

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

**ANALYTICAL REGULARIZATION METHOD FOR
SIMULATION OF ELECTROSTATIC FIELDS TO MODERN
ENGINEERING AND THEIR APPLICATIONS**

İBRAHİM EFE
A THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
DEPARTMENT OF ELECTRONICS ENGINEERING

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THESIS SUPERVISOR
ASSOC. PROF. DR. FATİH DİKMEN

GEBZE
2023

T.C.
GEBZE TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

ANALİTİK REGÜLARİZASYON
YÖNTEMİNİN MODERN MÜHENDİSLİK
UYGULAMALARINDA ELEKTROSTATİK
ALAN SİMÜLASYONU İÇİN KULLANIMI

İBRAHİM EFE
DOKTORA TEZİ
ELEKTRONİK MÜHENDİSLİĞİ ANABİLİM DALI

TEZ DANIŞMANI
DOÇ. DR. FATİH DİKMEN

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2023

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SUMMARY

Electrostatic fields created by electric charges in various environments have many practical applications in engineering. In order to develop these applications and simulate their behavior in the computer environment, the load distribution and the area created by this distribution should be calculated. However, it is not possible to make these calculations analytically if the geometry of the medium carrying the electric charge is outside of certain shapes. For this purpose, a solution is sought by reducing the problem to a system of linear equations with certain numerical methods (finite differences, finite elements, moment method, etc.). However, the equation system created here is a 1st type linear equation system and as the matrix size increases, numerical instability problems arise due to rounding errors [1]-[6]. Some improvement methods have been suggested in the literature (the best known is the Tikhonov regularization) to prevent such problems [8]-[11]. In this thesis, “Analytical Regularization Method” to prevent such instability problems will be discussed.

Primarily, Green's function and analytically regularized formulation of Laplace's equation are derived for surfaces satisfying the Dirichlet boundary condition in arbitrarily shaped open and closed geometries with axial symmetry.

Finally, sample engineering applications that require such calculations are discussed.

Key Words: Boundary Integral Equations, Analytical Regularization Method, Orthogonal Polynomials Method, Computational Electrostatic.

ÖZET

Çeşitli ortamlardaki elektrik yüklerinin oluşturduğu elektrostatik alanların mühendislikte birçok pratik uygulaması bulunmaktadır. Bu uygulamaların geliştirilebilmesi ve bilgisayar ortamında davranışlarının simüle edilebilmesi için yük dağılımının ve bu dağılımın oluşturduğu alanın hesaplanabilmesi gerekir. Ancak bu hesaplamaları analitik olarak yapmak elektrik yükünü taşıyan ortamın geometrisi belli başlı şekiller dışında ise mümkün olmamaktadır. Bu amaca yönelik belli başlı sayısal yöntemler (sonlu farklar, sonlu elemanlar, moment metodu vb.) ile problem doğrusal denklem sistemi haline indirgenerek probleme çözüm aranmaktadır. Ancak burada oluşturulan denklem sistemi 1. türden bir doğrusal denklem sistemidir ve matris boyutu büyüdükçe yuvarlama hatalarından kaynaklanan sayısal kararsızlık problemleri ortaya çıkmaktadır [1]-[6]. Bu türden problemleri önlemeye yönelik literatürde bazı iyileştirme yöntemleri (en iyi bilineni Tikhonov regülarizasyonu) önerilmiştir [8]-[11]. Tez kapsamında bu türden kararsızlık problemlerini önlemeye yönelik “Analitik Regülarizasyon Yöntemi” ele alınacaktır.

Öncelikle eksenel simetriye sahip keyfi şekilli açık ve kapalı geometrilerde Dirichlet sınır koşulunu sağlayan yüzeyler için Laplace denklemine ait Green fonksiyonu ve analitik regularize edilmiş formülasyonu çıkarılıp, sonrasında ise bu tür eksenel simetriden bazı geometrilerde yüzey elektrik yük dağılımı ve yüzey etrafındaki potansiyel ve alan dağılımları hesaplanacaktır.

Son olarak bu tür hesaplamalara ihtiyaç gerektiren örnek mühendislik uygulamalarından bahsedilecektir.

Anahtar Kelimeler: Sınır İntegrali Denklemleri, Analitik Regularizasyon Yöntemi, Ortogonal Polinomlar Yöntemi, Hesaplamalı Elektrostatik.

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LIST OF ABBREVIATIONS AND ACRONYMS

<u>Abbreviations and Acronyms</u>	<u>Explanations</u>
∇^2	: Laplace Operator
∇	: Del Operator
σ	: Surface charge distribution
$G(q, p)$	: Green's function
$L(q, p)$	: Logarithmic singularity of integral kernel
$K(\theta, \tau)$	: Smooth part of integral kernel
ARM	: Analytical Regularization Method
BOR	: Body of Revolution
OTS	: Open Toroidal Screen
CTS	: Closed Toroidal Screen
BIE	: Boundary Integral Equation
OPM	: Orthogonal Polynomial Method

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1. INTRODUCTION

The requirements for the analysis of electrostatic fields are encountered in a very wide area in today's engineering applications [1]. It is important to be able to calculate the potential distribution created by systems with axial symmetry, whose physical structure has complex geometry and does not allow calculation analytically [2]. The reliability of the method to be used is of great importance, especially in places where this calculation must be done with very high accuracy during the design phase, such as micromachines, particle accelerators, microprocessor junction zones, etc. As it is known, performing such a calculation with a computer, in its most general form requires the reduction of the differential or integral equation to the linear algebra system and the solution of the matrix equation [2]-[6]. The linear equation systems of the first type obtained in classical methods are unfortunately very sensitive to the calculation of the matrix inverse and their reliability is lower than the equations of the second type [7]. In order to eliminate these disadvantages, many methods using processes called regularization have been developed in the literature [8]-[11]. In the Analytical Regularization Method (ARM), which constitutes the scope of this thesis, Laplace's equation that we want to solve will be expressed as the Fredholm integral equation of the first kind using the corresponding kernel function and appropriate boundary conditions. Thus, the equation will be solved by transforming it into a linear algebra system of the second type with the transformation operations to be explained in detail.

2. ANALYTICAL REGULARIZATION METHOD

In the solution of the systems of equations related to the electrostatic fields mentioned in the previous section for complex geometries, we generally do not have a solution option other than numerical approaches. In the most general form, these approaches require Laplace's equation to be solved by discretizing all of the solution domain in the form of a differential equation (FDM, FEM, etc.) or to solve it at the boundaries in the form of integral equations with the boundary integral equation (BIE) approach [5], [6], [13]. In both approaches, the system of equations is reduced to a finite linear algebraic system and a solution is sought. However, these systems of equations are of the first kind and are generally numerically unstable [7], [13], [15]. This requires us to be skeptical as to whether the solution we obtained converges to the real solution we seek.

2.1. Ill-Posed Problems

The well-posed problem concept defined by Hadamard in 1923 [16]. Accordingly, let us consider equation (2.1).

$$Ax = b \tag{2.1}$$

Equation (2.1) is said to be well-posed when it satisfies the following three conditions.

Existence: There is a solution set x that satisfies the equation $Ax = b$.

Uniqueness: There is only one solution set x that satisfies this equation $Ax = b$.

Stability: The solution x depends continuously on b .

The system of equations to be solved is said to be ill-posed unless at least one of these conditions is met. The three conditions above are generally interdependent. For example, when A is a bounded linear operator, the stability condition is automatically satisfied if the existence and uniqueness conditions are met. Existence and uniqueness are well-studied for many physical and engineering problems. Therefore, instability is often a primary concern in the study of ill-posed problems. In

general, ill-posed problems where the stability condition is not met are extremely sensitive to matrix inversion, and the calculated result becomes numerically instability with increasing system size.

Since the matrix inversion of a system of the first kind is very sensitive to the number of truncations, it is necessary to transform this system of equations into a system of the second kind in order to obtain a new system insensitive to rounding errors [17], [18]. Although there are various regularization methods used for this purpose, the Analytical Regularization Method will be used due to its various advantages which will be explained in detail in this thesis.

Briefly, in this method, a new linear algebraic equation system is created by choosing a left-side operator (L) and a right-side operator (R) and applying them to the first kind of linear algebraic equation system. This new system is of the second kind and is stable regardless of the truncation number [14], [15].

Analytical regularization is a technique used to transform the Fredholm linear algebraic equation system of the first kind with logarithmic singularities into an equivalent Fredholm linear algebraic equation system of the second kind. Such a transformation offers an effective solution to eliminating the above-mentioned problems. Let A be an operator in the Banach (Hilbert) space and it is necessary to solve the equation of the first kind [19];

To find the unknown x , we need to solve a system of equations of the first kind containing the operator A (with supposing that inverse operator A^{-1} exists) defined in Hilbert space with supposing that inverse operator A^{-1} exists (problem has solution and solution is unique). In practice, when the equation (2.1) is rewritten with a limited system size with truncation number N , the equation is

$$A_N x_N = y_N \quad (2.2)$$

Of course, the solution set x_N is an approximate solution of the linear equation system. There are some questions we need to answer at this stage.

- i) Is the truncated solution x_N converge to x as N goes to infinity?

$$\lim_{N \rightarrow \infty} x_N \stackrel{?}{=} x \quad (2.3)$$

In general, the answer to this question is negative, for the reasons mentioned above.

ii) According to the definition of the condition number, is $\nu_N = \nu(A_N) \leq \text{const}$ when $N \rightarrow \infty$ where $\nu_N = \|A_N\|_2 \|A_N^{-1}\|_2$ and is the numerical process stable?

In general, the answer is negative too. System of the first kind is characterized by $\|A_N\|_2 \rightarrow \infty$ or $\|A_N^{-1}\|_2 \rightarrow \infty$ when $N \rightarrow \infty$. Consequently, $\nu_N \rightarrow \infty$ as $N \rightarrow \infty$ [20]

This means that a stable and correct solution cannot be obtained. At this stage, our goal is to equivalently transform the original problem into a system of equations of the second type. Suppose the A operator has the form: $A = I + H$, where I is the unit matrix, H is the compact operator, and $(I + H)^{-1}$ exists and its condition number bounded as;

$$\nu_\infty = \|I + H\|_2 \|(I + H)^{-1}\|_2 < \infty \quad (2.4)$$

and the system we have to solve is clearly

$$(I + H_N)x_N = b_N \quad (2.5)$$

truncated matrix operator H_N can be chosen as,

$$\|H - H_N\|_2 \rightarrow 0, \quad \text{as } N \rightarrow \infty \quad (2.6)$$

and all ν_N are uniformly bounded for sufficiently large N :

$$\nu_N = \lim_{N \rightarrow \infty} \|I + H_N\|_2 \|(I + H_N)^{-1}\|_2 \rightarrow \nu_\infty \quad (2.7)$$

If we ask the previous questions again:

i) Since (1.6) is valid, also $\|x - x_N\| \rightarrow 0$. So, the answer is positive.

$$\lim_{N \rightarrow \infty} x^N = x$$

ii) Because of the relation (2.7), the answer is positive too

Considering the definitions above, regularization for the operator of the equation of the first kind can be expressed as:

Based on the definitions above, regularization for the operator of the equation of the first kind can be expressed as follows: Let A be an operator on the pair of Banach spaces B_1 and B_2 , $A: B_1 \rightarrow B_2$ and there are two operators $L_0: B_2 \rightarrow B$ and $R_0: B \rightarrow B_1$ where B is Banach space too and the inverse operators L_0^{-1} and R_0^{-1} exist and bounded ones. This pair (L_0, R_0) is called a two-sided regularizator if where H is a compact operator in B . Consider the functional equation of the first kind:

$$Ax = b, \quad x \in B_1, y \in B_2 \quad (2.8)$$

Since R_0^{-1} is bounded, for any element $x \in B_1$ it is valid $x = R_0 y$ for some $y = R_0^{-1} x \in B$. Using this representation and acting on both sides of the equation, we can obtain

$$(I + H)y = L_0 b, \quad y \in B, L_0 b \in B \quad (2.9)$$

with the compact operator H in B .

3. ELECTRIC FIELD CALCULATION BY ARM

In this thesis, we will use the ARM to solve the Fredholm integral equation problems satisfying the Dirichlet boundary condition involving bodies of revolution (BOR), which comprise rotationally symmetric open and/or closed arbitrary cross-sectional toroidal objects [21]-[23].

Such geometries are frequently encountered in engineering applications, which will be discussed in the following sections. Solving an axially symmetric structure with the BOR approach instead of 3D solving, of course, provides great advantages in terms of using computer resources.

Consider a body of revolution such as the cone-sphere in Figure 3.1. Such an object can be identified by scanning all the points that make up the surface by rotating the bounding curve about the axis of symmetry.

3.1. Problem Formulation

Let us consider an infinitely thin arbitrarily shaped axially symmetric (toroidal) screen, and suppose that this screen charged with potential Ψ . For a charged perfectly conductive screen, the potential U is constant on a toroidal screen (TS).

However, due to the wide area of possible applications of the approach, we suppose that the potential $\Psi = \Psi(p)$ is a given smooth function of the point that belongs to TS, i.e., $p \in S$.

Let $U = U(q)$ be a function describing the electrostatic potential at the space observation point $q \in R^3$. This function satisfies the Laplace equation with the Dirichlet boundary condition [24].

$$\nabla^2 U(q) = 0, \quad q \in R^3 \setminus S \quad (3.1)$$

$$U(q) - \Psi(q) = 0, \quad q \in S \quad (3.2)$$

Also, for a physically reasonable solution, it is assumed that the potential smoothly converges to zero at infinity:

$$U(q) \rightarrow 0 \text{ as } q \rightarrow \infty \quad (3.3)$$

and the field energy integral must be finite in any confined spatial volume V including the edges:

$$\int_V |\nabla U|^2 dv < \infty \quad (3.4)$$

The enforcement of conditions (3.1)-(3.4) is adequate to ensure that a unique solution to the potential problem exists [18], [25].

Thus, the general solution to the problem at hand is expressed in the form of the surface integral [18], [25], [27], [28]: (for more detail see Appendix A)

$$U(q) = \iint_S \sigma_D(p) G(q, p) ds_p, \quad q \in R^3, \quad p \in S, \quad (3.5)$$

$$G(q, p) = -\frac{1}{4\pi} \frac{1}{R(q, p)}; \quad (3.6)$$

where $\sigma_D(p)$ is an unknown single-layer density (or the surface charge distribution), $G(q, p)$ is the Green's function for Laplace's equation in three-dimensional free space, and $R(q, p)$ is the distance between q and p points. Thus, the expression (3.5) with boundary condition (2.2) yields the Fredholm integral equation of the first kind

$$\iint_S \sigma_D(p) G(q, p) ds_p = \Psi(q), \quad q, p \in S, \quad (3.7)$$

for the unknown function $\sigma_D(p)$.

As the equation constructed as described in the previous section is of the first type, it is an ill-posed problem [7]-[12], [14]-[16]. In addition, contrary to the finite energy condition stated in (3.4) for an open-ended geometry, the charge distribution σ_D on the conductor has a singularity at the edges. In this case, condition (3.4) must be satisfied with the Meixner boundary condition [26].

$$\sigma_D(p) \sim [R(p, p_0)]^{-1/2}, \quad (3.8)$$

3.2. Problem Geometry and ARM Procedure

For a perfectly conductive object charged with a potential V_0 , symmetrical with respect to the z -axis shown in Figure 3.1, two different approaches will be examined, formed by the open and closed curves shown in Figure 3.2.

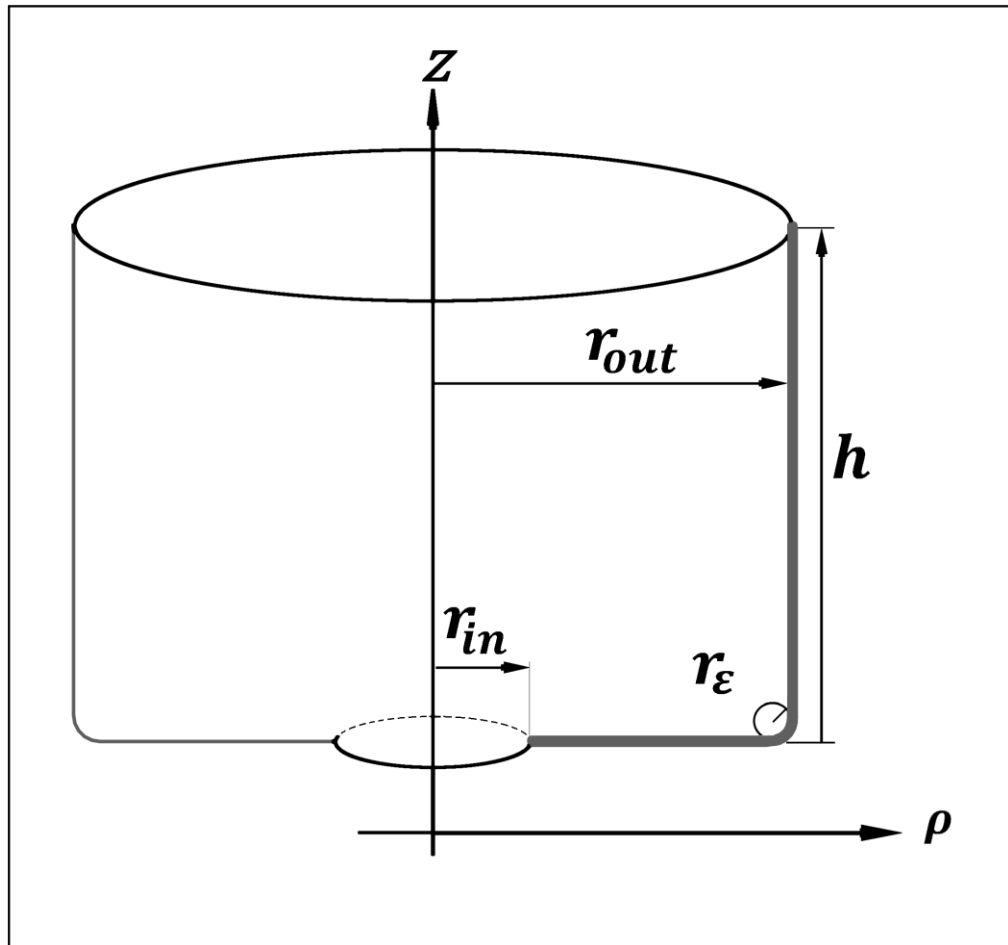


Figure 3.1: Surface of revolution.

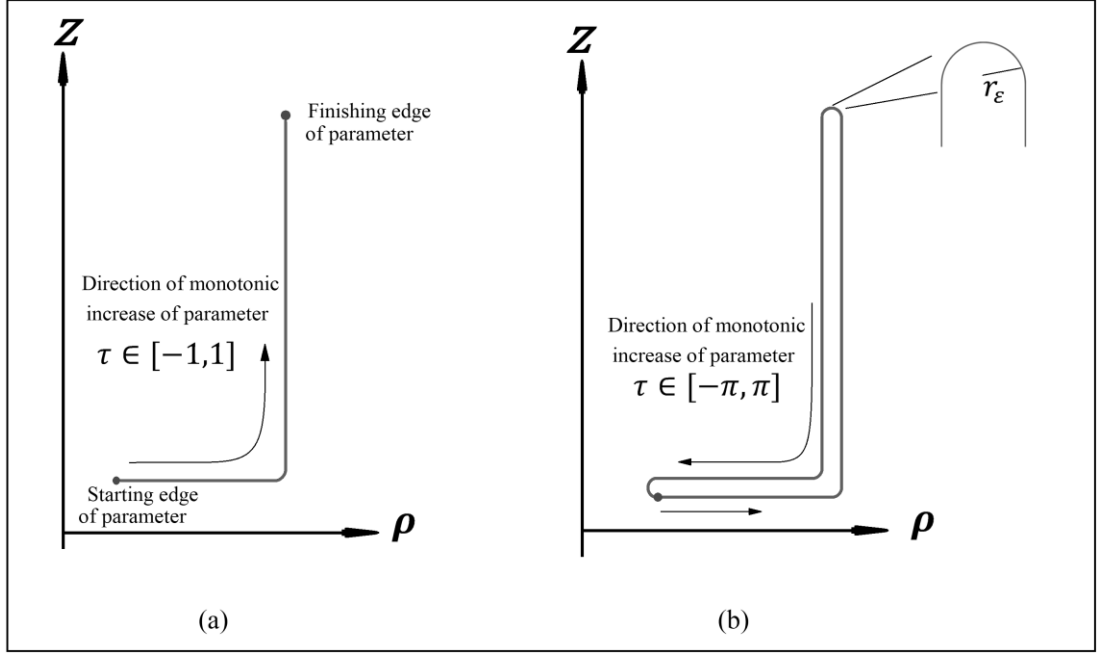


Figure 3.2: (a) Open Curve (b) Closed Curve.

Let the open curve, in Figure 3.2.a, l_o be specified in cross-sectional coordinates of the BOR with two given functions $z(\tau_o), \rho(\tau_o)$ of zonal variable $\tau_o \in [\alpha_1, -\alpha_2]$. With these functions the arc length of the cross-sectional contour

$$l(\tau) = \sqrt{\{z'(\tau)\}^2 + \{\rho'(\tau)\}^2}, \tau \in [\alpha_1, -\alpha_2] \quad (3.9)$$

Similar to l_o , the closed curve l_c in Figure 3.2.b, can be parametrized like two open curves intertwined with each other, and on the edges they closed with a semi-circle. Then the corresponding $z(\tau_c), \rho(\tau_c)$ of zonal variable $\tau_c \in [-\pi, \pi]$ is to be defined.

Let us first, define the distance between two points on the rotating surface and consider the corresponding integral representation

$$R(\theta, \tau) = \sqrt{(z(\theta) - z(\tau))^2 + (\rho(\theta) - \rho(\tau))^2 + 4\rho(\theta)\rho(\tau)\sin^2((\varphi(\theta) - \varphi(\tau))/2)} \quad (3.10)$$

The distance $|\theta - \tau|$ between two points $\theta, \tau \in S$, where $\theta = (\rho_\theta, \varphi_\theta, z_\theta)$ and $\tau = (\rho_\tau, \varphi_\tau, z_\tau)$. In the cylindrical coordinate system we can write

$$|q - p| = R(z_\theta - z_\tau, \rho_\theta - \rho_\tau, \varphi_\theta - \varphi_\tau) \quad (3.11)$$

$$ds = l(\tau)d\tau d\varphi_\tau$$

Then integral equation (3.7) can be rewritten as follows:

$$\int_{-d}^d \int_{-\pi}^{\pi} \sigma(z_\tau, \rho_\tau, \varphi_\tau) G(z_\theta - z_\tau, \rho_\theta - \rho_\tau, \varphi_\theta - \varphi_\tau) l(\tau) d\tau d\varphi_\tau = U(z_\theta, \rho_\theta, \varphi_\theta), \quad (3.12)$$

Here, parameterization $[-d, d]$ will be used by taking $[-1, 1]$ for l_o and $[-\pi, \pi]$ using the convenient present form for l_c . σ , G , and U functions are expanded to the Fourier-Chebyshev series for open curves and the Fourier series for closed curves, and then the integral equation is rewritten with these components [21]-[23].

In cases where the integration and observation points of p and q do not overlap, the integral kernel is a regular function, while a logarithmic singularity emerges as the point p approaches the point q , after a proper scaling as mentioned in Chapter 4 [21], [22].

In this case, considering that the integral kernel is the sum of a regular function and a function with a logarithmic singularity, we can write the current kernel as the sum of two separate functions. Accordingly, equation (3.7) will become

$$\int_{-d}^d \{L(\theta, \tau) + K(\theta, \tau)\} z(\tau) d\tau = b(\theta) \quad (3.13)$$

Here $L(\theta, \tau)$ is the most dominant singularity of the integral equation kernel which is a logarithmical function. $K(\theta, \tau)$ from other side is a more smooth function with first derivatives continuous and partial second derivatives are square integrable.

In the next section, we will consider the calculation and the ARM procedure separately for open and closed surfaces.

3.2.1. ARM Procedure for Open Contours

As mentioned before, the functions forming the equation (3.13) on an open curve are expanded into the Chebyshev series and the integral equation is rewritten in the interval $[-1,1]$. Therefore, a mapping function $\eta(\zeta)$ must be defined to convert the interval $[a,b]$ to the interval $[-1,1]$.

Here, a known formulation (3.14) can be used to expand the logarithmic function $L(q, p)$ to its Chebyshev series [31].

$$L(\xi, \zeta) = -\pi^{-1} \ln|\tau_q - \tau_p| = \sum_{n=0}^{\infty} \gamma_n^{-2} \hat{T}_n(\tau_q) \hat{T}_n(\tau_p), \quad (3.14)$$

$$\tau_q, \tau_p \in [-1,1]$$

Here, $\hat{T}_n$, is the orthonormal Chebyshev polynomial of the first kind and is defined as follows.

$$\hat{T}_n(x) = d_n^{-1} T_n(x), \quad x \in [-1,1] \quad (3.15)$$

In here $d_0 = \sqrt{\pi}$; $d_n = \sqrt{\pi/2}$, for $n \neq 0$

and $\gamma_0 = 1/\sqrt{\ln(2)}$; $\gamma_n = \sqrt{|n|}$ for $n \neq 0$

Since other functions that make up the integral are smooth, it is suitable for expanding to the Chebyshev series.

$$z(\tau_p) = \left[1/\sqrt{1-\tau_p^2}\right] \sum_{n=0}^{\infty} z_n \hat{T}_n(\tau_p), \quad \tau_p \in [-1,1], \quad (3.16)$$

$$b(\tau_q) = \sum_{n=0}^{\infty} b_n \hat{T}_n(\tau_q), \quad \tau_q \in [-1,1], \quad (3.17)$$

$$K(\tau_q, \tau_p) = \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} k_{sn} \hat{T}_s(\tau_q) \hat{T}_n(\tau_p), \quad \tau_q, \tau_p \in [-1,1], \quad (3.18)$$

Here z_n is unknown coefficients, b_n and k_{sn} are the Chebyshev expansions belonging to $b(\tau_q)$ and $K(\tau_q, \tau_p)$ functions. If we use these expansions and rewrite the integral equation and after that use the Kronecker δ definition

$$\delta_{s,n} = \int_{-1}^1 (1 - \zeta^2)^{-1/2} \hat{T}_n(\zeta) \hat{T}_s(\zeta) d\zeta = \begin{cases} 1; & n = s \\ 0; & n \neq s \end{cases}, \quad s, n = 0, 1, 2, 3, \dots \quad (3.19)$$

and using the orthogonality properties, we can obtain the system of equations

$$\frac{1}{\gamma_n^2} z_n + \sum_{s=0}^{\infty} k_{ns} z_s = b_n \quad n = 0, 1, 2, 3, \dots \quad (3.20)$$

3.2.2. ARM Procedure for Closed Contours

We will repeat the almost same operations as in the previous section for a closed curve. The main difference here is that instead of Chebyshev polynomials, complex exponentials are used for providing the orthogonal basis. Corresponding Fourier series definition within $[-\pi, \pi]$, we will write the integral in this range. Let us consider the Fourier expansions of the functions that make up the equation (3.13). This time the Fourier expansion for the logarithmic function $L(q, p)$ [31].

$$L(\theta, \tau) = -2 \ln \left(2 \left| \sin \left(\frac{\theta - \tau}{2} \right) \right| \right) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{in(\theta - \tau)}}{|n|}; \quad \theta, \tau \in [-\pi, \pi), \quad (3.21)$$

Since other functions that make up the integral are smooth, it is suitable for expanding to the Fourier series.

$$z(\theta) = \sum_{n=-\infty}^{\infty} z_n e^{in\theta}, \quad \theta \in [-\pi, \pi), \quad (3.22)$$

$$b(\tau) = \sum_{n=-\infty}^{\infty} b_n e^{in\tau}, \quad \tau \in [-\pi, \pi), \quad (3.23)$$

$$K(\theta, \tau) = \sum_{s=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} k_{sn} e^{i(n\theta + s\tau)}, \quad \theta, \tau \in [-\pi, \pi), \quad (3.24)$$

Here z_n is unknown coefficients, b_n and k_{sn} are the Fourier expansions belonging to $b(q)$ and $K(q, p)$ functions. As we did for the open curve if we use these expansions and rewrite the integral equation and after that use the Kronecker δ definition

$$\delta_{s,n} = \int_{-\pi}^{\pi} \frac{1}{2\pi} e^{i(n-s)\tau} d\tau = \begin{cases} 1; & n = s \\ 0; & n \neq s \end{cases}, \quad s, n = 0, \pm 1, \pm 2, \dots \quad (3.25)$$

and using the orthogonality properties, we can obtain the system of equations

$$\frac{1}{v_n^2} z_n + \sum_{s=-\infty}^{\infty} (k_{n,-s} + \delta_{-s,0} \delta_{n,0}) z_s = b_n \quad n = 0, 1, 2, 3, \dots \quad (3.26)$$

with $v_n = \max(1, \sqrt{|n|})$

The resulting equation is of the first type and has the aforementioned handicaps. At this stage, let's define new coefficients to make our equation of the second type by applying the ARM procedure.

$$y_n = \frac{z_n}{\kappa_n}, \quad \kappa_n = \begin{cases} \gamma_n, & \text{open contour} \\ v_n, & \text{closed contour} \end{cases}$$

coefficient defined and both sides of (3.20) or (3.26) are multiplied by κ_n

$$y_n + \sum_{s=0}^{\infty} \kappa_n \kappa_s k_{ns} y_n = \kappa_n b_n \quad n = 0, 1, 2, 3, \dots \quad (3.27)$$

is obtained. With new definitions $\hat{k}_{ns} = \kappa_n \kappa_s k_{ns}$ and $\hat{b}_n = \kappa_n b_n$, (3.27) transform to

$$y_n + \sum_{s=0}^{\infty} \hat{k}_{ns} y_s = \hat{b}_n \quad n = 0, 1, 2, 3 \dots \quad (3.28)$$

The functional equation for this is in the form of

$$(I + \hat{K})y = \hat{b} \quad y, \hat{b} \in l_2 \quad (3.29)$$

The important difference between equation (3.28) and equation (3.20) or (3.26) is that the $\hat{K}$ operator is compact in L_2 when the kernel $K(q, p)$ function is smooth enough. This means that equation (3.28) is numerically stable regardless of system size.

3.2.3. Parametrization

Suppose that the generating curve l located in the XOZ plane parametrized by the variable in T is smoothly parameterized with

$$T \in \begin{cases} (-\pi, \pi], & l_c \\ [-1, 1], & l_o \end{cases}$$

$$z = z(\tau), \quad \rho = \rho(\tau), \quad \tau \in T$$

The parameter τ varies in the interval T , where $z = z(\tau)$ and $\rho = \rho(\tau)$ are given functions of the zonal parameter τ , which define cylindrical coordinates of point $p = p(z, \rho, \varphi = 0) \in l$; see Figure 3.2 and definitions in section 3.2. The functions at hand are that an element of arc length

$$l(\tau) = \sqrt{\{z'(\tau)\}^2 + \{\rho'(\tau)\}^2}, \quad \tau \in T, \quad (3.30)$$

satisfies the condition $0 < l(\tau) < \infty$, which ensures one-to-one parameterization of the points on the curve.

Now TS formed with rotation of the generating curve l around OZ – axis is defined in cylindrical coordinates $\{z, \rho, \varphi\}$ by

$$x = \rho(\tau) \cos \varphi, \quad y = \rho(\tau) \sin \varphi, \quad z = z(\tau) \quad (3.31)$$

where $\tau \in T$ and $\varphi \in \Phi = [0, 2\pi]$. The surface element for TS in the cylindrical coordinates is $ds = \rho(\tau)l(\tau)d\tau d\varphi$. As a result of the axial symmetry of TS, the distance $R(q, p)$ for $q, p \in TS$ is a function of the square of the difference of the azimuthal angles $\varphi = \varphi_q - \varphi_p$:

$$R(\tau_q, \tau_p, \varphi) = \sqrt{d(\tau_q, \tau_p) + 4w(\tau_q, \tau_p)\sin^2\left(\frac{\varphi}{2}\right)}, \quad \tau \in T, \quad \varphi \in \Phi \quad (3.32)$$

where $d(\tau_q, \tau_p) = \{\rho(\tau_q) - \rho(\tau_p)\}^2 + \{z(\tau_q) - z(\tau_p)\}^2$ and $w(\tau_q, \tau_p) = \rho(\tau_q)\rho(\tau_p)$. For $\delta = \tau_q - \tau_p$, one can approximate $d(\tau_q, \tau_p - \delta) \approx [l(\tau_q)\delta + O(\delta)]^2$, when $|\delta| \ll 1$.

4. EXTRACTION OF THE INTEGRAL KERNEL SINGULARITY

In the previous section, the ARM procedure was given in general, and the singularity in the integral kernel for an axisymmetric geometry, which is the subject of our thesis, should be examined in detail.

Introduce the unknown function, describing the normalized single-layer density on TS

$$X(\tau, \varphi) = \xi(\tau)l(\tau)\sigma_D(\tau, \varphi), \quad \tau \in T, \quad \varphi \in \Phi \quad (4.1)$$

where $\xi(\tau) = \sqrt{\rho(\tau)}$ is the scale function. Utilizing the introduced notations, one can rewrite equation (4.1) in the following form

$$\int_{\Phi} \int_T \xi(\tau_p) G(\tau_q, \varphi_q; \tau_p, \varphi_p) X(\tau_p, \varphi_p) d\tau_p d\varphi_p = \psi(\tau_q, \varphi_q) \quad (4.2)$$

where $G(\tau_q, \varphi_q; \tau_p, \varphi_p)$ is the Green's function restricted onto surfaces TS, and $\psi(\tau_q, \varphi_q) = \psi(q)|_{q \in S}$

All $f(\tau, \varphi)$ functions of the azimuthal angle expanding in Fourier series that be formed

$$f(\tau, \varphi) = \sum_{m=-\infty}^{\infty} f^m(\tau) e^{im\varphi}, \quad f^m(\tau) = \frac{1}{2\pi} \int_0^{2\pi} f(\tau, \varphi) e^{-im\varphi} d\varphi \quad (4.3)$$

and integrate equation (4.3) with respect to φ , we obtain the set of decoupled integral equations

$$\begin{aligned} \int_{TS} \xi(\tau_p) G_L^m(\tau_q, \tau_p) X^m(\tau_p) d\tau_p &= \frac{1}{2\pi} \psi^m(\tau_q), \quad \tau_q \in T, \quad m \\ &= 0, \pm 1, \pm 2, \dots \end{aligned} \quad (4.4)$$

where the $G_L^m(\tau_q, \tau_p)$ are azimuthal Fourier coefficients of Green's function. Due to the axial symmetry of TS, the coefficients of kernels has symmetry

$$G_L^m(\tau_q, \tau_p) = G_L^{-m}(\tau_q, \tau_p).$$

In set (4.4) each equation containing the azimuthal Fourier harmonics corresponds to azimuthal index m . Thus, the initial surface integral equation (3.7) is separated for TS into an infinite set of decoupled integral equations over the zonal parameter.

Using the representation of Green's function for Laplace's equation, projected onto the axially symmetrical surface, through the Legendre functions [29], one can express its azimuthal Fourier coefficients in the following form:

$$G_L^m(\tau_q, \tau_p) = -\frac{1}{4\pi^2} \frac{1}{\sqrt{w(\tau_q, \tau_p)}} Q_{|m|-1/2}(\chi(\tau_q, \tau_p)), \quad q, p \in S, \quad (4.5)$$

$$\chi(\tau_q, \tau_p) = \frac{\rho(\tau_q)^2 + \rho(\tau_p)^2 + (z(\tau_q) - z(\tau_p))^2}{2\rho(\tau_q)\rho(\tau_p)} = 1 + \frac{1}{2} \frac{d(\tau_q, \tau_p)}{w(\tau_q, \tau_p)}$$

where $Q_{m-1/2}(\chi)$ is the odd-half-integer degree Legendre function of the second kind, which is a toroidal harmonic [31]. In accordance with definitions $d(\tau_q, \tau_p)$ and $w(\tau_q, \tau_p)$ in (2.32) it follows $\chi \in [1, \infty]$.

$$Q_{-\frac{1}{2}}(\chi) = \frac{\sqrt{\pi}\Gamma(\frac{1}{2})}{\sqrt{2\chi}\Gamma(1)} {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{1}{\chi^2}\right) = \frac{\pi}{\sqrt{2\chi}} {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{1}{\chi^2}\right) \quad (4.6)$$

When $\rho(\tau_q) \rightarrow \rho(\tau_p), z(\tau_q) \rightarrow z(\tau_p); \chi \rightarrow 1$. At this stage, there is no singularity in the $\frac{\pi}{\sqrt{2\chi}}$. If we examine the hypergeometric function in more detail [31], (see Appendix B for elliptic integral representation)

$${}_2F_1(\alpha, \beta; \gamma; \xi) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{\xi^n}{n!} \quad (4.7)$$

Here $(P)_n$ denotes the Pochhammer symbol described by

$$(P)_n = \begin{cases} 1, & n = 0 \\ p(p+1) \dots (p+n-1) & n > 0 \end{cases} \quad (4.8)$$

In the case of $\xi < 0.5$ formula (4.7) converges very well. The more appropriate form for the case $\xi > 0.5$

$${}_2F_1(\alpha, \beta; \gamma; \xi) = \frac{-\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \left[\begin{array}{c} \ln(1 - \xi) \quad {}_2F_1(\alpha, \beta; \gamma; 1 - \xi) \\ + \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(k!)^2} (1 - \xi)^k \left\{ \begin{array}{c} \psi(\alpha + k) \\ + \psi(\beta + k) \\ - 2\psi(1 + k) \end{array} \right\} \end{array} \right] \quad (4.9)$$

ψ is the digamma function described by

$$\psi(x) = \frac{d}{dx} \ln(\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)} \quad (4.10)$$

If we use our arguments in the hypergeometric formula we obtain

$$\begin{aligned} & {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{1}{\chi^2}\right) \\ &= \frac{-\Gamma(1)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} \left[\ln\left(1 - \frac{1}{\chi^2}\right) \quad {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \frac{1}{\chi^2}\right) + \phi \right] \end{aligned} \quad (4.11)$$

$$\phi = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^2} \left(1 - \frac{1}{\chi^2}\right)^n \left\{ \begin{array}{c} \psi\left(\frac{1}{4} + n\right) \\ + \psi\left(\frac{3}{4} + n\right) \\ - 2\psi(1 + n) \end{array} \right\} \quad (4.12)$$

If $\chi \rightarrow 1$ then ${}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \frac{1}{\chi^2}\right) \rightarrow {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 0\right) = 1$ and also summation process don't have any singularity. Only the $\ln\left(1 - \frac{1}{\chi^2}\right)$ part has singular behavior. To get rid of this singularity we can multiply (4.11) with some

logarithmic expression which we know Fourier coefficients and add and subtract from the formula (4.12). The subtracted part gives us a finite limit when the logarithm has a singularity. For simplicity $\rho(\tau_q) = \rho_q, z(\tau_q) = z_q, \theta(\tau_q) = \theta_q$

$$\begin{aligned}
& {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{1}{\chi^2}\right) \\
&= \frac{-\Gamma(1)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} \begin{bmatrix} \ln\left(1 - \frac{1}{\chi^2}\right) & {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \frac{1}{\chi^2}\right) + \phi \\ -\ln\left(\left|2\sin\left(\frac{\theta_q - \theta_p}{2}\right)\right|^2\right) & {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \frac{1}{\chi^2}\right) \\ +\ln\left(\left|2\sin\left(\frac{\theta_q - \theta_p}{2}\right)\right|^2\right) & {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \frac{1}{\chi^2}\right) \end{bmatrix} \quad (4.13)
\end{aligned}$$

$$\begin{aligned}
& {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{1}{\chi^2}\right) \\
&= \frac{-\Gamma(1)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} \begin{bmatrix} \ln\left(\frac{1 - \frac{1}{\chi^2}}{\left|2\sin\left(\frac{\theta_q - \theta_p}{2}\right)\right|^2}\right) & {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \frac{1}{\chi^2}\right) \\ +\ln\left(\left|2\sin\left(\frac{\theta_q - \theta_p}{2}\right)\right|^2\right) & {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \frac{1}{\chi^2}\right) + \phi \end{bmatrix} \quad (4.14)
\end{aligned}$$

Now logarithmic part has a limit value when $\chi \rightarrow 1$. Fourier coefficients of the other logarithmic identity

$$\ln\left(\left|2\sin\left(\frac{\theta_q - \theta_p}{2}\right)\right|^2\right) = \sum_{n=-\infty}^{\infty} a_n e^{-jn(\theta_q - \theta_p)} \quad (4.15)$$

$$a_n = -\frac{1}{2|n|}, \quad a_0 = 0 \quad (4.16)$$

$$G_0(\rho_q, z_q, \rho_p, z_p) = \frac{1}{4\pi^2 \sqrt{\rho_q \rho_p}} Q_{-\frac{1}{2}}(\chi) \quad (4.17)$$

$$= \frac{1}{4\pi\sqrt{\rho_q\rho_p}\sqrt{2\chi}} \frac{-\Gamma(1)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} \left[\begin{aligned} & \ln\left(\frac{1-\frac{1}{\chi^2}}{\left|2\sin\left(\frac{\theta_q-\theta_p}{2}\right)\right|^2}\right) A \\ & + 2\ln\left(\left|2\sin\left(\frac{\theta_q-\theta_p}{2}\right)\right|\right) A \\ & + \phi \end{aligned} \right] \quad (4.18)$$

$$A = {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \frac{1}{\chi^2}\right)$$

This expression is valid everywhere when χ defined. But for efficiency we can use formula (4.7) for $\frac{1}{\chi^2} < 0.5$

$$G_0(\rho_q, z_q, \rho_p, z_p) = \frac{1}{4\pi\sqrt{\rho_q\rho_p}\sqrt{2\chi}} \frac{-\Gamma(1)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{1}{\chi^2}\right) \quad (4.19)$$

After obtaining the singular behavior of the azimuthal harmonics of Green's function for $\tau_q \rightarrow \tau_p$

$$G_L^m(\tau_q, \tau_p) = -\frac{1}{4\pi^2} \frac{1}{\sqrt{\rho(\tau_q)\rho(\tau_p)}} \left\{ \ln\left(\frac{\sqrt{\rho(\tau_q)\rho(\tau_p)}}{l(\tau_q)|\delta|}\right) - \frac{1}{2}C^m \right\} + O(\delta^2 \ln|\delta|), \quad (4.20)$$

$$C^m = \psi\left(\frac{2m+3}{4}\right) + \psi\left(\frac{2m+1}{4}\right) + 2\gamma_e$$

where $|\delta| = |\tau_q - \tau_p| \ll 1$, $\psi(t)$ is the digamma function, and γ_e is the Euler–Mascheroni constant. For the zeroth harmonic $\psi^{m=0} = -6\ln(2)$ and therefore we have

$$G_L^0(\tau_q, \tau_p) = -\frac{1}{4\pi^2} \frac{1}{\sqrt{\rho(\tau_q)\rho(\tau_p)}} \ln \left(8 \frac{\sqrt{\rho(\tau_q)\rho(\tau_p)}}{l(\tau_q)|\delta|} \right) + O(\delta^2 \ln|\delta|), \quad (4.21)$$

$|\delta| \ll 1$

Let's choose the interval of the zonal parameter variation in depend upon a closed toroidal screen (CTS) or open toroidal screen (OTS) is under consideration: $\tau \in T = \{\bar{T} = [-\pi, \pi], \text{ for CTS; } \tilde{T} = [-1, 1], \text{ for OTS}\}$. The reason for such choice of interval T will be clarified below. Also introduce the notation

$$X^m(\tau) = \{\bar{X}^m(\tau), \text{ for CTS; } \tilde{X}^m(\tau), \text{ for OTS}\}$$

The unknowns in the perfectly conducting OTS in accordance with singular behaviors of the kind $\tilde{X}^m(\tau) = (1 - \tau^2)^{-1/2} h^m(\tau)$, where $h^m(\tau)$ are some smooth functions of Hölder class on the interval $\tau \in \tilde{T}$. In the case of CTS the unknowns $\bar{X}^m(\tau), \tau \in \bar{T}$ are smooth functions of Hölder class.

Now on the basis of the investigation of singular behavior, we introduce the splitting of azimuthal harmonics of the Green's function:

$$G_L^m(\tau_q, \tau_p) = -\frac{1}{4\pi\xi(\tau_q)\xi(\tau_p)} \{S(\tau_q, \tau_p) + D^m(\tau_q, \tau_p)\}, \quad (4.22)$$

$m = 0 \pm 1 \pm 2, \dots \quad \tau_q, \tau_p \in T$

$$\bar{S}(\tau_q, \tau_p) = -2\ln \left(2 \left| \sin \frac{\tau_q - \tau_p}{2} \right| \right), \quad \tau_q, \tau_p \in \bar{T}, \quad \text{for CTS} \quad (4.23.a)$$

$$\tilde{S}(\tau_q, \tau_p) = -\frac{1}{\pi} \ln |\tau_q - \tau_p|, \quad \tau_q, \tau_p \in \tilde{T}, \quad \text{for OTS} \quad (4.23.b)$$

where $S(\tau_q, \tau_p)$ is the function that contains the principal singularity of the azimuthal harmonics of $G_L(q, p)$, when $q, p \in S$ and $D^m(\tau_q, \tau_p)$ is the more smooth remainder function

$$D^m(\tau_q, \tau_p) = -\{4\pi\xi(\tau_q)\xi(\tau_p)G_L^m(\tau_q, \tau_p) + S(\tau_q, \tau_p)\},$$

$$m = 0, \pm 1, \pm 2, \dots \quad \tau_q, \tau_p \in T \quad (4.24)$$

In the case of coincidence of points, i.e. $\tau_q = \tau_p$, the exact expression for the remainder is obtained

$$\lim_{\tau_q \rightarrow \tau_p} D^m(\tau_q, \tau_p) = -\frac{1}{2\pi} \left[\ln \left(\frac{(\{\rho'(\tau_q)\}^2 + \{z'(\tau_q)\}^2) \delta^2}{\rho(\tau_q)\rho(\tau_p)} \right) + C^m \right]$$

$$+ 2 \ln \left(2 \left| \sin \frac{\tau_q - \tau_p}{2} \right| \right) + O(\delta^2 \ln |\delta|), \quad (4.25.a)$$

$$\tau_q \in \bar{T}, \quad \text{for CTS}$$

Noteworthy that introduced in such a way splitting of the integral equation kernel produces a quite smooth remainder function: $D^m(\tau_q, \tau_p)$ possesses only singular terms $O(\delta^l \ln |\delta|)$, where $l \geq 2$. It can be shown, that the reminder and its first derivatives are continuous: $D^m, \partial D^m / \partial \tau_q, \partial D^m / \partial \tau_p \in C(T \times T)$, and the second mixed derivatives of D^m are square-integrable functions $\partial^2 D^m / \partial \tau_q \partial \tau_p \in L_2(T \times T)$.

Now equation (3.2) can be rewritten as:

$$\int_T \{S(\tau_q, \tau_p) + D^m(\tau_q, \tau_p)\} \chi^m(\tau_p) d\tau_p = \psi^m(\tau_q), \quad \tau_q \in T, \quad m$$

$$= 0, \pm 1, \pm 2, \dots \quad (4.26)$$

where $\psi^m(\tau_q) = -2\xi(\tau_q)\Psi^m(\tau_q)$.

Thus, the kernel function is defined by dividing it into its singular and regular parts for an axisymmetric geometry, the second type of linear equation system is obtained by using the regularization operations described in the previous section.

$$(\mathbf{I} + \mathbf{D}^{\mathbf{m}})\mathbf{x}^{\mathbf{m}} = \boldsymbol{\psi}^{\mathbf{m}}, \quad \mathbf{x}^{\mathbf{m}}, \boldsymbol{\psi}^{\mathbf{m}} \in l_2 \quad (4.27)$$

$$\sum_{n=0}^{\infty} \sum_{s=0}^{\infty} (1+n)(1+s) \left| \widehat{\widehat{D}}_{ns}^m \right|^2 < \infty \quad (4.28)$$

and, consequently, operator $\mathbf{D}^{\mathbf{m}}$ is a compact and even Hilbert-Schmidt operator. Thus, (3.28) is an equation of the second kind.

5. NUMERICAL RESULTS

In this section, the charge distribution on the conductive bodies with axially symmetric open and closed geometric surfaces and the potential distribution they create around their surroundings will be examined using the ARM method.

The accuracy and possibilities of the ARM are demonstrated below by the examples of the various arbitrary CTS and OTS.

The circular OTS seen in Figure 3.2.a consists of two straight lines articulated with a quarter circle perpendicular to each other.

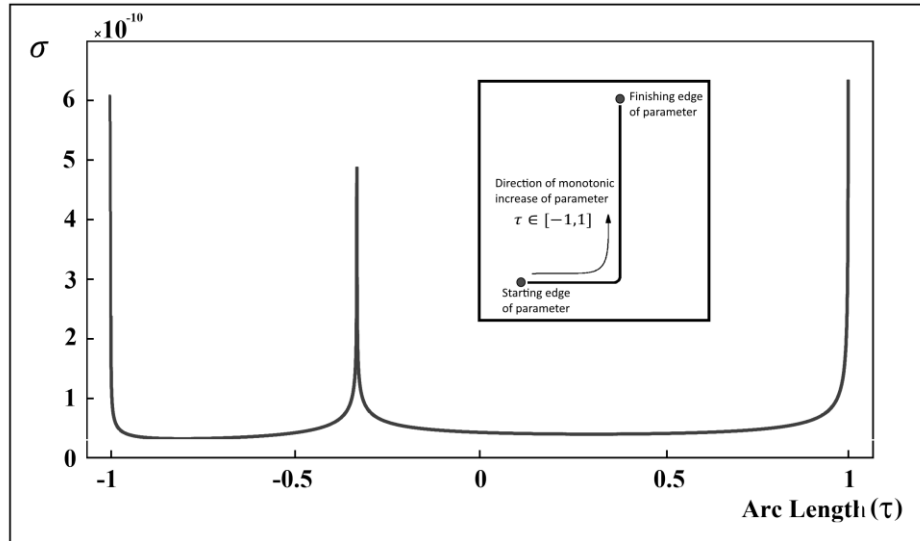
The CTS seen in Figure 3.2.b is created by closing the endpoints with a semicircle to form a closed geometry of the shape used in the OTS.

In Figure 5.3, a closed-surfaced star-like shape is defined by "superformula" and solved in a similar way.

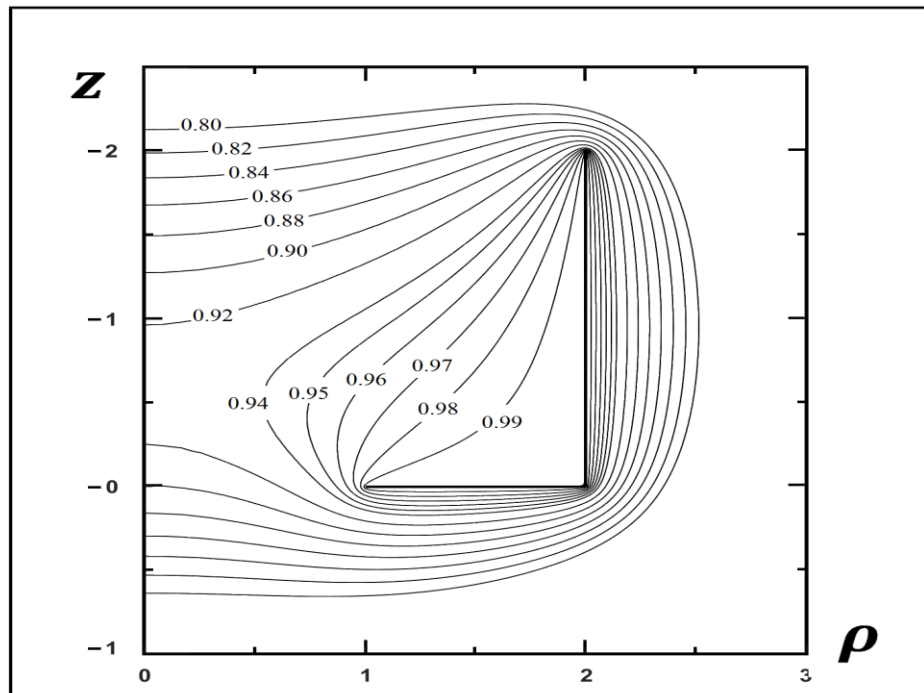
The charge density distributions on the OTS and CTS with the same geometrical parameters are shown in Figures 5.1(a) and 5.2(a) correspondingly. The TS is obtained by rotations of smooth contours composed of circular arcs. Thus, the sharp variations of surface are absent, and the charge density exposes irregularities only in the vicinity of the infinitely thin edge of OTS and roundings of CTS. As mentioned before, the Meixner condition is applied for the singularity at the edges of the OTS.

In the next section, the numerical effectiveness of the ARM is examined.

5.1. ARM Solution of the Dirichlet Boundary Value Problem for Laplace's Equation

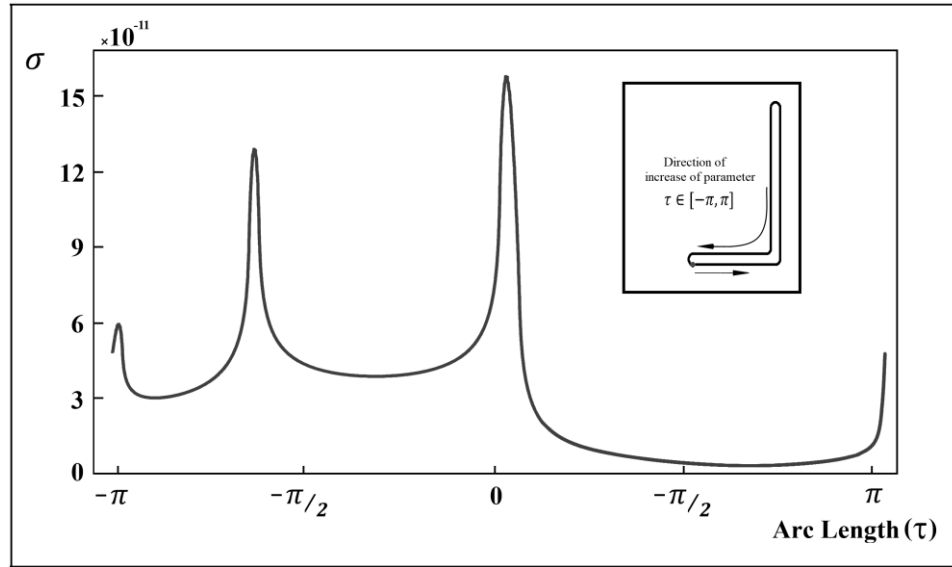


(a)

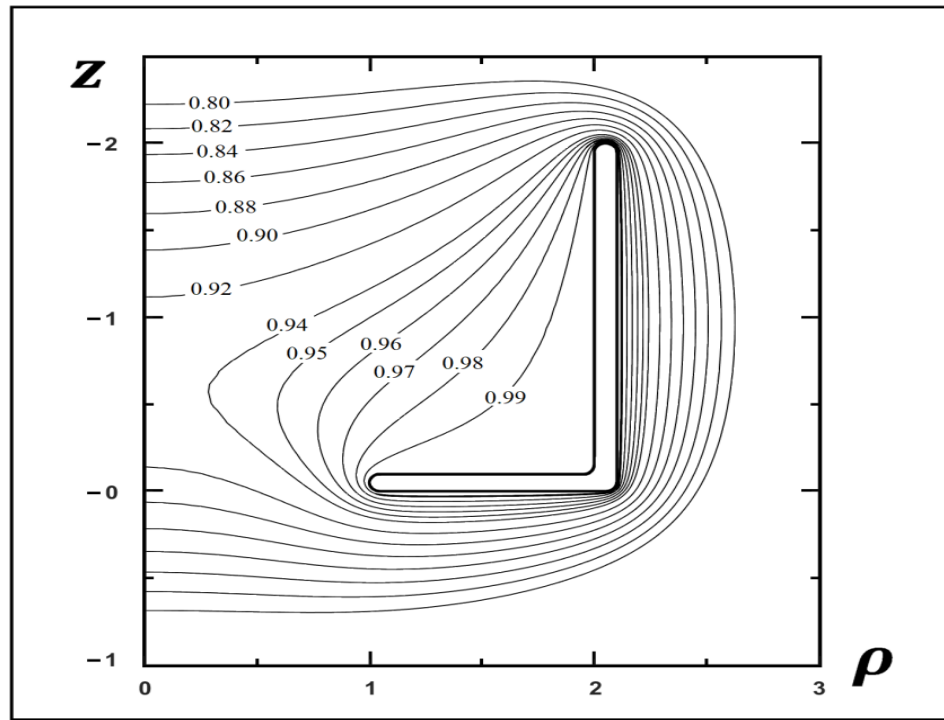


(b)

Figure.5.1: (a) Surface charge density σ versus $\tau \in [-1 + h, 1 - h]$ h is grid length (distance between each calculation point), (b) spatial potential distribution U for the circular OTS.



(a)



(b)

Figure 5.2: (a) Surface charge density σ versus τ , and (b) spatial potential distribution U for the circular CTS, $h = 0.01$.

The charge density distributions σ on the circular OTS and CTS with the same geometrical parameters (and $h = 0.01$ for CTS) are shown in (5.a) and (5.b) correspondingly. The TS at hand is obtained by rotations of smooth contours

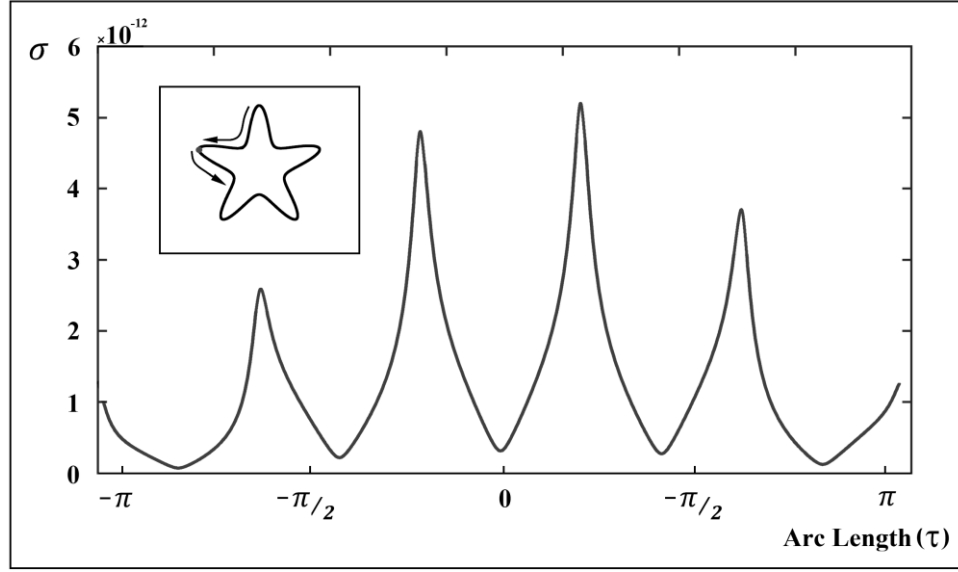
composed of circular arc/s. In this case, the sharp variations of surface are absent, and the charge density exposes irregularities only in the vicinity of the infinitely thin edge of OTS and roundings of CTS. The unlimited singularity at the infinitely thin edge of OTS is clearly seen even for relatively small truncation number $N = 15$ due to the extraction of this singularity $(1 - \tau^2)^{-1/2}$.

We can take the "Smooth Star" function rotated at a certain angle as another closed curve example. To derive this geometry, formula (5.1) was used with the parameters indicated in the Figure.5.3

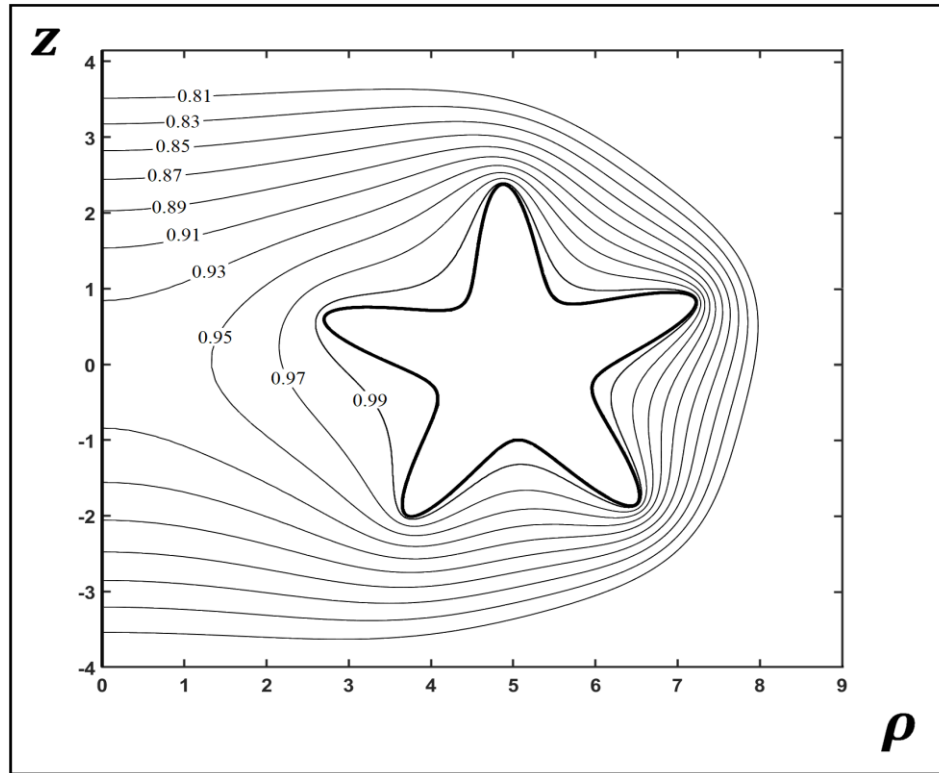
$$r(\varphi) = \left(\left| \frac{\cos\left(\frac{m\varphi}{4}\right)}{a} \right|^{n_2} + \left| \frac{\sin\left(\frac{m\varphi}{4}\right)}{b} \right|^{n_3} \right)^{-\frac{1}{n_1}} \quad (5.1)$$

where φ is a polar angle and by choosing different values for the parameters a, b, m, n_1, n_2 , and n_3 different shapes can be generated. As well we consider the potential problem for CTS generated by the rotation of the tilted closed "smooth star" contour around the oZ - axis, see Figure 5.3.

The results for CTS obtained by the rotation of a smooth star contour is presented in Figure 5.3. Because of the shape of the generating contour, the surface charge density σ as a function of parameter τ exhibits pronounced variations, see Figure 5.3.a. The minimums of σ dependence correspond to concave parts of the smooth star contour. For this charge density the equipotential lines of the electrostatic potential is presented in Figure 5.3.b. These simulations were performed with the same size as the truncated system ($N = 128$).



(a)



(b)

Figure 5.3: (a) Surface charge density σ versus τ , and (b) spatial potential distribution U for CTS obtained by the rotation of the tilted closed “Smooth Star” contour around the OZ –axis with parameters $a = 1$, $b = 1$, $m = 5$, $n_1 = 2$

$n_2 = 7$, $n_3 = 7$ RotationAngle = $3\pi/2$ and symmetry radius = 5

5.2. Numerical Effectiveness of the ARM Solution

The accuracy of the suggested method will be investigated by a potential problem shown in Figure 5.2. The system (3.29) can be solved by a truncated number. The solution of the system rapidly converges to the exact solution [14-15]. Efficiency of the method is shown by the behavior of the truncation error vs truncation number calculated in the maximum norm sense as [24]:

$$E(N) = \left(\frac{\max_{n \leq N} |x_n^{N+1} - x_n^N|}{\max_{n \leq N} |x_n^N|} \right)$$

If error and convergence analysis is performed for different thickness values of the calculated geometry as sizes denoted in Figure 5.2, it is seen that the method used converges to the calculated value with the increase in the truncation number. When the condition number of the algebraic system with the increase of the truncation number is examined, it is seen that it converges to a certain limit value. These results show us that the method used is numerically stable, produced algebraic systems that are well-conditioned and converge to the correct result as the system size increases.

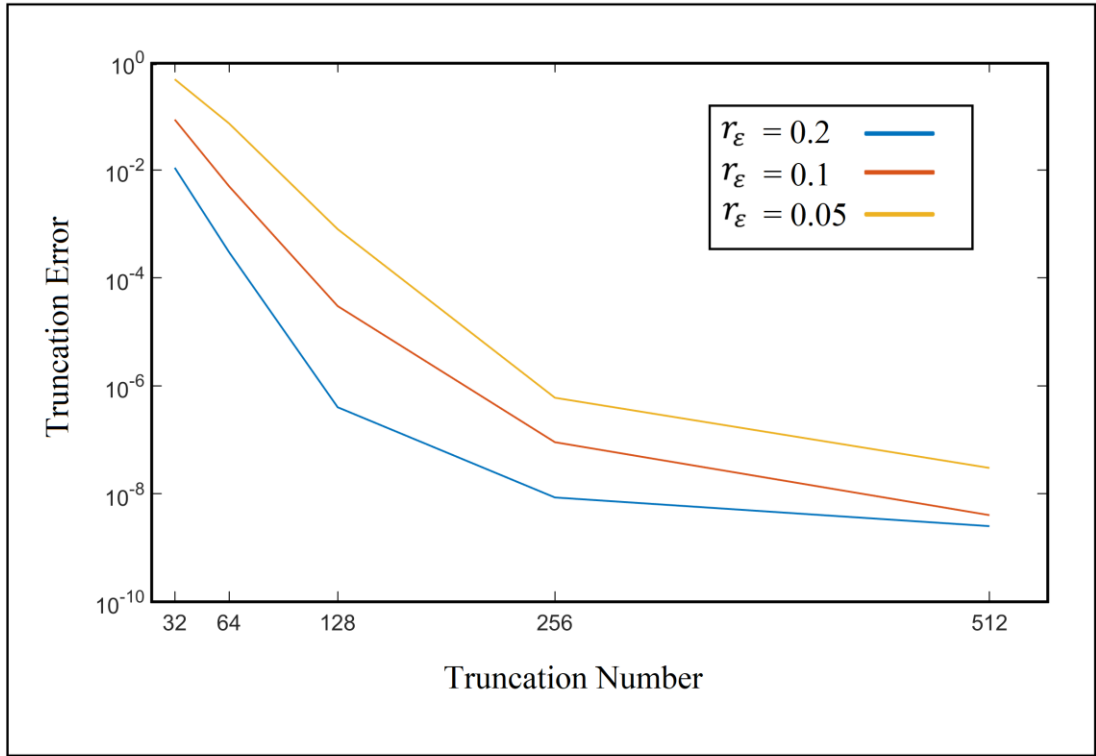


Figure 5.4: Truncation error vs truncation number.

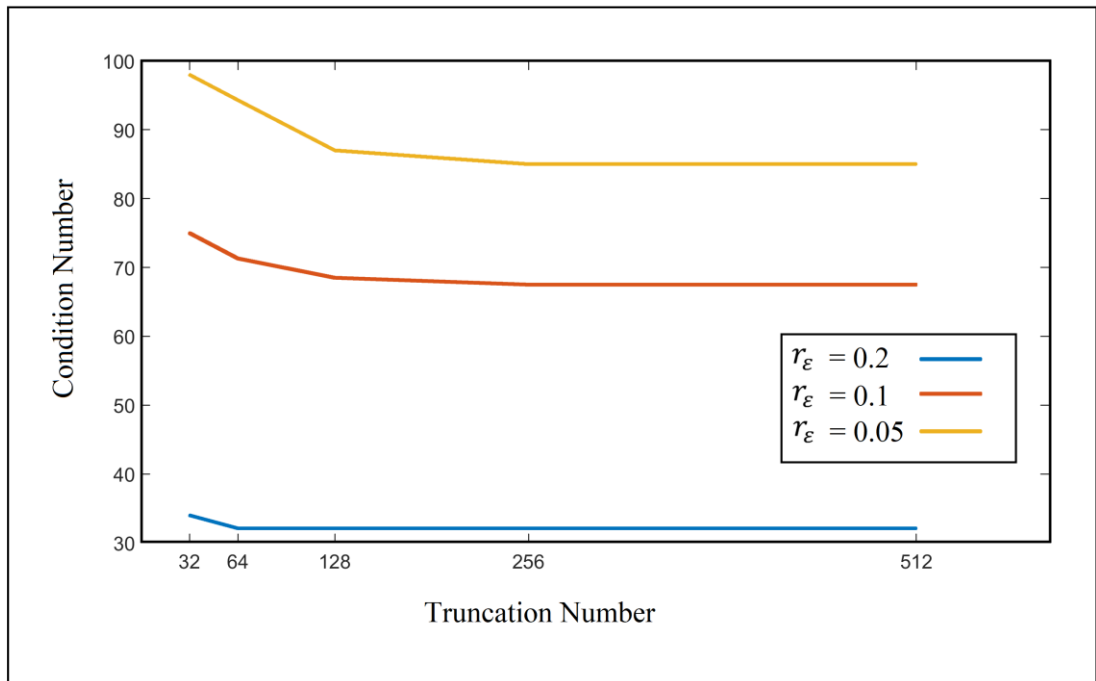


Figure 5.5: Condition number vs truncation number.

6. SOME ENGINEERING APPLICATIONS OF ELECTROSTATIC PHENOMENA AND THEIR ANALYSIS WITH ARM

6.1. Micro Electromechanical Systems

Today, with the development of production techniques, the production of electromechanical systems at micro scales has provided many practical application areas for such structures. It has found many applications, especially by minimizing accelerometers, sensors, actuators, and motors that direct mirrors or lenses, to make it possible to fit in mobile phones and many portable/wearable devices. In this section, an electrostatic micropump model for applications such as biomedical drug delivery will be presented and analyzed with ARM.

The structure we will examine in this section is a micropump system consisting of two electrodes, one fixed and the other movable positioned parallel to each other.

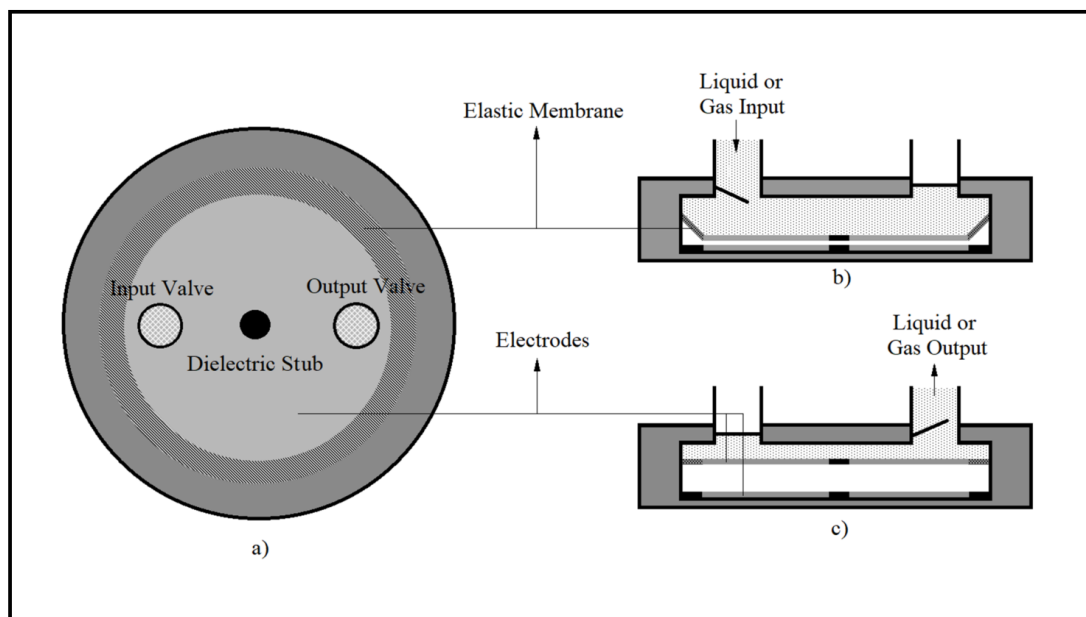


Figure 6.1: Sketch of electrostatic micropump structure: a) Top view, b) Sucking mode, c) Pumping mode.

In designing the proposed structure in the most appropriate dimensions, the system was analyzed with ARM and the results were evaluated. The main physical

characteristics of the suggested system and its deviation from that of infinitely large plane approximation are discussed.

Microfluidic systems based on a combination of mechanical and electrostatic phenomena are commonly encountered in biomedical micropump systems. [33]-[35]. Among them, electrostatically actuated microelectromechanical systems (EA-MEMS) are well known as structurally simple and functionally flexible [36]. We will examine a displacement type EA-MEMS with rigid electrodes [37] without flexible membranes [38], [39], they do not require the inclusion of the deflection phenomenon in their analysis, only the charges of the junction plates due to attraction and repulsion [38], [39]. Our model is elaborated through the first kind of boundary integral equation (BIE) formulation of Laplace's Equation [25]. This type of BIE analysis is very common in research on complex MEMS organizations. We solved the system of integral equations thus obtained by means of Orthogonal Polynomial Methods (OPM) including Chebyshev polynomials developed by ARM [40], [43].

The working principle is basically as follows: Electrostatic pulling and repulsive forces are created between rigid parallel plates, depending on the voltage applied to the plates [34]. The inner holes of the rings in Figure 6.1.b,c are filled with dielectric stub discs in Figure 6.1.a. Thus, the upper rigid electrode is mounted on the chamber walls with a flexible dielectric ring, allowing it to move up and down under electrostatic forces. The lower electrode is fixed to the floor of the chamber. In this way, the liquid can be filled from the inlet into the chamber above the moving electrode and flows out of the outlet. The amount of flow depends on the surface area of the plates, the distance between them and the applied voltage.

6.1.1. The Force Between The Plates

In such EA-MEMS setups it is very typical practice to calculate the charge on the parallel plates as a constant on the infinite plates, ie the fringe electrostatic field and equipotential lines are not taken into account. In this approach, the following simple formula is used for the magnitude of the force [1], [41],

$$F = \frac{\epsilon AV^2}{2d^2}, \quad (6.1)$$

where ε is the dielectric permittivity of the medium, d is the distance between the plates, V is the potential difference of plates, and A is the area of the electrodes. In fact, the force between the plates with fixed potentials is the gradient of the stored electrostatic energy. W_e , resulting in (6.1) under the idealization above [1]. When the fringing quantities are not omitted, we write the force as the gradient of W_e ,

$$F = \nabla W_e = z \frac{\varepsilon}{2} \int_v \left[\frac{\partial}{\partial z} |E|^2 \right] dv, \text{ where } E = -\nabla U, \quad (6.2)$$

U is the potential distribution between plates that are calculated rigorously throughout the boundary integral equation formulation in the next section.

6.1.2. Potential Distribution in Space With The Dirichlet Boundary Conditions

In (6.2), since the supply voltage V is constant on electrodes and the system is axially symmetrical, U does not change along the azimuthal angle, and forces along the radial direction are equal to zero. Therefore only the variation along the z -axis remains. $U(q)$ observed at point q , exists due to the surface charge density $\sigma_j(p)$ on the points p of the electrode numbered i ($i = 1, 2$) has the integral representation over the surfaces forming these electrodes Figure 6.1.b,c. The integral representation for the corresponding boundary value problem (BVP) contains the unknowns $\sigma_j(p)$ and the Green's function of Laplace equation: $G(q, p) = 1/(4\pi|q - p|)$. Under Dirichlet boundary conditions ($U_i(p) = -V_i, i = 1, 2$) we end up with the following integral equation.

$$\sum_{j=1}^2 \int_{S_j} \sigma_j(p) G(q_i, p) ds(p) = -V(q_i), \quad q_i \in C_i, \quad i = 1, 2 \quad (6.3)$$

It is well known that the singular part of Green's function of the scalar Helmholtz equation is $G(q, p)$ given above. $G(q, p)$ is sufficient for constructing the corresponding solution tailored to the considered problem. In fact, (6.3) can be

reduced to the following canonical integral equation, the solution procedure of which is similar to the algorithm established for scalar waves satisfying the Helmholtz equation [22].

$$\begin{aligned} & \int_{-1}^1 \left\{ -\frac{1}{\pi} \ln|u-v| + K_{ii}(u, v) \right\} \frac{m_i(v)}{\sqrt{1-v^2}} dv \\ & + \int_{-1}^1 K_{ij}(u, v) \frac{m_j(v)}{\sqrt{1-v^2}} dv = b_i(u), \end{aligned} \quad (6.4)$$

$$u \in [-1, 1], \quad i = 1, 2; \quad j = 2, 1$$

In (6.4), unknown $m_i(v)$ are smooth functions of Hölder class and all the kernels K_{ij} are smooth functions too. When $i \neq j$, the kernel of the second integral originates from the infinitely smooth function $G(q_i, p_j)$, i.e. the source and the observation points are on different plates. As $i = j$ the kernel originating from $G(q_i, p_j)$ leads us to the logarithmically singular kernel as in (6.4), with the steps thoroughly explained in [22] to be shortly outlined in next section.

Substitution of the Fourier series expansions with respect to the azimuthal coordinate φ of all functions in (6.3), as in [22], is appropriate due to the periodicity functions with respect to φ . The constant value of the right-hand side of (6.3) results to the independence of the whole equation from φ . Therefore, the equation is reduced to the so-called DC term, i.e. only $m = 0$ term presented, where m is the azimuthal index of the Fourier expansion. The Fourier coefficient for $m = 0$ of the kernels in (6.3) in cylindrical coordinates (ρ, φ, z) is evidently the following

$$G_0^{ij}(z, \rho) = \frac{1}{8\pi^2} \int_{-\pi}^{\pi} \frac{d\varphi}{4\pi \sqrt{(z_i - z_j)^2 + (\rho_i - \rho_j)^2 + 4\rho_i \rho_j \sin^2\left(\frac{\varphi}{2}\right)}}, \quad (6.5)$$

$$\varphi = \varphi_i - \varphi_j.$$

Points $(p(v))$ on the plate having constant z coordinate, inner and outer radii a and b respectively are parametrized as

$$\begin{aligned}
p(v) &= (z, \rho(v) = [(b-a)v + (b+a)]/2), \quad b > a, \quad v \in [-1, 1], \quad \rho \\
&\in [a, b] \\
ds(p(v)) &= \rho(v)\rho'(v)dv
\end{aligned} \tag{6.6}$$

The ordinary way of such equation solving is based on the utilization of OPM involving Chebyshev polynomials of the first kind. The polynomials are orthogonal with the weight $(1 - v^2)^{-1/2}$ which fits the Meixner edge condition for the unknown charge density $\sigma_j(p(v)) = (1 - v^2)^{-1/2}m_j(v)$, in (6.3) [22]. The Fourier-Chebyshev coefficients of the CLS in (6.4) are known [22]. This fact gives a possibility to construct the regularizer of the equation (6.4) based on the OPM technique [22]. This construction leads to the infinite system of linear algebraic equations of the second kind with uniformly bounded condition numbers of the finite system matrix for their increasing truncation numbers.

In brief, the kernel in (6.4) is separated with the help of the following relation [22].

$$\begin{aligned}
G_0^{ii}(\rho(u), \rho(v)) &= \frac{1}{4\pi\sqrt{\rho(u)\rho(v)}} Q_{-\frac{1}{2}}(\chi(u, v)), \quad \chi(u, v) \\
&= \frac{\rho^2(u) + \rho^2(v)}{\rho(u)\rho(v)},
\end{aligned} \tag{6.7}$$

where G_0^{ii} is the Fourier coefficient of the $G(q_i, p_j)$ defined by (6.5). It is given in the closed form (6.7) by means of the negative half-integer degree Legendre function of the second kind of order zero $Q_{-1/2}$, which is known as a toroidal harmonic [42]. As it is known, $Q_{-1/2}(\chi) = \pi(2\chi)^{-1/2} {}_2F_1(3/4, 1/4; 1, 1/\chi^2)$ where ${}_2F_1$ is Gauss hypergeometric function, power series expansions of which can be exploited [1], to validate the following limit

$$Q_{-\frac{1}{2}}(\chi(v + \delta, v)) = -\frac{1}{\pi} \ln \delta + K_{ii}(v + \delta, v), \quad \delta = |u - v| \tag{6.8}$$

Substituting (6.7) in (6.3) and appropriately scaling the result with $\sqrt{\rho(u)}$ leads to (4) ([22]).

Once $\sigma_j(p)$ are found from (6.4), the potential distribution $U(q)$, at a point q in space can be calculated as

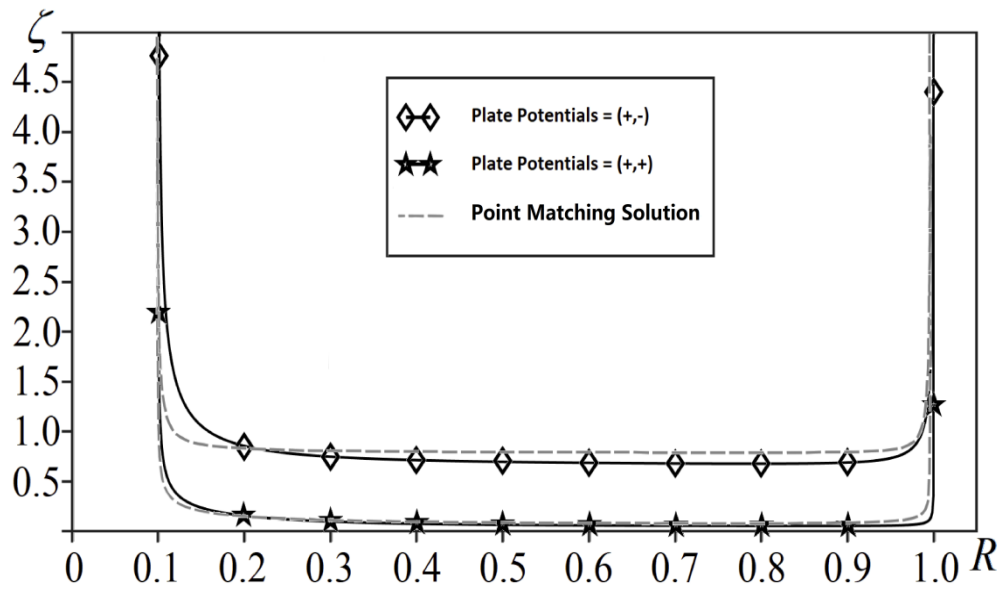
$$U(q) = \sum_{i=1}^2 \int_{S_i} \sigma_j(p) G(q, p) ds(p) \quad (6.9)$$

6.1.3. Results and Discussion of Numerical Modelling

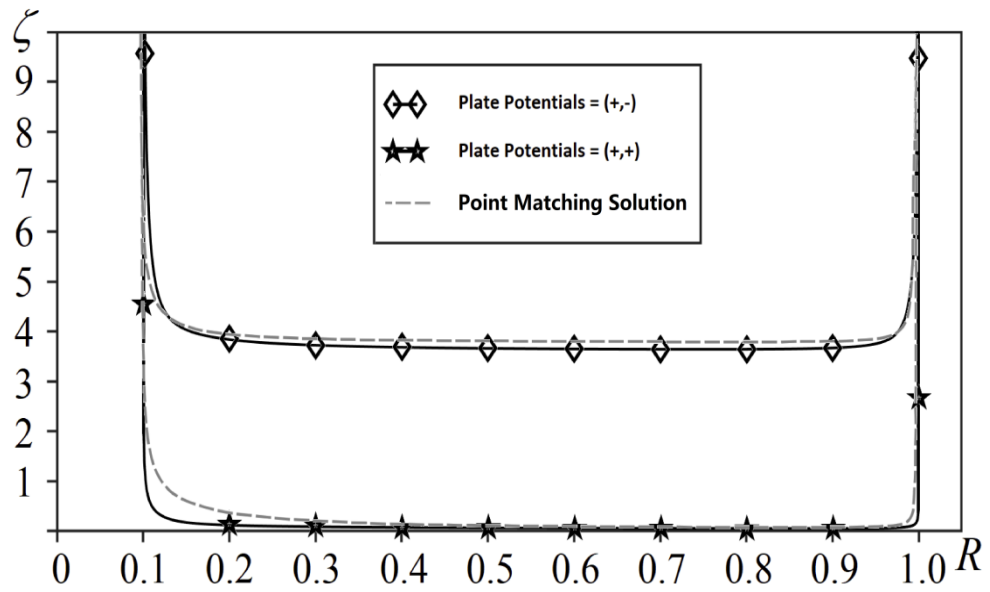
The physical characteristics of the pump model in Figure 6.1 are exposed by calculating the charge and force distributions on the plates, and the potential space distribution, as well as the capacitance of the plate system. All physical and geometrical values below are normalized as dimensionless ones. In particular, the outer radii of the ring both are equal to $R = 1$. The Inner radii of the rings are equal to r and given as fractions of R . Potentials values on plates are $V = \pm 1$. The values of the forces and capacities are normalized according to formulae (6.10)-(6.11)

Figure 6.2 depicts the calculated surface charge density distributions on the plates as functions of d . Figure 6.3 demonstrates the spatial potential distribution of the U around the plates.

Figure 6.2 shows the well-known square root singularity $\rho^{-1/2}$ of the charge densities, where ρ is the distance from the observation point on the plate to the ring edge. As well, the charge density is nearly constant, and its values close to that of an infinite plate system when the observation point is far enough from the edges. The mentioned singularities predefine the spatial distribution of potential U .



a) Distance between plates = 0.3



b) Distance between plates = 0.05

Figure 6.2: Charge distribution on the plates.

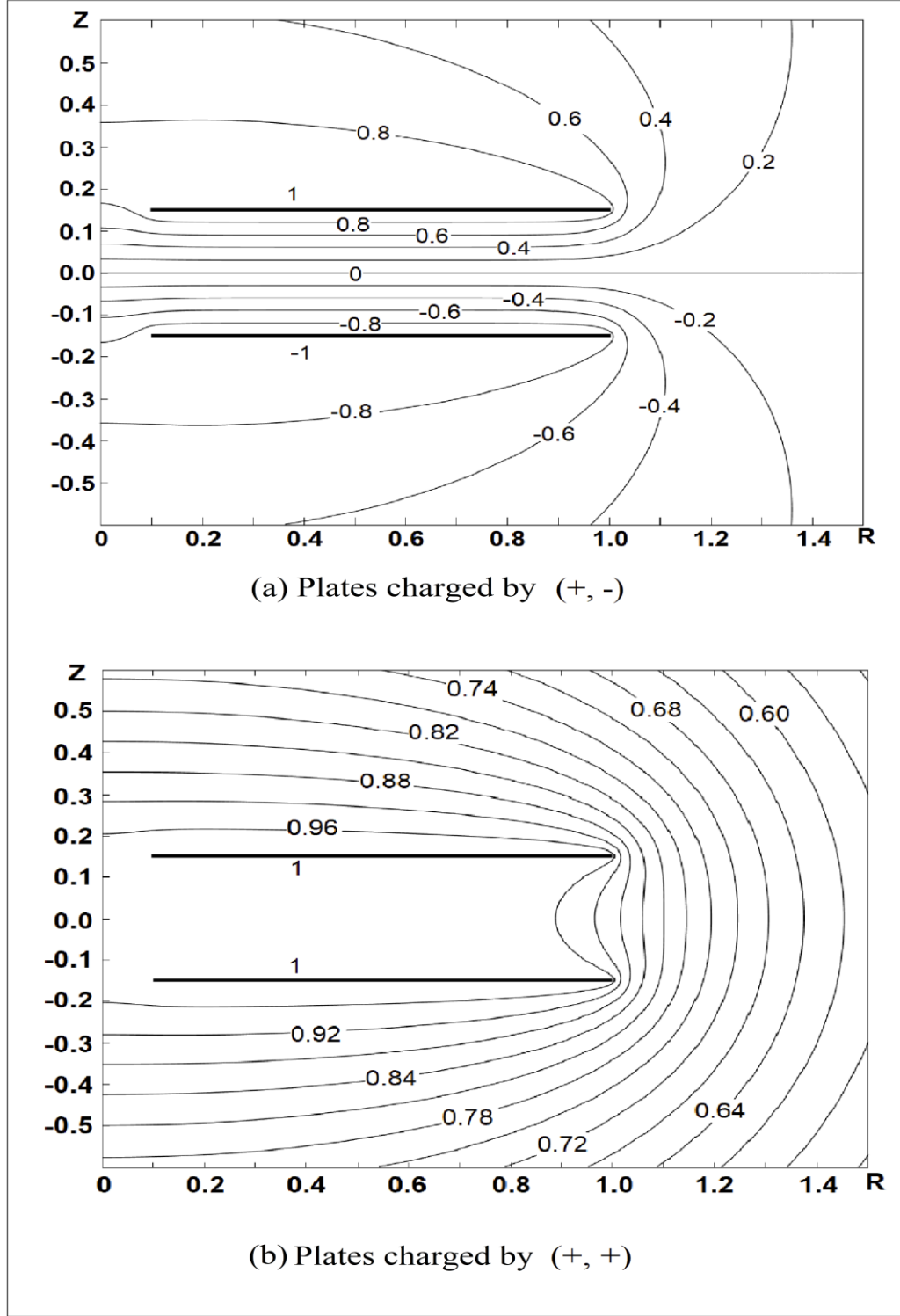


Figure 6.3: Potential distribution around plates.

The normalized force applied to the rings is $F_{norm} = (d^2/V^2)F_z$ where F_z is the modulus of z component of the force in (6.11) and (6.12), see Figure 6.4.a. In the formula (6.10) C is the capacitance. Notations F_{norm}^e and F_{norm}^a correspond to the

exact solution explained above and the approximative solution for infinite plates respectively. Thus,

$$F_{norm}^e = \frac{d^2}{2} \frac{dC}{dz}, \quad (6.10)$$

$$F_{norm}^a = \frac{\varepsilon A}{2}, \quad (6.11)$$

The approximated C^a and exact value C^e of C are calculated as follows

$$C^a = \frac{\varepsilon A}{d}, \quad (6.12)$$

$$C^e = \frac{\int_v \varepsilon |\nabla U|^2 dv}{V^2}, \quad (6.13)$$

In Figure 6.4.b the normalized capacitances $C_{norm} = (d/A)C$ are depicted corresponding to the cases in Figure 6.4.a. In both figures, approximative values of F_{norm} and C_{norm} do not depend on the distance between the rings. In both results of exact calculations performed while the inner radii of the rings vary with certain proportions of the outer radii. These variances show that the attractive and repulsive forces between the rings increase when the inner radii diminish. Accurate calculations show the essential influence of the parameter d . It means that this influence must be taken into account in the design of a real pump.

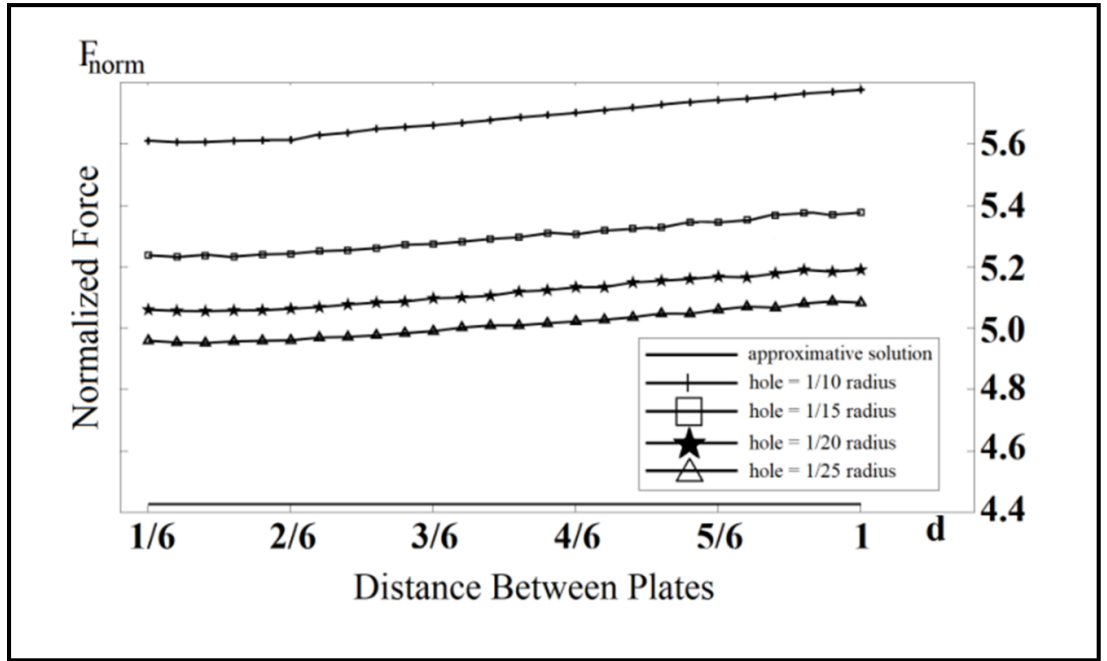


Figure 6.4: Normalized attraction forces F_n via distance between rings.

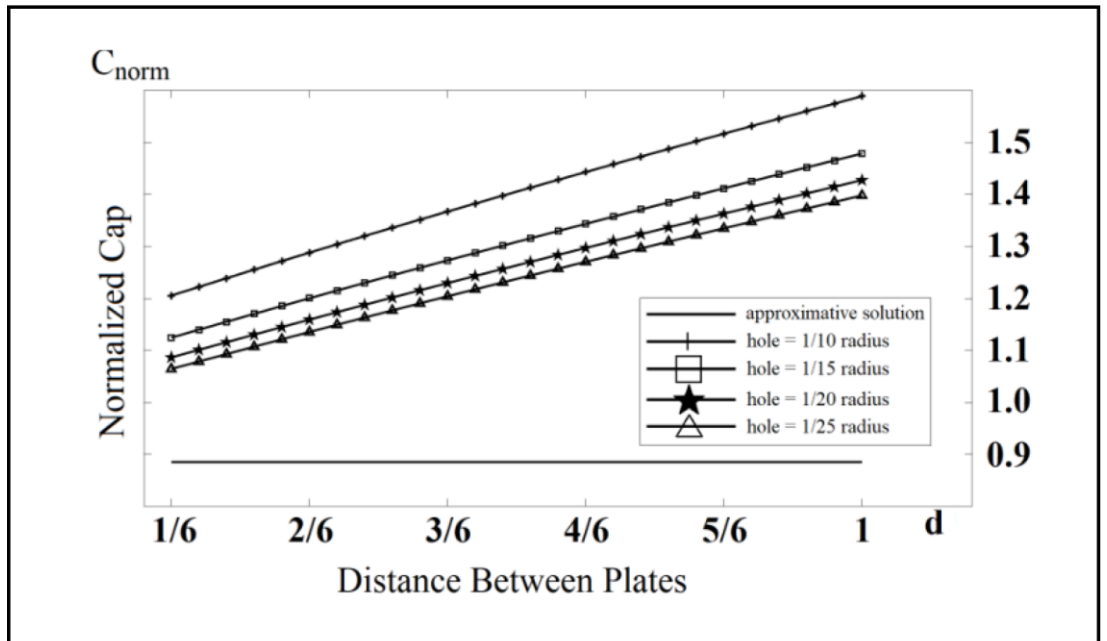


Figure 6.5: Normalized capacitances C_n via distance between rings.

It is seen in Figure 6.3 that the distribution of U between the rings is similar to that for infinite plates when the observation point is at some distance from the edges. Nevertheless, the presence of the edges essentially influences the distribution around the plates and, consequently, the capacitance of the system and the electrostatic

forces applied to the plates. This effect is the reason why the forces and capacitance in Figure 6.4. and Figure 6.5 depend on d and are different from the corresponding approximative values.

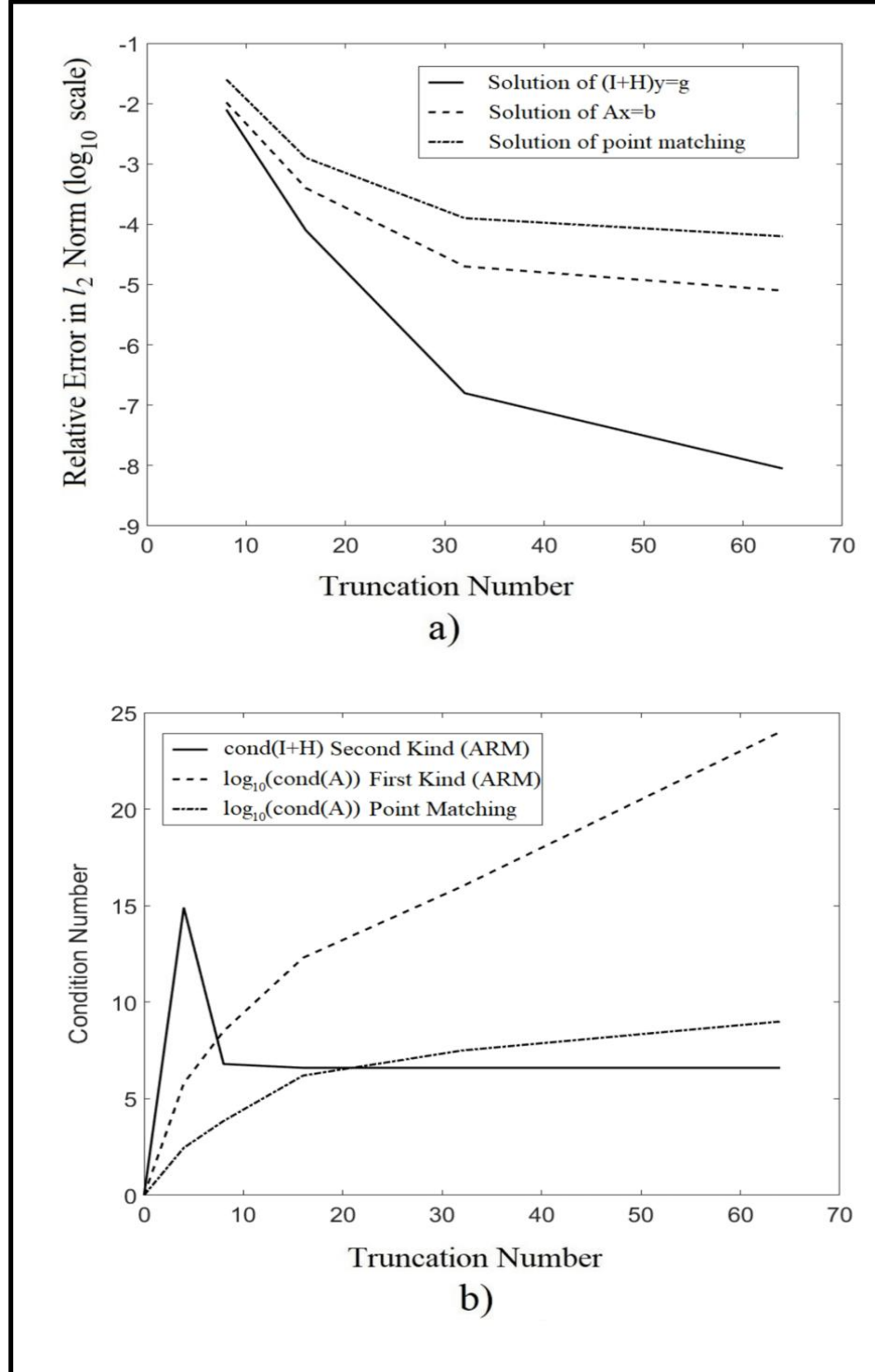


Figure 6.6: (a) Convergence Analysis (b) Condition Number.

7. CONCLUSION

Laplace's equation is the most common equation in physics that describes the behavior of many physical structures. The effort to solve this system of equations for complex geometries has led to the development of many numerical methods. The fact that the linear equation system causes instability with the increase in the number of truncation in classical numerical solution methods has been examined in detail in many studies. In this study, a numerically stable method for solving Laplace's equation under Dirichlet boundary condition is investigated for an axisymmetric structure. A rigorous, efficient and accurate method for the solution of the Laplace equation with the Dirichlet boundary condition for the open and closed axially symmetrical surfaces is constructed on the basis of the analytical regularization method.

This feature is the main advantage of the method used. This proposed approach gives high-accuracy results in the investigation of the electrostatic behavior of open and closed axisymmetric structures. In this study, the solution to a real engineering problem is discussed together with the open and closed geometries examined as examples. The results obtained form the basis for solving a wide variety of problems with more complex geometries. In addition, formulating this approach for Neumann and Robin boundary condition and extending it to the Helmholtz equation may be a suitable goal for the subject of future studies.

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BIOGRAPHY

İbrahim Efe received his B.Sc from Marmara University Electrical Education, and his M.Sc. from Gebze Institute of Technology Energy Systems Engineering. He works as a lecturer at MEB (Ministry of National Education) and various universities. He is interested in modeling physical systems in general, especially numerical methods for initial and boundary value problems.

APPENDICES

Appendix A: Derivation of Fredholm Integral Equation

Consider the potential distribution in space produced by an arbitrarily shaped conductor with a potential U_0 and a distributed electric charge $f(x)$.

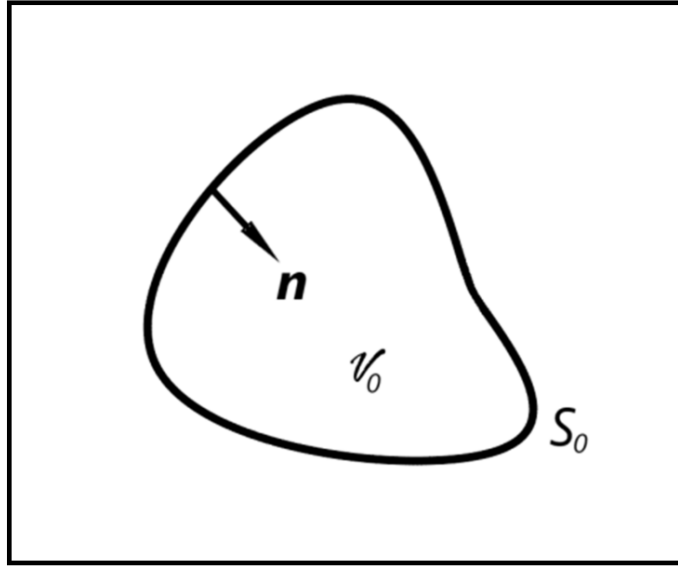


Figure A1.1: An object occupying volume V_0 .

The potential distribution produced by a charged object, $\psi(\vec{r})$ satisfies Poisson's equation in (A1.1) and the boundary value at infinity (A1.2).

$$\nabla^2 \Psi(\mathbf{r}) = -f(\mathbf{r}), \quad (\text{A1.1})$$

$$\Psi(\mathbf{r}) \rightarrow 0 \text{ as } \mathbf{r} \rightarrow \infty \quad (\text{A1.2})$$

An integral equation for this problem can be posed using the free space Green's function given in (2.6), which is the solution of (A1.3).

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r}, \mathbf{r}'), \quad (\text{A1.3})$$

First, multiply (A1.1) with $G(\mathbf{r}, \mathbf{r}')$, and (A1.3) with $\Psi(\mathbf{r})$.

$$[\nabla^2 \Psi(\mathbf{r}) = -f(\mathbf{r})]G(\mathbf{r}, \mathbf{r}') \quad (\text{A1.4})$$

$$[\nabla^2 G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r}, \mathbf{r}'),]\Psi(\mathbf{r}) \quad (\text{A1.5})$$

and, integrate the difference of the multiplications over the volume,

$$\int_v \nabla^2 \Psi(\mathbf{r}) G(\mathbf{r}, \mathbf{r}') dv - \int_v \nabla^2 G(\mathbf{r}, \mathbf{r}') \Psi(\mathbf{r}) dv \quad (\text{A1.6})$$

$$= - \int_v G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}) dv + \int_v \Psi(\mathbf{r}) \delta(\mathbf{r}, \mathbf{r}') dv$$

Applying the second Green's theorem (A1.7)

$$\int_v (a \nabla^2 b - b \nabla^2 a) dv = \int_s \left(a \frac{\partial b}{\partial n} - b \frac{\partial a}{\partial n} \right) dS \quad (\text{A1.7})$$

where S denotes the surface surrounding v , we can get

$$\int_s \left[G(\mathbf{r}, \mathbf{r}') \frac{\partial \Psi(\mathbf{r})}{\partial n} - \Psi(\mathbf{r}) \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} \right] dS - \int_v \Psi(\mathbf{r}) \delta(\mathbf{r}, \mathbf{r}') dv \quad (\text{A1.8})$$

$$= - \int_v G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}) dv$$

As a property of the Dirac delta function (A1.9)

$$\int_v \Psi(\mathbf{r}) \delta(\mathbf{r}, \mathbf{r}') dv = \begin{cases} \Psi(\mathbf{r}); & \mathbf{r}' \text{ in } v \\ 0; & \mathbf{r}' \text{ not in } v \end{cases} \quad (\text{A1.9})$$

$$\int_s \left[G(\mathbf{r}, \mathbf{r}') \frac{\partial \Psi(\mathbf{r})}{\partial n} - \Psi(\mathbf{r}) \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} \right] dS + \int_v G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}) dv \quad (\text{A1.10})$$

$$= \begin{cases} \Psi(\mathbf{r}') & ; \mathbf{r}' \text{ in } V_\infty \\ 0 & ; \mathbf{r}' \text{ in } V_0 \end{cases}$$

Exchanging $\mathbf{r}$ and $\mathbf{r}'$ and using the symmetry property of Green's function

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$$

$$\int_s \left[G(\mathbf{r}, \mathbf{r}') \frac{\partial \Psi(\mathbf{r})}{\partial n'} - \Psi(\mathbf{r}) \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} \right] dS' + \int_v G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}') dv' \quad (\text{A1.11})$$

$$= \begin{cases} \Psi(\mathbf{r}) & ; \mathbf{r} \text{ in } V_\infty \\ 0 & ; \mathbf{r} \text{ in } V_0 \end{cases}$$

When there is no source $f(\mathbf{r}) = 0$ at V_∞ , the volume integral will vanish.

$$\Psi(\mathbf{r}) = \int_s \left[G(\mathbf{r}, \mathbf{r}') \frac{\partial \Psi(\mathbf{r})}{\partial n'} - \Psi(\mathbf{r}) \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} \right] dS' \quad (\text{A1.12})$$

This equation shows that the field in a source-free region can be calculated by applying the potential value and the normal derivative to the surface surrounding the region. The second term will be zero at the boundary. If we define a new unknown function

$$\sigma(\mathbf{r}) = \frac{\partial \Psi(\mathbf{r})}{\partial n'} \quad (\text{A1.13})$$

then the integral equation will be

$$\Psi(q) = \iint_S \sigma_D(p) G(q, p) ds_p, \quad q, p \in S, \quad (\text{A1.14})$$

$$G(q, p) = -\frac{1}{4\pi} \frac{1}{R(q, p)}$$

Appendix B: Calculation of Kernel by Elliptic Integral Approximation

Let us write the Laplace equation in cylindrical coordinates (r, φ, z) . First, we define

$$R_{qp} = \sqrt{(z_q - z_p)^2 + (r_q - r_p)^2 + 4r_q r_p \sin^2\left(\frac{\varphi_q - \varphi_p}{2}\right)} \quad (\text{B1.1})$$

also define

$$\varphi_q - \varphi_p = \varphi \quad (\text{B1.2})$$

$$(z_q - z_p)^2 + (r_q - r_p)^2 = d_{qp}^2 \quad (\text{B1.3})$$

$$r_q r_p = W \quad (\text{B1.4})$$

and rewrite equation (1)

$$R_{qp} = \sqrt{d_{qp}^2} \sqrt{1 + 4 \frac{W}{d_{qp}^2} \sin^2\left(\frac{\varphi}{2}\right)} \quad (\text{B1.5})$$

$$4 \frac{W}{d_{qp}^2} = \alpha^2 \quad (\text{B1.6})$$

$$R_{qp} = \sqrt{d_{qp}^2} \sqrt{1 + \alpha^2 \sin^2\left(\frac{\varphi}{2}\right)} \quad (\text{B1.7})$$

$$\eta(\tau) = (x(\tau), y(\tau))$$

Green's function for the Laplace equation

$$G_{(q,p)} = -\frac{1}{4\pi R_{qp}} \quad (\text{B1.8})$$

$$\begin{aligned}
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} -\frac{1}{4\pi R_{qp}} e^{-im\varphi} d\varphi \\
&= -\frac{1}{8\pi^2} \int_{-\pi}^{\pi} \frac{e^{-im\varphi}}{R_{(\tau_q, \tau_p, \varphi)}} d\varphi
\end{aligned} \tag{B1.9}$$

Consider

$$g_{(\tau_q, \tau_p, \varphi)} = \frac{1}{\sqrt{d_{qp}} \sqrt{1 + \alpha^2 \sin^2 \left(\frac{\varphi}{2} \right)}} \tag{B1.10}$$

$$g^m_{(\tau_q, \tau_p)} = \frac{1}{2\pi \sqrt{d_{qp}}} \int_{-\pi}^{\pi} \frac{e^{-im\varphi}}{\sqrt{1 + \alpha^2 \sin^2 \left(\frac{\varphi}{2} \right)}} d\varphi \tag{B1.11}$$

$$g^0_{(\tau_q, \tau_p)} = \frac{1}{2\pi \sqrt{d_{qp}}} \int_{-\pi}^{\pi} \frac{1}{\sqrt{1 + \alpha^2 \sin^2 \left(\frac{\varphi}{2} \right)}} d\varphi \tag{B1.12}$$

When $q \rightarrow p$

$$q \rightarrow p \Rightarrow d_{qp} \rightarrow 0; \varphi \rightarrow 0; \alpha^2 \rightarrow \infty \tag{B1.13}$$

$$\alpha^2 \sin^2 \left(\frac{\varphi}{2} \right) \rightarrow ? \tag{B1.14}$$

If we consider symmetry

$$\begin{aligned}
g^0_{(\tau_q, \tau_p)} &= \frac{1}{2\pi\sqrt{d_{qp}}} \int_{-\pi}^{\pi} \frac{1}{\sqrt{1 + \alpha^2 \sin^2\left(\frac{\varphi}{2}\right)}} d\varphi \\
&= \frac{1}{\pi\sqrt{d_{qp}}} \int_0^{\pi} \frac{1}{\sqrt{1 + \alpha^2 \sin^2\left(\frac{\varphi}{2}\right)}} d\varphi
\end{aligned} \tag{B1.15}$$

for change boundary of integration

$$\int_0^{\pi} f(x) dx = \int_0^{\pi/2} 2f(y) dy \quad ; \quad y = \frac{x}{2} \tag{B1.16}$$

$$\frac{\varphi}{2} = \theta; d\varphi = 2d\theta, \quad \varphi \Big|_0^{\pi} \rightarrow \theta \Big|_0^{\pi/2} \tag{B1.17}$$

thus we have

$$g^0_{(\tau_q, \tau_p)} = \frac{1}{\pi\sqrt{d_{qp}}} \int_0^{\pi/2} \frac{1}{\sqrt{1 + \alpha^2 \sin^2 \theta}} d\theta \tag{B1.18}$$

definition of the complete elliptic integral of the first kind

$$K(\tau) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \tau^2 \sin^2 \theta}} \tag{B1.19}$$

as we see here if we write $\alpha^2 = -\tau^2$, we can describe our integral in (B18) as a complete elliptic integral of the first kind

$$K(i\tau) = \frac{1}{\sqrt{\tau^2 + 1}} K\left(\sqrt{\frac{\tau^2}{\tau^2 + 1}}\right) \tag{B1.20}$$

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\tau^2 \sin^2 \theta}} = \frac{1}{\sqrt{\tau^2+1}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{\tau^2}{\tau^2+1} \sin^2 \theta}} \quad (\text{B1.21})$$

$$g^0_{(\tau_q, \tau_p, \alpha)} = \frac{2}{\pi \sqrt{d_{qp}}} \frac{1}{\sqrt{1+\alpha^2}} K \left(\sqrt{\frac{\alpha^2}{\alpha^2+1}} \right) \quad (\text{B1.22})$$

We can rearrange the argument of the elliptic integral

$$\gamma = \frac{\alpha^2}{\alpha^2+1} = \frac{\frac{4W}{d}}{\frac{4W}{d}+1} = \frac{4W}{4W+d} \quad (\text{B1.23})$$

$$\sqrt{d+4W} = \sqrt{d} \sqrt{1+\alpha^2} \quad (\text{B1.24})$$

then we can define Green's function of Laplace equation for $m=0$

$$g^0_{(\tau_q, \tau_p)} = \frac{2}{\pi} \frac{1}{\sqrt{d+4W}} K \left(\sqrt{\frac{4W}{4W+d}} \right) \quad (\text{B1.25})$$

If we take into account hypergeometric functions, we can represent complete elliptic integral as a

$$K(\tau) = \frac{\pi}{2} {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1; \tau^2 \right) \quad (\text{B1.26})$$

$$K(\tau) = \int_0^{\pi/2} \frac{dz}{\sqrt{1-k^2 \sin^2(z)}} = \int_0^1 \frac{1}{2} \frac{du}{\sqrt{1-k^2 u^2} (1-u)u} ; u = \sin^2(z) \quad (\text{B1.27})$$

When q tends to p formula (B23) will tend to 1

$$\gamma = \frac{4W}{4W + d}; q \rightarrow p \Rightarrow d \rightarrow 0; \gamma \rightarrow 1 \quad (\text{B1.28})$$

for $\gamma \rightarrow 1$ we have asymptotic expansion

$$K(\gamma) \cong -\frac{1}{2} \log(1 - \gamma) \left[1 - \frac{\gamma - 1}{4} + \frac{9}{64} (\gamma - 1)^2 + \dots \right] \quad (\text{B1.29})$$

$$\log(4) + \frac{1}{4} (1 - \log(4))(z - 1) + \frac{3}{128} (6 \log(4) - 7)(z - 1)^2$$

$$1 - \gamma = \frac{d}{4W + d} \quad (\text{B1.30})$$

When $\tau_p - \tau_q$ tends to zero we can use Taylor expansion

$$\begin{aligned} \tau_p - \tau_q &= \varepsilon; \\ \tau_q &= \tau \\ \tau_p &= \tau + \varepsilon \\ r_q &= r(\tau_q) = r(\tau) \\ r_p &= r(\tau) + r'(\tau)\varepsilon + \frac{1}{2} r''(\tau)\varepsilon^2 + O(\varepsilon^2) \\ z_q &= z(\tau_q) = z(\tau) \\ z_p &= z(\tau) + z'(\tau)\varepsilon + \frac{1}{2} z''(\tau)\varepsilon^2 + O(\varepsilon^2) \end{aligned} \quad (\text{B1.31})$$

$$\begin{aligned} d &= \left(r'(\tau)\varepsilon + \frac{1}{2} r''(\tau)\varepsilon^2 \right)^2 + \left(z'(\tau)\varepsilon + \frac{1}{2} z''(\tau)\varepsilon^2 \right)^2 \\ &= (r_p - r_q)^2 + (z_p - z_q)^2 \end{aligned} \quad (\text{B1.32})$$

$$d = \varepsilon^2 (r'^2 + z'^2) + \varepsilon^3 (r' r'' + z' z'') + \varepsilon^4 \left(\frac{r''^2 + z''^2}{4} \right)$$

$$W = r_q r_p = r(\tau) \left[r(\tau) + r'(\tau)\varepsilon + \frac{r''(\tau)}{2}\varepsilon^2 + \frac{r'''(\tau)}{6}\varepsilon^3 + \frac{r^{1v}(\tau)}{24}\varepsilon^4 \right] \quad (\text{B1.33})$$

$$\begin{aligned} & d + 4W \\ &= 4r^2 \left[1 \right. \\ & \quad \left. + \frac{1}{4r^2} \left(\frac{4rr'\varepsilon + (r'^2 + z'^2 + 2rr'')\varepsilon^2}{+ (r'r'' + z'z'' + \frac{2}{3}rr''')} \varepsilon^3 + \left(\frac{r''^2 + z''^2}{4} + \frac{rr^{1v}}{6} \right) \varepsilon^4 \right) \right] \end{aligned} \quad (\text{B1.34})$$

$$\Delta = \frac{1}{4r^2} \left(\frac{4rr'\varepsilon + (r'^2 + z'^2 + 2rr'')\varepsilon^2}{+ (r'r'' + z'z'' + \frac{2}{3}rr''')} \varepsilon^3 + \left(\frac{r''^2 + z''^2}{4} + \frac{rr^{1v}}{6} \right) \varepsilon^4 \right) \quad (\text{B1.35})$$

we can write briefly

$$d + 4W = 4r^2(1 + \Delta) \quad (\text{B1.36})$$

If we put Δ definition in the γ formula (B28) we get this kind of structure

$$\gamma = \frac{W}{r^2} \frac{1}{(1 + \Delta)}$$

$$y = \frac{1}{1+x}; \quad y' = -(1+x)^{-2}; \quad y'' = 2(1+x)^{-3} \quad (\text{B1.37})$$

series expansion around zero

$$y = \frac{1}{1+x} \cong 1 - x + x^2 + O(x^3) \quad (\text{B1.38})$$

$$\gamma = \frac{W}{r^2} \frac{1}{(1 + \Delta)} = \frac{W}{r^2} \{1 - \Delta + \Delta^2\} \quad (\text{B1.39})$$

$$\frac{W}{r^2} = 1 + \frac{r' \varepsilon}{r} + \frac{r'' \varepsilon^2}{2r} + \frac{r''' \varepsilon^3}{6r} + \frac{r^{iv} \varepsilon^4}{24r} \quad (\text{B1.40})$$

$$\gamma = \frac{4W}{d + 4W} = \frac{w}{r^2} \frac{1}{(1 + \Delta)} = \frac{W}{r^2} (1 - \Delta + \Delta^2) \quad (\text{B1.41})$$

$$\gamma = 1 - \frac{1}{4r^2} (z'^2 + r'^2) \varepsilon^2 = 1 - \left(\frac{l\varepsilon}{2r} \right)^2 ; \quad l^2 = z'^2 + r'^2 \quad (\text{B1.42})$$

If we use this γ definition into elliptic integral asymptotic formula (B29)

$$\begin{aligned} K(\gamma) &\cong -\frac{1}{2} \log \left(\frac{l\varepsilon}{2r} \right)^2 \left[1 + \frac{1}{4} \left(\frac{l\varepsilon}{2r} \right)^2 + \frac{9}{64} \left(\frac{l\varepsilon}{2r} \right)^4 + \dots \right] \\ &+ \log(4) - \frac{1}{4} (1 - \log(4)) \left(\frac{l\varepsilon}{2r} \right)^2 + \frac{3}{128} (6 \log(4) - 7) \left(\frac{l\varepsilon}{2r} \right)^4 + \dots^4 \end{aligned} \quad (\text{B1.43})$$

Finally, we can use this description in Green's function formula (B25)

$$\begin{aligned} g^0_{(\tau_q, \tau_p)} &= \frac{2}{\pi} \frac{1}{\sqrt{\varepsilon^2 l + 4(r^2 + rr' \varepsilon + \frac{rr''}{2} \varepsilon^2)}} \\ &\left\{ \begin{aligned} &-\log \left(\frac{l\varepsilon}{2r} \right) \left[1 + \frac{1}{4} \left(\frac{l\varepsilon}{2r} \right)^2 + \frac{9}{64} \left(\frac{l\varepsilon}{2r} \right)^4 \right] \\ &+ \log(4) - \frac{1}{4} (1 - \log(4)) \left(\frac{l\varepsilon}{2r} \right)^2 \\ &+ \frac{3}{128} (6 \log(4) - 7) \left(\frac{l\varepsilon}{2r} \right)^4 \end{aligned} \right\} \quad (\text{B1.44}) \end{aligned}$$

Appendix C: Point Matching Method

For comparison our results, we used the standard point matching method with our kernel. We divide the parallel strips into N segments of width Δ_ρ with N basis functions.

$$\Delta_\rho = \frac{2(b-a)}{N} \quad (\text{C1.1})$$

The matrix elements can be given as follows.

$$Z_{mn} = \frac{\Delta_\rho}{4\pi} Q_{-\frac{1}{2}}(\chi(\rho_m, \varrho_n)) \quad (\text{C1.2})$$

This formula is valid for matrix elements other than diagonal terms. For diagonal terms (self terms) we need to consider singularity of the kernel. We can separate singular and smooth parts of Green's function.

$$Z_{mm} = \lim_{q \rightarrow p} \int_0^{\Delta_\rho} G(q, p) ds(p) \quad (\text{C1.3})$$

$$G(q, p) = \frac{1}{4\pi\sqrt{\rho_m \varrho_n}} \{S + D\} \quad (\text{C1.4})$$

$$S(\rho_m, \varrho_n) = -\frac{1}{\pi} \ln|\rho_m \varrho_n| + C_S \quad (\text{C1.5})$$

$$D(\rho_m, \varrho_n) = \frac{1}{2\pi} \left\{ \ln \frac{(b-a)^2}{\rho_m \varrho_m} \right\} + C_D \quad (\text{C1.6})$$

D is smooth part of kernel which we can integrate numerically. But singular part S need to evaluate analytically.

$$I = \int_{-\frac{\Delta_\rho}{2}}^{\frac{\Delta_\rho}{2}} \ln|\rho_m - \varrho_n| d\rho = \Delta_\rho \left(\ln(\Delta_\rho) \right) - (\ln 2 + 1) \quad (\text{C1.7})$$