

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

**NONLINEAR STATIC AND DYNAMIC ANALYSIS OF SHELLS OF
REVOLUTION**

ÖZGÜR KALBARAN
A THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
DEPARTMENT OF MECHANICAL ENGINEERING

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THESIS SUPERVISOR
PROF. DR. HASAN KURTARAN

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T.C.
GEBZE TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

DÖNEL KABUK YAPILARIN NONLİNEER
STATİK VE DİNAMİK DAVRANIŞLARININ
İNCELENMESİ

ÖZGÜR KALBARAN
DOKTORA TEZİ
MAKİNE MÜHENDİSLİĞİ ANABİLİM DALI

DANIŞMANI
PROF. DR. HASAN KURTARAN

GEBZE
2020

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JÜRİ

ÜYE

(TEZ DANIŞMANI)

: Prof. Dr. Hasan KURTARAN



ÜYE

: Doç. Dr. Ahmet Sinan ÖKTEM



ÜYE

: Prof. Dr. Halit Süleyman TÜRKMEN



ÜYE

: Prof. Dr. Mehmet Ali ARSLAN



ÜYE

: Prof. Dr. Ekrem TÜFEKÇİ



ONAY

Gebze Teknik Üniversitesi Fen Bilimleri Enstitüsü Yönetim Kurulu'nun

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SUMMARY

In this Ph.D. dissertation, nonlinear static and transient dynamic behavior of laminated composite shells of revolution structure(s) with variable thickness resting on elastic foundation is investigated using Generalized Differential Quadrature (GDQ) method. Winkler-Pasternak model is used to represent elastic foundation. Linear, arch, sine and cosine thickness functions are used to express the variable thicknesses. In static and transient dynamic analyses, First Order Shear Deformation Theory (FSDT) is used to consider the transverse shear effects. Nonlinearity is taken into account using Green-Lagrange nonlinear strain-displacement relations considering the deepness effect. As a novelty, Green-Lagrange strain-displacement relations for shells of revolution are derived retaining full non-linear terms in all degree of freedom parameters using theory of surfaces. Virtual work principle is used to derive the equations of motion and equilibrium equations. Partial derivatives in the equation of motion and equilibrium equations are expressed with GDQ method and time integration is carried out using Newmark average acceleration method for the transient analyses. Several problems are solved and compared with the finite element results in order to validate the proposed method. After validation, effects of thickness functions, thickness variation parameter, geometric characteristic parameter of shells of revolution, boundary conditions, elastic foundation parameters as well as composite lamination scheme on nonlinear static and transient dynamic behavior of shells of revolution structures are investigated.

Key Words: Shells of Revolution, Variable Thickness, Geometric Nonlinearity, Winkler-Pasternak Elastic Foundation, Generalized Differential Quadrature, Laminated Composite.

ÖZET

Bu doktora tez çalışmasında, yayılı yük altında elastik zemine oturan değişken kalınlığa sahip tabakalı kompozit dönel kabuk yapıların statik ve dinamik davranışı ‘Genelleştirilmiş Diferansiyel Kareleme’ yöntemi kullanılarak incelenmiştir. Ayrıca, elastik zeminin doğrusal (lineer) ve homojen olduğu varsayılmış, zeminin kiriş tepkisi iki parametrelili Winkler-Pasternak modeli kullanılarak modellenmiştir. Kalınlık değişimini ifade etmek için doğrusal, kemer, sinüs ve kosinüs formlarına sahip kalınlık fonksiyonları kullanılmıştır. Statik ve dinamik analizlerde, enine kayma etkilerini dikkate alabilmek için ‘Birinci Mertebe Kayma Deformasyon Teorisi’ kullanılmıştır. Geometrik nonlineerlik (doğrusalsızlık), doğrusal olmayan Green-Lagrange gerilme-yer değiştirme ilişkileri kullanılarak dikkate alınmıştır. Nonlinear Green-Lagrange gerilme-yer değiştirme ilişkileri yüzey teorisini kullanarak türetilmiştir. Bu çalışmada hareket ve denge denklemlerini elde etmek için virtüel iş prensibi kullanılmıştır. Hareket ve denge denklemlerindeki kısmi türevler Genelleştirilmiş Diferansiyel Kareleme yöntemi kullanılarak hesaplanmaktadır. Dinamik analizlerdeki zaman entegrasyonu için ise Newmark-Beta metodu kullanılmıştır. Önerilen yöntemi doğrulamak için çeşitli problemler çözülmüş ve ‘Sonlu Elemanlar Metodu’ sonuçlarıyla karşılaştırılmıştır. Doğrulamadan sonra, kalınlık fonksiyonlarının etkileri, kalınlık değişim parametresi, dönel kabuğun geometrik karakteristik parametresi, sınır şartları, elastik zemin parametreleri ve kompozit tabaka diziliminin dönel kabuk yapılarının doğrusal olmayan statik ve dinamik davranışları üzerine detaylı olarak incelemesi yapılmıştır.

Anahtar Kelimeler: Dönel Kabuk Yapılar, Değişken Kalınlık, Nonlinear Statik Dinamik Analiz, Winkler-Pasternak Elastik Zemin Modeli, Genelleştirilmiş Diferansiyel Kareleme Yöntemi, Tabakalı Kompozitler.

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LIST of ABBREVIATIONS and ACRONYMS

<u>Abbreviations</u>	<u>Explanations</u>
<u>and Acronyms</u>	
DQM	: Differential Quadrature Method
GDQ	: Generalized Differential Quadrature
FSDT	: First Order Shear Deformation Theory
HSDT	: Higher Order Shear Deformation Theory
PPRS	: Parabolic Panels of Revolution Structure(s)
EPRS	: Elliptic Panels of Revolution Structure(s)
FGM	: Functionally Graded Material

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1. INTRODUCTION

Shell structures have been widespread in many fields of engineering as they give rise to optimum conditions for static/dynamic behavior, strength and stability. A primary difference between a shell structure and a plate structure is that, in the unstressed state, the shell structure has curvature as opposed to the plates structure which is flat.

Shells of revolution structures, constitutes an important part of shell structures and they are one of the basic structural elements. These structures take many well-known geometrical shapes such as spherical, conical, paraboloids, hyperboloids and ellipsoids in engineering applications as shown in Figure 1.1.

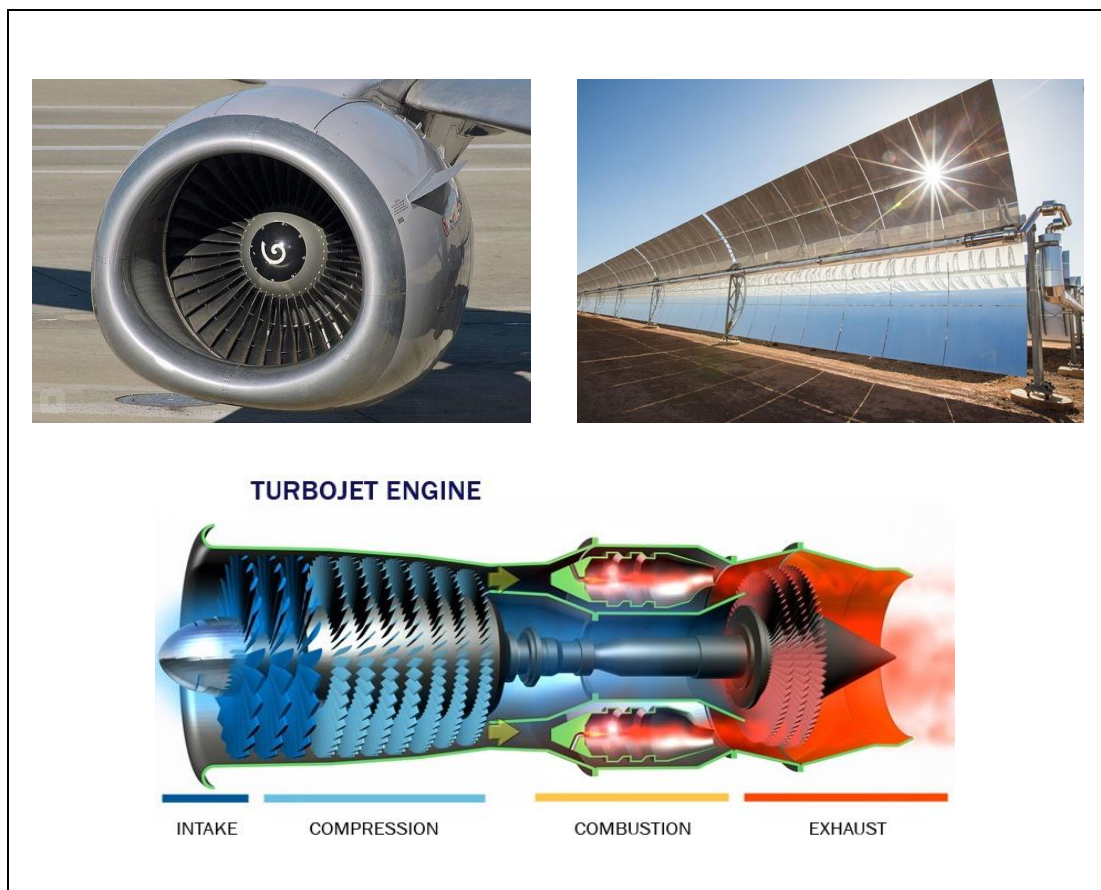


Figure 1.1: Some engineering applications of shells of revolution structures.

They support applied external forces efficiently thanks to their geometrical shape. The shells of revolution structures for their mechanical properties and good structural framework are commonly used in defense, automotive, marine and

aerospace industries in the form of buildings, pressure vessels, bridges, hangar, stadium covers, exhibition halls, auditoriums, aircraft, space stations, ships and submarines. Because of their extensive application areas, the problems related to the static and dynamic analyses of these structures became more complex and the demand for solutions of shell problems increased.

Composite materials are often used in most of these shells of revolution structures. The biggest advantage of composite materials is that they are light as well as strong. Composites also provide design flexibility for shells of revolution because many of them can be molded into complex shapes. Therefore, in this thesis, composite materials are used for shells of revolution structural materials. Composite shells of revolution structures can have variable thickness according to design, minimum weight and geometric necessity requirements. The effect of variable thickness is important in a design process. The wings, aileron, rudder and rotor blade of an air vehicle as shown in Figure 1.2 can be given as example for composite shells of revolution having variable thickness.

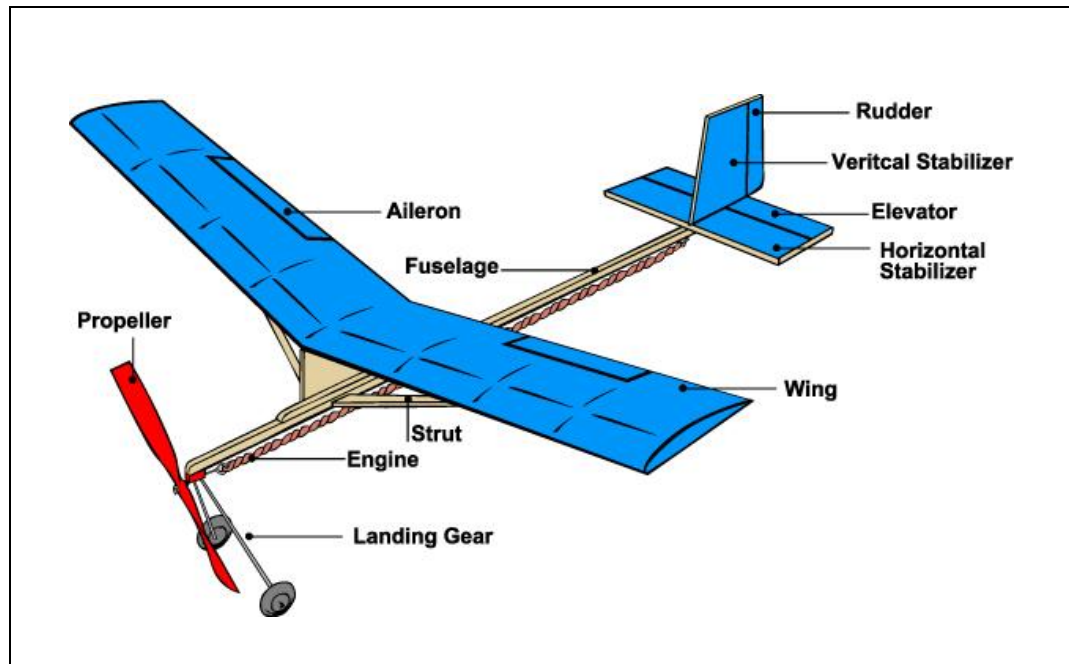


Figure 1.2: The parts of air vehicle with variable thickness.

This thesis presents the nonlinear static and transient dynamic responses of laminated composite shell of revolution structures subjected to distributed loads under the different boundary conditions using GDQ method.

The GDQ Method is a numerical technique for solving differential equations. This method was improved by [Shu, 2000] who introduced simple explicit formula for the weight coefficients. In GDQ method, selection of grid points for the weight coefficients is arbitrary. In addition, higher order derivatives can easily be calculated by using recursive formulations. The GDQ method has been applied to the free vibration, buckling, static and dynamic analysis of beams, arches, plates and shells in many areas and it has allowed to obtain good results using a few numbers of grid points [Bacciocchi et al., 2016], [Kurtaran, 2015a], [Kurtaran, 2015b], [Kurtaran, 2015c], [Tornabene, 2011a], [Tornabene et al., 2016], [Tornabene et al., 2015a], [Tornabene et al., 2015b]. Compared to other numerical techniques, the advantage of GDQ Method is to achieve high accuracy with less grid points.

1.1. Literature Review

Variety of composite shells of revolution structures are commonly utilized in defense, aerospace and construction fields in the forms of rocket heads, underwater vehicles, aero planes, stadium covers, exhibition halls, auditoriums etc. Therefore, analysis of such structures is important. Many books have been published on analysis of shell structures [Amabili, 2008], [Flügge, 1973], [Kovarík, 2013], [Libai and Simmonds, 2005], [Qatu, 2004], [Reddy, 2003], [Soedel, 2004], [Tornabene and Fantuzzi, 2014], [Vorovich, 1998]. In addition, many articles are available in literature on shells of revolution. Studies related to static, free vibration and buckling analyses of shells of revolution with variable thickness are given as follows.

[Kang, 2001] presented stability analysis of shallow spherical shells with variable thickness using modified method of multiple scales. [Eraslan and Orcan, 2002] explored stress analysis of rotating plastic and elastic solid disks with concave profile variable thickness using analytical method. [Jiang and Redekop, 2003] carried out a modal and static analysis of orthotropic toroidal shells of revolution with variable thickness based on the Sanders-Budiansky equations using differential quadrature method. [Grigorenko and Avramenko, 2008, Grigorenko and Yaremchenko, 2004] investigated strain-stress analysis of orthotropic noncircular closed conical shells based on spline-approximation and cylindrical shells based on a refined model of rectilinear element with variable thickness, respectively. [Kang and Leissa, 2005, Kang and Leissa, 2008] studied free vibration analysis of closed paraboloid and

ellipsoidal moderately thick shells of revolution having variable thickness based on a 3-dimensional theory using Ritz method. [Kang, 2007] developed energy functionals, equations of motion, field equations considering a three dimensional theory for thick shells of revolution having variable thickness. [Efraim and Eisenberger, 2010] investigated free vibration analysis of moderately thick spherical panels with variable thickness under various boundary conditions using dynamic stiffness method. [Vivio and Vullo, 2010] presented linear stress and bending analysis of circular plates with non-linear variable thickness using analytical and numerical methods. [Tajeddini et al., 2011] performed 3-dimensional free vibration analysis of annular and circular FG and isotropic plates with variable thickness using polynomial-Ritz method resting on Pasternak foundation. [Kang, 2012a, Kang, 2012b] examined 3-dimensional modal analysis of joined thick conical-cylindrical and hemispherical shells of revolution having variable thickness using Ritz method. [Awrejcewicz et al., 2013] investigated nonlinear free vibration behavior of orthotropic doubly-curved shells with variable thickness based on Donell's nonlinear shallow theory using Ritz method. [Alashti and Ahmadi, 2014] analyzed buckling behavior of FG cylindrical shells having variable thickness utilizing GDQ method. [Ataabadi et al., 2014] carried out modal analysis of orthotropic cylindrical shells with variable thickness based on linear theory using spline functions considering different types of boundary conditions. [Khalifa Ahmed, 2015] performed modal analysis of orthotropic oval shells of revolution having parabolic variable thickness based on Flügge's theory resting on Winkler and Pasternak foundations using transfer matrix method. [Awrejcewicz et al., 2015] presented nonlinear free vibration solution of laminated composite shells with variable thickness based on R-functions theory regarding various boundary conditions using numerical and analytical methods. [Dai et al., 2016] studied natural frequency analysis of truncated conical shells of revolution with parabolic and linear variable thickness utilizing Haar wavelet method with different types of boundary conditions. [Kang, 2015, Kang, 2016] conducted free vibration analysis of deep/shallow parabolic and hollow circular cross-section thick shells of revolution having variable thickness using Ritz method. [Ahmed, 2016] obtained natural frequencies and mode shapes of orthotropic cylindrical shells with variable thickness resting on Winkler-Pasternak elastic foundation using transfer matrix method. [Baccocchi et al., 2016] performed free vibration analysis of laminated composite plates and doubly-curved shells such as conical, spherical, hemi-spherical and paraboloidal panel with different variable

thickness form using the GDQ method. [Tornabene et al., 2016] investigated free vibration characteristics of several doubly-curved shells such as parabolic, helicoid and ribbed panels with variable thickness by taking various higher-order equivalent single layer theories into account using GDQ method. [Zheng et al., 2016] presented linear stress analysis of FG rotating disks having variable thickness utilizing Finite Difference Method (FDM). [Livani et al., 2016] carried out a study about the flutter behavior of sandwich doubly-curved panels with variable thickness based on a Higher-order Shear Deformation Theory (HSDT) under aero-thermoelastic loading using the Fourier series method. [Grigorenko et al., 2017] examined modal analysis of elliptical shells of revolution having variable thickness based on Timoshenko-Mindlin theory using the spline-approximation method. [Nejad et al., 2017] developed a disk formulation to investigate thermo-elastic behavior of functionally graded (FG) moderately thick shells of revolution having variable thickness based on HSDT. [Yang and Kang, 2017] investigated modal analysis results of a circular and hemi-ellipsoidal composite thick shells of revolution having variable thickness using Ritz method considering the various boundary conditions. [Tornabene et al., 2017] analyzed the natural frequencies of FG material sandwich shells of revolution with variable thickness based on HSDT utilizing GDQ method. [Ko and Kang, 2017] presented free vibration analysis of deep and shallow hemi-ellipsoidal shells of revolution with variable thickness based on HSDT using Ritz method considering the various boundary conditions. [Grigorenko et al., 2018] conducted strain-stress analysis of closed and open elliptic shells of revolution with variable thickness based on the Donell-Vlasov-Mushtari shell model approach using discrete orthogonalization and spline-collocation technics. [Akgün and Kurtaran, 2019] carried out transient dynamic analysis of super-elliptic shells made of FG materials using GDQ method.

1.2. Objectives and Content of the Thesis

From the literature review, it is seen that there are many studies on shells of revolution structures. The vast majority of these studies include free vibration analysis. However, there is no study available on nonlinear static and transient dynamic behavior of shells of revolution structures with variable thickness. Elliptic Panels of Revolution Structure(s) (EPRS) and Parabolic Panels of Revolution Structure(s) (PPRS) occupy important part of shells of revolutions. Shells of revolution can be

made with variable thickness due to design, geometric necessities, strength capacities and minimum weight requirements. Therefore, analysis of shells of revolution is important. In this thesis, nonlinear static and transient dynamic analysis of laminated composite shells of revolution with variable thickness and resting on Winkler-Pasternak elastic foundation is investigated for the clamped and simply supported boundary conditions using GDQ method.

The objective of this thesis is to develop a mathematical model and a computer program to investigate the nonlinear static and transient dynamic behavior of laminated composite shells of revolution structures under different boundary conditions. As a novelty, Green-Lagrange strain-displacement relations for shells of revolution were derived retaining full non-linear terms in all degree of freedom parameters using the theory of surfaces.

In the following sections, in Chapter 2 surface theory of shells of revolution structures are given and nonlinear strain theory is derived. Chapter 3, equations of motion and equilibrium equations for shells of revolution are expressed. In Chapter 4, solution procedure and GDQ method is summarized. In Chapter 5, validation of the code developed for nonlinear static and dynamic analysis is carried out and several examples are solved within the context of the parametric study.

1.3. Contribution of the Thesis

Analysis of non-complex shell structures is widely being studied by many researchers. When the shell geometry becomes more complex, studies on these structures begin to decrease. Linear analyzes (static, free vibration, buckling etc.) of these complex shells are generally found in the literature.

In this Ph.D. dissertation;

- Nonlinear static and transient dynamic behavior of laminated composite shells of revolution structures with variable thickness is investigated using GDQ method.
- Novel nonlinear Green-Lagrange strain-displacement relations for shells of revolution structures are derived utilizing the theory of surfaces.
- Linear, arch, sine and cosine thickness functions are used to express variable thickness in ϕ direction for shells of revolution.

- Effects of thickness functions, thickness variation parameter, geometric characteristic parameter of shells of revolution, boundary conditions, elastic foundation parameters as well as composite lamination scheme on nonlinear static and transient dynamic behavior of shells of revolution structures are investigated.

2. SHELLS of REVOLUTION EQUATIONS with VARIABLE THICKNESS

Shells of revolution are structures defined by curved surfaces and there are many geometry types of shells of revolution (toroidal, hyperbolic, parabolic and elliptical etc.) in the literature according to their curvature characteristic. Shells of revolution geometries have been widely used in designing engineering structures such as dome roofs, aircraft hulls, nuclear reactors, aerospace structures, pressure vessel components and flat-bottom tanks.

2.1. Equations of Parabolic/Elliptic Panels of Revolution Structure(s)

Shells of revolution with parabolic and elliptic geometry are taken into account in this study. Parabolic panels of revolution structure(s) (PPRS) and elliptic panels of revolution structure(s) (EPRS) as shown in Figure 2.1 and Figure 2.2 are considered. Surface of PPRS and EPRS are created with rotation of ϕ_1 - ϕ_2 parabolic and elliptic curve about the axis Z (revolution axis) through the angle θ as shown in Figure 2.1 and Figure 2.2, respectively. ϕ_1 and ϕ_2 are the starting and ending points of the shells of revolution structures in ϕ direction, respectively. R_ϕ and R_θ indicate the principal radii of curvature of the parabolic and elliptic surfaces. R_0 is the horizontal radius. Horizontal radius states the distance from revolution axis Z to each point of the reference surface. R_b is the shift of the geometric central axis Z' with reference to the revolution axis Z as shown in Figure 2.1 and Figure 2.2.

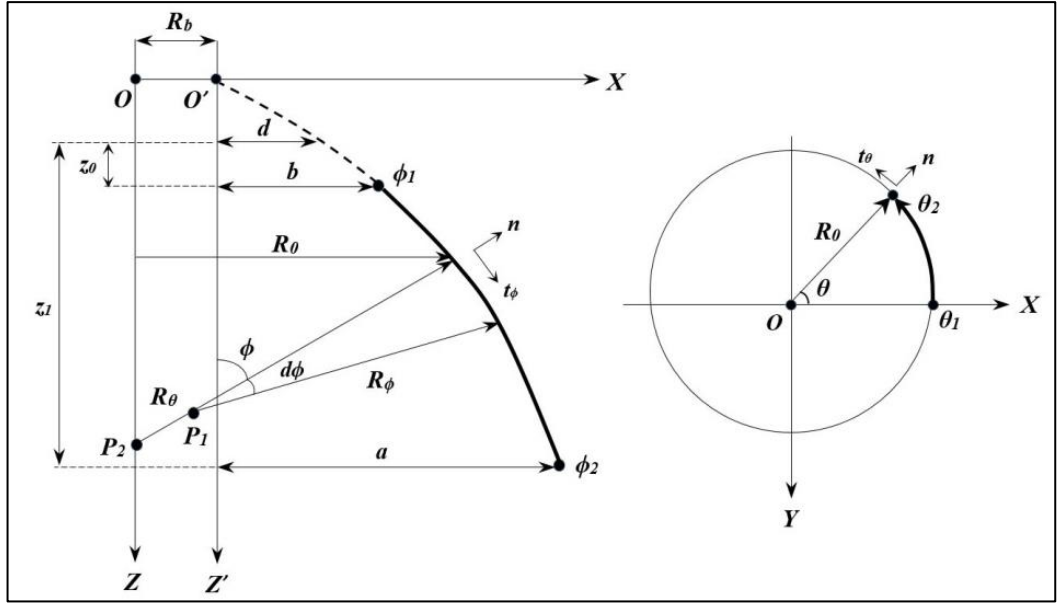


Figure 2.1: The geometric parameters of PPRS on global coordinate system (XYZ).

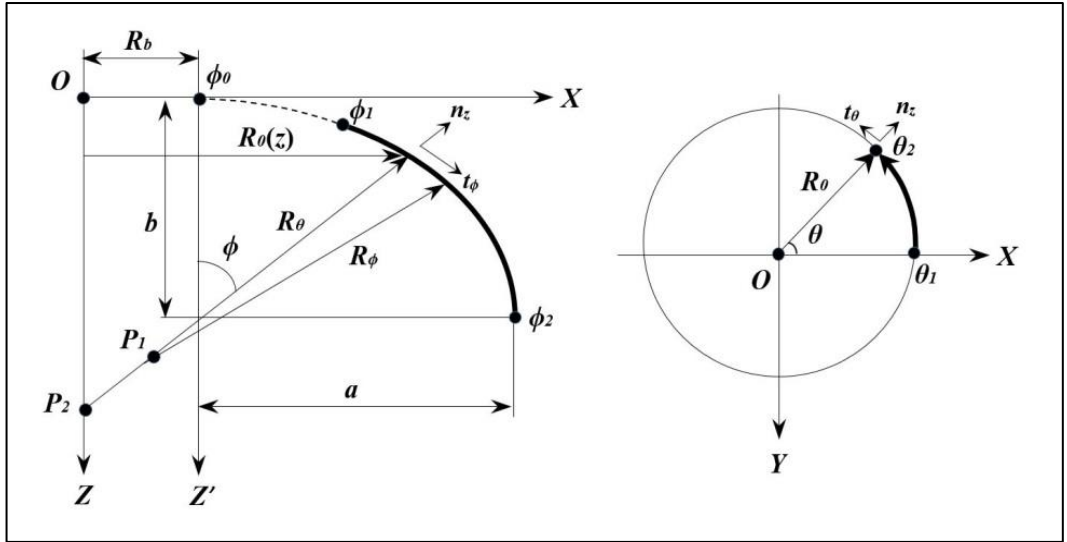


Figure 2.2: The geometric parameters of EPRS on global coordinate system (XYZ).

The orthogonal coordinates along the meridional, circumferential and normal directions of the reference surface of shells of revolution are taken as ϕ , θ and z respectively. $O'\phi\theta z$ denotes the orthogonal curvilinear coordinate system on the reference surface of shells of revolution which is shown in Figure 2.3.

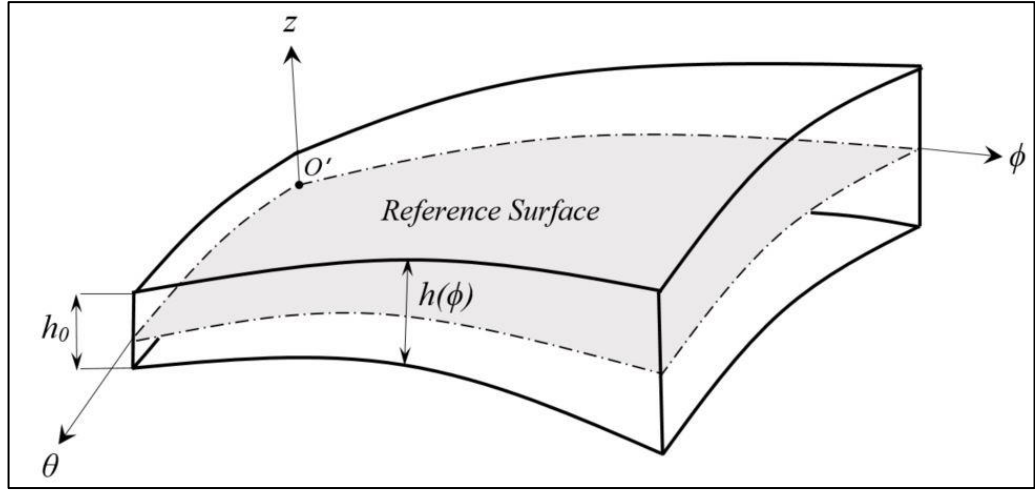


Figure 2.3: Geometry notations of shells of revolution with variable thickness.

Geometric expressions for PPRS as shown in Figure 2.1 can be defined below

$$(R_0 - R_b)^2 = kz \quad (2.1)$$

The horizontal radius of PPRS R_0 is

$$R_0(\phi) = R_b + \frac{k \tan(\phi)}{2} \quad (2.2)$$

where k is the characteristic parameter [Tornabene and Viola, 2008] of the PPRS which is given as

$$k = \frac{a^2 - d^2}{z_1} \quad (2.3)$$

The principal radii of curvature of PPRS at reference surface R_ϕ and R_θ in the meridional and circumferential directions respectively are written as

$$R_\phi(\phi) = \frac{k}{2 \cos^3(\phi)} \quad (2.4)$$

$$R_\theta(\phi) = \frac{Rb}{\sin(\phi)} + \frac{k}{2 \cos(\phi)} \quad (2.5)$$

Geometric expressions for EPRS [Tornabene, 2011b] as shown in Figure 2.2 can be stated below

$$(R_0 - R_b)^2 + k^2 (b - z)^2 = a^2 \quad (2.6)$$

where b and a are the length of the semi-minor and semi-major axes of the elliptic curve, respectively. Characteristic parameter of elliptic curve can be expressed as follows

$$k = \frac{a}{b} \quad (2.7)$$

The horizontal radius of EPRS R_0 is

$$R_0(\phi) = R_b + \frac{ak \tan(\phi)}{\sqrt{1 + k^2 \tan^2(\phi)}} \quad (2.8)$$

The radii of curvature R_θ and R_ϕ in the circumferential and meridional directions respectively can be described as follows:

$$R_\phi(\phi) = \frac{ak}{\cos^3(\phi) \sqrt{(1 + k^2 \tan^2(\phi))^3}} \quad (2.9)$$

$$R_\theta(\phi) = \frac{Rb}{\sin(\phi)} + \frac{ak}{\cos(\phi) \sqrt{1 + k^2 \tan^2(\phi)}} \quad (2.10)$$

Finally, the Gauss-Codazzi relation for shells of revolution can be stated as below

$$\frac{dR_0}{d\phi} = R_\phi \cos(\phi) \quad (2.11)$$

2.2. Variable Thickness Functions

In this study, linear, arch, sine and cosine thickness functions are used to express variable thickness in ϕ direction for shells of revolution. A laminated composite shells of revolution structure with variable thickness which has n plies is shown in Figure 2.4. $h(\phi)$ is the thickness function of the shells of revolution in Figure 2.4.

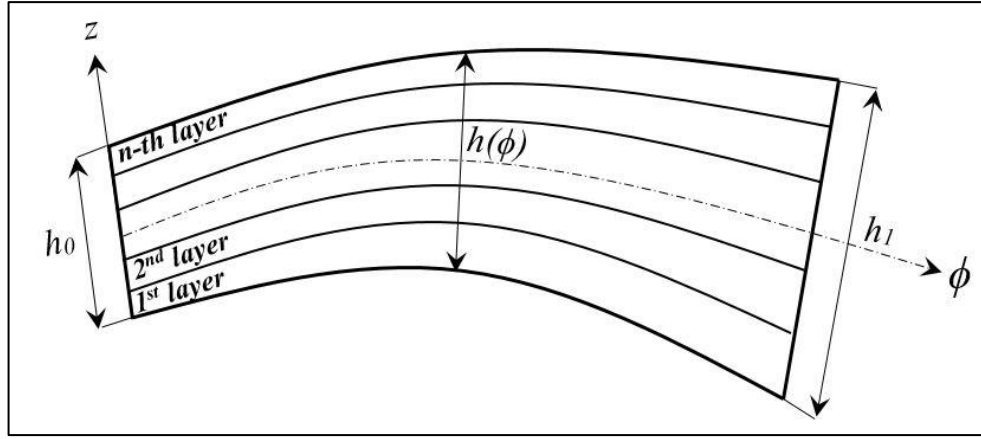


Figure 2.4: Thickness variation of laminated composite shells of revolution.

In this study, thickness of the shells of revolution only varies with respect to ϕ direction. In θ direction, thickness variation is not considered. The points, which have equal distances to the two inclined surfaces, form the reference surface of shells of revolution. Thickness function $h(\phi)$ is expressed for linear (Figure 2.5-a), arch (Figure 2.5-b), sine (Figure 2.5-c) and cosine (Figure 2.5-d) form of variable thickness in ϕ direction as below

$$\text{Linear Form} : h(\phi) = h_0 \left[1 + \beta \left(\frac{\phi - \phi_1}{\phi_2 - \phi_1} \right) \right] \quad (2.12)$$

$$\text{Arch Form} : h(\phi) = h_0 \left[1 + \beta \left(\frac{\phi - \phi_1}{\phi_2 - \phi_1} \right)^{0.5} \right] \quad (2.13)$$

$$\text{Sine Form} : h(\phi) = h_0 \left[1 + \beta \sin \left(\frac{\phi - \phi_1}{\phi_2 - \phi_1} \pi \right) \right] \quad (2.14)$$

$$\text{Cosine Form} : h(\phi) = h_0 \left[1 + \beta + \beta \cos \left(\left(\frac{\phi - \phi_1}{\phi_2 - \phi_1} + 0.5 \right) \pi \right) \right] \quad (2.15)$$

where h_0 is the thickness value of shells of revolution structure at the starting point ϕ_1 as shown in Figure 2.4 for linear and arch thickness forms and β is the thickness variation parameter governing the change in thickness, in ϕ direction. h_1 is the thickness value of the shells of revolution structure at the ending point ϕ_2 .

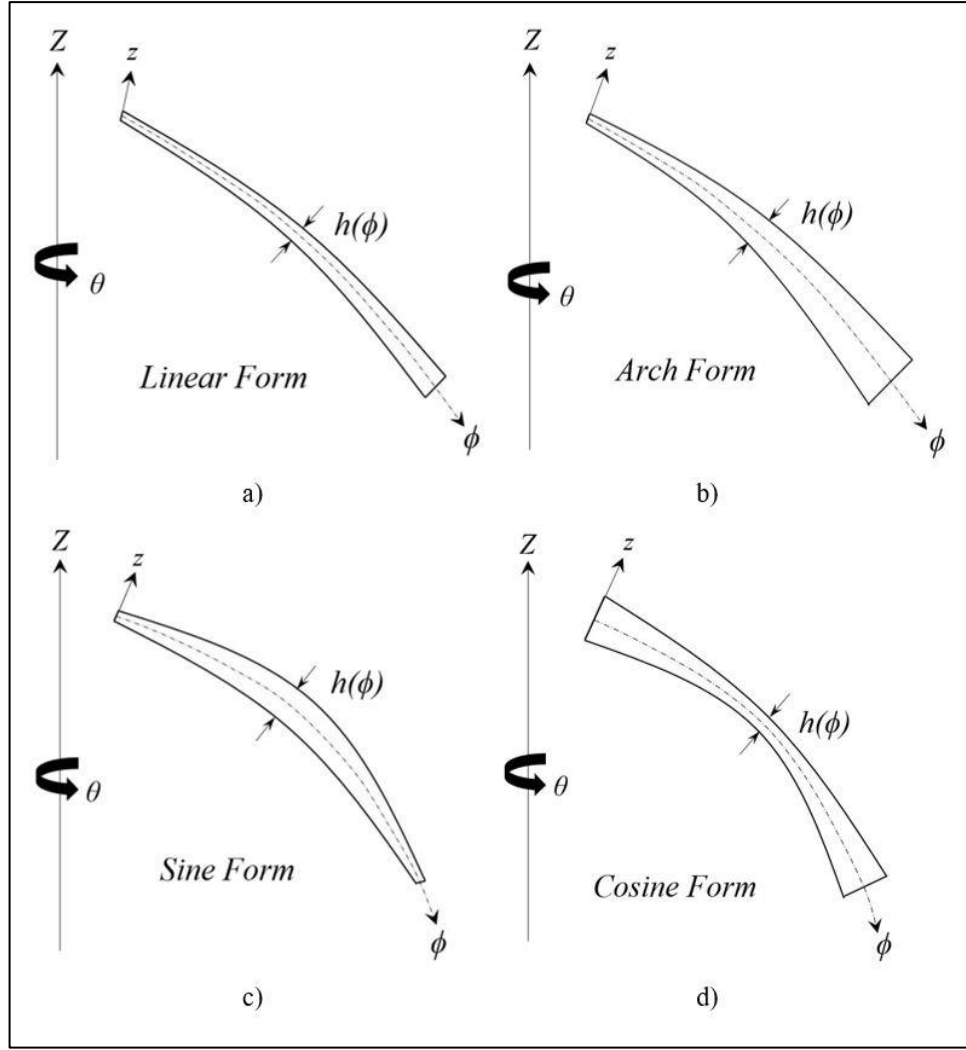


Figure 2.5: Variable thickness forms in ϕ direction for shells of revolution. a) Linear Form, b) Arch Form, c) Sine Form and d) Cosine Form.

2.3. Nonlinear Strain-Displacement Relationships for Shells of Revolution Structures

A shell structure may be defined as a body enclosed between two closely spaced and curved surfaces [Saada, 1974]. Reference surface in a shell structure can be determined in a rectangular cartesian coordinate system (XYZ) using ϕ and θ parameters as in following equations

$$X = r_x(\phi, \theta) \quad , \quad Y = r_y(\phi, \theta) \quad , \quad Z = r_z(\phi, \theta) \quad (2.16)$$

where r_x , r_y and r_z are single-valued continuous functions of the parameters ϕ and θ .

A point P on the reference surface in Figure 2.6 can be expressed by a position vector $\mathbf{r}(\phi, \theta)$ as below

$$\mathbf{r}(\phi, \theta) = r_x(\phi, \theta)\mathbf{i} + r_y(\phi, \theta)\mathbf{j} + r_z(\phi, \theta)\mathbf{k} \quad (2.17)$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denote unit vectors along the axes of the global reference coordinate system XYZ . The infinitesimal distance between points P and P' on the reference surface as shown in Figure 2.6, $d\mathbf{r}$ can be defined as

$$d\mathbf{r} = \frac{d\mathbf{r}}{d\phi} d\phi + \frac{d\mathbf{r}}{d\theta} d\theta \quad (2.18)$$

where $(d\mathbf{r}/d\phi)$ and $(d\mathbf{r}/d\theta)$ vectors are tangent to the ϕ and θ curves, respectively.

The magnitude of $d\mathbf{r}$ is obtained by

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = E(d\phi)^2 + 2Fd\phi d\theta + G(d\theta)^2 \quad (2.19)$$

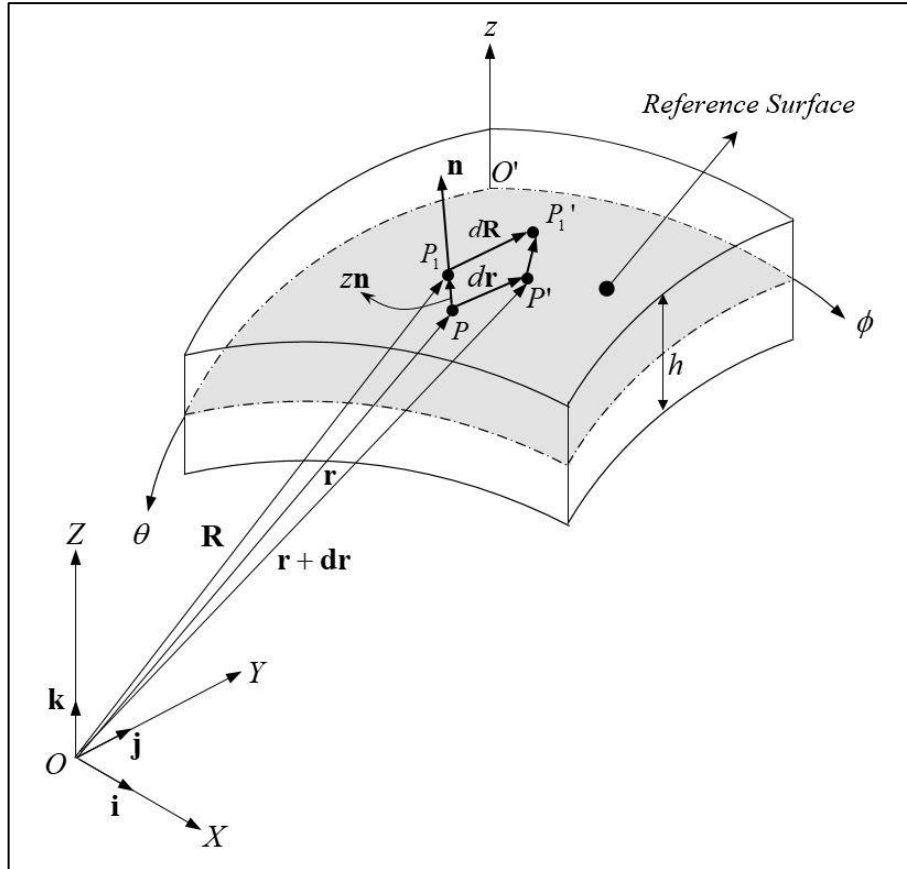


Figure 2.6: Coordinate system and position vector of a general point on shell reference surface.

Equation (2.19) indicates the first fundamental form of the reference surface. E , F and G denote magnitudes of first fundamental form or metric coefficients. They can be obtained as below

$$E = \frac{d\mathbf{r}}{d\phi} \frac{d\mathbf{r}}{d\phi}, \quad F = \frac{d\mathbf{r}}{d\phi} \frac{d\mathbf{r}}{d\theta}, \quad G = \frac{d\mathbf{r}}{d\theta} \frac{d\mathbf{r}}{d\theta} \quad (2.20)$$

If ϕ and θ curves are chosen as orthogonal (as in our case), then value of the metric F becomes 0. Therefore, Equation (2.19) becomes

$$(ds)^2 = A_\phi^2 (d\phi)^2 + A_\theta^2 (d\theta)^2 \quad (2.21)$$

where A_ϕ and A_θ are the Lamé parameters or fundamental form parameters. Lamé parameters for shells of revolution are expressed as

$$A_\phi = \sqrt{E} = R_\phi \quad (2.22)$$

$$A_\theta = \sqrt{G} = R_\theta \sin(\phi) \quad (2.23)$$

The infinitesimal distance between a generic point P_1 and P_1' is shown in Figure 2.6. P_1 is located at a distance z from the reference surface. The position of P_1 can be stated as following

$$\mathbf{R}(\phi, \theta, z) = \mathbf{r}(\phi, \theta) + z \mathbf{n}(\phi, \theta) \quad (2.24)$$

where $\mathbf{n}(x,y)$ is the unit vector normal to the reference surface. $d\mathbf{R}$ is the differential change between points P_1 and P_1' and it is given as following

$$d\mathbf{R} = d\mathbf{r} + z d\mathbf{n} + \mathbf{n}dz \quad (2.25)$$

The magnitude of $d\mathbf{R}$ is obtained by

$$(ds)^2 = d\mathbf{R}d\mathbf{R} = A_\phi^2 \left(1 + \frac{z}{R_\phi}\right)^2 (d\phi)^2 + A_\theta^2 \left(1 + \frac{z}{R_\theta}\right)^2 (d\theta)^2 + (dz)^2 \quad (2.26)$$

Eq. (2.26) can be expressed in terms of metric coefficients g_{ii} as following

$$(ds)^2 = g_{\phi\phi}(d\phi)^2 + g_{\theta\theta}(d\theta)^2 + g_{zz}(dz)^2 \quad (2.27)$$

The measure lengths are given by the metric coefficients g_{ii} . Metric coefficients can be expressed as following

$$\begin{aligned} g_{\phi\phi} &= h_\phi^2 \\ g_{\theta\theta} &= h_\theta^2 \\ g_{zz} &= h_z^2 \end{aligned} \quad (2.28)$$

where h_i is called the scale factors and they are given as

$$\begin{aligned} h_\phi &= A_\phi \left(1 + \frac{z}{R_\phi} \right) \\ h_\theta &= A_\theta \left(1 + \frac{z}{R_\theta} \right) \\ h_z &= 1 \end{aligned} \quad (2.29)$$

The strain-displacement relationships can be obtained by introducing the scale factors as [Saada, 1974]

$$\varepsilon_{ij} = \frac{\gamma_{ij}}{h_i h_j} \quad (2.30)$$

Strain-Displacement relationship considering nonlinear terms without any simplification for shell structures is shown below [Saada, 1974]

$$\begin{aligned} \gamma_{\phi\phi} &= h_\phi \frac{\partial u_\phi}{\partial \phi} + \frac{h_\phi u_\theta}{h_\theta} \frac{\partial h_\phi}{\partial \theta} + \frac{h_\phi u_z}{h_z} \frac{\partial h_\phi}{\partial z} + \\ &+ \frac{1}{2} \left(\frac{\partial u_\phi}{\partial \phi} + \frac{u_\theta}{h_\theta} \frac{\partial h_\phi}{\partial \theta} + \frac{u_z}{h_z} \frac{\partial h_\phi}{\partial z} \right)^2 + \frac{1}{2} \left(\frac{\partial u_\theta}{\partial \phi} - \frac{u_\phi}{h_\theta} \frac{\partial h_\phi}{\partial \theta} \right)^2 \\ &+ \frac{1}{2} \left(\frac{\partial u_z}{\partial \phi} - \frac{u_\phi}{h_z} \frac{\partial h_\phi}{\partial z} \right)^2 \end{aligned} \quad (2.31)$$

$$\begin{aligned}
\gamma_{\theta\theta} = & h_\theta \frac{\partial u_\theta}{\partial \theta} + \frac{h_\theta u_z}{h_z} \frac{\partial h_\theta}{\partial z} + \frac{h_\theta u_\phi}{h_\phi} \frac{\partial h_\theta}{\partial \phi} + \\
& + \frac{1}{2} \left(\frac{\partial u_\theta}{\partial \theta} + \frac{u_z}{h_z} \frac{\partial h_\theta}{\partial z} + \frac{u_\phi}{h_\phi} \frac{\partial h_\theta}{\partial \phi} \right)^2 + \frac{1}{2} \left(\frac{\partial u_z}{\partial \theta} - \frac{u_\theta}{h_z} \frac{\partial h_\theta}{\partial z} \right)^2 + \\
& + \frac{1}{2} \left(\frac{\partial u_\phi}{\partial \theta} - \frac{u_\theta}{h_\phi} \frac{\partial h_\theta}{\partial \phi} \right)^2
\end{aligned} \tag{2.32}$$

$$\begin{aligned}
\gamma_{\phi\theta} = & \frac{1}{2} \left(h_\phi \frac{\partial u_\phi}{\partial \theta} + h_\theta \frac{\partial u_\theta}{\partial \phi} - u_\theta \frac{\partial h_\theta}{\partial \phi} - u_\phi \frac{\partial h_\phi}{\partial \theta} \right) + \\
& + \frac{1}{2} \left(\frac{\partial u_\phi}{\partial \theta} - \frac{u_\theta}{h_\phi} \frac{\partial h_\theta}{\partial \phi} \right) \left(\frac{\partial u_\phi}{\partial \phi} + \frac{u_\theta}{h_\theta} \frac{\partial h_\phi}{\partial \theta} + \frac{u_z}{h_z} \frac{\partial h_\phi}{\partial z} \right) + \\
& + \frac{1}{2} \left(\frac{\partial u_\theta}{\partial \phi} - \frac{u_\phi}{h_\theta} \frac{\partial h_\phi}{\partial \theta} \right) \left(\frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{h_\phi} \frac{\partial h_\theta}{\partial \phi} + \frac{u_z}{h_z} \frac{\partial h_\theta}{\partial z} \right) + \\
& + \frac{1}{2} \left(\frac{\partial u_z}{\partial \phi} - \frac{u_\phi}{h_z} \frac{\partial h_\phi}{\partial z} \right) \left(\frac{\partial u_z}{\partial \theta} - \frac{u_\theta}{h_z} \frac{\partial h_\theta}{\partial z} \right)
\end{aligned} \tag{2.33}$$

$$\begin{aligned}
\gamma_{\phi z} = & \frac{1}{2} \left(h_z \frac{\partial u_z}{\partial \phi} + h_\phi \frac{\partial u_\phi}{\partial z} - u_\phi \frac{\partial h_\phi}{\partial z} - u_z \frac{\partial h_z}{\partial \phi} \right) + \\
& + \frac{1}{2} \left(\frac{\partial u_\phi}{\partial z} - \frac{u_z}{h_\phi} \frac{\partial h_z}{\partial \phi} \right) \left(\frac{\partial u_\phi}{\partial \phi} + \frac{u_z}{h_z} \frac{\partial h_\phi}{\partial z} + \frac{u_\theta}{h_\theta} \frac{\partial h_\phi}{\partial \theta} \right) + \\
& + \frac{1}{2} \left(\frac{\partial u_z}{\partial \phi} - \frac{u_\phi}{h_z} \frac{\partial h_\phi}{\partial z} \right) \left(\frac{\partial u_z}{\partial z} + \frac{u_\phi}{h_\phi} \frac{\partial h_z}{\partial \phi} + \frac{u_\theta}{h_\theta} \frac{\partial h_z}{\partial \theta} \right) + \\
& + \frac{1}{2} \left(\frac{\partial u_\theta}{\partial \phi} - \frac{u_\phi}{h_\theta} \frac{\partial h_\phi}{\partial \theta} \right) \left(\frac{\partial u_\theta}{\partial z} - \frac{u_z}{h_\theta} \frac{\partial h_z}{\partial \theta} \right)
\end{aligned} \tag{2.34}$$

$$\begin{aligned}
\gamma_{\theta z} = & \frac{1}{2} \left(h_z \frac{\partial u_z}{\partial \theta} + h_\theta \frac{\partial u_\theta}{\partial z} - u_\theta \frac{\partial h_\theta}{\partial z} - u_z \frac{\partial h_z}{\partial \theta} \right) + \\
& + \frac{1}{2} \left(\frac{\partial u_\theta}{\partial z} - \frac{u_z}{h_\theta} \frac{\partial h_z}{\partial \theta} \right) \left(\frac{\partial u_\theta}{\partial \theta} + \frac{u_z}{h_z} \frac{\partial h_\theta}{\partial z} + \frac{u_\phi}{h_\phi} \frac{\partial h_\theta}{\partial \phi} \right) + \\
& + \frac{1}{2} \left(\frac{\partial u_z}{\partial \theta} - \frac{u_\theta}{h_z} \frac{\partial h_\theta}{\partial z} \right) \left(\frac{\partial u_z}{\partial z} + \frac{u_\theta}{h_\theta} \frac{\partial h_z}{\partial \theta} + \frac{u_\phi}{h_\phi} \frac{\partial h_z}{\partial \phi} \right) + \\
& + \frac{1}{2} \left(\frac{\partial u_\phi}{\partial \theta} - \frac{u_\theta}{h_\phi} \frac{\partial h_\theta}{\partial \phi} \right) \left(\frac{\partial u_\phi}{\partial z} - \frac{u_z}{h_\phi} \frac{\partial h_z}{\partial \phi} \right)
\end{aligned} \tag{2.35}$$

where u_ϕ , u_θ and u_z define displacements in ϕ , θ and z directions at a general point respectively. By substituting Eq. (2.29) into Eqs. (2.31-2.35), Green-Lagrange strain-displacement relationships for shells of revolution considering deepness effect and full nonlinearity in transverse shear are obtained as below

$$\begin{aligned}
\varepsilon_{\phi\phi} = & \frac{1}{A_\phi} \frac{1}{\left(1 + \frac{z}{R_\phi}\right)} \left(u_z + \frac{du_\phi}{d\phi} \right) + \\
& + \frac{1}{2} \left(\frac{1}{A_\phi} \right)^2 \frac{1}{\left(1 + \frac{z}{R_\phi}\right)^2} \left[\left(\frac{du_\phi}{d\phi} + u_z \right)^2 + \left(\frac{du_\theta}{d\phi} \right)^2 + \left(\frac{du_z}{d\phi} - u_\phi \right)^2 \right]
\end{aligned} \tag{2.36}$$

$$\begin{aligned}
\varepsilon_{\theta\theta} = & \frac{1}{A_\theta} \frac{1}{\left(1 + \frac{z}{R_\theta}\right)} \left(u_\phi \cos(\phi) + \frac{du_\theta}{d\theta} + u_z \sin(\phi) \right) + \\
& + \left(\frac{1}{A_\theta} \right)^2 \frac{1}{2 \left(1 + \frac{z}{R_\theta}\right)^2} \left(\begin{aligned} & \left(\frac{du_\phi}{d\theta} - u_\theta \cos(\phi) \right)^2 \\ & + \left(u_\phi \cos(\phi) + \frac{du_\theta}{d\theta} + u_z \sin(\phi) \right)^2 \\ & + \left(\frac{du_z}{d\theta} - u_\theta \sin(\phi) \right)^2 \end{aligned} \right)
\end{aligned} \tag{2.37}$$

$$\begin{aligned}
\gamma_{\phi\theta} = & \frac{1}{A_\phi} \frac{1}{\left(1 + \frac{z}{R_\phi}\right)} \frac{du_\theta}{d\phi} + \frac{1}{A_\theta} \frac{1}{\left(1 + \frac{z}{R_\theta}\right)} \left(\frac{du_\phi}{d\theta} - u_\theta \cos(\phi) \right) + \\
& + \frac{1}{A_\phi} \frac{1}{A_\theta} \frac{1}{\left(1 + \frac{z}{R_\phi}\right)} \frac{1}{\left(1 + \frac{z}{R_\theta}\right)} \left[\left(\frac{du_\phi}{d\phi} \frac{du_\phi}{d\theta} \right) + \left(\frac{du_\theta}{d\phi} \frac{du_\theta}{d\theta} \right) + \left(\frac{du_z}{d\phi} \frac{du_z}{d\theta} \right) \right. \\
& \left. + u_z \frac{du_\phi}{d\theta} - u_\phi \frac{du_z}{d\theta} \right. \\
& \left. + \left(u_\phi \frac{du_\theta}{d\phi} - u_\theta \frac{du_\phi}{d\phi} - u_\theta u_z \right) \cos(\phi) \right. \\
& \left. + \left(u_z \frac{du_\theta}{d\phi} - u_\theta \frac{du_z}{d\phi} + u_\phi u_\theta \right) \sin(\phi) \right] \quad (2.38)
\end{aligned}$$

$$\begin{aligned}
\gamma_{\phi z} = & \frac{du_\phi}{dz} + \frac{1}{\left(1 + \frac{z}{R_\phi}\right)} \frac{1}{A_\phi} \left(\frac{du_z}{d\phi} - u_\phi \right) + \\
& + \frac{1}{\left(1 + \frac{z}{R_\phi}\right)} \frac{1}{A_\phi} \left[\left(\frac{du_\phi}{d\phi} \frac{du_\phi}{dz} \right) + \left(\frac{du_\theta}{d\phi} \frac{du_\theta}{dz} \right) + \left(\frac{du_z}{d\phi} \frac{du_z}{dz} \right) + u_z \frac{du_\phi}{dz} - u_\phi \frac{du_z}{dz} \right] \quad (2.39)
\end{aligned}$$

$$\begin{aligned}
\gamma_{\theta z} = & \frac{du_\theta}{dz} + \frac{1}{\left(1 + \frac{z}{R_\theta}\right)} \frac{1}{A_\theta} \left(\frac{du_z}{d\theta} - u_\theta \sin(\phi) \right) + \\
& + \frac{1}{\left(1 + \frac{z}{R_\theta}\right)} \frac{1}{A_\theta} \left[\left(\frac{du_\phi}{d\theta} \frac{du_\phi}{dz} \right) + \left(\frac{du_\theta}{d\theta} \frac{du_\theta}{dz} \right) + \left(\frac{du_z}{d\theta} \frac{du_z}{dz} \right) \right. \\
& \left. + \sin(\phi) \left(u_z \frac{du_\theta}{dz} - u_\theta \frac{du_z}{dz} \right) + \cos(\phi) \left(u_\phi \frac{du_\theta}{dz} - u_\theta \frac{du_\phi}{dz} \right) \right] \quad (2.40)
\end{aligned}$$

2.4. Governing Equations

Displacements at a general point (ϕ, θ, z) at time t (for static analysis $t=0$) considering FSDT can be stated as following

$$\begin{aligned}
u_\phi(\phi, \theta, z, t) &= u_\phi^0(\phi, \theta, t) + z\psi_\phi(\phi, \theta, t) \\
u_\theta(\phi, \theta, z, t) &= u_\theta^0(\phi, \theta, t) + z\psi_\theta(\phi, \theta, t) \\
u_z(\phi, \theta, z, t) &= u_z^0(\phi, \theta, t)
\end{aligned} \tag{2.41}$$

where u_ϕ^0 , u_θ^0 and u_z^0 correspond to mid-plane displacements in the meridional, circumferential and normal directions respectively. ψ_ϕ and ψ_θ correspond to the rotations about θ and ϕ axes respectively. Green-Lagrange nonlinear strains in terms of middle surface displacements and rotations can be written as below

for $\varepsilon_{\phi\phi}$;

$$\varepsilon_{\phi\phi} = \frac{1}{\left(1 + \frac{z}{R_\phi}\right)} \left(\varepsilon_{\phi_L}^0 + z\kappa_\phi \right) + \frac{1}{2 \left(1 + \frac{z}{R_\phi}\right)^2} \left(\varepsilon_{\phi_{NL}} \right) \tag{2.42}$$

$$\varepsilon_{\phi_L}^0 = \frac{1}{R_\phi} \left(u_z^0 + \frac{du_\phi^0}{d\phi} \right) \tag{2.43}$$

$$\kappa_\phi = \frac{1}{R_\phi} \left(\frac{d\psi_\phi}{d\phi} \right) \tag{2.44}$$

$$\varepsilon_{\phi_{NL}} = \left(\frac{1}{R_\phi} \right)^2 \left[\left(\frac{du_\phi^0}{d\phi} \right)^2 + \left(\frac{du_\theta^0}{d\phi} \right)^2 + \left(\frac{du_z^0}{d\phi} \right)^2 + \right. \\ \left. + \left(u_\phi^0 \right)^2 + \left(u_z^0 \right)^2 + 2u_z^0 \frac{du_\phi^0}{d\phi} - 2u_\phi^0 \frac{du_z^0}{d\phi} \right] \tag{2.45}$$

for $\varepsilon_{\theta\theta}$;

$$\varepsilon_{\theta\theta} = \frac{1}{\left(1 + \frac{z}{R_\theta}\right)} \left(\varepsilon_{\theta_L}^0 + z\kappa_\theta \right) + \frac{1}{2 \left(1 + \frac{z}{R_\theta}\right)^2} \left(\varepsilon_{\theta_{NL}} \right) \tag{2.46}$$

$$\varepsilon_{\theta_L}^0 = \frac{1}{R_\theta \sin(\phi)} \left(u_\phi^0 \cos(\phi) + \frac{du_\theta^0}{d\theta} + u_z^0 \sin(\phi) \right) \quad (2.47)$$

$$\kappa_\theta = \frac{1}{R_\theta \sin(\phi)} \left(\frac{d\psi_\theta}{d\theta} + \psi_\phi \cos(\phi) \right) \quad (2.48)$$

$$\varepsilon_{\theta_{NL}} = \left(\frac{1}{R_\theta \sin(\phi)} \right)^2 \left[\begin{aligned} & \left(\frac{du_\phi^0}{d\theta} \right)^2 + \left(\frac{du_\theta^0}{d\theta} \right)^2 + \left(\frac{du_z^0}{d\theta} \right)^2 + \\ & + \left((u_\phi^0)^2 + (u_\theta^0)^2 \right) \cos^2(\phi) + \left((u_\theta^0)^2 + (u_z^0)^2 \right) \sin^2(\phi) + \\ & + \left(2u_\phi^0 \frac{du_\theta^0}{d\theta} - 2u_\theta^0 \frac{du_\phi^0}{d\theta} \right) \cos(\phi) + \\ & + \left(2u_z^0 \frac{du_\theta^0}{d\theta} - 2u_\theta^0 \frac{du_z^0}{d\theta} \right) \sin(\phi) + 2u_\phi^0 u_z^0 \sin(\phi) \cos(\phi) + \end{aligned} \right] \quad (2.49)$$

for $\gamma_{\phi\theta}$;

$$\begin{aligned} \gamma_{\phi\theta} = & \frac{1}{\left(1 + \frac{z}{R_\phi} \right)} \left(\gamma_{\phi\theta_L}^0 + z\kappa_{\phi\theta} \right) + \frac{1}{\left(1 + \frac{z}{R_\theta} \right)} \left(\gamma_{\theta\phi_L}^0 + z\kappa_{\theta\phi} \right) + \\ & + \frac{1}{\left(1 + \frac{z}{R_\phi} \right)} \frac{1}{\left(1 + \frac{z}{R_\theta} \right)} \left(\gamma_{\phi\theta_{NL}} \right) \end{aligned} \quad (2.50)$$

$$\gamma_{\phi\theta_L}^0 = \frac{1}{R_\phi} \left(\frac{du_\theta^0}{d\phi} \right) \quad (2.51)$$

$$\kappa_{\phi\theta} = \frac{1}{R_\phi} \left(\frac{d\psi_\theta}{d\phi} \right) \quad (2.52)$$

$$\gamma_{\theta\phi_L}^0 = \frac{1}{R_\theta \sin(\phi)} \left(\frac{du_\phi^0}{d\theta} - u_\theta^0 \cos(\phi) \right) \quad (2.53)$$

$$\kappa_{\theta\phi} = \frac{1}{R_\theta \sin(\phi)} \left(\frac{d\psi_\phi}{d\theta} - \psi_\theta \cos(\phi) \right) \quad (2.54)$$

$$\gamma_{\phi\theta_{NL}} = \frac{1}{R_\phi} \frac{1}{R_\theta \sin(\phi)} \left[\begin{aligned} & \left(\frac{du_\phi^0}{d\phi} \frac{du_\phi^0}{d\theta} \right) + \left(\frac{du_\theta^0}{d\phi} \frac{du_\theta^0}{d\theta} \right) + \left(\frac{du_z^0}{d\phi} \frac{du_z^0}{d\theta} \right) + \\ & + u_z^0 \frac{du_\phi^0}{d\theta} - u_\phi^0 \frac{du_z^0}{d\theta} + \left(u_\phi^0 \frac{du_\theta^0}{d\phi} - u_\theta^0 \frac{du_\phi^0}{d\phi} - u_\theta^0 u_z^0 \right) \cos(\phi) + \\ & + \left(u_z^0 \frac{du_\theta^0}{d\phi} - u_\theta^0 \frac{du_z^0}{d\phi} + u_\phi^0 u_\theta^0 \right) \sin(\phi) \end{aligned} \right] \quad (2.55)$$

for $\gamma_{\phi z}$;

$$\gamma_{\phi z} = \gamma_{\phi z 0} + \frac{1}{\left(1 + \frac{z}{R_\phi}\right)} \gamma_{\phi z 1} + \frac{z}{\left(1 + \frac{z}{R_\phi}\right)} \gamma_{\phi z 2} \quad (2.56)$$

$$\gamma_{\phi z 0} = \psi_\phi \quad (2.57)$$

$$\gamma_{\phi z 1} = \gamma_{\phi z_L}^0 + \gamma_{\phi z_{NL}} \quad (2.58)$$

$$\gamma_{\phi z 2} = \kappa_{\phi z} \quad (2.59)$$

$$\gamma_{\phi z_L}^0 = \frac{1}{R_\phi} \frac{du_z^0}{d\phi} - \frac{u_\phi^0}{R_\phi} \quad (2.60)$$

$$\kappa_{\phi z} = -\frac{1}{R_\phi} \psi_\phi \quad (2.61)$$

$$\gamma_{\phi z_{NL}} = \frac{1}{R_\phi} \left[\frac{du_\phi^0}{d\phi} \psi_\phi + u_z^0 \psi_\phi + \frac{du_\theta^0}{d\phi} \psi_\theta \right] \quad (2.62)$$

for $\gamma_{\theta z}$;

$$\gamma_{\theta z} = \gamma_{\theta z 0} + \frac{1}{\left(1 + \frac{z}{R_\theta}\right)} \gamma_{\theta z 1} + \frac{z}{\left(1 + \frac{z}{R_\theta}\right)} \gamma_{\theta z 2} \quad (2.63)$$

$$\gamma_{\theta z 0} = \psi_\theta \quad (2.64)$$

$$\gamma_{\theta z 1} = \gamma_{\theta z_L}^0 + \gamma_{\theta z_{NL}} \quad (2.65)$$

$$\gamma_{\theta z 2} = \kappa_{\theta z} \quad (2.66)$$

$$\gamma_{\theta z_L}^0 = \frac{1}{R_\theta \sin(\phi)} \left(\frac{du_z^0}{d\theta} - u_\theta^0 \sin(\phi) \right) \quad (2.67)$$

$$\kappa_{\theta z} = \frac{-\psi_\theta}{R_\theta} \quad (2.68)$$

$$\gamma_{\theta z_{NL}} = \frac{1}{R_\theta \sin(\phi)} \left[\frac{du_\phi^0}{d\theta} \psi_\phi + \frac{du_\theta^0}{d\theta} \psi_\theta + u_z^0 \psi_\theta \sin(\phi) + (u_\phi^0 \psi_\theta - u_\theta^0 \psi_\phi) \cos(\phi) \right] \quad (2.69)$$

By keeping terms including z^2 and expanding deepness related terms in series as following

$$\frac{1}{\left(1 + \frac{z}{R_\phi}\right)} \cong 1 - \frac{z}{R_\phi} + \frac{z^2}{R_\phi^2} \quad (2.70)$$

$$\frac{1}{\left(1 + \frac{z}{R_\theta}\right)} \cong 1 - \frac{z}{R_\theta} + \frac{z^2}{R_\theta^2} \quad (2.71)$$

$$\frac{1}{2\left(1+\frac{z}{R_\phi}\right)^2} \cong \frac{1}{2}\left(1-2\frac{z}{R_\phi}+3\frac{z^2}{R_\phi^2}-4\frac{z^3}{R_\phi^3}\right) \cong \frac{1}{2}-\frac{z}{R_\phi}+\frac{3}{2}\frac{z^2}{R_\phi^2} \quad (2.72)$$

$$\frac{1}{2\left(1+\frac{z}{R_\theta}\right)^2} \cong \frac{1}{2}\left(1-2\frac{z}{R_\theta}+3\frac{z^2}{R_\theta^2}-4\frac{z^3}{R_\theta^3}\right) \cong \frac{1}{2}-\frac{z}{R_\theta}+\frac{3}{2}\frac{z^2}{R_\theta^2} \quad (2.73)$$

Strain-displacement relationships in Eqs. (2.42-2.69) can be stated for in-plane and transverse shear parts as below

$$\begin{bmatrix} \varepsilon_{\phi\phi} \\ \varepsilon_{\theta\theta} \\ \gamma_{\phi\theta} \end{bmatrix} = \begin{bmatrix} \lambda_\phi^{(0)} \\ \lambda_\theta^{(0)} \\ \lambda_{\phi\theta}^{(0)} \end{bmatrix} + z \begin{bmatrix} \lambda_\phi^{(1)} \\ \lambda_\theta^{(1)} \\ \lambda_{\phi\theta}^{(1)} \end{bmatrix} + z^2 \begin{bmatrix} \lambda_\phi^{(2)} \\ \lambda_\theta^{(2)} \\ \lambda_{\phi\theta}^{(2)} \end{bmatrix} + z^3 \begin{bmatrix} \lambda_\phi^{(3)} \\ \lambda_\theta^{(3)} \\ \lambda_{\phi\theta}^{(3)} \end{bmatrix} + z^4 \begin{bmatrix} 0 \\ 0 \\ \lambda_{\phi\theta}^{(4)} \end{bmatrix} \quad (2.74)$$

$$\begin{bmatrix} \gamma_{\theta z} \\ \gamma_{\phi z} \end{bmatrix} = \begin{bmatrix} \lambda_{\theta z}^{(0)} \\ \lambda_{\phi z}^{(0)} \end{bmatrix} + z \begin{bmatrix} \lambda_{\theta z}^{(1)} \\ \lambda_{\phi z}^{(1)} \end{bmatrix} + z^2 \begin{bmatrix} \lambda_{\theta z}^{(2)} \\ \lambda_{\phi z}^{(2)} \end{bmatrix} + z^3 \begin{bmatrix} \lambda_{\theta z}^{(3)} \\ \lambda_{\phi z}^{(3)} \end{bmatrix} \quad (2.75)$$

The terms in the Eq. (2.74) and Eq. (2.75) can be expressed as

$$\lambda_\phi^{(0)} = \varepsilon_{\phi_L}^0 + \frac{1}{2} \varepsilon_{\phi_{NL}} \quad (2.76)$$

$$\lambda_\phi^{(1)} = -\frac{\varepsilon_{\phi_L}^0}{R_\phi} + \kappa_\phi - \frac{\varepsilon_{\phi_{NL}}}{R_\phi} \quad (2.77)$$

$$\lambda_\phi^{(2)} = \frac{\varepsilon_{\phi_L}^0}{R_\phi^2} - \frac{\kappa_\phi}{R_\phi} + \frac{3}{2} \frac{\varepsilon_{\phi_{NL}}}{R_\phi^2} \quad (2.78)$$

$$\lambda_\phi^{(3)} = \frac{\kappa_\phi}{R_\phi^2} \quad (2.79)$$

$$\lambda_\theta^{(0)} = \varepsilon_{\theta_L}^0 + \frac{1}{2} \varepsilon_{\theta_{NL}} \quad (2.80)$$

$$\lambda_{\theta}^{(1)} = -\frac{\varepsilon_{\theta_L}^0}{R_{\theta}} + \kappa_{\theta} - \frac{\varepsilon_{\theta_{NL}}}{R_{\theta}} \quad (2.81)$$

$$\lambda_{\theta}^{(2)} = \frac{\varepsilon_{\theta_L}^0}{R_{\theta}^2} - \frac{\kappa_{\theta}}{R_{\theta}} + \frac{3}{2} \frac{\varepsilon_{\theta_{NL}}}{R_{\theta}^2} \quad (2.82)$$

$$\lambda_{\theta}^{(3)} = \frac{\kappa_{\theta}}{R_{\theta}^2} \quad (2.83)$$

$$\lambda_{\phi\theta}^{(0)} = \gamma_{\phi\theta_L}^0 + \gamma_{\theta\phi_L}^0 + \gamma_{\phi\theta_{NL}} \quad (2.84)$$

$$\lambda_{\phi\theta}^{(1)} = -\frac{\gamma_{\phi\theta_L}^0}{R_{\phi}} + \kappa_{\phi\theta} - \frac{\gamma_{\theta\phi_L}^0}{R_{\theta}} + \kappa_{\theta\phi} - \frac{\gamma_{\phi\theta_{NL}}}{R_{\theta}} - \frac{\gamma_{\phi\theta_{NL}}}{R_{\phi}} \quad (2.85)$$

$$\lambda_{\phi\theta}^{(2)} = \frac{\gamma_{\phi\theta_L}^0}{R_{\phi}^2} - \frac{\kappa_{\phi\theta}}{R_{\phi}} + \frac{\gamma_{\theta\phi_L}^0}{R_{\theta}^2} - \frac{\kappa_{\theta\phi}}{R_{\theta}} + \frac{\gamma_{\phi\theta_{NL}}}{R_{\theta}^2} + \frac{\gamma_{\phi\theta_{NL}}}{R_{\phi}R_{\theta}} + \frac{\gamma_{\phi\theta_{NL}}}{R_{\phi}^2} \quad (2.86)$$

$$\lambda_{\phi\theta}^{(3)} = \frac{\kappa_{\phi\theta}}{R_{\phi}^2} + \frac{\kappa_{\theta\phi}}{R_{\theta}^2} - \frac{\gamma_{\phi\theta_{NL}}}{R_{\phi}R_{\theta}^2} - \frac{\gamma_{\phi\theta_{NL}}}{R_{\phi}^2R_{\theta}} \quad (2.87)$$

$$\lambda_{\phi\theta}^{(4)} = \frac{\gamma_{\phi\theta_{NL}}}{R_{\phi}^2R_{\theta}^2} \quad (2.88)$$

$$\lambda_{\phi z}^{(0)} = \gamma_{\phi z 0} + \gamma_{\phi z 1} \quad (2.89)$$

$$\lambda_{\phi z}^{(1)} = -\frac{\gamma_{\phi z 1}}{R_{\phi}} + \gamma_{\phi z 2} \quad (2.90)$$

$$\lambda_{\phi z}^{(2)} = \frac{\gamma_{\phi z 1}}{R_{\phi}^2} - \frac{\gamma_{\phi z 2}}{R_{\phi}} \quad (2.91)$$

$$\lambda_{\phi z}^{(3)} = \frac{\gamma_{\phi z 2}}{R_{\phi}^2} \quad (2.92)$$

$$\lambda_{\theta z}^{(0)} = \gamma_{\theta z 0} + \gamma_{\theta z 1} \quad (2.93)$$

$$\lambda_{\theta z}^{(1)} = -\frac{\gamma_{\theta z1}}{R_\theta} + \gamma_{\theta z2} \quad (2.94)$$

$$\lambda_{\theta z}^{(2)} = \frac{\gamma_{\theta z1}}{R_\theta^2} - \frac{\gamma_{\theta z2}}{R_\theta} \quad (2.95)$$

$$\lambda_{\theta z}^{(3)} = \frac{\gamma_{\theta z2}}{R_\theta^2} \quad (2.96)$$

Nonlinear strain-displacement relationships for moderately thick shells of revolution in Eq. (2.74) and Eq. (2.75) can be stated in series form of z as following

$$\boldsymbol{\varepsilon}_b = \boldsymbol{\varepsilon}_{b0} + z\boldsymbol{\varepsilon}_{b1} + z^2\boldsymbol{\varepsilon}_{b2} + z^3\boldsymbol{\varepsilon}_{b3} + z^4\boldsymbol{\varepsilon}_{b4} \quad (2.97)$$

$$\boldsymbol{\varepsilon}_s = \boldsymbol{\varepsilon}_{s0} + z\boldsymbol{\varepsilon}_{s1} + z^2\boldsymbol{\varepsilon}_{s2} + z^3\boldsymbol{\varepsilon}_{s3} \quad (2.98)$$

where $\boldsymbol{\varepsilon}_b$ and $\boldsymbol{\varepsilon}_s$ indicate in-plane and transverse shear strains, respectively.

2.5. Constitutive Equations for Shells of Revolution

Lamina coordinate system can be observed in Figure 2.7. 1-2 axes specify the local coordinate system of lamina. α is the fiber angle of each laminate with the global panel axis ϕ direction. All results should be determined in one global coordinate system of the panel. The constitutive relations for a laminated composite shells of revolution can be expressed in terms of the in-plane moment and force resultants as

$$\mathbf{N} = \begin{Bmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \\ \mathbf{N}_3 \\ \mathbf{N}_4 \\ \mathbf{N}_5 \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} & \mathbf{E} \\ \mathbf{B} & \mathbf{C} & \mathbf{D} & \mathbf{E} & \mathbf{F} \\ \mathbf{C} & \mathbf{D} & \mathbf{E} & \mathbf{F} & \mathbf{G} \\ \mathbf{D} & \mathbf{E} & \mathbf{F} & \mathbf{G} & \mathbf{H} \\ \mathbf{E} & \mathbf{F} & \mathbf{G} & \mathbf{H} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}_{b0} \\ \boldsymbol{\varepsilon}_{b1} \\ \boldsymbol{\varepsilon}_{b2} \\ \boldsymbol{\varepsilon}_{b3} \\ \boldsymbol{\varepsilon}_{b4} \end{Bmatrix} \quad (2.99)$$

where

$$\mathbf{N}_1 = \begin{Bmatrix} N_{\phi 1} \\ N_{\theta 1} \\ N_{\phi \theta 1} \end{Bmatrix}, \quad \mathbf{N}_2 = \begin{Bmatrix} N_{\phi 2} \\ N_{\theta 2} \\ N_{\phi \theta 2} \end{Bmatrix}, \quad \mathbf{N}_3 = \begin{Bmatrix} N_{\phi 3} \\ N_{\theta 3} \\ N_{\phi \theta 3} \end{Bmatrix}, \quad \mathbf{N}_4 = \begin{Bmatrix} N_{\phi 4} \\ N_{\theta 4} \\ N_{\phi \theta 4} \end{Bmatrix}, \quad \mathbf{N}_5 = \begin{Bmatrix} N_{\phi 5} \\ N_{\theta 5} \\ N_{\phi \theta 5} \end{Bmatrix} \quad (2.100)$$

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{21} & A_{22} & A_{26} \\ A_{61} & A_{62} & A_{66} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{21} & B_{22} & B_{26} \\ B_{61} & B_{62} & B_{66} \end{bmatrix}, \quad \dots, \quad \mathbf{I} = \begin{bmatrix} I_{11} & I_{12} & I_{16} \\ I_{21} & I_{22} & I_{26} \\ I_{61} & I_{62} & I_{66} \end{bmatrix} \quad (2.101)$$

Laminate constitutive equation for transverse shear is stated as following

$$\mathbf{S} = \begin{Bmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \\ \mathbf{S}_3 \\ \mathbf{S}_4 \end{Bmatrix} = \begin{bmatrix} \mathbf{A}_s & \mathbf{B}_s & \mathbf{C}_s & \mathbf{D}_s \\ \mathbf{B}_s & \mathbf{C}_s & \mathbf{D}_s & \mathbf{E}_s \\ \mathbf{C}_s & \mathbf{D}_s & \mathbf{E}_s & \mathbf{F}_s \\ \mathbf{D}_s & \mathbf{E}_s & \mathbf{F}_s & \mathbf{G}_s \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}_{s0} \\ \boldsymbol{\varepsilon}_{s1} \\ \boldsymbol{\varepsilon}_{s2} \\ \boldsymbol{\varepsilon}_{s3} \end{Bmatrix} \quad (2.102)$$

where

$$\mathbf{S}_1 = \begin{Bmatrix} S_{\theta z 1} \\ S_{\phi z 1} \end{Bmatrix}, \quad \mathbf{S}_2 = \begin{Bmatrix} S_{\theta z 2} \\ S_{\phi z 2} \end{Bmatrix}, \quad \mathbf{S}_3 = \begin{Bmatrix} S_{\theta z 3} \\ S_{\phi z 3} \end{Bmatrix}, \quad \mathbf{S}_4 = \begin{Bmatrix} S_{\theta z 4} \\ S_{\phi z 4} \end{Bmatrix} \quad (2.103)$$

$$\mathbf{A}_s = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix}, \quad \mathbf{B}_s = \begin{bmatrix} B_{44} & B_{45} \\ B_{45} & B_{55} \end{bmatrix}, \quad \dots, \quad \mathbf{G}_s = \begin{bmatrix} G_{44} & G_{45} \\ G_{45} & G_{55} \end{bmatrix} \quad (2.104)$$

$A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}, F_{ij}, G_{ij}, H_{ij}, I_{ij}$ indicate laminate stiffness coefficients for in-plane, bending-stretching coupling, bending and transverse shear stiffnesses. Stiffness coefficients can be obtained as following

$$\begin{aligned} & \{A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}, F_{ij}, G_{ij}, H_{ij}, I_{ij}\} \\ &= \sum_{k=1}^n \int_{z_{k-1}(\phi)}^{z_k(\phi)} \{1, z, z^2, z^3, z^4, z^5, z^6, z^7, z^8\} \bar{Q}_{ij}^{(k)} A_{\phi} \left(1 + \frac{z}{R_{\phi}}\right) A_{\theta} \left(1 + \frac{z}{R_{\theta}}\right) dz \quad (i, j = 1, 2, 6) \end{aligned} \quad (2.105)$$

$$\begin{aligned} & \{A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}, F_{ij}, G_{ij}\} \\ &= \sum_{k=1}^n k_i k_j \int_{z_{k-1}(\phi)}^{z_k(\phi)} \{1, z, z^2, z^3, z^4, z^5, z^6\} \bar{Q}_{ij}^{(k)} A_{\phi} \left(1 + \frac{z}{R_{\phi}}\right) A_{\theta} \left(1 + \frac{z}{R_{\theta}}\right) dz \quad (i, j = 4, 5) \end{aligned} \quad (2.106)$$

where $\bar{Q}_{ij}^{(k)}$ indicates transformed stiffness coefficients of k -th layer and $k_i^2 = \frac{5}{6}$ ($i=4,5$)

denotes the shear correction factors. $\bar{Q}_{ij}^{(k)}$ is stated as following

$$\bar{Q}_{11} = Q_{11} \cos^4(\alpha) + 2[Q_{12} + 2Q_{66}] \sin^2(\alpha) \cos^2(\alpha) + Q_{22} \sin^4(\alpha) \quad (2.107)$$

$$\bar{Q}_{12} = [Q_{11} + Q_{22} - 4Q_{66}] \sin^2(\alpha) \cos^2(\alpha) + Q_{12} [\sin^4(\alpha) + \cos^4(\alpha)] \quad (2.108)$$

$$\bar{Q}_{22} = Q_{11} \sin^4(\alpha) + 2[Q_{12} + 2Q_{66}] \sin^2(\alpha) \cos^2(\alpha) + Q_{22} \cos^4(\alpha) \quad (2.109)$$

$$\bar{Q}_{16} = [Q_{11} - Q_{12} - 2Q_{66}] \sin(\alpha) \cos^3(\alpha) + [Q_{12} - Q_{22} + 2Q_{66}] \sin^3(\alpha) \cos(\alpha) \quad (2.110)$$

$$\bar{Q}_{26} = [Q_{11} - Q_{12} - 2Q_{66}] \sin^3(\alpha) \cos(\alpha) + [Q_{12} - Q_{22} + 2Q_{66}] \sin(\alpha) \cos^3(\alpha) \quad (2.111)$$

$$\bar{Q}_{66} = [Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}] \sin^2(\alpha) \cos^2(\alpha) + Q_{66} [\sin^4(\alpha) + \cos^4(\alpha)] \quad (2.112)$$

$$\bar{Q}_{44} = Q_{44} \cos^2(\alpha) + Q_{55} \sin^2(\alpha) \quad (2.113)$$

$$\bar{Q}_{45} = [Q_{55} - Q_{44}] \sin(\alpha) \cos(\alpha) \quad (2.114)$$

$$\bar{Q}_{55} = Q_{55} \cos^2(\alpha) + Q_{44} \sin^2(\alpha) \quad (2.115)$$

The components of transformed stiffness coefficients $\bar{Q}_{ij}^{(k)}$ are given below

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}} \quad (2.116)$$

$$Q_{12} = Q_{21} = \frac{\nu_{12}E_1}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_2}{1 - \nu_{12}\nu_{21}} \quad (2.117)$$

$$Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}} \quad (2.118)$$

$$Q_{66} = G_{12} \quad (2.119)$$

$$Q_{44} = G_{23} \quad (2.120)$$

$$Q_{55} = G_{13} \quad (2.121)$$

where E_1 and E_2 are the elastic modulus and ν_{12} and ν_{21} are the Poisson's ratios of composite layer in orthogonal directions. G_{12} , G_{23} and G_{13} are shear modulus.

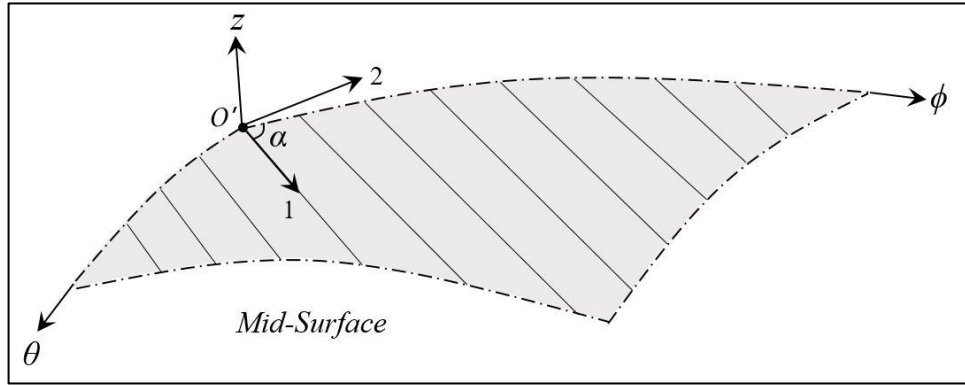


Figure 2.7: Lamina coordinate system.

In this thesis, weak form of equations of motion is derived for laminated composite shells of revolution structures having variable thickness resting on Winkler-Pasternak elastic foundation using the virtual work principle. Transient version of the virtual work principle can be expressed as virtual work sums of inertia, internal and elastic foundation forces are equal to the virtual work of external forces. Illustration of elastic foundation between the shells of revolution with variable thickness and the ground is shown in Figure 2.8.

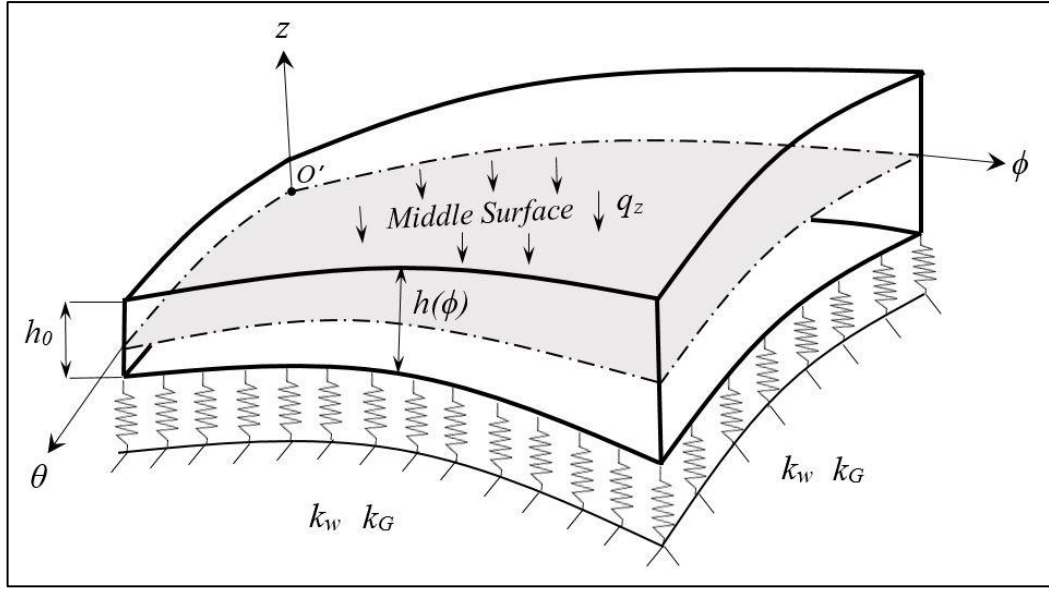


Figure 2.8: Shells of revolution having variable thickness resting on Winkler-Pasternak elastic foundation at the bottom surface.

Virtual work principle for the laminated composite shells of revolution with variable thickness resting on Winkler-Pasternak elastic foundation can be stated as below

$$\begin{aligned}
 & \sum_{k=1}^n \int_{z_{k-1}(\phi)}^{z_k(\phi)} \left[\int_{\Omega} \{ \sigma_{\phi\phi}^{(k)} \delta \varepsilon_{\phi\phi} + \sigma_{\theta\theta}^{(k)} \delta \varepsilon_{\theta\theta} + \tau_{\phi\theta}^{(k)} \delta \gamma_{\phi\theta} + \tau_{\phi z}^{(k)} \delta \gamma_{\phi z} + \tau_{\theta z}^{(k)} \delta \gamma_{\theta z} \} + \right. \\
 & + \rho^{(k)} \left\{ (\ddot{u}_0 + z \ddot{\psi}_{\phi}) (\delta u_0 + z \delta \psi_{\phi}) + (\ddot{v}_0 + z \ddot{\psi}_{\theta}) (\delta v_0 + z \delta \psi_{\theta}) + \ddot{w}_0 \delta w_0 \right\} \\
 & + \left. \left\{ k_w w_0 \delta w_0 + k_G \left(\frac{\partial w_0}{\partial \phi} \frac{\partial \delta w_0}{\partial \phi} + \frac{\partial w_0}{\partial \theta} \frac{\partial \delta w_0}{\partial \theta} \right) \right\} \right] A_{\phi} \left(1 + \frac{z}{R_{\phi}} \right) A_{\theta} \left(1 + \frac{z}{R_{\theta}} \right) d\phi d\theta dz \quad (2.122) \\
 & = \int_{\Omega} q_z \delta w_0 A_{\phi} \left(1 + \frac{z}{R_{\phi}} \right) A_{\theta} \left(1 + \frac{z}{R_{\theta}} \right) d\phi d\theta
 \end{aligned}$$

where k_w denotes Winkler modulus and k_G indicates Pasternak modulus. $\rho^{(k)}$ denotes density of the k -th layer. In Eq. (2.122) stresses correspond to 2nd Piola-Kirchhoff stresses. Distributed load q_z is applied on the middle surface ($z=0$) of shells of revolution structures as shown in Figure 2.8. Eq. (2.122) can be expressed in terms of moment and force resultants as follows

$$\begin{aligned}
& \int_{\Omega} \left[\left\{ \delta \mathbf{\varepsilon}_{b0}^T \mathbf{N}_1 + \delta \mathbf{\varepsilon}_{b1}^T \mathbf{N}_2 + \delta \mathbf{\varepsilon}_{b2}^T \mathbf{N}_3 + \delta \mathbf{\varepsilon}_{b3}^T \mathbf{N}_4 \right. \right. \\
& \quad \left. \left. + \delta \mathbf{\varepsilon}_{b4}^T \mathbf{N}_5 + \delta \mathbf{\varepsilon}_{s0}^T \mathbf{S}_1 + \delta \mathbf{\varepsilon}_{s1}^T \mathbf{S}_2 + \delta \mathbf{\varepsilon}_{s2}^T \mathbf{S}_3 + \delta \mathbf{\varepsilon}_{s3}^T \mathbf{S}_4 \right\} + \right. \\
& \quad \left. + \left\{ (I_0 \ddot{u}_0 + I_1 \ddot{\psi}_\phi) \delta u_0 + (I_0 \ddot{v}_0 + I_1 \ddot{\psi}_\theta) \delta v_0 + I_0 \ddot{w}_0 \delta w_0 \right\} + \right. \\
& \quad \left. + \left\{ (I_1 \ddot{u}_0 + I_2 \ddot{\psi}_\phi) \delta \theta_x + (I_1 \ddot{v}_0 + I_2 \ddot{\psi}_\theta) \delta \psi_\theta \right\} \right] \\
& + \left\{ k_w w_0 \delta w_0 + k_G \left[\frac{1}{R_\phi^2} \frac{\partial w_0}{\partial \phi} \frac{\partial \delta w_0}{\partial \phi} + \frac{1}{R_\theta^2} \frac{\partial w_0}{\partial \theta} \frac{\partial \delta w_0}{\partial \theta} \right] \right\} A_\phi A_\theta \Bigg] d\phi d\theta = \\
& = \int_{\Omega} q_z \delta w_0 A_\phi A_\theta d\phi d\theta
\end{aligned} \tag{2.123}$$

where I_0 , I_1 and I_2 are mass inertias and they can be obtained as following

$$\{I_0, I_1, I_2\} = \sum_{k=1}^n \int_{z_{k-1}(\phi)}^{z_k(\phi)} \{1, z, z^2\} A_\phi \left(1 + \frac{z}{R_\phi}\right) A_\theta \left(1 + \frac{z}{R_\theta}\right) \rho^{(k)} dz \tag{2.124}$$

In obtaining static equilibrium equations inertia terms are neglected in Eq. (2.122) and Eq. (2.123).

3. TECHNIQUE for SOLVING EQUATIONS of MOTION

Due to the non-uniform shape of shells of revolution structures, integrals in the equations of motion of shells in Eq. (2.123) can be computed easily using numerical analysis techniques such as Gauss Lobatto quadrature or Gauss quadrature rules.

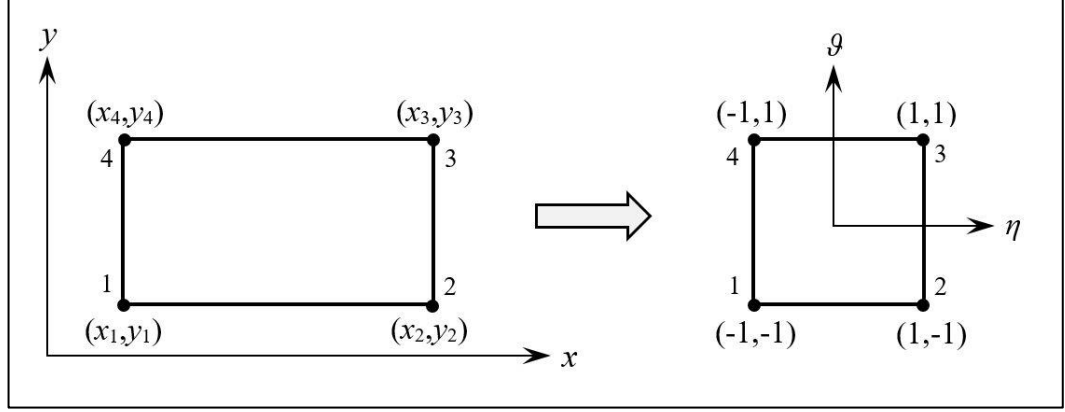


Figure 3.1: Geometric mapping transformation.

To do that, cartesian space is transformed into a bi-unit square domain as illustrated in Figure 3.1 using geometric mapping transformation as following

$$x(\eta, g) = \frac{1}{4}(\eta - 1)(g - 1)x_1 - \frac{1}{4}(\eta + 1)(g - 1)x_2 + \frac{1}{4}(\eta + 1)(g + 1)x_3 - \frac{1}{4}(\eta - 1)(g + 1)x_4 \quad (3.1)$$

$$y(\eta, g) = \frac{1}{4}(\eta - 1)(g - 1)y_1 - \frac{1}{4}(\eta + 1)(g - 1)y_2 + \frac{1}{4}(\eta + 1)(g + 1)y_3 - \frac{1}{4}(\eta - 1)(g + 1)y_4 \quad (3.2)$$

where η and g in Eq. (3.1) and Eq. (3.2) show the natural coordinates corresponding to x and y in the cartesian coordinates for bi-unit square domain. In terms of natural coordinates (η and g), the partial derivatives of a function $p(x, y)$ in cartesian coordinates can be written as follows

$$\frac{\partial p}{\partial x} = \frac{1}{J} \left(\frac{\partial y}{\partial \vartheta} \frac{\partial p}{\partial \eta} - \frac{\partial y}{\partial \eta} \frac{\partial p}{\partial \vartheta} \right) \quad (3.3)$$

$$\frac{\partial p}{\partial y} = \frac{1}{J} \left(\frac{\partial x}{\partial \eta} \frac{\partial p}{\partial \vartheta} - \frac{\partial x}{\partial \vartheta} \frac{\partial p}{\partial \eta} \right) \quad (3.4)$$

where J indicates the determinant of the Jacobian.

$$J = \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \vartheta} - \frac{\partial y}{\partial \eta} \frac{\partial x}{\partial \vartheta} \quad (3.5)$$

3.1. Generalized Differential Quadrature (GDQ) Method

Differential Quadrature Method (DQM) is a numerical technique for solving differential equations. This method, using less grid points with acceptable accuracy, was introduced by [Bellman and Casti, 1971] at the beginning of 1970s. Accuracy of DQM is affected by the calculation of the weight coefficients and the choice of grid points. In the original formulation of DQM, weight coefficients were calculated by an ill-conditioned algebraic equation system limiting the use of large grid numbers. Then, an improved version of DQM was proposed by Bellman and his colleagues. In this version, weighting coefficients are calculated without solving an algebraic system of equations. But, the distribution of grid points is restricted to the roots of the shifted Legendre polynomials.

Generalized Differential Quadrature (GDQ) method is an advanced form of the DQ method. In next times, the GDQ method was improved by [Shu, 2000] who introduced simple explicit formula for the weight coefficients. In GDQ method, selection of grid points for the weight coefficients is arbitrary. The GDQ method has been applied to the free vibration, buckling, static and dynamic analysis of beams, arches, plates and shells in many areas and it has allowed to obtain good results using a few numbers of grid points. Compared to other numerical techniques, the advantage of GDQ Method is to achieve high accuracy with less grid points.

GDQ method is employed in the computation of spatial derivatives of field variables in this thesis. Before GDQ method is implemented, laminated composite shells of revolution is discretized using grid points according to the Gauss Lobatto

distribution. In GDQ method, r -th derivative at a discrete point of the function $p(\eta)$ can be written as [Kurtaran, 2015b]

$$\left(\frac{\partial p^r(x)}{\partial \eta^r} \right)_{\eta_i} = \sum_{j=1}^n C_{ij}^{(r)} p_j \quad (3.6)$$

where $C_{ij}^{(r)}$ are the weighting coefficients and p_j are values of the function at each discrete point. n is total number of grid points. Weighting coefficient relation considering first-order derivative can be stated as below [Kurtaran, 2015b]

$$\bar{C}_{ij}^{(1)} = \frac{\Psi(\eta_i)}{(\eta_i - \eta_j)\Psi(\eta_j)} \quad (3.7)$$

where

$$\Psi(\eta_i) = \prod_{j=1}^n \eta_i - \eta_j \quad (i \neq j) \quad (3.8)$$

Higher order derivatives of weighting coefficient relations can be expressed as below

$$\bar{C}_{ij}^{(r)} = r \left[\bar{C}_{ii}^{(r-1)} \bar{C}_{ij}^{(r)} - \frac{\bar{C}_{ij}^{(r-1)}}{(\eta_i - \eta_j)} \right] \quad (i \neq j) \quad (3.9)$$

$$\bar{C}_{ii}^{(r)} = - \sum_{\substack{j=1 \\ i \neq j}}^n \bar{C}_{ij}^{(r)} \quad (3.10)$$

Partial derivatives of a two dimensional function $p(\eta, \vartheta)$ at the discrete node point for r -th and s -th orders in η and ϑ directions with GDQ method can be stated as

$$\left(\frac{\partial p^r(\eta, \vartheta)}{\partial \eta^r} \right)_{\eta_i, \vartheta_j} = \sum_{k=1}^{n_\eta} C_{ik}^{(r)} p_{kj} \quad (3.11)$$

$$\left(\frac{\partial p^s(\eta, \vartheta)}{\partial \vartheta^s} \right)_{\eta_i, \vartheta_j} = \sum_{m=1}^{n_g} C_{jm}^{(s)} p_{im} \quad (3.12)$$

$$\left(\frac{\partial p^{(r+s)}(\eta, \vartheta)}{\partial \eta^r \partial \vartheta^s} \right)_{\eta_i, \vartheta_j} = \frac{\partial^r}{\partial \eta^r} \left(\frac{\partial^s p}{\partial \vartheta^s} \right) = \sum_{k=1}^{n_\eta} C_{ik}^{(r)} \sum_{m=1}^{n_g} C_{jm}^{(s)} p_{km} \quad (3.13)$$

where n_η and n_g are total number of grid points in η and ϑ directions as shown in Figure 3.2, respectively. Partial derivatives at a discrete point (η_i, ϑ_i) for curvilinear coordinates can be expressed as

$$\left(\frac{\partial f}{\partial \phi} \right)_{ij} = \frac{1}{J_{ij}} \left[\left(\frac{\partial \theta}{\partial \vartheta} \right)_{ij} \left(\sum_{k=1}^{n_\phi} C_{ik}^{(1)} p_{kj} \right) - \left(\frac{\partial \theta}{\partial \eta} \right)_{ij} \left(\sum_{m=1}^{n_g} C_{jm}^{(1)} p_{im} \right) \right] \quad (3.14)$$

$$\left(\frac{\partial f}{\partial \theta} \right)_{ij} = \frac{1}{J_{ij}} \left[\left(\frac{\partial \phi}{\partial \eta} \right)_{ij} \left(\sum_{m=1}^{n_g} C_{jm}^{(1)} p_{im} \right) - \left(\frac{\partial \phi}{\partial \vartheta} \right)_{ij} \left(\sum_{k=1}^{n_\phi} C_{ik}^{(1)} p_{kj} \right) \right] \quad (3.15)$$

In this thesis, Gauss-Lobatto quadrature rule is employed in calculating spatial derivatives of field variables because of the rule creates grid points with non-uniform intervals. Gauss-Lobatto quadrature rule can locate grid points at the boundaries which allow the application of the boundary conditions easily. It provides a compromise between accuracy and ease of application of boundary conditions and therefore it is adopted in this study. Gauss-Lobatto points x_i can be calculated between the interval $[-1,1]$ by computing the roots of

$$\frac{d[P_{n-1}(x_i)]}{dx} = 0 \quad \text{for } i=2, \dots, (n-1) \quad (3.16)$$

where n is integration point in x direction. $P_n(x)$ denote Legendre polynomials and it can be expressed explicitly by Rodrigue's formula as below

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2 - 1)^n \right] \quad (3.17)$$

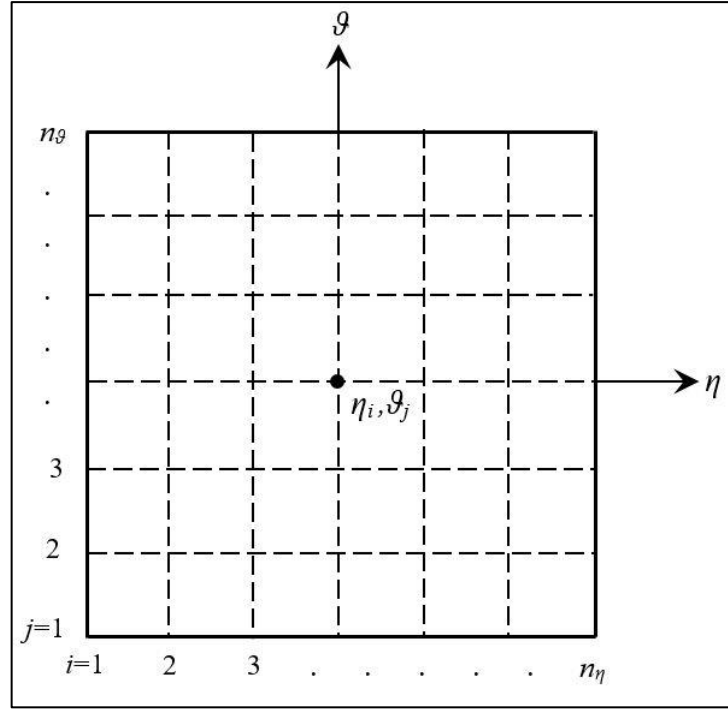


Figure 3.2: Grid points in 2-D natural coordinate system.

3.2. Solution of Equations of Motions

After discretization process of shells of revolution domain, matrix form of the equation of motion can be expressed as following

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{P} = \mathbf{F} \quad (3.18)$$

where $\mathbf{P}$, $\mathbf{F}$, $\ddot{\mathbf{U}}$ are internal, external force and acceleration vectors, respectively and $\mathbf{M}$ indicates mass matrix. Implicit Newmark constant average acceleration method is utilized to compute the time integration. In Newmark method, at $(n+1)$ -th time step (i.e. at time $(n+1)\Delta t$ or t_{n+1}), equation of motion is expressed as

$$\mathbf{M}\ddot{\mathbf{U}}_{n+1} + \mathbf{P}_{n+1} = \mathbf{F}_{n+1} \quad (3.19)$$

Acceleration and velocity expressions at t_{n+1} are given as

$$\ddot{\mathbf{U}}_{n+1} = C_0(\mathbf{U}_{n+1} - \mathbf{U}_n) - C_1\dot{\mathbf{U}}_n - \ddot{\mathbf{U}}_n \quad (3.20)$$

$$\dot{\mathbf{U}}_{n+1} = \dot{\mathbf{U}}_n + \Delta t \ddot{\mathbf{U}}_n + \frac{\Delta t}{2} (\dot{\mathbf{U}}_{n+1} - \ddot{\mathbf{U}}_n) \quad (3.21)$$

and substitution of Eq. (3.20) and Eq. (3.21) into Eq. (3.19) leads

$$C_0 \mathbf{M} \mathbf{U}_{n+1} + \mathbf{P}_{n+1} = \mathbf{F}_{n+1} + \mathbf{M} (C_0 \mathbf{U}_n + C_1 \dot{\mathbf{U}}_n + \ddot{\mathbf{U}}_n) \quad (3.22)$$

where $C_1 = 4 / \Delta t$ and $C_0 = 4 / \Delta t^2$. $\mathbf{U}_n$, $\dot{\mathbf{U}}_n$ and $\ddot{\mathbf{U}}_n$ indicates the displacements, velocities and accelerations at n -th time step. Internal force $\mathbf{P}_{n+1}$ term's being a function of unknown displacements $\mathbf{U}_{n+1}$ makes Eq. (3.22) nonlinear. To solve this nonlinear equation, an iterative method such as well-known Newton-Raphson method is utilized. In this method, Eq. (3.22) is rewritten in terms of error function or residual forces $\mathbf{R}_{n+1}$ as

$$\mathbf{R}_{n+1} = \mathbf{F}_{n+1} + \mathbf{M} (C_0 \mathbf{U}_n + C_1 \dot{\mathbf{U}}_n + \ddot{\mathbf{U}}_n) - C_0 \mathbf{M} \mathbf{U}_{n+1} - \mathbf{P}_{n+1} \quad (3.23)$$

Define $\mathbf{U}_{n+1}^i$ as an approximate solution for the i -th iteration and $\mathbf{R}_{n+1}^i$ of its error function. By equating linear Taylor series expansion of $\mathbf{R}_{n+1}^{i+1}$ to zero, one can obtain the updated solution $\mathbf{U}_{n+1}^{i+1}$ as follows

$$\mathbf{R}_{n+1}^{i+1} \approx \mathbf{R}_{n+1}^i + \mathbf{K}_{n+1}^i (\mathbf{U}_{n+1}^{i+1} - \mathbf{U}_{n+1}^i) = 0 \quad (3.24)$$

where $\mathbf{K}_{n+1}^i$ denotes tangent stiffness matrix. Eq. (3.24), can be expressed in incremental form as

$$\mathbf{K}_{n+1}^i \Delta \mathbf{U}_{n+1}^i = -\mathbf{R}_{n+1}^i \quad (3.25)$$

where $\Delta \mathbf{U}_{n+1}^i$ indicates displacement increment and given as

$$\Delta \mathbf{U}_{n+1}^i = \mathbf{U}_{n+1}^{i+1} - \mathbf{U}_{n+1}^i \quad (3.26)$$

Updated displacement for $(i+1)$ -th iteration can be expressed as following

$$\mathbf{U}_{n+1}^{i+1} = \mathbf{U}_{n+1}^i + \Delta \mathbf{U}_{n+1}^i \quad (3.27)$$

Considering initial accelerations $\ddot{\mathbf{U}}_0$ which is computed utilizing initial displacements $\mathbf{U}_0$ and velocities $\dot{\mathbf{U}}_0$ at time $t=0$. Newton-Raphson iteration procedure is started and repeated until $\mathbf{R}_{n+1}^{i+1}$ term converges enough to zero. In obtaining static solutions mass, inertia and velocity terms are neglected in Eqs. (3.18-3.27). The following convergence criteria is used in this study as below

$$\text{Conv} = \frac{R_{norm}}{1 + F_{norm}} \quad (3.28)$$

where F_{norm} and R_{norm} denote norm of external and residual forces, respectively. Convergence tolerance is used as $\text{Conv} \leq 0.005$ in this study.

3.3. Boundary Conditions

In this thesis, all edges simply-supported (SSSS) and all edges clamped (CCCC) boundary conditions of shells of revolution structures are considered. Generic edge is fully clamped (CCCC):

$$\begin{aligned} u_\phi^T = u_\theta^T = u_z^T = \psi_\phi^T = \psi_\theta^T = 0 & \quad \text{at} \quad \phi = \phi_1 \quad \text{or} \quad \phi = \phi_2 \quad \theta_1 \leq \theta \leq \theta_2 \\ u_\phi^T = u_\theta^T = u_z^T = \psi_\phi^T = \psi_\theta^T = 0 & \quad \text{at} \quad \theta = \theta_1 \quad \text{or} \quad \theta = \theta_2 \quad \phi_1 \leq \phi \leq \phi_2 \end{aligned}$$

Generic edge is fully simply-supported (SSSS):

$$\begin{aligned} u_\phi^T = u_\theta^T = u_z^T = 0 & \quad \text{at} \quad \phi = \phi_1 \quad \text{or} \quad \phi = \phi_2 \quad \theta_1 \leq \theta \leq \theta_2 \\ u_\phi^T = u_\theta^T = u_z^T = 0 & \quad \text{at} \quad \theta = \theta_1 \quad \text{or} \quad \theta = \theta_2 \quad \phi_1 \leq \phi \leq \phi_2 \end{aligned}$$

4. NUMERICAL EXAMPLES

In this thesis, static and transient dynamic response of several laminated composite parabolic/elliptic shells of revolution structure cases under uniformly distributed load are examined. Effects of thickness functions, thickness variation parameter (β), characteristic parameter of parabolic/elliptic shells of revolution (k), boundary conditions, elastic foundation parameters (k_w and k_G) as well as composite lamination scheme on nonlinear static and transient dynamic behavior are investigated in the solved examples. For this purpose, a computer program utilizing MATLAB commercial software for shells of revolution structure is built up to apply the GDQ method in the solution of equations of motion.

4.1. Validation and Convergence Analysis

In this study, displacement results are negative in tables and figures since uniformly distributed load is applied in negative z direction because of the selected coordinate system. Therefore absolute values should be considered in interpreting the magnitude of the results.

GDQ method is implemented to obtain nonlinear static and transient dynamic behavior of laminated composite parabolic/elliptic shells of revolution structures having variable thickness in this thesis. Static and transient dynamic responses of the shells of revolution structures are validated with commercial software Ansys Parametric Design Language (APDL), since there is no study related to nonlinear static and transient behavior of shells of revolution structures with variable thickness in the literature. All transient results in this thesis are given for center point of all the shells of revolution. Center point is defined by point of $\left(\frac{\phi_2 - \phi_1}{2}, \frac{\theta_2 - \theta_1}{2}\right)$.

In all examples, material properties are specified as $E_1=137.9$ GPa, $E_2=8.96$ GPa, $G_{23}=6.21$ GPa, $G_{12}=G_{13}=7.1$ GPa, $\rho=1450$ kg/m³, $\nu_{12}=0.3$ [Tornabene, 2011b] for laminated composite shells.

4.1.1. Grid Convergence Cases for Static and Transient Dynamic Analysis

To specify the sufficient grid points to be used in GDQ method, a grid convergence case is carried out before beginning elliptic shells of revolution studies for static analysis. Nondimensional static analysis results at the center of laminated composite elliptic shells of revolution structures having uniform thickness for fully clamped boundary condition are illustrated in Figure 4.1. Geometric parameters of EPRS are taken as $R_b=0$, $k=1$, $b=4$ m, $h_0=0.02$ m, and range of ϕ and θ are given as $[\phi_0, \phi_1] = [0, 11\pi/24]$, $[\phi_1, \phi_2] = [11\pi/24, 13\pi/24]$, $[\theta_1, \theta_2] = [0, \pi/12]$ for convergence case. Uniformly distributed load is $q_z = -2$ MPa and stacking sequence of laminated composite layers is taken $[0^\circ/45^\circ/90^\circ/45^\circ/0^\circ]$. As can be seen in Figure 4.1, static responses start to converge for grid numbers $n \geq 11$. In this thesis, in all static cases, 17×17 grids are used for GDQ method.

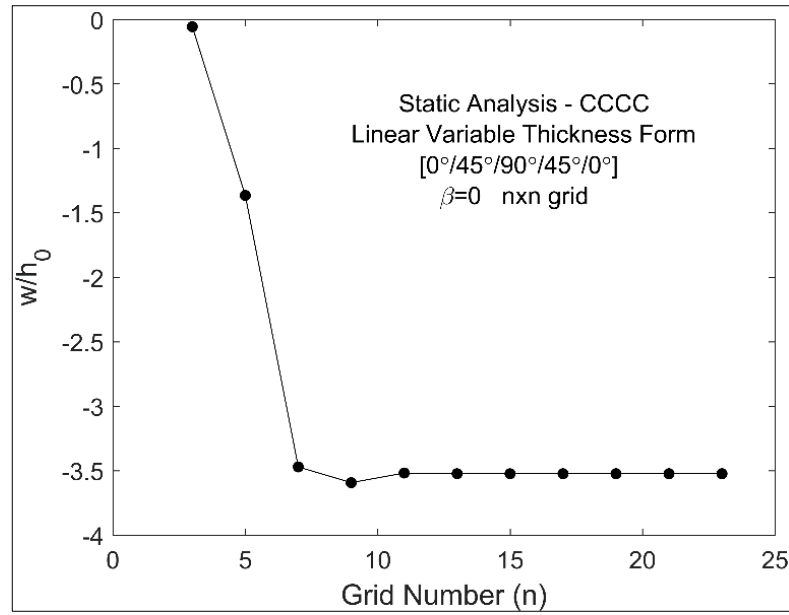


Figure 4.1: Grid convergence case studies for static analysis at the center of laminated composite EPRS regarding CCCC boundary condition.

Before starting transient dynamic cases, a grid convergence study is applied to determine appropriate grid numbers to be used in GDQ method. A laminated composite spherical panel is used for the grid convergence study. The GDQ code written for shells of revolution can be used easily for the spherical panel by adjusting the shells of revolution constants and radius. Transverse displacement (w/h_0) and stress

time histories at spherical panel center are shown in Figure 4.2 considering CCCC boundary condition. Geometric parameters of spherical panel are taken as $a=b=1$ m, $R=5$ m, and initial thickness value is taken as $h_0=0.03$ m. Uniformly distributed load (q_z) is applied as step load and $q_z=-3$ MPa and stacking sequence of laminated composite layers is taken $[0^\circ/90^\circ/45^\circ/90^\circ/0^\circ]$. Termination time of transient analysis is taken 5 ms and time step value Δt is 0.04 ms after convergence study.

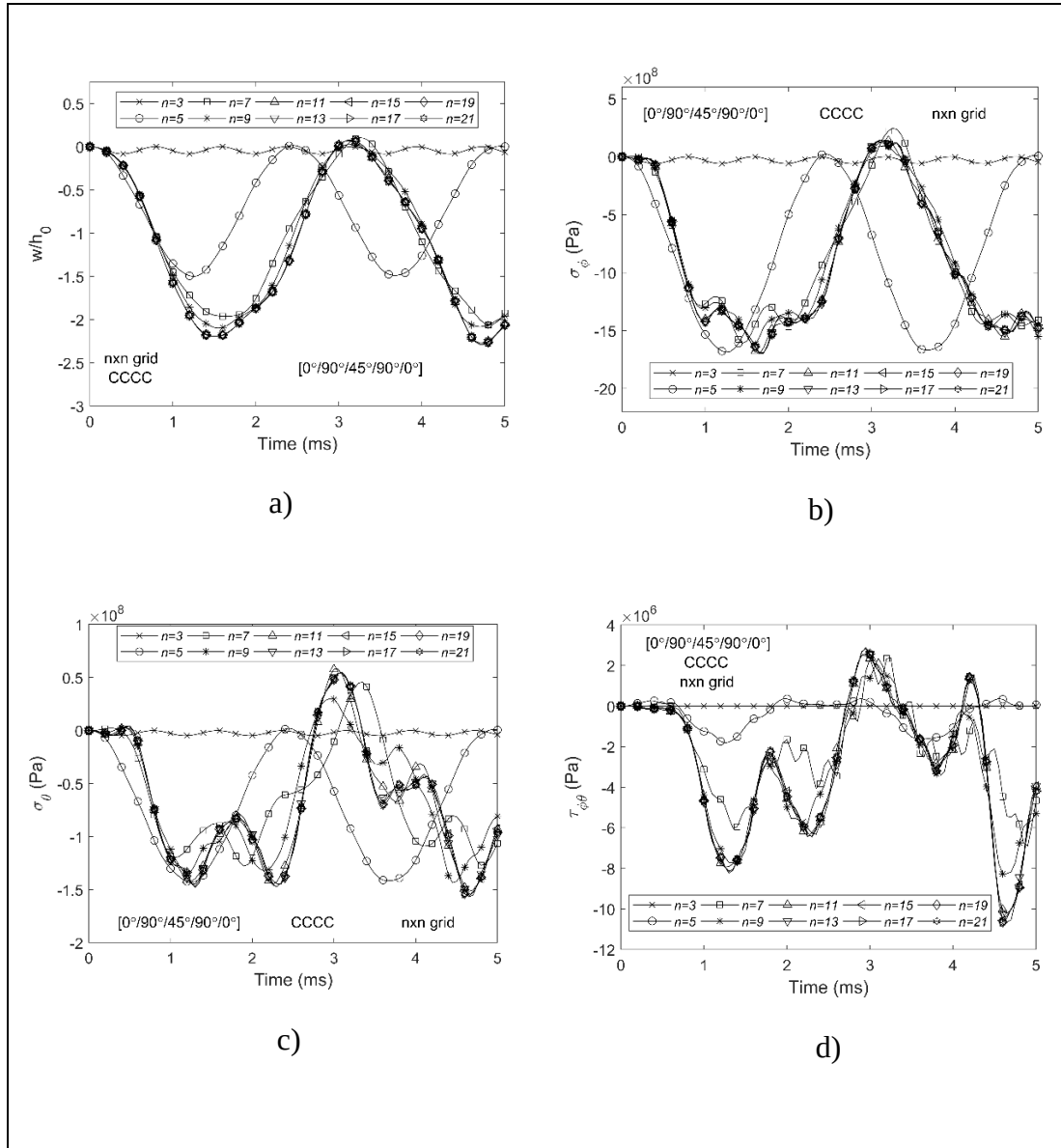


Figure 4.2: Grid convergence case studies of laminated composite spherical panel for CCCC boundary condition a) Non-dimensional transverse displacement time histories b) Stress time histories in ϕ direction, c) Stress time histories in θ direction, d) Stress time histories in $\phi\theta$ direction.

Figure 4.2 shows that, results begin to converge for grid numbers $n \geq 11$. In all examples in this study, 17×17 grids (17 grids in ϕ direction and 17 grids in θ direction) are found sufficient in converging results for transient dynamic results.

4.1.2. Validation Case for Static Analysis

For validation, nonlinear static responses of laminated composite EPRS having linear variable thickness form are investigated under the uniformly distributed load for both boundary conditions. Geometric parameters of EPRS used in this static validation example are identical with the previous example (Grid Convergence Case for static analysis). Uniformly distributed load is $q_z = -2$ MPa and stacking sequence of laminated composite layers is taken $[0^\circ/45^\circ/90^\circ/45^\circ/0^\circ]$.

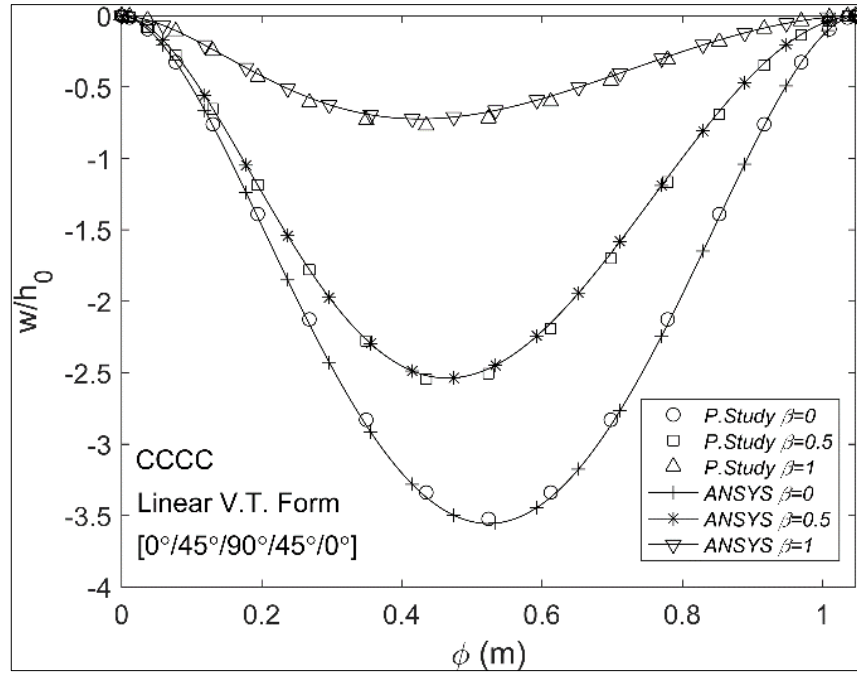


Figure 4.3: Non-dimensional static responses of fully clamped laminated composite EPRS having linear variable thickness form for different value of thickness variation parameter (β).

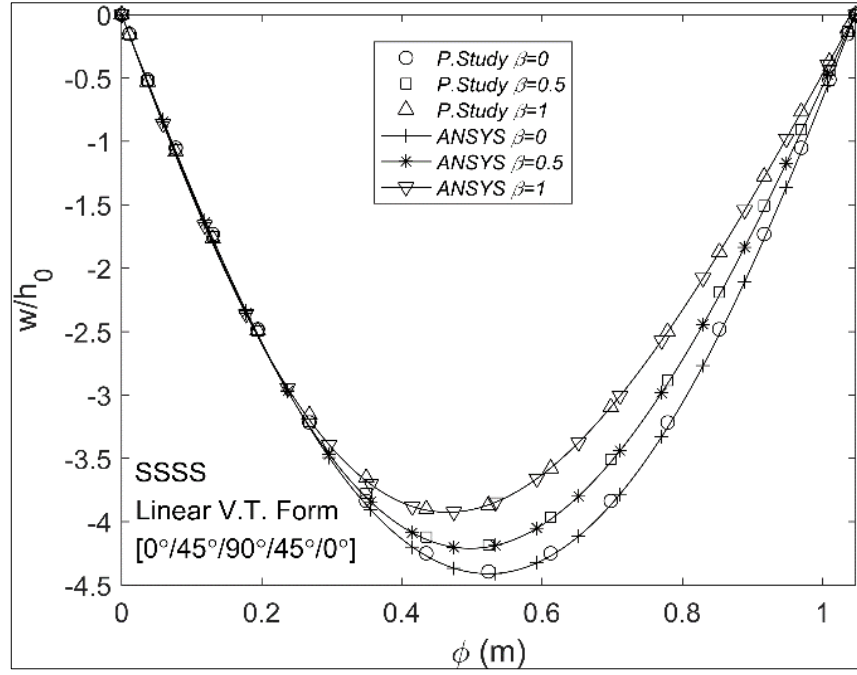


Figure 4.4: Non-dimensional static responses of fully simply-supported laminated composite EPRS having linear variable thickness form for different value of thickness variation parameter (β).

Table 4.1: Stress (in ϕ , θ , $\phi\theta$ directions and Von Mises Stress) responses of EPRS having uniform thickness for CCCC and SSSS boundary conditions ($[0^\circ/45^\circ/90^\circ/45^\circ/0^\circ]$).

Stress (Pa)	CCCC		SSSS	
	Present Study	Ansys	Present Study	Ansys
σ_ϕ	-0.1201e10	-0.1171e10	-0.7065e9	-0.6714e9
σ_θ	-0.65777e8	-0.64069e8	-0.4602e8	-0.4450e8
$\tau_{\phi\theta}$	-0.17948e8	-0.19495e8	-0.2595e8	-0.2787e8
σ_{vm}	0.1170e10	0.1148e10	0.6862e9	0.6523e9

As can be seen from the Figure 4.3 and Figure 4.4, non-linear static responses along the ϕ direction are in good agreement with the results obtained using ANSYS for different value of thickness variation parameter and boundary conditions. FE solutions (60x60 quadrilateral shell elements elements) are achieved utilizing commercial software Ansys Parametric Design Language (APDL). Also, non-linear stress responses at the panel center (stress in ϕ direction, stress in θ direction, stress in $\phi\theta$ direction and Von Mises Stress) are given for CCCC and SSSS boundary conditions

for $[0^\circ/45^\circ/90^\circ/45^\circ/0^\circ]$ stacking schemes, respectively in Table 4.1. Stress results are in good agreement with the ANSYS results.

Secondly, GDQ code is validated with linear static responses of laminated composite spherical panel for elastic foundation validation under uniformly distributed load considering simply-supported boundary condition in Table 4.2 which is available in the literature [Tornabene and Ceruti, 2013]. Geometric parameters of laminated composite spherical panel and applied load are taken as $R_\phi = R_\theta = 10$ m, $R_b = 0$ m, $\phi \in [\pi/3, 2\pi/3]$, $\theta \in [-\pi/6, \pi/6]$, $h = 0.09$ m, $q_z = -10$ kPa [Tornabene and Ceruti, 2013]. As can be seen from the comparisons in Table 4.2, GDQ method results are in good agreement with the results of [Tornabene and Ceruti, 2013].

Table 4.2. Linear static deflection of composite spherical panel resting on elastic foundation at panel center regarding SSSS boundary condition.

<i>Lamination Scheme</i>	$k_w = 0$			$k_w = 1.5e7 \text{ (N/m}^3\text{)}$			$k_w = 7.5e7 \text{ (N/m}^3\text{)}$		
	Present	Tornabene-Ceruti		Present	Tornabene-Ceruti		Present	Tornabene-Ceruti	
	Study (m)	GDQ (m)	FEM (m)	Study (m)	GDQ (m)	FEM (m)	Study (m)	GDQ (m)	FEM (m)
(30)	-4.579e-4	-4.589e-4	-4.418e-4	-2.492e-4	-2.503e-4	-2.444e-4	-9.796e-5	-9.899e-5	-9.758e-5
(0/90)	-5.860e-4	-5.911e-4	-5.862e-4	-3.214e-4	-3.257e-4	-3.229e-4	-1.110e-4	-1.129e-4	-1.119e-4
(30/45)	-4.475e-4	-4.476e-4	-4.276e-4	-2.303e-4	-2.303e-4	-2.237e-4	-9.066e-5	-9.102e-5	-8.940e-5
(0/90/0)	-5.870e-4	-5.922e-4	-5.875e-4	-3.220e-4	-3.263e-4	-3.235e-4	-1.111e-4	-1.131e-4	-1.121e-4
(30/45/70)	-4.405e-4	-4.413e-4	-4.253e-4	-2.410e-4	-2.420e-4	-2.361e-4	-9.431e-5	-9.507e-5	-9.351e-5
(0/90/90/0)	-5.858e-4	-5.909e-4	-5.862e-4	-3.221e-4	-3.264e-4	-3.236e-4	-1.111e-4	-1.130e-4	-1.120e-4
(30/45/70/90)	-3.846e-4	-3.859e-4	-3.764e-4	-2.348e-4	-2.364e-4	-2.319e-4	-9.573e-5	-9.686e-5	-9.553e-5

4.1.3. Validation Case for Transient Dynamic Analysis

A laminated composite spherical panel with variable thickness is used for validation. Nonlinear transient behavior of spherical panel with variable thickness considering linear, arch, sine and cosine form under distributed load for CCCC and SSSS boundary conditions is investigated in this validation example. Applied load, geometric and material properties of laminated composite spherical panel are specified as below

- Lengths of spherical panel are $a=b=1$ m,
- Radius of spherical panel is $R=5$ m,
- Initial thickness of spherical panel is $h_0=0.03$ m,
- Stacking sequence of composite layers is $[0^\circ/90^\circ/45^\circ/90^\circ/0^\circ]$
- Uniformly distributed load is $q_z = -3$ MPa and $q_z = -9$ MPa varies with step function in time domain and time step value Δt is 0.04 ms after convergence study.

In Figure 4.5, transverse displacement (w/h_0) and stress (stress in ϕ direction, stress in θ direction and stress in $\phi\theta$ direction) time histories at spherical panel center are investigated for SSSS boundary conditions considering uniform thickness. Good agreement is obtained between the results of GDQ method and finite element solutions (ANSYS).

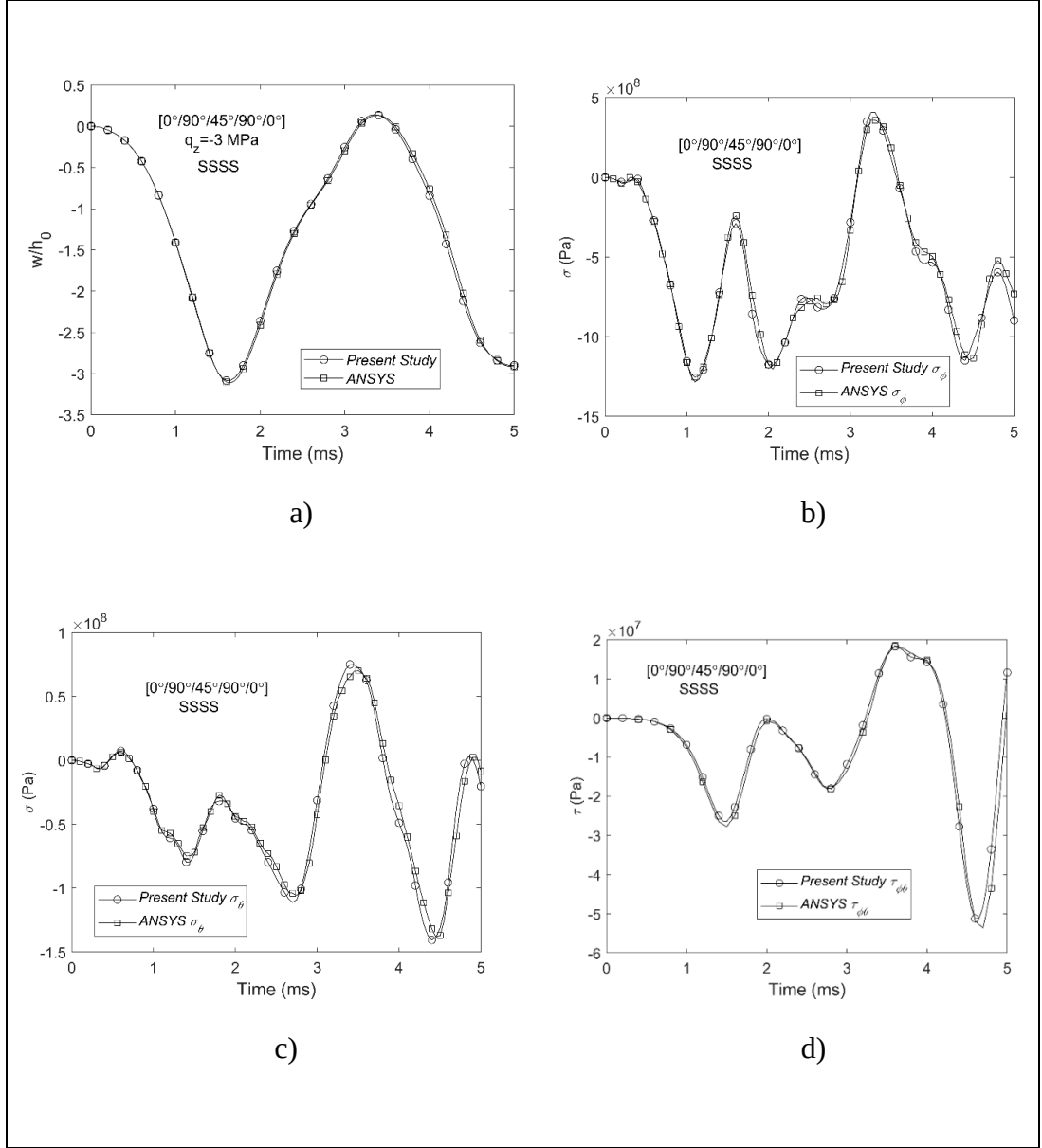


Figure 4.5: Transverse displacement and stress time histories at spherical panel center for SSSS boundary condition ($q_z = -3$ MPa) a) Non-dimensional transverse displacement time histories in ϕ direction b) Stress time histories in ϕ direction, c) Stress time histories in θ direction, d) Stress time histories in $\phi\theta$ direction.

Also, in Figures 4.6, non-dimensional transverse displacement histories of spherical panel with variable thickness considering linear, arch, sine and cosine forms at panel center are shown for different thickness variation parameter value (β) regarding CCCC boundary condition, respectively. A finite element code is developed using Ansys Parametric Design Language (APDL) to perform the finite element analysis of the structure. Four-noded quadrilateral shell elements (60x60 elements) are utilized. As seen in Figures 4.6, nonlinear GDQ results for spherical panel are in good agreement with the results obtained using ANSYS.

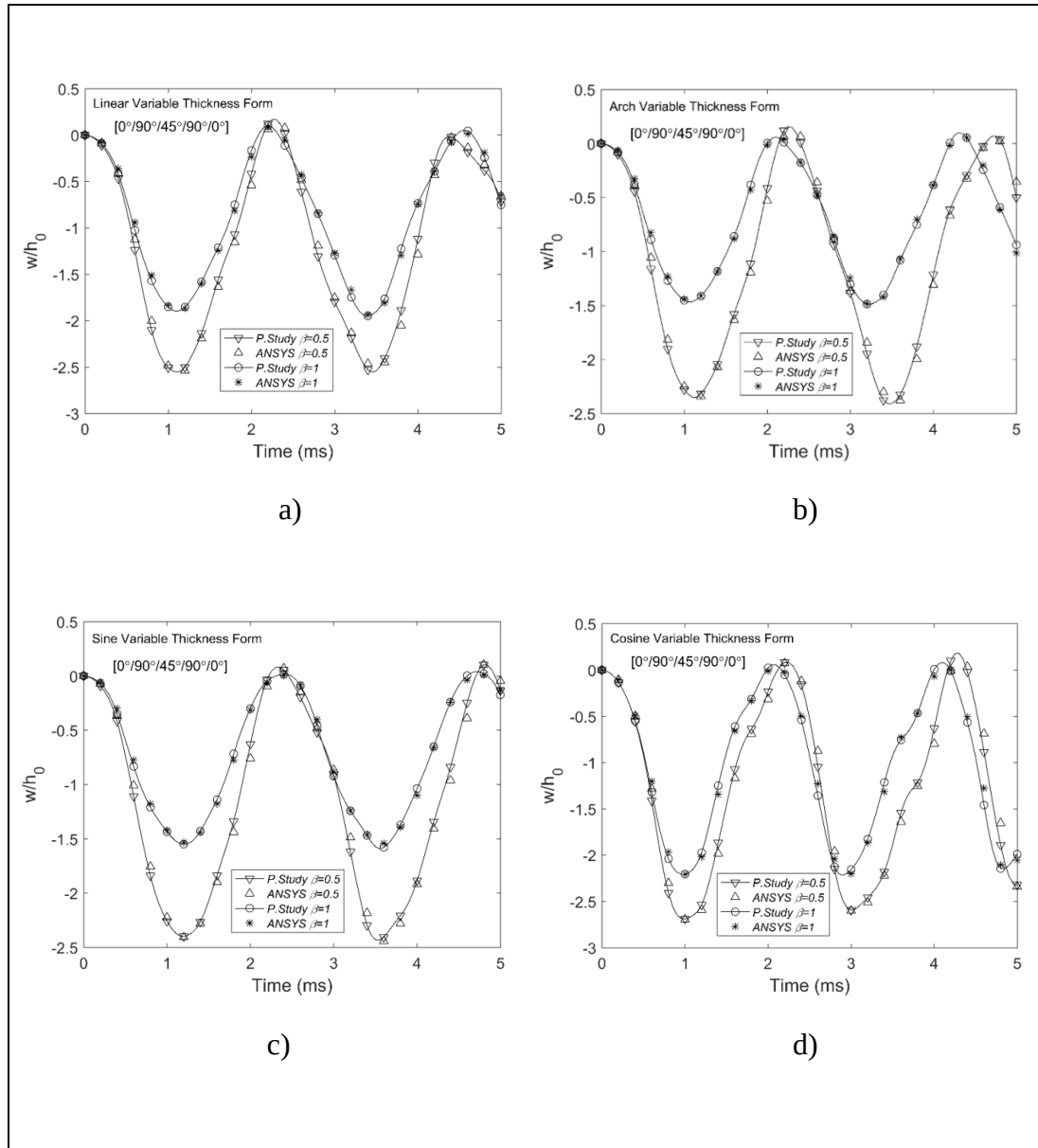


Figure 4.6: Non-dimensional transverse displacement histories at spherical panel center for variable thickness forms regarding CCCC boundary condition ($q_z=-9$ MPa) a) Linear Form, b) Arch Form, c) Sine Form and d) Cosine Form.

4.2. Nonlinear Static Analysis of Shells of Revolution Structures

In this chapter, some results and discussions about the nonlinear static analysis of laminated composite EPRS under the distributed load are presented in detail. Effects of thickness variation parameter, thickness functions, Winkler-Pasternak elastic foundation parameters, composite lamination scheme, characteristic parameter and boundary conditions on nonlinear static response are examined in the solved

examples. Geometric parameters of EPRS are taken as $R_b=0$, $b=4$ m and $h_0=0.02$ m in the examples.

4.2.1. Effect of Characteristic Parameter on Static Response

As a first step, the effect of the characteristic parameter on large displacement static response is investigated for clamped boundary condition. Three different characteristic parameter values (such as $k=0.75$, 1.00 and 1.25) of laminated composite EPRS having uniform and variable thickness forms are considered. Range of ϕ and θ are given as $[\phi_0, \phi_1] = [0, 5\pi/12]$, $[\phi_1, \phi_2] = [5\pi/12, \pi/2]$, $[\theta_1, \theta_2] = [0, \pi/12]$. Thickness variation parameter (β) is given 0, 0.25 and 1. Linear, arch, sine and cosine forms as seen in Figure 2.5 are used for thickness variation. Stacking sequence of laminated composite layers is taken $[0^\circ/45^\circ/90^\circ/45^\circ/0^\circ]$ and uniformly distributed load is $q_z=-2$ MPa.

Nonlinear static analysis results at the center of EPRS having uniform and variable thickness forms considering clamped boundary condition are shown in Table 4.3. Displacement values of EPRS on static responses in Table 4.3 are observed to be minimum for all variable thickness forms when the characteristic parameter is $k=1.25$. Maximum displacement values obtained in static analysis for EPRS occur when the characteristic parameter is $k=1$ considering $\beta=0$ and $\beta=0.25$. In Table 4.3, thickness variation parameter is taken as $\beta=0.25$ and it is seen that for the case of $k=0.75$ and 1 , the results are close to each other and much higher than those for the case of $k=1.25$.

Table 4.3: Nonlinear static analysis results at the center of EPRS having uniform and variable thickness considering CCCC boundary condition.

k	Uniform	Linear Variable		Sine Variable		Cosine Variable		Arch Variable	
	Thickness	Thickness Form (m)		Thickness Form (m)		Thickness Form (m)		Thickness Form (m)	
	$\beta = 0$	$\beta = 0.25$	$\beta = 1$	$\beta = 0.25$	$\beta = 1$	$\beta = 0.25$	$\beta = 1$	$\beta = 0.25$	$\beta = 1$
0.75	-0.0625	-0.0581	-0.0301	-0.0551	-0.0102	-0.0611	-0.0526	-0.0562	-0.0125
1.00	-0.0701	-0.0621	-0.0144	-0.0620	-0.0104	-0.0624	-0.0238	-0.0594	-0.0099
1.25	-0.0419	-0.0167	-0.0070	-0.0154	-0.0063	-0.0174	-0.0082	-0.0143	-0.0054

4.2.2. Effect of Thickness Form and Stacking Sequence on Static Response

Here, the effect of the thickness variation parameter and form on large displacement static response is examined for the clamped and the simply-supported boundary conditions. Geometric parameters, material properties, thickness forms and stacking sequence of laminated composite layers used in this example are identical with the previous example (Section 4.2.1). Uniformly distributed load is $q_z = -1.5$ MPa. To investigate the effect of thickness form, β 's are considered as 0, 0.25, 0.5 and 1.25.

Figure 4.7 and Figure 4.8 illustrate variation of displacement along ϕ direction for clamped and simply-supported boundary conditions at $\theta = \pi/24$. Figure 4.7 and Figure 4.8 demonstrate that as the thickness variation parameter increases, displacement values decrease for all thickness forms and for both boundary conditions. Also, thickness form has more influential effect on the displacement results for the clamped boundary condition compared to the simply-supported boundary condition. As seen in Figure 4.7 and Figure 4.8, sine and cosine thickness forms yield symmetrical displacement forms unlike the linear and arch forms. The β parameter does not have an important effect on displacements for cosine form considering the simply-supported boundary condition.

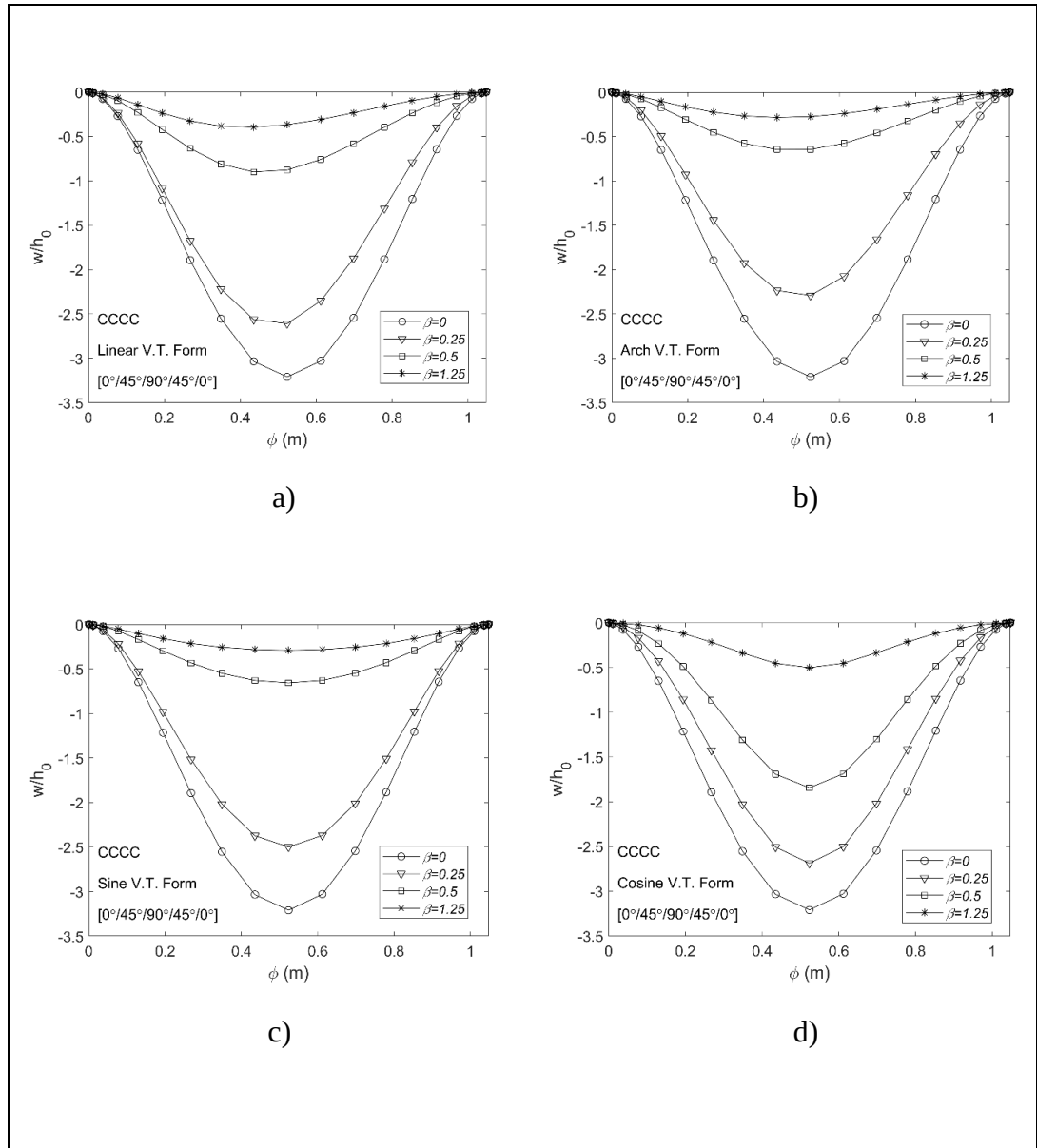


Figure 4.7: Non-dimensional static responses of clamped EPRS with variable thickness considering different thickness variation parameters along ϕ direction a) Linear Form, b) Arch Form, c) Sine Form and d) Cosine Form.

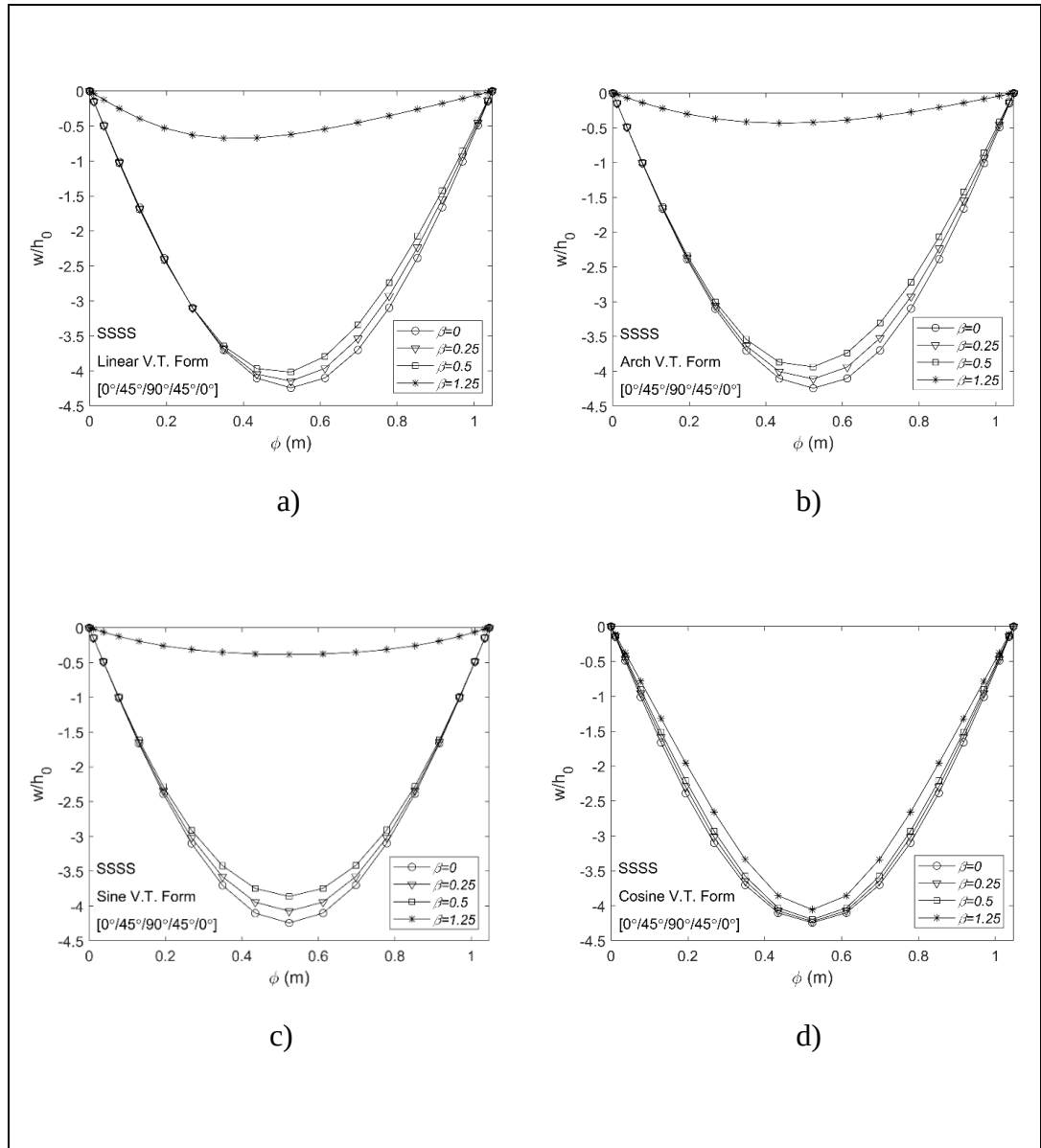


Figure 4.8: Non-dimensional static responses of simply-supported EPRS with variable thickness considering different thickness variation parameters along ϕ direction a) Linear Form, b) Arch Form, c) Sine Form and d) Cosine Form.

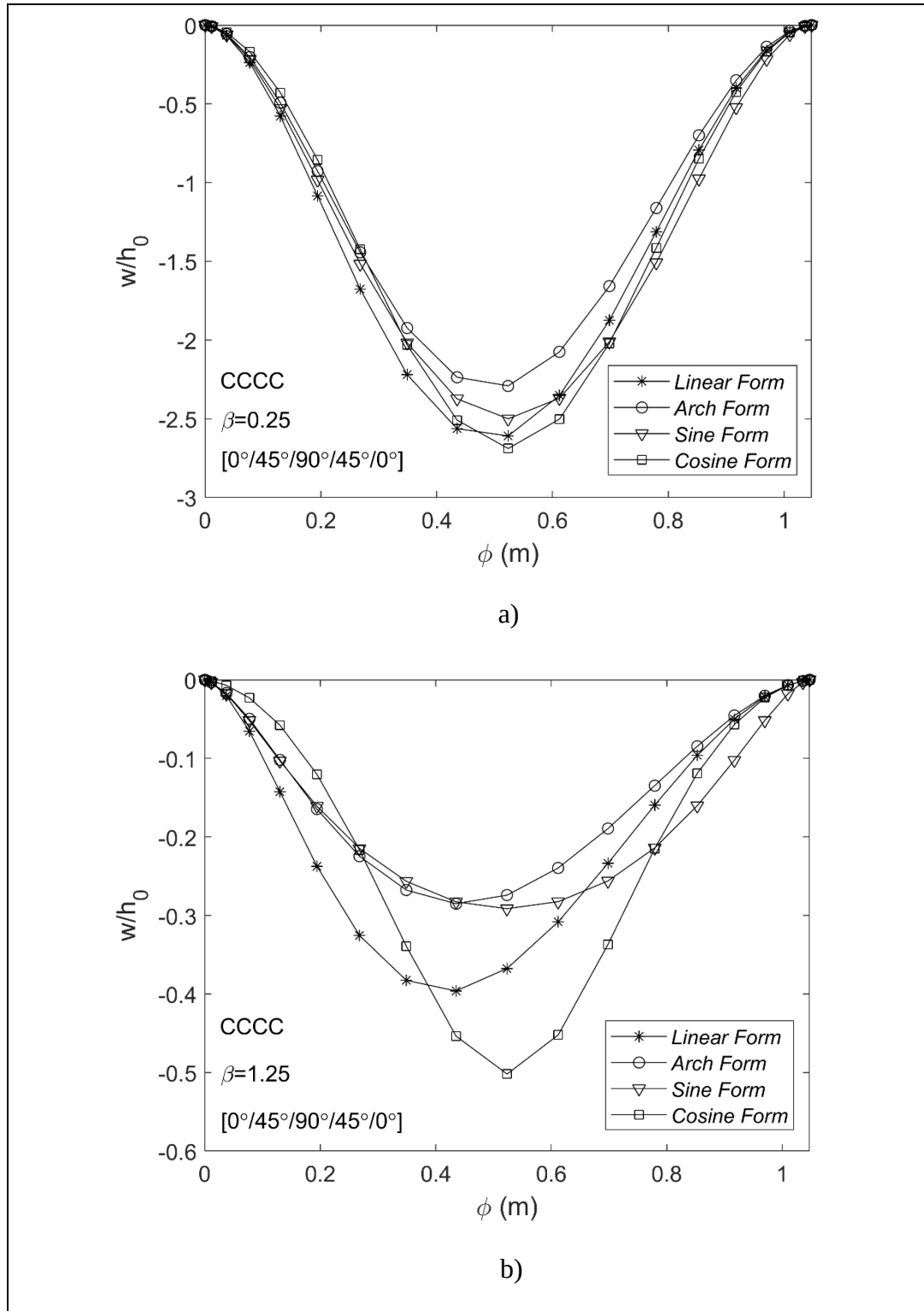


Figure 4.9: Non-dimensional static responses of clamped EPRS considering different thickness forms along ϕ direction a) $\beta=0.25$, b) $\beta=1.25$.

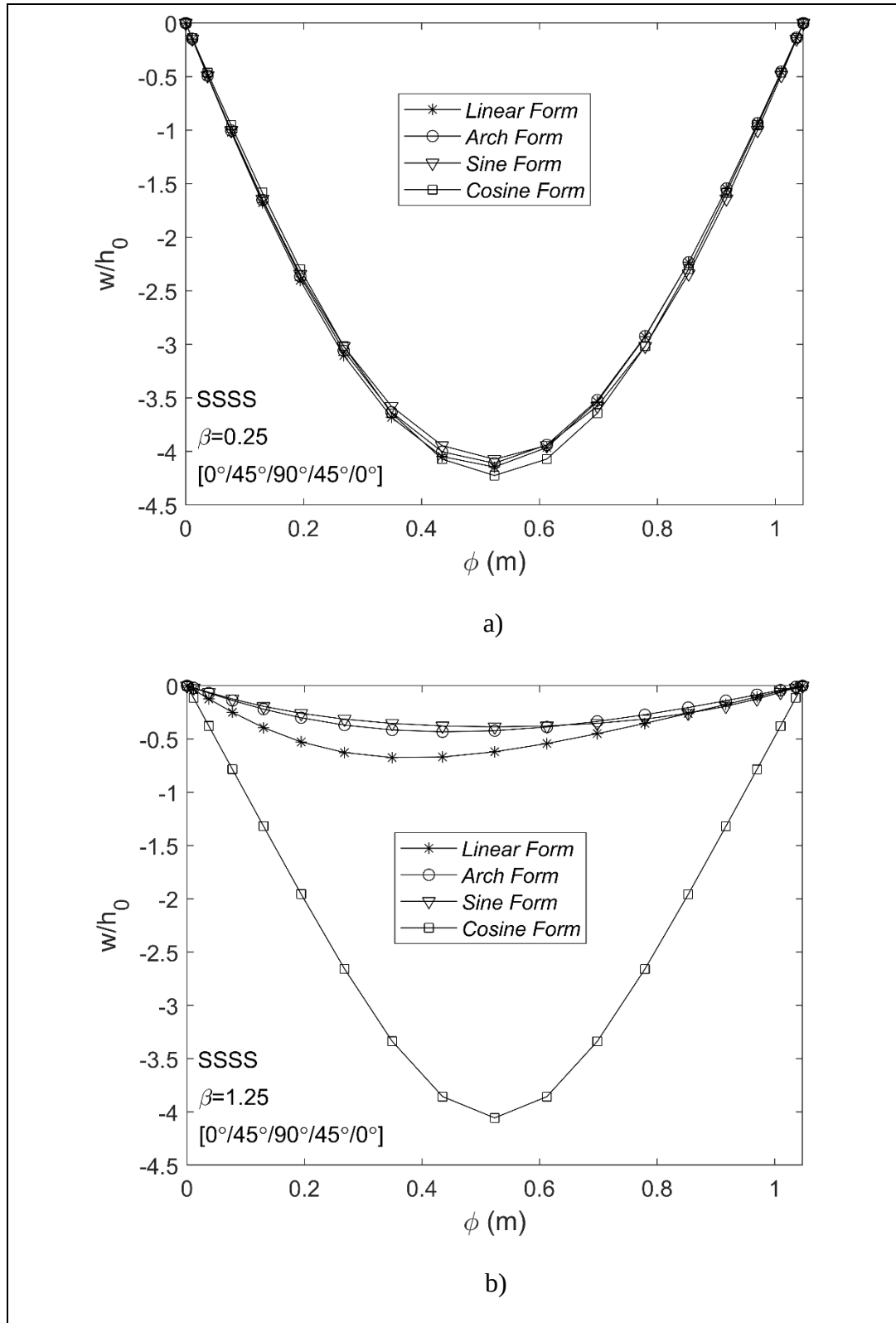


Figure 4.10: Non-dimensional static responses of simply-supported EPRS considering different thickness forms along ϕ direction a) $\beta=0.25$, b) $\beta=1.25$.

In Figures 4.9 and 4.10, static responses of EPRS with different thickness forms are compared for $\beta=0.25$ and $\beta=1.25$. When the thickness variation parameter is $\beta=0.25$ as seen in Figures 4.9-a and 4.10-a, the β parameter has a similar effect on displacement results for the considered thickness forms and boundary conditions. Conversely, the static responses differ for the four different thickness forms when the thickness variation parameter is $\beta=1.25$.

Displacement values of EPRS having arch variable thickness form in Figure 4.9 on static responses are observed to be minimum for CCCC boundary condition. Conversely, static responses for sine and arch variable thickness form are observed to be minimum considering SSSS boundary condition in Figure 4.10. In addition, the largest displacement values occur in cosine form for $\beta=1.25$. As can be seen from Figures 4.7-4.10, the displacement values of the clamped boundary condition are always lower than the simply-supported boundary condition. As can be seen from the Figure 4.9 and Figure 4.10, sine and cosine thickness forms yield symmetrical displacement forms unlike the linear and arch forms. Sine and cosine thickness forms are symmetric but linear and arch thickness forms are not symmetric in longitudinal direction (in ϕ direction). Therefore, the responses are not symmetric for linear and arch thickness forms.

Finally, the effect of the changing of composite angles on thickness forms is examined in Figure 4.11. Stacking sequence of angle-ply laminated composite layers is taken $[\alpha^\circ / -\alpha^\circ / \alpha^\circ / -\alpha^\circ / \alpha^\circ]$. α are considered as 30° , 60° and 90° and uniformly distributed load is $q_z = -2$ MPa. Figure 4.11 shows nonlinear static response of EPRS having variable thickness for four different stacking sequence considering the clamped boundary condition. As shown in Figure 4.11, different static responses have been obtained for four thickness forms when a different stacking sequence is used, for $\beta=0.25$. In addition, lowest displacement values occur when α is taken as 90° considering all variable thickness forms.

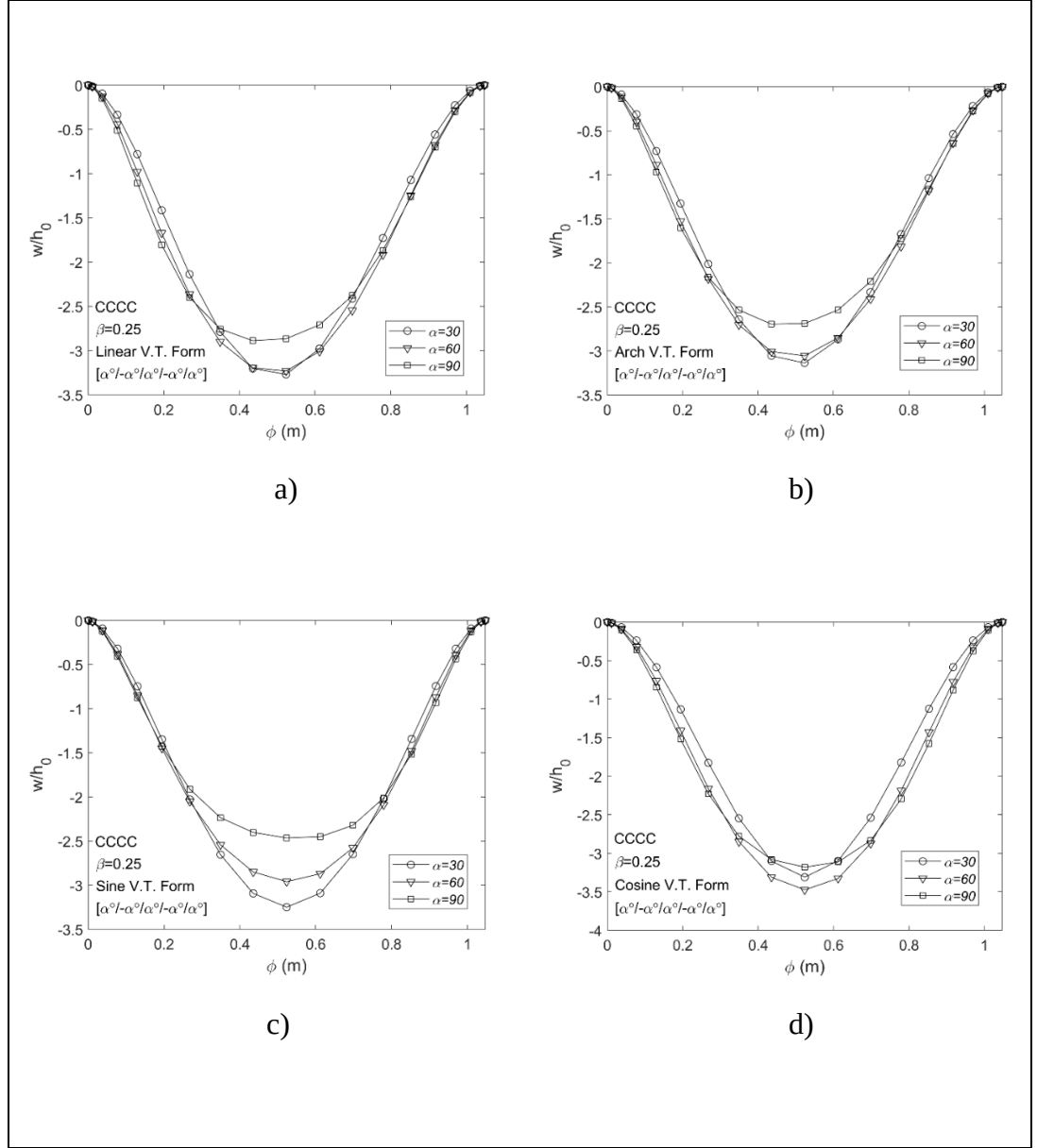


Figure 4.11: Non-dimensional static behavior of clamped angle-ply composite EPRS with variable thickness along ϕ direction ($q_z = -2$ MPa) a) Linear Form, b) Arch Form, c) Sine Form and d) Cosine Form.

4.2.3. Effect of Winkler-Pasternak Elastic Foundation on Static Response

Finally, the effect of Winkler-Pasternak elastic foundation on large displacement static response of laminated composite EPRS having linear variable thickness form is examined along ϕ direction at $\theta = \pi/24$. β is taken as 0.25 and 1. Geometric parameters and material properties used in this example are identical with the previous example (Section 4.2.1). Stacking sequence of laminated composite

layers is taken $[0^\circ/45^\circ/90^\circ/45^\circ/0^\circ]$ and uniformly distributed load is $q_z = -2$ MPa. Winkler modulus of $k_w=0$, $k_w=E_1/10^4$, $k_w=E_1/10^3$ and $k_w=E_1/10^2$ and Pasternak shear modulus of $k_G= G_{12}/10^4$, $k_G= G_{12}/10^3$ and $k_G= G_{12}/10^2$ are considered as elastic foundation parameters.

Figures 4.12 and 4.13 show non-dimensional static responses of EPRS having linear variable thickness resting on Winkler elastic foundation ($k_G= 0$) for clamped and simply-supported boundary conditions. As can be seen from Figures 4.12 and 4.13, displacement values decrease with increasing values of Winkler elastic foundation parameter for $\beta=0.25$ and $\beta=1$. Increasing Winkler modulus generally has important effect on displacement results for considered boundary conditions. However, this effect is more prominent for Winkler modulus higher than $E_1/10^4$.

Figures 4.14 and 4.15 demonstrate non-dimensional static responses of EPRS having linear variable thickness resting on Winkler-Pasternak elastic foundation ($k_w= E_1/10^4$) for clamped and simply-supported boundary conditions. As seen in Figures 4.14 and 4.15, displacement values decrease with increasing values of Pasternak modulus for $\beta=0.25$ and $\beta=1$. Pasternak modulus higher than $G_{12}/10^4$ has amplitude-reducing effect for both boundary conditions. Effect of Pasternak modulus on displacement is gradual considering clamped boundary condition.

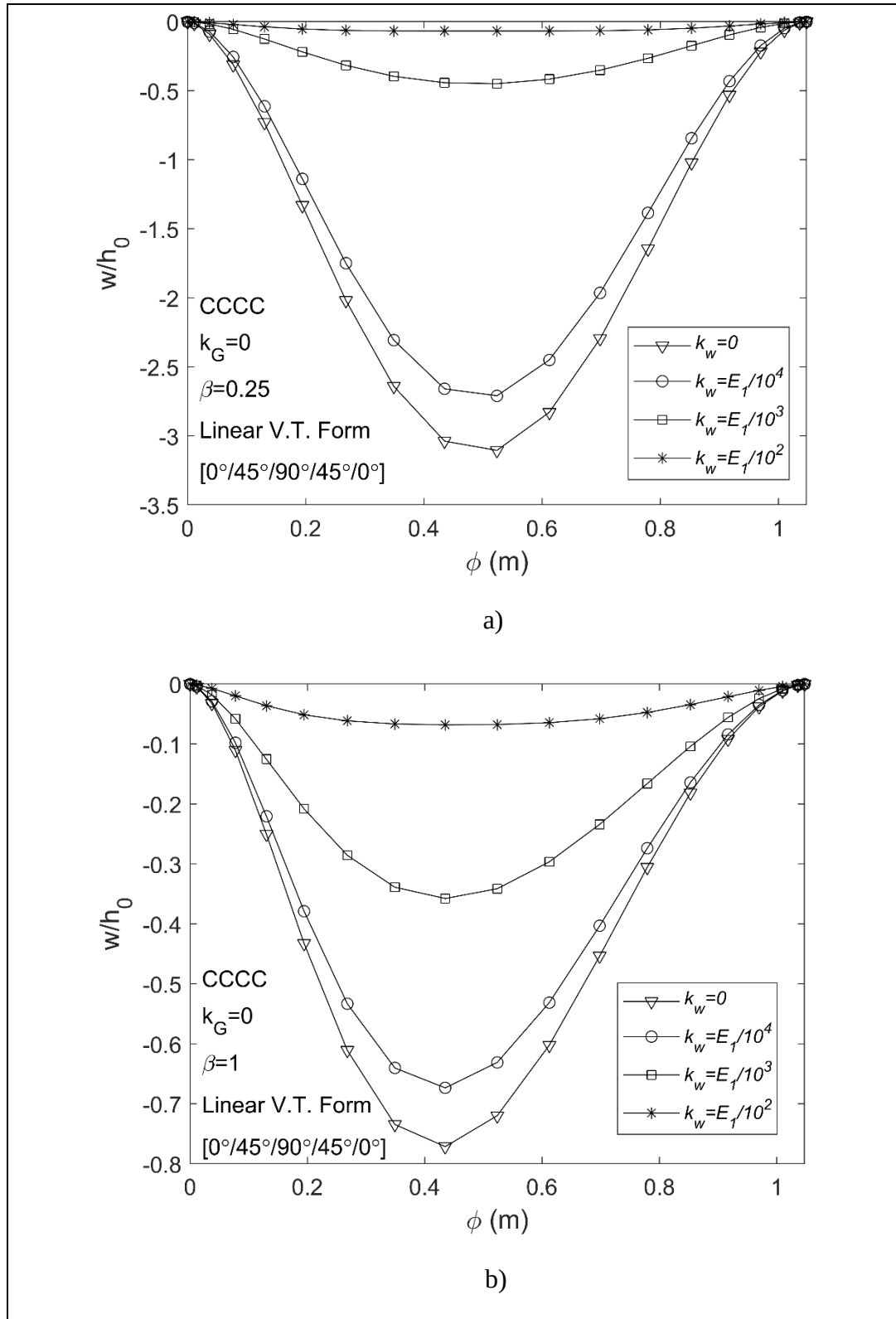


Figure 4.12: Non-dimensional static responses of EPRS having linear variable thickness resting on Winkler elastic foundation considering CCCC boundary condition a) $\beta=0.25$, b) $\beta=1$.

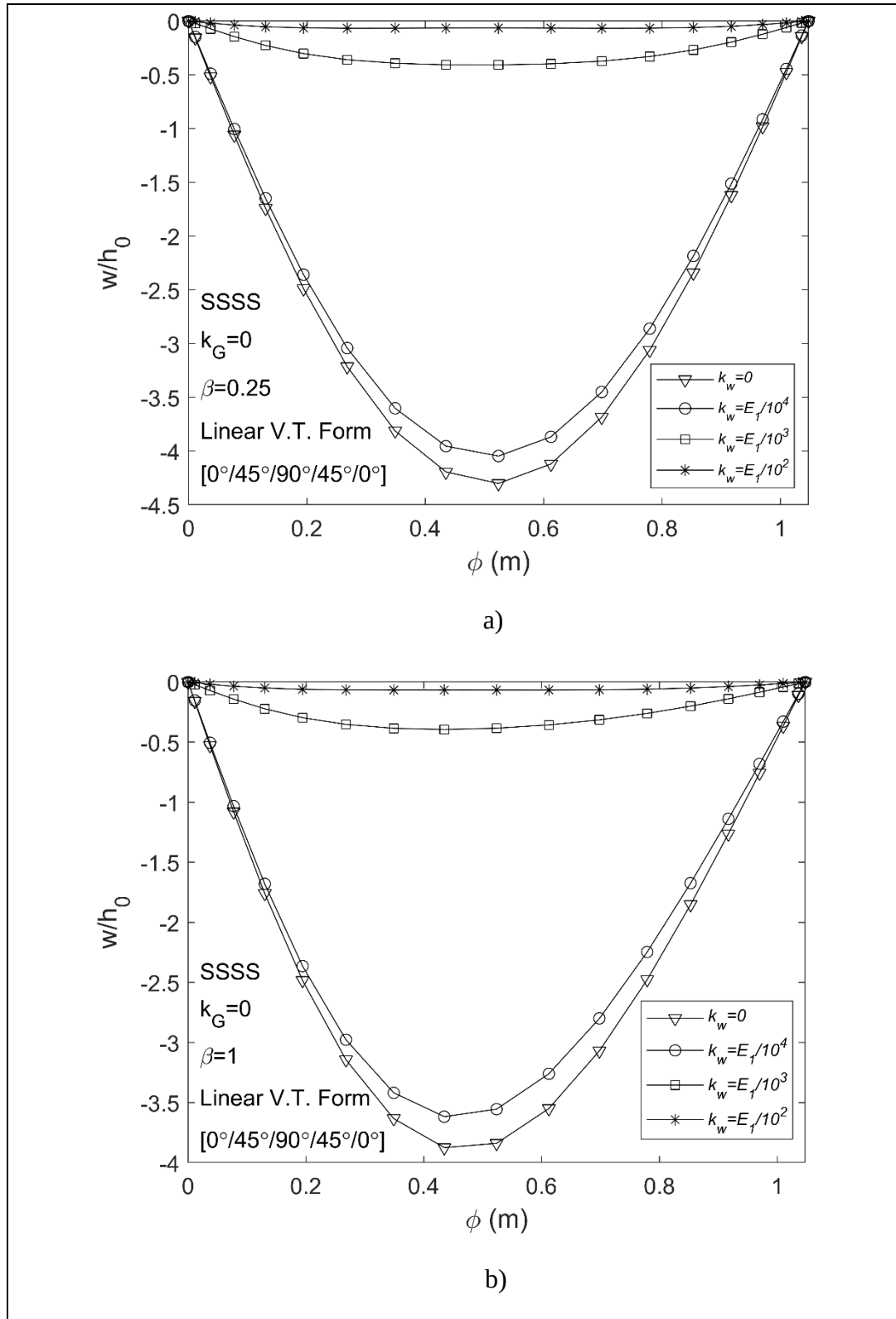


Figure 4.13: Non-dimensional static responses of EPRS having linear variable thickness resting on Winkler elastic foundation considering SSSS boundary condition a) $\beta=0.25$, b) $\beta=1$.

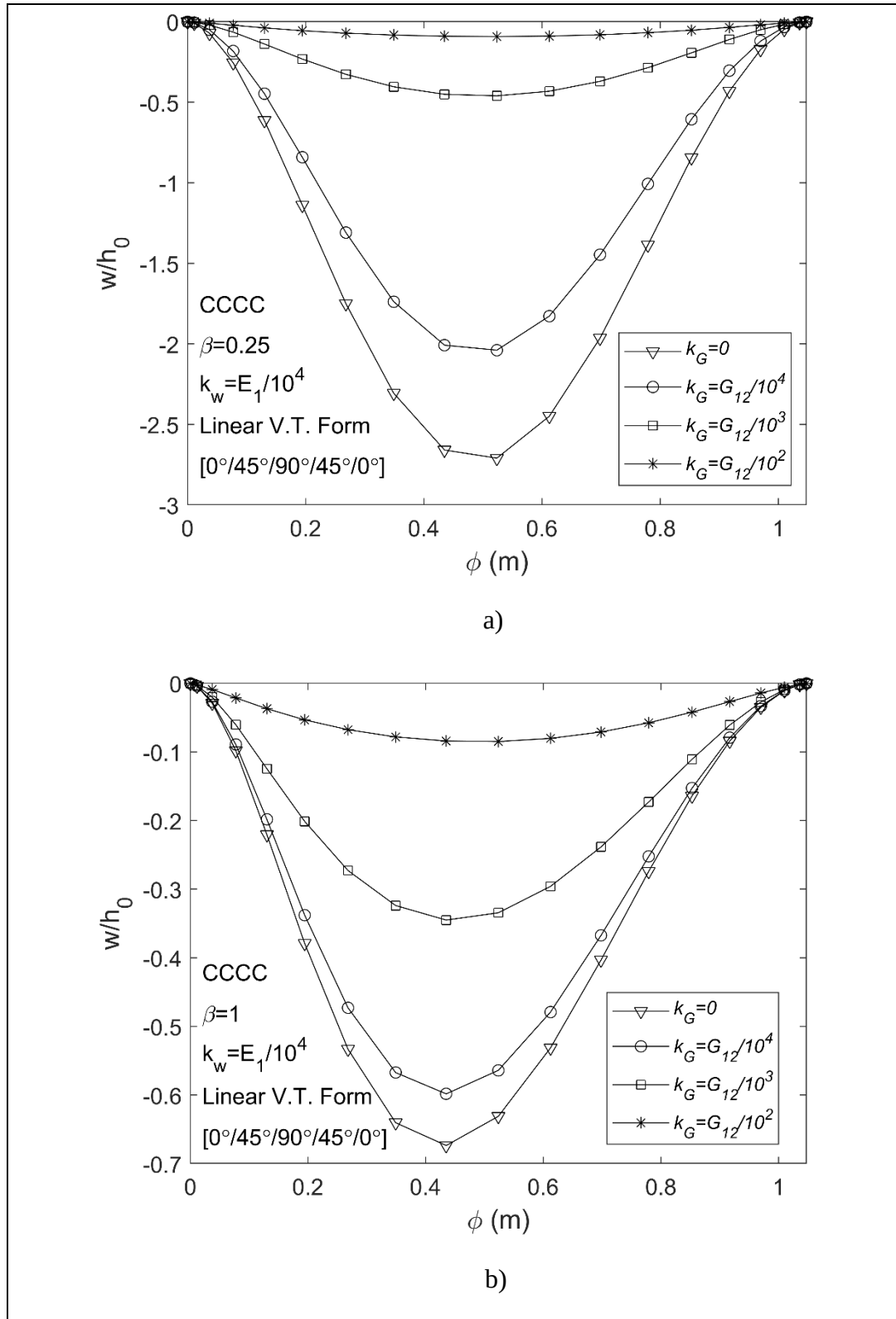


Figure 4.14: Non-dimensional static responses of EPRS having linear variable thickness resting on Winkler-Pasternak elastic foundation considering CCCC boundary condition a) $\beta=0.25$, b) $\beta=1$.

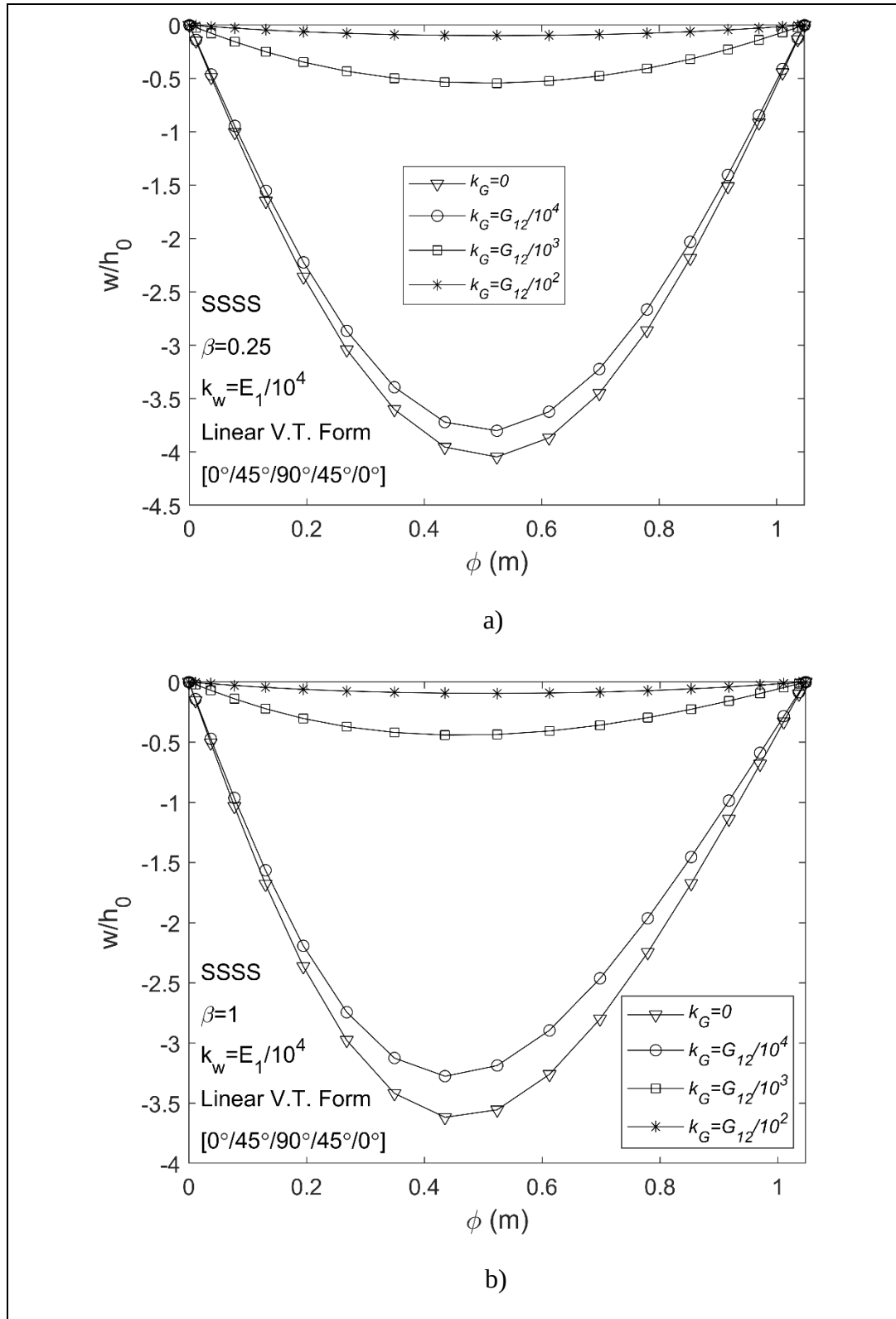


Figure 4.15: Non-dimensional static responses of EPRS having linear variable thickness resting on Winkler-Pasternak elastic foundation considering SSSS boundary condition a) $\beta=0.25$, b) $\beta=1$.

4.3. Nonlinear Transient Dynamic Analysis of Shells of Revolution Structures

In this section, transient behavior of several laminated composite PPRS cases under uniformly distributed load are examined. Effects of thickness functions, thickness variation parameter (β), characteristic parameter of PPRS (k), boundary conditions, elastic foundation parameters (k_w and k_G) as well as composite lamination scheme on nonlinear transient dynamic behavior are investigated in the solved examples. In all examples in this section, following values of geometric parameters (Figure 2.1) are utilized: $z_0=1$ m, $z_1=2$ m, $d=0$, $\theta_0=0$, $\theta=\pi/6$ rad, $R_b=0$ and initial thickness value is taken as $h_0=0.025$ m. Uniformly distributed load (q_z) is applied as step load and $q_z=-4$ MPa. Termination time of transient analysis is taken 5 ms and time step value Δt is 0.04 ms after convergence study.

4.3.1. Effect of Characteristic Parameter on Transient Dynamic Behaviour

Nonlinear transient dynamic response of laminated composite PPRS with variable thickness having sine form for different characteristic parameters of parabola is examined. Effects of characteristic parameter of PPRS on nonlinear transient behavior are investigated for uniform and non-uniform variable thickness considering CCCC and SSSS boundary conditions. Values of characteristic parameter of PPRS are taken as $k=1$, $k=2$ and $k=3$. Thickness variation parameters are considered $\beta=0$ for uniform thickness and $\beta=1$ for variable thickness. Thickness function having sine form in Eqs. (2.12-2.15) is used to express the variable thickness. Stacking schemes for composite layers is $[0^\circ/30^\circ/90^\circ/30^\circ/0^\circ]$.

Figure 4.16 a-b illustrates non-dimensional transverse displacement time histories at the center of PPRS with uniform thickness for CCCC and SSSS boundary conditions, respectively. In addition, displacement time histories for variable thickness of sine form are shown in Figure 4.17-a for CCCC boundary condition as well as in Figure 4.17-b for SSSS boundary condition. Figure 4.16 and Figure 4.17 show that as characteristic parameter k value increases, displacement values increase for both boundary conditions since the surface area becomes larger with higher values of k . In Figure 4.17, thickness variation parameter is taken as $\beta=1$ and it is seen that for the case of $k=2$ and 3, transient responses are close to each other and much higher

than those for the case of $k=1$ regarding SSSS boundary condition. However, displacement values for the case of $k=3$ are much higher than those for $k=2$ regarding CCCC boundary condition.

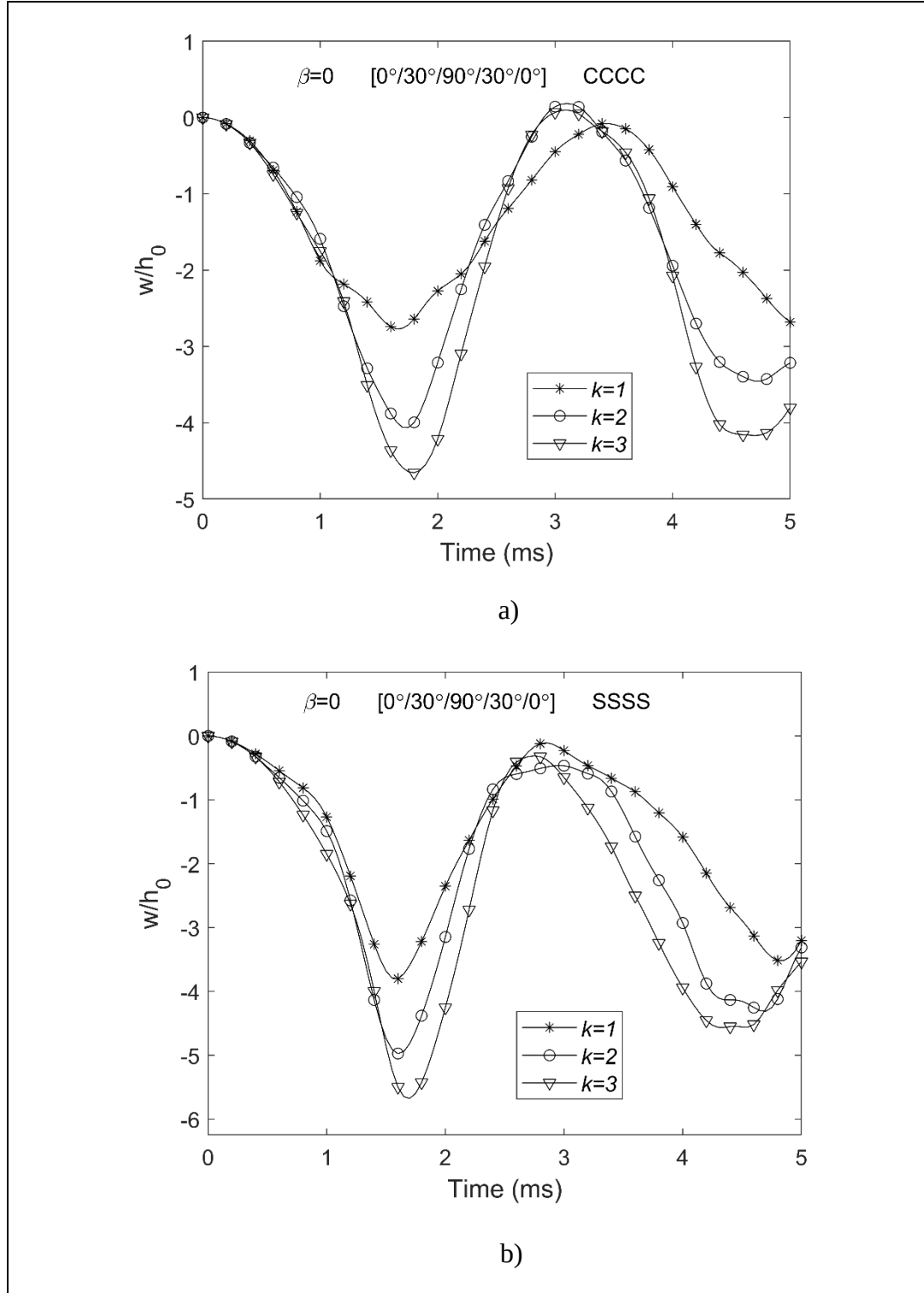


Figure 4.16: Non-dimensional transverse displacement histories at center of PPRS for different characteristic parameter and boundary conditions ($\beta=0$ uniform thickness) a) CCCC b) SSSS.

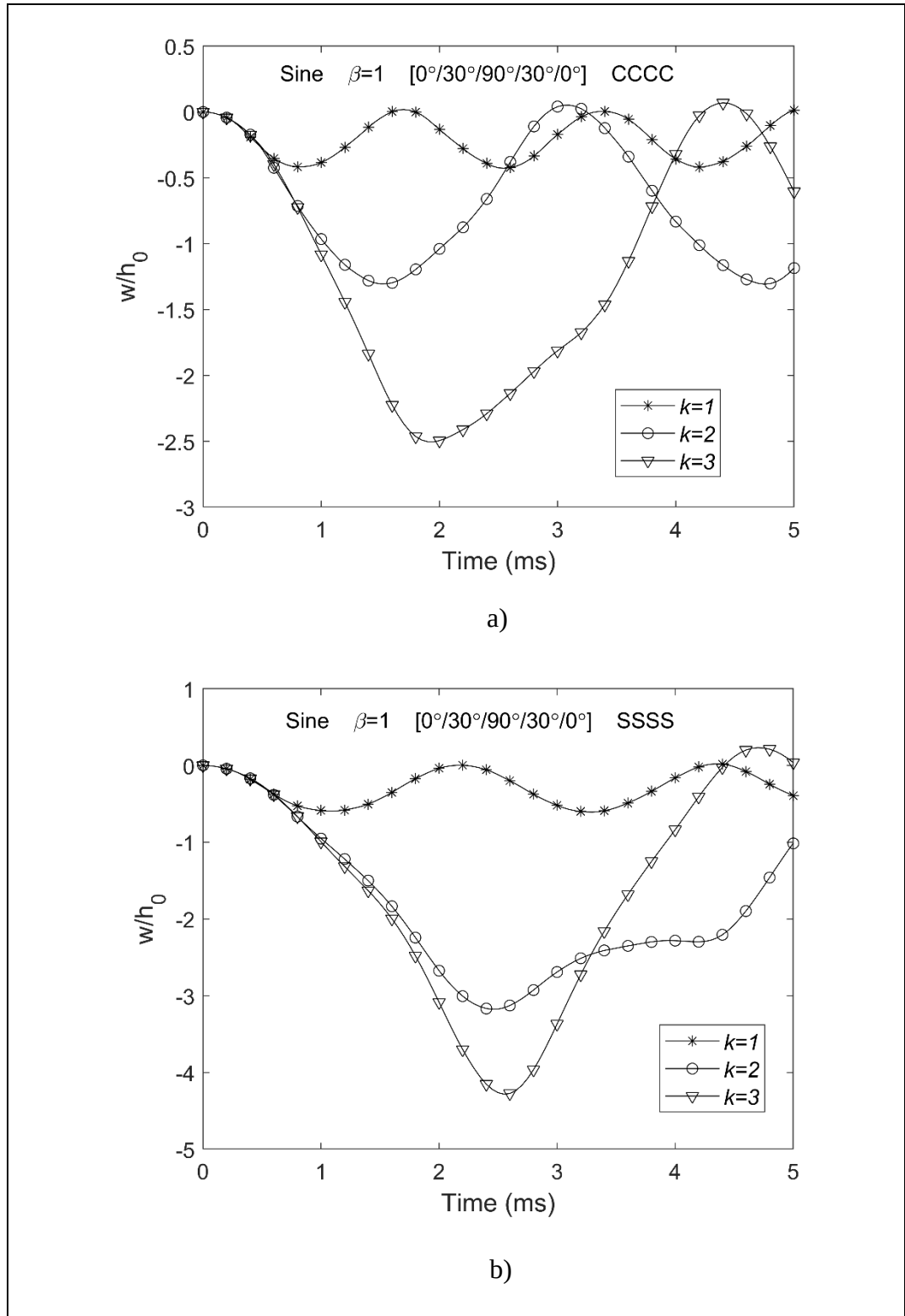


Figure 4.17: Non-dimensional transverse displacement histories at center of PPRS with variable thickness for different characteristic parameter and boundary conditions ($\beta=1$ sine form) a) CCCC b) SSSS.

4.3.2. Effect of Different Variable Thickness Forms and Thickness Variation Parameters on Transient Dynamic Behaviour

In this section, nonlinear transient dynamic response of laminated composite PPRS with variable thickness expressed with linear, arch, sine and cosine functions as in Eq. (2) is examined. Effects of thickness function and thickness variation parameter on nonlinear transient behavior are investigated considering CCCC and SSSS boundary conditions. Value of characteristic parameter of PPRS is considered as $k=2$. Stacking schemes for composite layers is $[0^\circ/30^\circ/90^\circ/30^\circ/0^\circ]$.

Figures 4.18-4.21 illustrates non-dimensional transverse displacement time histories at the center of PPRS considering different thickness function and thickness variation parameter for CCCC and SSSS boundary conditions, respectively. Thickness variation parameter values are taken as $\beta=0.25$, $\beta=0.5$, $\beta=1$ and $\beta=1.5$ to investigate the thickness variation parameter effects. Figure 4.19 and Figure 4.21 show that as the value of thickness variation parameter increases, displacement values decrease for CCCC and SSSS boundary conditions considering all types of variable thickness forms. The largest displacement values occur in cosine form for each β -value considering CCCC (in Figure 4.18) and SSSS (in Figure 4.20) boundary conditions. Displacement values of arch and sine forms which are lower than those of linear and cosine forms are close to each other for each β -value. In addition, for the case of $\beta=0.25$ displacement values are very close to each other regarding all types of variable thickness forms and boundary conditions. As the β -value increases, differences between the displacement values become more apparent. Figures 4.18-4.21 show that displacement values for SSSS boundary condition are higher than those for CCCC boundary condition for each β -value and variable thickness form.

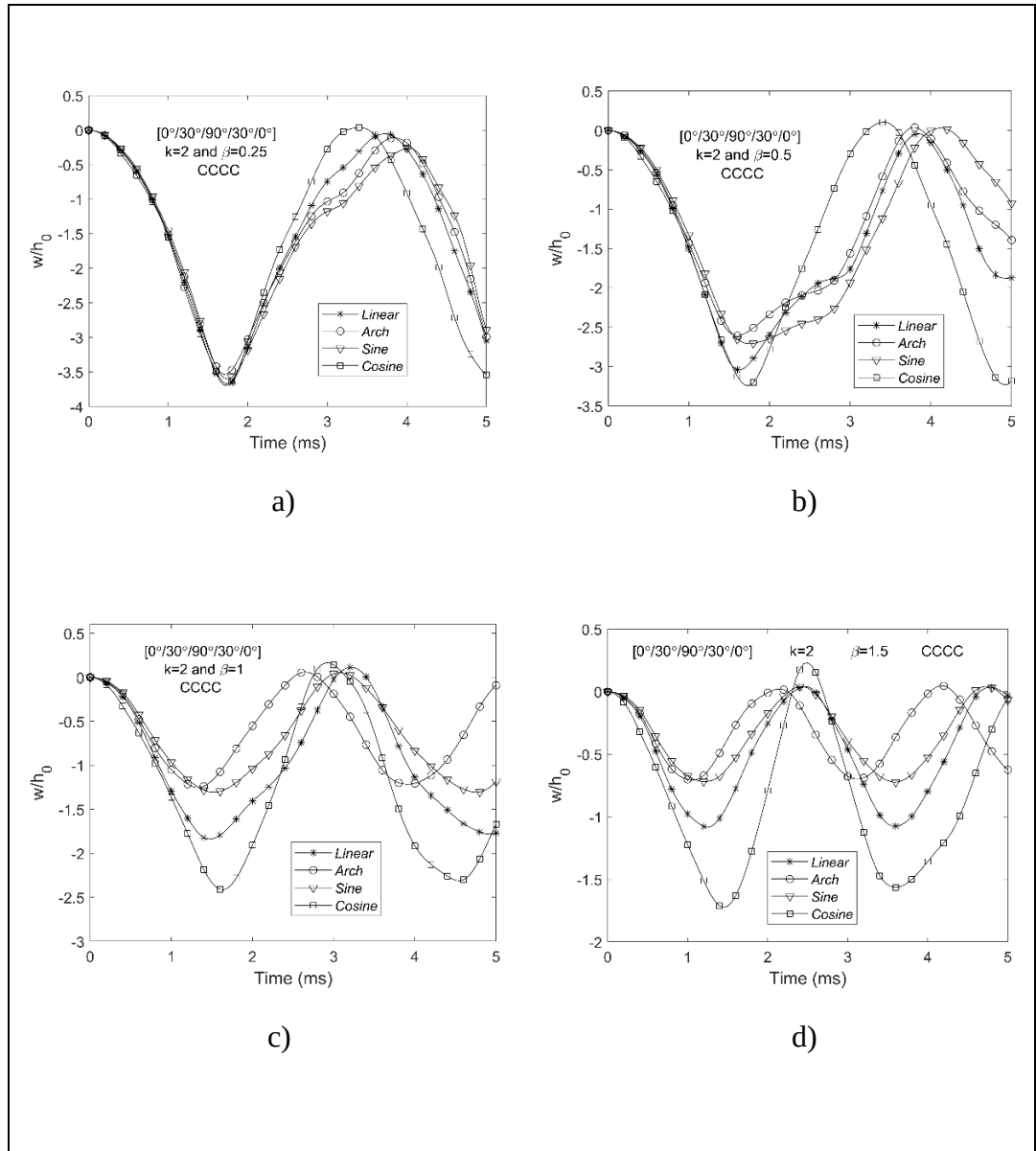


Figure 4.18: Non-dimensional transverse displacement histories at center of PPRS considering different variable thickness forms and thickness variation parameters for CCCC boundary condition a) $\beta=0.25$, b) $\beta=0.5$, c) $\beta=1$ and d) $\beta=1.5$.

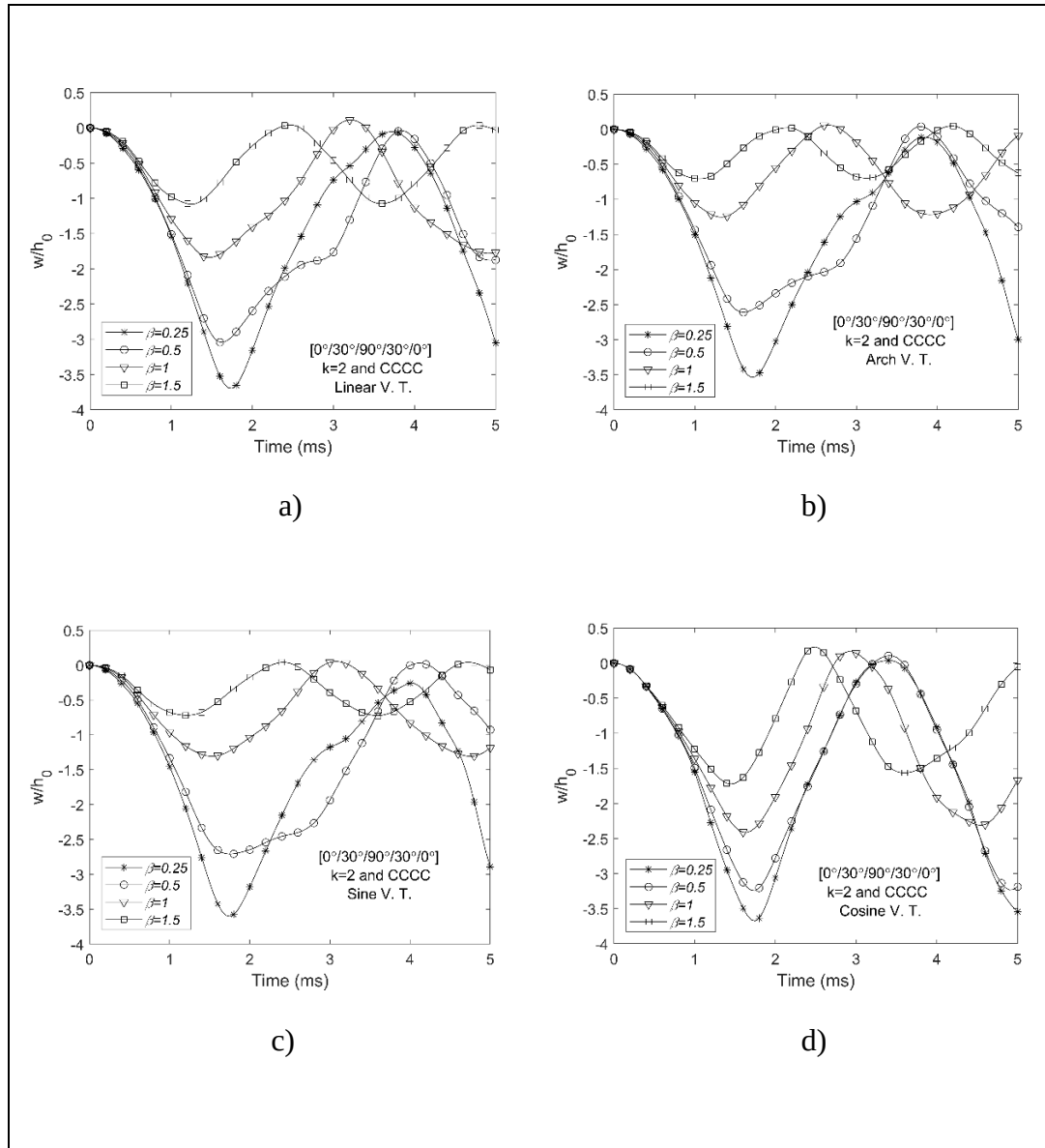


Figure 4.19: Non-dimensional transverse displacement histories at center of clamped PPRS for different thickness parameter a) Linear Form, b) Arch Form, c) Sine Form and d) Cosine Form.

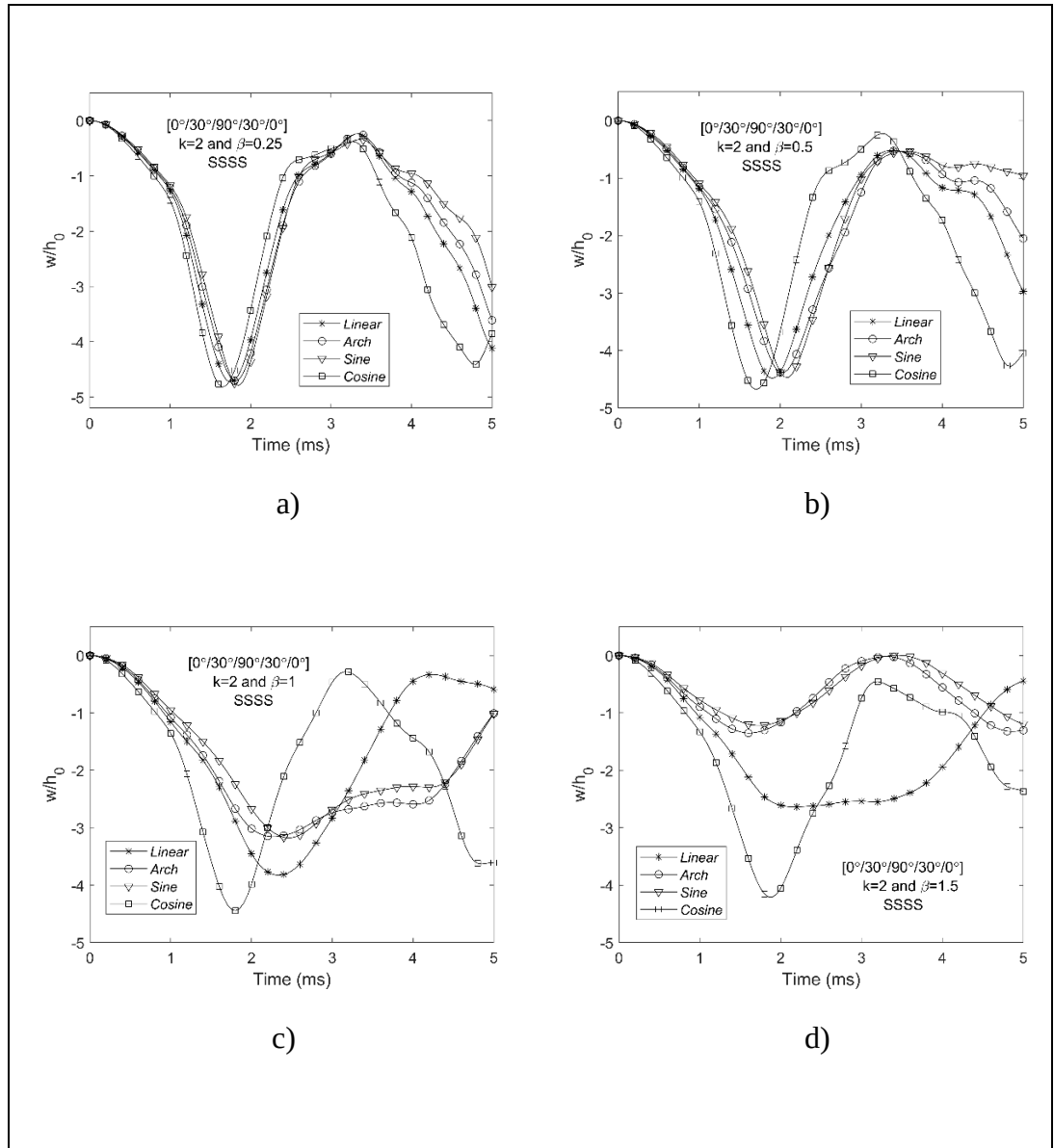


Figure 4.20: Non-dimensional transverse displacement histories at center of PPRS considering different variable thickness forms and thickness variation parameters for SSSS boundary condition a) $\beta=0.25$, b) $\beta=0.5$, c) $\beta=1$ and d) $\beta=1.5$.

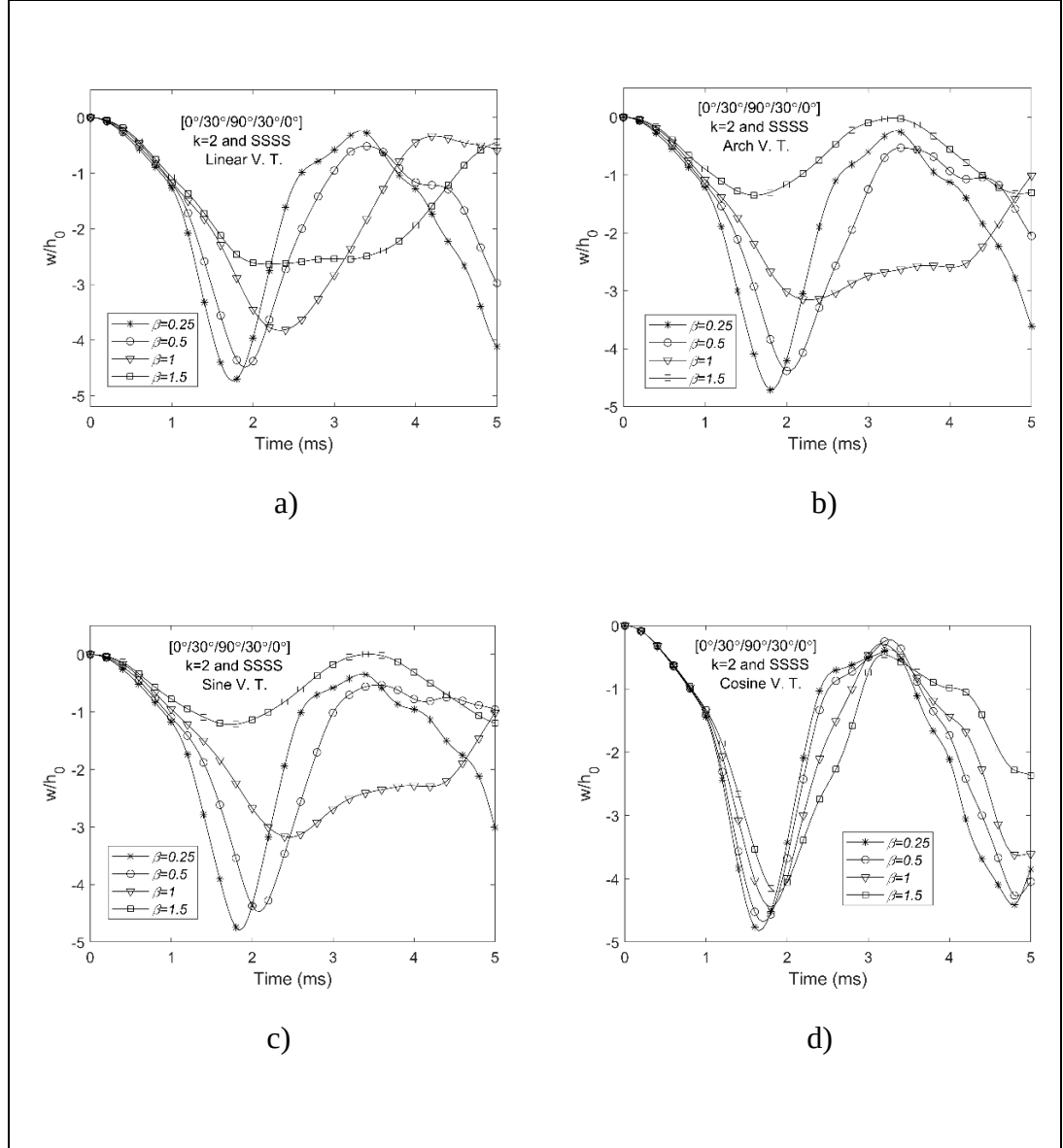


Figure 4.21: Non-dimensional transverse displacement histories at center of simply-supported PPRS for different thickness parameter a) Linear Form, b) Arch Form, c) Sine Form and d) Cosine Form.

4.3.3. Effect of Composite Stacking Schemes on Transient Dynamic Behaviour

In this section, nonlinear transient dynamic response of angle-ply laminated composite PPRS with variable thickness is examined for different stacking schemes $[\alpha^\circ/-\alpha^\circ/\alpha^\circ/-\alpha^\circ/\alpha^\circ]$. Effects of fiber orientation angle (α) on nonlinear transient behavior are investigated considering variable thickness functions of linear, arch, sine and cosine for $\beta=1$. Characteristic parameter value of PPRS is considered as $k=2$. CCCC boundary condition is considered.

Non-dimensional transverse displacement time histories at PPRS center are given for different values of layer orientation angles as $\alpha=30^\circ$, $\alpha=60^\circ$ and $\alpha=90^\circ$ in Figure 4.22. As shown in Figure 4.22, highest displacement values occur when $\alpha=30^\circ$ for all variable thickness forms. In contrast, lowest displacement values are observed for the case of $\alpha=90^\circ$. Transient responses of PPRS with cosine variable thickness form for the case of $\alpha=60^\circ$ in Figure 4.22-b and $\alpha=90^\circ$ in Figure 4.22c, differ from those with linear, arch and sine variable thickness forms for $\beta=1$. In addition, for the case of $\alpha=60^\circ$ and $\alpha=90^\circ$, displacement values of linear, arch and sine forms are close to each other.

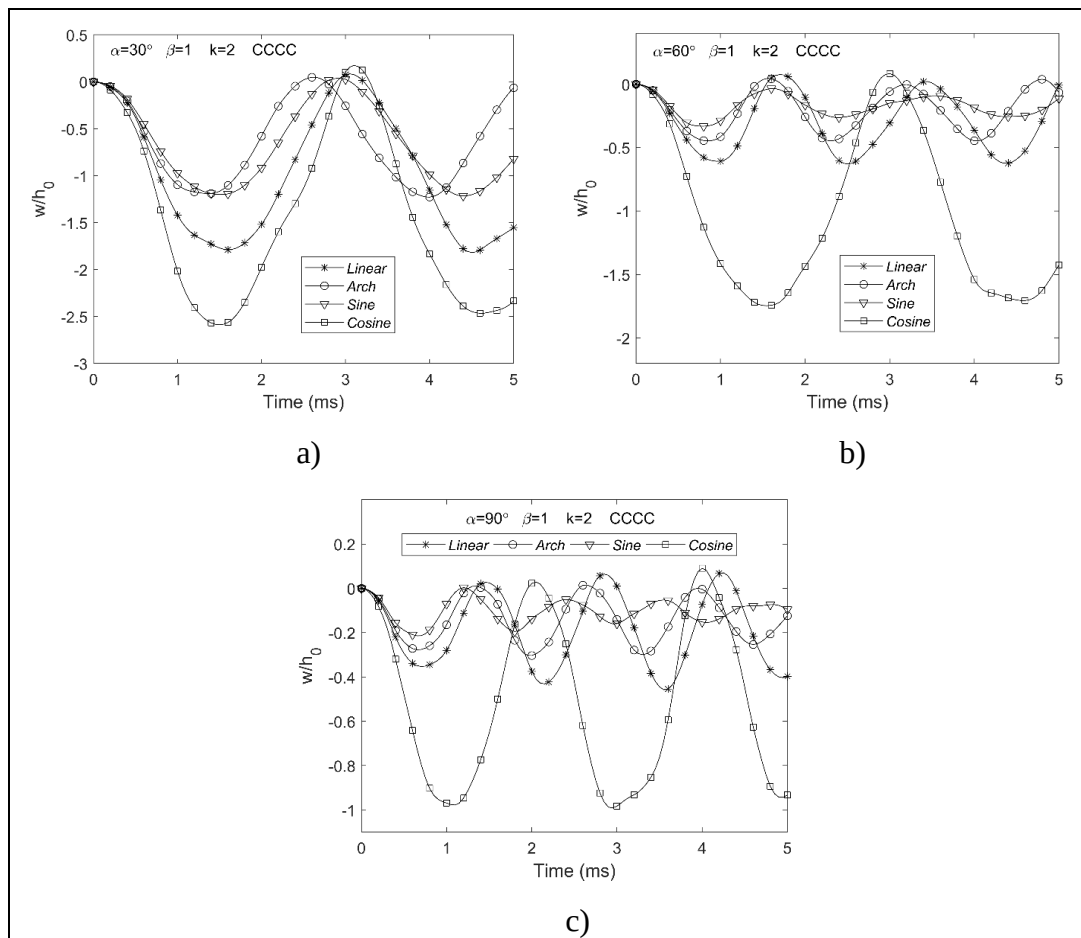


Figure 4.22: Non-dimensional transverse displacement histories at center of PPRS considering different variable thickness forms and stacking schemes for $\beta=1$. a) $\alpha=30^\circ$, b) $\alpha=60^\circ$ and c) $\alpha=90^\circ$.

4.3.4. Effect of Winkler-Pasternak Elastic Foundation on Transient Dynamic Behaviour

In this section, nonlinear transient dynamic response of laminated composite PPRS with variable thickness resting on Winkler-Pasternak elastic foundations is examined. Effects of Winkler and Pasternak elastic foundation parameters on nonlinear transient response are investigated regarding different forms of variable thickness and boundary conditions. Value of the characteristic parameter is considered as $k=2$. Stacking schemes for composite layers is chosen as $[30^\circ/-30^\circ/30^\circ/-30^\circ/30^\circ]$.

Non-dimensional transverse displacement time histories at the center of PPRS with uniform thickness ($\beta=0$) are illustrated in Figure 4.23 for different Winkler-Pasternak elastic foundation combinations regarding CCCC and SSSS boundary conditions, respectively. It is seen in Figure 4.23 displacements decrease with increasing Winkler-Pasternak elastic foundation parameters considering CCCC and SSSS boundary conditions. In addition, dynamic response characteristics are similar for both boundary conditions except maximum displacement values.

Figure 4.24 and Figure 4.25 show non-dimensional transverse displacement time histories at the center of PPRS resting on Winkler elastic foundation ($k_G=0$) for different thickness forms ($\beta=0.5$) considering both boundary conditions. As seen in Figure 4.24 and Figure 4.25, displacement values decrease with increasing values of Winkler elastic foundation parameter. While Winkler elastic foundation parameter (k_w) values that are less than $E_1/10^4$ do not have significant effect on the transient response, those that are higher than $E_1/10^4$ have amplitude-reducing effect. Besides, dynamic response characteristics are similar regarding all variable thickness forms for $\beta=0.5$.

In Figure 4.26 and Figure 4.27 effect of Winkler and Pasternak elastic foundation parameters on dynamic response is investigated for different thickness forms ($\beta=0.5$) considering CCCC boundary condition. When it is considered Figure 4.26 and Figure 4.27, influence of Pasternak elastic foundation parameter on dynamic behavior is more remarkable than that of Winkler elastic foundation parameter in terms of displacement values.

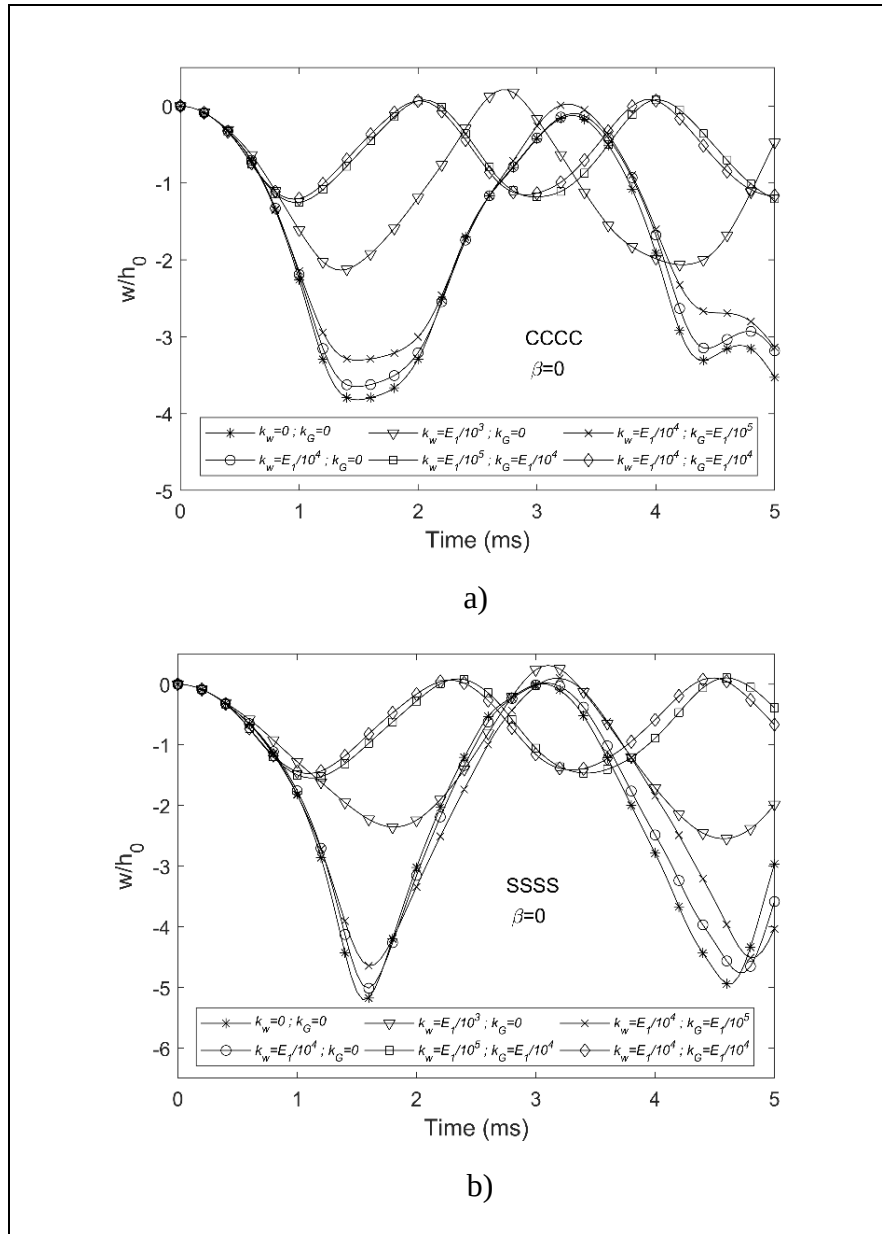


Figure 4.23: Non-dimensional transverse displacement histories at center of PPRS for different Winkler-Pasternak elastic foundation combinations ($\beta=0$ uniform thickness). a) CCCC, b) SSSS

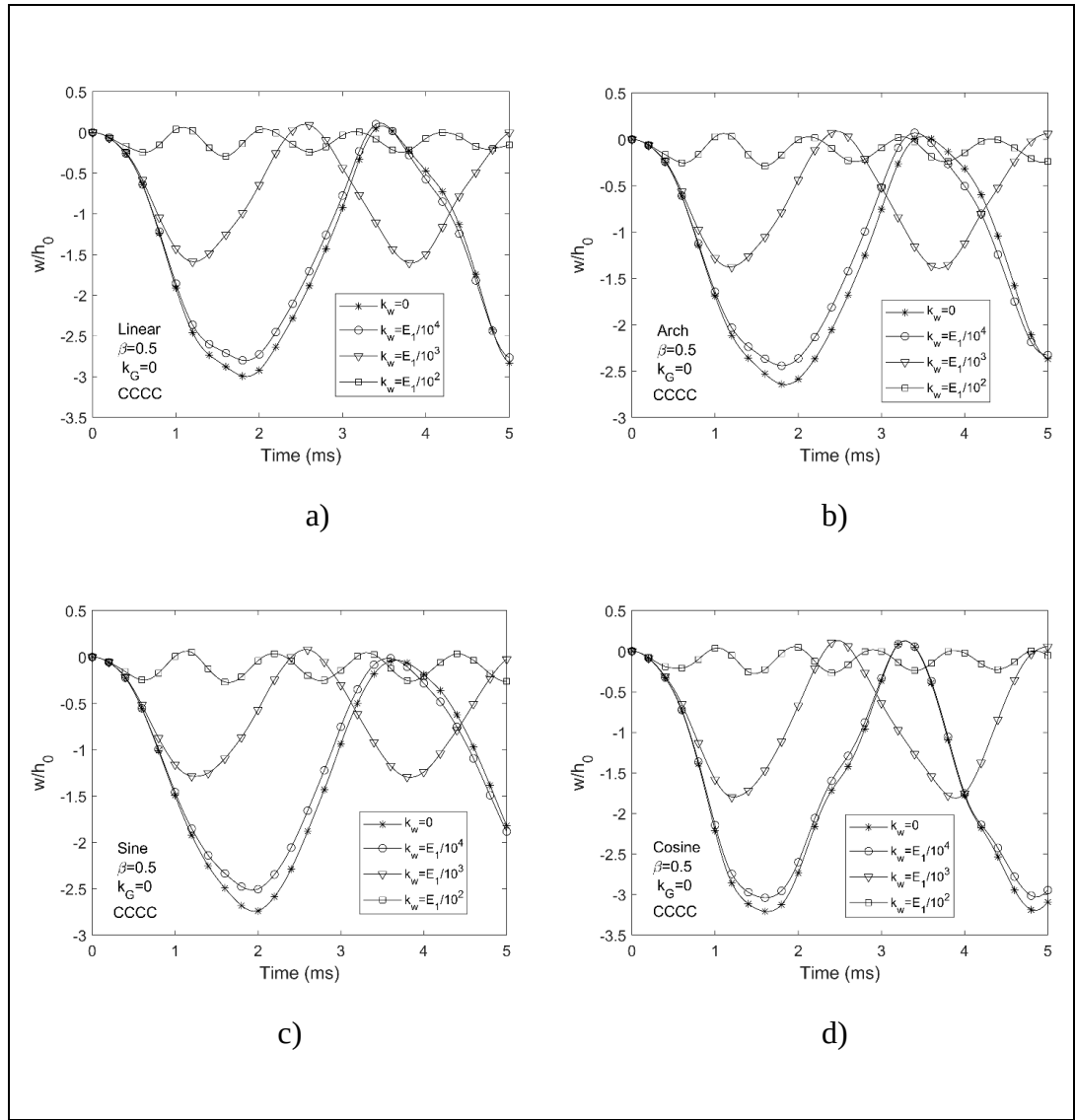


Figure 4.24: Non-dimensional transverse displacement histories at center of PPRS resting on Winkler elastic foundation ($k_G=0$) considering different variable thickness forms ($\beta=0.5$) and CCCC boundary condition. a) Linear Form, b) Arch Form, c) Sine Form and d) Cosine Form.

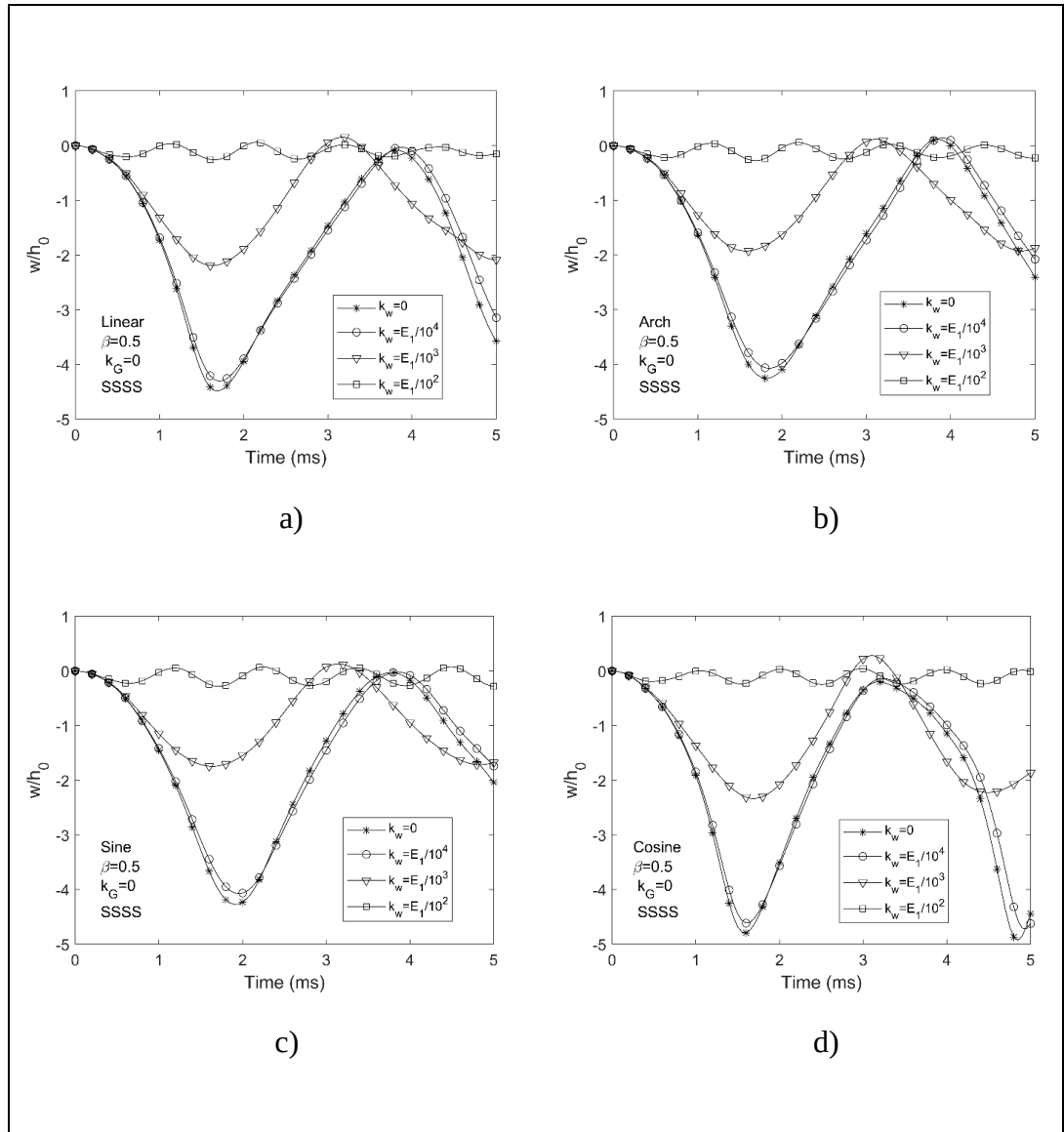


Figure 4.25: Non-dimensional transverse displacement histories at center of PPRS resting on Winkler elastic foundation ($k_G=0$) considering different variable thickness forms ($\beta=0.5$) and SSSS boundary condition. a) Linear Form, b) Arch Form, c) Sine Form and d) Cosine Form.

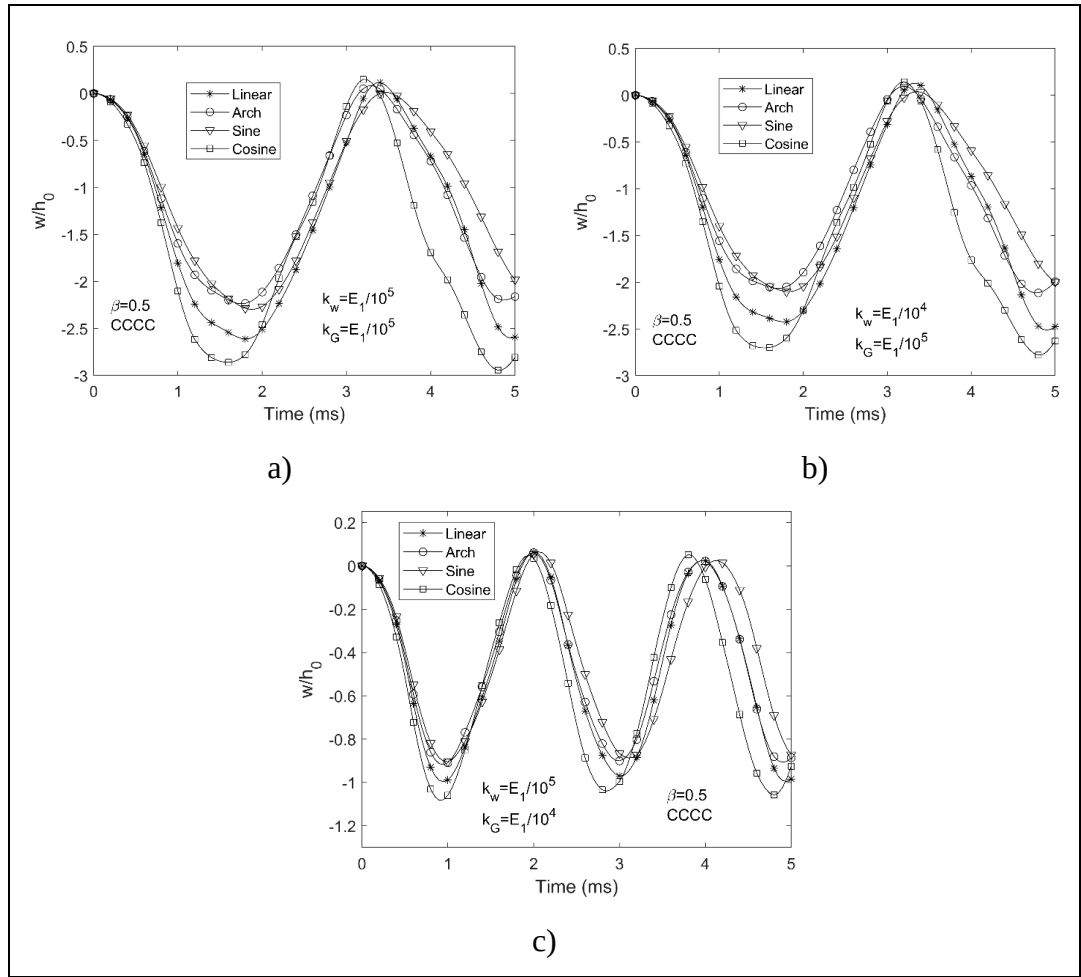


Figure 4.26: Non-dimensional transverse displacement histories at center of PPRS resting on Winkler-Pasternak elastic foundation considering different variable thickness forms ($\beta=0.5$) and CCCC boundary condition. a) $k_w = E_1/10^5$ and $k_G = E_1/10^5$, b) $k_w = E_1/10^4$ and $k_G = E_1/10^5$, c) $k_w = E_1/10^5$ and $k_G = E_1/10^4$

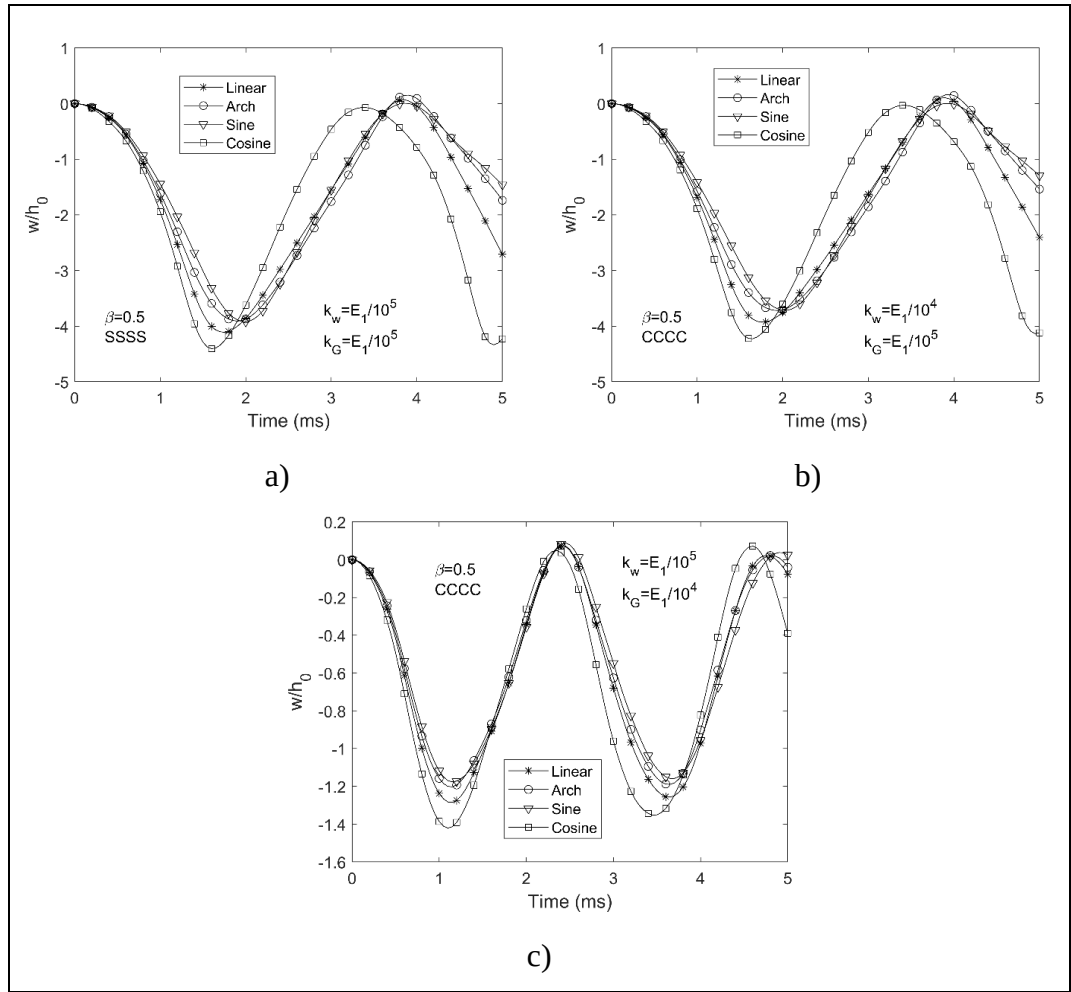


Figure 4.27: Non-dimensional transverse displacement histories at center of PPRS resting on Winkler-Pasternak elastic foundation considering different variable thickness forms ($\beta=0.5$) and SSSS boundary condition. a) $k_w= E_1/10^5$ and $k_G= E_1/10^5$, b) $k_w= E_1/10^4$ and $k_G= E_1/10^5$, c) $k_w= E_1/10^5$ and $k_G= E_1/10^4$

5. CONCLUSIONS

In this thesis, large displacement static and transient dynamic behavior of laminated composite shells of revolution with variable thickness resting on Winkler-Pasternak elastic foundation considering different types of boundary conditions was investigated utilizing GDQ method. Green-Lagrange nonlinear strain-displacement relations for shells of revolution were derived using the theory of surfaces. Transverse shear effect is considered through the FSDT. Equations of motion were obtained using the virtual work principle. Spatial derivatives in equation of motion was expressed with GDQ method. Newmark average acceleration method were utilized in time integration for transient analyses. Results obtained with GDQ code were validated with those of obtained with commercial software ANSYS and those in the literature. Good agreement was observed between the results. The influence of thickness functions, thickness variation parameter, geometric characteristic parameter of shells of revolution, boundary conditions, elastic foundation parameters as well as composite lamination scheme on nonlinear static and transient behaviour were investigated in detail.

Results of the static study can be summarized as follows:

- Displacement values of EPRS on static responses are observed to be minimum for all thickness forms when the characteristic parameter is $k=1.25$.
- The effect of thickness variation parameter on static responses is similar for linear, arch, cosine and sine thickness forms when β is 0.25 and stacking sequences is $[0^\circ/45^\circ/90^\circ/45^\circ/0^\circ]$.
- Static responses have been obtained for all thickness forms considering various stacking sequences ($\alpha=30, 60$ and 90) for $\beta=0.25$.
- Lowest displacement values for static analysis occur when α is taken as 90° considering all variable thickness forms for $\beta=0.25$.
- Displacement results of CCCC boundary condition are more effected by variation in thickness forms compared to those of SSSS boundary condition for static analysis.
- Static responses decrease for all considered boundary conditions and thickness forms, when the thickness variation parameter (β) increases.

- Sine and cosine thickness forms yield symmetrical displacement forms unlike the linear and arch forms on static analysis.
- Largest peak displacement values for static analysis occur in cosine thickness form.
- As the β -value increases, differences in static responses become more apparent.
- Displacement values decrease with increasing values of Winkler and Pasternak modulus.
- Pasternak modulus, which is higher than $G_{12}/10^4$, have significant amplitude-reducing effect.
- Winkler modulus, which is higher than $E_1/10^4$, have significant amplitude-reducing effect.

Results of the dynamic study can be summarized as follows:

- As characteristic parameter (k) of PPRS increases, transverse displacement values increase for CCCC and SSSS boundary conditions since the surface area becomes larger with higher values of k .
- Displacement values of arch and sine forms are close to each other for each thickness variation parameter (β) value and are lower than those of linear and cosine forms.
- As the value of thickness variation parameter (β) increases, displacement values decrease for both boundary conditions regarding all types of variable thickness forms.
- The largest displacement values occur in cosine form for each β -value considered in this study regarding both boundary conditions.
- As the β -value increases, differences between the displacement values become more apparent for all types of variable thickness forms.
- Dynamic responses of angle-ply laminated composite PPRS have highest displacement values when fiber orientation angle is $\alpha=30^\circ$ and lowest displacement values for the case of $\alpha=90^\circ$ for all thickness functions.
- Displacement values decrease with increasing values of Winkler elastic foundation parameter.

- Winkler elastic foundation parameter (k_w) values that are less than $E_1/10^4$ do not have significant effect on the transient response.
- When Winkler and Pasternak elastic foundations are applied together on PPRS, influence of Pasternak elastic foundation parameter on dynamic behavior is more remarkable than that of Winkler elastic foundation parameter in terms of displacement values.

Finally, this study can be extended to investigation of non-linear free vibration and stability analyses of shells of revolution structures as a future work. Also, these studies can be extended for FGM material.

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BIOGRAPHY

Özgür Kalbaran was born in 1986, Bandırma. He received her B.Sc. degree from Department of Mechanical Engineering, Istanbul University in 2009. In 2013, he completed his M.Sc. study at the Department of Mechanical Engineering, Istanbul Technical University. Between 2012 and 2019, he worked as a research assistant in Department of Mechanical Engineering, Gebze Technical University. Being as a research assistant, he has been pursuing his Ph.D. study under the supervision of Prof. Hasan Kurtaran at the Department of Mechanical Engineering, Institute of Natural and Applied Sciences, Gebze Technical University since 2013.

APPENDICES

Appendix A: Articles Published During the Thesis Study

Kalbaran, Ö., Kurtaran, H. (2019), “Nonlinear transient dynamic analysis of laminated composite parabolic panels of revolution with variable thickness resting on elastic foundation”, *Composite Structures*, 229, 111402.

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