

T.R.
GEBZE INSTITUTE of TECHNOLOGY
GRADUATE SCHOOL of ENGINEERING and SCIENCES

QUASILINEARIZATION METHODS for
CAUSAL DIFFERENTIAL EQUATIONS with
INITIAL TIME DIFFERENCE

ALİ ÖCAL ÖZLÜK
A THESIS SUBMITTED for THE DEGREE of
MASTER of SCIENCE
DEPARTMENT of MATHEMATICS

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SUMMARY

In this thesis, we endeavoured to exert Quasilinearization technique, classically nonlinear, which we use in the solution method of differential equation to causal differential equations with initial time difference, which is nonlinear.

In this work, similar results, given with either ordinary differential equations or fractional equations, have been attained for nonlinear differential equations results which are attained with exerting this method, in a way of being parallel to the conclusions having gained. So, monotone sequences have been attained given to the formal result of the nonlinear causal differential equations with initial time difference, converging uniformly and monotonically and being each element of the linear equation results, using quasilinearization technique under some conditions.

Key Words: Quasilinearization, Quadratic Convergence, Causal Differential Equations, Causal Operator, Initial Time Difference (ITD), Comparison Result.

ÖZET

Bu tez çalışmasında, klasik anlamda lineer olmayan diferansiyel denklemlerin çözüm yönteminde kullanılan kuasilineerizasyon tekniği, lineer olmayan başlangıç zaman farkı nedensel diferansiyel denklemlere uygulandı.

Sonuçta görüldü ki gerek klasik türevle gerekse fraksiyonel (kesirli mertebeden) türevle verilen ve lineer olmayan bir diferansiyel denklem için bu metodun uygulanmasıyla elde edilen sonuçlara paralel olacak şekilde benzer sonuçlar elde edilmiştir. Yani kuasilineerizasyon metodu kullanılarak belli koşullar altında verilen lineer olmayan başlangıç zaman farkı nedensel diferansiyel denklemin çözümüne düzgün, monoton ve kuadratik yakınsayan ve elemanlarından her biri lineer denklemin çözümleri olan monoton fonksiyon dizileri elde edilmiştir.

Anahtar Kelimeler: Kuasilineerizasyon, Kuadratik Yakınsaklık, Nedensel Diferansiyel Denklemler, Nedensel Operatör, Başlangıç Zaman Farkı (ITD), Karşılaştırma Metodu.

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LIST of ABBREVIATIONS and ACRONYMS

<u>Abbreviations</u> <u>and Acronyms</u>	<u>Explanations</u>
T, Z, P	: Causal Operator
CDE	: Causal Differential Equation
GYTE	: Gebze Institute of Technology
ITD	: Initial Time Difference
ODE	: Ordinary Differentian Equation

1. INTRODUCTION

The problems faced by modern society are in a wide range and contain a multidisciplinary perspective. Although the different aspects of problems, solutions produced in same field can be applicable to the other conditions. This amazing and productive method has become increasingly significance and has provided effective modern development using this different and dinamic system.

In this perspective, this study is dedicated to search the theory of causal differential equations or differential equations with causal operators. As it will be seen in the first chapter, causal differential equations include within the scope of dynamic systems and therefore this theory formed for CDEs (Causal Differential Equations) mainly.

When considered from this point of view, we see the importance of this study in the following chapters. Eventually, we submit a systematic account of new developments, explain the current state of this useful method, demonstrate the essential unity achieved and start different new extensions for other types of CDEs e.g., CDEs in infinite dimensional spaces with or without memory, fractional CDEs and set differential equations with causal.

2. FUNDAMENTAL THEOREMS and DEFINITIONS

Inverse causal operators are discussed and several examples which can be treated as causal operators are shown in the section 2.1, so one of them can formulate causal functional equations and integro-differential equations.

We collect several known basic results covering comparison results in section 2.2. The concept of Dini derivatives and their properties are shown. Section 2.3. is devoted to definition of the causal operator. In Section 2.4 and the concept of directional derivatives and their properties.

2.1. Basic Definition

If $E = C([0, \infty), R)$ and T , the operator defined by,

$$(Tx)(t) = x\left(\frac{t}{3}\right), \quad t \in [0, \infty), \quad (2.1)$$

continuous and has inverse in E , namely,

$$(T^{-1}y)(t) = y(3t), \quad t \in [0, \infty). \quad (2.2)$$

This operator is continuous, i.e., if $y_n \rightarrow y$ uniformly on any interval $[0, T]$, then $(T^{-1}y_n) \rightarrow T^{-1}(y)$ on any bounded interval of R_+ . But, as seen from (2.2) T^{-1} is not a causal operator. Hence, we cannot take it for granted the causality of the inverse of a causal operator.

However, with reference to the classical examples of causal operators, there are many situations where the inverse exists and is a causal operator. An example which plays a significant role in the study on second type linear integral equations (of Volterra)

$$x(t) = \int_{t_0}^t K(t, s)x(s)ds + f(t), \quad (2.3)$$

is provided by the operator

$$L(x)(t) = x(t) - \int_{t_0}^t K(t,s)x(s)ds \quad (2.4)$$

so that (2.3) can be written as

$$L(x)(t) = f(t), t \in [t_0, T]. \quad (2.5)$$

Choosing $E = C([t_0, T]), R^n$ and assuming that the kernel $K(t, s)$ is a continuous $n \times n$ matrix valued function on $t_0 \leq s \leq t \leq T$. This operator L is invertible on the space E and expressing $x = (L^{-1}f)$, we get the solution of (2.3) in terms of the resolvent formula given by

$$(L^{-1}f)(t) = x(t) = f(t) + \int_{t_0}^t R(t,s)f(s)ds, \quad (2.6)$$

where $R(t, s)$ is the resolvent kernel associated with $K(t, s)$ obtained by the method of successive approximations. Clearly L^{-1} is causal. Next, we shall provide examples of dynamic equations that can be included in causal differential equations of the form

$$x'(t) = T(x)(t), \quad x(t_0) = x_0, \quad (2.7)$$

clearly, the IVP

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0. \quad (2.8)$$

Can be considered as a CDE by identifying $f(t, x(t))$ with $(Tx)(t)$.

The next example is difference-differential IVP

$$\left. \begin{aligned} x'(t) &= f(t, x(t), x(t-\tau)), \quad \tau > 0 \\ x_{t_0} &= \phi_0(s), \quad t_0 - \tau \leq s \leq t_0 \end{aligned} \right\} \quad (2.9)$$

where ϕ_0 is the initial function at $t = t_0$. Clearly (2.9) is a causal differential equation since, we can make the identification

$$f(t, x(t), x(t - \tau)) \equiv (Tx)(t) \quad (2.10)$$

and because of the finite memory that is involved, we need the information on $x(t)$ for $[t_0 - \tau, t_0]$ given as the initial function $\phi_0(s)$. More general than (2.9) is known as the functional differential equation, given by

$$x(t) = f(t, x_t), x_{t_0} = \phi_0(s) \quad (2.11)$$

where $x_t = x_t(s) = x(t + s)$, $-\tau \leq s \leq 0$ (2.11) is also known as delay-differential equation and is clearly a causal differential equation with finite memory.

Consider the next IVP given by an integro-differential equation

$$x'(t) = \int_{t_0}^t K(t, s, x(s))ds, \quad x(t_0) = x_0. \quad (2.12)$$

One can incorporate finite memory and write (2.12) as

$$\left. \begin{aligned} x'(t) &= \int_{t-\tau}^t K(t, s, x(s))ds, \quad \tau > 0 \\ x(t_0 + s) &= x_{t_0} = \phi_0(s), \quad -\tau \leq s \leq t_0, \quad t_0 \geq 0. \end{aligned} \right\} \quad (2.13)$$

A more general problem than (2.12) is given by

$$\left. \begin{aligned} x'(t) &= f(t, x(t)) + \int_{t_0}^t K(t, s, x(s))ds \\ x(t_0) &= x_0, \end{aligned} \right\} \quad (2.14)$$

and if we incorporate delay, we get

$$\left. \begin{aligned} x'(t) &= f(t, x(t)) + \int_{t-\tau}^t K(t, s, x(s))ds \\ x(t_0 + s) &= x_{t_0} = \phi_0(s), \quad -\tau \leq s \leq t_0, \quad t_0 \geq 0, \end{aligned} \right\} \quad (2.15)$$

or in general

$$\left. \begin{aligned} x'(t) &= f(t, x_t) + \int_{t-\tau}^t K(t, s, x(s)) ds \\ x(t_0 + s) &= x_{t_0} = \phi_0(s), \quad -\tau \leq s \leq t_0, \quad t_0 \geq 0. \end{aligned} \right\} \quad (2.16)$$

The very general integro-differential equation can be written as

$$\left. \begin{aligned} x'(t) &= f(t, x(t)) + \int_{t_0}^t K(t, s, x(s)) ds, \\ x(t_0) &= x_0, \end{aligned} \right\} \quad (2.17)$$

which can be extended with memory as

$$\left. \begin{aligned} x'(t) &= f(t, x(t)), \int_{t-\tau}^t K(t, s, x(s)) ds \\ x_{t_0} &= \phi_0(s), \end{aligned} \right\} \quad (2.18)$$

or more generally,

$$\left. \begin{aligned} x'(t) &= f(t, x(t), x_t, \int_{t-\tau}^t K(t, s, x(s)) ds) \\ x_{t_0} &= \phi_0(s). \end{aligned} \right\} \quad (2.19)$$

All equations (2.12), (2.19) are examples of causal differential equations. In all of these equations where x_t occurs, x_t represents the graph of x in $[t - \tau, t]$ shifted to the interval $[-\tau, 0]$. We could have delay or memory of volterra type, namely, x_t is the graph of x on $[t_0 - \tau, t]$ in which case, an IVP could be of the form, for example,

$$x'(t) = f(t, x_t) + \int_{t_0-\tau}^t K(t, t, x(s)) ds, \quad x_{t_0} = \phi_0(s), \quad (2.20)$$

for example, the IVP

$$x'(t) = f(t, \max_{t_0 \leq s \leq t} x(s)), \quad x(t_0) = x_0 \quad (2.21)$$

or

$$x'(t) = f(t, x[t]), \quad x(t_0) = x_0. \quad (2.22)$$

Where the notation $x([t])$ represents the maximum value of $x(t)$ in each interval $[t_n, t_{n+1}]$, $n = 1, 2, \dots$.

Another possibility is difference equation

$$x_{n-1} - x_n = \sum_{i=1}^n K(i+1, i, x_i) x_{n_0} = x_0. \quad (2.23)$$

In this case, we see that the IVP for causal differential equation or causal equation a variety of problems, that we normally study independently in different branches of nonlinear analysis. But, all of these can be involved as special cases of causal differential or causal functional equations infinite.

The equations with causal operators also include integral equations of the type

$$x(t) = h(t) + \int_{t_0}^t K(t, s, x(s)) ds. \quad (2.24)$$

The functional differential equation can also contain advance argument as well as delay, such as

$$x'(t) = f(t, x(t), x(t - \tau_1), x(t + \tau_2)), \quad \tau_1, \tau_2 > 0 \quad (2.25)$$

a typical example with infinite delay is

$$x'(t) = \int_{-\infty}^t K(t, s, x(s)) ds, \quad (2.26)$$

and in general, functional differential equations with unbounded delay are also examples of causal differential equations.

2.2. Basic Results

We list, in this section, certain basic comparison results, that concern with estimating a function which satisfies a differential inequality by the extremal solutions of the corresponding differential equation. Sometimes, it is enough to have the differential inequality satisfied relative to only Dini derivatives. We adopt the following notation for Dini derivatives:

$$D^+u(t) = \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [u(t+h) - u(t)]; \quad (2.27)$$

$$D_+u(t) = \lim_{h \rightarrow 0^+} \inf \frac{1}{h} [u(t+h) - u(t)]; \quad (2.28)$$

$$D^-u(t) = \lim_{h \rightarrow 0^-} \sup \frac{1}{h} [u(t+h) - u(t)]; \quad (2.29)$$

$$D_-u(t) = \lim_{h \rightarrow 0^-} \inf \frac{1}{h} [u(t+h) - u(t)]; \quad (2.30)$$

where $u \in C((t_0, t_0 + \alpha), R)$. When $D^+u(t) = D_+u(t)$ the right derivative is denoted by $u'_+(t)$. Similarly $u'_-(t)$ denotes the left derivative when $D^-u(t) = D_-u(t)$. The following results are useful in the sequel.

Let us begin with a simple result.

Theorem 2.2.1: Suppose $m(t)$ is continuous on (a, b) . Then $m(t)$ is nondecreasing (nonincreasing) on (a, b) if and only if $D^+m(t) \geq 0$ (≤ 0) for every $t \in (a, b)$, where

$$D^+m(t) = \lim_{\delta \rightarrow 0^+} \sup \frac{1}{\delta} [m(t+\delta) - m(t)]. \quad (2.31)$$

Proof 2.2.1: The condition is obviously necessary. Let us prove that it is sufficient. Suppose that $D^+m(t) > 0$ on (a, b) . If there exists two points $\alpha, \beta \in (a, b)$, $\alpha < \beta$, $\alpha, \beta \in (a, b)$. Such that $m(\alpha) > m(\beta)$ then there exists a μ with $m(\alpha) > \mu > m(\beta)$ and some points $t \in [\alpha, \beta]$ such that $m(t) > \mu$. $t \in [\alpha, \beta]$ Such that $m(t) > \mu$. Let $\zeta = \sup\{t; m(t) > \mu, t \in [\alpha, \beta]\}$. Clearly, $\zeta \in (\alpha, \beta)$ and $m(\zeta) = \mu$.

Therefore, for every $t \in (\zeta, \beta)$, we have $\frac{m(t)-m(\zeta)}{t-\zeta} < 0$

which implies $D^+m(\zeta) \leq 0$. This is a contradiction and therefore the proof is complete.

Theorem 2.2.2: Let $v, w \in C([t_0, T], R)$ and for some fixed Dini derivative, $Dv(t) \leq w(t), t \in [t_0, T]$ then $D_-v(t) \leq w(t), t \in [t_0, T]$.

Proof 2.2.2: Define the function,

$$m(t) = v(t) \int_{t_0}^t w(s) ds. \quad (2.32)$$

It then follows, from the assumption, that,

$$Dm(t) = Dv(t) - w(t) \leq 0, t \in [t_0, T]. \quad (2.33)$$

Hence by Theorem 2.2.1, $m(t)$ is nonincreasing in t on $[t_0, T]$. Consequently

$$Dm(t) = D_-v(t) - w(t) \leq 0, t \in [t_0, T], \quad (2.34)$$

and the theorem is proved.

The existence theorem together with the extension result demonstrates the following.

Theorem 2.2.3: Let $g \in C(E, R)$, where E is an open (t, u) set in R^2 and $(t_0, u_0) \in E$.

Then, the IVP

$$u' = g(t, u), \quad u(t_0) = u_0. \quad (2.35)$$

As extremal solutions (that is, minimal and maximal solutions) which can be extended up to the boundary of E . The next theorem is useful in certain situations.

Theorem 2.2.4: Let the hypotheses of theorem 2.2.3 hold and let $[t_0, T)$ be the largest interval of existence of maximal solution $r(t)$ of IVP (2.35) suppose that $[t_0, t_1]$ is a

compact subinterval of $[t_0, T)$. Then there is an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the maximal solution $r(t, \varepsilon)$ of IVP

$$u' = g(t, u) + \varepsilon, u(t_0) = u_0 + \varepsilon, \quad (2.36)$$

exists over $[t_0, t_1]$ and

$$r(t) = \lim_{\varepsilon \rightarrow 0} r(t, \varepsilon) \quad (2.37)$$

uniformly on $[t_0, t_1]$.

Theorem 2.2.5: Let $g \in C[[t_0, t_0 + \alpha] \times R, R]$ and nondecreasing in u for each $t \in [t_0, t_0 + \alpha]$. Assume that

$$g(t, 0) \equiv 0,$$

$$|g(t, u)| \leq M \text{ on } [t_0, t_0 + \alpha] \times R. \quad (2.38)$$

And $u(t) \equiv 0$ is the unique solution of

$$u' = g(t, u), u(t_0) = 0 \quad (2.39)$$

on $[t_0, t_0 + \alpha]$. Then, the successive approximations

$$u_0 = M(t - t_0),$$

$$x'(t) = \int_{t_0}^t g(s, u_n(s)) ds \quad (2.40)$$

are well defined;

$$0 \leq u_{n+1}(t) \leq u_n(t) \text{ on } [t_0, t_0 + \alpha] \quad (2.41)$$

$$\lim_{n \rightarrow \infty} u_n(t) \equiv 0 \text{ uniformly on } [t_0, t_0 + \alpha]. \quad (2.42)$$

Moreover, for $n \geq 1$, the maximal solution $r_n(t)$ of

$$u' = g(t, u) + kg(t, u_{n-1}(t)), \quad u_n(t_0) = 0 \quad k > 0 \quad (2.43)$$

exists on $[t_0, t_0 + \alpha]$ and $\lim_{n \rightarrow \infty} r_n(t) \equiv 0$ uniformly on $[t_0, t_0 + \alpha]$.

One of the results which are commonly used is the following comparison theorem.

Theorem 2.2.6: Let E be an open (t, u) set in R^2 and let $g \in C[E, R]$. Suppose that $[t_0, t_0 + \alpha]$ is the largest interval in which the maximal solution $r(t)$ of (2.35) exists.

Let $m \in C[(t_0, t_0 + \alpha), R]$ ($m(t) \in E$ for $t \in [t_0, t_0 + \alpha)$, $m(t_0) \leq u_0$, and for a fixed Dini derivative,

$$Dm(t) \leq g(t, m(t)), \quad t \in [t_0, t_0 + \alpha), \quad (2.44)$$

Then,

$$m(t) \leq r(t), \quad t \in [t_0, t_0 + \alpha), \quad (2.45)$$

where $r(t) = r(t, t_0, u_0)$ is the maximal solution of IVP (2.35) existing on $[t_0, t_0 + \alpha)$.

2.3. Causal Operator

This is primarily a brief study about account of the investigation of equations with causal or nonanticipatory or Volterra operators. We use the word 'causal' to indicate such operators. The causal operators can be explained using several functions or functionals that result in the formulation of many dynamic and functional systems incl. discrete systems. For this reason, the theory of causal systems becomes very significant, because a single result which involves causal operators covers interesting corresponding results arising from many categories of dynamic systems, in this way avoiding duplication and monotony of repetition. Also, within this

general set up, one can imagine how much we can progress and potential barriers, at least. The researching of the theory of causal systems, dynamic or others, is a significant field of nonlinear analysis.

Firstly, let us explain the concept of the causal operator. In the formula $E = E(J, X)$, J is an appropriate time interval, X means either finite or infinite dimensional space, based on the requirement of the context, and E is a function space. The operator $T: E \rightarrow E$ is said to be a causal or nonanticipatory operator if the following property is satisfied.

For each couple elements x, y of E such that $x(s) = y(s)$ for $0 \leq t_0 \leq s \leq t$ we also have $(Tx)(s) = (Ty)(s)$ for $0 \leq t_0 \leq s \leq t$, $t < T$, T being arbitrary. Actually, this definition requires a little modification when E is a space of measurable functions on $[t_0, T)$, $t_0 \geq 0$, requiring property (*) to be valid almost everywhere on $[t_0, T]$. We want to focus on causal operators, a notation identical with what is encountered for a general equation which has a memory which may be specified as follows. A representation of the form

$$(Tx)(t) = T(x, x_t) \quad (2.46)$$

where for each $t \in [t_0, T)$, $T(t, x_t)$ is a functional on E which takes values in X , for each t , while the whole family of functionals, $t \in [t_0, T)$, define the operator from $E = C([t_0, T), X)$ to itself. Sometimes, this description is more clear in definition of the operator T . Numerous operators for function spaces have been identified and examined in connection with their use in the theory of functional or functional differential equations.

The sum and product of two causal operators $T: E \rightarrow E$ and $P: E \rightarrow E$ are causal. The other feature of causal operators is related to the convergence of a sequence of such operators. For instance, let us take $E = C([t_0, T), R^n)$ as the underlying space. Let $\{T_n\}$ be a sequence of causal operators on E such that

$$\lim_{n \rightarrow \infty} (T_n x)(t) = (Tx)(t) \quad (2.47)$$

for each $(t, x) \in [t_0, T) \times E$. The question is whether we can deduce that the limit $T: E \rightarrow E$ is also a causal operator. The answer will be yes because the causality of

$\{T_n\}$ implies that

$$T_n x(s) = (T_n y)(s), \quad s \in [t_0, T]. \quad (2.48)$$

If we let $n \rightarrow \infty$ on both sides, in the above relation and use (2.47) for each fixed $s \in [t_0, T]$, we obtain the causality of T .

2.4. Directional Derivatives

When we use $|x|$ or $|x|^2$ as a measure in subsequent estimates, we have to assume some conditions from the point of their one-sided directional derivatives. In this regard we identify and list some features of such derivatives.

Let $x, y \in E$, E being a real banach space with norm $|\cdot|$. Define

$$[x, y]_h = \frac{1}{h} (|x + hy| - |x|) \quad (2.49)$$

For any $h \in R$. Then we reached the following result.

Theorem 2.4.1:

- i) The limits $\lim_{h \rightarrow 0^+} [x, y]_h = [x, y]_+$ and $\lim_{h \rightarrow 0^-} [x, y]_h = [x, y]_-$ exist; and
- ii) $[x, y]_+$ is upper semicontinuous and $[x, y]_-$ is lower semicontinuous.

Proof 2.4.1: Let us first show that $[x, y]_h$ is monotone nondecreasing in h . Suppose $0 < h_1 < h_2$ and let $\beta \in (0, 1)$ be such that $h_1 = (1 - \beta)h_2$

Since

$$x + h_1 y = x + (1 - \beta)h_2 y = \beta x + (1 - \beta)(x + h_2 y), \quad (2.50)$$

we have

$$[x, y]_{h_1} = \frac{1}{h_1} (|x + h_1 y| - |x|)$$

$$\begin{aligned}
&= \frac{1}{h_1} \{ |\beta x + (1 - \beta)(x + h_2 y)| - |x| \} \\
&\leq \frac{1}{h_1} \{ |\beta x + (1 - \beta)(x + h_2 y)| - |x| \} = \frac{|x + h_2 y| - |x|}{h_2} = [x, y]_{h_2} \quad (2.51)
\end{aligned}$$

similarly, one gets $[x, y]_{h_1} \leq [x, y]_{h_2}$ if $h_1 < h_2 < 0$. If $h_1 < 0 < h_2$, we let $h = \min(-h_1, h_2)$ and note that

$$2|x| = |x + hy + x - hy| \leq |x + hy| + |x - hy| \quad (2.52)$$

this implies that $[x, y] - h \leq [x, y]h$, which in turn yields

$$[x, y]h_1 \leq [x, y] - h \leq [x, y]h_2 \quad (2.53)$$

proving that $[x, y]h$ is monotone. If $-1 \leq h_1 \leq h_2 \leq 1$, the monotone property of $[x, y]$ gives

$$[x, y]_{-1} \leq [x, y]h_1 \leq [x, y]h_2 \leq [x, y]_1 \quad (2.54)$$

and hence the limits

$$[x, y]_+ = \lim_{h \rightarrow 0^+} [x, y]_h \quad (2.55)$$

and

$$[x, y]_- = \lim_{h \rightarrow 0^-} [x, y]_h, \quad (2.56)$$

exist. To prove that $[x, y]_+$ is upper semicontinuous; let $\{x_n\}, \{y_n\}$ be two sequences in E such that $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$. Then for $h > 0$

$$[x_n, y_n]_+ \leq \frac{1}{h} \{ |x_n - hy_n| - |x_n| \} \text{ for all } n \geq 1. \quad (2.57)$$

Letting $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \sup [x_n, y_n]_+ \leq \frac{1}{h} \{|x + hy| - |x|\} \quad (2.58)$$

for all $h > 0$. We now let $h \rightarrow 0^+$, obtaining

$$\lim_{n \rightarrow \infty} \sup [x_n, y_n]_+ \leq [x, y]_+. \quad (2.59)$$

Since $[x, y]_- = [x, -y]_+$ by definition, lower semicontinuity of $[x, y]_-$.

3. FUNDAMENTAL THEORIES

The basic theory of causal differential equations (CDE) is discussed in this chapter and for this reason this chapter forms a basis for the next chapters. Section 3.1, 3.2 starts with causal differential inequalities and proves fundamental results including strict and nonstrict functional and causal differential inequalities, which are helpful to verify existence of extrem solutions and significant comparison results. Section 3.3, 3.4 is about local existence, global existence results including the existence of maximal and minimal solution and upper and lower solutions of CDEs, which are essential for next discussions. While manifesting the significant comparison results in this section, we notice that when imposing various conditions on the causal functions included, it is enough to assume such conditions on a reasonable subset of the function space, which is a certain benefit in searching the qualitative features of solutions of CDEs.

We discuss global existence by using two different approaches; one of them is based on the use of Tychonoff's fixed point theorem and the other on a direct approach using the comparison results. Section 3.5 examines the existence and uniqueness. Section 3.6 is a Classical Notions of Quasilinearization Method of CDEs

3.1. Causal Functional and Differential Inequalities

Recall that an operator $T: E \rightarrow E$, $E = C([t_0, T], R^n)$ is said to be causal if, for any $x, y \in E$ such that $x(s) = y(s)$, we have $(Tx)(s) = (Ty)(s)$, $t_0 \leq s < T$. Let us consider the causal functional equation

$$x(t) = (Tx)(t), \quad x(t_0) = x_0. \quad (3.1)$$

Where the causal operator $T: E \rightarrow E$ is continuous and $x(t_0) = x_0$, $t_0 \geq 0$, states the initial value for any $x \in E$. A fundamental result in causal functional inequalities for the scalar situation is the following.

Theorem 3.1.1: Assume that $T: E \rightarrow E$ is a continuous causal operator where $E = C([t_0, T], R)$ and let $v, w \in E$ satisfy

$$v(t) \leq (Tv)(t), \quad w(t) \geq (Tw)(t), \quad t_0 \leq t < T. \quad (3.2)$$

Suppose that T is semi-nondecreasing i.e.

$$x(t_1) = y(t_1), \quad x(t) < y(t), \quad t_0 \leq t < t_1 < T \quad (3.3)$$

implies

$$(Tx)(t_1) = (Ty)(t_1) \text{ and } (Tx)(t) \leq (Ty)(t), \quad t_0 \leq t < t_1 < T, \quad (3.4)$$

then,

$$v(t) < w(t), \quad t_0 \leq t < T \quad (3.5)$$

whenever

$$v(t_0) < w(t_0), \quad (3.6)$$

provided one of the inequalities in (3.2) is strict.

Proof 3.1.1: Suppose that the claim (3.5) is false and $w(t) > (Tw)(t)$. Then because of the continuity of the functions involved and (3.6), there would exist a $t_1 > t_0$ such that

$$v(t_1) = w(t_1), \quad v(t) < w(t), \quad t_0 \leq t < T \quad (3.7)$$

since T is assumed to be semi-nondecreasing, we have

$$v(t_1) \leq (Tv)(t_1) \leq (Tw)(t_1) < w(t_1). \quad (3.8)$$

This contradicts $v(t_1) = w(t_1)$ and therefore, proves the claim (3.5). The proof is complete.

The next result is for nonstrict inequalities that requires a one-sided Lipschitz condition.

Theorem 3.1.2: Assume further that

$$(Tx)(t) - (Ty)(t) \leq \max_{t_0 \leq s \leq t} [x(s) - y(s)] \quad (3.9)$$

whenever $x(s) \geq y(s)$ for $t_0 \leq s \leq t$ and $0 < L < 1$. Then

$$v(t_0) \leq w(t_0) \text{ Implies } v(t) \leq w(t), \quad t_0 \leq t < T \quad (3.10)$$

Proof 3.1.2: Set $w_\varepsilon(t) = w(t) + \varepsilon$, $\varepsilon > 0$ being arbitrary and small. Then we have

$$w_\varepsilon(t_0) \geq w(t_0) + \varepsilon \geq v(t_0) + \varepsilon > v(t_0) \quad (3.11)$$

and

$$w_\varepsilon(t) \geq w(t), \quad t_0 \leq t < T. \quad (3.12)$$

Now, utilizing the one-sided Lipschitz condition (3.9), we get

$$w_\varepsilon(t) \geq (Tw)(t) + \varepsilon \geq (Tw_\varepsilon)(t) - L_\varepsilon + \varepsilon > (Tw)(t) \quad (3.13)$$

for $t_0 \leq t < T$, because of the condition $0 \leq L < 1$. Now, applying Theorem 3.1.1 to $v(t)$ and $w_\varepsilon(t)$, we find that.

$$v(t) < w_\varepsilon(t), \quad t_0 \leq t < T \quad (3.14)$$

proving the stated conclusion (3.10).

We will now pay attention to the fundamental result on causal differential inequalities.

For that purpose, take notice the IVP for the causal differential equation.

$$x'(t) = (Tx)(t), \quad x(t_0) = x_0 \quad (3.15)$$

where $T : E \rightarrow E$ is a continuous causal operator. Relative to (3.15) we can prove the following basic inequalities result.

Theorem 3.1.3: Suppose that $v, w \in C([t_0, T], R)$ and

$$v'(t) \leq (Tv)(t), \quad w'(t) \geq (Tw)(t), \quad t_0 \leq t < T \quad (3.16)$$

then, $v(t_0) < w(t_0)$ implies

$$v(t) < w(t), \quad t_0 \leq t < T \quad (3.17)$$

provided one of the inequalities in (3.16) is strict and the causal operator T is semi-nondecreasing.

Proof 3.1.3: Suppose that the conclusion (3.17) is false and $w(t) > (Tw)(t)$. Then, the continuity of v, w and the fact $v(t_0) < w(t_0)$ yield that there exists a $t_1 \geq t_0$ such that

$$v(t_1) = w(t_1), \quad v(t) < w(t) \text{ on } t_0 \leq t < t_1 \quad (3.18)$$

the semi-nondecreasing nature of T and (3.18) give

$$(Tv)(t_1) \leq (Tw)(t_1). \quad (3.19)$$

In view of (3.18), we get for small $h > 0$,

$$v(t_1 - h) - v(t_1) < w(t_1 - h) - w(t_1) \quad (3.20)$$

and hence (3.16) and (3.19) show that

$$(Tv)(t_1) \geq v'(t_1) \geq w'(t_1) > (Tw)(t_1) \geq (Tv)(t_1) \quad (3.21)$$

this is a contradiction and so the claim (3.17) is valid. Proving process is complete.

As previously, for nonstrict differential inequalities, we demand a one-sided Lipschitz condition.

3.2. Existence and External Solutions

We shall consider the causal functional equation and causal differential equation respectively given by

$$x(t) = (Tx)(t), \quad x(t_0) = x_0 \quad (3.22)$$

and

$$x'(t) = (Tx)(t), \quad x(t_0) = x_0 \quad (3.23)$$

where $T: E \rightarrow E, E = C([t_0, T], R^n), x_0 \in R^n$. The problem (3.23) can be reduced to

$$x(t) = x_0 + \int_{t_0}^t (Tx)(s) ds. \quad (3.24)$$

We will examine the existence problem in (3.22) and (3.23). But, firstly we will handle the functional equation with causal operator by proving the following existence theorem.

Theorem 3.2.1: Assume that the causal operator $T: E \rightarrow E$ in equation (3.22) is continuous and compact, as well as the feature of the fixed initial value

$x(t_0) = x_0 \in R^n$. And then there exists a solution of (3.22) on $C([t_0, t_0 + \delta], R^n)$ for some $\delta > 0$ and $t_0 + \delta \leq T$

Proof 3.2.1: For any $x \in C([t_0, T], R^n)$, consider the initial value $x(t_0) = x_0$ and the functional equation

$$x(t) - x_0 = (Tx)(t) - x_0 \quad (3.25)$$

that is equivalent to (3.22). In consideration of the compactness assumption of T , if we fixed $r > 0$, one can write

$$|(Tx)(t) - x_0| \leq \beta \text{ for } t_0 \leq t \leq t_0 + \delta \quad (3.26)$$

and for all $x \in C([t_0, T], R^n)$ satisfying

$$|x(t) - x_0| \leq r, \quad t \in [t_0, T]. \quad (3.27)$$

If r is fixed, then δ depends on β . We can deem without loss of generality that $\beta \leq r$. Then states that

$$|(Tx)(t) - x_0| \leq \beta \leq r, \quad (3.28)$$

for all $x(t)$ with $|x(t) - x_0| \leq r$, on $[t_0, t_0 + \delta]$. This infers that in the space $E = C([t_0, t_0 + \delta], R^n)$, the ball of radius r with center x_0 is taken into itself by T .

Since T is deemed to be continuous and compact, Schauder's fixed point theorem applies in the Banach space $C([t_0, t_0 + \delta], R^n)$ and the ball $|x(t) - x_0| \leq r$ is clearly convex and closed. Thus there exists at least one solution $x(t)$ of (3.22) on the interval $[t_0, t_0 + \delta]$. Proven process is complete.

We will next prove the existence result for IVP.

Theorem 3.2.2: Pay attention to the causal differential equation (3.23) with the initial condition. Assume that T is a causal operator on $C([t_0, T], R^n)$, continuous and takes bounded sets into bounded sets. Then, there exists a solution $x(t)$ of (3.23) on $[t_0, t_0 + \delta], t_0 + \delta \leq T$

Proof 3.2.2: Since the IVP (3.23) is equivalent to the functional equation (3.24), it is enough to prove that the operator W given by

$$(Wx)(t) = x_0 + \int_{t_0}^t (Tx)(s)ds \quad (3.29)$$

satisfies the conditions of Theorem 3.2.1: the continuity and causality of the operator W is clear. Since the convergence in E is uniform convergence, the continuity of T implies the continuity of W . To prove the compactness of W in E , assume that $x \in B \subset E$ where B is a bounded set in E , with T bounded in E . In view of the assumption that T takes bounded sets into bounded sets, for each $x \in B$ we have

$$|(Tx)(t)| \leq K, t \in [t_0, T]. \quad (3.30)$$

Therefore, for each $x \in B$, one has

$$|(Wx)(t)| \leq |x_0| + K(t - t_0), t \in [t_0, T] \quad (3.31)$$

and

$$|(Wx)(t) - (Wx)(s)| \leq K|t - s|, \quad t, s \in [t_0, T]. \quad (3.32)$$

These inequalities state that the subset of E consisting of those functions that are of the form $(Wx)(t)$, $x \in B$, satisfies the conditions of Ascoli-Arzelà Theorem and therefore W is compact. For this reason, by Theorem 3.2.1 there exists a solution $x(t)$ of (3.24) on $[t_0, t_0 + \delta]$ and equivalently of (3.23). So the theorem is proved.

We will now examine the existence of extremal solutions for the IVP (3.23) i.e. maximal and minimal solutions in (3.23). We will do this only for the scalar situation.

Definition 3.2.1: Let $r(t)$ be a solution of IVP (3.23) on $[t_0, T]$. Then $r(t)$ is said to be the maximal solution of (3.23) if for every solution $x(t)$ of (3.23) existing on $[t_0, T]$ we have

$$x(t) \leq r(t), t_0 \leq t < T. \quad (3.33)$$

In a similar way, if $p(t)$ is a solution of (3.23) and for every solution $x(t)$ of (3.23) existing on $[t_0, T]$, we have $x(t) \geq p(t)$, $t_0 \leq t < T$, then $p(t)$ is said to be the

minimal solution of (3.23).

Theorem 3.2.3: Suppose that $m \in C^1([t_0, T], R^+)$ satisfies

$$m'(t) \leq (Tm)(t), \quad m(t_0) \leq x_0 \quad (3.34)$$

then

$$m(t) \leq r(t), \quad t \in [t_0, T], \quad (3.35)$$

where $r(t)$ is the maximal solution of (3.23) existing on $[t_0, T]$.

Proof 3.2.3: Let $x(t, \varepsilon) \equiv x_\varepsilon$ for $\varepsilon > 0$, be any solution of

$$x'(t, \varepsilon) = (Tx_\varepsilon) + \varepsilon, \quad x(t_0) = x_0 + \varepsilon \quad (3.36)$$

$$m(t) < x(t, \varepsilon), \quad t \in [t_0, T] \quad (3.37)$$

from which it follows that $m(t) \leq r(t)$ on $[t_0, T]$ and the proof is complete.

3.3. Global Existence

Pay attention the IVP for causal differential equation

$$x'(t) = (Tx)(t), \quad x(t_0) = x_0 \quad (3.38)$$

where $T \in E = C([t_0, \infty), R^n)$ is a continuous causal operator. We will first employ Tychonoff's fixed point theorem to get global existence of solutions in (3.38).

Theorem 3.3.1: Let $T \in E$ and satisfy the estimate

$$|(Tx)(t)| \leq g(t, |x(t)|), \quad x \in \Omega \quad (3.39)$$

where $\Omega = \{x \in E: \max_{t_0 \leq s \leq t} |x(s)| = |x(t)|\}$ and $g \in C([t_0, \infty) \times \mathbb{R}^+, \mathbb{R}^+)$, $g(t, u)$ is monotone nondecreasing in u for each $t \in [t_0, \infty)$. Suppose that, for every $u_0 > 0$, the scalar differential equation

$$u' = g(t, u), \quad u(t_0) = u_0 \geq 0, \quad (3.40)$$

has a solution $u(t)$ existing on $[t_0, \infty)$. And the every x_0 such that $|x_0| \leq u_0$, there exists a solution of $x(t)$ (3.38) on $[t_0, \infty)$ satisfying

$$|x(t)| \leq u(t), \quad t \in [t_0, \infty). \quad (3.41)$$

Proof 3.3.1: Pay attention to the space E , the topology on E being that induced by the family of pseudonorms $\{p_n(x)\}_{n=1}^\infty$ where for $x \in E$,

$$p_n(x) = \sup_{t_0 \leq t \leq n} |x(t)| \quad (3.42)$$

a basic system of neighborhoods is then given by $\{V_n(x)\}_{n=1}^\infty$ where

$$V_n(x) = \{x \in E: p_n(x) \leq 1\}. \quad (3.43)$$

Under this topology, E becomes a complete, locally convex linear space. Now, define a subset E_0 , $E_0 \subset E$ as follows

$$E_0 = \{x \in \Omega: |x(t)| \leq u(t), \quad t \geq t_0\}, \quad (3.44)$$

where $u(t)$ is a solution of (3.40) existing on $[t_0, \infty)$. It is quite clear that, in the topology of E , the set E_0 is closed, convex and bounded. Consider the integral operator defined by

$$(Tx)(t) = x_0 + \int_{t_0}^t (Tx)(s) ds \quad (3.45)$$

whose fixed point corresponds to the solution of (3.38). This operator T is compact in the topology of E and therefore closure of $T \in E_0$ is compact in respect of the boundedness of E_0 .

For proving the theorem, it remains to be shown that $TE_0 \subset E_0$. For this purpose, we observe that for any $x \in E_0$,

$$|(Tx)(t)| \leq |x_0| + \int_{t_0}^t |(Tx)(s)| ds \quad (3.46)$$

$$|x_0| + \int_{t_0}^t g(s, |x(s)|) ds \quad (3.47)$$

$$|x_0| + \int_{t_0}^t g(s, u(s)) ds, \quad (3.48)$$

here, we have used the monotone nature of $g(t, u)$, the definition of set E_0 and the fact that $u(t)$ is a solution of (3.40) with $|x_0| \leq u_0$. It then follows from (3.48) that

$$|(Tx)(t)| \leq u(t) \quad (3.49)$$

proving that $TE_0 \subset E_0$ and proving process is complete.

A direct proof of global existence is given by the following result

Theorem 3.3.2: Suppose that $T \in E$ and is smooth enough to guarantee local existence of (3.38) solutions for any (t_0, x_0) and satisfies the inequality (3.39) for $x \in \Omega$, where $g \in C(R_+^2, R_+)$, $g(t, u)$ is nondecreasing in u for each $t \in [t_0, \infty)$. Assume further that the maximal solution $r(t)$ of the scalar equation (3.40) exists on $[t_0, \infty)$. And then the largest interval of existence of any solution $x(t)$ of (3.38) such that $|x_0| \leq u_0$ is $[t_0, \infty)$.

Proof 3.3.2: Let $x(t)$ be any solution of (3.38) with $|x_0| \leq u_0$, which exists on $[t_0, \beta)$, for $t_0 < \beta < \infty$ and such that the value of β can not be increased. Define

$$m(t) = |x(t)|, \quad t_0 < t < \beta \quad (3.50)$$

then, using the assumption (3.39), we obtain

$$D^+m(t) \leq |x^t(t)| = |(Tx)(t)| \leq g(t, |x(t)|) \quad (3.51)$$

i.e.; we have the differential inequality

$$D^+m(t) \leq g(t, m(t)), m(t_0) \leq u_0 \quad (3.52)$$

$$m(t) \leq |x(t)| \leq r(t), t_0 < t < \beta \quad (3.53)$$

where $r(t)$ is the maximal solution of (3.40). For any t_1, t_2 such that $t_0 \leq t_1 < t_2 \leq \beta$

$$|x(t_1) - x(t_2)| = \left| \int_{t_1}^{t_2} (Tx)(s) ds \right| \quad (3.54)$$

$$\leq \int_{t_0}^t g(s, m(s)) ds \quad (3.55)$$

$$\leq \int_{t_0}^t g(s, r(s)) ds \quad (3.56)$$

$$= r(t_2) - r(t_1). \quad (3.57)$$

Here we used (3.39), (3.50) and the monotone character of $g(t, u)$. Since $\lim_{t \rightarrow \beta^-} r(t)$ exists and is finite, taking limits as $t_1, t_2 \rightarrow \beta^-$ and using Cauchy criterion for convergence, it follows that $\lim_{t \rightarrow \beta^-} x(t)$ exists. We define

$$x(\beta) = \lim_{t \rightarrow \beta^-} x(t) \quad (3.58)$$

and consider the IVP

$$x'(t) = (Tx)(t) \quad (3.59)$$

with initial value $x(\beta)$. By supposed local existence, we see that $x(t)$ can be continued beyond β , contradicting our assumption about its interval of existence. Therefore,

every solution of (3.38) with $|x_0| \leq u_0$ exists on $[t_0, \infty)$ and proving process is complete.

3.4. Upper and Lower Solutions

Assume the IVP for causal differential equation

$$x(t) = (Tx)(t), \quad x(t_0) = x_0, \quad t_0 \geq 0, \quad (3.60)$$

where the causal operator $T: E \rightarrow E$ is continuous, $E = C(J, R)$, $J = [t_0, T]$.

Definition 3.4.1: Let $v, w \in C(J, R)$. v, w are lower and upper solution of (3.60) if they satisfy the inequalities

$$\left. \begin{aligned} v'(t) &\leq (Tv)(t), & v(t_0) &\leq x_0 \\ w'(t) &\geq (Tw)(t), & w(t_0) &\geq x_0 \end{aligned} \right\} \quad (3.61)$$

for $t \in J$ respectively.

If we know the existence of lower and upper solutions of (3.60) such that $v(t) \leq w(t), t \in J$, then we can prove the existence of a solution of the IVP (3.60) in the closed set

$$\tilde{\Omega} = \{x \in E: v(t) \leq x(t) \leq w(t), t \in J\}. \quad (3.62)$$

Theorem 3.4.1: Let $v, w \in C(J, R)$ be lower and upper solutions of IVP (3.60) satisfying $v(t) \leq w(t), t \in J$. Suppose also that the operator T is bounded on $\tilde{\Omega}$. Then, there exists a solution $x(t)$ of (3.60) in the closed set $\tilde{\Omega}$, i.e. $v(t) \leq x(t) \leq w(t), t \in J$.

Proof 3.4.1: Let $P \in C(J, R)$ be defined by

$$(Px)(t) = \max[v(t), \min(x(t), w(t))]. \quad (3.63)$$

Then $T(Px)(t)$ defines a continuous extension of T on E which is also bounded since T is assumed to be bounded on. Hence there exists a solution of IVP

$$x'(t) = (TPx)(t), \quad x(t_0) = x_0. \quad (3.64)$$

For any $\varepsilon > 0$ consider

$$\left. \begin{aligned} w_\varepsilon(t) &= w(t) + \varepsilon(1 + t), \\ v_\varepsilon(t) &= v(t) - \varepsilon(1 + t). \end{aligned} \right\} \quad (3.65)$$

We then have, $v_\varepsilon(t_0) < x_0 < w_\varepsilon(t_0)$, since $v(t_0) < x_0 < w(t_0)$. We wish to show that

$$v_\varepsilon(t) < x(t) < w_\varepsilon(t) \text{ on } J, \quad (3.66)$$

if this is not true, then there exists a $t_1 \in (t_0, T]$ at which $x(t_1) = w_\varepsilon(t_1)$ and

$$v_\varepsilon(t) < x(t) < w_\varepsilon(t), \quad (t), \quad t_0 \leq t < t_1 \quad (3.67)$$

then

$$x(t_1) > w(t_1) \text{ and } (Px)(t_1) = w(t_1) \quad (3.68)$$

moreover,

$$v(t_1) \leq (Px)(t_1) \leq w(t_1) \quad (3.69)$$

hence,

$$w'(t_1) \geq (TPx)(t_1) = x'(t_1). \quad (3.70)$$

Since, $w'_\varepsilon(t_1) > w'(t_1)$ we have $w'(t_1) > t'(t_1)$. However, with $x(t_1) = w_\varepsilon(t_1)$ and, $x(t) < w_\varepsilon(t)$, $t_0 \leq t < t_1$ we have $w'_\varepsilon(t_1) \leq x'(t_1)$, which is a contradiction to $w_\varepsilon(t_1) > x'(t_1)$. Hence for all $t \in J$, $x(t) < w_\varepsilon(t)$ and consequently (3.66) holds on J . Letting $\varepsilon \rightarrow 0$ we get $v(t) \leq x(t) \leq w(t)$ on J . The proof is complete.

Next we shall present a simple result giving conditions that guarantee the existence of lower and upper solutions.

Theorem 3.4.2: Suppose that $(Tx)(t)$ is nonincreasing in $x \in C(J, R)$. Then there exists lower and upper solutions v_0, w_0 for the IVP (3.66) such that

$$v_0(t) \leq w_0(t), \text{ on } J. \quad (3.71)$$

Proof 3.4.2: Let $y(t)$ be the solution of

$$y'(t) = (T_0)(t), \quad y(0) = y_0 \quad (3.72)$$

Define $(v_0)(t) = -R_0 + y(t)$ and $w_0(t) = R_0 + y(t)$. Choose $R_0 > 0$ sufficiently large so that $v_0(t) \leq 0 \leq w_0(t)$ on J . Since T is nonincreasing, this implies that

$$w'_0(t) = y'(t) = (T_0)(t) \geq (Tw_0)(t) \quad (3.73)$$

and

$$v'_0(t) = y'(t) = (T_0)(t) \leq (Tv_0)(t), \quad t \in J \quad (3.74)$$

the functions $v_0(t), w_0(t)$ are desired lower and upper solutions of (3.66)

3.5. Existence and Uniqueness

Assume the IVP for causal differential equation

$$x'(t) = (Tx)(t), \quad x(t_0) = x_0 \quad (3.75)$$

where $T \in E = C(J, R^n)$, $T : E \rightarrow E$ and $J = [t_0, t_0 + a]$. This IVP is equivalent to

$$x(t) = x_0 + \int_{t_0}^t (Tx)(s) ds, t \in J. \quad (3.76)$$

Firstly we will discuss the uniqueness result for (3.75) under Lipschitz condition and apply the contraction mapping theorem to get local existence and uniqueness.

Theorem 3.5.1: Assume that T satisfies

$$|(Tx)(t) - (Ty)(t)| \leq L|x(t) - y(t)|, x, y \in \Omega, L > 0 \quad (3.77)$$

where $\Omega = \{x, y \in E : \max_{t_0 \leq s \leq t} |x(s) - y(s)| = |x(t) - y(t)|\}$

and then, there exists a unique solution $x(t)$ of IVP (3.75) on J if $a < \frac{1}{L}$.

Proof 3.5.1 Define the metric

$$|x - y|_0 = \max_{t_0 \leq t \leq t_0 + a} |x(t) - y(t)| \quad (3.78)$$

for all $x, y \in E$. For any $x \in E$, define operator T on J by the relation

$$Tx(t) = x_0 + \int_{t_0}^t (Tx)(s) ds, \quad (3.79)$$

so that $Tx \in E$ using (3.77), we get for $x, y \in \Omega$

$$|Tx(t) - Ty(t)| \leq \int_{t_0}^t |(Tx)(s) - (Ty)(s)| ds \quad (3.80)$$

$$\leq L \int_{t_0}^t \max_{t_0 \leq s \leq t} |x(s) - y(s)| ds \quad (3.81)$$

$$\leq aL|x - y|_0 \quad (3.82)$$

which implies

$$|Tx - Ty|_0 \leq La|x - y|_0. \quad (3.83)$$

Then, if $La < 1$, T is a contraction and the contraction mapping principle assures that there exists a unique fixed point of T , say x^* which demonstrate that $x^*(t)$ is the unique solution of IVP (3.75) on J . Proven process is complete.

If, on the other hand, we use the weighted metric,

$$|x - y|_* = \max_{t_0 \leq s \leq t_0 + a} |x(s) - y(s)|e^{-\lambda s}, \lambda > 0, \quad (3.84)$$

and λ to be chosen later suitably, we get from (3.82),

$$|Tx(t) - Ty(t)| \leq L \int_{t_0}^t \max_{t_0 \leq s \leq t} |x(s) - y(s)|e^{-\lambda s} e^{\lambda s} ds \leq L|x - y|_* \int_{t_0}^t e^{\lambda s} ds$$

so that,

$$e^{-\lambda t} |Tx(t) - Ty(t)| \leq L e^{-\lambda t} |x - y|_* \frac{e^{\lambda t}}{\lambda} = \frac{L}{\lambda} |x - y|_* \quad (3.85)$$

and

$$|Tx - Ty|_* \leq \frac{1}{2} |x - y|_* . \quad (3.86)$$

By choosing $\lambda = 2L$. Now, the contraction mapping principle demonstrate that there exists a unique fixed point of T , say x^* , that is unique solution of IVP (3.75). This approach avoids restriction $La < 1$.

We shall next discuss the convergence of successive approximations for IVP (3.75) under a condition more general than (3.75) and the proof is very instructive.

3.6. Classical Notions of Quasilinearization Method

If we use the technique of lower and upper solutions integrated with the method of quasilinearization and use the idea of Newton-Fourier, it will be possible to build concurrently upper and lower bounding monotone sequences whose elements are the solutions of linear initial value problems. Compared to monotone repetitive technique, the method of quasilinearization has the benefit that the sequences converge quadratically to the unique solution.

We assume the IVP for causal differential equation

$$x'(t) = (Zx)(t), \quad x(0) = x_0, \quad t \in J = [0, T] \quad (3.87)$$

where $Z : E \rightarrow E$ is a continuous operator, $E = C(J, R)$. We will prove the following simple result to bring out the ideas involved.

Theorem 3.6.1: Assume that

i) $\alpha_0, \beta_0 \in C(J, R)$ satisfy the inequalities

$$\alpha'_0(t) \leq (Z\alpha_0)(t), \quad \beta'_0(t) \geq (Z\beta_0)(t), \quad \alpha_0 \leq \beta_0, \quad t \in J; \quad (3.88)$$

ii) The second Frechet derivative of Z exists and satisfies $Z_x x(x) \geq 0$ and $Z_x(x)y$ is seminondecreasing in y for each x .

Then, there exists monotone sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ which converge uniquely and quadratically to the unique solution $x(t)$ of (3.87) on J .

Proof 3.6.1: In view of assumption ii), we obtain for $x \geq y$, $x, y \in C(J, R)$,

$$(Zx)(t) \geq (Zy)(t) + Z_x(y)(x - y)(t) \quad (3.89)$$

and for $x, y \in \Omega$,

$$(Zx)(t) - (Zy)(t) \leq L(x(t) - y(t)), \quad t \in J, \quad L > 0. \quad (3.90)$$

Since Z satisfies one-sided Lipschitz condition, it follows that (3.87) possesses the unique solution $x(t)$ on J . Define the iterates $\{\alpha_n(t)\}, \{\beta_n(t)\}$ as follows:

$$\alpha'_{n+1}(t) = (Z\alpha_n)(t) + (Zx\alpha_n)(\alpha_{n+1} - \alpha_n)(t), \alpha_{n+1}(0) = x_0 \quad (3.91)$$

$$\beta'_{n+1}(t) = (Z\beta_n)(t) + (Zx\beta_n)(\beta_{n+1} - \beta_n)(t), \beta_{n+1}(0) = x_0 \quad (3.92)$$

with $\alpha_0(0) \leq x_0 \leq \beta_0(0)$, $t \in J$.

We shall show that

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n \leq \beta_n \leq \dots \leq \beta_2 \leq \beta_1 \leq \beta_0, \text{ on } J. \quad (3.93)$$

Let us first prove that

$$\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0 \text{ on } J. \quad (3.94)$$

Set $p = \alpha_0 - \beta_1$ so that $p(0) \leq 0$ and by (3.91)

$$p' \leq (Z\alpha_0)(t) - [(Z\alpha_0)(t) + (Zx\alpha_0)(\alpha_1 - \alpha_0)(t)] \leq (Zx\alpha_0)p. \quad (3.95)$$

By theorem 3.1.1, it follows that $p(t) \leq 0$ on J which implies $\alpha_0(t) \leq \alpha_1(t)$ on J . Similarly, it can be shown that $\beta_1 \leq \beta_0$ on J . Now set $p = \alpha_1 - \beta_1$ so that $p(0) = 0$ and by (3.89), (3.91), and (3.92) $p' = [(Z\alpha_0) + (Zx\alpha_0)(\alpha_1 - \alpha_0)(t)] - [(Z\beta_0) + (Zx\beta_0)(\beta_1 - \beta_0)(t)] \leq (Zx(\alpha_0)(\alpha_0 - \beta_0) + Zx(\alpha_0)[\alpha_1 - \alpha_0] - Zx(\beta_0)[\beta_1 - \beta_0])$ which in view of hypothesis ii) regarding $Zx(x)y$ semi nondecreasing in y for each x , yields

$$p' \leq Zx(\alpha_0)p. \quad (3.96)$$

This shows that by theorem 3.1.1, $p(t) \leq 0$ which gives $\alpha_1 \leq \beta_1$ on J this establishes (3.94). Assuming, that for some integer k , the relation

$$\alpha_{k-1} \leq \alpha_k \leq \beta_k \leq \beta_{k-1} \quad (3.97)$$

is true, we can prove, by arguments similar to those in getting (3.94) that

$$\alpha_k \leq \alpha_{k+1} \leq \beta_{k+1} \leq \beta_k \quad (3.98)$$

and thus, by the principle of induction, (3.94) is true for all n .

Therefore, the iteration scheme described by (3.91) and (3.92) yield the monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$ satisfying (3.93). Employing standard arguments, it is now easy to prove that the entire sequences $\{\alpha_n\}$, $\{\beta_n\}$ converge uniformly and monotonically to the unique solution $x(t)$ of (3.87) on J .

We shall next show that the convergence is quadratic. For this purpose, consider

$$p_{n+1} = x - \alpha_{n+1} \geq 0, q_{n+1} = \beta_{n+1} - x \geq 0, \text{ on } J \quad (3.99)$$

and note that $p_{n+1}(0) = q_{n+1}(0) = 0$, $n \geq 1$. Now,

$$\begin{aligned} p'_{n+1} &= x'(t) - \alpha'_{n+1}(t) = (Zx)(t) \\ &\quad - [(Z\alpha_n(t) + (Z_x\alpha_n)(\alpha_{n+1} - \alpha_n)(t)] \\ &= Z_x(\eta)p_n - Z_x(\alpha_n)(p_n - p_{n-1}) \\ &\leq [Z_x(x) - Z_x(\alpha_n)]p_n + Z_x(\alpha_n)p_{n+1} \\ &= Z_{xx}(\sigma)p_n^2 + Z_x(\alpha_n)p_{n+1} \end{aligned} \quad (3.100)$$

where $\alpha_n \leq \eta \leq x$ and $\alpha_n \leq \eta \leq \beta_n$. Thus we get

$$p'_{n+1} \leq Mp_{n+1} + Np_n^2, \text{ on } J \quad (3.101)$$

where $|Z_x(x)| \leq M$, $|Z_{xx}(x)| \leq N$ on J . Consequently, applying Gronwall inequality, treating Np_n^2 as forcing term, it follows that

$$0 \leq x(t) - \alpha_{n+1}(t) \leq N \int_0^t [x_n(s) - \alpha_n(s)]^2 e^{M(t-s)} ds, \quad (3.102)$$

which yields the desired estimate

$$\max_j |x(t) - \alpha_{n+1}(t)| \leq \frac{N}{M} e^{MT} \max_j |x(t) - \alpha_n(t)|^2 \quad (3.103)$$

in a similar way, one can obtain, after some computation, the estimate

$$\max_j |\beta_{n+1} - x(t)| \leq \frac{3N}{2M} e^{MT} \max_j |\beta_n - x(t)|^2 + \frac{1N}{2M} e^{MT} \max_j |x(t) - \alpha_n(t)|^2$$

Therefore the proof is completed.

4. QUASILINEARIZATION METHODS for CAUSAL DIFFERENTIAL EQUATION with INITIAL TIME DIFFERENCE

4.1. Introductions

The most significant applications of the quasilinearization method in causal differential equations has been to get a sequence of lower and upper bounds that are the solutions of linear causal differential equations that converge highorder. After all, this method has become common in applied areas. But the convexity assumption that is required by the method of quasilinearization has been a barrier for further improvement of this theory. In the recent times, this method has become generalized, refined and extended in respect of several aspects, thus to be applicable to a much larger class of nonlinear problems by not requiring convexity feature. Also, other possibilities that have been examined make the method of generalized quasilinearization universally helpful in applications. In the searching of initial value problems of causal differential equations, we have been partial to initial time all along in the sense that we only perturb the space variable and keep the initial time unchanged.

But, it also appears significant to vary the initial time because it is impossible not to make errors in the starting time. In the recent times, the studies on initial value problems of causal differential equations where the initial time vary with each solution in addition to the change of spatial variable have been initiated and some results on the comparison theorems, global existence, the method of variation of parameters, the method of lower and upper solutions and the method of monotone repetitive techniques have achieved.

In this study, the generalized quasilinearization technique in causal differential equations is used to get upper and lower sequences in the respect of the solutions of linear causal differential equations that start at different initial times and bound the solutions of a given nonlinear causal differential equation.

4.2. Basic Definitions and Concepts

In this section, we specify some basic definitions and helpful theorems for the future reference to prove the main result. First one is comparison result and the second one is existence result in respect of the upper and lower solutions with initial time difference.

An operator $Z: E \rightarrow E$, $E = C[J; R^n]$ is said to be causal operator if, for any $x, y \in E$ such that $x(s) = y(s)$; we have $Z(x)(s) = Z(y)(s)$ for $t_0 \leq s < t_0 + T$;

Consider the casual functional equation

$$x(t) = (Zx)(t); \quad x(t_0) = x_0 \quad (4.1)$$

where the causal operator $Z: E \rightarrow E$ is continuous and $x(t_0) = x_0$ for $t_0 \geq 0$ states the initial value for any $x \in E$:

Let $\alpha_0, \beta_0 \in C^1[J; R]$ with $\alpha_0(t) \leq \beta_0(t)$ on $J = [t_0; t_0 + T]$; $t_0; T \in R^+$ and

$$\Omega = \{u \in E : \alpha_0(t) \leq u \leq \beta_0(t), t \in J\} \quad (4.2)$$

we assume the following initial value problem for casual differential equation

$$u'(t) = (Zu)(t); u(t_0) = u_0 \text{ for } t \geq t_0 \quad (4.3)$$

where $Z: E \rightarrow E$ continuous causal operator, $E = C[J; R]$ for $J = [t_0; t_0 + T]$; $t_0; T \in R^+$ and $\Omega \subseteq E$:

Definition 4.2.1: $\alpha_0, \beta_0 \in C^1[J; E]$ are said to be the natural lower and upper solutions of (4.3).

Respectively, if the the following inequalities are satisfied

$$\alpha'_0 \leq (Z\beta_0)(t), \alpha_0(t_0) \leq u_0 \text{ for } t \geq t_0 \quad (4.4)$$

$$\beta'_0 \geq (Z\alpha_0)(t), \beta_0(t_0) \geq u_0 \text{ for } t \geq t_0 \quad (4.5)$$

Definition 4.2.2: $\alpha_0, \beta_0 \in C^1 [J; E]$ are said to be the coupled lower and upper solutions of (4.3), respectively, if the the following inequalities are satisfied

$$\alpha'_0 \leq (Z\beta_0(t), \alpha_0(t_0) \leq u_0 \text{ for } t \geq t_0 \quad (4.6)$$

$$\beta'_0 \geq (Z\alpha_0(t), \beta_0(t_0) \geq u_0 \text{ for } t \geq t_0 \quad (4.7)$$

Definition 4.2.3: $Z: E \rightarrow E$ is said to be semi nondecreasing in t for each x if

$$(Zx)(t_1) = (Zy)(t_1) \text{ and } (Zx)(t) \leq (Zy)(t); \quad t_0 \leq t < t_1 < T + t_0 \quad (4.8)$$

for

$$x(t_1) = y(t_1); x(t) < y(t); t_0 \leq t < t_1 < T + t_0. \quad (4.9)$$

4.3. Quasilinearization Methods for Causal Differential Equations with Initial Time Difference

In this section, we will prove the principal theorem that gives several different conditions to apply the method of generalized quasilinearization to the nonlinear causal differential equations with initial time difference and demonstrate remarks and corollaries for special situations.

Theorem 4.3.1: Assume that

- i) $Z: E \rightarrow E$ is a continuous causal operator, $E = C [J; R]$ for $J = [t_0, t_0 + T]$; $t_0, \tau_0, T \in R^+$ and there exists a constant M such that $(Zu)(t) \leq M$ on $J \times \Omega$;
- ii) $(Zu)(t)$ is semi nondecreasing in u for each $t \in J$;
- iii) $\alpha_0 \in C^1 [[t_0, t_0 + T], E]$ and $\beta_0 \in C^1 [[\tau_0, \tau_0 + T], E]$ for $\tau_0 \geq t_0 > 0$ and $T > 0$;
 $\alpha'_0 \leq (Z\alpha_0(t))$ for $t_0 \leq t \leq t_0 + T$
 $\beta'_0 \leq (Z\beta_0(t))$ for $\tau_0 \leq t \leq \tau_0 + T$, where $\alpha_0(t_0) \leq \beta_0(\tau_0)$

- iv) $t_0 < s_0 < \tau_0$, $(Zu)(t)$ is nondecreasing in t for each u ; $\alpha_0, \beta_0 \in C^1[J, R]$ such that $\alpha'_0(t) \leq (Z\alpha_0)(t)$, $(Z\beta_0)(t) \leq \beta'_0(t)$ and $\alpha_0(t) \leq \beta_0(t)$, $t \in J$;
- v) The Fréchet derivative; $(Z_x x)(t)$ exists and is continuous and $(Z_x x)(t) \leq L_1$ for $(t, x) \in J \times \Omega$, for some $L_1 > 0$ and $(Zy)(t) \leq (Zx)(t) - (Z_x y)(t)(x - y)$ where $\alpha_0(t) \leq y \leq x \leq \beta_0(t)$, $t \in J$;
- vi) $(Z_x x)(t) - (Z_x y)(t)L^2(x - y)^2$, $t \in J$, where L_2 is a positive constant.

Then there exist monotone sequences $\{\tilde{\alpha}_n(t)\}$ and $\{\tilde{\beta}_n(t)\}$ which converge uniformly to the unique solution of (4.3) with $u(s_0) = x_0$ where s_0 is between initial time t_0 and τ_0 and the convergence is highorder.

Proof: 4.3.1: since $\tilde{\beta}_n(t) = \beta_0(t + \eta_1)$, $\eta_1 = \tau_0 - t_0$ we get $\tilde{\beta}_0(t_0) = \beta_0(\tau_0) \geq \alpha_0(t_0) = \tilde{\alpha}_n(t_0)$ and $\tilde{\beta}_0(t) \geq (Z\tilde{\beta}_0)(t + \eta_1)$ for $t_0 \leq t \leq t_0 + T$. Using assumptions (iv), it is clear that $Z(t, x)$ satisfies the Lipschitz condition in x for $(t, x) \in J \times \Omega$. Furthermore, we have the following inequalities

$$(Zx)(t) \geq (Zy)(t) + (Z_x y)(t)(x - y) \quad (4.10)$$

whenever $\tilde{\alpha}_0(t) \leq y \leq x \leq \tilde{\beta}_0(t)$ on J and also by using iv) we see that whenever $\tilde{\alpha}_0(t) \leq y \leq x \leq \tilde{\beta}_0(t)$

$$(Zx)(t) - (Zy)(t) \leq L(x - y) \quad (4.11)$$

for some $L > 0$.

Consider the linear IVPs of causal differential equations for $\eta_2 = s_0 - t_0$

$$\tilde{\alpha}_1' = (Z\tilde{\alpha}_0)(t + \eta_2) + (Z_x \tilde{\alpha}_0)(t + \eta_2)(\tilde{\alpha}_1 - \tilde{\alpha}_0), \tilde{\alpha}_1'(t_0) = u_0 \quad (4.12)$$

$$\tilde{\beta}_1' = (Z\tilde{\beta}_0)(t + \eta_2) + (Z_x \tilde{\alpha}_0)(t + \eta_2)(\tilde{\beta}_1 - \tilde{\beta}_0), \tilde{\beta}_1'(t_0) = u_0 \quad (4.13)$$

where $\tilde{\alpha}_0(t_0) \leq u_0 \leq \tilde{\beta}_0(t_0)$. We shall Show that $\tilde{\alpha}_0 \leq \tilde{\alpha}_1$ on J . To do this, let $p = \tilde{\alpha}_0 - \tilde{\alpha}_1(t)$ so that $p(t_0) \leq 0$. Then

$$p' = \tilde{\alpha}_0' - \tilde{\alpha}_1' \quad (4.14)$$

$$p' \leq (Z\tilde{\alpha}_0)(t + \eta_2) - [(Z\tilde{\alpha}_0)(t + \eta_2) + (Z_x\tilde{\alpha}_0)(t + \eta_2)(\tilde{\alpha}_1 - \tilde{\alpha}_0)] \times (Z_x\tilde{\alpha}_0)(t + \eta_2)p \quad (4.15)$$

$p(t) \leq 0$ on J proving that $\alpha'_0(t) \leq \alpha'_1(t)$ on J . Now set $p = \tilde{\alpha}_1 - \tilde{\beta}_0$ and note that $p(t_0) \leq 0$. Also, using $p' = \tilde{\alpha}_0 - \tilde{\alpha}_1$

$$\leq (Z\tilde{\alpha}_0)(t + \eta_2) + (Z_x\tilde{\alpha}_0)(t + \eta_2)(\tilde{\alpha}_1 - \tilde{\alpha}_0) - (Z\tilde{\beta}_0)(t + \eta_1) \quad (4.16)$$

$$\leq (Z\tilde{\beta}_0)(t + \eta_2) - (Z_x\tilde{\alpha}_0)(t + \eta_2)(\tilde{\beta}_0 - \tilde{\alpha}_0) + (Z_x\tilde{\alpha}_0)(t + \eta_2)(\tilde{\alpha}_1 - \tilde{\alpha}_0) - (Z\tilde{\beta}'_0)(t + \eta_2) \quad (4.17)$$

$$\leq (Z_x\tilde{\alpha}_0)(t + \eta_2)p \quad (4.18)$$

which again implies $\tilde{\alpha}_1(t) \leq \tilde{\beta}_0(t)$ on J .

Similarly, we can obtain that $\tilde{\alpha}_1(t) \leq \tilde{\beta}_1(t) \leq \tilde{\beta}_0(t)$ on J . In order to prove that $\tilde{\alpha}_1(t) \leq \tilde{\beta}_0(t)$ on J , we proceed as follows. Since

$$\tilde{\alpha}_0(t) \leq \tilde{\beta}_1(t) \leq \tilde{\beta}_0(t), \quad (4.19)$$

using (4.14), we see that

$$\tilde{\alpha}_1(t) = (Z\tilde{\alpha}_0)(t + \eta_2) + (Z_x\tilde{\alpha}_0)(t + \eta_2)(\tilde{\alpha}_1 - \tilde{\alpha}_0) \leq (Z\tilde{\alpha}_1)(t + \eta_2). \quad (4.20)$$

Similarly $(Z\tilde{\beta}_1)(t + \eta_2) \leq \tilde{\beta}_1'(t)$ it follows that $\tilde{\alpha}_1(t) \leq \tilde{\beta}_1'(t)$ on J which shows that

$$\tilde{\alpha}_0(t) \leq \tilde{\alpha}_1(t) \leq \tilde{\beta}_1(t) \leq \tilde{\beta}_0(t) \text{ on } J. \quad (4.21)$$

Assume that for some $n > 1$ $\tilde{\alpha}_n' \leq (Z\tilde{\alpha}_n)(t + \eta_2)$, $(Z\tilde{\beta}_n)(t + \eta_2) \leq \tilde{\beta}_n'$ and $\tilde{\alpha}_n(t) \leq \tilde{\beta}_n(t)$, $t \in J$.

We must show that

$$\tilde{\alpha}_n(t) \leq \tilde{\alpha}_{n+1}(t) \leq \beta_{n+1}(t) \leq \tilde{\beta}_n(t) \text{ on } J \quad (4.22)$$

where $\tilde{\alpha}_{n+1}(t)$ and $\tilde{\beta}_{n+1}(t)$ are the solutions of linear IVPs of casual differential equations as follows

$$\tilde{\alpha}'_{n+1} = (Z\tilde{\alpha}_n)(t + \eta_2) + (Z_x\tilde{\alpha}_n)(t + \eta_2)(\tilde{\alpha}_{n+1}(t) - \tilde{\alpha}_n), \tilde{\alpha}_{n+1}(t)(t_0) = u_0 \quad (4.23)$$

$$\tilde{\beta}'_{n+1} = (Z\tilde{\beta}_n)(t + \eta_2) + (Z_x\tilde{\alpha}_n)(t + \eta_2)(\tilde{\beta}_{n+1} - \tilde{\beta}_n), \tilde{\beta}'_{n+1}(t_0) = u_0 \quad (4.24)$$

hence setting $p = \tilde{\alpha}_n - \tilde{\alpha}_{n+1}(t)$, it follows as before $p' \leq (Z_x\alpha'_n)(t + \eta_2)p$ on J and hence $\tilde{\alpha}_n \leq \tilde{\alpha}_{n+1}(t) \leq \beta_n(t)$ on J . In a similar manner, we can prove that $\tilde{\alpha}_n(t) \leq \tilde{\beta}_{n+1}(t) \leq \beta_n(t)$ on J .

Using (4.22), we obtain,

$$\tilde{\alpha}'_{n+1}(t) = (Z\tilde{\alpha}_n)(t + \eta_2) + (Z_x\tilde{\alpha}_n)(t + \eta_2)(\tilde{\alpha}_{n+1} - \tilde{\alpha}_n) \quad (4.25)$$

$$\begin{aligned} \tilde{\alpha}'_{n+1}(t) &\leq (Z\tilde{\alpha}_{n+1})(t + \eta_2) - (Z_x\tilde{\alpha}_n)(t + \eta_2)(\tilde{\alpha}_{n+1} - \alpha_n) \\ &\quad + (Z_x\tilde{\alpha}_n)(t + \eta_2)(\tilde{\alpha}_{n+1} - \alpha_n) \end{aligned} \quad (4.26)$$

$$\tilde{\alpha}'_{n+1}(t) \leq (Z\alpha'_{n+1})(t + \eta_2). \quad (4.27)$$

Similar arguments yield $(Z\tilde{\beta}_{n+1})(t + \eta_2) \leq \tilde{\beta}'_{n+1}$ and show that $\tilde{\alpha}_{n+1}(t) \leq \tilde{\beta}_{n+1}(t)$ on J which proves (4.22) is true. So by using inductions we obtain

$$\tilde{\alpha}_0 \leq \tilde{\alpha}_1 \leq \dots \leq \tilde{\alpha}_n \leq \tilde{\alpha}_{n+1} \leq \tilde{\beta}_{n+1} \leq \tilde{\beta}_n \leq \dots \leq \tilde{\beta}_1 \leq \tilde{\beta}_0 \text{ on } J. \quad (4.28)$$

Now using standart arguments (arzela- ascoli and dini's theorems), it can be shown that the sequences $\{\tilde{\alpha}_n(t)\}$ and $\{\tilde{\beta}_n(t)\}$ converge uniformly and monotonically to the unique solution of $u(t)$ of (4.3) on J

$$\tilde{u}(t) = (Z\tilde{u})(t + \eta_2), \quad \tilde{u}(t_0) = u_0 \quad (4.29)$$

but letting $s = t + \eta_2$ and changing the variable, we can show that (4.29) is equivalent to

$$u'(s) = (Zu)(s), \quad u(s_0) = u_0. \quad (4.30)$$

Finally, to prove superlinear convergence, we let

$$p_n(t) = \tilde{u}(t) - \tilde{\alpha}_n(t) \text{ and } q_n(t) = \tilde{\beta}_n(t) - \tilde{u}(t) \quad (4.31)$$

note that,

$$p_n(t_0) = q_n(t_0) = 0 \quad (4.32)$$

$$\begin{aligned} p'_n(t) &= \tilde{u}'(t) - \tilde{\alpha}'_n(t) \\ &= Z(\tilde{u})(t + \eta_2) - [(\tilde{\alpha}_{n-1})(t + \eta_2) \\ &\quad + (Z_x \tilde{\alpha}_{n-1})(t + \eta_2)(\tilde{\alpha}_n - \tilde{\alpha}_{n-1})] \\ &= \int_0^1 (Z_x(s \tilde{u} + (1-s)(\tilde{\alpha}_{n-1}))(t + \eta_2)(\tilde{u} - \tilde{\alpha}_{n-1}) ds \\ &\quad - Z_x(\tilde{\alpha}_{n-1})(t + \eta_2)(\tilde{\alpha}_n - \tilde{\alpha}_{n-1}) \\ &= \int_0^1 (Z_x(s \tilde{u} + (1-s)(\tilde{\alpha}_{n-1}))(t + \eta_2) p_{n-1} ds \\ &\quad - Z_x(\tilde{\alpha}_{n-1})(t + \eta_2)(p_{n-1} - p_n) \\ &= \int_0^1 [(Z_x(s \tilde{u} + (1-s)(\tilde{\alpha}_{n-1}))(t + \eta_2) \\ &\quad - Z_x(\tilde{\alpha}_{n-1})(t + \eta_2) p_{n-1}] ds + Z_x(\tilde{\alpha}_{n-1})(t + \eta_2) p_n. \end{aligned} \quad (4.33)$$

From iv) and v), it follows that

$$\begin{aligned} \|p'_n(t)\| &\leq \int_0^1 L_2 \|s \tilde{u} + (1-s)(\tilde{\alpha}_{n-1}) - (\tilde{\alpha}_{n-1})\|^2 \|p_{n-1}\| ds + L_1 \|p_n\| \\ &\leq \int_0^1 L_2 \|s \tilde{u} + s(\tilde{\alpha}_{n-1})\|^2 \|p_{n-1}\| ds + L_1 \|p_n\| \\ &\leq \int_0^1 L_2 \|s(\tilde{p}_{n-1})\|^2 \|p_{n-1}\| ds + L_1 \|p_n\| \end{aligned} \quad (4.34)$$

then setting $\alpha_n = \|p_n\|$, we find

$$\alpha'_n \leq \|p'_n\| \leq L_2(\alpha_{n-1})^{2+1} + L_1\alpha_n. \quad (4.35)$$

Now Gronwall's inequality implies

$$0 \leq \alpha_n(t) \leq L_2 \int_0^t \exp[L_1(t-s)] (\alpha_n(s))^{2+1} ds \text{ on } J \quad (4.36)$$

which yields the estimate

$$\max_j \|p_n(t)\| \leq L_2 \frac{\exp(L_1 T)}{L_1} \leq \max_j \|p_{n-1}(t)\|^{2+1} \quad (4.37)$$

similarly,

$$\begin{aligned} q'_n(t) &= \tilde{\beta}'_n(t) - \tilde{u}'_n(t) \\ &= (Z\tilde{\beta}_{n-1})(t + \eta_2) + Z_x(\tilde{\alpha}_{n-1})(t + \eta_2)(\tilde{\beta}_n - \tilde{\beta}_{n-1}) - Z(\tilde{u}(t))(t + \eta_2) \\ &= \int_0^1 (Z_x(s\tilde{\beta}_{n-1} + (1-s)(\tilde{u}(t)))(t + \eta_2)(\tilde{\beta}_{n-1} - \tilde{u})ds \\ &\quad + Z_x(\tilde{\alpha}_{n-1})(t + \eta_2)(\tilde{\beta}_n - \tilde{\beta}_{n-1}) \\ &= \int_0^1 (Z(s\tilde{\beta}_{n-1} + (1-s)(\tilde{u}(t)))(t + \eta_2)q_{n-1}ds \\ &\quad + (\tilde{\beta}_{n-1} - \tilde{u})ds + Z_x(\tilde{\alpha}_{n-1})(t + \eta_2)(q_n - q_{n-1}) \\ &= \int_0^1 [(Z_x(s\tilde{\beta}_{n-1} + (1-s)(\tilde{u}(t)))(t + \eta_2) \\ &\quad - (Z_x\tilde{u})(t + \eta_2)]q_{n-1}ds \\ &\quad + [(Z_x\tilde{u})(t + \eta_2) - Z_x(\tilde{\alpha}_{n-1})(t + \eta_2)]q_{n-1} \\ &\quad + Z_x(\tilde{\alpha}_{n-1})(t + \eta_2)q_n. \end{aligned} \quad (4.38)$$

We find using iv) and v), that

$$\|q'_n(t)\| \leq \int_0^1 L_2 \|s\tilde{\beta}_{n-1} + (1-s)\tilde{u} - \tilde{u}\|^2 \|q_{n-1}\| ds$$

$$+L_2\|\tilde{u}-\tilde{\beta}_{n-1}\|^2\|q_{n-1}\|+L_1\|q_n\| \quad (4.39)$$

$$\|q'_n(t)\| \leq L_2\|q_{n-1}\|^{2+1} + L_2\|p_{n-1}\|^2\|q_{n-1}\| + L_1\|q_n\|. \quad (4.40)$$

Setting $b_n = \|q_n\|$ and $\alpha_{n+1} = \|p_{n-1}\|$, it easily follows that

$$b'_n \leq \|q'_n\| \leq L_2(b_{n-1})^2 b_{n-1} + L_1 b_n \quad (4.41)$$

similarly, an application of Gronwall's inequality yields

$$0 \leq \|q_n\| \leq L_2 \int_0^t \exp[L_1(t-s)] [\|q_{n-1}(s)\|^{2+1} + \|p_{n-1}(s)\|^2 \|q_{n-1}(s)\|] ds \text{ on } J$$

and hence

$$\max_j \|q_n(t)\| \leq L_2 \frac{\exp(L_1 T)}{L_1} [\max_j \|q_{n-1}(s)\|^{2+1} + \max_j \|p_{n-1}(s)\|^2 \|q_{n-1}(s)\|].$$

Therefore this completes the proof of the theorem.

5. CONCLUDING REMARKS

In this work, we have endeavoured to exert quasilinearization technique for nonlinear differential equations to causal differential equations with initial time difference by using the different upper and lower solutions and solutions of linear initial value problems with initial time difference.

Monotone sequences have been attained given to the formal result of the nonlinear causal differential equations with initial time difference, converging uniformly, monotonically and quadratically being each element of the linear equation results, using quasilinearization technique under some conditions.

This study, in the steps of the method will be generalized.

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BIOGRAPHY

Ali Öcal ÖZLÜK was born in Zonguldak on 1 January 1978. He completed his primary education in Pendik and secondary education in Kadıköy, İstanbul. In 2000, he graduated from Konya Selcuk University where he majored in teaching of Mathematics. In 2001, he started his master's degree in the Department of Mathematics which is located in the Institute of Science, Gebze Institute of Technology. Now, he is married and he is a father having two daughters. He is currently teaching in Kırımlı Fazilet Olcay Anatolian High school as a math teacher.