $f \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$.

We give some results concerning rd-continuous and regulated functions in the following theorem.

Theorem 2.14: Assume that $f : \mathbb{T} \longrightarrow \mathbb{R}$

- i) If f is continuous, then f is rd-continuous.
- ii) If f is rd-continuous, then f is regulated.
- iii) The jump operator σ is rd-continuous.

- iv) If f is regulated or rd-continuous. If $g : \mathbb{T} \longrightarrow \mathbb{R}$ is regulated or rd-continuous, then $f \circ g$ is regulated or rd-continuous.
- v) Every regulated function on a compact interval is bounded.

Remark 2.15: If f is continuous at each left-dense point and each right-dense point, then f is continuous on $\mathbb{T}$.

Theorem 2.16: If $c, d, e \in \mathbb{T}$ and $f, g \in C_{rd}[\mathbb{T}, \mathbb{R}]$, then

- i) $\int_c^d f(t) \Delta t = \int_c^e f(t) \Delta t + \int_e^d f(t) \Delta t$;
- ii) $\int_c^c f(t) \Delta t = 0$;
- iii) If $f(t) \geq 0$ for all $c \leq t \leq d$, then $\int_c^d f(t) \Delta t$;
- iv) $\int_c^d f(t) \Delta t = - \int_d^c f(t) \Delta t$;
- v) If $|f(t)| \leq g(t)$ on $[c, d]$, then

$$\left| \int_c^d f(t) \Delta t \right| \leq \int_c^d g(t) \Delta t. \quad (2.5)$$

Theorem 2.17: Let f and g be real valued functions defined on $\mathbb{T}$, both pre-differentiable with D . Then

$$|f^\Delta(t)| \leq g^\Delta(t) \quad \text{for all } t \in D \quad (2.6)$$

implies that

$$|f(b) - f(a)| \leq g(b) - g(a) \quad \text{for all } a, b \in \mathbb{T}, a \leq b. \quad (2.7)$$

Corollary 2.18:

- i) The mapping $f : \mathbb{T} \longrightarrow Y$ is pre-differentiable with D and N is some compact interval of $\mathbb{T}$ with endpoints a and b . Then we have

$$|f(b) - f(a)| \leq \sup_{t \in N^k \cap D} \{|f^\Delta(t)|\} |\mu(b, a)|. \quad (2.8)$$

- ii) If f be pre-differentiable with D and $f^\Delta(t) = 0$ for all $t \in D$, then f is constant.

In this regard, we give generalization of Theorem 2.17.

Theorem 2.19: Let f be a continuous function on $[a, b] \subset \mathbb{T}$ and be pre-differentiable on $[a, b)$ with region of differentiation D . Then there exist $\tau, \eta \in D$ such that

$$f^\Delta(\eta) \leq \frac{f(b) - f(a)}{b - a} \leq f^\Delta(\tau). \quad (2.9)$$

Theorem 2.20: Suppose that $f : \mathbb{T} \longrightarrow \mathbb{R}$ is a function and $t \in \mathbb{T}^k$. Then we have the following:

i) Every differentiable function is continuous.

ii) If f is continuous at t and $\mu^*(t) > 0$, then f is differentiable at t

$$f_t^\Delta = \frac{f(\sigma(t)) - f(t)}{\mu^*(t)} \quad (2.10)$$

iii) If f is differentiable at t and $\mu^*(t) = 0$, then

$$f'(t) = f_t^\Delta = \lim_{t \rightarrow t_0} \frac{f(t) - f(t_0)}{t - t_0} \quad (2.11)$$

iv) If f is differentiable at t , then

$$f(\sigma(t)) = f(t) + \mu(t)f_t^\Delta \quad (2.12)$$

v) Let $f, g : \mathbb{T} \longrightarrow X$ be differentiable $t \in \mathbb{T}^k$. Then for $\alpha, \beta \in k$

$$(\alpha f + \beta g)_t^\Delta = \alpha f_t^\Delta + \beta g_t^\Delta. \quad (2.13)$$

vi) Let $f, g : \mathbb{T} \longrightarrow X$ be differentiable in $t \in \mathbb{T}^k$. Then for the product rule, which is given by

$$(fg)_t^\Delta = f_t^\Delta g(t) + f(\sigma(t))g_t^\Delta. \quad (2.14)$$

and the quotient rule which is given by

$$\left(\frac{f}{g}\right)_t^\Delta = \frac{f_t^\Delta g(t) - f(t)g_t^\Delta}{g(t)g(\sigma(t))}. \quad (2.15)$$

where $g(t)g(\sigma(t)) \neq 0$.

Definition 2.21: A function $F : \mathbb{T}^k \longrightarrow \mathbb{R}$ is said to be anti-derivative of $f : \mathbb{T} \longrightarrow \mathbb{R}$ if it satisfies

$$F^\Delta(t) = f(t) \quad (2.16)$$

for all $t \in \mathbb{T}^k$.

Remark 2.22:

- *In this case, we give the delta integral of f by*

$$\int_c^t f(s) \Delta s = F(t) - F(c) \quad (2.17)$$

for $t \in \mathbb{T}$.

- *If $\mathbb{T} = \mathbb{R}$, then $g^\Delta(t) = g'(t)$, and if $\mathbb{T} = \mathbb{Z}$, then*

$$g^\Delta(t) = \Delta g(t) = g(t+1) - g(t). \quad (2.18)$$

We mention an important result of the induction principle that is as a basic tool in the argument.

Theorem 2.23: Let $t_0 \in \mathbb{T}$ and assume that $\{B(t) : t \in [t_0, \infty)\}$ is a family of statement satisfying:

- I) The statement $B(t_0)$ is true;*
- II) If $t \in [t_0, \infty)$ is right-scattered and $B(t)$ is true, then $B(\sigma(t))$ is also true;*
- III) If $t \in (t_0, \infty)$ is right-dense and $B(t)$ is true, then there is a neighborhood N of t such that $B(s)$ is true for all $s \in N \cap (t, \infty)$;*
- IV) If $t \in (t_0, \infty)$ is left-dense and $B(s)$ is true for all $s \in [t_0, t)$, then $B(t)$ is true.*

Remark 2.24: If $\mathbb{T} = \mathbb{N}$, then Theorem (2.23) reduces to the well-known principle of mathematical induction.

3. BASIC THEORY

In this chapter, we show an important results for dynamic system on time scale $\mathbb{T}$ which help us to prove the next results.

3.1. Local Existence and Uniqueness

We consider the local existence and uniqueness results for dynamic systems on $\mathbb{T}$ corresponding to Peanon's and Perron's Theorems [8].

Consider the dynamic initial value problem (IVP)

$$u^\Delta = f(t, u), \quad t \in \mathbb{T}^k, \quad u(t_0) = u_0 \quad (3.1)$$

where $f : \mathbb{T}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is rd-continuous on $\mathbb{T}^k \times \mathbb{R}^n$. The mapping $u : \mathbb{T}^k \longrightarrow \mathbb{R}^n$ is a solution of IVP, if $u(t)$ is an anti-derivative of $f(t, u(t))$ on $\mathbb{T}^k$ and satisfies $u(t_0) = u_0$.

Definition 3.1: The mapping $f : [t_0, t_0 + \alpha]_{\mathbb{T}^k} \times D \longrightarrow \mathbb{R}^n$ where $D \subseteq \mathbb{R}^n$ and $[t_0, t_0 + \alpha]$ be an arbitrary interval in $\mathbb{T}^k$. Then $u : [t_0, t_0 + \alpha]_{\mathbb{T}} \longrightarrow \mathbb{R}^n$ is a solution of (3.1) on $[t_0, t_0 + \alpha]_{\mathbb{T}}$, and u is delta-differentiable with $u^\Delta(t) = f(t, u(t))$, for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}^k}$, $u(t_0) = u_0$.

Lemma 3.2: If $f : [t_0, t_0 + \alpha]_{\mathbb{T}^k} \times D \longrightarrow \mathbb{R}^n$ be rd-continuous. Then a function $u(t)$ is a solution of (3.1) on $[t_0, t_0 + \alpha]_{\mathbb{T}}$ if and only if it satisfies the delta integral equation

$$u(t) = u_0 + \int_{t_0}^t f(s, u(s)) \Delta s, \quad \forall t \in [t_0, t_0 + \alpha]_{\mathbb{T}}. \quad (3.2)$$

Proof. Let u be a solution of (3.1) on $[t_0, t_0 + \alpha]_{\mathbb{T}}$. Then u is delta differentiable on $[t_0, t_0 + \alpha]_{\mathbb{T}^k}$, and so that it is continuous on $[t_0, t_0 + \alpha]_{\mathbb{T}^k}$. Now, integration of equation (3.1) on $[t_0, t_0 + \alpha]_{\mathbb{T}}$, gives

$$\int_{t_0}^t u^\Delta(s) \Delta s = \int_{t_0}^t f(s, u(s)) \Delta s, \quad \forall t \in [t_0, t_0 + \alpha]_{\mathbb{T}}. \quad (3.3)$$

Then

$$u(t) = u_0 + \int_{t_0}^t f(s, u(s)) \Delta s, \quad \forall t \in [t_0, t_0 + \alpha]_{\mathbb{T}}. \quad (3.4)$$

Hence, u is the solution of (3.2).

Conversely, let u satisfies (3.2). differentiating the equation(3.2) with respect to t , gives

$$u^\Delta(t) = f(t, u(t)) \quad (3.5)$$

for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}^k}$ and $u(t_0) = u_0$. Furthermore, since u is continuous on $t \in [t_0, t_0 + \alpha]_{\mathbb{T}^k}$ and f is rd-continuous on $t \in [t_0, t_0 + \alpha]_{\mathbb{T}^k} \times D$, then u^Δ is rd-continuous on $t \in [t_0, t_0 + \alpha]_{\mathbb{T}^k}$ such that $u(t) \in D$. Hence, u is a solution of (3.1). $\square$

We consider Peano's type existence result.

Theorem 3.3: Let $f \in C_{rd}[\mathbb{R}^0, \mathbb{R}^n]$ where $\mathbb{R}^0 = [t_0, t_0 + a] \times A$, $[t_0, t_0 + a]$ is understood as $[t_0, t_0 + a] \cap \mathbb{T}^k$ and $A = \{u \in \mathbb{R}^n : |u - u_0| \leq b\}$ and $|f(t, u)| \leq K$ on $\mathbb{R}^0$. Then the dynamic initial value problem (3.1) has at least one solution $u(t)$ on $[t_0, t_0 + \tau]$ where $\tau = \min(a, \frac{b}{K})$.

Proof. See in [7]. $\square$

We consider Perron's type uniqueness result.

Theorem 3.4: Assume that

- i) $g \in C_{rd}[[t_0, t_0 + a] \times [0, 2b], \mathbb{R}^+]$ and for every $t_1, t_0 \leq t_1 \leq t_0 + a$, $x(t) \equiv 0$ is the only solution of

$$x^\Delta = g(t, x), \quad x(t_1) = 0, \text{ on } [t_1, t_0 + a]. \quad (3.6)$$

- ii) $f \in C_{rd}[\mathbb{R}_0, \mathbb{R}^n]$ and for each $t \in [t_0, t_0 + a]$, there exists a compact neighborhood N_t such that $f^{[t]}$ in $N_t \times A$ satisfies

$$|f(t, u) - f(t, v)| \leq g(t, |u - v|), \quad (t, u), (t, v) \in N_t \times B. \quad (3.7)$$

Then the dynamic initial value problem (3.1) has a unique solution $u(t)$ on $[t_0, t_0 + a]$.

Proof. See in [7]. □

Remark 3.5: If $\mathbb{T} = \mathbb{R}$ and f satisfies the following conditions

- i) $f(t, u)$ is continuous on t for all $t \in \mathbb{R}$,
- ii) $|f_u(t, u)| \leq K$.

Then $f(t, u)$ is continuous on (t, u) and $|f_u(t, u)| \leq K$ implies that the existence and uniqueness of solutions in the classical case [25].

3.2. Dynamic Inequalities

We show a fundamental results on scalar dynamic inequalities, which will be used throughout the thesis. In this regard, we proceed to define the following concepts.

Definition 3.6: Let f be a continuous function on $[c, d]$ that is differentiable on $[c, d]$. If $f^\Delta(t) > 0$ and $f^\Delta(t) \geq 0$ for all $t \in [c, d]$, then f is increasing and non-decreasing on $[c, d]$, respectively.

Definition 3.7: Let f be a continuous function on $[c, d]$ that is differentiable on $[c, d]$. If $f^\Delta(t) < 0$ and $f^\Delta(t) \leq 0$ for all $t \in [c, d]$, then f is decreasing and non-increasing on $[c, d]$, respectively.

Definition 3.8: Let $f \in C[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$, if $u \leq v$ and $u_j = v_j$, for some $1 \leq j \leq n$ implies $f_j(t, u) \leq f_j(t, v)$ for $t \in \mathbb{T}$, then f is quasi-monotone non-decreasing.

Definition 3.9: A function $f : [t_0, t_0 + a]_{\mathbb{T}}^k \times D \rightarrow \mathbb{R}^n$, $D \subset \mathbb{R}^n$ is said to satisfy Lipschitz condition in a region $[t_0, t_0 + a]_{\mathbb{T}}^k \times D$, if there exists a positive constant K such that

$$|f(t, u) - f(t, v)| \leq K|u - v| \quad (3.8)$$

whenever the points (t, u) and (t, v) both lie in $[t_0, t_0 + a]_{\mathbb{T}}^k \times D$.

The constant K is called a Lipschitz constant for f on $[t_0, t_0 + a]_{\mathbb{T}}^k \times D$.

Definition 3.10: Let $f : [t_0, t_0 + a]_{\mathbb{T}}^k \times D \longrightarrow \mathbb{R}^n$ be rd-continuous. If there exists $K > 0$ such that f satisfies

$$f(t, u) - f(t, v) \leq K(u - v), \text{ for all } u > v \quad (3.9)$$

for $(t, u), (t, v) \in [t_0, t_0 + a]_{\mathbb{T}}^k \times D$, then f is said to satisfy a one-sided Lipschitz condition on $[t_0, t_0 + a]_{\mathbb{T}}^k \times D$.

Consider the dynamic initial value problem (IVP)

$$u^\Delta = f(t, u), u(t_0) = u_0. \quad (3.10)$$

Now, we show a result relative to a system of strict dynamic inequalities.

Theorem 3.11: Let $\mathbb{T}$ be the time scale with $t_0 \geq 0$ minimal element and no maximal element, such that

i) $\alpha, \beta \in C_{rd}^1[\mathbb{T}, \mathbb{R}^n]$ for each $t \in \mathbb{T}$ and satisfy

$$\alpha^\Delta(t) \leq f(t, \alpha(t)), \beta^\Delta(t) > f(t, \beta(t)), \quad t \in \mathbb{T} \quad (3.11)$$

ii) $f \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$, $f(t, u)$ is quasi-monotone non-decreasing in u , and for $1 \leq j \leq n$, $f_j(t, u)\mu^*(t) + u_j$ is non-decreasing in u_j for each $t \in \mathbb{T}$.

Then $\alpha(t_0) < \beta(t_0)$ implies $\alpha(t) < \beta(t)$ for $t \in \mathbb{T}$.

Proof. Applying the induction principle to the statement:

$$\{B(t) : \alpha(t) < \beta(t), t \in \mathbb{T}\}$$

I) $B(t_0)$ is clearly satisfied since $\alpha(t_0) < \beta(t_0)$.

II) Let t be right-scattered and $B(t)$ is true. We need to show that $B(\sigma(t))$ is true. It follows from $f(t, u)$ is non-decreasing in u and $\alpha(t) < \beta(t)$ that

$$f(t, \alpha(t)) < f(t, \beta(t)). \quad (3.12)$$

By using Theorem 2.20 ii, we get

$$\alpha^\Delta(t) = \frac{\alpha(\sigma(t)) - \alpha(t)}{\mu^*(t)}, \quad (3.13)$$

$$\beta^\Delta(t) = \frac{\beta(\sigma(t)) - \beta(t)}{\mu^*(t)}. \quad (3.14)$$

Then

$$\alpha^\Delta(t) - \beta^\Delta(t) = \frac{\alpha(\sigma(t)) - \alpha(t) - \beta(\sigma(t)) + \beta(t)}{\mu^*(t)}, \quad (3.15)$$

we reduces to

$$\alpha(\sigma(t)) - \beta(\sigma(t)) = [\alpha^\Delta(t) - \beta^\Delta(t)]\mu^*(t) + \alpha(t) - \beta(t). \quad (3.16)$$

Since $f(t, u)\mu^*(t) + u$ is non-decreasing in u , $t \in \mathbb{T}$ and $B(t)$ is true, then we have

$$\alpha(\sigma(t)) - \beta(\sigma(t)) < [f(t, \alpha(t)) - f(t, \beta(t))]\mu^*(t) + \alpha(t) - \beta(t) < 0. \quad (3.17)$$

Therefore

$$\alpha(\sigma(t)) < \beta(\sigma(t)). \quad (3.18)$$

Hence, $B(\sigma(t))$ is true.

III) Let t be right-dense and N be a neighborhood of t . Assume that $B(t)$ is true. We want to show that $B(s)$ is true for $s \geq t, s \in N$.

Suppose not, then there exists $s_0 \in N$ with $s_0 > t$ such that

$$\alpha(s_0) \leq \beta(s_0) \quad \text{and} \quad \alpha^\Delta(s_0) \geq \beta^\Delta(s_0). \quad (3.19)$$

Since $f(t, u)$ is non-decreasing in u , then

$$f(s_0, \alpha(s_0)) \leq f(s_0, \beta(s_0)). \quad (3.20)$$

Thus,

$$\alpha^\Delta(s_0) - f(s_0, \alpha(s_0)) \geq \beta^\Delta(s_0) - f(s_0, \beta(s_0)) \quad (3.21)$$

which contradiction to (3.11).

IV) Let t be left-dense such that $B(s)$ is true for $s < t$. We want to show that $B(t)$ is true. By using definition of rd-continuous of α and β , it follows that

$$\alpha(t) = \lim_{s \rightarrow t^-} \alpha(s) \leq \lim_{s \rightarrow t^-} \beta(s) = \beta(t). \quad (3.22)$$

It remains to show that $\alpha(t) \neq \beta(t), t \in \mathbb{T}$. Assume on the contrary that

$$\alpha(t) = \beta(t). \quad (3.23)$$

Then

$$\alpha^\Delta(t) - \beta^\Delta(t) < f(t, \alpha(t)) - f(t, \beta(t)) = 0 \quad (3.24)$$

By using Definition 2.9, we get for $1 < i < m$,

$$\alpha_i^\Delta(\sigma(t)) - \alpha_i(s) - \alpha_i^\Delta(t)(\sigma(t) - s) \leq \frac{\epsilon}{2}|\sigma(t) - s| \quad (3.25)$$

$$\beta_i^\Delta(\sigma(t)) - \beta_i(s) - \beta_i^\Delta(t)(\sigma(t) - s) \geq \frac{\epsilon}{2}|\sigma(t) - s| \quad (3.26)$$

implies that

$$\alpha^\Delta(\sigma(t)) - \alpha_i(s) - \frac{\epsilon}{2}|\sigma(t) - s| \leq \alpha_i^\Delta(t)(\sigma(t) - s) \quad (3.27)$$

$$-\beta_i^\Delta(\sigma(t)) + \beta_i(s) + \frac{\epsilon}{2}|\sigma(t) - s| \leq -\beta_i^\Delta(t)(\sigma(t) - s). \quad (3.28)$$

We reduces to

$$(\alpha_i^\Delta(t) - \beta_i^\Delta(t))(\sigma(t) - s) \geq \alpha_i^\Delta(\sigma(t)) - \beta_i^\Delta(\sigma(t)) - \alpha_i(s) + \beta_i(s). \quad (3.29)$$

Since $\sigma(t) - s \geq t - s > 0$, $\alpha(t) = \beta(t)$ and $B(s)$ is true, then we arrive at

$$\alpha_i^\Delta(t) - \beta_i^\Delta(t) > 0 \quad (3.30)$$

which contradicts with (3.24). Hence, we conclude that

$$\alpha(t) < \beta(t), t \in \mathbb{T}. \quad (3.31)$$

□

The next result is for non-strict dynamic inequalities which requires a one-sided Lipschitz-type condition.

Theorem 3.12: Let $\mathbb{T}$ be the time scale with $t_0 \geq 0$ minimal element but not maximal element and such that

i) $\alpha, \beta \in C_{rd}^1[\mathbb{T}, \mathbb{R}^n]$ for each $t \in \mathbb{T}$ and satisfy

$$\begin{aligned} \alpha^\Delta(t) &\leq f(t, \alpha(t)), & \alpha(t_0) &\leq u_0 \\ \beta^\Delta(t) &\geq f(t, \beta(t)), & \beta(t_0) &\geq u_0. \end{aligned} \quad (3.32)$$

ii) $f \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$, $f(t, u)$ is quasi-monotone non-decreasing in u and $f_j(t, u)\mu^*(t) + u_j$ is non-decreasing in u_j for each j , $1 \leq j \leq n$, for $t \in \mathbb{T}$.

Moreover, let $f(t, u)$ satisfies the following condition

$$f_j(t, u) - f_j(t, v) \leq K \sum_{j=1}^m (u_j - v_j) \quad \text{for } u \geq v, K > 0. \quad (3.33)$$

Then $\alpha(t_0) \leq \beta(t_0)$ implies that $\alpha(t) \leq \beta(t)$ for $t \in \mathbb{T}$.

Proof. Let us apply the induction principle to the statement :

$$\{B(t) : \alpha(t) \leq \beta(t), t \in \mathbb{T}\}$$

I) $B(t_0)$ is clearly true since $\alpha(t_0) \leq \beta(t_0)$.

II) Let t be right-scattered and $B(t)$ is true. We need to show that $B(\sigma(t))$ is true.

By using Theorem 2.20, we find

$$\alpha^\Delta(t) = \frac{\alpha(\sigma(t)) - \alpha(t)}{\mu^*(t)} \quad (3.34)$$

$$\beta^\Delta(t) = \frac{\beta(\sigma(t)) - \beta(t)}{\mu^*(t)}, \quad (3.35)$$

we reduce to

$$\alpha(\sigma(t)) - \beta(\sigma(t)) = [\alpha^\Delta(t) - \beta^\Delta(t)]\mu^*(t) + \alpha(t) - \beta(t). \quad (3.36)$$

Since $f(t, u)\mu^*(t) + u$ is non-decreasing in u , $B(t)$ is true and (3.32), implies that

$$\alpha(\sigma(t)) - \beta(\sigma(t)) \leq [f(t, \alpha(t)) - f(t, \beta(t))]\mu^*(t) + \alpha(t) - \beta(t) \leq 0. \quad (3.37)$$

Thus,

$$\alpha(\sigma(t)) \leq \beta(\sigma(t)). \quad (3.38)$$

Hence, $B(\sigma(t))$ is true.

III) Let t be right-dense and N be a neighborhood of t . Assume that $B(t)$ is true. We want to show that $B(s)$ is true for $s \geq t, s \in N$. Consider

$$\beta_0(t) = \beta(t) + \epsilon e^{(n+1)K(t-s)} \quad (3.39)$$

where ϵ is the n -dimensional vector $(\epsilon, \epsilon, \dots, \epsilon)$, $\epsilon > 0$. We want to show that $\alpha(s) < \beta_0(s)$ for $s \geq t, s \in N$.

If this is not true, there exists $s_0 > t, s_0 \in N$ and an index $i, 1 \leq i \leq m$ such that

$$\alpha_i(s_0) = \beta_{0i}(s_0), \quad (3.40)$$

$$\alpha_j(s_0) \leq \beta_{0j}(s_0) \quad (3.41)$$

$j \neq i$ and

$$\alpha_j(s) < \beta_{0j}(s), \quad s \in [t, s_0]. \quad (3.42)$$

Since

$$\alpha(s_0 - \mu^*) \leq \beta_{0i}(s_0 - \mu^*), \quad (3.43)$$

then

$$\alpha(s_0 - \mu^*) - \alpha(s_0) \leq \beta_{0i}(s_0 - \mu^*) - \beta_{0i}(s_0) \quad (3.44)$$

Multiplying the equation (3.44) by $(\frac{1}{\mu^*})$ to both sides, gives

$$\alpha^\Delta(s_0) \geq \beta_{0i}^\Delta(s_0). \quad (3.45)$$

Since (3.39), we obtain

$$\beta_{0i}^\Delta(s_0) \geq f(t, \beta_i(s_0)) + \epsilon K(m+1)e^{(m+1)K(s-s_0)}. \quad (3.46)$$

By using quasi-monotone property of f and (3.33), implies that

$$\begin{aligned} & f_i(s_0, \alpha_1(s_0), \dots, \alpha_i(s_0), \dots, \alpha_m(s_0)) \\ & \geq f_i(s_0, \beta_1(s_0), \dots, \beta_i(s_0), \dots, \beta_m(s_0)) + \epsilon K(m+1)e^{K(m+1)(s_0-s)} \\ & \geq f_i(s_0, \beta_{01}(s_0), \dots, \beta_{0i}(s_0), \dots, \beta_{0n}(s_0)) + K\epsilon e^{K(m+1)(s_0-s)} \\ & > f_i(s_0, \alpha_1(s_0), \dots, \alpha_i(s_0), \dots, \alpha_m(s_0)) \end{aligned} \quad (3.47)$$

This contradiction proves

$$\beta_0(s) > \alpha(s). \quad (3.48)$$

Since ϵ is arbitrary, we get letting $\epsilon \rightarrow 0$, $\alpha(s) \leq \beta(s)$, $s \in N$, $s \geq t$ and therefore $B(s)$ is true.

IV) Let t be left-dense such that $B(s)$ is true for $s < t$. We need to show that $B(t)$ is true. By using definition of rd-continuous of α and β , it follows that

$$\alpha(t) = \lim_{s \rightarrow t^-} \alpha(s) \leq \lim_{s \rightarrow t^-} \beta(s) = \beta(t) \quad (3.49)$$

Hence, $B(t)$ is true. $\square$

We conclude the following corollaries which are useful to prove the next theorems.

Corollary 3.13: Let $q \in C_{rd}^1[\mathbb{T}, \mathbb{R}^n]$ such that

$$q^\Delta(t) \leq 0; \quad q(t_0) \leq 0. \quad (3.50)$$

Then $q(t) \leq 0$ on $\mathbb{T}$. Similarly, if

$$q^\Delta(t) \geq 0; \quad q(t_0) \geq 0. \quad (3.51)$$

Then $q(t) \geq 0$ on $\mathbb{T}$.

Corollary 3.14: Let $q \in C_{rd}^1[\mathbb{T}, \mathbb{R}^n]$ such that

$$q^\Delta(t) \leq -Lq(t), \quad q(t_0) \leq 0, \quad (3.52)$$

for $t \in \mathbb{T}$ and $L > 0$. Then $q(t) \leq 0$ for $t \in \mathbb{T}$. Similarly, if

$$q^\Delta(t) \geq -Lq(t); \quad q(t_0) \geq 0. \quad (3.53)$$

for $t \in \mathbb{T}$ and $L > 0$. Then $q(t) \geq 0$ for $t \in \mathbb{T}$.

3.3. Existence of Extremal Solutions

We consider the existence of extremal solutions for dynamic equations on time scale $\mathbb{T}$ under the hypothesis of Peano's existence result.

Definition 3.15: Let $r(t)$ be the solution of

$$u^\Delta = f(t, u), \quad u(t_0) = u_0 \quad (3.54)$$

for $t \geq t_0$, $t \in [t_0, t_0 + c]_{\mathbb{T}}$. Then $r(t)$ is said to be a maximal solution of IVP (3.54), if $u(t)$ is any solution of (3.54) on $t \in [t_0, t_0 + c]_{\mathbb{T}}$ satisfying the inequality $u(t) \leq r(t)$, $t \in [t_0, t_0 + c]_{\mathbb{T}}$.

Definition 3.16: Let $\rho(t)$ be the solution of

$$u^\Delta = f(t, u), \quad u(t_0) = u_0 \quad (3.55)$$

for $t \geq t_0$, $t \in [t_0, t_0 + c]_{\mathbb{T}}$. Then $\rho(t)$ is said to be a minimal solution of IVP (3.55), if $u(t)$ is any solution of (3.55) on $[t_0, t_0 + c]_{\mathbb{T}}$ satisfying the inequality $\rho(t) \leq u(t)$, $t \in [t_0, t_0 + c]_{\mathbb{T}}$.

Consider the initial value problem (IVP) for dynamic equation

$$u^\Delta = f(t, u), \quad u(t_0) = u_0 \quad (3.56)$$

where $u_0 > 0$, $f \in C_{rd}[\mathbb{R}^0, \mathbb{R}^n]$, $\mathbb{R}^0 = \{([t_0, t_0 + a]) \times A\}$, $A = \{u \in \mathbb{R}^n : |u - u_0| \leq b\}$, and $|f(t, u)| \leq K$ on $\mathbb{R}^0$.

Theorem 3.17: Suppose that

- i) $f(t, u)$ is quasi-monotone non-decreasing in u .
- ii) for each j , $1 \leq j \leq m$, and $f_j(t, u)\mu^*(t) + u_j$ is non-decreasing in u_j for each $t \in \mathbb{T}$.

Then there exist maximal and minimal solutions of (3.56) on $I = [t_0, t_0 + \tau] \cap \mathbb{T}$ where $\tau = \min(a, \frac{b}{2K+b})$.

Proof. We show the existence of maximal solution only, because the case of the minimal solution is very similar.

Let $0 < \epsilon < \frac{b}{2}$ and consider the dynamic equation with IVP as follows

$$u^\Delta(t, \epsilon) = f(t, u) + \epsilon, \quad u(t_0, \epsilon) = u_0 + \epsilon. \quad (3.57)$$

Observing that

$$f_\epsilon(t, u) = f(t, u) + \epsilon \quad (3.58)$$

since $f_\epsilon(t, u)$ is defined and rd-continuous on $R_\epsilon = \{([t_0, t_0 + a] \cap \mathbb{T}) \times A_\epsilon\}$ where $A_\epsilon = \{u \in \mathbb{R}^n : |u - (u_0 + \epsilon)| \leq \frac{b}{2}\}$, $R_\epsilon \subseteq \mathbb{R}^0$ and $|f_\epsilon(t, u)| \leq K + \frac{b}{2}$ on R_ϵ . We conclude from Theorem 3.3 that the initial value problem (3.57) has at least one solution $u(t, \epsilon)$ on the interval I . For $0 < \epsilon_2 < \epsilon_1 \leq \epsilon$, we find

$$u^\Delta(t, \epsilon_2) \leq f(t, u(t, \epsilon_2)) + \epsilon_2 \quad (3.59)$$

and since

$$\epsilon_1 > \epsilon_2. \quad (3.60)$$

We add $f(t, u(t, \epsilon_1))$ to both sides in (3.60), we obtain

$$f(t, u(t, \epsilon_1)) + \epsilon_1 > f(t, u(t, \epsilon_1)) + \epsilon_2 \quad (3.61)$$

implies that

$$u^\Delta(t, \epsilon_1) > f(t, u(t, \epsilon_1)) + \epsilon_2 \quad (3.62)$$

and we know that

$$u(t_0, \epsilon_1) = u_0 + \epsilon_1 > u(t_0, \epsilon_2) = u_0 + \epsilon_2 \quad (3.63)$$

from (3.59), (3.62), (3.63), and applying Theorem 3.11, to get

$$u(t, \epsilon_2) < u(t, \epsilon_1) \quad (3.64)$$

for $t \in I$, $|f(t, u)| \leq K$ on $\mathbb{R}^0$, the family of functions $\{u(t, \epsilon_m)\}_{m>0}$ is uniformly bounded and equicontinuous, then by applying Arzela-Ascoli's Theorem, there exists $\{u(t, \epsilon_m)\}_{m>0}$ converges uniformly on I to an rd-continuous function, say $r(t)$. Hence,

$$\lim_{m \rightarrow \infty} u(t, \epsilon_m) = r(t) \quad \text{with} \quad r(t_0) = u_0$$

since $f(t, u)$ is rd-continuous on I , term by term integration is possible which implies that $r(t) = (r(t_1), r(t_2), \dots, r(t_m))$ is a solution of (3.57). To show that $r(t)$ is the required maximal solution of (3.57) on I , we have to show that if $u(t)$ is any solution of (3.57) on I , then $u(t) \leq r(t)$ on I . For any $t \in I$ and $\epsilon \leq \frac{b}{2}$, we find

$$u(t_0) = u_0 < u_0 + \epsilon = u(t_0, \epsilon) \quad (3.65)$$

$$u^\Delta(t) \leq f(t, u) \quad (3.66)$$

$$u^\Delta(t, \epsilon) = f(t, u) + \epsilon > f(t, u(t, \epsilon)). \quad (3.67)$$

from (3.65), (3.66), (3.67), and by using Theorem 3.11, we get that $u(t) < u(t, \epsilon)$, $t \in I$. Taking the limit on each side as $\epsilon \rightarrow 0$, gives

$$u(t) \leq \lim_{\epsilon \rightarrow 0} u(t, \epsilon) = r(t). \quad (3.68)$$

□

3.4. Comparison Results

An important technique in the theory of dynamic equations is discussed the estimating a function satisfying the dynamic inequality by minimal and maximal solutions of the corresponding dynamic equation. Comparison theorem is one of results that is widely utilized.

Theorem 3.18: Suppose that

- i) $f(t, u)$ is quasi-monotone non-decreasing in u .
- ii) $f_j(t, u)\mu^*(t) + u_j$ is non-decreasing in u_j for each $t \in \mathbb{T}$, for each j , $1 \leq j \leq m$.
- iii) $v : I \rightarrow \mathbb{R}^n$ be a mapping that is differentiable for each $t \in \mathbb{T}$ and it satisfies

$$v^\Delta(t) \leq f(t, v(t)), t \in I \quad (3.69)$$

where $I = [t_0, t_0 + a) \cap \mathbb{T}$. Then $v(t_0) \leq u_0$ implies that $v(t) \leq r(t)$, $t \in I$, where $r(t)$ is the maximal solution of (3.56) existing on I .

Proof. Set

$$u^\Delta(t) = f(t, u) + \epsilon, \quad u(t_0) = u_0 + \epsilon, \quad \epsilon > 0 \quad (3.70)$$

since $f_\epsilon(t, u) = f(t, u) + \epsilon$ is defined and rd-continuous on $R_\epsilon = I \times A_\epsilon$ where

$A_\epsilon = \{u \in \mathbb{R}^n : |u - (u_0 - \epsilon)| \leq \frac{b}{2}\}$ and $R_\epsilon \subseteq R^0$, we conclude by Theorem 3.17 that (3.70) has a maximal solution $r(t, \epsilon)$ on I where $\tau = \min(a, \frac{b}{2K+b})$. For $\epsilon > 0$, we find

$$u^\Delta(t, \epsilon) = f(t, u(t, \epsilon)) + \epsilon > f(t, u(t, \epsilon)) \quad (3.71)$$

$$v^\Delta(t) \leq f(t, v(t)) \quad (3.72)$$

and

$$u(t_0, \epsilon) = u_0 + \epsilon > u_0 \quad (3.73)$$

since $v(t_0) \leq u_0 < u(t_0, \epsilon)$, then $v(t_0) < u(t_0, \epsilon)$. Now, from (3.71), (3.72), (3.73) and by Theorem 3.11, we get that

$$v(t) < u(t, \epsilon) \leq r(t, \epsilon) \quad t \in I. \quad (3.74)$$

Then

$$v(t) < r(t, \epsilon). \quad (3.75)$$

Passing limit on each side as $\epsilon \rightarrow 0$, gives

$$\lim_{\epsilon \rightarrow 0} r(t, \epsilon) = r(t) \quad \text{uniformly on } I.$$

Hence, $v(t) \leq r(t)$ for $t \in I$.

□

Definition 3.19: Let N be a neighborhood of t , for each $t \in \mathbb{T}$. Then the Dini derivative of f define as follows; for any $\epsilon > 0$, there exists a right neighborhood $N_\epsilon \subset N$ of t such that for all $s \in N_\epsilon, s > t$,

$$D^+ f^\Delta(t) > \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}, \quad (3.76)$$

in other words,

$$D^+ f^\Delta(t) = \begin{cases} \lim_{h \rightarrow 0^+, h+t \in \mathbb{T}} \sup \frac{f(t+h) - f(t)}{h}, & \text{if } t \text{ is right-dense;} \\ \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}, & \text{if } t \text{ is right-scattered.} \end{cases} \quad (3.77)$$

Definition 3.20: Let $y \in C_{rd}[\mathbb{T}, \mathbb{R}^n]$, given any $\epsilon > 0$, if there exists a neighborhood N_ϵ of $t \in \mathbb{T}$ satisfying

$$\frac{1}{\mu(t, s)}[|y(\sigma(t))| - |y(\sigma(t)) - \mu(t, s)y^\Delta(t)|] < [y, y^\Delta]_+ + \epsilon \quad (3.78)$$

for each $s \in N_\epsilon$ and $s > t$, where $\mu(t, s) = \sigma(t) - s$, then we say that $[y, y^\Delta]_+$ is generalized derivative of $y(t)$.

Remark 3.21: If t is right-scattered, then we have

$$[y, y^\Delta]_+ = \frac{1}{\mu^*(t)}[|y(\sigma(t))| - |y(t)|] \quad (3.79)$$

where $\mu^*(t) = \mu(\sigma(t), t)$.

In this regard, we can show the following comparison result of the generalized derivative of $|y|$.

Theorem 3.22: Assume that $[y, y^\Delta]_+ \leq f(t, |y|)$ on $\mathbb{T} \times \mathbb{R}$ where $f \in C_{rd}[\mathbb{T} \times \mathbb{R}^+, \mathbb{R}]$ and $f(t, u)\mu^*(t) + u$ is non-decreasing in u for each $t \in \mathbb{T}$, where $y : \mathbb{T} \rightarrow \mathbb{R}^n$ is any rd-continuously differentiable function such that $|y_0| \leq u_0$. Then $|y(t)| \leq r(t)$, $t \in \mathbb{T}$, where $r(t) = r(t, t_0, u_0)$ is the maximal solution of (3.56) existing on $\mathbb{T}$.

Proof. Set

$$v(t) = |y(t)|$$

and note that $v(t_0) \leq u_0$. Now, let us apply the induction principle to the statement

$$\{B(t) : v(t) \leq r(t), t \in \mathbb{T}\}.$$

I) $B(t_0)$ is true, because $|y_0| \leq u_0$.

II) Let $t \in \mathbb{T}$ be right-scattered and $B(t)$ be true. We shall show that $B(\sigma(t))$. By definition of right scattered derivative and since $f(t, u)\mu^*(t) + u$ is non-decreasing in u , we get

$$v(\sigma(t)) - r(\sigma(t)) = (D^+v^\Delta(t) - r^\Delta(t))\mu^*(t) + v(t) - r(t) \quad (3.80)$$

$$\leq (f(t, v(t)) - f(t, r(t)))\mu^*(t) + v(t) - r(t) \leq 0. \quad (3.81)$$

Hence, $B(\sigma(t))$ is true.

III) Let t is right-dense, $B(t)$ be true and let N be the right neighborhood of $t \in \mathbb{T}$.

We need to show that $B(s)$ is true for $s \in N$, and $s > t$. We obtain

$$D^+v^\Delta(t) \leq f(s, v(s)), t \leq s, s \in N. \quad (3.82)$$

Then by using Theorem 3.18, we find

$$v(s) \leq r(s), t \leq s, s \in N. \quad (3.83)$$

This shows that $B(s)$ is true, because $B(t)$ is true.

IV) Let t be left-dense such that $B(s)$ is true for $s < t$. We need to show that $B(t)$ is true. By definition of rd-continuous of the functions, we get

$$\lim_{s \rightarrow t^-} v(s) \leq \lim_{s \rightarrow t^-} r(s). \quad (3.84)$$

Thus, $B(t)$ is true for all $t \in \mathbb{T}$. □

3.5. Linear Variation of Parameters

Let $(\mathbb{T}, \mu, Y)$ be a dynamical triple, and $B(Y)$ be a Banach algebra with unity of the continuous endomorphism on a Banach space Y .

Definition 3.23: The mapping $M : \mathbb{T} \longrightarrow B(Y)$ is said to be regressive, if

$$M(t)\mu^*(t) + 1 \neq 0 \quad \text{for each } t \in \mathbb{T}^k.$$

The set of all regressive and rd-continuous functions $M : \mathbb{T} \longrightarrow \mathbb{R}$ is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$, and the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ is given by

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) := \{M \in \mathcal{R} : 1 + \mu^*(t)M(t) > 0, \text{ for all } t \in \mathbb{T}\}$$

Remark 3.24:

- *If $\mathbb{T} = \mathbb{R}$, then any $M(t)$ is regressive.*
- *If $\mathbb{T} = \mathbb{Z}$, then $M(t)$ is regressive when $|M(t)| < 1$.*

Suppose $M \in \mathcal{R}(\mathbb{T}, B(Y))$ and $f : \mathbb{T} \longrightarrow Y$ is rd-continuous, then $u : \mathbb{T} \longrightarrow Y$ is called a solution of dynamic initial value problem

$$u^\Delta = M(t)u + f(t), \quad u(\tau) = \lambda. \quad (3.85)$$

Consider the linear dynamic initial value problem (IVP) in the Banach algebra $B(Y)$,

$$u^\Delta = M(t)u, \quad u(\tau) = 1 \quad (3.86)$$

where I is the unity of $B(Y)$.

So, we present what we call the Hilger complex plane.

Definition 3.25: For $h > 0$, the Hilger complex numbers, the Hilger real axis, the Hilger alternating axis, and the Hilger imaginary circle are respectively given by

- $\mathbb{C}_h := \{z \in \mathbb{C} : z \neq \frac{1}{h}\},$
- $\mathbb{R}_h := \{z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z > -\frac{1}{h}\},$
- $\mathbb{A}_h := \{z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z < -\frac{1}{h}\},$
- $\mathbb{I}_h := \{z \in \mathbb{C}_h : |z + \frac{1}{h}| = \frac{1}{h}\}.$

For $h > 0$, let Z_h be the strip $Z_h := \{z \in \mathbb{C} : -\frac{\pi}{h} < \text{Im}(z) < \frac{\pi}{h}\}$. For $h = 0$, let $\mathbb{C}_0 := \mathbb{C}, \mathbb{R}_0 := \mathbb{R}, \mathbb{I}_0 = i\mathbb{R}, \mathbb{A}_0 := \emptyset$, and $Z_h := \mathbb{C}$.

Definition 3.26: For $h > 0$, the cylinder transformation $\xi_h : \mathbb{C}_h \longrightarrow \mathbb{Z}_h$ is defined by

$$\xi_h(z) = \frac{1}{h} \text{Log}(1 + zh),$$

where Log is the principle logarithm function.

For $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbb{C}$. We call ξ_h the cylinder transformation because when $h > 0$ we can view $\mathbb{Z}_h$ as a cylinder if we paste the boundary lines $\text{Im}(z) = -\frac{1}{h}$ and $\text{Im}(z) = \frac{1}{h}$ of $\mathbb{Z}_h$ together to form a cylinder. Now we are ready to present the exponential function $e_M(t, r)$ on $\mathbb{T}$.

Definition 3.27: If $M \in \mathcal{R}$, then the exponential function is given by

$$e_M(t, r) := \exp\left(\int_r^t \xi_{\mu^*(s)}(M(s)) \Delta s\right) \text{ for } s, t \in \mathbb{T}$$

where the cylinder transformation $\xi_h(z)$ is introduced in Definition 3.26 and $\mu^*(t) > 0$.

Let us give some an important properties of the exponential function in the following theorem.

Theorem 3.28: If M and N are regressive and rd-continuous, $a \in \mathbb{R}$. Then

- i) $e_0(t, r) \equiv 1$ and $e_M(t, t) = 1$;
- ii) $e_M(\sigma(t), r) = (1 + M(t)\mu(t))e_M(t, r)$;
- iii) $e_M(t, r) = \frac{1}{e_M(r, t)} = e_{\ominus M}(a, t)$, where $\ominus M = \frac{-M}{1+\mu^*M}$;
- iv) $e_M(t, r)e_M(r, b) = e_M(t, b)$;
- v) $\left(\frac{1}{e_M(., a)}\right)^\Delta = \frac{-M(t)}{e_M^a(., a)}$;
- vi) $(e_M(t, r))^\Delta = M(t)e_M(t, r)$;
- vii) $e_{a \odot M} = e_M^a$, where $a \odot M = aM(t) \int_0^1 (1 + \mu^*(t)M(t)h)^{a-1} dh$, $h \in [0, 1]$ such that $1 + \mu^*(t)M(t)h > 0$;
- viii) $(e_M^a(., r))^\Delta = a \odot M(e_M^a(., r))$.

Let $e_M(t, \xi)$ be the solution of (3.86), then $e_M(t, \tau)$ satisfies

$$e_M^\Delta(t, \tau) = M(t)e_M(t, \tau), \quad e_M(\tau, \tau) = 1. \quad (3.87)$$

The next result which is the well known Gronwall's inequality.

Lemma 3.29: Let f, g, h be continuous functions form $\mathbb{T}^k \longrightarrow \mathbb{R}^+$. Then the inequality

$$f(t) \leq g(t) + \int_{\tau}^t h(s)f(s)\Delta s \quad \text{for all } t \in \mathbb{T}^k, \quad (3.88)$$

implies that

$$f(t) \leq g(t) + \int_{\tau}^t h(s)e_h(t, \sigma(s))g(s)\Delta s \quad \text{for all } t \in \mathbb{T}^k. \quad (3.89)$$

Theorem 3.30: Suppose $M \in \mathcal{R}$ and f is rd-continuous. Then the unique solution of (3.85) is given by

$$u(t) = e_M(t, \tau)\lambda + \int_{\tau}^t e_M(t, \sigma(s))f(s)\Delta s. \quad (3.90)$$

Proof. Let

$$u(t) = e_M(t, \tau)z(t). \quad (3.91)$$

By using the product rule of (3.91), gives

$$u^{\Delta} = e_M(\sigma(t), \tau)z^{\Delta}(t) + e_M^{\Delta}(t, \tau)z(t). \quad (3.92)$$

Now, by substituting in (3.85), gets

$$e_M(\sigma(t), \tau)z^{\Delta}(t) = f(t). \quad (3.93)$$

Then

$$\begin{aligned} z^{\Delta}(t) &= e_M^{-1}(\sigma(t), \tau)f(t) \\ &= e_M(\tau, \sigma(t))f(t). \end{aligned} \quad (3.94)$$

Taking delta integral to both sides from ξ into t , gives

$$z(t) = \lambda + \int_{\tau}^t e_M(\tau, \sigma(s))f(s)\Delta s. \quad (3.95)$$

By substituting in (3.91), we find

$$u(t) = e_M(t, \tau)\lambda + \int_{\tau}^t e_M(t, \tau)e_M(\tau, \sigma(s))f(s)\Delta s. \quad (3.96)$$

Hence,

$$u(t) = e_M(t, \tau)\lambda + \int_{\tau}^t e_M(t, \sigma(s))f(s)\Delta s. \quad (3.97)$$

□

Remark 3.31:

- If $M(t)$ does not satisfy regressivity condition in the previous theorem, then this theorem is false. The dynamic IVP

$$u^{\Delta} = -u, \quad u(2) = \lambda \quad (3.98)$$

in $\mathbb{R}$ over the time scale $\mathbb{N}$ has infinitely many solutions for $\lambda = 0$ and none for $\lambda \neq 0$. The regressivity condition is violated since this mapping $M(t)\mu^* + 1 = -1 \cdot 1 + 1 = 0$.

- If $f(t) = 0$, then there exists the solution of (3.85) satisfies

$$u(t) = e_M(t, \tau)\lambda. \quad (3.99)$$

4. THE METHOD OF MONOTONE ITERATIVE TECHNIQUE ON TIME SCALE

4.1. Existence Via Upper and Lower Solutions

We investigate the method of lower and upper solutions has applied for dynamic system on time scale.

Let us consider the initial value problem

$$u^\Delta(t) = f(t, u(t)), \quad u(0) = u_0 \quad (4.1)$$

where $f \in C_{rd}[\mathbb{T} \times \mathbb{R}, \mathbb{R}]$ and $\mathbb{T}$ is a time scale such that $t = 0$ is the minimal element and $t = \mathbb{T}$ is the maximal element which is not left-scattered.

We now need to define the following definitions before we proceed further.

Definition 4.1: A function $\beta \in C_{rd}^1[\mathbb{T}, \mathbb{R}]$ is said to be an upper solution of (4.1), if β satisfies the inequality

$$\beta^\Delta \geq f(t, \beta), t \in \mathbb{T}, \beta(0) \geq u_0. \quad (4.2)$$

Definition 4.2: A function $\alpha \in C_{rd}^1[\mathbb{T}, \mathbb{R}]$ is said to be a lower solution of (4.1), if α satisfies the inequality

$$\alpha^\Delta \leq f(t, \alpha), t \in \mathbb{T}, \alpha(0) \leq u_0. \quad (4.3)$$

We have to give the concept of an equicontinuous and uniformly bounded family of functions.

Definition 4.3: Let $\mathbb{T}$ be a time scale and $f_n : \mathbb{T} \longrightarrow \mathbb{R}$, for each $n \in \mathbb{N}$. The sequence of functions $\{f_n\}_{n \in \mathbb{N}}$ is said to be equicontinuous on $\mathbb{T}$, if for every $\epsilon > 0$, there exist $\delta = \delta(\epsilon) > 0$ such that

$$|f_n(t) - f_n(s)| < \epsilon \quad (4.4)$$

for all $s, t \in \mathbb{T}$ with $|t - s| < \delta$ and all $n \in \mathbb{N}$.

Definition 4.4: Let $\mathbb{T}$ be a time scale and $f_n : \mathbb{T} \longrightarrow \mathbb{R}$, for each $n \in \mathbb{N}$. The sequence of functions $\{f_n\}_{n \in \mathbb{N}}$ is said to be uniformly bounded on $\mathbb{T}$, if there exists $L > 0$ such that $|f_n(t)| \leq L$ for all $t \in \mathbb{T}$ and all $n \in \mathbb{N}$.

The following theorem prove the fundamental property of such a family of functions.

Theorem 4.5: Let $f_n(t)$ be a sequence of functions defined on a compact set $\mathbb{T} \subset \mathbb{R}^n$, which is equicontinuous and uniformly bounded on $\mathbb{T}$. Then there exists a subsequence $\{f_{n_k}\}$, $n_k = 1, 2, \dots$, which is uniformly convergent on $\mathbb{T}$.

If we know the existence of upper and lower solutions β, α such that $\alpha \leq \beta$, then we can show the existence of a solution in the closed set

$$Q = [(t, u) : \alpha(t) \leq u \leq \beta(t), t \in \mathbb{T}]. \quad (4.5)$$

Proposition 4.6: Consider the system (4.1) where $f \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$ and $f(t, u)$ is bounded everywhere. Then there exists a solution $u(t)$ on $\mathbb{T}$ of (4.1).

Theorem 4.7: Suppose that

- i) $\alpha, \beta \in C_{rd}^1[\mathbb{T}, \mathbb{R}]$ are lower and upper solutions of (4.1) such that $\alpha(t) \leq \beta(t)$ on $\mathbb{T}$
- ii) $f \in C_{rd}[Q, \mathbb{R}]$ with f is bounded on Q .

Then there exists a solution $u(t)$ of (4.1) such that $\alpha(t) \leq u(t) \leq \beta(t)$ on $\mathbb{T}$, provided that $\alpha(0) \leq u_0 \leq \beta(0)$.

Proof. Let $H : \mathbb{T} \times \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$H(t, u) = \max[\alpha(t), \min(u, \beta(t))]. \quad (4.6)$$

Then $f(t, H(t, u))$ defines a rd-continuous extension of f to $\mathbb{T} \times \mathbb{R}$ which is also bounded, since f is bounded on Q . Now, by using proposition 4.6, then

$$u^\Delta = f(t, H(t, u)), \quad u(0) = u_0 \quad (4.7)$$

has a solution on $\mathbb{T}$.

Next, we need to prove this solution satisfies that $\alpha(t) \leq u(t) \leq \beta(t)$. For $\epsilon > 0$, consider $\beta_\epsilon(t) = \beta(t) + \epsilon(1+t)$ and $\alpha_\epsilon(t) = \alpha(t) - \epsilon(1+t)$. To prove this, we use the induction principle to the statement

$$\{B(t) : \alpha_\epsilon(t) < u(t) < \beta_\epsilon(t), \quad t \in \mathbb{T}\}$$

I) The statement $u(0)$ is clearly satisfies $\alpha_\epsilon(0) < u_0 < \beta_\epsilon(0)$. Since β and α are upper and lower solutions of (4.1) respectively, then $\alpha(0) < u_0 < \beta(0)$, we have $\alpha_\epsilon(0) \leq \alpha(0) < u_0 < \beta(0) \leq \beta_\epsilon(0)$.

II) If t is right-scattered, and $B(t)$ is true. We have to prove that $B(\sigma(t))$ is true. If $\alpha_\epsilon(\sigma(t)) < u(\sigma(t)) < \beta_\epsilon(\sigma(t))$ is not true on $t \in \mathbb{T}$, then suppose that for $\sigma(t) \in \mathbb{T}$, we have $\alpha_\epsilon(t) < u(t) < \beta_\epsilon(t)$ on $[0, \sigma(t)]$ and $u(\sigma(t)) = \beta_\epsilon(\sigma(t))$. We get $u(\sigma(t)) > \beta(\sigma(t))$ and so

$$H(\sigma(t), u(\sigma(t))) = \max[\alpha(\sigma(t)), \min(u(\sigma(t)), \beta(\sigma(t)))] = \beta(\sigma(t)) \quad (4.8)$$

and also

$$\alpha(\sigma(t)) \leq H(\sigma(t), u(\sigma(t))) \leq \beta(\sigma(t)). \quad (4.9)$$

Hence, since β is an upper solution,

$$\beta^\Delta(\sigma(t)) \geq f(\sigma(t), H(\sigma(t), u(\sigma(t)))) = u^\Delta(\sigma(t)) \quad (4.10)$$

since

$$\beta_\epsilon^\Delta(\sigma(t)) > \beta^\Delta(\sigma(t)), \quad (4.11)$$

we have

$$\beta_\epsilon^\Delta(\sigma(t)) > u^\Delta(\sigma(t)) \quad (4.12)$$

which gives

$$\frac{\beta_\epsilon(\sigma(t)) - \beta_\epsilon(\sigma(t) - \mu^*(t))}{\mu^*(t)} > \frac{u(\sigma(t)) - u(\sigma(t) - \mu^*(t))}{\mu^*(t)} \quad (4.13)$$

since $\beta_\epsilon(\sigma(t)) = u(\sigma(t))$, we get

$$\beta_\epsilon(\sigma(t) - \mu^*(t)) < u(\sigma(t) - \mu^*(t)) \quad (4.14)$$

this contradicts with the assumption $u(t) < \beta_\epsilon(t)$ for $t \in [0, \sigma(t)]$. Consequently,

$$\alpha_\epsilon(\sigma(t)) < u(\sigma(t)) < \beta_\epsilon(\sigma(t)) \quad (4.15)$$

and $B(\sigma(t))$ is true.

III) If t is right-dense, choosing t_1 , is a neighborhood N to the right of t . Suppose on the contrary, suppose $t_1 \in \mathbb{T}$ such that $\alpha_\epsilon(t) \leq u(t) \leq \beta_\epsilon(t)$ on $t \in [0, t_1]$ and $u(t_1) = \beta(t_1)$ also, $\alpha(t_1) \leq H(t_1, u(t_1)) \leq \beta(t_1)$. Hence, β is upper solution,

$$\beta^\Delta(t_1) > f(t_1, H(t_1, u(t_1))) = u^\Delta(t_1) \quad (4.16)$$

since $\beta_\epsilon^\Delta(t_1) > \beta^\Delta(t_1)$, we have

$$\beta_\epsilon^\Delta(t_1) > u^\Delta(t_1) \quad (4.17)$$

which gives

$$\frac{\beta_\epsilon(t_1) - \beta_\epsilon(t_1 - \mu^*(t))}{\mu^*(t)} > \frac{u(t_1) - u(t_1 - \mu^*(t))}{\mu^*(t)} \quad (4.18)$$

since $\beta_\epsilon(t_1) = u(t_1)$, we get

$$\beta_\epsilon(t_1 - \mu^*(t)) < u(t_1 - \mu^*(t)) \quad (4.19)$$

this contradiction with the assumption $u(t) < \beta_\epsilon(t)$ for $t \in [0, t_1]$.

We find that, the conclusion $\alpha_\epsilon(t) < u(t) < \beta_\epsilon(t)$ on all of N . Hence, $B(s)$ is true for all $s \in N, s \geq t$.

IV) If t be a left-dense, and $B(s)$ is true for all $s \in [0, t)$, since u, α, β are all rd-continuous the limit from left exists

$$\lim_{s \rightarrow t^-} \alpha_\epsilon(s) < \lim_{s \rightarrow t^-} u(s) < \lim_{s \rightarrow t^-} \beta_\epsilon \quad (4.20)$$

implies that $\alpha_\epsilon(t) \leq u(t) \leq \beta_\epsilon(t)$. Hence, we have $B(t)$ is true on $\mathbb{T}$. Letting $\epsilon \rightarrow 0^+$, we get

$$\alpha(t) \leq u(t) \leq \beta(t) \quad (4.21)$$

on $\mathbb{T}$. □

4.2. Monotone Iterative Technique with IVP

In this section, we use the method of lower and upper solutions to show the existence of monotone sequences for a considered problem which converge uniformly and monotonically to the extremal solutions.

Theorem 4.8: Let $f \in C_{rd}[\mathbb{T} \times \mathbb{R}, \mathbb{R}]$, $\alpha_0, \beta_0 \in C_{rd}^1[\mathbb{T}, \mathbb{R}]$ be lower and upper solutions of (4.1) respectively, such that $\alpha_0 \leq \beta_0$ on $\mathbb{T}$. Moreover, suppose that f satisfies

$$f(t, u) - f(t, v) \geq -K(u - v), \text{ for } \alpha_0 \leq v \leq u \leq \beta_0 \quad (4.22)$$

with $K \geq 0$ for t right-dense and $K \leq 0$ for t right-scattered. Then there exist monotone sequences $\{\alpha_m(t)\}, \{\beta_m(t)\}$ such that $\alpha_m \rightarrow \rho$ and $\beta_m \rightarrow r$ as $m \rightarrow \infty$ locally uniformly and monotonically on $\mathbb{T}$ and that ρ, r are minimal and maximal solutions for (4.1), respectively.

Proof. For any $\tau \in C_{rd}[\mathbb{T}, \mathbb{R}]$ such that $\alpha_0 \leq \tau \leq \beta_0$, we consider the linear dynamic equation

$$u^\Delta = f(t, \tau) - K(u - \tau), \quad u(0) = u_0. \quad (4.23)$$

We replace v by τ in (4.22), which gives

$$f(t, u) - f(t, \tau) \geq -K(u - \tau) \quad (4.24)$$

implies that

$$f(t, u) \geq f(t, \tau) - K(u - \tau) = u^\Delta. \quad (4.25)$$

Then there exists a unique solution u of (4.23) on $\mathbb{T}$. Setting a mapping A by $A\tau = u$. This mapping will be used to define the sequences $\{\alpha_n\}, \{\beta_n\}$. Let us show

that,

$$a) \alpha_0 \leq A\alpha_0, \quad \beta_0 \geq A\beta_0.$$

b) A is monotone operation on the segment

$$[\alpha_0, \beta_0] = \{u \in C_{rd}[\mathbb{T}, \mathbb{R}] : \alpha_0 \leq u \leq \beta_0\}. \quad (4.26)$$

To show (a), set $A\alpha_0 = \alpha_1$ where α_1 is a unique solution of (4.23) with $\tau = \alpha_0$.

It follows from (4.23) that

$$\alpha_1^\Delta = f(t, \alpha_0) - K(\alpha_1 - \alpha_0), \quad \alpha_1(0) = u_0. \quad (4.27)$$

Setting $\varphi = \alpha_1 - \alpha_0$, since α_0 a lower solution, we note that

$$\varphi^\Delta = \alpha_1^\Delta - \alpha_0^\Delta \geq f(t, \alpha_0) - K(\alpha_1 - \alpha_0) - f(t, \alpha_0) = -K\varphi \quad (4.28)$$

and $\varphi(0) \geq 0$. Consequently, we get from Corollary 3.13 that $\varphi(t) \geq 0$ on $\mathbb{T}$, which gives $\alpha_0 \leq A\alpha_0$.

Now, we need to show that $\beta_0(t) \geq \beta_1(t)$, set $A\beta_0 = \beta_1$ where β_1 is a unique solution of (4.23) with $\tau = \beta_0$. Setting $\varphi = \beta_0 - \beta_1$, then by using (4.23) and β_0 is a upper solution, we have

$$\varphi^\Delta = \beta_0^\Delta - \beta_1^\Delta \geq f(t, \beta_0) - f(t, \beta_0) + K(\beta_1 - \beta_0) = K\varphi, \quad (4.29)$$

and $\varphi(0) \geq 0$. Thus, we obtain from Corollary 3.14, that $\varphi(t) \geq 0$, which gives $\beta_0 \geq A\beta_0$.

To show (b), set $\tau_1, \tau_2 \in [\alpha_0, \beta_0]$ such that $\tau_1 \leq \tau_2$. Suppose that $u_1 = A\tau_1$ and $u_2 = A\tau_2$. Let us show that $u_1 \leq u_2$, setting $\varphi = u_2 - u_1$, then it follows from (4.23) that

$$u_1 = f(t, \tau_1) + K(u_1 - \tau_1), \quad u_1(0) = u_0, \quad (4.30)$$

$$u_2 = f(t, \tau_2) - K(u_2 - \tau_2), \quad u_2(0) = u_0. \quad (4.31)$$

Then

$$\begin{aligned}\varphi^\Delta &= u_2^\Delta - u_1^\Delta = f(t, \tau_2) - K(u_2 - \tau_2) - f(t, \tau_1) + K(u_1 - \tau_1) \\ &\geq -K(\tau_2 - \tau_1) - K(u_2 - \tau_2) + K(u_1 - \tau_1) = -K\varphi\end{aligned}\quad (4.32)$$

and $\varphi(0) = 0$. We obtain $\varphi(t) \geq 0$. Hence, $A\tau_2 \geq A\tau_1$.

Next, we can define the sequences $\alpha_m = A\alpha_{m-1}$, $\beta_m = A\beta_{m-1}$ and conclude from the previous argument

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_m \leq \beta_m \leq \cdots \leq \beta_1 \leq \beta_0 \quad (4.33)$$

on $\mathbb{T}$.

We have to show that the sequences converge uniformly on $\mathbb{T}$. We shall show these sequences are uniformly bounded and equicontinuous on $\mathbb{T}$. By the assumption both of α_0, β_0 are bounded on $\mathbb{T}$, then there exists $L_1, L_2 > 0$ such that for $t \in \mathbb{T}$, $|\alpha_0(t)| \leq L_1$ and $|\beta_0(t)| \leq L_2$. Since $\alpha_0(t) \leq \beta_m(t) \leq \beta_0(t)$ for each $m > 0$, consequently $\{\beta_m\}$ is uniformly bounded. By similar argument, we can show that $\{\alpha_m(t)\}$ is also uniformly bounded. To show that $\{\beta_m(t)\}$ is equicontinuous, let $0 \leq t_1 \leq t_2$ on $\mathbb{T}$. Then for $m > 0$,

$$\begin{aligned}|\beta_m(t_1) - \beta_m(t_2)| &= |x_0 + \int_0^{t_1} f(s, \beta_{m-1}(s)) - K(\beta_m(s) - \beta_{m-1}(s))\Delta s \\ &\quad - (x_0 + \int_0^{t_2} f(s, \beta_{m-1}(s)) - K(\beta_m(s) - \beta_{m-1}(s))\Delta s)| \\ &= | \int_0^{t_1} f(s, \beta_{m-1}(s)) - K(\beta_m(s) - \beta_{m-1}(s))\Delta s \\ &\quad - \int_0^{t_2} f(s, \beta_{m-1}(s)) + K(\beta_m(s) - \beta_{m-1}(s))\Delta s | \\ &\leq \int_0^{t_1} |f(s, \beta_{m-1}(s))| + |K(\beta_m(s) - \beta_{m-1}(s))|\Delta s \\ &\quad + \int_0^{t_2} |f(s, \beta_{m-1}(s))| + |K(\beta_m(s) - \beta_{m-1}(s))|\Delta s,\end{aligned}\quad (4.34)$$

since $\{\alpha_m(t)\}$ and $\{\beta_m(t)\}$ are uniformly bounded and , there exist $\bar{L}$ independent of

m such that

$$|\beta_m(t_1) - \beta_m(t_2)| \leq \bar{L}|t_1 - t_2|. \quad (4.35)$$

Thus, for $\epsilon > 0$ there exists $\delta = \frac{\epsilon}{L} > 0$,

$$|\beta_m(t_1) - \beta_m(t_2)| < \epsilon \quad \text{whenever} \quad |t_1 - t_2| < \delta. \quad (4.36)$$

By similar argument, we can show that $\{\alpha_m(t)\}$ is also equicontinuous. Then by Arzela- Ascoli's Theorem 4.5, there exist subsequences $\{\alpha_{m_k}(t)\}$ and $\{\beta_{m_k}(t)\}$ which converge uniformly to $\rho(t)$ and $r(t)$, respectively on $\mathbb{T}$. Thus, $\alpha_m(t), \beta_m(t)$ locally uniformly and monotonically to $\rho(t)$ and $r(t)$ on $\mathbb{T}$, respectively ; that is,

$$\begin{aligned} \lim_{m \rightarrow \infty} \alpha_m(t) &= \rho(t), \\ \lim_{m \rightarrow \infty} \beta_m(t) &= r(t). \end{aligned} \quad (4.37)$$

After that, we shall show that $\rho(t), r(t)$ are solutions of (4.1). Observing that $\alpha_{m+1}(t), \beta_{m+1}$ are constructed as follows

$$\alpha_{m+1}^\Delta(t) = f(t, \alpha_m(t)) - K(\alpha_{m+1}(t) - \alpha_m(t)), \quad \alpha_{m+1}(0) = u_0. \quad (4.38)$$

$$\beta_{m+1}^\Delta(t) = f(t, \beta_m(t)) - K(\beta_{m+1}(t) - \beta_m(t)), \quad \beta_{m+1}(0) = u_0. \quad (4.39)$$

Then taking delta integral for above equations (4.38) and (4.39) from 0 into t , we obtain

$$\alpha_{m+1}(t) = u_0 + \int_0^t f(s, \alpha_m(s)) - K(\alpha_{m+1}(s) - \alpha_m(s)) \Delta s, \quad (4.40)$$

and also

$$\beta_{m+1}(t) = u_0 + \int_0^t f(s, \beta_m(s)) - K(\beta_{m+1}(s) - \beta_m(s)) \Delta s. \quad (4.41)$$

Passing limit as $m \rightarrow \infty$, and then taking delta derivative, so that we get the

result $\rho(t), r(t)$ are solution of (4.1); that is,

$$\rho^\Delta(t) = f(t, \rho(t)), \quad \rho(0) = u_0, \quad (4.42)$$

$$r^\Delta(t) = f(t, r(t)), \quad r(0) = u_0. \quad (4.43)$$

Finally, we have to show that $\rho(t), r(t)$ are minimal and maximal solutions of (4.1); that is, we shall prove that if $u(t)$ is any solution of (4.1) on $\mathbb{T}$, then $\rho(t) \leq u(t) \leq r(t)$ on $\mathbb{T}$. To show this, we know that $\alpha_0 \leq u \leq \beta_0$. Now, suppose that for some $m > 1$, $\alpha_m \leq u \leq \beta_m$ on $\mathbb{T}$, set $\varphi = u - \alpha_{m+1}$ and by using (4.22), gives

$$\begin{aligned} \varphi^\Delta &= u^\Delta - \alpha_{m+1}^\Delta = f(t, u) - f(t, \alpha_m) + K(\alpha_{m+1} - \alpha_m) \\ &\geq -K(u - \alpha_m) + K(\alpha_{m+1} - \alpha_m) = -K(u - \alpha_m) = -K\varphi, \end{aligned} \quad (4.44)$$

since $\varphi(0) = 0$, by Corollary 3.14 which implies that $\alpha_{m+1} \leq u$ on $\mathbb{T}$. Now, set $\varphi = \beta_{m+1} - u$ then by using (4.22), we obtain

$$\begin{aligned} \varphi^\Delta &= \beta_{m+1}^\Delta - u^\Delta = f(t, \beta_m) - K(\beta_{m+1} - \beta_m) - f(t, u) \\ &\geq -K(\beta_m - u) - K(\beta_{m+1} - \beta_m) = -K\varphi. \end{aligned} \quad (4.45)$$

We get from $\varphi(0) = 0$ and using Corollary 3.13 that $u \leq \beta_{m+1}$. Hence, $\alpha_{m+1} \leq u \leq \beta_{m+1}$ on $\mathbb{T}$. Letting $m \rightarrow \infty$, we conclude that $\rho(t) \leq u(t) \leq r(t)$ on $\mathbb{T}$. $\square$

Corollary 4.9: If in addition to the assumptions of Theorem 4.8, we assume

$$f(t, u) - f(t, v) \leq K(u - v), \quad \alpha_0 \leq v \leq u \leq \beta_0. \quad (4.46)$$

Then $\rho(t) = u(t) = r(t)$ is the unique solution of initial value problem (IVP) (4.1).

Proof. We know that $\rho(t) \leq r(t)$, we need to show that $r(t) \leq \rho(t)$. Letting $p(t) = r(t) - \rho(t)$, then we find that

$$p^\Delta(t) = r^\Delta(t) - \rho^\Delta(t) \quad (4.47)$$

$$\begin{aligned}
p^\Delta(t) &= f(t, r(t)) - f(t, \rho(t)) \\
&\leq K(r(t) - \rho(t)) \\
&= Kp
\end{aligned} \tag{4.48}$$

since $p(0) = 0$, consequently by Corollary 3.13, we get $\rho(t) \geq r(t)$. Thus, $\rho(t) = u(t) = r(t)$. $\square$

4.3. Generalized Monotone Iterative Techniques for Dynamic Equations with the Initial Value Problem on Time Scale

We generalized monotone method on time scale, given below, by using coupled upper and lower solutions and the corresponding initial value problem (IVP). Moreover, $f_1(t, u)$ and $f_2(t, u)$ can be non-decreasing.

Consider the initial value problem,

$$u^\Delta(t) = f_1(t, u) - f_2(t, u), \quad u(0) = u_0. \tag{4.49}$$

In this stage, we provide the following definitions.

Definition 4.10: Let $\alpha_0, \beta_0 \in C_{rd}^1[\mathbb{T}, \mathbb{R}]$. Then α_0 and β_0 are said to be

i) *natural lower and upper solutions of (4.49), if*

$$\begin{aligned}
\alpha_0^\Delta(t) &\leq f_1(t, \alpha_0(t)) - f_2(t, \alpha_0(t)), \quad \alpha_0(0) \leq u_0 \\
\beta_0^\Delta(t) &\geq f_1(t, \beta_0(t)) - f_2(t, \beta_0(t)), \quad \beta_0(0) \geq u_0,
\end{aligned} \tag{4.50}$$

on $t \in \mathbb{T}$.

ii) *coupled lower and upper solutions of Type I of (4.49), if*

$$\begin{aligned}
\alpha_0^\Delta(t) &\leq f_1(t, \alpha_0(t)) - f_2(t, \beta_0(t)), \quad \alpha_0(0) \leq u_0 \\
\beta_0^\Delta(t) &\geq f_1(t, \beta_0(t)) - f_2(t, \alpha_0(t)), \quad \beta_0(0) \geq u_0,
\end{aligned} \tag{4.51}$$

on $t \in \mathbb{T}$.

iii) *coupled lower and upper solutions of Type II of (4.49), if*

$$\begin{aligned}\alpha_0^\Delta(t) &\leq f_1(t, \beta_0(t)) - f_2(t, \alpha_0(t)), & \alpha_0(0) &\leq u_0 \\ \beta_0^\Delta(t) &\geq f_1(t, \alpha_0(t)) - f_2(t, \beta_0(t)), & \beta_0(0) &\geq u_0,\end{aligned}\tag{4.52}$$

on $t \in \mathbb{T}$.

iv) *coupled lower and upper solutions of Type III of (4.49), if*

$$\begin{aligned}\alpha_0^\Delta(t) &\leq f_1(t, \beta_0(t)) - f_2(t, \beta_0(t)), & \alpha_0(0) &\leq u_0 \\ \beta_0^\Delta(t) &\geq f_1(t, \alpha_0(t)) - f_2(t, \alpha_0(t)), & \beta_0(0) &\geq u_0,\end{aligned}\tag{4.53}$$

on $t \in \mathbb{T}$.

Definition 4.11: A function $x, y \in C_{rd}^1[\mathbb{T}, \mathbb{R}]$ are said to be

i) *natural solutions, if*

$$\begin{aligned}x^\Delta(t) &= f_1(t, x(t)) - f_2(t, x(t)), & x(0) &= x_0 \\ y^\Delta(t) &= f_1(t, y(t)) - f_2(t, y(t)), & y(0) &= y_0;\end{aligned}\tag{4.54}$$

ii) *coupled solutions of Type I, if*

$$\begin{aligned}x^\Delta(t) &= f_1(t, x(t)) - f_2(t, y(t)), & x(0) &= x_0 \\ y^\Delta(t) &= f_1(t, y(t)) - f_2(t, x(t)), & y(0) &= y_0;\end{aligned}\tag{4.55}$$

iii) *coupled solutions of Type II, if*

$$\begin{aligned}x^\Delta(t) &= f_1(t, y(t)) - f_2(t, x(t)), & x(0) &= x_0 \\ y^\Delta(t) &= f_1(t, x(t)) - f_2(t, y(t)), & y(0) &= y_0;\end{aligned}\tag{4.56}$$

iv) *coupled solutions of Type III, if*

$$\begin{aligned}x^\Delta(t) &= f_1(t, y(t)) - f_2(t, y(t)), & x(0) &= x_0 \\ y^\Delta(t) &= f_1(t, x(t)) - f_2(t, x(t)), & y(0) &= y_0.\end{aligned}\tag{4.57}$$

Definition 4.12: Coupled solution $\rho, r \in C_{rd}^1[\mathbb{T}, \mathbb{R}]$ are said to be coupled minimal and maximal solution of (4.49), if for any coupled x, y such that $x \geq \rho, r \geq y$.

In the next theorem, we obtain natural sequences that converge uniformly and monotonically to coupled extremal solutions of Type I for (4.49) by utilizing coupled lower and upper solutions of Type I on $\mathbb{T}$.

Theorem 4.13: Suppose that

- A1) α_0, β_0 are coupled lower and upper solutions of Type I for (4.49) with $\alpha_0(t) \leq u \leq \beta_0(t)$ on $t \in \mathbb{T}$;
- A2) $f_1, f_2 \in C_{rd}[\mathbb{T}^k \times \mathbb{R}, \mathbb{R}]$, $f_1(t, u)$ and $f_2(t, u)$ are non-decreasing in u for each $t \in \mathbb{T}^k$.

Then there exist monotone sequences $\{\alpha_m(t)\}$ and $\{\beta_m(t)\}$ such that $\alpha_m(t) \rightarrow \rho(t)$ and $\beta_m(t) \rightarrow r(t)$ uniformly and monotonically, such that $\rho(t), r(t)$ are coupled minimal and maximal solutions of (4.49) respectively; such that, they satisfy the coupled system,

$$\begin{aligned} \rho^\Delta(t) &= f_1(t, \rho(t)) - f_2(t, r(t)), & \rho(0) &= u_0 \\ r^\Delta(t) &= f_1(t, r(t)) - f_2(t, \rho(t)), & r(0) &= u_0 \end{aligned} \tag{4.58}$$

on $\mathbb{T}$.

Proof. For each $m \geq 0$, consider the system

$$\begin{aligned} \alpha_{m+1}^\Delta &= f_1(t, \alpha_m) - f_2(t, \beta_m), & \alpha_{m+1}(0) &= u_0, \\ \beta_{m+1}^\Delta &= f_1(t, \beta_m) - f_2(t, \alpha_m), & \beta_{m+1}(0) &= u_0. \end{aligned} \tag{4.59}$$

By the assumption $\alpha_0 \leq u \leq \beta_0$ is true. It follows from (4.59) that

$$\begin{aligned} \alpha_1^\Delta &= f_1(t, \alpha_0) - f_2(t, \beta_0), & \alpha_1(0) &= u_0, \\ \beta_1^\Delta &= f_1(t, \beta_0) - f_2(t, \alpha_0), & \beta_1(0) &= u_0. \end{aligned} \tag{4.60}$$

Now, we need to show that $\alpha_0 \leq \alpha_1 \leq u \leq \beta_1 \leq \beta_0$ on $\mathbb{T}$. To show this, let $\varphi = \beta_1 - \beta_0$, then $\varphi(0) \leq 0$ and

$$\begin{aligned}
\varphi^\Delta &= \beta_1^\Delta - \beta_0^\Delta \\
&\leq f_1(t, \beta_0) - f_2(t, \alpha_0) - f_1(t, \beta_0) + f_2(t, \alpha_0) \\
&\leq 0.
\end{aligned} \tag{4.61}$$

We have $\varphi^\Delta(t) \leq 0$ and $\varphi(0) \leq 0$. Then by Corollary 3.13, we get $\varphi(t) \leq 0$. Hence, $\beta_1(t) \leq \beta_0(t)$ on $\mathbb{T}$. By similar argument, we can show that $\alpha_1 \geq \alpha_0$ on $\mathbb{T}$. Let $\varphi = u - \beta_1$ on $\mathbb{T}$, then $\varphi(0) = 0$, and

$$\begin{aligned}
\varphi^\Delta &= u^\Delta - \beta_1^\Delta \\
&= f_1(t, u) - f_2(t, u) - f_1(t, \beta_0) + f_2(t, \alpha_0) \\
&\leq 0
\end{aligned} \tag{4.62}$$

since f_1 and f_2 are non-decreasing in u for each t and by the fact that $\alpha_0 \leq u \leq \beta_0$. Thus, $u(t) \leq \beta_1(t)$ on $\mathbb{T}$. Similarly, we can show that $u(t) \geq \alpha_1(t)$ on $\mathbb{T}$. Hence, $\alpha_0 \leq \alpha_1 \leq u \leq \beta_1 \leq \beta_0$ is true for $k = 1$. Suppose that for some $k > 1$,

$$\alpha_{k-1} \leq \alpha_k \leq u \leq \beta_k \leq \beta_{k-1} \text{ on } \mathbb{T}. \tag{4.63}$$

is true on $\mathbb{T}$. Next, we wish to show that

$$\alpha_k \leq \alpha_{k+1} \leq u \leq \beta_{k+1} \leq \beta_k \text{ on } \mathbb{T}. \tag{4.64}$$

To prove this, let $\varphi = \beta_{k+1} - \beta_k$ on $\mathbb{T}$. Then $\varphi(0) = 0$ and

$$\begin{aligned}
\varphi^\Delta &= \beta_{k+1}^\Delta - \beta_k^\Delta \\
&= f_1(t, \beta_k) - f_2(t, \alpha_k) - f_1(t, \beta_{k-1}) + f_2(t, \alpha_{k-1}) \\
&\leq 0.
\end{aligned} \tag{4.65}$$

We obtain $\varphi(t) \leq 0$ which shows this $\beta_{k+1} \leq \beta_k$ on $\mathbb{T}$. Similarly, we can show that $\alpha_k \leq \alpha_{k+1}$ on $\mathbb{T}$. Let $\varphi = u - \beta_{k+1}$.

Then $\varphi(0) = 0$ and

$$\begin{aligned}\varphi^\Delta &= u^\Delta - \beta_{k+1}^\Delta \\ &= f_1(t, u) - f_2(t, u) - f_1(t, \beta_k) + f_2(t, \alpha_k) \\ &\leq 0.\end{aligned}\tag{4.66}$$

Thus, $\varphi(t) \leq 0$, this shows $u(t) \leq \beta_{k+1}(t)$ on $\mathbb{T}$. Similarly, we can show that $u(t) \geq \alpha_{k+1}(t)$ on $\mathbb{T}$. Hence, we conclude by induction this the relation

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_m \leq u \leq \beta_m \leq \cdots \leq \beta_1 \leq \beta_0 \text{ on } \mathbb{T}.\tag{4.67}$$

Now, we need to show that the sequences are uniformly bounded and equicontinuous to show these sequences converge uniformly. To show the sequences $\{\alpha_m(t)\}, \{\beta_m(t)\}$ are uniformly bounded, then by the assumption both of $\alpha_0(t), \beta_0(t)$ are bounded on $\mathbb{T}$ and we proved that $\alpha_0(t) \leq \beta_m(t) \leq \beta_0(t)$ for each $m > 0$. Then

$$0 \leq |\beta_m(t) - \alpha_0(t)| \leq |\beta_0(t) - \alpha_0(t)| \text{ on } \mathbb{T}.$$

Consequently, the sequence $\{\beta_m(t)\}$ is uniformly bounded on $\mathbb{T}$. Similarly, we can show that the sequence $\{\alpha_m(t)\}$ is also uniformly bounded on $\mathbb{T}$. To show that $\{\beta_m(t)\}$ is equicontinuous, let $0 \leq t_1 \leq t_2$ on $\mathbb{T}$. Then for $m > 0$,

$$\begin{aligned}|\beta_m(t_1) - \beta_m(t_2)| &= |u_0 + \int_0^{t_1} [f_1(s, \beta_{m-1}(s)) - f_2(s, \alpha_{m-1}(s))] \Delta s \\ &\quad - u_0 - \int_0^{t_2} [f_1(s, \beta_{m-1}(s)) - f_2(s, \alpha_{m-1}(s))] \Delta s| \\ &= \left| \int_0^{t_1} [f_1(s, \beta_{m-1}(s)) - f_2(s, \alpha_{m-1}(s))] \Delta s \right. \\ &\quad \left. - \int_0^{t_2} [f_1(s, \beta_{m-1}(s)) - f_2(s, \alpha_{m-1}(s))] \Delta s \right| \\ &= \left| \int_{t_2}^{t_1} [f_1(s, \beta_{m-1}(s)) - f_2(s, \alpha_{m-1}(s))] \Delta s \right| \\ &\leq \int_{t_2}^{t_1} |f_1(s, \beta_{m-1}(s)) - f_2(s, \alpha_{m-1}(s))| \Delta s.\end{aligned}\tag{4.68}$$

Since $\{\alpha_m(t)\}$ and $\{\beta_m(t)\}$ are uniformly bounded and $f_1(t, u(t)), f_2(t, u(t))$ are bounded on $\mathbb{T}$, there exist $L > 0$ independent of n such that

$$|\beta_m(t_1) - \beta_m(t_2)| \leq L|t_1 - t_2|. \quad (4.69)$$

Thus, for any $\epsilon > 0$, there exist $\delta = \frac{\epsilon}{L} > 0$ independent of m such that for each m ,

$$|\beta_m(t_1) - \beta_m(t_2)| \leq \epsilon \quad \text{whenever} \quad |t_1 - t_2| \leq \delta. \quad (4.70)$$

Similarly, we can show that $\{\alpha_m(t)\}$ is also equicontinuous. Hence, by using Arzela-Ascoli's Theorem 4.5, there exist subsequences $\{\alpha_{m_k}(t)\}$ and $\{\beta_{m_k}(t)\}$ which converge uniformly to $\rho(t)$ and $r(t)$, respectively. But the monotonicity property of $\{\alpha_m(t)\}$, $\{\beta_m(t)\}$ implies that $\{\alpha_m(t)\}$ and $\{\beta_m(t)\}$ converge uniformly and monotonically to $\rho(t)$ and $r(t)$, respectively. After that, we need to show that $\rho(t)$ and $r(t)$ are coupled solution of Type I for (4.49). By using definition of (4.59) and taking delta integral from 0 into t , we obtain

$$\begin{aligned} \alpha_{m+1} &= u_0 + \int_0^t f_1(s, \alpha_m) - f_2(s, \beta_m) \Delta s, \\ \beta_{m+1} &= u_0 + \int_0^t f_1(s, \beta_m) - f_2(s, \alpha_m) \Delta s. \end{aligned} \quad (4.71)$$

Passing limit as $m \rightarrow \infty$ and taking delta derivative, give

$$\begin{aligned} \rho^\Delta &= f_1(t, \rho) - f_2(t, r), \quad \rho(t_0) = u_0, \\ r^\Delta &= f_1(t, r) - f_2(t, \rho), \quad r(t_0) = u_0. \end{aligned} \quad (4.72)$$

Finally, to show that ρ, r are coupled minimal and maximal solutions of Type I for (4.49). In view of $\alpha_0 \leq \alpha_m \leq u \leq \beta_m \leq \beta_0$ on $\mathbb{T}$ for all m , and passing limit as $m \rightarrow \infty$, we get the result $\alpha_0 \leq \rho \leq u \leq r \leq \beta_0$ on $\mathbb{T}$, which proves that ρ, r are coupled minimal and maximal solutions of Type I for (4.49), respectively. $\square$

Corollary 4.14: If in addition to assumption of previous theorem, if for $u_1 \geq u_2$ and $f_1(t, u(t))$ and $f_2(t, u(t))$ satisfy

$$f_1(t, u_1) - f_1(t, u_2) \leq -K_1(u_1 - u_2) \quad (4.73)$$

$$f_2(t, u_1) - f_2(t, u_2) \leq K_2(u_1 - u_2), \quad (4.74)$$

where $K_1, K_2 > 0$. Then $\rho(t) = u(t) = r(t)$ is the unique solution.

Proof. Since $\rho(t) \leq r(t)$, it is enough to prove that $r(t) \leq \rho(t)$. Let $p(t) = r(t) - \rho(t)$, then we find that $p(0) = 0$, by using (4.73) and 4.74, we get

$$\begin{aligned} p^\Delta(t) &= r^\Delta(t) - \rho^\Delta(t) \\ &= f_1(t, r(t)) - f_2(t, \rho(t)) - f_1(t, \rho(t)) + f_2(t, r(t)) \\ &\leq -(K_1 + K_2)p. \end{aligned} \quad (4.75)$$

Since $p(0) = 0$ and by Corollary 3.14; consequently, $p(t) \leq 0$. Thus $r(t) \leq \rho(t)$.

Hence, $\rho(t) = u(t) = r(t)$. □

Corollary 4.15: If $f_2 \equiv 0$ in the assumption of Theorem 4.13, then the conclusion of Theorem 4.13 holds.

The next theorem, we get monotone sequences that converge uniformly to coupled extremal solutions of Type I for (4.49) when we used coupled lower and upper solutions of Type II on $\mathbb{T}$.

Theorem 4.16: Assume that

- C1) α_0, β_0 are coupled lower and upper solutions of Type II for (4.49) with $\alpha_0 \leq \beta_0$ on $t \in \mathbb{T}$;
- C2) $f_1, f_2 \in C_{rd}[\mathbb{T}^k \times \mathbb{R}^n, \mathbb{R}^n]$, $f_1(t, u)$ and $f_2(t, u)$ are non-decreasing in u for each $t \in \mathbb{T}$.

Then there exist two monotone sequences $\{\alpha_m(t)\}$ and $\{\beta_m(t)\}$ such that

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_m \leq \beta_m \leq \cdots \leq \beta_1 \leq \beta_0 \text{ on } \mathbb{T} \quad (4.76)$$

provided $\alpha_0 \leq \alpha_1$ and $\beta_1 \leq \beta_0$ on $t \in \mathbb{T}$, where the sequences are given by

$$\begin{aligned} \alpha_{m+1}^\Delta &= f_1(t, \alpha_m) - f_2(t, \beta_m), & \alpha_{m+1}(0) &= u_0 \\ \beta_{m+1}^\Delta &= f_1(t, \beta_m) - f_2(t, \alpha_m), & \beta_{m+1}(0) &= u_0. \end{aligned} \quad (4.77)$$

Furthermore, the monotone sequences $\{\alpha_m(t)\}$ and $\{\beta_m(t)\}$ converge uniformly to $\rho(t), r(t)$ respectively, where $\rho(t), r(t)$ are coupled minimal and maximal solutions of (4.49), they satisfy the coupled system,

$$\begin{aligned}\rho^\Delta &= f_1(t, \rho) - f_2(t, r), & \rho(0) &= u_0 \\ r^\Delta &= f_1(t, r) - f_2(t, \rho), & r(0) &= u_0\end{aligned}\tag{4.78}$$

on $\mathbb{T}$.

We get the intertwined sequences that converge uniformly to coupled extremal solutions of Type I for (4.49) in the following theorem, by employing coupled lower and upper solutions of Type I.

Theorem 4.17: Suppose that the conditions (A1) and (A2) of Theorem (4.13) hold. Then the iterative scheme given by

$$\begin{aligned}\alpha_{m+1}^\Delta &= f_1(t, \beta_m) - f_2(t, \alpha_m), & \alpha_{m+1}(0) &= u_0, \\ \beta_{m+1}^\Delta &= f_1(t, \alpha_m) - f_2(t, \beta_m), & \beta_{m+1}(0) &= u_0,\end{aligned}\tag{4.79}$$

we have iterates monotone sequences $\{\alpha_{2m}(t)\}, \{\alpha_{2m+1}(t)\}, \{\beta_{2m}(t)\}$ and $\{\beta_{2m+1}(t)\}$ satisfy

$$\alpha_0 \leq \beta_1 \leq \cdots \leq \alpha_{2m} \leq \beta_{2m+1} \leq \alpha_{2m+1} \leq \beta_{2m} \leq \cdots \leq \alpha_1 \leq \beta_0\tag{4.80}$$

for each $m \geq 1$ on $\mathbb{T}$ provided that $\alpha_0 \leq u \leq \beta_0$. Moreover, the monotone sequences $\{\alpha_{2m}, \beta_{2m+1}\} \longrightarrow \rho$, and $\{\alpha_{2m+1}, \beta_{2m}\} \longrightarrow r$, where ρ and r are coupled minimal and maximal solutions of Type I for (4.49), respectively such that ρ and r satisfy the coupled system

$$\begin{aligned}\rho^\Delta &= f_1(t, \rho) - f_2(t, r), & \rho(0) &= u_0, \\ r^\Delta &= f_1(t, r) - f_2(t, \rho), & r(0) &= u_0.\end{aligned}\tag{4.81}$$

Proof. By the hypotheses $\alpha_0 \leq u \leq \beta_0$ on $\mathbb{T}$, it follows from (4.79) that

$$\begin{aligned}\alpha_1^\Delta &= f_1(t, \beta_0) - f_2(t, \alpha_0), \alpha_1(0) = u_0, \\ \beta_1^\Delta &= f_1(t, \alpha_0) - f_2(t, \beta_0), \beta_1(0) = u_0,\end{aligned}\tag{4.82}$$

and we know that α_0 and β_0 are coupled lower and upper solutions of Type I on $\mathbb{T}$. Now, let us to show $\beta_1 \leq \beta_0$ on $\mathbb{T}$ and in the same way, we can also show that $\alpha_0 \leq \alpha_1$ on $\mathbb{T}$.

Let $\varphi = \beta_1 - \beta_0$, then $\varphi(0) \leq 0$ and

$$\begin{aligned}\varphi^\Delta &= \beta_1^\Delta - \beta_0^\Delta \\ &\leq f_1(t, \alpha_0) - f_2(t, \beta_0) - f_1(t, \beta_0) + f_2(t, \alpha_0) \\ &\leq 0.\end{aligned}\tag{4.83}$$

We obtain $\varphi^\Delta(t) \leq 0$ and $\varphi(0) \leq 0$. We conclude by using Corollary 3.13 that $\varphi(t) \leq 0$ which implies that $\beta_1 \leq \beta_0$ on $\mathbb{T}$. Next, we need to show that $\alpha_0 \leq \beta_1 \leq \alpha_2 \leq \beta_3 \leq \alpha_3 \leq \beta_2 \leq \alpha_1 \leq \beta_0$ is true for $k = 1$ on $\mathbb{T}$. To show this, let $\varphi = \alpha_1 - \beta_0$, so that $\varphi(0) \leq 0$ and

$$\begin{aligned}\varphi^\Delta &= \alpha_1^\Delta - \beta_0^\Delta \\ &\leq f_1(t, \beta_0) - f_2(t, \alpha_0) - f_1(t, \beta_0) + f_2(t, \alpha_0) \\ &\leq 0.\end{aligned}\tag{4.84}$$

By Corollary 3.13, we have $\varphi(t) \leq 0$. Hence, $\alpha_1 \leq \beta_0$ on $\mathbb{T}$. Similarly, we can show that $\alpha_0 \leq \beta_1$ on $\mathbb{T}$. Let $\varphi = \beta_1 - \alpha_1$, so that $\varphi(0) = 0$ and

$$\begin{aligned}\varphi^\Delta &= \beta_1^\Delta - \alpha_1^\Delta \\ &= f_1(t, \alpha_0) - f_2(t, \beta_0) - f_1(t, \beta_0) + f_2(t, \alpha_0) \\ &\leq 0,\end{aligned}\tag{4.85}$$

since $\alpha_0 \leq \beta_0$ and f_1, f_2 are non-decreasing on $\mathbb{T}$. Then $\varphi(t) \leq 0$. This proves $\beta_1 \leq \alpha_1$ on $\mathbb{T}$. In the same way, we can show that $\alpha_2 \leq \beta_2$ and $\beta_3 \leq \alpha_3$ on $\mathbb{T}$.

Let $\varphi = \beta_2 - \alpha_1$ on $\mathbb{T}$, then $\varphi(0) = 0$ and

$$\varphi^\Delta = \beta_2^\Delta - \alpha_1^\Delta\tag{4.86}$$

$$\begin{aligned}\varphi^\Delta &\leq f_1(t, \alpha_1) - f_2(t, \beta_1) - f_1(t, \beta_0) + f_2(t, \alpha_0) \\ &\leq 0,\end{aligned}\tag{4.87}$$

we get $\varphi(t) \leq 0$ which gives that $\beta_2 \leq \alpha_1$ on $\mathbb{T}$. By similar argument, we can show that $\alpha_1 \leq \beta_2$ on $\mathbb{T}$. Assume that for some $k > 1$,

$$\alpha_{2k-2} \leq \beta_{2k-1} \leq \alpha_{2k} \leq \beta_{2k} \leq \alpha_{2k-1} \leq \beta_{2k-2} \text{ on } \mathbb{T}.\tag{4.88}$$

We wish to prove that

$$\alpha_{2k} \leq \beta_{2k+1} \leq \alpha_{2k+2} \leq \beta_{2k+2} \leq \alpha_{2k+1} \leq \beta_{2k} \text{ on } \mathbb{T}.\tag{4.89}$$

To prove this, let $\varphi = \alpha_{2k} - \beta_{2k+1}$. Then $\varphi(0) = 0$ and

$$\begin{aligned}\varphi^\Delta &= \alpha_{2k}^\Delta - \beta_{2k+1}^\Delta \\ &= f_1(t, \beta_{2k-1}) - f_2(t, \alpha_{2k-1}) - f_1(t, \alpha_{2k}) + f_2(t, \beta_{2k}) \\ &\leq 0\end{aligned}\tag{4.90}$$

since $\beta_{2k-1} \leq \alpha_{2k} \leq \beta_{2k} \leq \alpha_{2k-1}$ is true on $\mathbb{T}$, and by monotone nature of f_1, f_2 on $\mathbb{T}$, we get $\varphi(t) \leq 0$ on $\mathbb{T}$ implies that $\alpha_{2k} \leq \beta_{2k+1}$ on $\mathbb{T}$. Similarly, we can show that $\alpha_{2k+1} \leq \beta_{2k}$, $\beta_{2k+1} \leq \alpha_{2k+2}$ and $\beta_{2k+2} \leq \alpha_{2k+1}$ on $\mathbb{T}$. Let $\varphi = \beta_{2k+1} - \alpha_{2k+1}$, then $\varphi(0) = 0$. By the same steps as before, we get $\varphi(t) \leq 0$, which gives that $\beta_{2k+1} \leq \alpha_{2k+1}$ on $\mathbb{T}$. Similarly, we can show that $\alpha_{2k+2} \leq \beta_{2k+2}$ on $\mathbb{T}$. We conclude by induction this the relation

$$\alpha_0 \leq \beta_1 \leq \cdots \leq \alpha_{2m} \leq \beta_{2m+1} \leq \alpha_{2m+1} \leq \beta_{2m} \leq \cdots \leq \alpha_1 \leq \beta_0 \text{ on } \mathbb{T}.\tag{4.91}$$

We shall show uniformly bounded of the sequence $\{\alpha_{2m}(t)\}$ only, since the cases of the sequences $\{\beta_{2m+1}(t)\}$, $\{\alpha_{2m+1}(t)\}$ and $\{\beta_{2m}(t)\}$ are very similar. By the assumption of both $\alpha_0(t)$ and $\beta_0(t)$ are bounded on $\mathbb{T}$, it follows from the relation (4.91) that $\{\alpha_{2m}(t)\}$ is uniformly bounded. And also, $\{\beta_{2m+1}(t)\}$, $\{\alpha_{2m+1}(t)\}$ and $\{\beta_{2m}(t)\}$ are also uniformly bounded on $\mathbb{T}$. We can show equicontinuous of the sequences $\{\alpha_{2m}(t)\}$, $\{\beta_{2m+1}(t)\}$, $\{\alpha_{2m+1}(t)\}$ and $\{\beta_{2m}(t)\}$ as we did that in Theorem 4.13. Hence, we conclude that the sequences $\{\alpha_{2m}(t), \beta_{2m+1}(t)\}$, and $\{\alpha_{2m+1}(t), \beta_{2m}(t)\}$

are uniformly bounded and equicontinuous. Then by using the Arzela-Ascoli's theorem, there exist subsequences $\{\alpha_{2m_k}(t), \beta_{2m_k+1}(t)\}$ and $\{\alpha_{2m_k+1}(t), \beta_{2m_k}(t)\}$ which converge uniformly to ρ and r respectively, on $\mathbb{T}$. Since the sequences $\{\alpha_{2m}(t), \beta_{2m+1}(t)\}$ and $\{\alpha_{2m+1}(t), \beta_{2m}(t)\}$ are monotone, then the entire sequences converge uniformly and monotonically to ρ and r respectively, on $\mathbb{T}$. We need to prove that ρ and r satisfy (4.81).

Observe that, by since each $\{\alpha_{2m+1}(t)\}$ and $\{\beta_{2m}(t)\}$ are constructed as following

$$\begin{aligned}\alpha_{2m+1}^\Delta &= f_1(t, \beta_{2m}) - f_2(t, \alpha_{2m}), & \alpha_{2m+1}(0) &= u_0, \\ \beta_{2m}^\Delta &= f_1(t, \alpha_{2m+1}) - f_2(t, \beta_{2m+1}), & \beta_{2m}(0) &= u_0.\end{aligned}\tag{4.92}$$

Now, we take delta integral of (4.92) on each side from 0 into t , then we get that

$$\begin{aligned}\alpha_{2m+1} &= u_0 + \int_0^t [f_1(s, \beta_{2m}(s)) - f_2(s, \alpha_{2m}(s))] \Delta s, \\ \beta_{2m} &= u_0 + \int_0^t [f_1(s, \alpha_{2m+1}(s)) - f_2(s, \beta_{2m+1}(s))] \Delta s.\end{aligned}\tag{4.93}$$

Taking limit as $m \rightarrow \infty$ on each side, we conclude that

$$\begin{aligned}r &= u_0 + \int_{t_0}^t [f_1(s, r(s)) - f_2(s, \rho(s))] \Delta s, \\ \rho &= u_0 + \int_{t_0}^t [f_1(s, \rho(s)) - f_2(s, r(s))] \Delta s.\end{aligned}\tag{4.94}$$

Then

$$\begin{aligned}r^\Delta &= f_1(t, r) - f_2(t, \rho), & r(0) &= u_0, \\ \rho^\Delta &= f_1(t, \rho) - f_2(t, r), & \rho(0) &= u_0.\end{aligned}\tag{4.95}$$

Finally, it remains to prove that ρ, r are coupled minimal and maximal solutions of Type I for (4.49) on $\mathbb{T}$. To show this, suppose that $\alpha_{2k-2} \leq \beta_{2k-1} \leq \alpha_{2k} \leq u \leq \beta_{2k} \leq \alpha_{2k+1} \leq \beta_{2k+2}$ is true, for some $k > 1$. We need to show that $\alpha_{2k} \leq \beta_{2k+1} \leq \alpha_{2k+2} \leq u \leq \beta_{2k+2} \leq \alpha_{2k+1} \leq \beta_{2k}$ on $\mathbb{T}$.

Let $\varphi = \beta_{2k+1} - u$ on $\mathbb{T}$. Then $\varphi(0) = 0$ on $\mathbb{T}$, and

$$\varphi^\Delta = \beta_{2k+1}^\Delta - u^\Delta\tag{4.96}$$

$$\begin{aligned}\varphi^\Delta &= f_1(t, \alpha_{2k}) - f_2(t, \beta_{2k}) - f_1(t, u) + f_2(t, u) \\ &\leq 0\end{aligned}\tag{4.97}$$

since $\beta_{2k-1} \leq \alpha_{2k} \leq u \leq \beta_{2k} \leq \alpha_{2k-1}$ on $\mathbb{T}$ is true and by monotone nature of f_1, f_2 on $\mathbb{T}$, we get by Corollary 3.13 that $\varphi(t) \leq 0$ which proves this $\beta_{2k+1} \leq u$ on $\mathbb{T}$. By similarly argument, we can show that $\alpha_{2k+1} \geq u$, $\beta_{2k+2} \geq u$ and $\alpha_{2k+2} \leq u$ on $\mathbb{T}$. We obtain this relation

$$\alpha_0 \leq \beta_1 \leq \cdots \leq \alpha_{2m} \leq \beta_{2m+1} \leq u \leq \alpha_{2m+1} \leq \beta_{2m} \leq \cdots \leq \alpha_1 \leq \beta_0 \quad \text{on } \mathbb{T}.$$

Taking the limit as $m \rightarrow \infty$, then we get that $\alpha_0 \leq \rho \leq u \leq r \leq \beta_0$ on $\mathbb{T}$, showing that ρ and r are coupled minimal and maximal solutions of Type I for (4.49) on $\mathbb{T}$. $\square$

Remark 4.18: In Theorem (4.17),

- if $f_1(t, u) - f_2(t, u)$ is non-increasing on $\mathbb{T}$, then there exists a unique solution on $\mathbb{T}$.
- if $f_2 \equiv 0$, then the result of Theorem 4.17 is true.

Theorem 4.19: Suppose that

- H1) α_0, β_0 are coupled lower and upper solutions of Type III for (4.49) with $\alpha_0 \leq u \leq \beta_0$ on $t \in \mathbb{T}$;
- H2) $f_1, f_2 \in C_{rd}[\mathbb{T}^k \times \mathbb{R}, \mathbb{R}]$, $f_1(t, u)$ and $f_2(t, u)$ are non-decreasing in u for each t in $\mathbb{T}$.

Then the sequences defined by

$$\begin{aligned}\alpha_{m+1}^\Delta &= f_1(t, \beta_m) - f_2(t, \alpha_m), \quad \alpha_{m+1}(0) = u_0 \\ \beta_{m+1}^\Delta &= f_1(t, \alpha_m) - f_2(t, \beta_m), \quad \beta_{m+1}(0) = u_0.\end{aligned}\tag{4.98}$$

give alternating monotone sequences $\{\alpha_{2m}, \beta_{2m+1}\}$ and $\{\beta_{2m}, \alpha_{2m+1}\}$ in $C_{rd}^1[\mathbb{T}, \mathbb{R}^n]$ where the sequences are given by

$$\alpha_0 \leq \beta_1 \leq \cdots \leq \alpha_{2m} \leq \beta_{2m+1} \leq u \leq \alpha_{2m+1} \leq \beta_{2m} \leq \cdots \leq \alpha_1 \leq \beta_0 \quad \text{on } \mathbb{T} \tag{4.99}$$

provided that $\alpha_0 \leq \beta_1 \leq u \leq \alpha_1 \leq \beta_0$ on $t \in \mathbb{T}$.

Furthermore, the monotone sequences $\{\alpha_{2m}, \beta_{2m+1}\}$ converge to ρ and $\{\beta_{2m}, \alpha_{2m+1}\}$ converge to r , where ρ, r are coupled minimal and maximal solutions of (4.49) respectively; such that, they satisfy the coupled system,

$$\begin{aligned}\rho^\Delta &= f_1(t, \rho) - f_2(t, r), & \rho(0) &= u_0 \\ r^\Delta &= f_1(t, r) - f_2(t, \rho), & r(0) &= u_0\end{aligned}\tag{4.100}$$

on $\mathbb{T}$.

We give some examples which are application of Theorems (4.16) and (4.19).

Example 4.20: Let us consider the dynamic initial value problems(IVP)

$$u^\Delta(t) = \frac{u}{2} - \frac{u^2}{4}, \quad u(0) = 1, \quad t \in [0, 1].\tag{4.101}$$

Let

$$\alpha_0(t) = -1 \quad \beta_0(t) = 2 \quad \text{for all } t \in [0, 1].$$

Then

$$\left\{ \begin{array}{l} \alpha_0^\Delta = 0 \\ \alpha_0(0) = -1 \end{array} \right. \leq \left\{ \begin{array}{l} f_1(t, \beta_0) - f_2(t, \alpha_0) = \frac{3}{4}; \\ u(0) = 1, t \in [0, 1]. \end{array} \right. , \quad \left\{ \begin{array}{l} \beta_0^\Delta = 0 \\ \beta_0(0) = 2 \end{array} \right. \geq \left\{ \begin{array}{l} f_1(t, \alpha_0) - f_2(t, \beta_0) = -\frac{3}{2}; \\ u(0) = 1, t \in [0, 1] \end{array} \right.$$

This implies that α_0, β_0 are coupled lower and upper solutions of Type II for problem (4.101). Consider the functions $\alpha_1(t) = -t^2$ and $\beta_1(t) = \frac{t^2}{2} + \frac{7}{5}$ satisfy $\alpha_0 \leq \alpha_1, \beta_1 \leq \beta_0$, respectively. Then all the conditions of Theorem (4.16) are satisfied. Thus, applying Theorem (4.16) and using (4.59), we obtain the existence of monotone sequences which converge to minimal and maximal solutions in the sector $[-1, 2]$.

Example 4.21: Let us consider the dynamic initial value problems as follows,

$$u^\Delta = u^2 - 2 - u, \quad u(0) = 1 \quad t \in [0, 1].\tag{4.102}$$

Let

$$\alpha_0 = -1, \quad \beta_0 = e.$$

We obtain that

$$\left\{ \begin{array}{l} \alpha_0^\Delta = 0 \leq f_1(t, \beta_0) - f_2(t, \beta_0) = e^2 - 2 - e; \\ \alpha_0(0) \leq u(0), t \in [0, 1]. \end{array} \right., \left\{ \begin{array}{l} \beta_0^\Delta = 0 \geq f_1(t, \alpha_0) - f_2(t, \alpha_0) = 0; \\ \beta_0(0) \geq u(0), t \in [0, 1]. \end{array} \right.$$

This gives that $\alpha_0(t)$ and $\beta_0(t)$ are coupled lower and upper solutions of Type III for problem (4.102). Then all the conditions of Theorem (4.19) are all satisfied. Let $\alpha_1(t) = t^7 + \frac{5}{3}$ and $\beta_1(t) = -t^7$ such that they satisfy $\alpha_0 \leq \beta_1 \leq \alpha_1 \leq \beta_0$, we can applying Theorem (4.19) and using (4.98), then we get the existence of alternating monotone sequences which converge to minimal and maximal solutions in the sector $[-1, e]$.

4.4. The Method of Mixed Monotony

In this section, we discuss the method of mixed monotone such that

- we first need to assume at least mixed monotone property to obtain this generalization.
- if we does not have any monotone property, we can construct monotone sequences to the systems by using a method known as the method of mixed monotonicity.

Consider dynamic initial value problem

$$u^\Delta = g(t, u), \quad u(0) = u_0, \quad t \in \mathbb{T}^k \quad (4.103)$$

where $g : C_{rd}[\mathbb{T}^k \times \mathbb{R}^n, \mathbb{R}^n]$. A function $u : \mathbb{T}^k \longrightarrow \mathbb{R}^n$ is a solution of IVP (4.103), if $u(t)$ is an anti-derivative of $g(t, u)$ on $\mathbb{T}^k$ and satisfies $u(t_0) = u_0$. In this section, let $\mathbb{T}^k = [0, J_0] \cap \mathbb{T}$, $J_0 > 0$.

Definition 4.22: Let $G : C_{rd}[\mathbb{T}^k \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$. Then we say that G is mixed-monotone, if there exists a continuous map $g : \mathbb{T}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfying:

- $G(t, u, u) = g(t, u)$;
- $\forall v \in \mathbb{R}^n, u_1, u_2 \in \mathbb{T}^k, u_1 \leq u_2 \Rightarrow G(t, u_1, v) \leq G(t, u_2, v)$;
- $\forall u \in \mathbb{R}^n, v_1 \leq v_2 \Rightarrow G(t, u, v_1) \geq G(t, u, v_2)$.

In short, $G(t, u, v)$ is monotonically non-decreasing in u and monotonically non-increasing in v .

Theorem 4.23: Let a function $G(t, u, v) \in C_{rd}[\mathbb{T}^k \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n]$, such that

- i) $G(t, u, v)$ is non-decreasing in u and non-increasing in v for $t \in \mathbb{T}^k$,
- ii) $G(t, u, u) = g(t, u)$,
- iii) $\alpha_0^\Delta \leq G(t, \alpha_0, \beta_0)$, $\beta_0^\Delta \geq G(t, \beta_0, \alpha_0)$, $\alpha_0(t) \leq \beta_0(t)$, $t \in \mathbb{T}^k$,

where $\alpha_0, \beta_0 : \mathbb{T}^k \rightarrow \mathbb{R}^n$ are differentiable for $t \in \mathbb{T}^k$. Then there exist monotone sequences $\{\alpha_m(t)\}$, $\{\beta_m(t)\}$ such that $\alpha_m \rightarrow \rho$, $\beta_m \rightarrow r$ as $m \rightarrow \infty$ locally uniformly and monotonically on $\mathbb{T}^k$ and that ρ, r satisfy

$$\rho^\Delta = G(t, \rho, r), \quad \rho(0) = u_0, \quad (4.104)$$

$$r^\Delta = G(t, r, \rho), \quad r(0) = u_0. \quad (4.105)$$

If in addition, we suppose that

$$|G(t, u, v) - G(t, v, u)| \leq K|u - v|. \quad (4.106)$$

Then $u(t) = \rho(t) = r(t)$ is the unique solution of (4.103) on $\mathbb{T}^k$.

Proof. Define $\alpha_m(t), \beta_m(t)$ as follows

$$\alpha_{m+1}^\Delta = G(t, \alpha_m, \beta_m), \quad \alpha_{m+1}(0) = u_0, \quad (4.107)$$

$$\beta_{m+1}^\Delta = G(t, \beta_m, \alpha_m), \quad \beta_{m+1}(0) = u_0 \quad (4.108)$$

since $\alpha_0^\Delta \leq G(t, \alpha_0, \beta_0) = \alpha_1^\Delta$, $\beta_0^\Delta \geq G(t, \beta_0, \alpha_0) = \beta_1^\Delta$ and by (4.23), it is clear that $\alpha_0(t) \leq \alpha_1(t)$, $\beta_0(t) \geq \beta_1(t)$ on $\mathbb{T}^k$, we know that the fact $\alpha_0(0) \leq u_0 \leq \beta_0(0)$.

Assume that for some $k > 1$,

$$\alpha_{k-1}(t) \leq \alpha_k(t), \quad \beta_k(t) \leq \beta_{k-1}(t) \quad \text{on } \mathbb{T}. \quad (4.109)$$

Then by using (i), we have

$$\alpha_k^\Delta(t) = G(t, \alpha_{k-1}, \beta_{k-1}) \leq G(t, \alpha_k, \beta_k) = \alpha_{k+1}^\Delta(t), \quad (4.110)$$

where $\alpha_{k+1}(0) = \alpha_k(0) = u_0$; and also

$$\beta_{k+1}^\Delta = G(t, \beta_k, \alpha_k) \leq G(t, \beta_{k-1}, \alpha_{k-1}) = \beta_k^\Delta \quad (4.111)$$

where $\beta_{k+1}(0) = \beta_k(0) = u_0$; which implies that $\alpha_k(t) \leq \alpha_{k+1}(t)$, $\beta_k \geq \beta_{k+1}$, $t \in \mathbb{T}^k$.

Hence, by induction we obtain this relation

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_m \leq \beta_m \leq \cdots \leq \beta_1 \leq \beta_0 \quad (4.112)$$

on $\mathbb{T}^k$.

Clearly, the constructed $\{\alpha_m(t)\}, \{\beta_m(t)\}$ are uniformly bounded and since G is bounded, we have $|\alpha_m^\Delta(t)|$ and $|\beta_m^\Delta(t)|$ are also bounded. The sequences $\{\alpha_m\}, \{\beta_m\}$ are equicontinuous. Then by using Arzela-Ascoli Theorem, there exist subsequences $\{\alpha_{m_k}(t)\}$ and $\{\beta_{m_k}(t)\}$ which converge uniformly on $\mathbb{T}^k$. $\{\alpha_m(t)\}, \{\beta_m(t)\}$ locally uniformly and monotonically on $\mathbb{T}^k$. It then follows by the standard argument that

$$\lim_{m \rightarrow \infty} \alpha_m(t) = \rho(t) \quad (4.113)$$

$$\lim_{m \rightarrow \infty} \beta_m(t) = r(t) \quad (4.114)$$

We need to prove that ρ, r satisfy

$$\begin{aligned} \rho^\Delta &= G(t, \rho, r), \quad \rho(0) = u_0, \\ r^\Delta &= G(t, r, \rho), \quad r(0) = u_0. \end{aligned} \quad (4.115)$$

Taking delta integral to (4.107) and (4.108) on each side from 0 into t . We find that

$$\alpha_{m+1}(t) = \alpha_{m+1}(0) + \int_0^t G(s, \alpha_m(s), \beta_m(s)) \Delta s. \quad (4.116)$$

$$\beta_{m+1}(t) = \beta_{m+1}(0) + \int_0^t G(s, \beta_m(s), \alpha_m(s)) \Delta s. \quad (4.117)$$

Now, we will take limit on each side as $m \longrightarrow \infty$ and since $\{\alpha_m\}, \{\beta_m\}$ are uniformly convergent, therefore we get that

$$\rho(t) = u_0 + \int_0^t G(s, \rho(s), r(s)) \Delta s. \quad (4.118)$$

$$r(t) = u_0 + \int_0^t G(s, r(s), \rho(s)) \Delta s. \quad (4.119)$$

Taking delta derivative to (4.118) and (4.119), we obtain

$$\rho^\Delta(t) = G(t, \rho(t), r(t)), \quad \rho(0) = u_0. \quad (4.120)$$

$$r^\Delta(t) = G(t, r(t), \rho(t)), \quad r(0) = u_0. \quad (4.121)$$

To prove $u = \rho = r$, letting $N = |r - \rho|$, by using (4.106). Then

$$\begin{aligned} q^\Delta(t) &= |r^\Delta(t) - \rho^\Delta(t)| \\ &= |G(t, r(t), \rho(t)) - G(t, \rho(t), r(t))| \\ &\leq Kq, \end{aligned} \quad (4.122)$$

since $q(0) = 0$, we get by Corollary 3.13 that $q(t) \leq 0$. Thus $r = \rho = u$ on $\mathbb{T}^k$. Since $G(t, u, u) = g(t, u)$, we note that $u = \rho = r$ is the unique solution of (4.103) on $\mathbb{T}^k$.

This proof is complete. □

Theorem 4.24: Suppose that

i) let $\alpha, \beta : \mathbb{T}^k \longrightarrow \mathbb{R}^n$ are differentiable for $t \in \mathbb{T}^k$ such that $\alpha^\Delta \leq g(t, \alpha) - A(\beta - \alpha)$, $\beta^\Delta \geq g(t, \beta) + A(\beta - \alpha)$, $\alpha(t) \leq \beta(t)$ on $t \in \mathbb{T}^k$,

ii) $|g(t, u) - g(t, y)| \leq A(u - y)$ whenever $\alpha(t) \leq y \leq u \leq \beta(t)$,

where A is matrix $m \times m$ of non-negative elements. Then the system (4.103) admits the

method of mixed monotony.

Proof. We define

$$\begin{aligned} G(t, v, z) &= \frac{1}{2}[g(t, v) + g(t, z) + A(v - z)]. \\ G(t, \bar{v}, \bar{z}) &= \frac{1}{2}[g(t, \bar{v}) + g(t, \bar{z}) + A(\bar{v} - \bar{z})]. \end{aligned} \quad (4.123)$$

It easy to note that $G(t, v, z)$ is mixed monotone and locally Lipschitzian. Then

$$-A(v - \bar{v}) \leq g(t, v) - g(t, \bar{v}) \leq A(v - \bar{v}) \quad (4.124)$$

$$-A(z - \bar{z}) \leq g(t, z) - g(t, \bar{z}) \leq A(z - \bar{z}). \quad (4.125)$$

We reduce to

$$-A(z - \bar{z}) - A(v - \bar{v}) \leq g(t, v) - g(t, \bar{v}) + g(t, z) - g(t, \bar{z}) \leq A(v - \bar{v}) + A(z - \bar{z}). \quad (4.126)$$

Now, multiply the equation (4.126) by $\frac{1}{2}$ on each side, after that we will add $\frac{1}{2}A(v - \bar{v}) - \frac{1}{2}A(z - \bar{z})$ to both sides. We obtain that

$$-A(z - \bar{z}) \leq G(t, v, z) - G(t, \bar{v}, \bar{z}) \leq A(v - \bar{v}). \quad (4.127)$$

whenever $\alpha(t) \leq \bar{v} \leq v \leq \beta(t)$ and $\alpha(t) \leq \bar{z} \leq z \leq \beta(t)$. in particular, if $\bar{v} = z$, $\bar{z} = v$ and substitution in (4.127), then we have that

$$-A(z - v) \leq G(t, v, z) - G(t, z, v) \leq A(v - z). \quad (4.128)$$

Then

$$G(t, v, z) - G(t, z, v) = A(v - z). \quad (4.129)$$

It follows from (4.129) that $G(t, \alpha, \beta) - G(t, \beta, \alpha) = A(\alpha - \beta)$, and from (4.123), we get that

$$G(t, \beta, \alpha) = \frac{1}{2}[g(t, \beta) + g(t, \alpha) + A(\beta - \alpha)]. \quad (4.130)$$

It follows from (ii) that

$$g(t, \beta) \geq g(t, \alpha) - A(\beta - \alpha). \quad (4.131)$$

By employing substitution in (4.130), gets

$$G(t, \beta, \alpha) \geq g(t, \alpha), \quad (4.132)$$

then multiplying the equation (4.132) by (-1) , we find

$$-G(t, \beta, \alpha) \leq -g(t, \alpha) \quad (4.133)$$

adding $G(t, \alpha, \beta)$ to both sides in the equation (4.133). Therefore

$$-A(\beta - \alpha) \leq G(t, \alpha, \beta) - g(t, \alpha) \quad (4.134)$$

Thus,

$$\alpha^\Delta \leq G(t, \alpha, \beta). \quad (4.135)$$

Similary, we can prove $\beta^\Delta \geq G(t, \beta, \alpha)$. It follows from (4.129) that

$$G(t, \beta, \alpha) - G(t, \alpha, \beta) = A(\beta - \alpha) \quad (4.136)$$

and we know that

$$G(t, \alpha, \beta) = \frac{1}{2}[g(t, \alpha) + g(t, \beta) + A(\alpha - \beta)]. \quad (4.137)$$

By employing (ii), we have

$$g(t, \alpha) \leq g(t, \beta) - A(\alpha - \beta). \quad (4.138)$$

By substitution in (4.137), we obtain

$$G(t, \alpha, \beta) \leq g(t, \beta). \quad (4.139)$$

Multiplying the equation (4.139) by (-1) , we have

$$-G(t, \alpha, \beta) \geq -g(t, \beta). \quad (4.140)$$

Adding $G(t, \beta, \alpha)$ to both sides in the equation (4.140), then

$$A(\beta - \alpha) \geq G(t, \beta, \alpha) - g(t, \beta). \quad (4.141)$$

Thus,

$$\beta^\Delta \geq G(t, \beta, \alpha). \quad (4.142)$$

Finally, setting $u = v = z$, then we get that

$$G(t, u, u) = g(t, u). \quad (4.143)$$

□

5. THE METHOD OF QUASILINEARIZATION ON TIME SCALE

In this chapter, by utilizing the method of lower and upper solutions together with the method of quasilinearization, it is probable to construct two monotone sequences that converge super-linear to the unique solution of the considered problem. We devote a method of quasilinearization to the dynamic equation on $\mathbb{T}$, that is,

$$u^\Delta = g(t, u), \quad u(0) = u_0 \quad (5.1)$$

where $g \in C_{rd}[\Omega_0, \mathbb{R}^n]$, $\Omega_0 = \{(t, u) : \tilde{\alpha}(t) \leq u(t) \leq \tilde{\beta}(t), t \in \mathbb{T}\}$ and $\tilde{\alpha}, \tilde{\beta} \in C_{rd}^1[\mathbb{T}, \mathbb{R}^n]$ such that $\tilde{\alpha}(t) \leq \tilde{\beta}(t), t \in \mathbb{T}$. The time scale $\mathbb{T}$ is supposed to have $t = 0$ as a minimal element and $t = T_0$ as a maximal element which is not left-scattered.

Let $\Omega = \{(t, u, v) : \tilde{\alpha}(t) \leq u, v \leq \tilde{\beta}(t), t \in \mathbb{T}\}$. Let us show the following theorem.

Theorem 5.1: Suppose a function $R \in C_{rd}[\Omega, \mathbb{R}^n]$ such that

- i) $\tilde{\alpha}, \tilde{\beta} \in C_{rd}[\mathbb{T}, \mathbb{R}^n]$ with $\tilde{\alpha}(t) \leq \tilde{\beta}(t)$ and $\tilde{\alpha}^\Delta \leq R(t, \tilde{\alpha}, \tilde{\alpha}), \quad \tilde{\beta}^\Delta \geq R(t, \tilde{\beta}, \tilde{\beta});$
- ii) R is bounded on Ω , $R(t, u, v)$ is quasi-monotone non-decreasing in u and $R_j(t, u, v)\mu^*(t) + u_j$ is non-decreasing in u_j , for each $1 \leq j \leq m, \quad t \in \mathbb{T}$ and $\tilde{\alpha}(t) \leq v \leq \tilde{\beta}(t);$
- iii) $R(t, u, u) = g(t, u)$ in Ω and $R(t, u, v) \leq R(t, u, u), \quad u \geq v, t \in \mathbb{T};$
- iv) $R_j(t, u_1, v) - R_j(t, u_2, v) \leq L \sum_{j=1}^n (u_{1j} - u_{2j}),$ whenever $u_1 \geq u_2$ and $\tilde{\alpha}(t) \leq v \leq \tilde{\beta}(t), t \in \mathbb{T};$
- v) $R(t, u, u) - R(t, v, x) \leq A(u - x)^{1+\beta} + B(u - v),$ whenever $\tilde{\alpha}(t) \leq u, v, x \leq \tilde{\beta}(t), t \in \mathbb{T},$ where A, B are $k \times k$ non-negative matrices.

Then there exists monotone sequence $\{\tilde{\alpha}_m(t)\}$ which converges locally uniformly to the unique solution of (5.1) on $\mathbb{T}$ and this convergence is super-linear.

Proof. Set $\tilde{\alpha} = \tilde{\alpha}_0, \tilde{\beta} = \tilde{\beta}_0$ and

$$\tilde{\alpha}_1^\Delta = R(t, \tilde{\alpha}_1, \tilde{\alpha}_0), \quad \tilde{\alpha}_1(0) = u_0, \quad (5.2)$$

since $\tilde{\alpha}_0^\Delta \leq R(t, \tilde{\alpha}_0, \tilde{\alpha}_0)$, $\tilde{\alpha}_0(t) \leq \tilde{\beta}_0(t)$ and by (iii), we have that $\tilde{\beta}_0^\Delta \geq R(t, \tilde{\beta}_0, \tilde{\beta}_0) \geq R(t, \tilde{\beta}_0, \tilde{\alpha}_0)$. By using Theorem 4.7 there exists a solution $\tilde{\alpha}_1(t)$ of (5.2) such that $\tilde{\alpha}_0 \leq \tilde{\alpha}_1 \leq \tilde{\beta}_0$ on $\mathbb{T}$. Applying the Theorem 3.12, which shows that $\tilde{\alpha}_1$ is the unique solution of (5.2). Then due to (iii), we find

$$\tilde{\alpha}_1^\Delta = R(t, \tilde{\alpha}_1, \tilde{\alpha}_0) \leq R(t, \tilde{\alpha}_1, \tilde{\alpha}_1).$$

Set

$$\tilde{\alpha}_2^\Delta = R(t, \tilde{\alpha}_2, \tilde{\alpha}_1), \quad \tilde{\alpha}_2(0) = u_0 \quad (5.3)$$

since $\tilde{\alpha}_1^\Delta \leq R(t, \tilde{\alpha}_1, \tilde{\alpha}_1)$, $\tilde{\beta}_0 \geq \tilde{\alpha}_1$ and by (iii), then

$$\tilde{\beta}_0^\Delta \geq R(t, \tilde{\beta}_0, \tilde{\beta}_0) \geq R(t, \tilde{\beta}_0, \tilde{\alpha}_1).$$

By using Theorem 4.7, there exists a solution $\tilde{\alpha}_2(t)$ of (5.3) such that $\tilde{\alpha}_0 \leq \tilde{\alpha}_1 \leq \tilde{\alpha}_2$ on $\mathbb{T}$. By Theorem 3.12, proves that $\tilde{\alpha}_2$ is the unique solution of (5.3) implies that $\tilde{\alpha}_0 \leq \tilde{\alpha}_1 \leq \tilde{\alpha}_2 \leq \tilde{\beta}_0$ on $\mathbb{T}$. Continue in the same way, let

$$\tilde{\alpha}_m^\Delta = R(t, \tilde{\alpha}_m, \tilde{\alpha}_{m-1}), \quad \tilde{\alpha}_m(0) = u_0. \quad (5.4)$$

As before, $\tilde{\alpha}_{m-1}^\Delta \leq R(t, \tilde{\alpha}_{m-1}, \tilde{\alpha}_{m-1})$, $\tilde{\beta}_0 \leq \tilde{\alpha}_{m-1}$, then we get

$$\tilde{\beta}_0^\Delta \geq R(t, \tilde{\beta}_0, \tilde{\alpha}_{m-1}).$$

Then there exists a solution $\tilde{\alpha}_m(t)$ of (5.4) such that $\tilde{\alpha}_0 \leq \tilde{\alpha}_{m-1} \leq \tilde{\alpha}_m$ on $\mathbb{T}$. Then $\tilde{\alpha}_m$ is the unique solution of (5.4) implies that

$$\tilde{\alpha}_0 \leq \dots \tilde{\alpha}_{m-1} \leq \tilde{\alpha}_m \leq \tilde{\beta}_0 \text{ on } \mathbb{T}. \quad (5.5)$$

We note that

$$\tilde{\alpha}_0 \leq \tilde{\alpha}_m \leq \tilde{\beta}_0$$

and $\tilde{\alpha}_0, \tilde{\beta}_0$ are bounded. Thus, $\{\tilde{\alpha}_m(t)\}$ is uniformly bounded on $\mathbb{T}$. In view the fact $\tilde{\alpha}_0, \tilde{\beta}_0$ are bounded and R is bounded, then we have $\{\tilde{\alpha}_m^\Delta(t)\}$ is also uniformly bounded on $\mathbb{T}$. This proves that $\{\tilde{\alpha}_m(t)\}$ is equicontinuous on $\mathbb{T}$. Applying Arzela- Ascoli's Theorem, then we obtain $\{\tilde{\alpha}_m(t)\}$ converge uniformly to $u(t)$ and monotonically on $\mathbb{T}$. It is easy to note that $u(t)$ is the unique solution of $u^\Delta(t) = R(t, u, u) = g(t, u)$.

Finally, to show that the convergence $\{\tilde{\alpha}_m(t)\}$ to u is super-linear, let

$$\varphi_{m+1}(t) = u(t) - \tilde{\alpha}_{m+1}(t), \text{ and so } \varphi_{m+1}(0) = 0.$$

Differentiating above equation with respect to t , gives

$$\begin{aligned} \varphi_{m+1}^\Delta(t) &= u^\Delta(t) - \tilde{\alpha}_{m+1}^\Delta(t) \\ &= R(t, u, u) - R(t, \tilde{\alpha}_{m+1}(t), \tilde{\alpha}_m(t)). \end{aligned} \quad (5.6)$$

We note that $\varphi_{m+1}(t) \geq 0$. Then by (v), we find

$$\begin{aligned} \varphi_{m+1}^\Delta(t) &\leq B(u(t) - \tilde{\alpha}_{m+1}(t)) + A(u(t) - \tilde{\alpha}_m(t))^{1+\beta} \\ &\leq B\varphi_{m+1}(t) + A\varphi_m(t)^{1+\beta}. \end{aligned} \quad (5.7)$$

By employing the method of variation of parameters formula (VPF) for dynamic equation to solve above equation, let

$$\varphi_{m+1}(t) = e_B(t, 0)w(t). \quad (5.8)$$

Then by using product rule, we find

$$\varphi_{m+1}^\Delta(t) = Be_B(t, 0)w(t) + e_B(\sigma(t), 0)w^\Delta(t). \quad (5.9)$$

Then by substituting in (5.7), we get

$$\begin{aligned} Be_B(t, 0)w(t) + e_B(\sigma(t), 0)w^\Delta(t) &\leq Be_B(t, 0)w(t) + A\varphi_m(t)^{1+\beta} \\ e_B(\sigma(t), 0)w^\Delta(t) &\leq A\varphi_m(t)^{1+\beta} \\ w^\Delta(t) &\leq e_B^{-1}(\sigma(t), 0)A\varphi_m(t)^{1+\beta} \\ w^\Delta(t) &\leq e_B(0, \sigma(t))A\varphi_m(t)^{1+\beta} \\ \int_0^t w^\Delta(s)\Delta s &\leq A \int_0^t e_B(0, \sigma(s))\varphi_m(s)^{1+\beta}\Delta s \\ w(t) &\leq A \int_0^t e_B(0, \sigma(s))\varphi_m(s)^{1+\beta}\Delta s. \end{aligned} \quad (5.10)$$

Then, by substituting in (5.8), we have that

$$\begin{aligned}\varphi_{m+1}(t) &\leq e_B(t, 0)A \int_0^t e_B(0, \sigma(s))\varphi_m(s)^{1+\beta} \Delta s. \\ &\leq A \int_0^t e_B(t, \sigma(s))\varphi_m(s)^{1+\beta} \Delta s,\end{aligned}\tag{5.11}$$

where $e_B(t, s)$ is the solution of

$$Y^\Delta = BY, \quad Y(0) = I(\text{identity matrix}).\tag{5.12}$$

$$\begin{aligned}\max_{\mathbb{T}} |\varphi_{m+1}(t)| &\leq \max_{\mathbb{T}} |A \int_0^t e_B(t, \sigma(s))\varphi_m(s)^{1+\beta} \Delta s| \\ &\leq \max_{\mathbb{T}} |\varphi_m(t)^{1+\beta}| A \int_0^{T_0} |e_B(t, \sigma(s)) \Delta s|,\end{aligned}\tag{5.13}$$

where $W = A \int_0^{T_0} |e_B(t, \sigma(s)) \Delta s|$. Thus,

$$\max_{\mathbb{T}} |\varphi_{m+1}(t)| \leq W \max_{\mathbb{T}} |\varphi_m(t)^{1+\beta}|.\tag{5.14}$$

This shows the convergence of $\{\tilde{\alpha}_m(t)\}$ to u is super-linear. $\square$

In the next theorem, when $g(t, u)$ is convex function, then one can give lower and upper bounds, at the same time they converge quadratically to unique solution of the IVP (5.1).

Theorem 5.2: Suppose that

- C1) $\tilde{\alpha}_0, \tilde{\beta}_0 \in C_{rd}[\mathbb{T}, \mathbb{R}]$ with $\tilde{\alpha}_0^\Delta \leq g(t, \tilde{\alpha}_0)$ and $\tilde{\beta}_0^\Delta \geq g(t, \tilde{\beta}_0)$ such that $\tilde{\alpha}_0(t) \leq \tilde{\beta}_0(t)$ on $\mathbb{T}$;
- C2) g is bounded on Ω_0 where $\Omega_0 = \{(t, u) : \tilde{\alpha}_0(t) \leq u \leq \tilde{\beta}_0(t), t \in \mathbb{T}\}$ such that $g(t, u)\mu^*(t) + u$ is non-decreasing in u for each $t \in \mathbb{T}$ and $g_u(t, u), g_{uu}(t, u)$ exists and rd-continuous on Ω_0 ;
- C3) $g(t, u)$ is convex function on Ω_0 .

Then there exists monotone sequences $\tilde{\alpha}_m(t) \rightarrow \rho(t)$ and $\tilde{\beta}_m(t) \rightarrow r(t)$ which converge uniformly to unique solution $\rho(t) = r(t) = u(t)$ of IVP (5.1) on $\mathbb{T}$ and the convergence is quadratic.

Proof. By assumption (C3),

$$\begin{aligned}
\int_v^u g_{uu}(t, u) &\geq 0 \\
g_u(t, u) - g_u(t, v) &\geq 0 \\
\int_v^u g_u(t, s) \Delta s &\geq \int_v^u g_u(t, v) \Delta s \\
g(t, u) - g(t, v) &\geq g_u(t, v)(u - v).
\end{aligned} \tag{5.15}$$

We get that

$$g(t, u) \geq g(t, v) + g_u(t, v)(u - v), \quad u \geq v, \tag{5.16}$$

Thus, $g(t, u)$ satisfies Lipschitz condition. Consider linear dynamic equation as follows

$$\tilde{\alpha}_{m+1}^\Delta = g(t, \tilde{\alpha}_m) + g_u(t, \tilde{\alpha}_m)(\tilde{\alpha}_{m+1} - \tilde{\alpha}_m), \quad \tilde{\alpha}_{m+1}(0) = u_0 \tag{5.17}$$

$$\tilde{\beta}_{m+1}^\Delta = g(t, \tilde{\beta}_m) + g_u(t, \tilde{\alpha}_m)(\tilde{\beta}_{m+1} - \tilde{\beta}_m), \quad \tilde{\beta}_{m+1}(0) = u_0 \tag{5.18}$$

and $\tilde{\alpha}_0(0) \leq u_0 \leq \tilde{\beta}_0(0)$. Now, we shall show that $\tilde{\alpha}_0 \leq \tilde{\alpha}_1 \leq \tilde{\beta}_1 \leq \tilde{\beta}_0$ is true for $h = 1$, on $\mathbb{T}$. It follows from (5.17) and (5.18) that

$$\tilde{\alpha}_1^\Delta = g(t, \tilde{\alpha}_0) + g_u(t, \tilde{\alpha}_0)(\tilde{\alpha}_1 - \tilde{\alpha}_0), \quad \tilde{\alpha}_1(0) = u_0, \tag{5.19}$$

$$\tilde{\beta}_1^\Delta = g(t, \tilde{\beta}_0) + g_u(t, \tilde{\alpha}_0)(\tilde{\beta}_1 - \tilde{\beta}_0), \quad \tilde{\beta}_1(0) = u_0. \tag{5.20}$$

Let $\varphi = \tilde{\alpha}_0 - \tilde{\alpha}_1$, so that $\varphi(0) \leq 0$ and

$$\begin{aligned}
\varphi^\Delta &= \tilde{\alpha}_0^\Delta - \tilde{\alpha}_1^\Delta \\
&\leq g(t, \tilde{\alpha}_0) - g(t, \tilde{\alpha}_0) - g_u(t, \tilde{\alpha}_0)(\tilde{\alpha}_1 - \tilde{\alpha}_0) \\
&= g_u(t, \tilde{\alpha}_0)(\tilde{\alpha}_0 - \tilde{\alpha}_1) \\
&= g_u(t, \tilde{\alpha}_0)\varphi.
\end{aligned} \tag{5.21}$$

By using Corollary 3.14, we obtain that $\varphi(t) \leq 0$, which shows this $\tilde{\alpha}_0 \leq \tilde{\alpha}_1$ on $\mathbb{T}$. Similarly, we can also prove $\tilde{\beta}_0 \geq \tilde{\beta}_1$ on $\mathbb{T}$. Since $\tilde{\beta}_0 \geq \tilde{\alpha}_0$ and by (5.16), then

$$g(t, \tilde{\beta}_0) \geq g(t, \tilde{\alpha}_0) + g_u(t, \tilde{\alpha}_0)(\tilde{\beta}_0 - \tilde{\alpha}_0).$$

Now, let $\varphi = \tilde{\alpha}_1 - \tilde{\beta}_1$, then $\varphi(0) = 0$ and

$$\begin{aligned}
\varphi^\Delta &= \tilde{\alpha}_1^\Delta - \tilde{\beta}_1^\Delta \\
&= g(t, \tilde{\alpha}_0) + g_u(t, \tilde{\alpha}_0)(\tilde{\alpha}_1 - \tilde{\alpha}_0) - g(t, \tilde{\beta}_0) - g_u(t, \tilde{\alpha}_0)(\tilde{\beta}_1 - \tilde{\beta}_0) \\
&\leq g_u(t, \tilde{\alpha}_0)(\tilde{\alpha}_0 - \tilde{\beta}_0) + g_u(t, \tilde{\alpha}_0)(\tilde{\alpha}_1 - \tilde{\alpha}_0) - g_u(t, \tilde{\alpha}_0)(\tilde{\beta}_1 - \tilde{\beta}_0) \\
&\leq g_u(t, \tilde{\alpha}_0)(\tilde{\alpha}_0 - \tilde{\beta}_0 + \tilde{\alpha}_1 - \tilde{\alpha}_0 - \tilde{\beta}_1 + \tilde{\beta}_0) \\
&\leq g_u(t, \tilde{\alpha}_0)(\tilde{\alpha}_1 - \tilde{\beta}_1) \\
&\leq g_u(t, \tilde{\alpha}_0)\varphi,
\end{aligned} \tag{5.22}$$

by Corollary 3.14, we find that $\varphi(t) \leq 0$. Thus, $\tilde{\alpha}_1 \leq \tilde{\beta}_1$ on $\mathbb{T}$.

Suppose that for some $h > 1$,

$$\tilde{\alpha}_0 \leq \tilde{\alpha}_1 \leq \cdots \leq \tilde{\alpha}_{h-1} \leq \tilde{\alpha}_h \leq \tilde{\beta}_h \leq \tilde{\beta}_{h-1} \leq \cdots \leq \tilde{\beta}_1 \leq \tilde{\beta}_0 \text{ on } \mathbb{T}. \tag{5.23}$$

Next, we wish to show that

$$\tilde{\alpha}_0 \leq \tilde{\alpha}_1 \leq \cdots \leq \tilde{\alpha}_h \leq \tilde{\alpha}_{h+1} \leq \tilde{\beta}_{h+1} \leq \tilde{\beta}_h \leq \cdots \leq \tilde{\beta}_1 \leq \tilde{\beta}_0 \text{ on } \mathbb{T}. \tag{5.24}$$

Put $u = \tilde{\alpha}_h$, $v = \tilde{\alpha}_{h-1}$ and by (5.16), we find that

$$g(t, \tilde{\alpha}_h) \geq g(t, \tilde{\alpha}_{h-1}) + g_u(t, \tilde{\alpha}_{h-1})(\tilde{\alpha}_h - \tilde{\alpha}_{h-1}). \tag{5.25}$$

Since $\tilde{\alpha}_{h-1} \leq \tilde{\alpha}_h$ and $g_u(t, u)$ is non-decreasing, then

$$g_u(t, \tilde{\alpha}_{h-1}) \leq g_u(t, \tilde{\alpha}_h). \tag{5.26}$$

Let $\varphi = \tilde{\alpha}_h - \tilde{\alpha}_{h+1}$, so that $\varphi(0) = 0$. Then

$$\begin{aligned}
\varphi^\Delta &= \tilde{\alpha}_h^\Delta - \tilde{\alpha}_{h+1}^\Delta \\
&= g(t, \tilde{\alpha}_{h-1}) + g_x(t, \tilde{\alpha}_{h-1})(\tilde{\alpha}_h - \tilde{\alpha}_{h-1}) - g(t, \tilde{\alpha}_h) - g_u(t, \tilde{\alpha}_h)(\tilde{\alpha}_{h+1} - \tilde{\alpha}_h) \\
&\leq g_u(t, \tilde{\alpha}_h)\varphi.
\end{aligned} \tag{5.27}$$

Then $\tilde{\alpha}_h \leq \tilde{\alpha}_{h+1}$ on $\mathbb{T}$. Similarly, we can also prove that $\tilde{\beta}_{h+1} \leq \tilde{\beta}_h$ on $\mathbb{T}$.

Put $u = \tilde{\beta}_{h-1}$, $v = \tilde{\beta}_h$, then

$$g(t, \tilde{\beta}_{h-1}) \geq g(t, \tilde{\beta}_h) + g_u(t, \tilde{\beta}_h)(\tilde{\beta}_{h-1} - \tilde{\beta}_h). \quad (5.28)$$

Let $\varphi = \tilde{\beta}_{h+1} - \tilde{\beta}_h$, so that $\varphi(0) = 0$. Then

$$\begin{aligned} \varphi^\Delta &= \tilde{\beta}_{h+1}^\Delta - \tilde{\beta}_h^\Delta \\ &= g(t, \tilde{\beta}_h) + g_u(t, \tilde{\alpha}_h)(\tilde{\beta}_{h+1} - \tilde{\beta}_h) - g(t, \tilde{\beta}_{h-1}) - g_u(t, \tilde{\alpha}_{h-1})(\tilde{\beta}_h - \tilde{\beta}_{h-1}) \\ &\leq g_u(t, \tilde{\beta}_h)(\tilde{\beta}_h - \tilde{\beta}_{h-1}) + g_u(t, \tilde{\alpha}_h)(\tilde{\beta}_{h+1} - \tilde{\beta}_h) + g_u(t, \tilde{\alpha}_h)(\tilde{\beta}_{h-1} - \tilde{\beta}_h) \\ &\leq g_u(t, \tilde{\alpha}_h)(\tilde{\beta}_h - \tilde{\beta}_{h-1} + \tilde{\beta}_{h+1} - \tilde{\beta}_h + \tilde{\beta}_{h-1} - \tilde{\beta}_h) \\ &\leq g_u(t, \tilde{\alpha}_h)\varphi. \end{aligned} \quad (5.29)$$

Thus, we have $\varphi(t) \leq 0$ which shows that $\tilde{\beta}_{h+1} \leq \tilde{\beta}_h$ on $\mathbb{T}$. Now, let $\varphi = \tilde{\alpha}_{h+1} - \tilde{\beta}_{h+1}$, then $\varphi(0) = 0$ and so that

$$\begin{aligned} \varphi^\Delta &= \tilde{\alpha}_{h+1}^\Delta - \tilde{\beta}_{h+1}^\Delta \\ &= g(t, \tilde{\alpha}_h) + g_u(t, \tilde{\alpha}_h)(\tilde{\alpha}_{h+1} - \tilde{\alpha}_h) - g(t, \tilde{\beta}_h) - g_u(t, \tilde{\alpha}_h)(\tilde{\beta}_{h+1} - \tilde{\beta}_h) \\ &\leq g_u(t, \tilde{\alpha}_h)(\tilde{\alpha}_h - \tilde{\beta}_h) + g_u(t, \tilde{\alpha}_h)(\tilde{\alpha}_{h+1} - \tilde{\alpha}_h) - g_u(t, \tilde{\alpha}_h)(\tilde{\beta}_{h+1} - \tilde{\beta}_h) \quad (5.30) \\ &\leq g_u(t, \tilde{\alpha}_h)(\tilde{\alpha}_h - \tilde{\beta}_h + \tilde{\alpha}_{h+1} - \tilde{\alpha}_h - \tilde{\beta}_{h+1} + \tilde{\beta}_h) \\ &\leq g_u(t, \tilde{\alpha}_h)\varphi, \end{aligned}$$

since g_u is non-decreasing and by (5.16) when we choose $u = \tilde{\beta}_h$ and $v = \tilde{\alpha}_h$. We get that $\varphi(t) \leq 0$. Hence, $\tilde{\alpha}_{h+1} \leq \tilde{\beta}_{h+1}$ on $\mathbb{T}$.

Consequently, we conclude by mathematical induction that

$$\tilde{\alpha}_0 \leq \tilde{\alpha}_1 \leq \cdots \leq \tilde{\alpha}_m \leq \tilde{\beta}_m \leq \cdots \leq \tilde{\beta}_1 \leq \tilde{\beta}_0 \text{ on } \mathbb{T}. \quad (5.31)$$

Here, we need to prove that $\{\tilde{\alpha}_m(t)\}$, $\{\tilde{\beta}_m(t)\}$ are uniformly convergent and monotonically on $\mathbb{T}$. Clearly, these sequences $\{\tilde{\alpha}_m(t)\}$ and $\{\tilde{\beta}_m(t)\}$ are uniformly bounded, because of (5.31) on $\mathbb{T}$. Since $g(t, u)$ is bounded, then we have that $\{\tilde{\alpha}_m^\Delta(t)\}$ and $\{\tilde{\beta}_m^\Delta(t)\}$ are also uniformly bounded on $\mathbb{T}$. This shows that $\{\tilde{\alpha}_m(t)\}$ and $\{\tilde{\beta}_m(t)\}$ are equicontinuous on $\mathbb{T}$. By employing Arzela- Ascoli Theorem 4.5, there exists

subsequences $\{\tilde{\alpha}_{m_k}(t)\}$ and $\{\tilde{\beta}_{m_k}(t)\}$ which converge uniformly on $\mathbb{T}$. Because of the monotonicity of $\{\tilde{\alpha}_m(t)\}$ and $\{\tilde{\beta}_m(t)\}$ implies that $\{\tilde{\alpha}_m(t)\}$ and $\{\tilde{\beta}_m(t)\}$ converge uniformly and monotonically to $\rho(t)$ and $r(t)$, respectively as $m \rightarrow \infty$.

Now, let us show that $\rho(t)$ and $r(t)$ are solutions of IVP (5.1) on $\mathbb{T}$. Taking delta integral to both sides of (5.17), (5.18), then we find

$$\begin{aligned} \int_0^t \tilde{\alpha}_{m+1}^\Delta(s) \Delta s &= \int_0^t [g(s, \tilde{\alpha}_m(s)) + g_u(s, \tilde{\alpha}_m(s))(\tilde{\alpha}_{m+1}(s) - \tilde{\alpha}_m(s))] \Delta s \\ \tilde{\alpha}_{m+1}(t) &= u_0 + \int_0^t [g(s, \tilde{\alpha}_m(s)) + g_u(s, \tilde{\alpha}_m(s))(\tilde{\alpha}_{m+1}(s) - \tilde{\alpha}_m(s))] \Delta s, \end{aligned} \quad (5.32)$$

$$\begin{aligned} \int_0^t \tilde{\beta}_{m+1}^\Delta(s) \Delta s &= \int_0^t [g(s, \tilde{\beta}_m(s)) + g_u(s, \tilde{\alpha}_m(s))(\tilde{\beta}_{m+1}(s) - \tilde{\beta}_m(s))] \Delta s \\ \tilde{\beta}_{m+1}(t) &= u_0 + \int_0^t [g(s, \tilde{\beta}_m(s)) + g_u(s, \tilde{\alpha}_m(s))(\tilde{\beta}_{m+1}(s) - \tilde{\beta}_m(s))] \Delta s. \end{aligned} \quad (5.33)$$

We have $\{\tilde{\alpha}_m(t)\}$, $\{\tilde{\beta}_m(t)\}$ converge uniformly to ρ, r respectively on $\mathbb{T}$ and passing limit to both side as $m \rightarrow \infty$, then we obtain that

$$\begin{aligned} \lim_{m \rightarrow \infty} \tilde{\alpha}_{m+1}(t) &= u_0 + \int_0^t [g(s, \lim_{m \rightarrow \infty} \tilde{\alpha}_m(s)) \\ &\quad + g_u(s, \lim_{m \rightarrow \infty} \tilde{\alpha}_m(s))(\lim_{m \rightarrow \infty} \tilde{\alpha}_{m+1}(s) - \lim_{m \rightarrow \infty} \tilde{\alpha}_m(s))] \Delta s \\ \rho(t) &= u_0 + \int_0^t g(s, \rho(s)) \Delta s \\ \rho^\Delta &= g(t, \rho), \quad \rho(0) = u_0. \end{aligned} \quad (5.34)$$

And also,

$$\begin{aligned} \lim_{m \rightarrow \infty} \tilde{\beta}_{m+1}(t) &= u_0 + \int_0^t [g(s, \lim_{m \rightarrow \infty} \tilde{\beta}_m(s)) \\ &\quad + g_u(s, \lim_{m \rightarrow \infty} \tilde{\alpha}_m(s))(\lim_{m \rightarrow \infty} \tilde{\beta}_{m+1}(s) - \lim_{m \rightarrow \infty} \tilde{\beta}_m(s))] \Delta s \\ r(t) &= u_0 + \int_0^t g(s, r(s)) \Delta s \\ r^\Delta &= g(t, r), \quad r(0) = u_0. \end{aligned} \quad (5.35)$$

Hence, $\rho(t)$ and $r(t)$ are solution of IVP(5.1). Because of $g(t, u)$ satisfies one side of Lipschitz condition and $g_u(t, u)$ is bounded on Ω_0 , this shows that there is a

unique solution of IVP such that $\rho(t) = r(t) = u(t)$.

Finally, we shall show the convergence of $\{\tilde{\alpha}_m(t)\}$ and $\{\tilde{\beta}_m(t)\}$ to the unique solution $u(t)$ is quadratic; that is, we need to show that

$$\max_{\mathbb{T}} |u(t) - \tilde{\alpha}_{m+1}(t)| \leq V \max_{\mathbb{T}} |u(t) - \tilde{\alpha}_m|^2. \quad (5.36)$$

Let $\varphi_{m+1} = u - \tilde{\alpha}_{m+1}$, $\varphi_{m+1}(0) = 0$. We note that

$$\varphi_{m+1}(t) \geq 0. \quad (5.37)$$

Then by using mean value theorem, we find

$$\begin{aligned} \varphi_{m+1}^\Delta &= u^\Delta - \tilde{\alpha}_{m+1}^\Delta \\ &= g(t, u) - g(t, \tilde{\alpha}_m) - g_u(t, \tilde{\alpha}_m)(\tilde{\alpha}_{m+1} - \tilde{\alpha}_m) \\ &\leq g_u(t, \zeta)(u - \tilde{\alpha}_m) + g_u(t, \tilde{\alpha}_m)(\tilde{\alpha}_{m+1} - \tilde{\alpha}_m) \\ &\leq g_u(t, \zeta)\varphi_m + g_u(t, \tilde{\alpha}_m)\varphi_{m+1} - g_u(t, \tilde{\alpha}_m)\varphi_m \\ &\leq g_{uu}(t, \eta)(\zeta - \tilde{\alpha}_m)\varphi_m + g_u(t, \tilde{\alpha}_m)\varphi_{m+1} \\ &\leq g_{uu}(t, \eta)\varphi_m^2 + g_u(t, \tilde{\alpha}_m)\varphi_{m+1} \end{aligned} \quad (5.38)$$

where $\tilde{\alpha}_m < \zeta < u$, $\tilde{\alpha}_m < \eta < \zeta$. Since $g_u(t, u)$, $g_{uu}(t, u)$ exist on Ω_0 , then $Z \geq |g_u(t, u)|$ and $K \geq |g_{uu}(t, u)|$, $(t, u) \in \Omega_0$. We get that

$$\varphi_{m+1}^\Delta(t) \leq K\varphi_m^2(t) + Z\varphi_{m+1}(t). \quad (5.39)$$

Utilizing the method of variation of parameters formula(VPF) for dynamic equation, let

$$\varphi_{m+1}(t) = e_Z(t, 0)p(t), \quad (5.40)$$

then $p(0) = 0$. By product rule of (5.40), gives

$$\varphi_{m+1}^\Delta(t) = Ze_Z(t, 0)p(t) + \Phi_Z(\sigma(t), 0)p^\Delta(t). \quad (5.41)$$

By substituting in (5.39), gives

$$\begin{aligned}
Ze_Z(t, 0)p(t) + e_Z(\sigma(t), 0)p^\Delta(t) &\leq K\varphi_m^2(t) + Ze_Z(t, 0)p(t) \\
e_Z(\sigma(t), 0)p^\Delta(t) &\leq K\varphi_m^2(t) \\
p^\Delta(t) &\leq e_Z^{-1}(\sigma(t), 0)K\varphi_m^2(t) \\
\int_0^t p^\Delta(s)\Delta s &\leq \int_0^t e_Z(0, \sigma(s))K\varphi_m^2(s)\Delta s \\
p(t) &\leq K \int_0^t e_Z(0, \sigma(s))\varphi_m^2(s)\Delta s.
\end{aligned} \tag{5.42}$$

So that

$$\begin{aligned}
\varphi_{m+1}(t) &\leq e_Z(t, 0)K \int_0^t e_Z(0, \sigma(s))\varphi_m^2(s)\Delta s \\
&\leq K \int_0^t e_Z(t, \sigma(s))\varphi_m^2(s)\Delta s,
\end{aligned} \tag{5.43}$$

where $e_Z(t, s)$ is solution of

$$Y^\Delta = ZY, \quad Y(0) = I. \tag{5.44}$$

Therefore

$$\begin{aligned}
\max_{\mathbb{T}} |\varphi_{m+1}(t)| &\leq K \max_{\mathbb{T}} |e_Z(t, \sigma(s)) \int_0^t \varphi_m^2(s)\Delta s| \\
&\leq K \max_{\mathbb{T}} |\varphi_m^2(t)| \int_0^t |e_Z(t, \sigma(s))\Delta s|.
\end{aligned} \tag{5.45}$$

Let $V = K \int_0^t |e_Z(t, \sigma(s))\Delta s|$, we conclude that

$$\max_{\mathbb{T}} |\varphi_{m+1}(t)| \leq V \max_{\mathbb{T}} |\varphi_m^2(t)|. \tag{5.46}$$

Hence, the convergence of $\{\tilde{\alpha}_m(t)\}$ to u is quadratic. In the same way, we can show the convergence of $\{\tilde{\beta}_m(t)\}$ to u is quadratic. Let $\psi_{m+1} = \tilde{\beta}_{m+1} - u$, $\psi(0) = 0$.

We see that

$$\psi_{m+1}(t) \geq 0 \tag{5.47}$$

Using mean value theorem, gets

$$\begin{aligned}
\psi_{m+1}^\Delta &= \tilde{\beta}_{m+1}^\Delta - u^\Delta \\
&= g(t, \tilde{\beta}_m(t)) + g_u(t, \tilde{\alpha}_m(t))(\tilde{\beta}_{m+1}(t) - \tilde{\beta}_m(t)) - g(t, u) \\
&\leq g_u(t, \sigma)\psi_m + g_u(t, \tilde{\alpha}_m(t))(\tilde{\beta}_{m+1}(t) - \tilde{\beta}_m(t)) \\
&\leq g_u(t, \tilde{\beta}_m)\psi_m + g_u(t, \tilde{\alpha}_m(t))(\tilde{\beta}_{m+1}(t) - \tilde{\beta}_m(t)) \\
&\leq \frac{3}{2}g_{uu}(t, \delta)\psi_m^2 + \frac{1}{2}g_{uu}(t, \delta)\varphi_m^2 + g_u(t, \tilde{\alpha}_m(t))\psi_{m+1}
\end{aligned} \tag{5.48}$$

where $u < \sigma < \tilde{\beta}_m$ and $\tilde{\alpha}_m < \delta < \tilde{\beta}_m$. We obtain that

$$\psi_{m+1}^\Delta(t) \leq \frac{3}{2}K\psi_m^2(t) + \frac{1}{2}K\varphi_m^2(t) + Z\psi_{m+1}(t) \tag{5.49}$$

Now, by using the method of variation of parameters formula 3.30 for dynamic equation, we get that

$$\psi_{m+1}(t) \leq \frac{3}{2}KV \max_{\mathbb{T}} |\psi_m^2(t)| + \frac{1}{2}KV \max_{\mathbb{T}} |\varphi_m^2(t)| \tag{5.50}$$

Therefore this proof completes this theorem. $\square$

In the next theorem, we generalized the method of quasilinearization such that $g(t, u)$ is no convex function, but imposing a less restrictive assumption; that is; $g(t, u) + \phi(t, u)$ are convex function for some convex function $\phi(t, u)$.

Theorem 5.3: Suppose that conditions (C1), (C2) of Theorem 5.2 are satisfied and C3*) $g(t, u) + \phi(t, u) \geq 0$ where $\phi \in C_{rd}[\Omega_0, \mathbb{R}]$ such that ϕ_u, ϕ_{uu} exists and rd-continuous and $\phi_{uu}(t, u) \geq 0$ on Ω_0 .

Then the conclusion of Theorem 5.2 is true.

Now, we need to use Theorems 3.12, 4.7 to improve Theorem (5.1) to obtain two monotone sequences which converge to unique solution and also the convergence is super-linear in the following result.

Theorem 5.4: Suppose that there exists a function $\tilde{\alpha}, \tilde{\beta}, R_i$ for each $i = 1, 2$ such that the following conditions are satisfied:

- A0) $R_i \in C_{rd}[\mathbb{T}, \mathbb{R}^n]$, R_i bounded on Q where , $Q = \{(t, u, v, w) : \tilde{\alpha}(t) \leq u, v, w \leq \tilde{\beta}(t), t \in \mathbb{T}\}$, $R_i(t, u, v, w)$ is quasi-monotone non-decreasing in u and $R_i(t, u, v, w)\mu^*(t) + u_i$ is non-decreasing in u_i , for each $1 \leq i \leq m$, and $\tilde{\alpha}(t) \leq v, w \leq \tilde{\beta}(t)$;
- A1) $\tilde{\alpha}, \tilde{\beta} \in C_{rd}^1[\mathbb{T}, \mathbb{R}^n]$ satisfying $\tilde{\alpha}^\Delta \leq R(t, \tilde{\alpha}, \tilde{\alpha}, \tilde{\beta})$, $\tilde{\beta}^\Delta \geq R(t, \tilde{\beta}, \tilde{\alpha}, \tilde{\beta})$ and $\tilde{\alpha}(t) \leq \tilde{\beta}(t)$ on $\mathbb{T}$;
- A2) $R_1(t, v, v, w) = g(t, v)$, $R_2(t, w, v, w) = g(t, w)$ and $R_i(t, u, u, u) = g(t, u)$;
- A3) if $v \leq u$, for each $\tilde{\alpha}(t) \leq w \leq \tilde{\beta}(t)$, then $R_1(t, u, v, w) \leq R_1(t, u, u, w)$, and also if $u \leq w$, for each $\tilde{\alpha}(t) \leq v \leq \tilde{\beta}(t)$, then $R_2(t, u, v, w) \geq R_2(t, u, v, u)$;
- A4) $R_i(t, u_1, v, w) - R_i(t, u_2, v, w) \leq L \sum_{j=1}^m (u_{1j} - u_{2j})$ whenever $\tilde{\alpha}(t) \leq u_2 \leq u_1 \leq \tilde{\beta}(t)$, for some $L \geq 0$;
- A5) $|R_i(t, u, u, u) - R_i(t, x, y, z)| \leq B|u - x| + A[|u - y|^{1+\lambda} + |u - z|^{1+\lambda}]$, for $t \in \mathbb{T}, u, x, y, z \in Q$ where $0 < \lambda \leq 1$, B and A are $k \times k$ non-negative constant matrices.

Then there exist two monotone sequences $\{\tilde{\alpha}_m(t)\}$ and $\{\tilde{\beta}_m(t)\}$ which converge uniformly, locally to the unique solution of (5.1) on $\mathbb{T}$ and this convergence is super-linear.

Proof. Let $\tilde{\alpha}(t) = \tilde{\alpha}_0(t)$ and $\tilde{\beta}(t) = \tilde{\beta}_0(t)$, consider dynamic IVPs

$$\tilde{\alpha}_{m+1}^\Delta = R_1(t, \tilde{\alpha}_{m+1}, \tilde{\alpha}_m, \tilde{\beta}_m), \quad \tilde{\alpha}_{m+1}(0) = u_0, \quad (5.51)$$

$$\tilde{\beta}_{m+1}^\Delta = R_2(t, \tilde{\beta}_{m+1}, \tilde{\alpha}_m, \tilde{\beta}_m), \quad \tilde{\beta}_{m+1}(0) = u_0. \quad (5.52)$$

Firstly, let us show that

$$\tilde{\alpha}_0 \leq \tilde{\alpha}_1 \leq \tilde{\beta}_1 \leq \tilde{\beta}_0 \text{ on } \mathbb{T}. \quad (5.53)$$

It follows from (5.51) and (5.52) that

$$\tilde{\alpha}_1^\Delta = R_1(t, \tilde{\alpha}_1, \tilde{\alpha}_0, \tilde{\beta}_0), \quad \tilde{\alpha}_1(0) = u_0, \quad (5.54)$$

$$\tilde{\beta}_1^\Delta = R_2(t, \tilde{\beta}_1, \tilde{\alpha}_0, \tilde{\beta}_0), \quad \tilde{\beta}_1(0) = u_0, \quad (5.55)$$

such that $\tilde{\alpha}_0(0) \leq u_0 \leq \tilde{\beta}_0(0)$. In view of (A0) and (A2), we obtain

$$\tilde{\alpha}_0^\Delta \leq g(t, \tilde{\alpha}_0) = R_1(t, \tilde{\alpha}_0, \tilde{\alpha}_0, \tilde{\beta}_0) \quad (5.56)$$

$$\tilde{\beta}_0^\Delta \geq g(t, \tilde{\beta}_0) = R_1(t, \tilde{\beta}_0, \tilde{\alpha}_0, \tilde{\beta}_0). \quad (5.57)$$

By using Theorems 3.12 and 4.7, we have the existence of a unique solution $\tilde{\alpha}_1(t)$ of (5.54) and (5.55) on $\mathbb{T}$ satisfying

$$\tilde{\alpha}_0(t) \leq \tilde{\alpha}_1(t) \leq \tilde{\beta}_0(t) \text{ on } \mathbb{T}. \quad (5.58)$$

As before, due to (A0) and (A2), we also obtain

$$\tilde{\alpha}_0^\Delta \leq g(t, \tilde{\alpha}_0) = R_2(t, \tilde{\alpha}_0, \tilde{\alpha}_0, \tilde{\beta}_0) \quad (5.59)$$

$$\tilde{\beta}_0^\Delta \geq g(t, \tilde{\beta}_0) = R_2(t, \tilde{\beta}_0, \tilde{\alpha}_0, \tilde{\beta}_0) \quad (5.60)$$

By similar argument, by using the same Theorems 3.12 and 4.7, we have the existence of a unique solution $\tilde{\beta}_1(t)$ of (5.55) on $\mathbb{T}$ satisfying

$$\tilde{\alpha}_0(t) \leq \tilde{\beta}_1(t) \leq \tilde{\beta}_0(t) \text{ on } \mathbb{T}. \quad (5.61)$$

Now, we need to show that $\tilde{\alpha}_1 \leq \tilde{\beta}_1$. Because of $\tilde{\alpha}_0 \leq \tilde{\alpha}_1$ and $\tilde{\beta}_1 \leq \tilde{\beta}_0$, by (A3), then

$$\begin{aligned} \tilde{\alpha}_1 &= R_1(t, \tilde{\alpha}_1, \tilde{\alpha}_0, \tilde{\beta}_0) \leq R_1(t, \tilde{\alpha}_1, \tilde{\alpha}_1, \tilde{\beta}_0), \\ \tilde{\beta}_1 &= R_2(t, \tilde{\beta}_1, \tilde{\alpha}_0, \tilde{\beta}_0) \geq R_2(t, \tilde{\beta}_1, \tilde{\alpha}_0, \tilde{\beta}_1) \end{aligned} \quad (5.62)$$

where $\tilde{\alpha}_1(0) = u_0 = \tilde{\beta}_1(0)$. Applying Theorem 3.12, we get that

$$\tilde{\alpha}_1(t) \leq \tilde{\beta}_1(t) \text{ on } \mathbb{T}.$$

Hence, the relation (5.53) is true for $h = 1$ on $\mathbb{T}$

$$\tilde{\alpha}_0 \leq \tilde{\alpha}_1 \leq \tilde{\beta}_1 \leq \tilde{\beta}_0. \quad (5.63)$$

We wish to show that

$$\tilde{\alpha}_h \leq \tilde{\alpha}_{h+1} \leq \tilde{\beta}_{h+1} \leq \tilde{\beta}_h \text{ on } \mathbb{T}. \quad (5.64)$$

Suppose for some $h > 1$ and $t \in \mathbb{T}$,

$$\tilde{\alpha}_h^\Delta \leq g(t, \tilde{\alpha}_h), \quad \tilde{\beta}_h^\Delta \geq g(t, \tilde{\beta}_h) \quad (5.65)$$

where $\tilde{\alpha}_h(t) \leq \tilde{\beta}_h(t)$, from (5.51) and (5.52), we get that

$$\tilde{\alpha}_{h+1}^\Delta = R_1(t, \tilde{\alpha}_{h+1}, \tilde{\alpha}_h, \tilde{\beta}_h), \quad \tilde{\alpha}_{h+1}(0) = u_0, \quad (5.66)$$

$$\tilde{\beta}_{h+1}^\Delta = R_2(t, \tilde{\beta}_{h+1}, \tilde{\alpha}_h, \tilde{\beta}_h), \quad \tilde{\beta}_{h+1}(0) = u_0. \quad (5.67)$$

It is clearly that

$$\tilde{\alpha}_h^\Delta \leq g(t, \tilde{\alpha}_h) = R_1(t, \tilde{\alpha}_h, \tilde{\alpha}_h, \tilde{\beta}_h) \quad (5.68)$$

$$\tilde{\beta}_h^\Delta \geq g(t, \tilde{\beta}_h) = R_1(t, \tilde{\beta}_h, \tilde{\alpha}_h, \tilde{\beta}_h). \quad (5.69)$$

Applying Theorem 3.12 and Theorem 4.7, then there exists a unique solution $\tilde{\alpha}_{h+1}(t)$ of (5.66) which satisfies

$$\tilde{\alpha}_h(t) \leq \tilde{\alpha}_{h+1}(t) \leq \tilde{\beta}_h(t) \text{ on } \mathbb{T}. \quad (5.70)$$

In the same way, we can also show that $\tilde{\alpha}_h(t) \leq \tilde{\beta}_{h+1} \leq \tilde{\beta}_h(t)$ on $\mathbb{T}$.

$$\tilde{\alpha}_h^\Delta \leq g(t, \tilde{\alpha}_h) = R_2(t, \tilde{\alpha}_h, \tilde{\alpha}_h, \tilde{\beta}_h) \quad (5.71)$$

$$\tilde{\beta}_h^\Delta \geq g(t, \tilde{\beta}_h) = R_2(t, \tilde{\beta}_h, \tilde{\alpha}_h, \tilde{\beta}_h). \quad (5.72)$$

Hence, as before, we give that

$$\tilde{\alpha}_h(t) \leq \tilde{\beta}_{h+1} \leq \tilde{\beta}_h(t). \quad (5.73)$$

Next, since $\tilde{\alpha}_h \leq \tilde{\alpha}_{h+1}$ and $\tilde{\beta}_h \geq \tilde{\beta}_{h+1}$ on $\mathbb{T}$ and by (A3), we obtain that

$$\tilde{\alpha}_{h+1}^\Delta = R_1(t, \tilde{\alpha}_{h+1}, \tilde{\alpha}_h, \tilde{\beta}_h) \leq R_1(t, \tilde{\alpha}_{h+1}, \tilde{\alpha}_{h+1}, \tilde{\beta}_h), \quad (5.74)$$

$$\tilde{\beta}_{h+1}^\Delta = R_2(t, \tilde{\beta}_{h+1}, \tilde{\alpha}_h, \tilde{\beta}_h) \geq R_2(t, \tilde{\beta}_{h+1}, \tilde{\alpha}_h, \tilde{\beta}_{h+1}). \quad (5.75)$$

Therefore

$$\tilde{\alpha}_{h+1} \leq \tilde{\beta}_{h+1} \text{ on } \mathbb{T}. \quad (5.76)$$

We conclude by induction this relation for all m , on $\mathbb{T}$

$$\tilde{\alpha}_0 \leq \tilde{\alpha}_1 \leq \cdots \leq \tilde{\alpha}_m \leq \tilde{\beta}_m \leq \cdots \leq \tilde{\beta}_1 \leq \tilde{\beta}_0 \text{ on } \mathbb{T}. \quad (5.77)$$

We need to show that $\{\tilde{\alpha}_m(t)\}, \{\tilde{\beta}_m(t)\}$ are uniformly bounded and equicontinuous on $\mathbb{T}$. Clearly $\{\tilde{\alpha}_m(t)\}$ and $\{\tilde{\beta}_m(t)\}$ are uniformly bounded on $\mathbb{T}$. Since R_i is bounded on Q for each $i = 1, 2$, then we get that $\{\tilde{\alpha}_m^\Delta(t)\}$ and $\{\tilde{\beta}_m^\Delta(t)\}$ are equicontinuous on $\mathbb{T}$. Applying Arzela-Ascoli's Theorem 4.5, there exist subsequences $\{\tilde{\alpha}_{m_k}(t)\}, \{\tilde{\beta}_{m_k}(t)\}$ which converge to $\rho(t), r(t)$, respectively on $\mathbb{T}$. Since the sequences are monotone, the entire sequences $\{\tilde{\alpha}_m(t)\}, \{\tilde{\beta}_m(t)\}$ converge uniformly and monotonically to $\rho(t), r(t)$, respectively as $m \rightarrow \infty$. Let us show that ρ, r are solution of IVP (5.1) on $\mathbb{T}$. Integrating of the equations (5.51), (5.52), gives that

$$\begin{aligned} \int_0^t \tilde{\alpha}_{m+1}^\Delta(s) \Delta s &= \int_0^t R_1(s, \tilde{\alpha}_{m+1}(s), \tilde{\alpha}_m(s), \tilde{\beta}_m(s)) \Delta s \\ \tilde{\alpha}_{m+1}(t) &= u_0 + \int_0^t R_1(s, \tilde{\alpha}_{m+1}(s), \tilde{\alpha}_m(s), \tilde{\beta}_m(s)) \Delta s. \end{aligned} \quad (5.78)$$

And also,

$$\begin{aligned} \int_0^t \tilde{\beta}_{m+1}^\Delta(s) \Delta s &= \int_0^t R_2(s, \tilde{\beta}_{m+1}(s), \tilde{\alpha}_m(s), \tilde{\beta}_m(s)) \Delta s \\ \tilde{\beta}_{m+1}(t) &= u_0 + \int_0^t R_2(s, \tilde{\beta}_{m+1}(s), \tilde{\alpha}_m(s), \tilde{\beta}_m(s)) \Delta s. \end{aligned} \quad (5.79)$$

Because of $\{\tilde{\alpha}_m(t)\}, \{\tilde{\beta}_m(t)\}$ converge uniformly and by (A3), so that we find

$$\begin{aligned}\lim_{m \rightarrow \infty} \tilde{\alpha}_{m+1}(t) &= u_0 + \int_0^t R_1(s, \lim_{m \rightarrow \infty} \tilde{\alpha}_{m+1}(s), \lim_{m \rightarrow \infty} \tilde{\alpha}_m(s), \lim_{m \rightarrow \infty} \tilde{\beta}_m(s)) \Delta s \\ \rho(t) &= u_0 + \int_0^t R_1(s, \rho(s), \rho(s), r(s)) \Delta s \\ &= u_0 + \int_0^t g(s, \rho(s)) \Delta s.\end{aligned}\tag{5.80}$$

$$\begin{aligned}\lim_{m \rightarrow \infty} \tilde{\beta}_{m+1}^\Delta(t) &= u_0 + \int_0^t R_2(s, \lim_{m \rightarrow \infty} \tilde{\beta}_{m+1}(s), \lim_{m \rightarrow \infty} \tilde{\alpha}_m(s), \lim_{m \rightarrow \infty} \tilde{\beta}_m(s)) \Delta s \\ r(t) &= u_0 + \int_0^t R_2(s, r(s), \rho(s), r(s)) \Delta s \\ &= u_0 + \int_0^t g(s, r(s)) \Delta s.\end{aligned}\tag{5.81}$$

Therefore

$$\rho^\Delta(t) = g(t, \rho(t)), \quad \rho(0) = u_0,\tag{5.82}$$

$$r^\Delta(t) = g(t, r(t)), \quad r(0) = u_0.\tag{5.83}$$

Since R_i for each $i = 1, 2$ satisfies one side of Lipschitz condition, then it is easy to see that u is a unique solution of $u^\Delta = R(t, u, u, u) = g(t, u)$.

In this step, we shall show that the convergence of $\{\tilde{\alpha}_m(t)\}$ to u is super-linear.

Letting

$$\varphi_{m+1}(t) = u(t) - \tilde{\alpha}_{m+1}(t),\tag{5.84}$$

so that

$$\varphi(t) \geq 0, \varphi(0) = 0$$

Employing (A5) and differentiating it with respect to t , we arrive at

$$\begin{aligned}\varphi_{m+1}^\Delta(t) &= u^\Delta(t) - \tilde{\alpha}_{m+1}^\Delta(t) \\ &= R_1(t, u, u, u) - R_1(t, \tilde{\alpha}_{m+1}, \tilde{\alpha}_m, \tilde{\beta}_m) \\ &\leq B|u(t) - \tilde{\alpha}_{m+1}(t)| + A[|u(t) - \tilde{\alpha}_m(t)|^{1+\lambda} + |u(t) - \tilde{\beta}_m(t)|^{1+\lambda}] \\ &\leq B\varphi_{m+1}(t) + A[|\varphi_m(t)|^{1+\lambda} + |\psi_m(t)|^{1+\lambda}].\end{aligned}\tag{5.85}$$

Now, we use the method of variation of parameters formula for dynamic equations.

Let

$$\varphi_{m+1}(t) = e_B(t, 0)z(t), \quad z(0) = 0. \quad (5.86)$$

Differentiating it with respect to t and using product rule, we get

$$\varphi_{m+1}^\Delta(t) = B e_B(t, 0)z(t) + e_B(\sigma(t), 0)z^\Delta(t). \quad (5.87)$$

Substituting in equation (5.85), we obtain

$$B e_B^\Delta(t, 0)z(t) + e_B(\sigma(t), 0)z^\Delta(t) \leq B e_B(t, 0)z(t) + A[|\varphi_m(t)|^{1+\lambda} + |\psi_m(t)|^{1+\lambda}]. \quad (5.88)$$

Then

$$\begin{aligned} e_B(\sigma(t), 0)z^\Delta(t) &\leq A[|\varphi_m(t)|^{1+\lambda} + |\psi_m(t)|^{1+\lambda}] \\ z^\Delta(t) &\leq e_B(0, \sigma(t))A[|\varphi_m(t)|^{1+\lambda} + |\psi_m(t)|^{1+\lambda}] \\ \int_0^t z^\Delta(s)\Delta s &\leq A \int_0^t e_B(0, \sigma(s))[|\varphi_m(s)|^{1+\lambda} + |\psi_m(s)|^{1+\lambda}]\Delta s \\ z(t) &\leq A \int_0^t e_B(0, \sigma(s))[|\varphi_m(s)|^{1+\lambda} + |\psi_m(s)|^{1+\lambda}]\Delta s. \end{aligned} \quad (5.89)$$

Substituting in equation (5.86), we get

$$\begin{aligned} \varphi_{m+1}(t) &\leq e_B(t, 0)A \int_0^t e_B(0, \sigma(s))[|\varphi_m(s)|^{1+\lambda} + |\psi_m(s)|^{1+\lambda}]\Delta s \\ &\leq A \int_0^t e_B(t, \sigma(s))[|\varphi_m(s)|^{1+\lambda} + |\psi_m(s)|^{1+\lambda}]\Delta s, \end{aligned} \quad (5.90)$$

where $e_B(t, s)$ is exponential function such that it is solution of

$$Y^\Delta = BY, \quad Y(0, 0) = I. \quad (5.91)$$

Therefore

$$\begin{aligned} \max_{\mathbb{T}} |\varphi_{m+1}(t)| &\leq A \max_{\mathbb{T}} (|\varphi_m(t)|^{1+\lambda} + |\psi_m(t)|^{1+\lambda}) \int_0^t |e_B(t, \sigma(s))\Delta s| \\ &\leq V \max_{\mathbb{T}} (|\varphi_m(t)|^{1+\lambda} + |\psi_m(t)|^{1+\lambda}) \end{aligned} \quad (5.92)$$

where

$$V = A \int_0^t |e_B(t, \sigma(s)) \Delta s|.$$

Similarly, we will also prove that the convergence of $\{\tilde{\beta}_m(t)\}$ to u is super-linear.

$$\psi_{m+1}(t) = \tilde{\beta}_{m+1}(t) - u(t), \quad (5.93)$$

we see that

$$\psi(t) \geq 0, \psi(0) = 0.$$

Differentiating above equation (5.93) with respect to t and by (A5), gives as

$$\begin{aligned} \psi_{m+1}^\Delta(t) &= \tilde{\beta}_{m+1}^\Delta(t) - u^\Delta(t) \\ &= R_2(t, \tilde{\beta}_{m+1}, \tilde{\beta}_m, \tilde{\alpha}_m) - R_2(t, u, u, u) \\ &\leq B|u - \tilde{\beta}_{m+1}| + A[|u - \tilde{\beta}_m|^{1+\lambda} + |u - \tilde{\alpha}_m|^{1+\lambda}] \\ &\leq B|\psi_{m+1}(t)| + A[|\psi_m(t)|^{1+\lambda} + |\varphi_m(t)|^{1+\lambda}]. \end{aligned}$$

Employing the method of variation of parameters formula, setting

$$\psi_{m+1}(t) = e_B(t, 0)w(t), \quad w(0) = 0$$

differentiating it with respect to t , gets

$$\psi_{n+1}^\Delta(t) = B e_B(t, 0)w(t) + e_B(\sigma(t), 0)w^\Delta(t). \quad (5.94)$$

So that

$$\begin{aligned} B e_B(t, 0)w(t) + e_B(\sigma(t), 0)w^\Delta(t) &\leq B e_B(t, 0)w(t) + A[|\psi_m(t)|^{1+\lambda} + |\varphi_m(t)|^{1+\lambda}] \\ e_B(\sigma(t), 0)w^\Delta(t) &\leq A[|\psi_m(t)|^{1+\lambda} + |\varphi_m(t)|^{1+\lambda}] \\ w^\Delta(t) &\leq e_B^{-1}(\sigma(t), 0)A[|\psi_m(t)|^{1+\lambda} + |\varphi_m(t)|^{1+\lambda}], \end{aligned}$$

integrating it, we find

$$w(t) \leq A \int_0^t e_B(0, \sigma(s)) [|\psi_m(s)|^{1+\lambda} + |\varphi_m(s)|^{1+\lambda}] \Delta s. \quad (5.95)$$

We obtain that

$$\psi_{m+1}(t) \leq A \int_0^t e_B(t, \sigma(s)) [|\psi_m(s)|^{1+\lambda} + |\varphi_m(s)|^{1+\lambda}] \Delta s \quad (5.96)$$

Therefore

$$\max_{\mathbb{T}} |\psi_{m+1}(t)| \leq P \max_{\mathbb{T}} [|\psi_m(s)|^{1+\lambda} + |\varphi_m(s)|^{1+\lambda}] \quad (5.97)$$

where

$$P = A \int_0^t |e_B(t, \sigma(s)) \Delta s|.$$

Hence, both of $\{\tilde{\alpha}_m(t)\}$ and $\{\tilde{\beta}_m(t)\}$ super-linear converge to u . $\square$

As an application of Theorem 5.4, there are conditions which guarantee the existence of R_i , for each $i = 1, 2$ verifies the hypotheses of Theorem 5.4. Suppose that

- i) $\tilde{\alpha}, \tilde{\beta} \in C_{rd}[\mathbb{T}, \mathbb{R}^n]$ with $\tilde{\alpha}^\Delta \leq g(t, \tilde{\alpha})$, $\tilde{\beta}^\Delta \geq g(t, \tilde{\beta})$ and also $\tilde{\alpha} \leq \tilde{\beta}$ on $\mathbb{T}$.
- ii) $g \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$ and on Ω , $g(t, u)$ is quasi-monotone non-decreasing in u .

Moreover, $g(t, u)$ is uniformly convex in x and $g_u(t, u)$ exists and

$$|g_u(t, u) - g_u(t, v)| \leq K|u - v|^\eta \text{ for } 0 < \eta \leq 1 \quad (5.98)$$

where K is an matrix $l \times l$ and $|u| = (|u_1|, \dots, |u_m|)$, such that we consider

$$R_1(t, u, v, w) = g(t, u) + g_u(t, v)(u - v) \quad (5.99)$$

$$R_2(t, u, v, w) = g(t, w) + g_u(t, v)(u - w) \quad (5.100)$$

for (t, u, v, w) on Q .

Let us show the following lemma which help us to satisfy that the conditions (A0) to (A4) of Theorem 5.4 are satisfied by R_i , for each $i = 1, 2$.

Lemma 5.5: Suppose $g \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$, $g_u(t, u)$ exists and continuous. If $g(t, u)$ is uniformly convex in u , then

$$g(t, u + h) - g(t, u) \geq g_u(t, u)h, \quad (5.101)$$

where the inequalities are component wise.

Proof. Since $g_u(t, u)$ exists and continuous, we have

$$g(t, u + h) = g(t, u) + g_u(t, u)h + |h|\delta(t, u, h), \quad h, u \in \mathbb{R}^n \quad (5.102)$$

where $|\delta(t, x, y)| \rightarrow 0$ as $|h| \rightarrow 0$. Since $g(t, u)$ is uniformly convex, so that $g(t, u)$ is convex,

$$g(t, \tau u + (1 - \tau)z) \leq \tau g(t, u) + (1 - \tau)g(t, z), \quad 0 < \tau < 1. \quad (5.103)$$

Let

$$z = u + \frac{h}{1 - \tau}, \quad 0 < \tau < 1$$

so that

$$(1 - \tau) + \tau u = u + h.$$

Substituting by $u + h$ in the equation(5.102), gives as

$$\begin{aligned} g(t, \tau u + (1 - \tau)z) &= g(t, u) + g_u(t, u)h + |h|\delta(t, u, h) \\ &= \tau g(t, u) + (1 - \tau)g(t, u) + g_u(t, u)h + |h|\delta(t, u, h). \end{aligned} \quad (5.104)$$

Substituting in (5.103), we get

$$\begin{aligned} \tau g(t, u) + (1 - \tau)g(t, u) + g_u(t, u)h + |h|\delta(t, u, h) &\leq \tau g(t, u) + (1 - \tau)g(t, z) \\ |h|\delta(t, u, h) &\leq (1 - \tau)[g(t, z) - g(t, u)] - g_u(t, u)h \frac{1 - \tau}{1 - \tau} \\ &\leq (1 - \tau)[g(t, z) - g(t, u) - g_u(t, u)\frac{h}{1 - \tau}]. \end{aligned} \quad (5.105)$$

It follows from (5.102) that

$$|h|\delta(t, u, h) \leq (1 - \tau)[|\frac{h}{1 - \tau}|\delta(t, u, \frac{h}{1 - \tau})] = \delta(t, u, \frac{h}{1 - \tau})|h|. \quad (5.106)$$

Consequently, we have

$$\delta(t, u, h) \leq \delta(t, u, \frac{h}{1-\tau}), \text{ for } 0 < \tau < 1. \quad (5.107)$$

Now, let $h = \frac{y}{n}$, $y \in \mathbb{R}^n$ and $\tau = 1 - \frac{1}{n}$, we obtain

$$\delta(t, u, \frac{y}{n}) \leq \delta(t, u, y). \quad (5.108)$$

Taking limit as $n \rightarrow \infty$, we find

$$\delta(t, u, v) \geq 0. \quad (5.109)$$

Hence, the inequality (5.101) is true. This proof is complete. $\square$

Table 5.1: Comparison between the method of monotone iterative technique and the method of quasilinearization.

	Monotone Iterative Technique	Quasilinearization
The method of lower and upper solutions	exist	exist
(t, u) defined on	$\mathbb{T}^k \times \mathbb{R}^n$	$\Omega = \{(t, u) : \tilde{\alpha}(t) \leq u(t) \leq \tilde{\beta}(t)\}$
Convergence of monotone sequences	uniformly convergent	quadratic convergent
Uniqueness of solutions	may be there are a unique solutions or Not	there are a unique solutions
Monotone property of a function	nondecreasing	nondecreasing
The existence of extremal solutions	exist such that $\rho(t) \leq u(t) \leq r(t)$	exist such that $\rho(t) = u(t) = r(t)$
Construction of Monotone sequences corresponding from	linear dynamic equations	linear or nonlinear dynamic equations
Use	showing existence of solutions of nonlinear dynamic equations	obtaining approximate solutions of nonlinear dynamic equations

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Nour Alsharif was born on the 27th of September 1994 in Gaza southwest Palestine. She began her undergraduate degree in Mathematics in October 2012 at the Al-Aqsa University. After graduation, she moved to Turkey to pursue her master studies through a Turkish Government scholarship. In 2017, she obtained a diploma in Turkish language from Gebze Technical University, Turkey. In the same year, she started her M. Sc. in mathematics at the Graduate School of Natural and Applied Science in Gebze Technical University, Turkey.