

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

**CHARACTERIZATION OF SEMI-TRIPOTENT AND
NILPOTENT ELEMENTS OF A RING**

TÜLAY YILDIRIM TURAN
A THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
DEPARTMENT OF MATHEMATICS

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2019

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GEBZE TEKNİK ÜNİVERSİTESİ
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SUMMARY

An associative ring with unity is called semi-tripotent if every element is a sum of a tripotent and element from the Jacobson radical. In this thesis, many of the useful basic properties of semi-tripotent rings are shown and several examples of some important classes of rings (such as Boolean rings, strongly nil-clean rings, strongly 2-nil-clean rings and semiboolean rings) are satisfied the semi-tripotent property. Necessary and sufficient conditions for a Morita context, respectively for a group ring of an abelian group or a locally finite nilpotent group to be semi-tripotent are obtained. Furthermore for $n \geq 2$, $P_n(R)$ is defined for $a^n - a$ is nilpotent for all $a \in R$ and $Q_n(R)$ is defined for $R/J(R)$ has identity $x^n = x$ and $J(R)$ is nil. Then rings R for which $P_n(R)$ holds are completely characterized for integers n up to 12, for $n = 2^k$ with k odd, and for $n = 2^k p$ where $k > 0$ and $p = 3, 5, 7$ or 9 . Finally for an arbitrary integer $n \geq 2$, it is deduced that $P_n(R)$ holds if and only if $Q_n(R)$ holds for all rings R if and only if n is even with $n \equiv 1 \pmod{3}$, or n is odd with $n \not\equiv 1 \pmod{3}$ and $n \not\equiv 1 \pmod{8}$.

Keywords: Boolean Ring, Semi-Boolean Ring, Idempotents Lifting Modulo Jacobson Radical, Ring with Identity $x^n = x$ Modulo Jacobson Radical.

ÖZET

Birimli ve deęişmeli bir halkanın, her elemanı bir tripotent eleman ve Jacobson radikalinden bir elemanın toplamıysa, bu halka yarı-tripotent halka olarak adlandırılır. Bu tezde, yarı-tripotent halkaların temel özelliklerinin birçoęu gösterilmiştir ve bazı önemli sınıf halkalarının (Boolean halkaları, güçlü nil-temiz halkaları, kuvvetli 2-nil-temiz halkaları ve yarı boolean halkaları gibi) birkaç örneęi yarı-tripotent özellięini sağlamıştır. Bir Morita bağlamının, bir abelian grubunun grup halkasının veya lokal sonlu bir nilpotent grubunun yarı tripotent olması için gerekli ve yeterli koşullar elde edilmiştir. Ayrıca $n \geq 2$ ve tüm $a \in R$ ler için $a^n - a$ nilpotent olacak şekilde $P_n(R)$ tanımlanmıştır ve $x^n - x$ birimli $R/J(R)$ ve $J(R)$ nil olacak şekilde $Q_n(R)$ tanımlanmıştır. Daha sonra, n tam sayısından 12 ye kadar olan tam sayılar, k tek sayısı olduęunda $n = 2^k$ ve $p = 3, 5, 7$ veya 9 , $k > 0$ olduęunda $n = 2^k p$ durumlarında $P_n(R)$ nin sağlandığı tamamiyle karakterize edilmiştir. Son olarak, keyfi $n \geq 2$ tam sayısı ve tüm R halkarı için $P_n(R)$ sağlanır ancak ve ancak $Q_n(R)$ sağlanır olması için gerek ve yeter koşulun n çift iken $n \equiv 1 \pmod{3}$, yada n tek iken $n \not\equiv 1 \pmod{3}$ ve $n \not\equiv 1 \pmod{8}$ olması gerektięi sonucuna varılmıştır.

Anahtar Kelimeler: Boole halkası, Yarı-Boole Halkası, İdempotentlerin Jacobson Radikaline Modüler Yükseltilmesi, $x^n = x$ Modüler Jacobson Radikal Birimli Halka.

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LIST of ABBREVIATIONS and ACRONYMS

p	: The Prime Numbers
w	: The Augmentation Map
R	: The Ring
S	: The Subring of Ring
I	: The Ideal of Ring
$\mathbb{Q}$	: The Rational Numbers
$\mathbb{C}$	: The Complex Numbers
D	: The Divison Ring
G	: The Group
RG	: The Group Ring
$J(R)$	: The Jacobson Radical of R
$U(R)$	: The Group of Units of R
$N(R)$	: The Set of Nilpotent Elements of R
$Id(R)$	: The Set of Idempotent Elements of R
M_n	: The Ring of $n \times n$ Matrices
Z_n	: The Ring of Integer Modulo n
$\mathbb{F}_q$	: The Field of q -elements
C_n	: The Cyclic Group of Order n
$R[x]$	: The Ring of Polynomials Over R
$R[[x]]$	: The Ring of Power Series Over R
$\triangle$	: The Augmentation Ideal
$o(G)$	: The Set of Order G

1. INTRODUCTION

An element a of a ring R is defined to be clean if there is an idempotent $e \in R$ such that $a - e$ is invertible in R . If each element of R is clean, then R is called clean ring [Nicholson, 1977]. Nicholson improved that these rings as a natural rings in which exchange property holds. In [Nicholson, 1999], he also defined strongly clean rings such that a ring R is called strongly clean if each element of R can be written as a sum of an idempotent and a unit element of R that commute. These rings are considered as a natural generalization of strongly π -regular rings. In [Boroah, 2005], it is shown that every strongly π -regular element of R is strongly clean. For more details see [Camillo, 2006], [Yang, 2009], [Chen, 2013], [Diesl, 2013].

Following [Diesl, 2013], Diesl studied on a general theory of (strongly) clean rings by using the idea of direct sum decompositions. This leads him to define nil clean rings and strongly nil clean rings. A ring R is called (strongly) nil clean if each element of R is a sum of an idempotent and a nilpotent (that commute). He analyzed basic properties of (strongly)nil-clean rings and behavior of these properties under some ring extentions. In [Kosan et al., 2016], Kosan et al. continue the study on the structure of (strongly) nil clean rings . They observed that an element $a \in R$ is strongly nil clean if and only if a is strongly clean and $a - a^2$ is a nilpotent and this ring R is strongly nil clean if and only if $R/J(R)$ is boolean and $J(R)$ is nil, where $J(R)$ is Jacobson radical of R . Also they answer the question of when a matrix ring is nil clean by proving that the matrix ring over a 2-primal ring R is nil clean if and only if $R/J(R)$ is boolean and $J(R)$ is nil, i.e. R is strongly nil clean. The subject of (strongly) nil-clean rings is interesting for many mathematicians [Chen, 2011], [Diesl, 2013], [Hirano et al., 1988], [Kosan et al., 2016].

Idempotent elements of a ring has an important role in understanding the structure of that ring. If a ring has a wealthy supply idempotent elements, then one could look forward to something good about the structure of the ring. A high case is the notion of Boolean rings. A ring is called Boolean if each of its elements is an idempotent. For an idempotent element $e \in R$, both e and $-e$ are tripotent, i. e., the elements equal to their cubes. Boolean rings are significant in algebra and applications in other areas. In the ring theory, several researchers have the origin in

Boolean rings. For instance in 1988, as a generalization of Boolean rings, Hirano and Tominaga investigated the ring R for which every element is a sum of two idempotents, and they showed the following equivalent conditions: every element of a commutative ring R is the sum of two idempotents if and only if every element of the ring R is the sum of two commuting idempotents if and only if R has the identity $a^3 = a$ [Hirano and Tominaga, 1988].

Up to 2016, there is no structure theorem for the rings for which every element of a ring is a sum of two idempotents. Thus Ying et al. naturally motivated the following results: (i) each element of ring R is a sum of a tripotent and an idempotent if and only if R has the identity $a^6 = a^4$ if and only if $R = A \times B$, where $A/J(A)$ is Boolean with $U(A)$ a group of exponent 2, and B is a subdirect product of $\mathbb{Z}_3$'s or zero. (ii) each element of ring R is a sum or difference of two commuting idempotent if and only if $R = A \times B$, where $A/J(A)$ is Boolean with $J(A) = \{0, 2\}$ or $J(A) = 0$, and B is a subdirect product of $\mathbb{Z}_3$'s or zero. (iii) each element of ring R is a sum of two commuting tripotents if and only if $R = A \times B \times C$, where $A/J(A)$ is Boolean with $U(A)$ a group of exponent 2, B is subdirect product of $\mathbb{Z}_3$'s or zero, and C is a subdirect product of $\mathbb{Z}_5$'s or zero [Ying et al., 2016].

In the recent paper [Chen and Sheibani, 2017], Chen and Sheibani defined the interesting notion of strongly 2-nil clean rings: A ring is called a strongly 2-nil clean if every element is a sum of two idempotents and a nilpotent that commute. As a new type of rings, this ring generalize the notion of (strongly)nil-clean rings. They also proved the followings: (i) for all $a \in R$, R is strongly 2-nil clean if and only if R is a sum of a nilpotent and tripotent that commute. (ii) R is strongly 2-nil clean if and only if $R/J(R)$ has identity $a^3 = a$ and $J(R)$ is nil if and only if $R \cong A, B, A \times B$ where $A/J(A)$ is a Boolean ring and $J(A)$ is nil, $B/J(B)$ is a Yaqub ring and $J(B)$ is nil, where a ring R is called Yaqub ring if it is the subdirect product of $\mathbb{Z}_3$'s. (iii) each element of a ring R is a sum of a nilpotent and tripotent that commute if and only if each element of a ring R is a sum of a nilpotent and two idempotents that commute.

Also they proved that every element of a ring R is a sum of a nilpotent and tripotent (that commute with each other) if and only if every element of a ring R is a sum of a nilpotent and two idempotents (that commute with each other) and identified the rings with the equivalent statements.

In the recent paper [Zhou, 2018], Zhou studied the rings in which every element is a sum of a nilpotent, an idempotent and a tripotent that commute with each other, and also rings for which every element is a sum of a nilpotent and two tripotents that commute with each other.

In this thesis we closely study the semi-tripotent rings for which each element of a ring R is a sum of a tripotent and an element from $J(R)$, or equivalently $R/J(R)$ has identity $a^3 = a$ and 3-potents lift modulo $J(R)$ in R . It is observed that semi-tripotent rings are naturally connected to Boolean rings, semiboolen rings, strongly nil clean rings and strongly 2-nil clean rings. We continue the study of semi-tripotent rings with the focus on the structure and construction of these rings. We also give the necessary and sufficient conditions for the question of when a Morita context, group of an abelian group or a locally finite nilpotent group are semi-tripotent.

Finally for an integer $n \geq 2$ and a ring R , we defined the following two notions:

- $P_n(R) : a^n - a \text{ is nilpotent for all } a \in R$
- $Q_n(R) : R/J(R) \text{ has identity } a^n = a \text{ and } J(R) \text{ is nil}$

For a ring R , it is shown that $P_n(R)$ is entirely described for integers n up to 12, for $n = 2^k p$, $p = 3, 5, 7$, or 9 , $k > 0$ and for $n = 2^k$ where k is odd. Moreover, for $n > 1$ it is proved that for all rings R , $P_n(R) \Leftrightarrow Q_n(R)$ if and only if n is odd with $n \not\equiv 1 \pmod{3}$ and $n \not\equiv 1 \pmod{8}$ or n is even with $n \equiv 1 \pmod{3}$.

This thesis is organized as follows:

In chapter 2, we give some background material from the ring theory and the theory of group rings. We also discuss some properties of (strongly)nil-clean rings.

In chapter 3, as natural generalizations of Boolean rings, rings R for which $R/J(R)$ is Boolean and $J(R)$ is nil and, respectively, rings R for which $R/J(R)$ is Boolean and idempotents lift modulo $J(R)$ have been well studied by taken into consideration to the literature. The former is the characterization of (strongly)nil-clean rings, and the latter is the notion of semiboolen rings, which are exactly those rings R whose elements are the sum of an idempotent and an element from $J(R)$. The main purpose of this chapter is to view semiboolen rings as a natural generalization of (strongly)nil-clean rings, with " $J(R)$ is nil" being replaced by "idempotents lift modulo $J(R)$ ".

Furthermore, we motivated the following questions: let p be a prime, (i) what can be said about rings R for which $R/J(R)$ has identity $a^p = a$ and $J(R)$ is nil? and (ii) what can be said about rings R for which $R/J(R)$ has identity $a^p = a$ and idempotents lift modulo $J(R)$? Answers to these questions are known for $p = 2$ as mentioned above. Moreover, question (i) was answered for $p = 3$ in [Chen and Sheibani, 2017] as mentioned above, and further for $p = 5$ in [Zhou, 2018]. In this thesis, we provide an answer to question (ii) for $p = 3$. Let $n > 1$ be an integer. An element $a \in R$ is called an n -potent if $a^n = a$, and a 3-potent is usually called a tripotent. We call a ring R semi- n -potent if $R/J(R)$ has identity $a^n = a$ and n -potents lift modulo $J(R)$ or equivalently every element of R is a sum of an n -potent and an element from $J(R)$, and a semi-3-potent ring is called a semi-tripotent ring. In the following chapter, some basic properties of semi- n -potent rings are proved. For instance, the corner rings of a semi- n -potent ring are again semi- n -potent, and a sufficient and necessary condition for a Morita context to be semi- n -potent is obtained. Also, various characterizations of semi-tripotent rings are obtained. Some new equivalent conditions of a semiboolean ring are also presented. At the end of the chapter, we determine when the group ring of an abelian group or a locally finite nilpotent group is semi-tripotent.

In chapter 4, we focus on the rings R for which $a^n - a$ is nilpotent for all $a \in R$, where $n > 1$ is a fixed integer. The structures of these rings are known if $n = 2, 3$ or 5 . By [Kosan et al., 2016, Theorem 2.7] and [Sahinkaya et.al, 2017, Lemma 2.4], for a ring R , $a^2 - a$ is nilpotent for all $a \in R$ if and only if $R/J(R)$ is Boolean and $J(R)$ is nil, if and only if every element of R is a sum of an idempotent and a nilpotent that commute. Such rings are called strongly nil-cleans by Diesl [Diesl, 2013]. In [Chen and Sheibani, 2017], it is showed that $a^3 - a$ is nilpotent for all $a \in R$ if and only if $R/J(R)$ has identity $a^3 = a$ and $J(R)$ is nil, if and only if every element of R is a sum of a tripotent and a nilpotent that commute. In [Zhou, 2018], it is proved that $a^5 - a$ is nilpotent for all $a \in R$ if and only if $R/J(R)$ has identity $a^5 = a$ and $J(R)$ is nil, if and only if every element of R is a sum of a nilpotent and two tripotents that commute with one another.

For an integer $n > 1$ and a ring R , we fix two notations for convenience: $P_n(R)$ denotes that $a^n - a$ is nilpotent for all $a \in R$ and $Q_n(R)$ denotes that $R/J(R)$ has identity $a^n = a$ and $J(R)$ is nil. Our interest is in characterizing rings

R such that $P_n(R)$ holds where $n > 1$ is a fixed integer. In this chapter, rings R with $P_n(R)$ are completely characterized for integers n up to 12, for $n = 2^k$ with k odd, and for $n = 2^k p$ where $k > 0$ and $p = 3, 5, 7$ or 9 . For an arbitrary integer $n > 1$, $Q_n(R)$ implies $P_n(R)$ trivially. It is proved that if n is even with $n \equiv 1 \pmod{3}$, or n is odd with $n \not\equiv 1 \pmod{3}$ and $n \not\equiv 1 \pmod{8}$, then $P_n(R) \Leftrightarrow Q_n(R)$. This sufficient condition is also necessary. Indeed, we prove that, for an integer $n > 1$, $P_n(R)$ holds if and only if $Q_n(R)$ holds for all rings R if and only if n is even with $n \equiv 1 \pmod{3}$, or n is odd with $n \not\equiv 1 \pmod{3}$ and $n \not\equiv 1 \pmod{8}$.

2. PRELIMINARIES

In this chapter, we give the necessary background on the ring theory. For the convenience of the reader, we also give the fundamental definitions which will be used frequently. We refer reader to [Hirano and Tominaga, 1988], [Diesl, 2013], [Kosan et al., 2016] and [Chen and Sheibani, 2017] for more details about the clean and (strongly) nil-clean rings as well as to [Connell, 1963], [Nicholson and Zhou, 2005] and [Sahinkaya et.al, 2017] for the theory of group rings. Throughout, R is an associative ring with identity. The Jacobson radical, the group of units and the set of nilpotent and idempotent elements of R are denoted by $J(R)$, $U(R)$, $N(R)$ and $Id(R)$, respectively. We write $\mathbb{Z}_n$ for the ring of integers modulo n , $\mathbb{F}_q$ for the field of q elements, and $M_n(R)$ for the ring of $n \times n$ matrices over R whose identity is denoted by I_n and $R[x]$ (respectively, $R[[x]]$) for the ring of polynomials (respectively, power series) over R . When we say that S is a subring of a ring R , we do not insist that S and R share the same identity.

2.1. Preliminaries on (Strongly) Nil-Clean Rings

Definition 2.1: Let R be a ring,

- an element of R is called strongly π -regular if $a^n \in Ra^{n+1} \cap a^{n+1}R$ for some $n \geq 1$,
- an element of R is called strongly regular if $a \in Ra^2 \cap a^2R$.

Equivalently for each $a \in R$, there exists $n \geq 1$ such that $a^n = ue = eu$ for some $e^2 = e$ and some $u \in U(R)$. If every element of a ring is strongly π -regular, then the ring is called a strongly π -regular ring. It is easily seen that, if R is a strongly π -regular ring with $J(R) = 0$, then every nonzero right ideal contains a nonzero idempotent.

A Boolean ring R is a ring for which every element is an idempotent. This ring has an extremely large role in algebra and have many applications in other areas. In [Hirano and Tominaga, 1988], Hirano and Tominaga consider generalization of Boolean rings such that for all $a \in R$, $a = e + f$ where $e, f \in Id(R)$ and determined

the rings for which every element is a sum of two commuting idempotents.

Theorem 2.1: The following conditions are equivalent:

- i) *R is a commutative ring in which every element is the sum of two idempotents.*
- ii) *R is a ring in which every element is the sum of two commuting idempotents.*
- iii) *R satisfies the identity $x^3 = x$.*

Proof See Theorem 1 in [Hirano and Tominaga, 1988] for more details.

Definition 2.2: Let R be a ring, an element $r \in R$ is called nil clean if there exists $e \in Id(R)$ and $b \in N(R)$ such that $r = e + b$; if further $eb = be$ the element $r \in R$ is called strongly nil clean. A ring R is called (strongly) nil-clean if each of its elements is (strongly) nil-clean.

Proposition 2.1: Let R be a ring, and let I be any nil ideal of R .

- i) *R is nil clean if and only if R/I is nil clean.*
- ii) *If R is a nil clean ring, then $J(R)$ is nil.*

Proof (i) The necessity is clear. For the sufficiency, we have idempotents lift modulo any nil ideal I of R and $(1-e)$ can be lifted to an idempotent e of R so that proof is easily seen.

(ii) See Proposition 3.16 in [Diesl, 2013].

Corollary 2.1: Let R be a ring,

- i) *R is nil clean if and only if $J(R)$ is nil and $R/J(R)$ is nil clean.*
- ii) *R is nil clean if and only if $R/J(R)$ is Boolean and $J(R)$ is nil.*

Proof See Corollary 3.17 and 3.20 in [Diesl, 2013].

In [Kosan et al., 2016], it is provided that strongly nil-clean elements are strongly clean elements a with $a - a^2 \in N(R)$ and so they obtained some equivalent conditions of a strongly nil-clean element. These conditions constructed the structure of a strongly nil-clean ring.

Theorem 2.2: [Kosan et al., 2016] Let R be a ring, an element $a \in R$ is strongly nil-clean if and only if a is strongly clean in R and $a - a^2 \in N(R)$.

Proof ($\Rightarrow$) Let $a = e + b$ be a strongly nil-clean decomposition in R . Then $a = (1 - e) + (2e - 1 + b)$ is a strongly clean decomposition in R . Further, $a^2 = e + 2eb + b^2$, and so that $a - a^2 = (1 - 2e - b)b$ is a nilpotent.

($\Leftarrow$) Now let $a = e + u$ be a strongly clean decomposition in R and $a - a^2$ be a nilpotent. Then $a^2 = e + 2eu + u^2$ and so $a - a^2 = (1 - 2e - u)u$ and so we have $1 - 2e - u$ is a nilpotent. Therefore $a = (1 - e) + (-1 + 2e + u)$ is a strongly nil-clean decomposition in R .

By Corollary 3.11 in [Diesl, 2013], characterization of a strongly nil-clean ring is given such that a ring R is strongly nil-clean if and only if R is a strongly π -regular ring and $U(R) = 1 + Nil(R)$. In the following theorem, structure of a strongly nil-clean ring is discussed:

Theorem 2.3: A ring R is strongly nil-clean if and only if $R/J(R)$ is boolean and $J(R)$ is nil.

Proof See Theorem 2.7 in [Kosan et al., 2016].

Definition 2.3: A Morita context is a 4-tuple $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$, where A, B are rings, ${}_A M_B$ and ${}_B N_A$ are bimodules, and there exist context products $M \times N \rightarrow A$ and $N \times M \rightarrow B$ written multiplicatively as $(w, z) \mapsto wz$ and $(z, w) \mapsto zw$, such that $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is an associative ring with the obvious matrix operations.

Theorem 2.4: i) Let $R := \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a Morita context, and let $\begin{pmatrix} A/J(A) & M/M_0 \\ N/N_0 & B/J(B) \end{pmatrix}$. Then $R/J(R) \cong \begin{pmatrix} A/J(A) & M/M_0 \\ N/N_0 & B/J(B) \end{pmatrix}$.
ii) Let $R := \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a Morita context. Then $R/J(R)$ is boolean if and only if $A/J(A)$, $B/J(B)$ are boolean, $MN \subseteq J(A)$ and $NM \subseteq J(B)$.
iii) Let $R := \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a Morita context. If R is a strongly nil clean ring, then A, B are strongly nil clean rings, $MN \subseteq J(A)$ and $NM \subseteq J(B)$; the converse holds if MN, NM are nilpotent ideal of A and B , respectively.

Proof For more details about proofs see Lemma 3.2, Lemma 3.3 and Theorem 3.4, respectively in [Kosan et al., 2016].

In the next lemma, we have the well-known property for lifting idempotents modulo a nil ideal:

Lemma 2.1: [Ying et al., 2016] Let $a \in R$. If $a^2 - a$ is nilpotent, then there exists a monic polynomial $\phi(t) \in \mathbb{Z}[t]$ such that $\phi(a)^2 = \phi(a)$ and $a - \phi(a)$ is nilpotent.

[Ying et al., 2016] Let R be a ring. Then every element of R is a difference of two commuting idempotents if and only if R has the identity $x^3 = x$.

In [Ying et al., 2016], it is observed which rings have the property that every element is a sum of an idempotent and a tripotent that commute and which rings have the property that every element is a sum of two commuting tripotents.

Definition 2.4: A ring R is called a (strong) SIT-ring if every element is a sum of an idempotent and a tripotent (that commute).

Theorem 2.5: [Ying et al., 2016][Theorem 3.10] Every element of R is a sum of an idempotent and a tripotent that commute if and only if R has the identity $x^6 = x^4$ if and only if $R/J(R)$ is Boolean and $U(R)$ is a group of exponent 2 or R is a subdirect product of $\mathbb{Z}_3$'s or $R \cong R_1 \times R_2$, where $R_1/J(R_1)$ is Boolean with $U(R_1)$ a group of exponent 2 and R_2 is zero or a subdirect product of $\mathbb{Z}_3$'s.

Theorem 2.6: [Ying et al., 2016][Theorem 4.4] Every element of R is either a sum or a difference of two commuting idempotents if and only if $R/J(R)$ is Boolean with $\{J(R) = 0 \text{ or } J(R) = \{0, 2\}\}$ or R is a subdirect product of $\mathbb{Z}_3$'s or $R \cong R_1 \times R_2$, where $R_1/J(R_1)$ is Boolean with $\{J(R_1) = 0 \text{ or } J(R_1) = \{0, 2\}\}$, and R_2 is zero or a subdirect product of $\mathbb{Z}_3$'s.

Theorem 2.7: [Ying et al., 2016][Theorem 5.2] Every element of R is a sum of two commuting tripotents if and only if $R \cong R_1 \times R_2 \times R_3$, where $R_1/J(R_1)$ is Boolean with $U(R_1)$ a group of exponent 2, R_2 is zero or a subdirect product of $\mathbb{Z}_3$'s, and R_3 is zero or a subdirect product of $\mathbb{Z}_5$'s.

A ring R is a Yaqub ring if it is the subdirect product of $\mathbb{Z}_3$'s. In the [Chen and Sheibani, 2017], Chen and Sheibani introduce the strongly 2 nil-clean rings as a new type of rings which generalize the ideal of (strongly) nil-clean rings and entirely define the structure of such rings. By the following results, they obtained basic properties of

strongly 2 nil-clean rings.

Theorem 2.8: [Chen and Sheibani, 2017][Theorem 2.3] Let R be a ring. Then the following are equivalent:

- i) R is strongly 2-nil-clean.*
- ii) For all $a \in R$, $a - a^3 \in N(R)$.*
- iii) For all $a \in R$, $a^2 \in R$ is strongly nil-clean.*

Theorem 2.9: [Chen and Sheibani, 2017][Theorem 3.3] A ring R is strongly 2-nil-clean if and only if

- i) $J(R)$ is nil;*
- ii) $R/J(R)$ is tripotent.*

Theorem 2.10: [Chen and Sheibani, 2017][Theorem 4.5] A ring R is strongly 2-nil-clean if and only if $R \cong R_1, R_2$ or $R_1 \times R_2$, where

- i) $R_1/J(R_1)$ is a Boolean ring and $J(R_1)$ is nil;*
- ii) $R_2/J(R_2)$ is a Yaquab ring and $J(R_2)$ is nil.*

In [Zhou, 2018], Zhou studied on the rings for which every element is a sum of a nilpotent, an idempotent and a tripotent that commute with one another, and the rings for which every element is a sum of a nilpotent and two tripotents that commute with one another.

Definition 2.5: A ring R is called an SNTT-ring if every element of R is a sum of a nilpotent and two tripotents, and R is called a strong SNTT-ring if every element of R is a sum of a nilpotent and two tripotents that commute with one another.

Theorem 2.11: [Zhou, 2018][Theorem 2.5] Let R be a ring with $2 \in N(R)$. The following are equivalent:

- i) R is a strong SNTT-ring.*
- ii) $a^2 - a$ is nilpotent for all $a \in R$.*
- iii) $R/J(R)$ is Boolean with $J(R)$ nil.*
- iv) Every element of R is a sum of a nilpotent and a tripotent that commute.*

v) Every element of R is a sum of a nilpotent and two idempotents that commute with one another.

vi) Every element of R is a sum of a nilpotent, an idempotent and a tripotent that commute with one another.

Lemma 2.2: If $2 \in U(R)$ and $a^3 - a$ is nilpotent, then there exists a polynomial $\phi(t) \in \mathbb{Z}[t]$ such that $\phi(a)^3 = \phi(a)$ and $a - \phi(a)$ is nilpotent.

Theorem 2.12: [Zhou, 2018][Proposition 2.8] Let R be a ring with $3 \in N(R)$. The following are equivalent:

- i) R is a strong SNTT-ring.
- ii) $a^3 - a$ is nilpotent for all $a \in R$.
- iii) $R/J(R)$ is a subdirect product of $\mathbb{Z}_3$'s. with $J(R)$ nil.
- iv) Every element of R is a sum of a nilpotent and a tripotent that commute.
- v) Every element of R is a sum of a nilpotent and two idempotents that commute with one another.
- vi) Every element of R is a sum of a nilpotent, an idempotent and a tripotent that commute with one another.

Theorem 2.13: [Zhou, 2018][Theorem 2.11] The followings are equivalent for a ring R :

- i) Every element of R is a sum of a nilpotent and two tripotents that commute with one another.
- ii) $R \cong A \oplus B \oplus C$, where A is zero or $A/J(A)$ is Boolean with $J(A)$ nil, B is zero or $B/J(B)$ is a subdirect product of $\mathbb{Z}_5$'s with $J(C)$ nil.
- iii) $J(R)$ is nil and $R/J(R)$ has identity $x^5 = x$.
- iv) $a^5 - a$ is nilpotent for all $a \in R$.

Theorem 2.14: [Zhou, 2018][Theorem 2.12] Let R be a ring, the following are equivalent:

- i) Every element of R is a sum of a nilpotent, an idempotent and a tripotent that commute with one another.
- ii) Every element of R is a sum of a nilpotent, and two idempotents that commute with one another.

- iii) Every element of R is a sum of a nilpotent and tripotent that commute.
- iv) $R \cong R_1 \oplus R_2$, where A is zero or $R_1/J(R_1)$ is Boolean with $J(R_1)$ nil, and R_2 is zero or $R_2/J(R_2)$ is a subdirect product of $\mathbb{Z}_3$'s with $J(R_2)$ nil.
- v) $R/J(R)$ has the identity $x^3 = x$ and $J(R)$ is nil.
- vi) $a^3 - a$ is nilpotent for all $a \in R$.

2.2. Preliminaries on Group Rings

Let R be a ring and G a group, RG denotes the group ring of the group G over R . The ring homomorphism $\omega : RG \rightarrow R$, $\sum r_g g \mapsto \sum r_g$ is called the augmentation map, and the kernel $\ker(\omega)$ is called the augmentation ideal of the group ring RG and is denoted by $\Delta(RG)$. Note that $\Delta(RG)$ is an ideal of RG generated by the set $\{1 - g : g \in G\}$.

Theorem 2.15: Let R be a ring and G be a group.

- i) If RG is a semiboolean ring, then R is a semiboolean ring and G is a 2-group.
- ii) If R is a semiboolean ring and G is a locally finite 2-group, then RG is a semiboolean ring.

Proof (i) Clearly, R (as an image of RG) is semiboolean and so Z_2G is semiboolean. This ring is also uniquely clean and so G is a 2-group.

(ii) See Theorem 4.4 in [Kosan et al., 2016] for more details.

A ring R is regular (in the sense of von Neumann regular) if for every $x \in R$ there exists a $y \in R$ such that $xyx = x$. A group is locally finite if every finite subset generates a finite subgroup.

Theorem 2.16: Let R and S be two rings, then R is regular if and only if

- i) S is regular,
- ii) G is locally finite,
- iii) each $n \in o(G)$ is a unit in S where $o(G)$ denote the set of orders of all finite subgroups in G .

Proof We refer reader to see Theorem 3.3 in [Connell, 1963].

A group G is called a p -group if the order of each element of G is a power of p .

Theorem 2.17: Let p be a prime with $p \in J(R)$. If G is a locally finite p -group, then $\Delta(RG) \subseteq J(RG)$.

Proof See Lemma 2 in [Zhou, 2010].

3. SEMI-TRIPOTENT RINGS

This chapter is about rings R for which every element is a sum of a tripotent and an element from the Jacobson radical $J(R)$. These rings are called semi-tripotent rings. Examples include Boolean rings, strongly nil-clean rings, strongly 2-nil-clean rings, and semiboolean rings. Here many characterizations of semi-tripotent rings are obtained. Necessary and sufficient conditions for a Morita context, respectively for a group ring of an abelian group or a locally finite nilpotent group, to be semi-tripotent are proved.

3.1. Semi-n-potent rings

In this section, $n \geq 2$ is a fixed integer.

Lemma 3.1: Let R be a ring and $x \in R$. The following are equivalent:

- i) $x^n = x$.
- ii) $x^2 = vx$, where $v^{n-1} = 1$.
- iii) $x^2 = \varphi(x)x$, where $\varphi(t) \in \mathbb{Z}[t]$ with $\varphi(x)^{n-1} = 1$.
- iv) $x^2 = vx$, where $v^{n-1} = 1$ and $vx = xv$.
- v) $x = eu$, where $e^2 = e$, $u^{n-1} = 1$ and $eu = ue$.
- vi) $x = eu$, where $e^2 = e$, $u^{n-1} = 1$ and $eue = ue$.

Proof i) $\Rightarrow$ iii). If $x^n = x$, let $\varphi(t) = 1 + t - t^{n-1} \in \mathbb{Z}[t]$. Then $x^2 = \varphi(x)x$, and $\varphi(x)^k = 1 + x^k - x^{n-1}$ for $k = 1, 2, \dots, n$. In particular, $\varphi(x)^{n-1} = 1$.

iii) $\Rightarrow$ iv) and (v) $\Rightarrow$ vi). They are trivial.

iv) $\Rightarrow$ v). Given (iv), we see $x = v^{-1}x^2 = v^{n-2}x^2 = v(v^{n-3}x^2)$ with $(v^{n-3}x^2)^2 = v^{n-3}x^2$.

vi) $\Rightarrow$ ii). Given (vi), we have $x^2 = eueu = ux$.

ii) $\Rightarrow$ i). Given (ii), we have $x^n = vx \cdot x^{n-2} = vx^{n-1} = v \cdot vx \cdot x^{n-3} = v^2x^{n-2} = \dots = v^{n-1}x = x$. ■

Definition 3.1: Let I be an ideal of a ring R . We say that n -potents lift modulo I in R if whenever $a^n - a \in I$, there exists $e^n = e \in R$ such that $a - e \in I$.

Definition 3.2: A ring R is called a semi- n -potent ring if every element of R is a sum of an n -potent and an element from $J(R)$, equivalently if $R/J(R)$ has identity $x^n = x$ and n -potents lift modulo $J(R)$ in R .

The next example can be easily verified.

Example 3.1: Let R, S be rings, V an (R, S) -bimodule, M an R -bimodule, and $m \geq 1$.

- i) If R is semi- n -potent, then so is every homomorphic image of R .
- (i) A direct product $\prod R_\alpha$ of rings is semi- n -potent if and only if every R_α is semi- n -potent.
- iii) The formal triangular matrix ring $\begin{pmatrix} R & V \\ 0 & S \end{pmatrix}$ is semi- n -potent if and only if R and S are semi- n -potent.
- iv) $\mathbb{T}_m(R)$ is semi- n -potent if and only if R is semi- n -potent.
- v) The trivial extension $R \ltimes M$ is semi- n -potent if and only if R is semi- n -potent.
- vi) $R[x]/(x^m)$ is semi- n -potent if and only if R is semi- n -potent.
- vii) $R[[x]]$ is semi- n -potent if and only if R is semi- n -potent.

A subring of a semi- n -potent ring need not be semi- n -potent: $\mathbb{Z}_2[[x]]$ is semi- n -potent, but $\mathbb{Z}_2[x]$ is not semi- n -potent.

Lemma 3.2: Let I be an ideal of a ring R . The following are equivalent:

- i) If $a^n - a \in I$, then there exists $e^n = e \in aR$ such that $a - e \in I$.
- ii) If $a^n - a \in I$, then there exists $e^n = e \in aRa$ such that $a - e \in I$.
- iii) If $a^n - a \in I$, then there exists $e^n = e \in Ra$ such that $a - e \in I$.

Proof Write $r \equiv s$ to mean that $r - s \in I$, so that $r \equiv s$ implies that $xry \equiv xsy$ for all $x, y \in R$. It suffices to show the implication “(i) $\Rightarrow$ (ii)”. Suppose that (i) holds. If $a^n \equiv a$, then $(a^n)^n \equiv a^n$. Choose $f^n = f \in a^n R$ such that $a^n \equiv f$, so $f \equiv a$. Write $f = a^n x$ with $x \in R$. We may assume that $x = x f^{n-1}$. Let $e = a^{n-1} x a \in aRa$. Then $e^n = a^{n-1} x (a^n x)^{n-1} a = a^{n-1} x f^{n-1} a = a^{n-1} x a = e$. Moreover, $e = a^{n-1} x a \equiv (a^n)^{n-1} x a = (a^n)^{n-2} (a^n x) a \equiv (a^n)^{n-2} f a \equiv a^{n-2} f a \equiv a^n \equiv a$. So (ii) holds. ■

We say that n -potents lift strongly modulo I if the equivalent conditions of Lemma 3.2 hold. The following result is known when $n = 2$ (see [Nicholson and Zhou, 2005, 2, Lemma 5]).

Proposition 3.1: *Let R be a ring. If n -potents lift modulo $J(R)$, then they lift strongly modulo $J(R)$.*

Proof Let $a^n - a \in J(R)$. Choose $f^n = f \in R$ such that $f - a \in J(R)$. Then $f^{n-1} - a^{n-1} \in J(R)$, so $u := 1 - (f^{n-1} - a^{n-1}) \in U(R)$, and $uf = a^{n-1}f \in aR$. So $e := uf u^{-1} \in aR$ and $e^n = e$. As $\bar{u} = \bar{1}$ in $R/J(R)$, $\bar{e} = \bar{u}\bar{f}\bar{u}^{-1} = \bar{f} = \bar{a}$. ■

Corollary 3.1: *Let R be a ring with $e^2 = e \in R$. If n -potents lift modulo $J(R)$ in R , then n -potents lift modulo $J(eRe)$ in eRe .*

Proof Let $a^n - a \in J(eRe)$ where $a \in eRe$. Then $a^n - a \in eJ(R)e \subseteq J(R)$, so, by Proposition 3.1, there exists $f^n = f \in aRa$ such that $a - f \in J(R)$. As $f \in aRa \subseteq eRe$, $a - f \in J(R) \cap eRe = J(eRe)$. ■

Corollary 3.2: *Let $e^2 = e \in R$. If R is semi- n -potent, then so is eRe .*

Proof As n -potents lift modulo $J(R)$ in R , n -potents lift modulo eJe in eRe by Corollary 3.1. Moreover, $eRe/J(eRe) = eRe/eJe \cong \bar{e}\bar{R}\bar{e} \subseteq \bar{R}$, where $\bar{R} = R/J(R)$. As $\bar{R}$ has identity $x^n = x$, $\bar{e}\bar{R}\bar{e}$, and hence $eRe/J(eRe)$ has identity $x^n = x$. So eRe is semi- n -potent. ■

It is easy to see that no proper matrix ring can be semi- n -potent. Next we consider when a Morita context is a semi- n -potent ring.

A Morita context is a 4-tuple $R := \begin{pmatrix} A & M \\ N & B \end{pmatrix}$, where A, B are rings ${}_A M_B$ and ${}_B N_A$ are bimodules, and there exist context products $M \times N \rightarrow A$ and $N \times M \rightarrow B$ written multiplicatively as $(w, z) \mapsto wz$ and $(z, w) \mapsto zw$, such that $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is an associative ring with the obvious matrix operations.

The next result (in fact, a more general result) can be found in [Sands, 1973].

Lemma 3.3: [Sands, 1973] *Let $R := \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a Morita context. Then $J(R) = \begin{pmatrix} J(A) & M_0 \\ N_0 & J(B) \end{pmatrix}$, where $M_0 = \{x \in M : xN \subseteq J(A)\}$ and $N_0 = \{y \in N : yM \subseteq J(B)\}$.*

Theorem 3.1: Let $R := \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a Morita context. Then R is semi- n -potent if and only if A, B are semi- n -potent, $MN \subseteq J(A)$ and $NM \subseteq J(B)$.

Proof ($\Rightarrow$). As R is semi- n -potent, $R/J(R)$ has identity $x^n = x$. Especially, $R/J(R)$ is reduced. So, by Lemma 3.3, $M = M_0$ and $N = N_0$, and it follows that $MN \subseteq J(A)$ and $NM \subseteq J(B)$. By Corollary 3.2, A, B are semi- n -potent.

($\Leftarrow$). As $MN \subseteq J(A)$ and $NM \subseteq J(B)$, we have $J(R) = \begin{pmatrix} J(A) & M \\ N & J(B) \end{pmatrix}$ by Lemma 3.3. For $\alpha := \begin{pmatrix} a & x \\ y & b \end{pmatrix} \in R$, write $a = e + j_A$ and $b = f + j_B$ where $e^n = e \in A$, $f^n = f \in B$, $j_A \in J(A)$ and $j_B \in J(B)$. Then $\alpha = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} + \begin{pmatrix} j_A & x \\ y & j_B \end{pmatrix}$ is a sum of an n -potent and an element from $J(R)$. So R is semi- n -potent. ■

As a consequence of Theorem 3.1 and Corollary 3.2, the following improvement is observed:

Corollary 3.3: Let $e^2 = e \in R$. Then R is semi- n -potent if and only if eRe and $(1-e)R(1-e)$ are semi- n -potent, and $eR(1-e)$ and $(1-e)Re$ both are contained in $J(R)$.

Proof Consider the Pierce decomposition $R = \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}$. By Theorem 3.1, R is semi- n -potent if and only if eRe and $(1-e)R(1-e)$ are semi- n -potent, $eR(1-e)Re \subseteq eJ(R)e$ and $(1-e)ReR(1-e) \subseteq (1-e)J(R)(1-e)$. Note that $eR(1-e)Re \subseteq eJ(R)e$ if and only if $eR(1-e)Re \subseteq J(R)$, if and only if $(eR(1-e)R)^2 \subseteq J(R)$, if and only if $eR(1-e)R \subseteq J(R)$, if and only if $eR(1-e) \subseteq J(R)$. Similarly, $(1-e)ReR(1-e) \subseteq (1-e)J(R)(1-e)$ if and only if $(1-e)Re \subseteq J(R)$. ■

3.2. Characterization of Semi-tripotent rings

For $n \geq 3$, if n -potents lift modulo $J(R)$ in a ring R , then idempotents lift modulo $J(R)$. Indeed, if $a^2 - a \in J(R)$, then $a - e \in J(R)$ where $e^n = e \in R$. So $a - e^{n-1} = (a - a^{n-1}) + (a^{n-1} - e^{n-1}) \in J(R)$ with e^{n-1} an idempotent. This raises the question whether the converse holds. The next example shows that, for each integer

$n \geq 4$, there exists a ring R such that idempotents lift modulo $J(R)$ but n -potents do not.

Example 3.2: Let $n \geq 4$, and let $p(t) \in \mathbb{R}[t]$ be any irreducible polynomial of degree 2 which divides $t^{n-1} - 1$. (For example, if $n = 4$ then take $p(t) = t^2 + t + 1$.) Let $R = \mathbb{R}[t]_{(p(t))}$, the localization of $\mathbb{R}[t]$ at the maximal ideal generated by $p(t)$. Then $J(R)$ is generated by $p(t)$, so $R/J(R) \cong \mathbb{R}[t]/(p(t)) \cong \mathbb{C}$, the field of complex numbers. Hence idempotents trivially lift modulo $J(R)$.

The only n -potents in R are 0, 1, and possibly -1 when n is odd. (This is because R is a subring of the field of rational functions over $\mathbb{R}$.) Since neither t nor $t - 1$ nor $t + 1$ lies in $J(R)$, it follows that t is an n -potent modulo $J(R)$ (since $t^n - t \in J(R)$) which cannot be lifted to an n -potent in R .

For $n = 3$, the next lemma gives a partial answer to the question above. Note that square roots of 1 lift modulo the Jacobson radical exactly when idempotents lift, provided 2 is a unit (this is proved in [Khurana et al., 2016]), and this result can be used to give a quick proof of the next lemma. But here we give a direct, self-contained proof.

Lemma 3.4: Let R be a ring with $2 \in U(R)$. Then idempotents lift modulo $J(R)$ if and only if tripotents lift modulo $J(R)$.

Proof The sufficiency is noticed above. For the necessity, suppose $a^3 - a \in J(R)$. Let $b = \frac{1}{2}(a^2 + a)$ and $c = \frac{1}{2}(a^2 - a)$. Then $a = b - c$, $b - b^2 = \frac{1}{4}(a + 2)(a - a^3) \in J(R)$ and $c - c^2 = \frac{1}{4}(a - 2)(a - a^3) \in J(R)$. Thus, in $R/J(R)$, $\bar{a} = \bar{b} + \bar{c}$, $\bar{b}^2 = \bar{b}$ and $\bar{c}^2 = \bar{c}$. Moreover, $\bar{b}\bar{c} = \bar{c}\bar{b} = \frac{1}{4}(\bar{a}^4 - \bar{a}^2) = \bar{0}$. Since idempotents lift modulo $J(R)$, $\bar{b}$ and $\bar{c}$ can be lifted to orthogonal idempotents f and g in R . Let $e = f - g$. Then $e^3 = e$ and $\bar{a} = \bar{b} - \bar{c} = \bar{f} - \bar{g} = \bar{e}$. Hence, tripotents lift modulo $J(R)$. ■

Lemma 3.5: The following are equivalent for a ring:

- i) For each $a \in R$, $a = ve$ where $e^2 = e$ and $v^2 = 1$.
- ii) For each $a \in R$, $a = fw$ where $f^2 = f$ and $w^2 = 1$.
- iii) R has identity $x^3 = x$.

Proof (i) $\Rightarrow$ (iii). By (i), R is a unit-regular ring. We show that R is reduced. Assume that $r^2 = 0$ with $0 \neq r \in R$. By [Levitzki, 1953, Theorem 2.1], there

exists $0 \neq e^2 = e \in RrR$ such that $eRe \cong \mathbb{M}_2(S)$ for a non-trivial ring S . Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{M}_2(S)$ and then $A^2 = A + I_2 \neq I_2$. So, there exists $u \in U(eRe)$ such that $u^2 \neq e$. Thus $y := u + (1 - e) \in U(R)$ and $y^2 = u^2 + (1 - e) \neq 1$, contradicting (1). So R is reduced, and hence is abelian. Therefore, for each $a \in R$, write $a = ve$ with $e^2 = e$ and $v^2 = 1$, and we see $a^3 = v^3e^3 = ve = a$; this proves (3).

(iii) $\Rightarrow$ (i). It is clear by Lemma 3.1. ■

Lemma 3.6: Suppose that, for each $a \in R$, $a = b + j_1 + ev = b + j_2 + ve$, where $j_1, j_2 \in J(R)$, $e^2 = e \in R$, $v^2 = 1$, $b \in N(R)$ with $ab = ba$. Then $6 \in J(R)$, idempotents lift modulo $J(R)$, and $R/J(R)$ has identity $x^3 = x$.

Proof Let $J = J(R)$, and $\bar{R} = R/J$. Assume that $a^2 - a \in J$, and write $a = b + j_1 + ev = b + j_2 + ve$, where $j_1, j_2 \in J$, $e^2 = e \in R$, $v^2 = 1$, $b \in N(R)$ with $ab = ba$. Then

$$\begin{aligned} a^2 &= (b + j_1 + ev)(b + j_2 + ve) \\ &= [ba + (a - b)b] + [j_1(j_2 + ve) + evj_2] + e. \end{aligned} \tag{3.1}$$

Let $c = ba + (a - b)b$. Then $c \in N(R)$ and $ac = ca$. So, in $\bar{R}$, $\bar{a} = \bar{a}^2 = \bar{c} + \bar{e}$, and it follows that $\bar{c} + \bar{e} = (\bar{c} + \bar{e})^2 = \bar{c}^2 + 2\bar{c}\bar{e} + \bar{e}$. Thus, $\bar{c} = \bar{c}^2 + 2\bar{c}\bar{e}$, i.e., $\bar{c}(\bar{1} - 2\bar{e}) = \bar{c}^2$. So $\bar{c} = \bar{c}^2(\bar{1} - 2\bar{e})^{-1} = \bar{c}^2(\bar{1} - 2\bar{e})$. As $\bar{c}$ is nilpotent, it must be that $\bar{c} = \bar{0}$. Hence $\bar{a} = \bar{e}$.

Write $2 = b + j_1 + ev = b + j_2 + ve$, where $j_1, j_2 \in J$, $e^2 = e \in R$, $v^2 = 1$ and $b \in N(R)$. Then $\bar{2} = \bar{b} + \bar{e}\bar{v} = \bar{b} + \bar{v}\bar{e}$, so

$$\bar{4} = (\bar{b} + \bar{e}\bar{v})(\bar{b} + \bar{v}\bar{e}) = \bar{b}(\bar{b} + \bar{v}\bar{e}) + \bar{e}\bar{v}\bar{b} + \bar{e} = \bar{c} + \bar{e}, \tag{3.2}$$

where $\bar{c} = \bar{b}(\bar{b} + \bar{v}\bar{e}) + \bar{e}\bar{v}\bar{b}$ is nilpotent. So, $\bar{4}^2 = (\bar{c} + \bar{e})^2 = \bar{c}^2 + 2\bar{c}\bar{e} + \bar{e}$, and hence

$$\bar{12} = \bar{4}^2 - \bar{4} = (\bar{c}^2 + 2\bar{c}\bar{e} + \bar{e}) - (\bar{c} + \bar{e}) = \bar{c}(\bar{c} + 2\bar{e} - \bar{1}) \tag{3.3}$$

is nilpotent in $\bar{R}$. It follows that $\bar{6}$ is nilpotent in $\bar{R}$, so $\bar{6} = \bar{0}$, or $6 \in J$.

As $6 \in J$, $\bar{R} = R_1 \times R_2$, where $2 = 0$ in R_1 and $3 = 0$ in R_2 . Note that, by hypothesis, for any $a \in R$, $\bar{a} = \bar{b} + \bar{e}\bar{v} = \bar{b} + \bar{v}\bar{e}$, where $\bar{e}^2 = \bar{e}$, $\bar{v}^2 = \bar{1}$ and $\bar{b}$ is nilpotent with $\bar{a}\bar{b} = \bar{b}\bar{a}$, so $\bar{v}\bar{e} = \bar{e}\bar{v}$, and hence $(\bar{v}\bar{e})^3 = \bar{v}\bar{e}$. So, R_1 is Boolean by [Zhou, 2018, Proposition 2.5] and R_2 is zero or a subdirect product of $\mathbb{Z}_3$'s by [Zhou, 2018, Proposition 2.8]. It follows that R/J has identity $x^3 = x$. ■

Some characterizations are obtained for semi-tripotent rings.

Theorem 3.2: *The following are equivalent for a ring R :*

- i) *For each $a \in R$, $a = j + f$ where $j \in J(R)$ and $f^3 = f$, i.e., R is semi-tripotent.*
- ii) *For each $a \in R$, $a = b + j + f$ where $j \in J(R)$, $f^3 = f$ and $b \in N(R)$ with $ab = ba$.*
- iii) *For each $a \in R$, $a = j + ev$, where $j \in J(R)$, $e^2 = e \in R$ and $v^2 = 1$.*
- iv) *For each $a \in R$, $a = j + ve$, where $j \in J(R)$, $e^2 = e \in R$ and $v^2 = 1$.*
- v) *$R/J(R)$ has identity $x^3 = x$ and idempotents lift modulo $J(R)$.*
- vi) *$R/J(R) = R_1 \times R_2$, where R_1 is a Boolean ring, R_2 is zero or a subdirect product of $\mathbb{Z}_3$'s, and idempotents lift modulo $J(R)$.*

Proof The implication (i) $\Rightarrow$ (ii) is obvious. The implications (i) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv) follow from Lemma 3.1.

(j) $\Rightarrow$ (v): $j = 2, 3, 4$. First assume (iii) holds. For each $a \in R$, $\bar{a} = \bar{e}\bar{v}$ where $e^2 = e$ and $v^2 = 1$. By Lemma 3.5, (iii) implies that R/J has identity $x^3 = x$; so R/J is abelian. Therefore, $\bar{a} = \bar{e}\bar{v} = \bar{v}\bar{e}$. It follows that $a = j_1 + ev = j_2 + ve$ for some $j_1, j_2 \in J$. Similarly, (iv) implies that, for each $a \in R$, $a = j_1 + ev = j_2 + ve$ where $e^2 = e, v^2 = 1$ and $j_1, j_2 \in J$. Hence, in view of Lemma 3.1, either of (ii), (iii) and (iv) implies that, each $a \in R$, $a = b + j_1 + ev = b + j_2 + ve$, where $j_1, j_2 \in J(R)$, $e^2 = e \in R, v^2 = 1, b \in N(R)$ with $ab = ba$. Thus, (v) holds by Lemma 3.6.

(v) $\Leftrightarrow$ (vi). It is clear.

(v) $\Rightarrow$ (i). Let $a \in R$. Since $\bar{R} := R/J(R)$ has identity $x^3 = x$, by [Hirano and Tominaga, 1988, Theorem 1] $\bar{1} + \bar{a} = y + z$ for some commuting idempotents y and z . By [Mazurek et al., 2015, Theorem 2.1], one can lift the idempotents y, z to commuting idempotents f, g in R . Thus, $1 + a = f + g + j$ for some $j \in J(R)$,

and hence $a = f - (1 - g) + j$ where, as a difference of two commuting idempotents, $f - (1 - g)$ is a tripotent. ■

We remark that an element that is a product of an idempotent and a square root of 1 in a ring R may not be a sum of a tripotent and an element from $J(R)$. To see this, consider $R = \mathbb{M}_2(\mathbb{Q})$. Then, in R , $a := \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, a product of an idempotent and a square root of 1, but a can not be a sum of a tripotent and an element from $J(R)$.

We also remark that Theorem 3.2(6) can not be replaced by the condition that $R = A \times B$, where $A/J(A)$ is a Boolean ring and idempotents lift modulo $J(A)$, and B is zero or $B/J(B)$ is a subdirect product of $\mathbb{Z}_3$'s and idempotents lift modulo $J(B)$. To see this, consider the formal matrix ring $R = \begin{pmatrix} \mathbb{Z}_{(2)} & \mathbb{Q} \\ 0 & \mathbb{Z}_{(3)} \end{pmatrix}$, where $\mathbb{Z}_{(2)}, \mathbb{Z}_{(3)}$ are the localizations of $\mathbb{Z}$ at 2 and 3. Then R is indecomposable, but idempotents lift modulo $J(R)$ and $R/J(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$.

A ring R is called semiboolean if every element of R is a sum of an idempotent and an element from $J(R)$. It is known that a ring R is semiboolean if and only if $R/J(R)$ is Boolean and idempotents lift modulo $J(R)$ (see [Nicholson and Zhou, 2005, Lemma 2.4]). A semiboolean ring was also termed as a J -clean ring in [Chen and Sheibani, 2017], [Kosan et al., 2018] and [Matczuk, 2017]. We note that J -clean ring had been used differently in [Burgess and Raphael, 2007]. We can add some new conditions to the equivalence list for semiboolean rings.

Corollary 3.4: *The following are equivalent for a ring R :*

- i) R is semiboolean.
- ii) For each $a \in R$, $a = b + j + e$, where $b \in N(R)$, $j \in J(R)$, $e^2 = e \in R$ and $ab = ba$.
- iii) $R/J(R)$ is Boolean and idempotents lift modulo $J(R)$.
- iv) R is semi-tripotent and $2 \in J(R)$.

Proof i) $\Rightarrow$ ii). It is obvious.

ii) $\Rightarrow$ iii). By Theorem 3.2, we have $R/J = R_1 \times R_2$, where R_1 is a Boolean ring and R_2 is zero or a subdirect product of $\mathbb{Z}_3$'s. Assume on the contrary that R/J is

not Boolean. Then $R_2 \neq 0$, so R has a quotient ring isomorphic to $\mathbb{Z}_3$. As any quotient ring of R still satisfies (2), we may assume that $R = \mathbb{Z}_3$. So, since $N(R) = J(R) = 0$, 2 is an idempotent. Thus, $2 = 1$, a contradiction.

$i) \Leftrightarrow iii)$. This is by [Nicholson and Zhou, 2005, Lemma 24] (also see [Kosan et al., 2018, Theorem 3.2]).

$i) \Rightarrow iv)$. It is easy to see.

$iv) \Rightarrow iii)$. For $a \in R$, $a^3 - a \in J(R)$ as $R/J(R)$ has identity $x^3 = x$. So $(a^2 - a)^2 = a^4 - 2a^3 + a^2 = a(a^3 - a) + 2(a^2 - a^3) \in J(R)$, and hence $a^2 - a \in J(R)$. So $R/J(R)$ is Boolean. ■

The assumption that “ $ab = ba$ ” in Theorem 3.2 and Corollary 3.4 can not be removed: For $k \geq 2$ and $n \geq 1$, the ring $R := \mathbb{M}_k(\mathbb{Z}_{2^n})$ is a nil-clean ring (see [Diesl, 2013, Corollary 3.17 and Example 4.5]), that is, every element of R is a sum of a nilpotent and an idempotent, but $R/J(R)$ is neither a Boolean ring nor a subdirect product of a Boolean ring and a direct product of $\mathbb{Z}_3$ ’s. We also remark that there exists a ring R such that $R/J(R)$ is Boolean, but idempotents do not lift modulo $J(R)$ (see [Lee and Zhou, 2016, Example 15]).

None of the conditions of Theorem 3.2 can be replaced by “For each $a \in R$, $a = b + j + ev$, where $j \in J(R)$, $e^2 = e$, $v^2 = 1$, and $b \in N(R)$ with $ab = ba$ ”: One can easily check that $R = \mathbb{M}_2(\mathbb{Z}_2)$ satisfies the latter condition, but R is not semi-tripotent.

Theorem 3.3: The following are equivalent for a ring R :

- i) R is semi-tripotent.
- ii) For each $a \in R$, $a = j + e + f$, where $j \in J(R)$, $e^2 = e$, $f^2 = f$ and $ef = fe$.
- iii) For each $a \in R$, $a = j + e + f$, where $j \in J(R)$, $e^2 = e$, $f^3 = f$ and $ef = fe$.
- iv) For each $a \in R$, $a = j + b + e + f$, where $j \in J(R)$, $b \in N(R)$, $e^2 = e$, $f^2 = f$, and b, e, f commute with one another.
- v) For each $a \in R$, $a = j + b + e + f$, where $j \in J(R)$, $b \in N(R)$, $e^2 = e$, $f^3 = f$, and b, e, f commute with one another.

Proof $i) \Rightarrow ii)$. By Theorem 3.2, $R/J = A/J \oplus B/J$, where A/J is a Boolean ring and B/J is a subdirect product of $\mathbb{Z}_3$ ’s, and idempotents lift modulo J . Write $\bar{1} = \alpha + \beta$

where $\alpha \in A/J$ and $\beta \in B/J$. We may assume that, for some $e^2 = e \in R$, $\alpha = \bar{e}$ and $\beta = \overline{1-e}$. Let $f = 1 - e$. Then $A/J = (eRe + J)/J$ and $B/J = (fRf + J)/J$. Let $x \in R$ be an arbitrary element, and write $\bar{x} = \bar{a} + \bar{b}$ where $\bar{a} \in A/J$ and $\bar{b} \in B/J$. We may assume that $a \in eRe$ and $b \in fRf$.

As A/J is Boolean, $a^2 - a \in J$. So, by [Nicholson and Zhou, 2005, 2, Lemma 5], there exists $e_1^2 = e_1 \in aRa \subseteq eRe$ such that $a - e_1 \in J$.

By Corollary 3.1, idempotents lift modulo $J(fRf)$ in fRf . As $fRf/J(fRf) \cong B/J$ is a subdirect product of $\mathbb{Z}_3$'s, $b - b^3 \in J(fRf)$, and $2 \in U(fRf)$. So, by Lemma 3.4, there exists $e_2^3 = e_2 \in fRf$ such that $b - e_2 \in J(fRf) \subseteq J$. Then $x = j + e_1 + e_2$ with $e_1e_2 = e_2e_1 = 0$ and to finish the proof, let $g = e_2^2$. Then $g^2 = g$, and $(e_2 - g)^2 - (e_2 - g) = -3(e_2 - g) \in J(fRf) \subseteq J$. So by [Nicholson and Zhou, 2005, 2, Lemma 5], there exists $h^2 = h \in (e_2 - g)R(e_2 - g)$ such that $e_2 - g - h \in J$. Let $j' = e_2 - g - h$ and write $h = (e_2 - g)r(e_2 - g)$ with $r \in R$. Then $gh = h = hg$. As $h, g \in fRf$, $e_1g = ge_1 = 0$ and $e_1h = he_1 = 0$. It follows that $e_1 + g$ and h are commuting idempotents and $x = (j + j') + (e_1 + g) + h$. This verifies (ii).

$ii) \Rightarrow iii) \Rightarrow (v)$ and $(ii) \Rightarrow (iv) \Rightarrow (v)$. The implications are clear.

$v) \Rightarrow i)$. By (v), every element of $R/J(R)$ is a sum of a nilpotent, an idempotent and a tripotent that commute with one another other. So, by [Zhou, 2018, Theorem 2.12], $R/J(R)$ has identity $x^3 = x$, and hence $N(R) \subseteq J(R)$. By Theorem 3.2, to show (1) it remains to show that idempotents lift modulo $J(R)$. Assume that $a - a^2 \in J(R)$. Write $a = j + b + e + f$ as in (5). Then $j + b \in J(R)$, so we may assume that $b = 0$. Thus $a - a^2 = (f - f^2 - 2ef) + (j - ja - (e + f)j)$, so $f - f^2 - 2ef \in J(R)$, and hence $f^2 - f - 2ef^2 = (f - f^2 - 2ef)f \in J(R)$. Let $g = e + f^2 - 2ef^2$. Then $a - g = j + (f - f^2 + 2ef^2) \in J(R)$ and $g^2 = g$. ■

3.3. Group rings

Semi-tripotent rings can be constructed via Morita contexts by Theorem 3.1. In this section, we discuss when a group ring is semi-tripotent. A group G is called locally finite if every finitely generated subgroup of G is finite. For a prime number p , a group

is called a p -group if the order of each of its elements is a power of p . A group of exponent p is a non-trivial group in which every element has order p . We write C_n for the cyclic group of order n .

Lemma 3.7: Let R be a ring with $2 \in J(R)$ and G a group. If RG is semi-tripotent, then G is a 2-group.

Proof As RG is semi-tripotent, $(R/J)G \cong RG/JG$ is semi-tripotent. As $2 \in J(R)$, $2 = 0$ in $(R/J)G$, so $(R/J)G$ is semiboolean by Corollary 3.4. Therefore, G is a 2-group by [Kosan et al., 2016, Theorem 4.4]. ■

Lemma 3.8: Let R be a ring with $3 \in J(R)$ and G a locally finite p -group with p a prime. If RG is semi-tripotent, then either G is a 3-group or G is a group of exponent 2.

Proof As RG is semi-tripotent, R is semi-tripotent, so by Theorem 3.2 $R/J = A \times B$ where A is a Boolean ring and B is zero or a subdirect product of $\mathbb{Z}_3$'s. As $3 \in J(R)$, $A = 0$, so $R/J = B$ which has identity $x^3 = x$. If $p \neq 3$, then by [Connell, 1963, Theorem 3], BG is (von Neumann) regular, so $J(BG) = 0$. But, as an image of RG , BG is semi-tripotent, so BG has identity $x^3 = x$. In particular, $g^2 = 1$ for all $g \in G$. So, G is a group of exponent 2. ■

Lemma 3.9: Let R be a ring and G a locally finite group. If RG is semi-tripotent and $2 \notin J(R)$ and $3 \notin J(R)$, then G is a group of exponent 2.

Proof As RG is semi-tripotent, R is semi-tripotent, so by Theorem 3.2 $R/J = A \times B$ where A is a Boolean ring and B is zero or a subdirect product of $\mathbb{Z}_3$'s. As $2 \notin J(R)$ and $3 \notin J(R)$, $A \neq 0$ and $B \neq 0$. As AG is semi-tripotent and $2 = 0$ in A , G is a 2-group by Lemma 3.7. As BG is semi-tripotent and $3 = 0$ in B , G is a group of exponent 2 by Lemma 3.8. ■

Lemma 3.10: [Connell, 1963, Proposition 9] If R is a ring and G is a locally finite group then $J(R) = J(RG) \cap R$. In particular, $J(R)(RG) \subseteq J(RG)$.

Lemma 3.11: If R is a semi-tripotent ring with $3 \in J(R)$ and G is a group of exponent 2, then RG is semi-tripotent.

Proof Let $J = J(R)$ and $\alpha \in RG$. Then there exists a finite subgroup H of G such that $\alpha \in RH$. Here H is a direct product of finite copies of C_2 . As $2 \in U(R)$, RH is a direct sum of finite copies of R , so RH is semi-tripotent. So, α is semi-tripotent in RH . We show that α is semi-tripotent in RG . By Lemma 3.10, $JH \subseteq J(RH)$. As R is semi-tripotent with $3 \in J(R)$, R/J has identity $x^3 = x$ with $2 \in U(R/J)$. So, $(R/J)H$ is a commutative von Neumann regular ring by [Connell, 1963, Theorem 3]. As $(R/J)H \cong RH/JH$, it follows that $JH = J(RH)$. So $J(RH) = JH \subseteq J(RG)$ by Lemma 3.10. Hence, α semi-tripotent in RH implies that α is semi-tripotent in RG . ■

Lemma 3.12: If R is a semi-tripotent ring with $3 \in J(R)$ and G is a locally finite 3-group, then RG is semi-tripotent.

Proof By [Zhou, 2010, Lemma 2], $\Delta(RG) \subseteq J(RG)$. As $RG/\Delta(RG) \cong R$ is semi-tripotent, idempotents lift modulo J in R by Theorem 3.2, and moreover $RG/J(RG)$ has identity $x^3 = x$. By [Nicholson, 1977, Proposition 1.5], R is a clean ring, i.e., every element is a sum of an idempotent and a unit. Hence, by [Zhou, 2010, Theorem 4], RG is a clean ring, so idempotent lift modulo $J(RG)$ in RG . Hence, RG is semi-tripotent by Theorem 3.2. ■

A group is said to be nilpotent if it has a central series.

Theorem 3.4: Let R be a ring and G be a locally finite, nilpotent group. Then RG is semi-tripotent if and only if R is semi-tripotent and one of the following holds:

- i) $2 \in J(R)$ and G is a 2-group.
- ii) $3 \in J(R)$ and G is a direct product of a group of exponent 2 and a 3-group.
- iii) $2 \notin J(R)$ and $3 \notin J(R)$, and G is a group of exponent 2.

Proof ($\Rightarrow$). It is known that every locally finite nilpotent group is a direct product of its p -subgroups. So G is a direct product of p -groups G_p where p runs over all primes. Since RG is semi-tripotent, RG_p is also semi-tripotent for each p and R is semi-tripotent. So, by Theorem 3.2, $R/J = A \times B$, where A is Boolean and B is zero or a subdirect product of $\mathbb{Z}_3$'s.

If $2 \in J(R)$, then G is a 2-group by Lemma 3.7.

If $3 \in J(R)$, then either $p = 3$ or $p = 2$ with G_2 a group of exponent 2 by Lemma 3.8. So G is a direct product of a group of exponent 2 and a 3-group.

If $2 \notin J(R)$ and $3 \notin J(R)$, then $2 \neq 0$ in R/J and $3 \neq 0$ in R/J . So $A \neq 0$ and $B \neq 0$. As images of RG , AG and BG are semi-tripotent. Since $2 = 0$ in A , G is a 2-group by Lemma 3.7. As $3 = 0$ in B , G must be of exponent 2 by Lemma 3.8.

($\Leftarrow$). Suppose that R is semi-tripotent. If (i) holds, then R is semiboolean by Corollary 3.4. So RG is semiboolean (and hence semi-tripotent) by [Kosan et al., 2016, Theorem 4.4].

If (ii) holds, then $G = H_1 \times H_2$, where H_1 is a group of exponent 2 and H_2 is a 3-group. So $RG \cong (RH_2)H_1$. By Lemma 3.12, RH_2 is semi-tripotent, and so $(RH_2)H_1$ is semi-tripotent by Lemma 3.11.

Suppose that (iii) holds. As R is semi-tripotent, $R/J = (X/J) \oplus (Y/J)$, where X/J is Boolean and Y/J is zero or a subdirect product of $\mathbb{Z}_3$'s. As idempotents lift modulo J , there exist $e^2 = e \in X$ and $f^2 = f \in Y$ such that $e + f = 1$, $X/J = (eRe + J)/J$ and $Y/J = (fRf + J)/J$. So, $R = eRe + fRf + J$, and hence $RG = (eRe)G + (fRf)G + JG$. Moreover, by Corollary 3.2 eRe and fRf are semi-tripotent. So, by Corollary 3.4, eRe is semiboolean.

If $\alpha \in RG$, write $\alpha = y + z + w$ where $y \in (eRe)G$, $z \in (fRf)G$ and $w \in JG$. Write $y = \sum a_i g_i$ where $a_i \in eRe$ and $g_i \in G$ for each i . As $2 = 0$ in eRe/eJe , $2e \in eJe$, so $(e(1 + g_i))^2 = e(1 + 2g_i + g_i^2) = e(2 + 2g_i) = 2e(1 + g_i) \in (eJe)G$. Hence $e(1 + g_i) \in (eJe)G \subseteq JG$, as $e(1 + g_i)$ is central in $(eRe)G$ (for all i). Thus, $y = \sum a_i g_i + \sum a_i + (-\sum a_i) = \sum a_i e(1 + g_i) + (-\sum a_i)$, where $\sum a_i e(1 + g_i) \in JG$. As eRe is semiboolean, write $-\sum a_i = j_1 + h_1$ where $j_1 \in eJe \subseteq J$ and $h_1^2 = h_1 \in eRe$. So $j_1 \in J(RG)$ by Lemma 3.11. As $3 \in J(fRf)$, $(fRf)G$ is semi-tripotent by Lemma 3.11. So $z = j_2 + h_2$ where $j_2 \in J((fRf)G)$ and $h_2^3 = h_2 \in (fRf)G$. As $3 = 0$ in fRf/fJf and fRf/fJf has identity $x^3 = x$, $(fRf)G/(fJf)G \cong (fRf/fJf)G$ is semiprimitive (indeed, commutative von Neumann regular by [Connell, 1963, Theorem 3]). It follows that $J((fRf)G) \subseteq (fJf)G \subseteq J(RG)$. So $j_2 \in J(RG)$. Let $\beta = \sum a_i e(1 + g_i) + j_1 + j_2 + w$ and $\gamma = h_1 + h_2$. Then $\beta \in J(RG)$. As $h_1 h_2 = h_2 h_1 = 0$, $\gamma^3 = h_1^3 + h_2^3 = h_1 + h_2 = \gamma$. So $\alpha = \beta + \gamma$ is semi-tripotent in RG . ■

Theorem 3.5: Let R be a ring and G be an abelian group. Then RG is semi-tripotent if and only if R is semi-tripotent and one of the following holds:

- i) $2 \in J(R)$ and G is a 2-group.
- ii) $3 \in J(R)$ and G is a direct product of a group of exponent 2 and a 3-group.
- iii) $2 \notin J(R)$ and $3 \notin J(R)$, and G is a group of exponent 2.

Proof ($\Rightarrow$). By Theorem 3.4, it suffices to show that G is torsion. As RG is semi-tripotent, R and hence $R/J(R)$ are semi-tripotent. So, by Theorem 3.2, $R/J(R) = A \times B$, where A is Boolean and B is zero or a subdirect product of $\mathbb{Z}_3$'s.

If $2 \in J(R)$, then G is a 2-group by Lemma 3.7.

If $3 \in J(R)$, then $3 = 0$ in $R/J(R)$, so $A = 0$. As $\mathbb{Z}_3$ is an image of $R/J(R)$, $\mathbb{Z}_3G$ is an image of RG and hence it is semi-tripotent. Assume that G is not torsion. Then $G/T(G)$ is non-trivial torsion-free where $T(G)$ is the torsion subgroup of G , and $\mathbb{Z}_3(G/T(G))$ is semi-tripotent (being an image of $\mathbb{Z}_3G$). So we can assume that G is torsion-free. If G has rank greater than 1, then G has a torsion-free quotient G' of rank 1. Since $\mathbb{Z}_3G'$ is semi-tripotent again, we can assume that G is of rank 1. So G is isomorphic to a subgroup of $(\mathbb{Q}, +)$. Take $g \in G$ such that $g^{-1} \neq g$. Since $g + g^{-1}$ is semi-tripotent in $\mathbb{Z}_3G$, there exist $j \in J(\mathbb{Z}_3G)$ and $b^3 = b \in \mathbb{Z}_3G$ such that $g + g^{-1} = j + b$. There exists a finitely generated subgroup G_1 of G such that $g, j, b, (1+j)^{-1} \in \mathbb{Z}_3G_1$. Because every finitely generated subgroup of $(\mathbb{Q}, +)$ is cyclic, G_1 is cyclic. Write $G_1 = \langle h \rangle$. Then $g = h^k, g^{-1} = h^{-k}$ for some positive integer k . There is a natural isomorphism $\mathbb{Z}_3\langle h \rangle \cong \mathbb{Z}_3[x, x^{-1}]$ with $h^k + h^{-k} \longleftrightarrow x^k + x^{-k}$. As $h^k + h^{-k} - b + 1 = 1 + j$ is a unit in $\mathbb{Z}_3\langle h \rangle$, $x^k + x^{-k} - f + 1$ is a unit in $\mathbb{Z}_3[x, x^{-1}]$, where $f^3 = f \in \mathbb{Z}_3[x, x^{-1}]$. But this is impossible because the tripotents of $\mathbb{Z}_3[x, x^{-1}]$ are in $\mathbb{Z}_3$ and the units of $\mathbb{Z}_3[x, x^{-1}]$ are in $\{ax^i : 0 \neq a \in \mathbb{Z}_3, i \in \mathbb{Z}\}$. The contradiction shows that G is torsion.

If $2 \notin J(R)$ and $3 \notin J(R)$, then $2 \neq 0$ and $3 \neq 0$ in $R/J(R)$, so $A \neq 0$ and $B \neq 0$. As an image of RG , AG is semi-tripotent. As $2 = 0$ in A , G is a 2-group by Lemma 3.7. As BG is semi-tripotent and $3 = 0$ in B , G is a group of exponent 2 by Lemma 3.8.

($\Leftarrow$). This follows from Theorem 3.4. ■

4. RINGS WITH $X^N - X$ IS NILPOTENT

In this chapter, we introduce $P_n(R)$ means that $a^n - a$ is nilpotent for all $a \in R$, and $Q_n(R)$ means that $R/J(R)$ has identity $x^n = x$ and $J(R)$ is nil, where $J(R)$ is the Jacobson radical of R for $n \geq 2$. We refer reader to [Diesl, 2013], [Chen and Sheibani, 2017] and [Zhou, 2018], for more details. The main purpose of this chapter is to characterize $P_n(R)$ for integers n up to 12 for $n = 2^k$ with k odd, and for $n = 2^k p$ where $k > 0$ and $p = 3, 5, 7$ or 9 . Also, for an arbitrary integer $n \geq 2$, we prove that $P_n(R)$ implies that $Q_n(R)$ and vice versa if and only if n is even with $n \equiv 1 \pmod{3}$, or n is odd with $n \not\equiv 1 \pmod{3}$ and $n \not\equiv 1 \pmod{8}$.

4.1 When $P_n(R) \Leftrightarrow Q_n(R)$ for every ring R ?

In this section, we determine the integers n such that $P_n(R) \Leftrightarrow Q_n(R)$ for every ring R . We begin with examples of rings R such that $P_n(R)$ and $Q_n(R)$ are not equivalent. Note that, if $A \in \mathbb{M}_n(\mathbb{F})$ is a nilpotent matrix where $\mathbb{F}$ is a field, then $A^n = 0$. Indeed, A is similar to a nilpotent Jordan matrix B which is strictly upper triangular, so $B^n = 0$ and hence, $A^n = 0$.

Example 4.1: Let $n > 1$ be an integer with $n \equiv 1 \pmod{3}$. If $R = \mathbb{M}_2(\mathbb{Z}_2)$, then $P_n(R)$ holds but $Q_n(R)$ fails.

Proof It is clear that $Q_n(R)$ fails. Write $n = 3k + 1$ where k is a positive integer. Let $A \in R$. We verify $(A^{3k+1} - A)^2 = 0$. As R is of characteristic 2, we have $(A^{3k+1} - A)^2 = A^2(A^{6k} - I_2)$. If A is nilpotent, then $A^2 = 0$. If A is invertible, then $A^{6k} - I_2 = 0$ as the unit group of units in R has exponent 6. If A is neither a unit nor a nilpotent, then A is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, so $A^2 = A$ and hence, $A^{3k+1} - A = 0$.

Example 4.2: Let $n > 1$ be an odd integer with $n \equiv 1 \pmod{8}$. If $R = \mathbb{M}_2(\mathbb{Z}_3)$, then $P_n(R)$ holds but $Q_n(R)$ fails.

Proof It is clear that $Q_n(R)$ fails. Write $n = 8k + 1$ where k is a positive integer. Let $A \in R$. We verify $(A^{8k+1} - A)^3 = 0$. As R is of characteristic 3, we have $(A^{8k+1} - A)^3 = A^3(A^{24k} - I_2)$. If A is nilpotent, then $A^2 = 0$. If A is invertible,

then $A^{24k} - I_2 = 0$ as the unit group of units in R has exponent 24. If A is neither a unit or a nilpotent, then A is similar to $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, where $0 \neq a \in Z_3$, so $A^3 = A$, i.e. $A(A^2 - I_2) = 0$. Hence, $A^{8k+1} - A = A((A^2)^{4k} - I_2) = A(A^2 - I_2)((A^2)^{4k-1} + \dots + A^2 + I_2) = 0$.

Lemma 4.1: [Levitzki, 1953, Theorem 2.1] Let R be a ring with $J(R) = 0$ such that every nonzero right ideal contains a nonzero idempotent. If $a^n = 0$ but $a^{n-1} \neq 0$, then there exists $e^2 = e \in RaR$ such that $eRe \cong \mathbb{M}_n(T)$ for some non-trivial ring T .

Lemma 4.2: Let R be a ring and $n > 1$. If $P_n(R)$ holds then $J(R)$ is nil and R is strongly π -regular. If in addition $Q_n(R)$ fails, then $R/J(R)$ contains a subring isomorphic to $\mathbb{M}_2(S)$ for some non-trivial ring S . The ring S can be chosen to be $\mathbb{Z}_2$ if n is even.

Proof Let $a \in R$. Then $(a - a^n)^k = 0$ for some $k \geq 1$, so $a^k(1 - a^{n-1})^k = 0$. It follows that $a^k \in a^{k+1}R \cap Ra^{k+1}$. So a is strongly π -regular. Thus R is strongly π -regular, so $J(R)$ is nil. Suppose that $Q_n(R)$ fails to hold. Then $\bar{a} - \bar{a}^n \neq \bar{0}$ for some $a \in R$. But $\bar{a} - \bar{a}^n$ is nilpotent in $\bar{R} := R/J(R)$. Thus, there exists $0 \neq y \in \bar{R}$ such that $y^2 = 0$. As $\bar{R}$ is strongly π -regular and semiprimitive, by Lemma 4.1 there exists $z^2 = z \in \bar{R}$ such that $z\bar{R}z \cong \mathbb{M}_2(S)$ for a non-trivial ring S .

If n is even, then $2 = (-1)^n - (-1)$ is nilpotent in R , so $2 = 0$ in $\bar{R}$ and in S . Thus, $\mathbb{M}_2(\mathbb{Z}_2)$ is a subring of $\mathbb{M}_2(S)$ and hence of $\bar{R}$. ■

By Examples 4.1 and 4.2, a necessary condition for an odd integer n to have the equivalence $P_n(R) \Leftrightarrow Q_n(R)$ for all rings R is $n \not\equiv 1 \pmod{3}$ and $n \not\equiv 1 \pmod{8}$.

Lemma 4.3: Let $n > 1$ be an odd integer with $n \not\equiv 1 \pmod{3}$. If R is a ring with $P_n(R)$, then $R = R_1 \times R_2$, where 3 is nilpotent in R_1 and $Q_n(R_2)$ holds.

Proof As $2^n - 2 > 1$ is nilpotent in R and n is odd, we can assume that $3^s k$ is nilpotent in R , where $s > 0$ and $k > 0$ is relatively prime to 3, so $3k$ is nilpotent. Therefore, by Chinese Remainder Theorem, $R = R_1 \times R_2$, where 3 is nilpotent in R_1 and k is nilpotent in R_2 . As $\gcd(3, k) = 1$, $3 \in U(R_2)$. Next we show that $Q_n(R_2)$ holds. Note that $P_n(R_2)$ holds. Assume on the contrary that $Q_n(R_2)$ fails. Then, by Lemma 4.2, $R_2/J(R_2)$ contains a subring isomorphic to $\mathbb{M}_2(S)$ for a non-trivial ring S . Let $w = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{M}_2(S)$. Then $w^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ and $w^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

If $n = 3l + 2$ where $l \geq 0$, then $w^n - w = (w^3)^l w^2 - w = w^2 - w = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$, so $(w^n - w)^2 = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$, which is not nilpotent (as $3 \in U(R_2)$), a contradiction to $P_n(R_2)$. If $n = 3l$ where $l \geq 1$, then $w^n - w = (w^3)^l - w = I_2 - w = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$ and $(w^n - w)^2 = \begin{pmatrix} 3 & -3 \\ 3 & 0 \end{pmatrix}$, which is not nilpotent, a contradiction to $P_n(R_2)$. Hence, $Q_n(R_2)$ holds. ■

Theorem 4.1: Let $n \geq 2$ be an integer.

- i) If n is even, then $P_n(R) \Leftrightarrow Q_n(R)$ for every ring R if and only if $n \not\equiv 1 \pmod{3}$.
- ii) If n is odd, then $P_n(R) \Leftrightarrow Q_n(R)$ for every ring R if and only if $n \not\equiv 1 \pmod{3}$ and $n \not\equiv 1 \pmod{8}$.

Proof (i) The necessity follows from Example 4.1. For the sufficiency, assume on the contrary that $P_n(R) \not\Leftrightarrow Q_n(R)$ for some ring R . Then $P_n(R)$ holds but $Q_n(R)$ fails. As n is even, $R/J(R)$ contains $\mathbb{M}_2(\mathbb{Z}_2)$ as a subring by Lemma 4.2. Let $w = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}_2)$. Then $w^3 = I_2$. As $n \not\equiv 1 \pmod{3}$, $n = 3k$ or $n = 3k + 2$ for some $k \geq 0$.

If $n = 3k$, then $w^n - w = (w^3)^k - w = I_2 - w = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, which is not nilpotent, a contradiction to $P_n(R)$. If $n = 3k + 2$, then $w^n - w = (w^3)^k w^2 - w = w^2 - w = I_2$, which is not nilpotent, again a contradiction to $P_n(R)$. Therefore, we have $P_n(R) \Leftrightarrow Q_n(R)$.

(ii) The necessity follows from Examples 4.1 and 4.2. For the sufficiency, suppose R is a ring with $P_n(R)$. As $n \not\equiv 1 \pmod{3}$, by Lemma 4.3 $R = R_1 \times R_2$ where 3 is nilpotent in R_1 , $P_n(R_1)$ and $Q_n(R_2)$ hold. So to show that $Q_n(R)$ holds, we just need to show that $Q_n(R_1)$ holds. Therefore, without loss of generality, we can assume that 3 is nilpotent in R . Thus $3 = 0$ in $R/J(R)$. Assume on the contrary that $Q_n(R)$ fails. By Lemma 4.2, $R/J(R)$ contains a subring isomorphic to $\mathbb{M}_2(S)$ for a non-trivial ring S . As $n \not\equiv 1 \pmod{8}$, we have $n \equiv 3 \pmod{4}$ or $n \equiv 5 \pmod{8}$.

- **Case 1:** $n \equiv 3 \pmod{4}$. Write $n = 4k + 3$ where $k \geq 0$. Let $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{M}_2(S)$. Then $w^2 = -I_2$, so $w^n - w = (w^2)^{2k+1} w - w = -I_2 w - w = w$, which is not nilpotent, a contradiction.

• *Case 2: $n \equiv 5 \pmod{8}$. Write $n = 5 + 8k$ with $k \geq 1$. Let $w = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbb{M}_2(S)$. Then $w^5 = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$, and $w^8 = I_2$. So $w^n - w = (w^8)^k w^5 - w = w^5 - w = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, which is a unit in $\mathbb{M}_2(S)$, a contradiction.*

4.2 Characterization of rings R with $P_n(R)$

In case n is even with $n \not\equiv 1 \pmod{3}$ or odd with $n \not\equiv 1 \pmod{3}$ and $n \not\equiv 1 \pmod{8}$, Theorem 4.1 characterizes rings R such that $a^n - a$ is nilpotent for all $a \in R$ as those rings R such that $R/J(R)$ has identity $x^n = x$ and $J(R)$ is nil. But, if n is not one of these integers, the characterization of rings R with $P_n(R)$ is unknown. In this section, for integers $n \leq 12$ or $n = 2^k$ with k odd or $n = 2^k p$ where $k \geq 1$ and $p = 3, 5, 7$ or 9 , we present characterizations for rings R such that $a^n - a$ is nilpotent for all $a \in R$. Theorem 4.1(1) with $n = 2$ gives an immediate consequence.

Corollary 4.1: [Kosan et al., 2016] Let R be a ring. Then $a^2 - a$ is nilpotent for all $a \in R$ if and only if $J(R)$ is nil and $R/J(R)$ is Boolean.

Lemma 4.4: Let p be a prime. The following are equivalent for a ring R :

- i) $p \in N(R)$ and $a^p - a$ is nilpotent for all $a \in R$.*
- ii) $J(R)$ is nil and $R/J(R)$ is a subdirect product of $\mathbb{Z}_p$'s.*

Proof (ii) $\Rightarrow$ (i). It is clear.

(i) $\Rightarrow$ (ii). The implication is true for $p = 2$ by Corollary 4.1. So we may assume that $p \geq 3$. As $p = 0$ in $R/J(R)$, to show (2), it suffices to show that $Q_p(R)$ holds by [Lam, 1991, Ex.12.11]. Assume on the contrary that $Q_p(R)$ fails. Then, by Lemma 4.2, $R/J(R)$ contains a subring isomorphic to $\mathbb{M}_2(S)$ for some non-trivial ring S . As $p = 0$ in S , S contains $\mathbb{Z}_p$ as a subring, so $R/J(R)$ contains a subring isomorphic to $\mathbb{M}_2(\mathbb{Z}_p)$. Let $k = \frac{p-1}{2}$. As there are at most k elements in $\mathbb{Z}_p$ whose k^{th} power is 1, there exists $0 \neq m \in \mathbb{Z}_p$ such that $m^k \neq 1$. Let $w = \begin{pmatrix} 0 & m \\ 1 & 0 \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}_p)$. Then $w^2 = mI_2$, so $w^p - w = w(w^{2k} - I_2) = w((w^2)^k - I_2) = w((mI_2)^k - I_2) = w(m^k - 1)I_2$, which is a unit in $\mathbb{M}_2(\mathbb{Z}_p)$, so $w^p - w$ is not nilpotent, a contradiction. ■

Lemma 4.5: Let $a \in R$ with $2 \in N(R)$ and let $n > 1$ be an odd number. Then $a^n - a$ is nilpotent if and only if $a^{\frac{n+1}{2}} - a$ is nilpotent.

Proof Note that $(a^{\frac{n+1}{2}} - a)^2 = a(a^n - a) + 2(a^2 - a^{\frac{n+3}{2}})$ and $a^n - a = (a^{\frac{n+1}{2}} - a)(a^{\frac{n-1}{2}} + 1)$. So the equivalence “ $\Leftrightarrow$ ” follows. ■

Corollary 4.2: [Chen and Sheibani, 2017] Let R be a ring. The following are equivalent:

- i) $a^3 - a$ is nilpotent for all $a \in R$.
- ii) $R = R_1 \times R_2$, where $R_1/J(R_1)$ is Boolean with $J(R_1)$ nil, and R_2 is zero or $R_2/J(R_2)$ is a subdirect product of $\mathbb{Z}_3$'s with $J(R_2)$ nil.

Proof ii) $\Rightarrow$ i). It is clear.

i) $\Rightarrow$ ii). As $6 = 2^3 - 2$ is nilpotent in R , $2^n 3^n = 0$ for some $n > 0$. So $R = R_1 \times R_2$ where $R_1 \cong R/2^n R$ and $R_2 \cong R/3^n R$. Thus, $2 \in N(R_1)$ and $3 \in N(R_2)$. As $a^3 - a$ is nilpotent for all $a \in R_2$, by Lemma 4.4 $J(R_2)$ is nil and $R_2/J(R_2)$ is a subdirect product of $\mathbb{Z}_3$'s. As $a^3 - a$ is nilpotent for all $a \in R_1$ and $2 \in N(R_1)$, $a^2 - a$ is nilpotent for all $a \in R_1$ by Lemma 4.5. So, by Corollary 4.1, $J(R_1)$ is nil and $R_1/J(R_1)$ is Boolean. ■

We see from Theorem 4.1(1) that $n = 4$ is the smallest positive even number such that $P_n(R) \not\cong Q_n(R)$ for some ring R . Note that, for a prime p and a division ring D of characteristic p , if $a^{p^k} = a$ for all $a \in D$, then D is a field by Jacobson's theorem on commutativity and is finite. Moreover, $D = \mathbb{F}_{p^s}$ with $s|k$. Also note that, if R is a right primitive ring that is not simple Artinian, then for any integer $k > 0$ there exists a subring R_k of R such that R_k has an image isomorphic to $\mathbb{M}_k(D)$ where D is a division ring.

Theorem 4.2: The following are equivalent for a ring R :

- i) $a^4 - a$ is nilpotent for all $a \in R$.
- ii) $J(R)$ is nil and $R/J(R)$ is a subdirect product of rings isomorphic to $\mathbb{Z}_2$ or $\mathbb{F}_4$ or $\mathbb{M}_2(\mathbb{Z}_2)$.

Proof (i) $\Rightarrow$ (ii). It follows from (i) that $J(R)$ is nil. As $2 = (-1)^4 - (-1)$ is nilpotent, $2 = 0$ in $\bar{R} = R/J(R)$. Let $\bar{R}$ be a subdirect product of right primitive rings R_α . If R_α is not simple Artinian, then, by the structure theorem for right primitive rings, $\mathbb{M}_3(D)$ is an image of a subring of R_α (where D is a division ring). For $x = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathbb{M}_3(D)$, then $x^4 - x = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ which is not nilpotent (it is indeed a unit in $\mathbb{M}_3(D)$). This contradiction shows that each R_α is a simple Artinian ring, say $R_\alpha = \mathbb{M}_n(D)$ with D a division ring. As seen above, it must be that $n \leq 2$.

If $n = 1$, then $R_\alpha = D$. As $a^4 = a$ for all $a \in D$, D is a field by a Jacobson's Theorem on commutativity and is finite. As D is a vector space over $\mathbb{Z}_2$, $D = \mathbb{F}_{2^s}$ with $s|2$. This is, $D \cong \mathbb{Z}_2$ or $D \cong \mathbb{F}_4$.

If $n = 2$, then $R_\alpha \cong \mathbb{M}_2(D)$. As D is isomorphic to a subring of $\mathbb{M}_2(D)$, $a^4 = a$ for all $a \in D$. As above, $D \cong \mathbb{Z}_2$ or $D \cong \mathbb{F}_4$. We next show that $D \not\cong \mathbb{F}_4$. Assume that $D \cong \mathbb{F}_4$. Take $a \in D$ such that $a^2 = a + 1$, and let $x = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{M}_2(D)$. Then $x^4 - x = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} - \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, which is not nilpotent. The contradiction shows that $D \not\cong \mathbb{F}_4$.

(ii) $\Rightarrow$ (i). If $R = \mathbb{Z}_2$ or $R = \mathbb{F}_4$ or $R = \mathbb{M}_2(\mathbb{Z}_2)$, then $a^4 = a$ for all $a \in R$, so (i) follows.

Corollary 4.3: [Zhou, 2018] Let R be a ring. The following are equivalent:

- i) $a^5 - a$ is nilpotent for all $a \in R$.
- ii) $R = R_1 \times R_2 \times R_3$, where $R_1/J(R_1)$ is Boolean with $J(R_2)$ nil, R_2 is zero or $R_2/J(R_2)$ is a subdirect product of $\mathbb{Z}_3$'s with $J(R_2)$ nil, and R_3 is zero or $R_3/J(R_3)$ is a subdirect product of $\mathbb{Z}_5$'s with $J(R_3)$ nil.

Proof ii) $\Rightarrow$ i). It is clear.

i) $\Rightarrow$ ii). As $30 = 2^5 - 2$ is nilpotent in R , $2^n 3^n 5^n = 0$ for some $n > 0$. So $R = R_1 \times R_2 \times R_3$ where $R_1 \cong R/2^n R$, $R_2 \cong R/3^n R$ and $R_3 \cong R/5^n R$. Thus, $2 \in N(R_1)$, $3 \in N(R_2)$ and $5 \in N(R_3)$.

As $a^5 - a$ is nilpotent for all $a \in R_3$, by Lemma 4.4 $J(R_3)$ is nil and $R_3/J(R_3)$ is a subdirect product of $\mathbb{Z}_5$'s.

As $a^5 - a$ is nilpotent for all $a \in R_1$ and $2 \in N(R_1)$, $a^3 - a$ is nilpotent for all $a \in R_1$ by Lemma 4.4, and further $a^2 - a$ is nilpotent for all $a \in R_1$ again by Lemma 4.4. So, by Corollary 4.1, $J(R_1)$ is nil and $R_1/J(R_1)$ is Boolean.

For $a \in R_2$, $[(a+1)^5 - (a+1)] - (a^5 - a) = (-a^4 + a^3 + a^2 - a) + 3(2a^4 + 3a^3 + 3a^2 + 2a)$. It follows that $-a^4 + a^3 + a^2 - a$ is nilpotent for all $a \in R_2$. So $-(a+1)^4 + (a+1)^3 + (a+1)^2 - (a+1)$ is nilpotent for all $a \in R_2$. But $-(a+1)^4 + (a+1)^3 + (a+1)^2 - (a+1) = -(a^4 - a^2) - 3(a^3 + a^2)$, so $a^4 - a^2$ is nilpotent for all $a \in R_2$. Thus, since $(a^3 - a)^2 = (a^4 - a^2) + (a^5 - a)a + 3(a^2 - a^4)$, we see that $a^3 - a$ is nilpotent for all $a \in R_2$. So, by Lemma 4.4, R_2 is zero or $R_2/J(R_2)$ is a subdirect product of $\mathbb{Z}_3$'s with $J(R_2)$ nil. ■

For an ideal I of a ring R and $a, b \in R$, we write $a \equiv b \pmod{I}$ to mean that $a - b \in I$.

Corollary 4.4: Let $n = 2^k 3$ with $k > 0$. The following are equivalent for a ring R :

- i) $a^n - a$ is nilpotent for all $a \in R$.
- ii) $a^2 - a$ is nilpotent for all $a \in R$.
- iii) $J(R)$ is nil and $R/J(R)$ is Boolean.

Proof i) $\Rightarrow$ ii). By (1), $2 = (-1)^n - (-1)$ is nilpotent in R . For $a \in R$,

$$\begin{aligned}
((a+1)^n - (a+1)) - (a^n - a) &\equiv ((a+1)^{2^k})^3 + a^n + 1 \pmod{2R} \\
&\equiv (a^{2^k} + 1)^2(a^{2^k} + 1) + a^n + 1 \pmod{2R} \\
&\equiv (a^{2^{k+1}} + 1)(a^{2^k} + 1) + a^n + 1 \pmod{2R} \quad (4.1) \\
&\equiv a^{2^{k+1}} + a^{2^k} \pmod{2R} \\
&\equiv (a^2 - a)^{2^k} \pmod{2R},
\end{aligned}$$

so $a^2 - a$ is nilpotent.

ii) $\Leftrightarrow$ iii). This is Corollary 4.1.

iii) $\Rightarrow$ i). This is because $a^n - a = (a^2 - a)(a^{n-2} + a^{n-3} + \cdots + a + 1)$ for all $a \in R$. ■

By the previous corollary, for a ring R , $a^6 - a$ is nilpotent for all $a \in R$ if and only if $J(R)$ is nil and $R/J(R)$ is Boolean.

By Theorem 4.1, $n = 7$ is the smallest positive odd number such that $P_n(R) \not\Rightarrow Q_n(R)$ for some ring R .

Corollary 4.5: *The following are equivalent for a ring R :*

- i) $a^7 - a$ is nilpotent for all $a \in R$.
- ii) $R = R_1 \times R_2 \times R_3$, where R_1 is zero or $R_1/J(R_1)$ is a subdirect product of rings isomorphic to $\mathbb{Z}_2$ or $\mathbb{F}_4$ or $\mathbb{M}_2(\mathbb{F}_2)$ with $J(R_1)$ nil, R_2 is zero or $R_2/J(R_2)$ is a subdirect product of $\mathbb{Z}_3$'s with $J(R_2)$ nil, and R_3 is zero or $R_3/J(R_3)$ is a subdirect product of $\mathbb{Z}_7$'s with $J(R_3)$ nil.

Proof ii) $\Rightarrow$ i). It is clear.

i) $\Rightarrow$ ii). As $2 \cdot 3^2 \cdot 7 = 2^7 - 2 \in R$ is nilpotent, $2^n 3^n 7^n = 0$ for some $n > 0$, so $R = R_1 \times R_2 \times R_3$ where $R_1 \cong R/2^n R$, $R_2 \cong R/3^n R$ and $R_3 \cong R/7^n R$. Thus, $2 \in N(R_1)$, $3 \in N(R_2)$ and $7 \in N(R_3)$.

As $a^7 - a$ is nilpotent for all $a \in R_3$, by Lemma 4.4 $J(R_3)$ is nil and $R_3/J(R_3)$ is a subdirect product of $\mathbb{Z}_7$'s.

By Lemma 4.5, $a^4 - a$ is nilpotent for all $a \in R_1$. So, by Theorem 4.2, R_1 is zero or $R_1/J(R_1)$ is a subdirect product of rings isomorphic to $\mathbb{Z}_2$ or $\mathbb{F}_4$ or $\mathbb{M}_2(\mathbb{Z}_2)$ with $J(R_1)$ nil.

For $a \in R_2$, $(a^3 - a)^3 = (a^7 - a)a^2 + 3(a^5 - a^7)$, so it follows that $a^3 - a$ is nilpotent for all $a \in R_2$. Hence, by Lemma 4.4, R_2 is zero or $R_2/J(R_2)$ is a subdirect product of $\mathbb{Z}_3$'s with $J(R_2)$ nil. ■

Corollary 4.6: *The following are equivalent for a ring R :*

- i) $a^8 - a$ is nilpotent for all $a \in R$.
- ii) $J(R)$ is nil and $R/J(R)$ is a subdirect product of rings isomorphic to $\mathbb{Z}_2$ or $\mathbb{F}_8$.

Proof ii) $\Rightarrow$ i). It is clear.

i) $\Rightarrow$ ii). It follows from (i) that $J(R)$ is nil. As $2 = (-1)^8 - (-1)$ is nilpotent, $2 = 0$ in $\bar{R} = R/J(R)$. Let $\bar{R}$ be a subdirect product of right primitive rings R_α . If R_α

is not simple Artinian, then, by the structure theorem for right primitive rings, $\mathbb{M}_2(D)$ is an image of a subring of R_α (where D is a division ring of characteristic 2). For $x = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{M}_2(D)$, then $x^8 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, so $x^8 - x = I_2$ which is not nilpotent. This contradiction shows that each R_α is a simple Artinian ring, say $R_\alpha = \mathbb{M}_n(D)$ with D a division ring of characteristic 2. As seen above, it must be that $n = 1$. So, $R_\alpha = D$. As $a^8 = a$ for all $a \in D$, D is a field by a Jacobson's theorem on commutativity and it is finite. As D is a vector space over $\mathbb{F}_2$, $D = \mathbb{F}_{2^s}$ with $s|3$. This is, $D \cong \mathbb{Z}_2$ or $D \cong \mathbb{F}_8$. ■

Corollary 4.7: The following are equivalent for a ring R :

- i) $a^9 - a$ is nilpotent for all $a \in R$.
- ii) $R = R_1 \times R_2 \times R_3$, where R_1 is zero or $R_1/J(R_1)$ is Boolean with $J(R_1)$ nil, R_2 is zero or $R_2/J(R_2)$ is a subdirect product of rings isomorphic to $\mathbb{Z}_3$ or $\mathbb{F}_9$ with $J(R_2)$ nil, and R_3 is zero or $R_3/J(R_3)$ is a subdirect product of $\mathbb{Z}_5$'s with $J(R_3)$ nil.

Proof ii) $\Rightarrow$ i). It is clear.

i) $\Rightarrow$ ii). As $2^9 - 2 = 2 \cdot 3 \cdot 5 \cdot 17 \in R$ is nilpotent and $3^9 - 3 = 2^5 \cdot 3 \cdot 5 \cdot 41 \in R$ is nilpotent, it follows that $2 \cdot 3 \cdot 5$ is nilpotent in R . So $2^n 3^n 5^n = 0$ for some $n > 0$, and hence $R = R_1 \times R_2 \times R_3$ where $R_1 \cong R/2^n R$, $R_2 \cong R/3^n R$ and $R_3 \cong R/5^n R$. Thus, $2 \in N(R_1)$, $3 \in N(R_2)$ and $5 \in N(R_3)$.

As $a^9 - a$ is nilpotent for all $a \in R_1$ and $2 \in N(R_1)$, $a^5 - a$ is nilpotent for all $a \in R_1$ by Lemma 4.5, and further $a^3 - a$ is nilpotent for all $a \in R_1$ again by Lemma 4.5. So, by Corollary 4.2, R_1 is zero or $R_1/J(R_1)$ is Boolean with $J(R_1)$ nil.

By Lemma 4.2, $J(R_2)$ is nil. Let $R_2/J(R_2)$ be a subdirect product of right primitive rings R_α . If R_α is not simple Artinian, then $\mathbb{M}_2(D)$ is an image of a subring of R_α (where D is a division ring). Let $w = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathbb{M}_3(D)$. Then $x^9 - x = \begin{pmatrix} 0 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, which is not nilpotent (it is indeed a unit in $\mathbb{M}_3(D)$). This contradiction shows that each R_α is a simple Artinian ring, say $R_\alpha = \mathbb{M}_n(D)$ with D a division ring. As seen above, it must be that $n \leq 2$. If $n = 1$, then $R_\alpha = D$. As $a^9 = a$ for all $a \in D$, $D = \mathbb{F}_3$ with $s|2$. This is, $D \cong \mathbb{Z}_3$ or $D \cong \mathbb{F}_9$. If

$n = 2$, then $R_\alpha \cong \mathbb{M}_2(D)$. As D is isomorphic to a subring of $\mathbb{M}_2(D)$, $a^9 = a$ for all $a \in D$. As above, $D \cong \mathbb{Z}_3$ or $D \cong \mathbb{F}_9$. We next show that $D \not\cong \mathbb{F}_9$. Assume that $D \cong \mathbb{F}_9$. Take $a \in D$ such that $a^2 = 2a + 1$, and let $x = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \in \mathbb{M}_2(D)$. Then $x^9 - x = \begin{pmatrix} -a & -1 \\ 1 & a \end{pmatrix}$ which is not nilpotent (it is indeed a unit). The contradiction shows that $D \not\cong \mathbb{F}_9$.

For $a \in R_3$, we have

$$(a+1)^9 - (a+1) \equiv a^9 - a^8 + a^7 - a^6 + a^5 + a^4 - a^3 + a^2 + 3a \pmod{5R}, \text{ so}$$

$$[(a+1)^9 - (a+1)] - (a^9 - a) \equiv -a^8 + a^7 - a^6 + a^5 + a^4 - a^3 + a^2 - a \pmod{5R}. \quad (4.2)$$

Thus,

$$\begin{aligned} & [(a+1)^9 - (a+1)] + \{[(a+1)^9 - (a+1)] - (a^9 - a)\}a \\ & \equiv (a^9 - a^8 + a^7 - a^6 + a^5 + a^4 - a^3 + a^2 + 3a) \\ & \quad + (-a^8 + a^7 - a^6 + a^5 + a^4 - a^3 + a^2 - a)a \pmod{5R} \\ & = 2a^5 + 3a \equiv 2(a^5 - a) \pmod{5R}. \end{aligned} \quad (4.3)$$

So $a^5 - a$ is nilpotent for all $a \in R_3$. Hence, by Lemma 4.4, R_3 is zero or $R_3/J(R_3)$ is a subdirect product of $\mathbb{Z}_5$'s with $J(R_3)$ nil. ■

For $x, y \in R$, we use $x \smile y$ to mean that $x - y$ is nilpotent. Note that if $x \smile y \smile z$ and $x - y, y - z$ commute, then $x \smile z$.

Corollary 4.8: Let $n = 2^k 5$ with $k > 0$.

- i) If k is even, then $a^n - a$ is nilpotent for all $a \in R$ if and only if $a^2 - a$ is nilpotent for all $a \in R$, i.e., $J(R)$ is nil and $R/J(R)$ is Boolean.
- ii) If k is odd, then $a^n - a$ is nilpotent for all $a \in R$ if and only if $a^4 - a$ is nilpotent for all $a \in R$, i.e., $J(R)$ is nil and $R/J(R)$ is a subdirect product of rings isomorphic to $\mathbb{Z}_2$ or $\mathbb{F}_4$ or $\mathbb{M}_2(\mathbb{Z}_2)$.

Proof Suppose that $a^n - a$ is nilpotent for all $a \in R$. Then $2 = (-1)^n - (-1)$ is nilpotent in R . Write $n = m(4 + 1)$ where $m = 2^k$. For $a \in R$,

$$\begin{aligned}
((a+1)^n - (a+1)) - (a^n - a) &\equiv ((a+1)^m)^{4+1} + a^n + 1 \pmod{2R} \\
&\equiv (a^m + 1)^4(a^m + 1) + a^n + 1 \pmod{2R} \\
&\equiv (a^{4m} + 1)(a^m + 1) + a^n + 1 \pmod{2R} \quad (4.4) \\
&\equiv a^{4m} + a^m \pmod{2R} \\
&\equiv (a^4 - a)^{2^k} \pmod{2R},
\end{aligned}$$

so $a^4 - a$ is nilpotent. Thus $a^5 \smile a^2$, and we have $a^n = (a^5)^{2^k} \smile (a^2)^{2^k} = (a^4)^{2^{k-1}} \smile a^{2^{k-1}} = (a^2)^{2^{k-2}}$. So $a^n \smile (a^2)^{2^k} \smile (a^2)^{2^{k-2}} \smile \dots \smile (a^2)^{2^{k-2t}}$ for any t with $2t < k$. If k is even, take t such that $k - 2t = 2$, so $(a^2)^{2^{k-2t}} = (a^2)^{2^2} = (a^4)^2 \smile a^2$. As $a \smile a^n$, it follows that $a^2 - a$ is nilpotent.

For the converse, note that $a^2 - a$ nilpotent clearly implies that $a^n - a$ is nilpotent. If $a^4 - a$ is nilpotent and k is odd, take t such that $k - 2t = 1$, so $(a^2)^{2^{k-2t}} = (a^2)^2 = a^4 \smile a$. As argued above, $a^n \smile a$, so $a^n - a$ is nilpotent. ■

By the previous corollary, for a ring R , $a^{10} - a$ is nilpotent for all $a \in R$ if and only if $J(R)$ is nil and $R/J(R)$ is a subdirect product of rings isomorphic to $\mathbb{Z}_2$ or $\mathbb{F}_4$ or $\mathbb{M}_2(\mathbb{Z}_2)$, and $a^{20} - a$ is nilpotent for all $a \in R$ if and only if $J(R)$ is nil and $R/J(R)$ is Boolean.

Corollary 4.9: The following are equivalent for a ring R :

- i) $a^{11} - a$ is nilpotent for all $a \in R$.
- ii) $R = R_1 \times R_2 \times R_3$, where R_1 is zero or $R_1/J(R_1)$ is Boolean with $J(R_1)$ nil, R_2 is zero or $R_2/J(R_2)$ is a subdirect product of $\mathbb{Z}_3$'s with $J(R_2)$ nil, and R_3 is zero or $R_3/J(R_3)$ is a subdirect product of $\mathbb{Z}_{11}$'s with $J(R_3)$ nil.

Proof ii) $\Rightarrow$ i). It is clear.

i) $\Rightarrow$ ii). As $2^{11} - 2 = 2 \cdot 3 \cdot 11 \cdot 31 \in R$ is nilpotent and $3^{11} - 3 = 2^3 \cdot 3 \cdot 11^2 \cdot 61 \in R$ is nilpotent, it follows that $2 \cdot 3 \cdot 11$ is nilpotent in R . So $2^n 3^n 11^n = 0$ for some $n > 0$, and hence $R = R_1 \times R_2 \times R_3$ where $R_1 \cong R/2^n R$, $R_2 \cong R/3^n R$ and $R_3 \cong R/11^n R$. Thus, $2 \in N(R_1)$, $3 \in N(R_2)$ and $11 \in N(R_3)$.

As $a^{11} - a$ is nilpotent for all $a \in R_1$ and $2 \in N(R_1)$, $a^6 - a$ is nilpotent for all $a \in R_1$ by Lemma 4.5. So, by Corollary 4.4, R_1 is zero or $R_1/J(R_1)$ is Boolean with $J(R_1)$ nil.

For $a \in R_2$, we have

$$(a+1)^{11} - (a+1) \equiv a^{11} - a^{10} + a^9 + a^2 + a \pmod{3R}, \text{ and} \quad (4.5)$$

$$[(a-1)^{11} - (a-1)] - (a^{11} - a) \equiv a^{10} + a^9 - a^2 - a \pmod{3R}.$$

Thus,

$$\begin{aligned} & -[(a+1)^{11} - (a+1)] + (1+a)\{[(a-1)^{11} - (a-1)] - (a^{11} - a)\} \\ & \equiv -(a^3 - a) \pmod{3R}. \end{aligned} \quad (4.6)$$

So $a^3 - a$ is nilpotent for all $a \in R_2$. Hence, by Lemma 4.4, R_2 is zero or $R_2/J(R_2)$ is a subdirect product of $\mathbb{Z}_3$'s with $J(R_2)$ nil.

As $a^{11} - a$ is nilpotent for all $a \in R_3$ and $11 \in N(R_3)$, R_3 is zero or $R_3/J(R_3)$ is a subdirect product of $\mathbb{Z}_{11}$'s with $J(R_3)$ nil by Lemma 4.4. ■

Corollary 4.10: Let $n = 2^k 7$ with $k > 0$.

- i) If k is odd, then $a^n - a$ is nilpotent for all $a \in R$ if and only if $a^2 - a$ is nilpotent for all $a \in R$, i.e., $J(R)$ is nil and $R/J(R)$ is Boolean.
- ii) If k is even, then $a^n - a$ is nilpotent for all $a \in R$ if and only if $a^4 - a$ is nilpotent for all $a \in R$, i.e., $J(R)$ is nil and $R/J(R)$ is a subdirect product of rings isomorphic to $\mathbb{Z}_2$ or $\mathbb{F}_4$ or $M_2(\mathbb{Z}_2)$.

Proof Suppose that $a^n - a$ is nilpotent for all $a \in R$. Then $2 = (-1)^n - (-1)$ is nilpotent in R . Write $n = m(4 + 2 + 1)$ where $m = 2^k$. For $a \in R$,

$$[(a+1)^n - (a+1)](1+a^m) - (a^n - a)a^m \equiv a^m + a \pmod{2R}, \quad (4.7)$$

so $a^m - a$ is nilpotent. It follows that $a^{7m} - a^7 = (a^m)^7 - a^7$ is nilpotent for all $a \in R$. So $a^7 - a = (a^7 - a^{7m}) + (a^{7m} - a)$ is nilpotent. As $2 \in N(R)$, $a^4 - a$ is nilpotent for all $a \in R$ by Lemma 4.5. Thus $a^4 \smile a$, and we have $a^n = (a^7)^{2^k} \smile a^{2^k} = (a^4)^{2^{k-2}} \smile$

$a^{2^{k-2}}$. A similar argument shows $a^n \smile a^{2^k} \smile a^{2^{k-2}} \smile \dots \smile a^{2^{k-2t}}$ for any t with $2t < k$. If k is odd, take t such that $k - 2t = 1$, so $a^{2^{k-2t}} = a^2$. As $a \smile a^n$, it follows that $a^2 - a$ is nilpotent.

For the converse, note that $a^2 - a$ nilpotent clearly implies that $a^n - a$ is nilpotent. If $a^4 - a$ is nilpotent and k is even, take t such that $k - 2t = 2$, so $a^{2^{k-2t}} = a^{2^2} = a^4 \smile a$. As $a^n \smile a^{2^{k-2t}}$, we have $a^n \smile a$, i.e., $a^n - a$ is nilpotent. ■

By the previous corollary, for a ring R , $a^{14} - a$ is nilpotent for all $a \in R$ if and only if $J(R)$ is nil and $R/J(R)$ is Boolean.

Corollary 4.11: Let $n = 2^k 9$ with $k > 0$.

- i) If $k \equiv 2 \pmod{3}$, then $a^n - a$ is nilpotent for all $a \in R$ if and only if $a^8 - a$ is nilpotent for all $a \in R$, i.e., R is zero or $R/J(R)$ is a subdirect product of rings isomorphic to $\mathbb{Z}_2$ or $\mathbb{F}_8$ with $J(R)$ nil.
- ii) If $k \not\equiv 2 \pmod{3}$, then $a^n - a$ is nilpotent for all $a \in R$ if and only if $a^2 - a$ is nilpotent for all $a \in R$, i.e., R is zero or $R/J(R)$ is Boolean with $J(R)$ nil.

Proof Suppose that $a^n - a$ is nilpotent for all $a \in R$. Then $2 = (-1)^n - (-1)$ is nilpotent in R . Write $n = m(8 + 1)$ where $m = 2^k$. For $a \in R$,

$$\begin{aligned}
[(a + 1)^n - (a + 1)] - (a^n - a) &\equiv ((a + 1)^m)^9 + a^n + 1 \pmod{2R} \\
&\equiv (a^m + 1)^8(a^m + 1) + a^n + 1 \pmod{2R} \\
&\equiv (a^{8m} + 1)(a^m + 1) + a^n + 1 \pmod{2R} \quad (4.8) \\
&\equiv a^{8m} + a^m \pmod{2R} \\
&\equiv (a^8 - a)^m \pmod{2R}
\end{aligned}$$

so $a^8 - a$ is nilpotent. Thus $a^8 \smile a$, and we have $a^n = (a^9)^{2^k} \smile (a^2)^{2^k} = a^{2^{k+1}} = (a^8)^{2^{(k+1)-3}} \smile a^{2^{(k+1)-3}}$. A similar argument shows $a^n \smile a^{2^{k+1}} \smile a^{2^{(k+1)-3}} \smile \dots \smile a^{2^{(k+1)-3t}}$ for any t with $3t < k + 1$.

Suppose $k \not\equiv 2 \pmod{3}$. Then $k = 3l$ or $k = 3l + 1$ for some $l > 0$. If $k = 3l$, then $a \smile a^n \smile a^{2^{(k+1)-3l}} = a^2$, so $a^2 - a$ is nilpotent.

If $k = 3l + 1$, then $a \smile a^n \smile a^{2^{(k+1)-3l}} = a^4$, so $a^4 - a$ is nilpotent. Thus, as $a^2 - a \smile (a^8 - a) - (a^4 - a)^2$, $a^2 - a$ is nilpotent.

For the converse, note that $a^2 - a$ nilpotent clearly implies that $a^n - a$ is nilpotent. If $a^8 - a$ is nilpotent and $k = 3l + 2$ for some $l > 0$, then $a^n \smile a^{2^{(k+1)-3l}} = a^8 \smile a$. So, $a^n \smile a$, i.e., $a^n - a$ is nilpotent. ■

Corollary 4.6 is a special case of the next proposition.

Proposition 4.1: Let $n = 2^k$ where $k > 0$ is odd. The following are equivalent for a ring R :

- i) $a^n - a$ is nilpotent for all $a \in R$.
- ii) $J(R)$ is nil and $R/J(R)$ is a subdirect product of rings isomorphic to $\mathbb{F}_{2^s}$ where $s|k$.

Proof ii) $\Rightarrow$ i). Write $k = ms$. If $a \in \mathbb{F}_{2^s}$, then $a^{2^s} = a$, so $a^{2^k} = a^{2^{ms}} = (a^{2^s})^{2^{(m-1)s}} = a^{2^{(m-1)s}} = \dots = a^{2^s} = a$. Therefore, if $a \in R$, then $a^n - a \in J(R)$, so $a^n - a$ is nilpotent.

i) $\Rightarrow$ ii). It follows from (i) that $J(R)$ is nil. As $2 = (-1)^n - (-1)$ is nilpotent, $2 = 0$ in $\bar{R} = R/J(R)$. Let $\bar{R}$ be a subdirect product of right primitive rings R_α . If R_α is not simple Artinian, then $\mathbb{M}_2(D)$ is an image of a subring of R_α (where D is a division ring of characteristic 2). For $x = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{M}_2(D)$, we have $x^3 = I_2$. As k is odd, $2^k = 3l + 2$ for some $l \geq 0$. Thus $x^{2^k} = (x^3)^l x^2 = x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, so $x^{2^k} - x = I_2$ which is not nilpotent. This contradiction shows that each R_α is a simple Artinian ring, say $R_\alpha = \mathbb{M}_l(D)$ with D a division ring of characteristic 2. As seen above, it must be that $l = 1$. So, $R_\alpha = D$. As $a^n = a$ for all $a \in D$, $D \cong \mathbb{F}_2$ with $s|k$. ■

For $n = 2^k$ with k even, we do not have a characterization of rings R with $P_n(R)$. Next we extend the argument in proving Theorem 4.2 to characterize rings R such that $a^{2^4} - a$ is nilpotent for all $a \in R$.

Proposition 4.2: The following are equivalent for a ring R :

- i) $a^{16} - a$ is nilpotent for all $a \in R$.

ii) $J(R)$ is nil and $R/J(R)$ is a subdirect product of rings isomorphic to either of the rings $\mathbb{Z}_2, \mathbb{F}_4, \mathbb{F}_{16}, \mathbb{M}_2(\mathbb{F}_2)$ and $\mathbb{M}_2(\mathbb{F}_4)$.

Proof ii) $\Rightarrow$ i). In view of Theorem 4.2, it suffices to show that $x^{16} - x$ is nilpotent for all $x \in \mathbb{M}_2(\mathbb{F}_4)$. It is known that x is similar to y where $y = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ or $y = \begin{pmatrix} 0 & u \\ 1 & v \end{pmatrix}$. As $u, v \in \mathbb{F}_4, u^4 = u$ and $v^4 = v$.

If $y = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$, then $y^{16} - y = 0$ is nilpotent, so $x^{16} - x$ is nilpotent.

If $y = \begin{pmatrix} 0 & u \\ 1 & v \end{pmatrix}$, we have $y^{16} = \begin{pmatrix} u^2 + u^2v^3 & uv \\ v^3 & u^2v^3 + u^2 + v \end{pmatrix}$. When $v = 0$, $y^{16} = \begin{pmatrix} u^2 & 0 \\ 0 & u^2 \end{pmatrix}$, $y^{16} - y = \begin{pmatrix} u^2 & u \\ 1 & u^2 \end{pmatrix}$, and so $(y^{16} - y)^2 = \begin{pmatrix} u^4 + u & u^3 + u^3 \\ u^2 + u^2 & u^4 + u \end{pmatrix} = 0$; so $y^{16} - y$ is nilpotent and hence $x^{16} - x$ is nilpotent. When $v \neq 0$, then $v^3 = 1$, and $y^{16} = \begin{pmatrix} 0 & uv \\ 1 & v \end{pmatrix}$, and so $y^{16} - y = \begin{pmatrix} 0 & uv + u \\ 0 & 0 \end{pmatrix}$ is nilpotent; so $x^{16} - x$ is nilpotent. Therefore, $x^{16} - x$ is nilpotent for all $x \in \mathbb{M}_2(\mathbb{F}_4)$.

i) $\Rightarrow$ ii). It follows from (i) that $J(R)$ is nil. As $2 = (-1)^{16} - (-1)$ is nilpotent, $2 = 0$ in $\bar{R} = R/J(R)$. Let $\bar{R}$ be a subdirect product of right primitive rings R_α . If R_α is not simple Artinian, then, by the structure theorem for right primitive rings, $\mathbb{M}_3(D)$ is an image of a subring of R_α (where D is a division ring). Let $x = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathbb{M}_3(D)$. Then $x^7 = I_3$. So $x^{16} - x = (x^7)^2 x^2 - x = x^2 - x = x(x - I_3)$, which is not nilpotent (it is indeed a unit, as both x and $x - I_3$ are units). This contradiction shows that each R_α is a simple Artinian ring, say $R_\alpha = \mathbb{M}_l(D)$ with D a division ring. As seen above, it must be that $l \leq 2$.

If $l = 1$, then $R_\alpha = D$. As $a^{16} = a$ for all $a \in D$, $D \cong \mathbb{F}_{2^s}$ with $s|4$.

If $l = 2$, then $R_\alpha \cong \mathbb{M}_2(D)$. As D is isomorphic to a subring of $\mathbb{M}_2(D)$, $a^{16} = a$ for all $a \in D$. As above, $D \cong \mathbb{F}_{2^s}$ where $s = 1, 2$ or 4 . We next show that $D \not\cong \mathbb{F}_{2^4}$. Assume that $D \cong \mathbb{F}_{2^4}$. Take $a \in D$ be a primitive element. Then $\phi(t) = (t - a)(t - a^2)(t - a^4)(t - a^{16}) = t^4 + t + 1$ is the minimal polynomial of a . That is, $a^4 = a + 1$. Let $x = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{M}_2(D)$. Then $x^{16} = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$, so $x^{16} - x = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, which is not nilpotent. The contradiction shows that $D \not\cong \mathbb{F}_{2^4}$. ■

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BIOGRAPHY

Tülay YILDIRIM TURAN was born in İstanbul, Turkey in 1990. She received the B.Sc. degrees in Mathematics from the Gebze Institute of Technology, Turkey in 2013. She graduated as a top college student. She also received two M.Sc degrees in Mathematics, one of them is on the homotopy theory with group actions from the Gebze Technical University and second one is on the various finite difference solutions of option pricing models from the Yıldız Technical University. She has been working as a research assistant at Gebze Technical University since 2013. Also, she has been a PhD student of Mathematics Department in Gebze Technical University Graduate School of Nature and Applied Science since 2016. Her research interests includes rings, modules and related areas.

APPENDICES

Appendix : Publications

Kosan M. T., Yildirim T., Zhou Y., (2020) “Rings with $x^n - x$ is nilpotent”, J. Algebra and Appl, DOI:10.1142/S0219498820500656.

Kosan M. T., Yildirim T., Zhou Y., “Rings whose elements are the sum of a tripotent and an element from the Jacobson radical”, Canadian Mathematical Bulletin., DOI: 10.4153/S0008439519000092.