

**T.C.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL**

**ELECTROMAGNETIC WAVE SCATTERING FROM
MULTIPLE DIELECTRIC CYLINDERS WITH INTEGRAL
EQUATION METHODS**

MURAT ENES HATİPOĞLU

**A THESIS OF DOCTORATE
DEPARTMENT OF ELECTRONICS ENGINEERING**

ADVISOR: ASSOC. PROF. DR FATİH DİKMEN

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T.C.
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*To my
Family ...*

ABSTRACT

The 2.5-dimensional (2.5-D) scattering solutions of an infinitely long dielectric cylindrical body illuminated by a three-dimensional finite source, such as a dipole, can be expressed as a linear superposition of line-source-type fields. For electric or magnetic dipole excitations with arbitrary orientation, the problem reduces to a two-dimensional coupled formulation involving electric and magnetic line sources, whose radial and longitudinal spectral dependences are cylindrical and obliquely planar, respectively. However, for cylindrical scatterers with arbitrary cross sections, the T-matrix method applied to each spectral component of the excitation is inherently ill-conditioned. Moreover, when the excitation source is located close to the scatterer, increasing the number of cylindrical harmonics does not necessarily improve the accuracy of the solution.

On the other hand, for scattering problems involving multiple scatterers, replacing the conventional scattering-matrix implementation of the T-matrix method with an iterative matrix–vector product accelerated by fast multipole approximations for far interactions significantly reduces the computational cost, but does not alleviate the ill-conditioning of the resulting system. To address this issue, this thesis starts from three-dimensional integral representations and develops 2–2.5D boundary and null-field integral equation formulations for obliquely incident wave excitation. An analytical regularization strategy is then introduced to render the T-matrix method well-conditioned for scatterers with arbitrary cross sections. In addition, extensions of the T-matrix method for dipole sources located in the near vicinity of cylindrical scatterers are investigated, and the impact of the proposed regularization on the accelerated formulation is examined and discussed in detail.

Keywords: Integral Equation Methods, Well-Conditioned T Matrix Method, Analytical Regularization, FMM Acceleration, Dipole Excitation, Microwave and Light Scattering.

ÖZET

Dipol gibi üç boyutlu sonlu bir kaynak tarafından aydınlatılan sonsuz uzun dielektrik silindirik bir cismin 2.5 boyutlu (2.5-D) saçılma çözümleri, çizgi kaynak tipi alanların doğrusal bir süperpozisyonu olarak ifade edilebilir. Keyfi yönelimli elektrik veya manyetik dipol uyarımları için problem, radyal ve boylamsal spektral bağımlılıkları sırasıyla silindirik ve eğik düzlemsel olan elektrik ve manyetik çizgi kaynaklarını içeren iki boyutlu bir birleşik formülasyona indirgenir. Bununla birlikte, keyfi kesitlere sahip silindirik saçıcılar için, uyarımın her bir spektral bileşenine uygulanan T-matris yöntemi doğası gereği kötü koşulludur. Ayrıca, uyarım kaynağı saçıcıya yakın olduğunda, silindirik harmoniklerin sayısını artırmak çözümün doğruluğunu mutlaka iyileştirmez.

Öte yandan, birden fazla saçıcı içeren saçılma problemleri için, T-matris yönteminin geleneksel saçılma-matris uygulamasını, uzak etkileşimler için hızlı çok kutuplu yaklaşımlarla hızlandırılmış bir yinelemeli matris-vektör çarpımı ile değiştirmek, hesaplama maliyetini önemli ölçüde azaltır, ancak sonuçta ortaya çıkan sistemin kötü koşullanmasını hafifletmez. Bu sorunu ele almak için, bu tez üç boyutlu integral gösterimlerinden başlamakta ve eğik gelen dalga uyarımı için 2-2.5D sınır ve boş alan integral denklem formülasyonları geliştirmektedir. Daha sonra, T-matris yöntemini keyfi kesitlere sahip saçıcılar için iyi koşullandırılmış hale getirmek için analitik bir düzenleme stratejisi tanıtılmıştır. Buna ek olarak, silindirik saçıcıların yakın çevresinde bulunan dipol kaynaklar için T-matris yönteminin uzantıları araştırılmış ve önerilen düzenlileştirilmenin hızlandırılmış formülasyon üzerindeki etkisi ayrıntılı olarak incelenmiş ve tartışılmıştır.

Anahtar Kelimeler: İntegral Denklem Yöntemleri, İyi Koşullu T Matrisi Yöntemi, Analitik Regülerleştirme, HÇY Hızlandırması, Dipol Uyarımı, Mikrodalga ve Işık Saçılması.

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LIST OF SYMBOLS AND ABBREVIATIONS

∇	: Operator Nabla
k	: Wavenumber
γ	: Euler-Mascheroni Constant
$\mathbf{E}$	: Electric field vector
$\mathbf{H}$	: Magnetic field vector
ε	: Dielectric permittivity
μ	: Magnetic permeability
ARM	: Analytical Regularization Method
BEM	: Boundary Element Method
BIE	: Boundary Integral Equations
CN	: Condition Number
CWS	: Cylindrical Wave Spectrum
CWH	: Cylindrical Wave Harmonics
EBCM	: Extended Boundary Condition Method
EFIE	: Electric Field Integral Equations
FMM	: Fast Multipole Method
Hz	: Hertz
GA	: Genetic Algorithms
LAES	: Linear Algebraic Equation System
l-PWS	: Longitudinal Plane Wave Spectrum
MFIE	: Magnetic Field Integral Equations
MoM	: Method of Moments
MLFMM	: Multilevel Fast Multipole Method
PW	: Plane Wave
R-TMM	: Regularized T Matrix Method
SIE	: Surface Integral Equations
TMM	: T Matrix Method
TE	: Transverse Electric
TM	: Transverse Magnetic
t-PWS	: Transverse Plane Wave Spectrum

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1. INTRODUCTION

Scattering problems of electromagnetic waves from particles or surfaces are of great importance in areas such as modeling the scattering properties of artificial surfaces or metamaterials such as near-zero index materials, remote sensing, radar metrology, controlling electromagnetic waves, beam forming, beam focusing and optical device design with photonic crystal structures. Such modern scattering problems in electromagnetic engineering require good analytical treatment of the problem and a numerically robust approach to obtain accurate solutions.

Modern solutions to monochromatic electromagnetic scattering problems have been dominated by efforts to find the electromagnetic fields satisfying Maxwell's equations and the related Helmholtz equation using surface integral equations (SIEs) established by the surface equivalence principle in the space separated by the scatterer surfaces. Based on this approach, the Boundary Element Method (BEM), known in the electromagnetics community as the Method of Moments (MoM), and the Multilevel Fast Multipole Method (MLFMM), which provides fast solutions for many scattering problems, including scattering problems at high frequencies and involving electrically large-sized objects, remain as one of the most widely used and studied methods.

Moreover, when it concerns the study of electromagnetic waves scattered from multiple objects, another method with a history of more than half a century continues to be a field of interest and research topic today. The T Matrix Method (TMM), first introduced by Waterman, like other widely used and studied methods, has become an effective and robust method for solving electromagnetic wave scattering problems from single, multiple, or composite particles.

Calculating the unknown fields with the MoM and the fast algorithm constructed on the MLFMM requires testing the surface integral equations on the boundary surface of the scatterers. However, the TMM is based on the solution of the null field integral equations valid in the volume where the total field is zero in the space separated by the surface of interest, the so-called Extended Boundary Conditions Method, or Extinction Theorem.

In methods based on the solution of Boundary Integral Equations (BIE), solutions are sought by selecting suitable test and basis functions. In contrast, in the TMM, the solutions are eigenfunctions of the Helmholtz equation consisting of cylindrical harmonics in two dimensions and spherical harmonics in three dimensions. Additionally, it is assumed that the surface current densities sought by surface integral equations where the fields are zero in the interior region of an arbitrarily shaped object can be represented by an infinite series in terms of these harmonics. By expressing the incident field in terms of these harmonics, the unknown series coefficient vector of the surface current densities can be described as a matrix product of a T matrix and a vector containing the known series coefficient of the incident field. In this respect, the TMM can be preferred as a very practical, convenient, and fast method for solving a boundary integral equation, despite its two apparent problems.

One of these problems is that even if the arbitrary scattering surface is canonical (circle in two dimensions, sphere in three dimensions), the TMM generates an ill-conditioned matrix-equation system. Another problem is the decreasing accuracy of this method in determining the fields on the boundary of an arbitrarily shaped scatterer, except for canonical geometries. In particular, the TMM does not give accurate results for bodies with corrugated and elongated geometries.

Previous studies pointed out that one of the reasons for the ill-conditioned matrix equation system obtained by the TMM is indicated by the fact that the surface integral equations used are Fredholm integral equations of the first kind. Another reason is the expansion of the fields for the canonical surface to determine the unknown field on an arbitrary surface that differs from this canonical surface. It implies that the TMM does not operate with a bounded invertible operator and fails to select the basis and test functions suitable for the surface of interest. The study of the relationship between the solutions of the surface integral equation realized by methods such as the method of moments at the relevant boundary and the TMM based on the null field approximation according to the equivalence principle is the key to improvements in this aspect.

Albeit not more evident than these limitations of the method, other constraints need to be studied. For instance, in multiple scattering problems, the addition theorem is used to express field representations in the local coordinate system of one scatterer in the local coordinates of another scatterer. When the scatterers are very close to each other, the series expression obtained from the addition theorem, which translates the outgoing

waves into standing waves, converges slowly. This requires taking too many terms from this series to get accurate solutions, and in some cases, the number of truncations of the series to be chosen for convergence may not be feasible. For this reason, in problems with multiple scatterers close to each other, the accuracy and speed of the method are compromised, and the technique's working could be improved.

On the other hand, in the series expression for transforming outgoing waves into standing waves in problems with many scatterers covering an electrically large area, the situation as mentioned above persists for scatterers very close to each other. However, although there is no slow convergence of this series in calculating the interactions of far scatterers, the considerable distance between the origin of the local coordinate systems of two far scatterers requires that the truncation number of this series be chosen sufficiently large. In addition, a fast and effective way to solve this problem is to place the scatterers on a grid whose width varies with this truncation number and to handle the far interactions with the FMM approach.

In the literature, although studies deal with the near and far interactions of cylindrical scatterers separately, it is still necessary to consider these interactions together comprehensively for scattering problems involving near and far cylindrical scatterers. It was mentioned earlier that the matrix equations obtained by the TMM need to be regularized and well-conditioned. It will be interesting to see to what extent an analytical regularization operation to correct for this ill-conditioned nature, combined with approaches to improve the interactions of close and far cylindrical scatterers for such scattering problems, will improve the system of equations and to what extent the effectiveness, accuracy, and speed of the method will be affected. These studies to remove the limitations of the TMM for cylindrical problems will shed light on removing the limits of the TMM for three-dimensional scattering problems.

Even though the solution of scattering problems from cylindrical scatterers of various lengths by surface integral equations is well known, the solution of multiple scattering problems involving such scatterers in three dimensions can take days and create a time bottleneck. To solve this scattering problem faster, the TMM in three dimensions seems inappropriate due to the series expansion of the fields in terms of spherical harmonics in the TMM in three dimensions and the limitations of the TMM mentioned earlier. Therefore, if the cylindrical scatterers are long enough, this scattering problem can be expressed in two dimensions as a scattering problem from infinitely long

cylinders. Then, this reduced 2D scattering problem can be solved by methods such as the MoM and the FMM based on the solution of boundary integral equations or by the TMM in two dimensions based on the null field approach, where the fields are expressed in terms of cylindrical harmonics. It is also interesting how much faster the solution of this scattering problem in two dimensions is compared to its solution in three dimensions.

In the case of excitation by a field in three dimensions, such as an electric dipole, to reduce the problem to two dimensions, the fields can be expressed as a superposition of plane waves in the direction of elongation of the cylinder. In other words, if the elongation direction of the cylinder is the z -axis, it is possible to express the z -dependence of the fields as a superposition of obliquely incident plane waves in the z -direction, denoted by $e^{jk_z z}$. From this point of view, it can be said that the solution of the scattering problem in two dimensions for an oblique incident wave excitation on a cylindrical scatterer enlightens the solution of the scattering problem from cylinders in three dimensions by reducing it to two dimensions. For this reason, the solution of the scattering problem from cylindrical scatterers of arbitrary cross-section under line source and oblique incident plane wave illumination by boundary integral equations and the TMM in two dimensions is of great importance.

When the cylindrical scatterers are not long enough for two-dimensional analysis and are assumed to be infinitely long, it is also a matter of interest whether the field distribution obtained with 2D solvers is correlated with the field distribution obtained with 3D solvers. If there is a correlation, solutions obtained in two dimensions can be used at each step during optimization to speed up the process in optimization problems of electromagnetic fields, or these solutions can be used to speed up the three-dimensional solvers. Furthermore, the analysis of the scattering problem from finite cylindrical scatterers in two dimensions without assuming that the cylinders are infinitely long and the advantages and disadvantages of this analysis compared to three-dimensional solvers is another issue that needs to be studied.

The thesis proposes an efficient, fast, numerically stable, and accurate method based on integral equations in two dimensions for scattering problems from multiple dielectric cylindrical bodies with the abovementioned motivation. In the thesis, the TMM in two dimensions for cylindrical scatterers with arbitrary cross sections, the ill-

conditionality of the resulting system of equations, its limitations will be discussed and demonstrated.

In addition, a methodology for the interaction between scatterer with arbitrary cross section and source placed its proximity will be discussed, and the FMM accelerated version of the TMM will be revisited to extend the scope of the literature. The thesis also aims to make the method well-conditioned by Analytical Regularization for cylindrical scatterers with arbitrary cross-sections and to discuss and demonstrate the effect of applied Analytical Regularization on the accelerated system.

The Regularized (Well-Conditioned) FMM Accelerated TMM obtained by achieving these objectives will be applied to the problem of a cylindrical scatterer with multiple dielectric arbitrary cross sections under various excitations. It also aims to show that the numerical results obtained with this method are consistent with those obtained with boundary integral equation methods. To achieve this goal, it is also aimed to perform a singularity analysis of the kernels in all components of boundary integral equations arising in the coupling case and to show the procedure for their correct treatment.

Thus, an efficient, fast, numerically stable, and accurate method based on the boundary integral equations and the null-field integral equations for scattering from cylindrical scatterers with multiple arbitrary cross-sections under the excitation of a field expressed in three dimensions is presented.

1.1. Literature Review

The foundations of the TMM, also known as the extended boundary conditions method, which can be traced back to the Rayleigh hypothesis, were first proposed by Waterman in 1965 and 1969 [1], [2]. Following this, discussions based on the equivalence of the TMM with the Rayleigh hypothesis started. Published in 1969, Burrows' paper [3] is one of the papers in this area. Another article by Waterman [4], published in 1971, presents numerical and experimental results on applying the TMM to arbitrary cross sections, while [5], published in 1972, presents results on Gaussian beam scattering from cylindrical scatterers. In 1973 and 1974, Peterson and Ström in [6], [7] extended the TMM to multiple scatterers and layers. In [8], also in 1974, a numerical comparison of the solutions obtained with Green's function, the TMM, and the Rayleigh hypothesis was made by Bolomey for cylinders of arbitrary cross-section,

and indications of the need for regularization of the T matrix were given. Following these studies, in 1975, Bates [9] discussed the analytical limitations associated with the TMM for calculating electromagnetic fields. In 1977, Bates and Wall, in their three-part study [10]-[12], gave extensive coverage to the null field approximation for scalar wave scattering problems and, naturally, to the T matrix approximation and the application of the method to inverse and flat scattering problems. While the analytical studies on the TMM continued, in van den Berg and Fokkema's paper [13] published in 1979, discussions on the cases where the Rayleigh hypothesis is valid were continued.

On the other hand, an application of the T matrix approach for nonlinear media was presented in [14] in 1988. In 1988, Varadan V.V. et al. in [15] commented on the criticisms of the TMM. This paper mentions that all numerical methods, including the TMM, have limitations and application areas. It is also mentioned in this paper that the method is a simple and powerful tool and the fact that the technique will naturally evolve as the computer age comes, so when comparing the TMM with other methods, it should be compared with the latest versions of the technique. In the studies [16], [17], also in 1988, the scattering problem from two dielectric cylinders under oblique incidence excitation was investigated analytically and numerically.

In the studies on the TMM up to 1988, the method and its limitations were discussed analytically. Although computers were not as advanced in those years as today, it was considered a powerful and effective method. From 1989 to 1995, the method was mainly studied by Chew et al. These years, recursive implementation and its use as a numerical method were emphasized. In [18], published in 1989, the TMM was presented as a method with computational complexity N^2 for multiple scattering problems. In [19]-[23], published in 1990, the recursive version of the TMM was presented as an efficient method for cylinders, spheres, and strips with low algorithmic complexity. In 1991, Gürel and Chew in [24] discussed the connection between the T matrix and integral equations, and in [25], a recursive T matrix algorithm for 1D and 2D strips was presented. In 1992, a recursive algorithm for scattering problems in two dimensions was also studied by Chew et al. [26]. The recursive version of the TMM appeared as a fast and efficient method for multiple scattering problems in those years when the memory capacity of computers was not high. This year, the addition theorem's slow convergence when dealing with multiple scatterers' interactions was

pointed out. The windowed addition theorem was proposed and applied in [27], [28] as a solution. In [29]-[32], the recursive T matrix algorithm continued to be applied for multiple spheres, strips, and patches in three dimensions. While recursive applications of the TMM in two and three dimensions continued, Gaussian beam scattering from an array of conducting and dielectric cylinders was studied in [33], [34]. In 1995, Chew et al. introduced the FFT – TMM for microwave scattering from random scatterers [35], one of the first studies to accelerate the method. In 1999, a generalized description of the TMM and its recursive version for solving problems in two dimensions and three dimensions was collected by Chew in [36].

The TMM, which continued to develop in literature until the end of the 1990s, was accepted for practical applications of scattering problems from cylindrical and spherical scatterers in the 2000s. Still, it continued to be specialized for these applications, and the numerical limitations of the method became more evident with the development of computers. In 1996, the problem of scattering from multiple spheres was considered by Mackowski and Mischenko [37]. In the same year, a comprehensive literature review was conducted on the problem of light scattering from non-spherical particles [38]. Efforts to develop the T matrix for scattering from non-spherical particles and to compare it with the geometrical optics method were presented in [39]. This year, scattering problems from multiple cylindrical scatterers of arbitrary cross-sections were presented in [40]. In [41], [42], published in 1997 and 1998, respectively, a solution to the slow convergence problem of the recursive TMM for scattering problems from metallic and dielectric cylinders was proposed. In [43], the scattering problem under a Gaussian beam incident on multilayer cylindrical scatterers was investigated. In 1999 and 2000 [44]-[48], the TMM was applied to 2D photonic crystal structures and scattering problems from periodic cylindrical scatterers. In [49], published in 2002, the application of the TMM to 2D photonic crystal structures was again discussed.

From 2002 to the present, the focus has been on applying the TMM to spheres and non-spherical particles in three dimensions. In this period, its application to scattering problems from periodic cylindrical arrays and photonic crystals in two dimensions is noteworthy. In [50], published in 2003, the connection between integral equations and the TMM was discussed. In the same year, Toyama [51] considered the problem of multiple and layered cylindrical scatterers using the TMM. Also, this year, a different

variation of the TMM, the recursive centered TMM, was discussed in [52]. In 2004, a comprehensive reference database on the TMM was established at the initiative of Mishchenko et al. in [53]. In 2004 and 2005, [54]-[58] studied scattering from cylindrical arrays in two dimensions.

Studies in 2007 are valuable for applying the TMM to cylindrical scatterers under oblique incident wave excitation and accelerating the method. In [59], [60], Henin et al. studied the problem of scattering from layered cylindrical scatterers and cylindrical arrays under oblique incident wave excitation. In [61], the acceleration of the TMM with the FMM approach is discussed. In 2007 and 2009, Waterman revisited the TMM in [62], [63]. In 2010, a review of the current perspective of the TMM can be found in [64]. This year, the TMM in three dimensions was discussed as an efficient numerical solver [65]. In 2011, scattering from layered cylindrical arrays was the subject of [66]-[68]. In the same year, the improvement of the numerical stability of the TMM was discussed in [69]. In 2012, a comprehensive literature review of the TMM for 2009-2011 was updated in [70]. In [71], the acceleration of the T matrix with GPU for light scattering was discussed.

In 2013, a new numerically stable implementation of the TMM was presented for the problem of scattering from spherical particles [72]. In [73], the MoM using the Rao-Wilton-Glisson (RWG) basis function of the T matrix is directly determined by the impedance matrix. In [74], published in 2015, an accurate and convergent TMM for light scattering from spherical particles was proposed. In [75], the recursive TMM was consistent with experimental results at microwave frequencies. In [76], Dikmen, Sever et al. considered the problem of scattering from multiple and layered circular cylindrical scatterers and showed the ill-conditioning of the resulting system of equations and the analytical regularization of the system. In 2016, in [77], the problem of scattering from two impedance cylinders was studied, and the ill-conditioning of the resulting system of equations is shown. In 2017, a comprehensive literature review of the TMM, updated for 2017-2019, was published in [78]. In [79], Ganesh and Hawkins introduced the T matrix reduced order model software (TMATROM) as a numerical solver. In 2018, the T matrix solver for fast scattering modeling from multiple PEC scatterers was studied [80]. In [81]-[83], a numerically stable algorithm for scattering problems from circular cylinders is discussed. In 2019, a recursive formulation of the TMM was studied in [84]. In 2020, a literature review on the TMM

was given in [85], covering the years 2017-2019. In 2021, conference papers [86], [87] include efforts to apply the TMM to multiple scatterers and photonic crystal structures in two dimensions and to accelerate simulations in three dimensions with results in two dimensions. More comprehensive versions of these studies will be presented in this thesis. In [88], published in the same year, the problem of scattering from periodic circular cylindrical arrays was discussed.

The MoM and the TMM hybridization were discussed in [89], recently published in 2022. In [90], Ganesh and Hawkins presented a numerically stable TMM for the problem of scattering from non-spherical particles. In [91], an application of the TMM to the problem of scattering from close arbitrary bodies is presented.

It is understood that TMM is indispensable in literature. In previous studies, the TMM has been applied to spherical and non-spherical particles in three dimensions, cylindrical arrays, and layered cylindrical structures in two dimensions. The equivalence of the TMM with the Rayleigh hypothesis, the accuracy, numerical stability, and acceleration of the TMM has been discussed until today. Although there are studies that handle each constraint of the method for different problems, there is no generalized TMM that includes all these improvements, especially for cylindrical scatterers with arbitrary cross-sections under oblique incidence or electric dipole excitation.

1.2. Contributions of the Thesis

Robust treatment of scattering problems from cylindrical scatterers is of great importance for topics such as accelerating 3D optimization of rod array structures with genetic algorithms [86], optimization of the cross-sectional area of cylindrical scatterers with genetic algorithms [98], electronic band gap structures [99], designing optical devices with photonic crystals [100],[101], designing optical logic gates [102],[103], investigation of scattering from zero refractive index materials [104] and light scattering from nanorods [105],[106].

For this reason, the potential contribution of this thesis is to present an integral equation-based numerical solver specialized for cylindrical scatterers that can be effective in 5G applications, mm-wave applications, light scattering from particles, determination of scattering characteristics and optical device design, which is analytically strong, numerically stable, fast, and accurate.

In particular, the numerical results in [86] inspire the acceleration of optimizations in 3D MLFMM simulations when there is a time bottleneck during genetic optimization since the fast solution produced by 2D solvers is correlated with the 3D solution. The optimizations to be performed by genetic algorithm in rod array problems can be achieved faster with 2D solvers.

On the other hand, the improvements to the TMM for 2.5D scattering problems and the studies to be carried out with integral equations within the scope of this thesis are aimed at finding their place in literature and bringing a different perspective that builds on previous studies.

The 2-2.5D integral equation-based numerical solution technique and solver that will result from the outputs of this thesis will benefit researchers studying scattering from particles in different disciplines or researchers studying the control of electromagnetic waves with a structure containing cylindrical arrays such as photonic crystal structures.

This thesis focuses on the Extended Boundary Condition Method (EBCM) also known as Null Field Approach and the efforts to improve the TMM based on this approach by eliminating its limitations for scattering problems from multiple arbitrary cylindrical bodies. The TMM, also known as the scattering matrix method or the EBCM, can be called a standard method for multiple scattering problems due to the ongoing intensive efforts in this research area. This method determines the unknown

field coefficients by performing a series expansion with eigenfunctions appropriate to the unknown scattered field. If the elongation or corrugation does not violate the symmetry of these expansions, this method is a practical tool.

The unique aspect of this thesis is that the TMM is subjected to a regularization specific to the cylindrical scattering problem with arbitrary cross sections, which corrects the ill-conditioning of the TMM. In addition, another unique aspect of this thesis is the demonstration of the effect of analytical regularization on the acceleration of the TMM with FMM.

Furthermore, the novelty of this thesis is the introduction of a fast, numerically stable, rapidly convergent, accurate, and efficient method based on integral equations under a three-dimensional source excitation to accelerate the three-dimensional analysis of structures involving arbitrary placement of cylindrical arrays, especially photonic crystal structures.

2. INTEGRAL EQUATIONS

In this section, well-known three-dimensional integral representations are revisited as a basis for formulating two and two and a half dimensional (2–2.5D) scattering problems using boundary and null field integral equations.

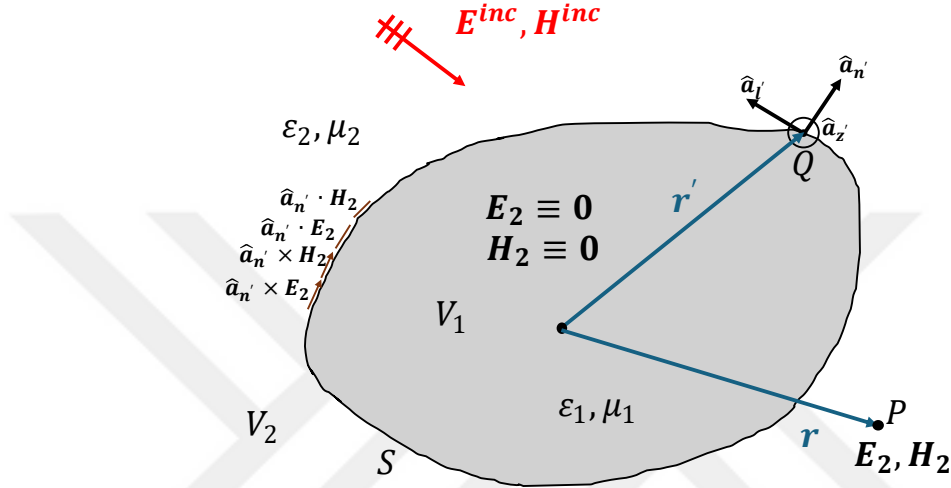


Figure 2.1: Definition of the exterior domain, fields, and surface sources employed for the exterior integral representations.

The traditional integral representations in three dimensions for the homogeneous dielectric scatterer illustrated in **Figure 2.1**, excited from its outer region with the observation point lying outside the scatterer, are as follows:

$$\mathbf{E}_2 = \mathbf{E}^{inc} + \int_S \left\{ -j\omega\mu_2 \mathbf{J}_2 G_2 - \mathbf{M}_2 \times \nabla' G_2 - \frac{j}{\omega\epsilon_2} (\mathbf{J}_2 \cdot \nabla') \nabla' G_2 \right\} ds', \mathbf{r} \in V_2 \quad (2.1)$$

$$\mathbf{H}_2 = \mathbf{H}^{inc} + \int_S \left\{ -j\omega\epsilon_2 \mathbf{M}_2 G_2 + \mathbf{J}_2 \times \nabla' G_2 - \frac{j}{\omega\mu_2} (\mathbf{M}_2 \cdot \nabla') \nabla' G_2 \right\} ds', \mathbf{r} \in V_2 \quad (2.2)$$

Here, \$(\mathbf{E}^{inc}, \mathbf{H}^{inc})\$ denote the incident fields and represent the primary sources. The secondary sources, namely the equivalent surface sources, are associated with the currents induced on the scatterer surface due to excitation by the primary sources and are given as follows:

$$-i_n \times \mathbf{E}_2 = \mathbf{M}_1 \quad (2.3)$$

$$i_n \times \mathbf{H}_2 = \mathbf{J}_2 \quad (2.4)$$

$$i_n \cdot \mathbf{E}_2 = -\frac{\nabla_s \cdot \mathbf{J}_2}{j\omega\epsilon_2} \quad (2.5)$$

$$i_n \cdot \mathbf{H}_2 = -\frac{\nabla_s \cdot \mathbf{M}_2}{j\omega\mu_2} \quad (2.6)$$

where $\nabla_s = \nabla - i_n \frac{\partial}{\partial n}$.

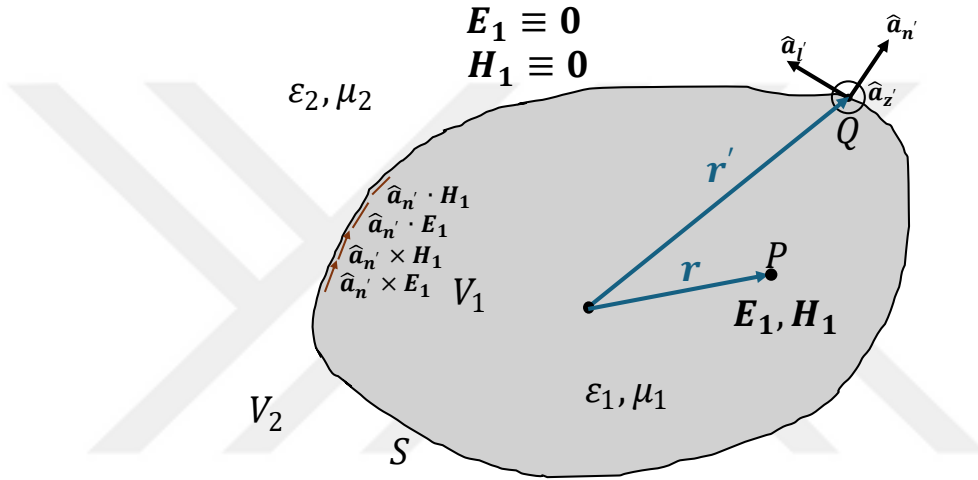


Figure 2.2: Definition of the interior domain, fields, and surface sources employed for the interior integral representations.

When the observation point lies inside the homogeneous dielectric, as shown in **Figure 2.2**, the interior integral equations take the following form:

$$\mathbf{E}_1 = \int_S \left\{ -j\omega\mu_1 \mathbf{J}_1 G_1 - \mathbf{M}_1 \times \nabla' G_1 - \frac{j}{\omega\epsilon_1} (\mathbf{J}_1 \cdot \nabla') \nabla' G_1 \right\} ds', \mathbf{r} \in V_1 \quad (2.7)$$

$$\mathbf{H}_1 = \int_S \left\{ -j\omega\epsilon_1 \mathbf{M}_1 G_1 + \mathbf{J}_1 \times \nabla' G_1 - \frac{j}{\omega\mu_1} (\mathbf{M}_1 \cdot \nabla') \nabla' G_1 \right\} ds', \mathbf{r} \in V_1 \quad (2.8)$$

where the equivalent surface sources are given by:

$$-i_n \times \mathbf{H}_1 = \mathbf{J}_1 \quad (2.9)$$

$$i_n \times \mathbf{E}_1 = \mathbf{M}_1 \quad (2.10)$$

$$i_n \cdot \mathbf{E}_1 = \frac{\nabla_s \cdot \mathbf{J}_1}{j\omega\epsilon_1} \quad (2.11)$$

$$i_n \cdot \mathbf{H}_2 = \frac{\nabla_s \cdot \mathbf{M}_2}{j\omega\mu_1} \quad (2.12)$$

$\mathbf{J}_i$ and $\mathbf{M}_i$ denote the equivalent electric and magnetic surface currents, and if there are no actual surface currents on S , the following relation holds:

$$\mathbf{J}_1 = -\mathbf{J}_2 \quad (2.13)$$

$$\mathbf{M}_1 = -\mathbf{M}_2 \quad (2.14)$$

Since none of the equivalent surface currents $\mathbf{J}_i$ and $\mathbf{M}_i$ in (2.1), (2.2), (2.7), and (2.8) possesses a component normal to the surface S , ∇' may be replaced by ∇'_s in the terms $\mathbf{J}_i \cdot \nabla'_s$ and $\mathbf{M}_i \cdot \nabla'_s$, without altering the validity of the formulations. A key property of (2.1), (2.2), (2.7), and (2.8) is that the normal components of the fields no longer appear in the integrands. Instead, the integrands involve second order partial derivatives of the Green's functions. Integral representations that involve only first order partial derivatives of Green's functions and are not dependent on the normal field components are given below:

$$\begin{aligned} \mathbf{E}_2 = \mathbf{E}^{\text{inc}} + \int_S \{ & -j\omega\mu_2 \mathbf{J}_2 G_2 - \mathbf{M}_2 \times \nabla' G_2 \\ & + \frac{j}{\omega\epsilon_2} (\nabla'_s \cdot \mathbf{J}_2) \nabla' G_2 \} ds', \mathbf{r} \in V_2 \end{aligned} \quad (2.15)$$

$$\begin{aligned} \mathbf{H}_2 = \mathbf{H}^{\text{inc}} + \int_S \{ & -j\omega\epsilon_2 \mathbf{M}_2 G_2 + \mathbf{J}_2 \times \nabla' G_2 \\ & + \frac{j}{\omega\mu_2} (\nabla'_s \cdot \mathbf{M}_2) \nabla' G_2 \} ds', \mathbf{r} \in V_2 \end{aligned} \quad (2.16)$$

$$\begin{aligned} \mathbf{E}_1 = \int_S \{ & -j\omega\mu_1 \mathbf{J}_1 G_1 - \mathbf{M}_1 \times \nabla' G_1 \\ & + \frac{j}{\omega\epsilon_1} (\nabla'_s \cdot \mathbf{J}_1) \nabla' G_1 \} ds', \mathbf{r} \in V_1 \end{aligned} \quad (2.17)$$

$$\mathbf{H}_1 = \int_S \left\{ -j\omega\epsilon_1 \mathbf{M}_1 G_1 + \mathbf{J}_1 \times \nabla' G_1 + \frac{j}{\omega\mu_1} (\nabla'_s \cdot \mathbf{M}_1) \nabla' G_1 \right\} ds', \mathbf{r} \in V_1 \quad (2.18)$$

where $G_1 = \frac{e^{-jk_i|r-r'|}}{4\pi|r-r'|}$. Equations (2.15) - (2.18) provide a more convenient formulation for numerical computation by avoiding second order derivatives of the Green's function that lead to hypersingular kernels.

The integral representations given above are written for the total field in the interior and exterior regions of the scatterer. When these fields are observed in the complementary region separated by the boundary of the object, the surface sources cause the fields to vanish. For instance, in the representations (2.1) and (2.2), the fields $(\mathbf{E}_2, \mathbf{H}_2)$ are generated in the observation region by the contributions of the incident field and the surface sources. However, when these representations are evaluated in V_1 , i.e., for $\mathbf{r} \in V_1$, the surface sources cancel the incident field, resulting in zero fields, as illustrated in **Figure 2.1**. Similarly, in the representations (2.7) and (2.8), the surface sources produce the fields $(\mathbf{E}_1, \mathbf{H}_1)$ in V_1 , whereas the fields vanish when the observation point lies in V_2 , as illustrated in **Figure 2.2**. This behavior is a direct consequence of the extinction theorem, and the corresponding integral representations in which the fields are identically zero are referred to as null field integral representations.

Accordingly, for an observation point located in V_j , the integral representations valid for i^{th} region can be summarized in the operator form given below:

$$\hat{\delta}_{i,j} \mathbf{E}_i = \hat{\delta}_{i,2} \mathbf{E}^{inc} + \frac{j}{\omega\epsilon_i} \mathcal{L}_i \{\mathbf{J}_i\} - \mathcal{K}_i \{\mathbf{M}_i\}, \quad \mathbf{r} \in V_j \quad (2.19)$$

$$\hat{\delta}_{i,j} \mathbf{H}_i = \hat{\delta}_{i,2} \mathbf{H}^{inc} + \frac{j}{\omega\mu_i} \mathcal{L}_i \{\mathbf{M}_i\} + \mathcal{K}_i \{\mathbf{J}_i\}, \quad \mathbf{r} \in V_j \quad (2.20)$$

where $\hat{\delta}_{i,j}$ is the Kronecker delta function. The subindices i and j can be 1 or 2 and denotes the internal and external medium of the scatterer. The integral operators $\mathcal{L}_i$ and $\mathcal{K}_i$ for the region of interest are as in [92] and can also be readily identified from the above integral representations. In the representations (2.19) and (2.20), the

observation point is subsequently brought to the scatterer boundary, where the boundary conditions are enforced and solutions are sought using well-established methods in the literature, such as the Method of Moments (MoM), under a source excitation expressed in three dimensions through three-dimensional integral equations or suitable combinations thereof, including the PMCHWT, N-Müller, or JM-Combined Field Integral Equation (JM-CFIE) formulations reported in [107], together with appropriate choices of test and basis functions.

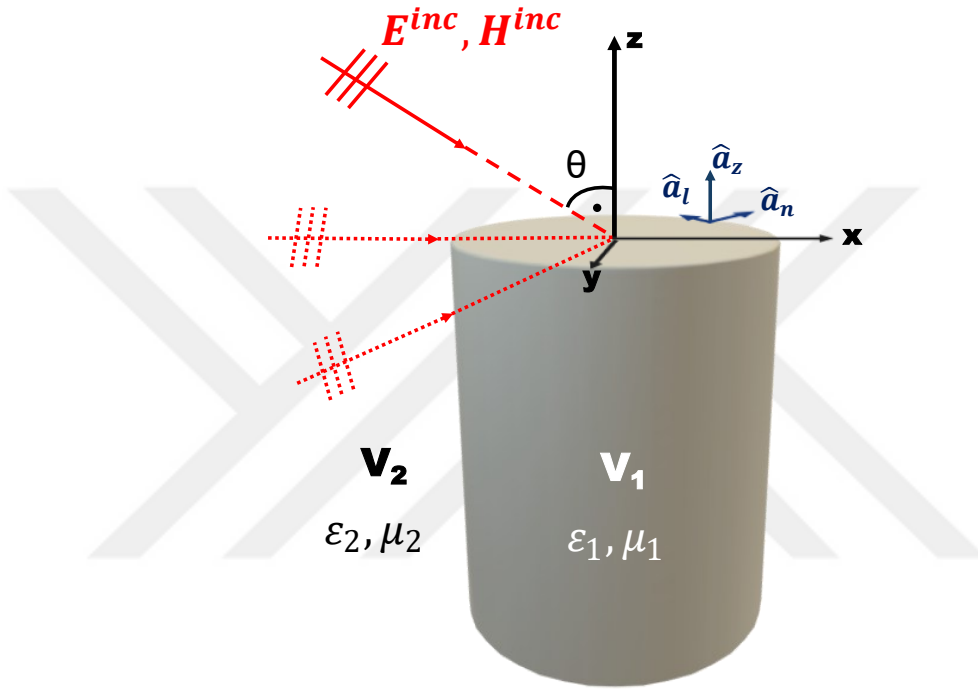


Figure 2.3: A cylindrical scatterer excited with oblique incident plane wave.

If such a scattering problem involves one or more cylindrical scatterers that are long with respect to the wavelength (10 times or longer than the wavelength), it is also possible to search for solutions for surface current densities by assuming uniformity of the scattering problem in the z -direction and infinite length of the cylindrical scatterers. This approach leads to the reduction of the representations (2.1), (2.2), (2.7), and (2.8) to 2.5D integral representations by accounting for the z dependence of the current densities and fields in the form $\begin{bmatrix} \mathbf{J}(\mathbf{r}) \\ \mathbf{M}(\mathbf{r}) \end{bmatrix} = \begin{bmatrix} \mathbf{K}(\boldsymbol{\rho}) \\ \mathbf{K}_m(\boldsymbol{\rho}) \end{bmatrix} e^{j\beta_0 z}$, $\mathbf{F}(\mathbf{r}) = \tilde{\mathbf{F}}(\boldsymbol{\rho}) e^{j\beta_0 z}$ and by considering that, for an infinitely long cylindrical scatterer, the surface integral can be decomposed as $\int_S ds' = \int_C dl' \int_{-\infty}^{\infty} dz'$, thereby resulting in a 2.5D formulation.

With these four reduced integral equations for $i=1,2$, it is possible to solve such a scattering problem under a single wave excitation oblique to the z axis. If the incident fields can be expressed as a superposition of multiple oblique incident fields, then these equations, which are not yet in k_z domain, must be evaluated in k_z domain together with the incident field. By multiplying both sides of the equations by $e^{jk_z z}$ and integrating over z , the set of equations can be easily moved to k_z domain by applying Fourier transform. Considering that the excitation and the right-hand side of the equations are sampled by the impulse train for different β_s , which are elements of the continuous k_z spectrum, the Dirac delta function $\delta(k_z - \beta_0)$ appearing on both sides of the equations is replaced by $\delta(k_z - \beta_s)$ and the equations written in k_z domain can be generalized by performing an infinite sum over s . This infinite sum becomes a continuous convolution integral of the form $\int_{-\infty}^{\infty} A(\beta)\delta(k_z - \beta)d\beta = A(k_z) * \delta(k_z)$ in the limit case with infinitesimal step β_s .

Equalizing the parts convolved with $\delta(k_z)$ on both sides of the resulting equation or considering that $A(k_z) * \delta(k_z) = A(k_z)$, we arrive at the following 2.5D integral representations:

$$\begin{aligned} \tilde{\mathbf{E}}_2 = \tilde{\mathbf{E}}^{\text{inc}} + \int_C \left[-j\omega\mu_2 \mathbf{K}_2 - \mathbf{K}_{m2} \times (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \right. \\ \left. - \frac{j}{\omega\varepsilon_2} \{ \mathbf{K}_2 \cdot (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \} (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \right] \tilde{G}_2 dl', \boldsymbol{\rho} \in V_2 \end{aligned} \quad (2.21)$$

$$\begin{aligned} \tilde{\mathbf{H}}_2 = \tilde{\mathbf{H}}^{\text{inc}} + \int_C \left[-j\omega\varepsilon_2 \mathbf{K}_{m2} + \mathbf{K}_2 \times (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \right. \\ \left. - \frac{j}{\omega\mu_2} \{ \mathbf{K}_{m2} \cdot (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \} (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \right] \tilde{G}_2 dl', \boldsymbol{\rho} \in V_2 \end{aligned} \quad (2.22)$$

$$\begin{aligned} \tilde{\mathbf{E}}_1 = \int_C \left[-j\omega\mu_1 \mathbf{K}_1 - \mathbf{K}_{m1} \times (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \right. \\ \left. - \frac{j}{\omega\varepsilon_1} \{ \mathbf{K}_1 \cdot (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \} (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \right] \tilde{G}_1 dl', \boldsymbol{\rho} \in V_1 \end{aligned} \quad (2.23)$$

$$\begin{aligned} \tilde{\mathbf{H}}_1 = \int_C \left[-j\omega\varepsilon_1 \mathbf{K}_{m1} + \mathbf{K}_1 \times (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \right. \\ \left. - \frac{j}{\omega\mu_1} \{ \mathbf{K}_{m1} \cdot (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \} (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \right] \tilde{G}_1 dl', \boldsymbol{\rho} \in V_1 \end{aligned} \quad (2.24)$$

where $\tilde{G}_i$ is 2-2.5D Green's function defined as $\tilde{G}_i = -\frac{j}{4}H_0^{(2)}(k_{\rho,i}|\boldsymbol{\rho} - \boldsymbol{\rho}'|)$ with $k_{\rho,i} = \sqrt{k_i^2 - k_z^2}$. In addition, $k_z = k_2 \cos \theta$ and θ is the spherical polar angle of the wave vector with respect to the z-axis, as illustrated in **Figure 2.3**. Applying the same procedure to the representations (2.15) – (2.18) yields their 2.5D counterparts are obtained as follows:

$$\begin{aligned} \tilde{\mathbf{E}}_2 = \tilde{\mathbf{E}}^{\text{inc}} + \int_C \left[-j\omega\mu_2 \mathbf{K}_2 - \mathbf{K}_{m2} \times (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \right. \\ \left. + \frac{j}{\omega\varepsilon_2} \left\{ \frac{\partial}{\partial l'} (\hat{\mathbf{a}}_{l'} \cdot \mathbf{K}_2) + jk_z (\hat{\mathbf{a}}_z \cdot \mathbf{K}_2) \right\} (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \right] \tilde{G}_2 dl', \boldsymbol{\rho} \in V_2 \end{aligned} \quad (2.25)$$

$$\begin{aligned} \tilde{\mathbf{H}}_2 = \tilde{\mathbf{H}}^{\text{inc}} + \int_C \left[-j\omega\varepsilon_2 \mathbf{K}_{m2} + \mathbf{K}_2 \times (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \right. \\ \left. + \frac{j}{\omega\mu_2} \left\{ \frac{\partial}{\partial l'} (\hat{\mathbf{a}}_{l'} \cdot \mathbf{K}_{m2}) + jk_z (\hat{\mathbf{a}}_z \cdot \mathbf{K}_{m2}) \right\} (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \right] \tilde{G}_2 dl', \boldsymbol{\rho} \in V_2 \end{aligned} \quad (2.26)$$

$$\begin{aligned} \tilde{\mathbf{E}}_1 = \int_C \left[-j\omega\mu_1 \mathbf{K}_1 - \mathbf{K}_{m1} \times (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \right. \\ \left. + \frac{j}{\omega\varepsilon_1} \left\{ \frac{\partial}{\partial l'} (\hat{\mathbf{a}}_{l'} \cdot \mathbf{K}_1) + jk_z (\hat{\mathbf{a}}_z \cdot \mathbf{K}_1) \right\} (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \right] \tilde{G}_1 dl', \boldsymbol{\rho} \in V_1 \end{aligned} \quad (2.27)$$

$$\begin{aligned} \tilde{\mathbf{H}}_1 = \int_C \left[-j\omega\varepsilon_1 \mathbf{K}_{m1} + \mathbf{K}_1 \times (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \right. \\ \left. + \frac{j}{\omega\mu_1} \left\{ \frac{\partial}{\partial l'} (\hat{\mathbf{a}}_{l'} \cdot \mathbf{K}_{m1}) + jk_z (\hat{\mathbf{a}}_z \cdot \mathbf{K}_{m1}) \right\} (\nabla'_t - \hat{\mathbf{a}}_z jk_z) \right] \tilde{G}_1 dl', \boldsymbol{\rho} \in V_1 \end{aligned} \quad (2.28)$$

Equations (2.21) - (2.24) and (2.25) - (2.28) are 2.5D coupled integral representations valid in the continuous k_z spectrum for an infinitely long cylindrical scatterer excited by an arbitrary source in three dimensions. These representations in the k_z spectrum can be concisely expressed in terms of operators to include the null-field formulation, as given below:

$$\begin{aligned} \hat{\delta}_{i,j} \tilde{\mathbf{E}}_i(k_z) = \hat{\delta}_{i,2} \tilde{\mathbf{E}}^{\text{inc}}(k_z) + \frac{j}{\omega\varepsilon_i} \tilde{\mathcal{L}}_i \{ \mathbf{K}_i(k_z), k_z \} \\ - \tilde{\mathcal{K}}_i \{ \mathbf{K}_{mi}(k_z), k_z \}, \quad \boldsymbol{\rho} \in V_j \end{aligned} \quad (2.29)$$

$$\begin{aligned} \hat{\delta}_{i,j} \tilde{\mathbf{H}}_i(k_z) = \hat{\delta}_{i,2} \tilde{\mathbf{H}}^{\text{inc}}(k_z) + \frac{j}{\omega\mu_i} \tilde{\mathcal{L}}_i \{ \mathbf{K}_{mi}(k_z), k_z \} \\ + \tilde{\mathcal{K}}_i \{ \mathbf{K}_i(k_z), k_z \}, \quad \boldsymbol{\rho} \in V_j \end{aligned} \quad (2.30)$$

We consider an orthogonal curvilinear coordinate system defined by $\hat{\mathbf{a}}_n = \hat{\mathbf{a}}_l \times \hat{\mathbf{a}}_z$ as shown in **Figure 2.3**. By formulating the problem in this coordinate system and expressing the integral representations through the decomposition of the currents and fields into their l and z components, the following formulations derived from Equations (2.21) - (2.24) are obtained:

$$\hat{\delta}_{i,j} \tilde{E}_{iz} = \hat{\delta}_{i,2} \tilde{E}_z^{\text{inc}} + \int_C \left[\frac{-jk_{\rho,i}^2}{\omega \varepsilon_i} K_{iz} + K_{mil} \frac{\partial}{\partial n'} - \frac{k_z}{\omega \varepsilon_i} K_{il} \frac{\partial}{\partial l'} \right] \tilde{G}_i dl', \boldsymbol{\rho} \in V_j \quad (2.31)$$

$$\hat{\delta}_{i,j} \tilde{H}_{iz} = \hat{\delta}_{i,2} \tilde{H}_z^{\text{inc}} + \int_C \left[\frac{-jk_{\rho,i}^2}{\omega \mu_i} K_{miz} - K_{il} \frac{\partial}{\partial n'} - \frac{k_z}{\omega \mu_i} K_{mil} \frac{\partial}{\partial l'} \right] \tilde{G}_i dl', \boldsymbol{\rho} \in V_j \quad (2.32)$$

$$\begin{aligned} \hat{\delta}_{i,j} \tilde{E}_{il} = \hat{\delta}_{i,2} \tilde{E}_l^{\text{inc}} + \int_C & \left[\frac{k_z}{\omega \varepsilon_i} K_{iz} \frac{\partial}{\partial l} + j k_z K_{mil} (\hat{\mathbf{a}}_l \cdot \hat{\mathbf{a}}_{n'}) \right. \\ & \left. - \frac{j}{\omega \varepsilon_i} K_{il} \left\{ \frac{\partial^2}{\partial n \partial n'} + k_z^2 (\hat{\mathbf{a}}_n \cdot \hat{\mathbf{a}}_{n'}) \right\} + K_{miz} \frac{\partial}{\partial n} \right] \tilde{G}_i dl', \boldsymbol{\rho} \in V_j \end{aligned} \quad (2.33)$$

$$\begin{aligned} \hat{\delta}_{i,j} \tilde{H}_{il} = \hat{\delta}_{i,2} \tilde{H}_l^{\text{inc}} + \int_C & \left[\frac{k_z}{\omega \mu_i} K_{miz} \frac{\partial}{\partial l} - j k_z K_{il} (\hat{\mathbf{a}}_l \cdot \hat{\mathbf{a}}_{n'}) \right. \\ & \left. - \frac{j}{\omega \mu_i} K_{mil} \left\{ \frac{\partial^2}{\partial n \partial n'} + k_z^2 (\hat{\mathbf{a}}_n \cdot \hat{\mathbf{a}}_{n'}) \right\} - K_{iz} \frac{\partial}{\partial n} \right] \tilde{G}_i dl', \boldsymbol{\rho} \in V_j \end{aligned} \quad (2.34)$$

Here, $\tilde{E}_{il}$ and $\tilde{H}_{il}$ are defined as follows:

$$\tilde{E}_{il} = \frac{j}{k_{\rho,i}^2} \left(k_z \frac{\partial \tilde{E}_{iz}}{\partial l} + \omega \mu_i \frac{\partial \tilde{H}_{iz}}{\partial n} \right) \quad (2.35)$$

$$\tilde{H}_{il} = \frac{j}{k_{\rho,i}^2} \left(k_z \frac{\partial \tilde{H}_{iz}}{\partial l} - \omega \varepsilon_i \frac{\partial \tilde{E}_{iz}}{\partial n} \right) \quad (2.36)$$

Substitution of these expressions into (2.31) and (2.32) results in (2.33) and (2.34). In addition, if it is preferred to look for the fields themselves and their derivatives as unknowns in the equation instead of the currents, then the following relationship between currents and fields should be used:

$$K_{mil} = (-1)^i \tilde{E}_{iz}; K_{miz} = (-1)^{i+1} \tilde{E}_{il} \quad (2.37)$$

$$K_{il} = (-1)^{i+1} \tilde{H}_{iz}; K_{iz} = (-1)^i \tilde{H}_{il} \quad (2.38)$$

and it is well known that the tangential components of the electric and magnetic fields are continuous across the boundary C , i.e.,

$$\tilde{E}_{2z} = \tilde{E}_{1z} ; \tilde{E}_{2l} = \tilde{E}_{1l} \quad (2.39)$$

$$\tilde{H}_{2z} = \tilde{H}_{1z} ; \tilde{H}_{2l} = \tilde{H}_{1l} \quad (2.40)$$

By setting $i=1$ in (2.35) and (2.36), using (2.37) - (2.40) and then substituting them into (2.31) and (2.32), the following integral equations with unknowns in terms of interior fields are obtained:

$$\begin{aligned} \delta_{2,j} \tilde{E}_{2z} = \tilde{E}_z^{\text{inc}} + \int_C \left[-\frac{k_{\rho,2}^2 \varepsilon_1}{k_{\rho,1}^2 \varepsilon_2} \frac{\partial \tilde{E}_{1z}}{\partial n'} + \tilde{E}_{1z} \frac{\partial}{\partial n'} \right. \\ \left. + \frac{(k_1^2 - k_2^2)}{\omega \varepsilon_2 k_{\rho,1}^2} k_z \tilde{H}_{1z} \frac{\partial}{\partial l'} \right] \tilde{G}_2 dl' ; \quad \mathbf{p} \in V_j \end{aligned} \quad (2.41)$$

$$\begin{aligned} \delta_{2,j} \tilde{H}_{2z} = \tilde{H}_z^{\text{inc}} + \int_C \left[-\frac{k_{\rho,2}^2 \mu_1}{k_{\rho,1}^2 \mu_2} \frac{\partial \tilde{H}_{1z}}{\partial n'} + \tilde{H}_{1z} \frac{\partial}{\partial n'} \right. \\ \left. - \frac{(k_1^2 - k_2^2)}{\omega \mu_2 k_{\rho,1}^2} k_z \tilde{E}_{1z} \frac{\partial}{\partial l'} \right] \tilde{G}_2 dl' ; \quad \mathbf{p} \in V_j \end{aligned} \quad (2.42)$$

$$\delta_{1,j} \tilde{E}_{1z} = \int_C \left[\frac{\partial \tilde{E}_{1z}}{\partial n'} - \tilde{E}_{1z} \frac{\partial}{\partial n'} \right] \tilde{G}_1 dl' ; \quad \mathbf{p} \in V_j \quad (2.43)$$

$$\delta_{1,j} \tilde{H}_{1z} = \int_C \left[\frac{\partial \tilde{H}_{1z}}{\partial n'} - \tilde{H}_{1z} \frac{\partial}{\partial n'} \right] \tilde{G}_1 dl' ; \quad \mathbf{p} \in V_j \quad (2.44)$$

Equations (2.41) - (2.44) are equivalent to (2.31) - (2.32), the only difference being that the unknowns are expressed in terms of the fields and their derivatives instead of the currents. Applying (2.35) and (2.36) to these equations readily yields the formulations for E_l and H_l .

Equations (2.31) - (2.34) and (2.41) - (2.44) provide a convenient formulation for the analysis of electromagnetic scattering from a cylinder under single or multiple obliquely incident TM or TE wave excitations. The inverse Fourier transform of the obtained field quantities over k_z leads to field quantities in three dimensions. It is important to follow an integration path like the one given in [36,108] when performing

this transformation. It is also possible to search for solutions with the TMM detailed in [36] by moving from the null field versions of equations (2.41) and (2.42) in which the z components of the fields are unknowns and expanding the fields in these equations into cylindrical harmonics.

In this case, the isolated T matrix of the scatterer consists of four block matrices containing TM, TE and coupling data, in contrast to the non-coupled case (perpendicular incidence). For a scattering problem involving photonic crystal structures modeled as an array of N cylindrical scatterers (2.41) and (2.42) are formulated with the source points located on the boundary of each scatterer, while the scattered field contributions from the boundaries of all scatterers are evaluated at observation points lying in the interior region of the scatterer of interest.

The null-field versions of equations (2.41) – (2.44), which also allow the T-matrix formulation, are written as follows:

$$\tilde{E}_z^{\text{inc}} = \int_C \left[\frac{k_{\rho,2}^2 \varepsilon_1}{k_{\rho,1}^2 \varepsilon_2} \frac{\partial \tilde{E}_{1z}}{\partial n'} - \tilde{E}_{1z} \frac{\partial}{\partial n'} - \frac{(k_1^2 - k_2^2)}{\omega \varepsilon_2 k_{\rho,1}^2} k_z \tilde{H}_{1z} \frac{\partial}{\partial l'} \right] \tilde{G}_2 dl' ; \quad \mathbf{\rho} \in V_2 \quad (2.45)$$

$$\tilde{H}_z^{\text{inc}} = \int_C \left[\frac{k_{\rho,2}^2 \mu_1}{k_{\rho,1}^2 \mu_2} \frac{\partial \tilde{H}_{1z}}{\partial n'} - \tilde{H}_{1z} \frac{\partial}{\partial n'} + \frac{(k_1^2 - k_2^2)}{\omega \mu_2 k_{\rho,1}^2} k_z \tilde{E}_{1z} \frac{\partial}{\partial l'} \right] \tilde{G}_2 dl' ; \quad \mathbf{\rho} \in V_2 \quad (2.46)$$

$$0 = \int_C \left[\frac{\partial \tilde{E}_{1z}}{\partial n'} - \tilde{E}_{1z} \frac{\partial}{\partial n'} \right] \tilde{G}_1 dl' ; \quad \mathbf{\rho} \in V_1 \quad (2.47)$$

$$0 = \int_C \left[\frac{\partial \tilde{H}_{1z}}{\partial n'} - \tilde{H}_{1z} \frac{\partial}{\partial n'} \right] \tilde{G}_1 dl' ; \quad \mathbf{\rho} \in V_1 \quad (2.48)$$

If the observation point is brought to the scatterer boundary and the integral equations are to be evaluated on the boundary, (2.41) – (2.44) take the form of the following boundary integral equations with using (2.39) and (2.40):

$$\tilde{E}_z^{\text{inc}} = \frac{1}{2} \tilde{E}_{1z} + \int_C \left[\frac{k_{\rho,2}^2 \varepsilon_1}{k_{\rho,1}^2 \varepsilon_2} \frac{\partial \tilde{E}_{1z}}{\partial n'} - \tilde{E}_{1z} \frac{\partial}{\partial n'} \right] \tilde{G}_2 dl' \quad (2.49)$$

$$\begin{aligned}
& -\frac{(k_1^2 - k_2^2)}{\omega \varepsilon_2 k_{\rho,1}^2} k_z \tilde{H}_{1z} \frac{\partial}{\partial l'} \Big] \tilde{G}_2 dl' ; \boldsymbol{\rho} \in \mathcal{C} \\
\tilde{H}_z^{\text{inc}} = & \frac{1}{2} \tilde{H}_{1z} + \int_C \left[\frac{k_{\rho,2}^2 \mu_1}{k_{\rho,1}^2 \mu_2} \frac{\partial \tilde{H}_{1z}}{\partial n'} - \tilde{H}_{1z} \frac{\partial}{\partial n'} \right. \\
& \left. + \frac{(k_1^2 - k_2^2)}{\omega \mu_2 k_{\rho,1}^2} k_z \tilde{E}_{1z} \frac{\partial}{\partial l'} \right] \tilde{G}_2 dl' ; \boldsymbol{\rho} \in \mathcal{C}
\end{aligned} \tag{2.50}$$

$$0 = -\frac{1}{2} \tilde{E}_{1z} + \int_C \left[\frac{\partial \tilde{E}_{1z}}{\partial n'} - \tilde{E}_{1z} \frac{\partial}{\partial n'} \right] \tilde{G}_1 dl' ; \boldsymbol{\rho} \in \mathcal{C} \tag{2.51}$$

$$0 = -\frac{1}{2} \tilde{H}_{1z} + \int_C \left[\frac{\partial \tilde{H}_{1z}}{\partial n'} - \tilde{H}_{1z} \frac{\partial}{\partial n'} \right] \tilde{G}_1 dl' ; \boldsymbol{\rho} \in \mathcal{C} \tag{2.52}$$

Using the relations $\mathbf{K} = \mathbf{K}_2 = -\mathbf{K}_1$ and $\mathbf{K}_m = \mathbf{K}_{m2} = -\mathbf{K}_{m1}$, the boundary integral equations obtained from the integral representations in which the unknowns are sought in terms of the currents, and which are equivalent to (2.49) – (2.52), are given below in terms of both the l- and z-components:

$$\hat{\delta}_{i,2} \tilde{E}_z^{\text{inc}} = \frac{(-1)^i}{2} K_{ml} + \int_C \left[\frac{jk_{\rho,i}^2}{\omega \varepsilon_i} K_z - K_{ml} \frac{\partial}{\partial n'} + \frac{k_z}{\omega \varepsilon_i} K_l \frac{\partial}{\partial l'} \right] \tilde{G}_i dl', \boldsymbol{\rho} \in \mathcal{C} \tag{2.53}$$

$$\hat{\delta}_{i,2} \tilde{H}_z^{\text{inc}} = \frac{(-1)^{i+1}}{2} K_l + \int_C \left[\frac{jk_{\rho,i}^2}{\omega \mu_i} K_{mz} + K_l \frac{\partial}{\partial n'} + \frac{k_z}{\omega \mu_i} K_{ml} \frac{\partial}{\partial l'} \right] \tilde{G}_i dl', \boldsymbol{\rho} \in \mathcal{C} \tag{2.54}$$

$$\begin{aligned}
\hat{\delta}_{i,2} \tilde{E}_l^{\text{inc}} = & \frac{(-1)^{i+1}}{2} K_{mz} + \int_C \left[-\frac{k_z}{\omega \varepsilon_i} K_{iz} \frac{\partial}{\partial l} - j k_z K_{mil} (\hat{\mathbf{a}}_l \cdot \hat{\mathbf{a}}_{n'}) \right. \\
& \left. + \frac{j}{\omega \varepsilon_i} K_{il} \left\{ \frac{\partial^2}{\partial n \partial n'} + k_z^2 (\hat{\mathbf{a}}_n \cdot \hat{\mathbf{a}}_{n'}) \right\} - K_{miz} \frac{\partial}{\partial n} \right] \tilde{G}_i dl', \boldsymbol{\rho} \in \mathcal{C}
\end{aligned} \tag{2.55}$$

$$\begin{aligned}
\hat{\delta}_{i,2} \tilde{H}_l^{\text{inc}} = & \frac{(-1)^i}{2} K_z + \int_C \left[-\frac{k_z}{\omega \mu_i} K_{mz} \frac{\partial}{\partial l} + j k_z K_l (\hat{\mathbf{a}}_l \cdot \hat{\mathbf{a}}_{n'}) \right. \\
& \left. + \frac{j}{\omega \mu_i} K_{ml} \left\{ \frac{\partial^2}{\partial n \partial n'} + k_z^2 (\hat{\mathbf{a}}_n \cdot \hat{\mathbf{a}}_{n'}) \right\} + K_z \frac{\partial}{\partial n} \right] \tilde{G}_i dl', \boldsymbol{\rho} \in \mathcal{C}
\end{aligned} \tag{2.56}$$

Consequently, equations (2.49) – (2.52) and (2.53) - (2.56) are boundary integral equations and can be solved by choosing the suitable test and basis functions. By choosing these functions as complex exponentials, it is possible to obtain super algebraically convergent and numerically stable solutions [93]. However, following

this solution path requires removing the singularities of the kernels of the equations and knowing the Fourier coefficients of the decomposed term analytically. On the other hand, equations (2.45) - (2.48) are null field integral equations and can be solved by the TMM, by expansion of the fields into cylindrical harmonics and by using the addition theorem.

These equations can be solved separately for the TM and TE components of the excitation in the decoupling case. These equations decouple in the following three cases as given in [96]:

- i. Normal (Perpendicular) incidence ($\theta = \frac{\pi}{2}, k_z = 0$)
- ii. E_{1z} and E_{2z} (or H_{1z} and H_{2z}) are independent of l , i.e, either they are zero (scatterer is a perfect electric conductor or perfect magnetic conductor), or the cylinder surface is a plane perpendicular to the plane of incidence, in which case the boundary value problem becomes a problem of Fresnel reflection and refraction coefficients for a plane wave incident on a planar interface separating two different planes.
- iii. Isorefractive / Anti-isorefractive scatterer i.e, the relative refractive index of the cylindrical scatterer $n = \pm 1$. This case states that the scatterer is either isorefractive ($n = +1$) or anti-isorefractive to the surrounding free space medium. In the most general case, the expressions $k_1^2 = k_2^2$ and $Z_1 \neq Z_2$ hold for any iso/anti-isorefractive scatterer in any surrounding medium.

In these three cases, the last term in equations (2.31), (2.32), (2.45), (2.46), (2.49) and (2.50) disappear and the unknowns can be found by solving the EFIE_z for the TM component of the excitation and the MFIE_z for the TE component.

2.1. Analysis of the Kernels of the Boundary Integral Equation for Singularity Extraction

As a result of previous section, the four integral equations (2.53) - (2.56) to be solved for the unknown surface currents at the scatterer boundary, involving four different kernel operators $\mathcal{S}, \mathcal{D}, \mathcal{H}$ and $\mathcal{R}$, are given below:

$$\hat{\delta}_{i,2} E_z^{inc}(k_z) = \frac{(-1)^i}{2} K_{ml}(k_z) + \frac{j}{\omega \epsilon_i} \mathcal{S}_i\{\mathbf{K}(k_z), k_z\} - \mathcal{D}_i\{\mathbf{K}_m(k_z), k_z\} \quad (2.57)$$

$$\hat{\delta}_{i,2} H_z^{inc}(k_z) = \frac{(-1)^{i+1}}{2} K_l(k_z) + \frac{j}{\omega \mu_i} \mathcal{S}_i\{\mathbf{K}_m(k_z), k_z\} + \mathcal{D}_i\{\mathbf{K}(k_z), k_z\} \quad (2.58)$$

$$\hat{\delta}_{i,2} E_l^{inc}(k_z) = \frac{(-1)^{i+1}}{2} K_{mz}(k_z) + \frac{j}{\omega \epsilon_i} \mathcal{H}_i\{\mathbf{K}(k_z), k_z\} - \mathcal{R}_i\{\mathbf{K}_m(k_z), k_z\} \quad (2.59)$$

$$\hat{\delta}_{i,2} H_l^{inc}(k_z) = \frac{(-1)^i}{2} K_z(k_z) + \frac{j}{\omega \mu_i} \mathcal{H}_i\{\mathbf{K}_m(k_z), k_z\} + \mathcal{R}_i\{\mathbf{K}(k_z), k_z\} \quad (2.60)$$

Here the operators $\mathcal{S}, \mathcal{D}, \mathcal{H}$ and $\mathcal{R}$ are given by:

$$\begin{aligned} \mathcal{S}_i\{\mathbf{X}(\boldsymbol{\rho}, k_z), k_z\} = PV \int_C \left[k_{\rho,l}^2 X_z(\boldsymbol{\rho}', k_z) \right. \\ \left. + j k_z X_l(\boldsymbol{\rho}', k_z) \frac{\partial}{\partial l'} \right] G_i(\boldsymbol{\rho}, \boldsymbol{\rho}', k_z); \boldsymbol{\rho} \in C \end{aligned} \quad (2.61)$$

$$\mathcal{D}_i\{\mathbf{X}(\boldsymbol{\rho}, k_z), k_z\} = \int_C \left[X_l(\boldsymbol{\rho}', k_z) \frac{\partial}{\partial n'} \right] G_i(\boldsymbol{\rho}, \boldsymbol{\rho}', k_z) dl'; \boldsymbol{\rho} \in C \quad (2.62)$$

$$\begin{aligned} \mathcal{H}_i\{\mathbf{X}(\boldsymbol{\rho}, k_z), k_z\} = -PV \int_C \left[j k_z X_z(\boldsymbol{\rho}', k_z) \frac{\partial}{\partial l} \right] G_i(\boldsymbol{\rho}, \boldsymbol{\rho}', k_z) dl' \\ + FP \int_C X_l(\boldsymbol{\rho}', k_z) \left[\frac{\partial^2}{\partial n \partial n'} + j k_z^2 (\hat{\mathbf{a}}_n \cdot \hat{\mathbf{a}}_{n'}) \right] G_i(\boldsymbol{\rho}, \boldsymbol{\rho}', k_z) dl'; \boldsymbol{\rho} \in C \end{aligned} \quad (2.63)$$

$$\begin{aligned} \mathcal{R}_i\{\mathbf{X}(\boldsymbol{\rho}, k_z), k_z\} = \int_C \left[X_z(\boldsymbol{\rho}', k_z) \frac{\partial}{\partial n} \right. \\ \left. + j k_z (\hat{\mathbf{a}}_l \cdot \hat{\mathbf{a}}_{n'}) X_l(\boldsymbol{\rho}', k_z) \right] G_i(\boldsymbol{\rho}, \boldsymbol{\rho}', k_z) dl'; \boldsymbol{\rho} \in C \end{aligned} \quad (2.64)$$

The 2.5D integral equations in the above operator form are written in the k_z domain, and for notational convenience, the tilde ($\tilde{E}, \tilde{H}$) notation has been omitted.

(2.57) and (2.58) are EFIE_z and MFIE_z respectively, while (2.59) and (2.60) are EFIE_l and MFIE_l respectively. The kernels of equations arising in the four equations that are the z and l components of the electric and magnetic field integral equations are given in the table below:

Table 2.1: Equation kernels and the integral equation it appears in.

Kernel	IE where the kernel appears
$G_i ; \frac{\partial}{\partial n'} G_i ; \frac{\partial}{\partial l'} G_i$	EFIE _z , MFIE _z
$\frac{\partial}{\partial n} G_i ; \frac{\partial}{\partial l} G_i$	EFIE _l , MFIE _l
$\frac{\partial}{\partial n} \frac{\partial}{\partial n'} G_i ; \frac{\partial}{\partial l} \frac{\partial}{\partial l'} G_i$	EFIE _l , MFIE _l
$\frac{\partial}{\partial l} \frac{\partial}{\partial n'} G_i ; \frac{\partial}{\partial n} \frac{\partial}{\partial l'} G_i$	EFIE _l , MFIE _l

In solving boundary integral equations (2.57) – (2.60), it is important and a matter of investigation whether these kernels of equations are singular as the observation point goes to the source point. By choosing the test and basis functions as complex exponentials, the integrals can be directly discretized numerically with the 2D FFT if there is no singularity when observing at the same location as the source point. However, if there is a singularity, the singular term is decomposed by adding and subtracting a term that removes the singularity in the kernel. The Fourier coefficients of this decomposed term must be known analytically. The smooth part can then be discretized by the 2D FFT, and the part with analytically known Fourier coefficients can be added to the result to compute the whole matrix equation system. Due to the logarithmic singularity of the Green's function, this term can be expressed as a Clausen function to remove the singularity. Clausen functions are related to the polylogarithm function and if complex exponentials are given as arguments in the polylogarithm function, the series expression of this function can be treated as a Fourier series expansion. The Clausen function of order n and its relationship with the polylogarithm function is as follows:

$$Cl_n(x) = \begin{cases} \frac{j}{2} [Li_n(e^{-jx}) - Li_n(e^{jx})] & ; \quad n \text{ is even} \\ \frac{1}{2} [Li_n(e^{-jx}) + Li_n(e^{jx})] & ; \quad n \text{ is odd} \end{cases} \quad (2.65)$$

$$Li_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \quad (2.66)$$

The Clausen function of the same behavior to be used to extract the logarithmic singularity is of order 1 and is expressed as follows:

$$Cl_1(x) = -\ln \left(\left| 2 \sin \left(\frac{x}{2} \right) \right| \right) \quad (2.67)$$

By substituting $\phi - \phi'$ for x , the function and its expression in polylogarithms, where ϕ and ϕ' are the azimuthal angles for the observation and source points respectively, can be written as follows:

$$Cl_1(\phi - \phi') = -\ln \left(\left| 2 \sin \left(\frac{\phi - \phi'}{2} \right) \right| \right) \quad (2.68)$$

$$Cl_1(\phi - \phi') = \frac{1}{2} [Li_1(e^{-j(\phi - \phi')}) + Li_1(e^{j(\phi - \phi')})] \quad (2.69)$$

Substituting the series expression in (2.66) into (2.69) and adjusting the index of one of the sums from $-\infty$ to -1 , the following Fourier series expansion is obtained:

$$Cl_1(\phi - \phi') = \sum_{k=-\infty}^{\infty} a_k e^{jk(\phi - \phi')} \quad (2.70)$$

$$a_k = \frac{1}{2|k|}, k \in \mathbb{Z} \setminus \{0\}; \quad a_0 = 0 \quad (2.71)$$

In addition, by formally differentiating both sides of the (2.70), the first and second order derivatives of $Cl_1(\phi - \phi')$ can be expressed as follows:

$$\frac{\partial}{\partial \phi} Cl_1(\phi - \phi') = \sum_{k=1}^{\infty} b_k e^{jk(\phi - \phi')} \quad (2.72)$$

$$b_k = \frac{j}{2} \text{sgn}(k) \quad (2.73)$$

$$\frac{\partial^2}{\partial \phi \partial \phi'} C l_1(\phi - \phi') = \sum_{k=1}^{\infty} c_k e^{jk(\phi - \phi')} \quad (2.74)$$

$$c_k = \frac{1}{2} |k| \quad (2.75)$$

for all $k \in \mathbb{Z}$ where $\text{sgn}(k)$ denotes the sign function, with $\text{sgn}(0) = 0$ by convention. Considering that the singularity of the Hankel function of the second kind, which is the homogeneous medium Green's function of the present problem, is logarithmic, the first and second order derivatives of the Green's function with respect to l and l' also imply a singularity of the logarithm of the order of the corresponding derivative. The Clausen functions in (2.72) and (2.74) should be used to eliminate these singularities. It is helpful to consider the following relationships to understand this:

$$\frac{\partial}{\partial l'} G_i = \frac{1}{l(\phi')} \frac{\partial}{\partial \phi'} G_i; \quad \frac{\partial}{\partial l} G_i = \frac{1}{l(\phi)} \frac{\partial}{\partial \phi} G_i; \quad (2.76)$$

$$l(\phi) = \sqrt{x^{(r)}(\phi)^2 + y^{(r)}(\phi)^2}; \quad l(\phi') = \sqrt{x'^{(r)}(\phi')^2 + y'^{(r)}(\phi')^2} \quad (2.77)$$

where $x^{(r)}(\phi)$ and $y^{(r)}(\phi)$ are the derivatives of x and y with respect to ϕ , respectively. Similarly, $x'^{(r)}(\phi')$ and $y'^{(r)}(\phi')$ are the derivatives of x' and y' with respect to ϕ' , respectively. The prime notation here denotes differentiation and should not be confused with the superscript notation used for source coordinates. To avoid confusion, the prime symbol denoting first-order derivatives is enclosed in parentheses when used in expressions involving superscripted source coordinates.

The kernels with first order derivatives on n and n' do not have singularity. On the other hand, first order tangential derivatives of the Green's function with respect to the l and l' directions give rise to strongly singular kernels that are well defined in the Cauchy principal value (PV) sense. Accordingly, the corresponding singular integrals in (2.61) and (2.63) are understood in the PV sense. In contrast, second-order normal derivatives with respect to n and n' lead to hypersingular kernels exhibiting a $\frac{1}{R^2}$ type singularity, which requires interpretation in the sense of the Hadamard finite part (FP) integral [92]. Consequently, the hypersingular contribution in (2.63) is evaluated as a

Hadamard finite-part integral. Moreover, the derivatives $\frac{\partial}{\partial l} \frac{\partial}{\partial n'}$ and $\frac{\partial}{\partial n} \frac{\partial}{\partial l'}$ appear in equations as differences of each other. The difference of these derivatives of the kernel does not lead to a singular kernel. It is also noted that second-order tangential derivatives with respect to l and l' lead to hypersingular kernels. To recognize these, it is useful to consider the following relations:

$$\left(\frac{\partial}{\partial n} \frac{\partial}{\partial n'} + \frac{\partial}{\partial l} \frac{\partial}{\partial l'} \right) G_i = k_{\rho,i}^2 (\hat{\mathbf{a}}_n \cdot \hat{\mathbf{a}}_{n'}) G_i \quad (2.78)$$

$$\left(\frac{\partial}{\partial n} \frac{\partial}{\partial l'} - \frac{\partial}{\partial l} \frac{\partial}{\partial n'} \right) G_i = k_{\rho,i}^2 (\hat{\mathbf{a}}_n \cdot \hat{\mathbf{a}}_{l'}) G_i = -k_{\rho,i}^2 (\hat{\mathbf{a}}_l \cdot \hat{\mathbf{a}}_{n'}) G_i \quad (2.79)$$

$$\lim_{\rho \rightarrow \rho'} k_{\rho,i}^2 (\hat{\mathbf{a}}_n \cdot \hat{\mathbf{a}}_{l'}) G_i = \lim_{\rho \rightarrow \rho'} -k_{\rho,i}^2 (\hat{\mathbf{a}}_l \cdot \hat{\mathbf{a}}_{n'}) G_i = 0 \quad (2.80)$$

In addition, we rearrange (2.78) to understand the singularity of the $\frac{\partial}{\partial l} \frac{\partial}{\partial l'}$ kernel:

$$\left(\frac{\partial}{\partial l} \frac{\partial}{\partial l'} \right) G_i = \left(k_{\rho,i}^2 (\hat{\mathbf{a}}_n \cdot \hat{\mathbf{a}}_{n'}) - \frac{\partial}{\partial n} \frac{\partial}{\partial n'} \right) G_i \quad (2.81)$$

$$\lim_{\rho \rightarrow \rho'} (\hat{\mathbf{a}}_n \cdot \hat{\mathbf{a}}_{n'}) = 1 \quad (2.82)$$

If the kernels of these equations can be expressed as a sum of a singular and a regular function, and if the Fourier coefficients of the singular part are analytically known, or alternatively, if the singular part can be written as the product of a function whose Fourier coefficients are analytically known and a regular function, then a system of equations similar to that proposed in [83],[93],[94] can be constructed, yielding super algebraically convergent solutions.

Accordingly, it is essential that the kernel K be represented in the form of a regular part R and a singular part S , as indicated in (2.83) and (2.84), with the Fourier series coefficients of S being available in closed form:

$$K(\boldsymbol{\rho}, \boldsymbol{\rho}') = R(\boldsymbol{\rho}, \boldsymbol{\rho}') + S(\boldsymbol{\rho}, \boldsymbol{\rho}') \quad (2.83)$$

$$K(\boldsymbol{\rho}, \boldsymbol{\rho}') = R_1(\boldsymbol{\rho}, \boldsymbol{\rho}') + R_2(\boldsymbol{\rho}, \boldsymbol{\rho}') S(\boldsymbol{\rho}, \boldsymbol{\rho}') \quad (2.84)$$

Achieving such representations for all kernels encountered in the integral equations is possible by identifying and extracting the functions responsible for the singular behavior of the kernels. The functions and limiting forms required to extract the singularities of each first order derivative singular kernel are provided below:

$$\lim_{\rho \rightarrow \rho'} \left[G_i(\rho, \rho') - \frac{1}{2\pi} C l_1(\phi - \phi') \right] = - \left[\frac{j}{4} + \frac{\gamma}{2\pi} + \frac{1}{2\pi} \ln \left(\frac{k_{\rho,i} l(\phi)}{2} \right) \right] \quad (2.85)$$

$$\lim_{\rho \rightarrow \rho'} \left[G_i(\rho, \rho') - \frac{1}{2\pi} C l_1(\phi - \phi') J_0(k_{\rho,i} |\rho - \rho'|) \right] = - \left[\frac{j}{4} + \frac{\gamma}{2\pi} + \frac{1}{2\pi} \ln \left(\frac{k_{\rho,i} l(\phi)}{2} \right) \right] \quad (2.86)$$

$$\lim_{\rho \rightarrow \rho'} \left[\frac{\partial}{\partial l'} G_i(\rho, \rho') - \frac{1}{2\pi l(\phi')} \frac{\partial}{\partial \phi'} C l_1(\phi - \phi') \right] = - \frac{1}{4\pi} \left[\frac{x'(\phi) x''(\phi) + y'(\phi) y''(\phi)}{l(\phi)^3} \right] \quad (2.87)$$

$$\lim_{\rho \rightarrow \rho'} \left[\frac{\partial}{\partial l'} G_i(\rho, \rho') - \frac{1}{2\pi l(\phi')} \frac{\partial}{\partial \phi'} \left(C l_1(\phi - \phi') J_0(k_{\rho,i} |\rho - \rho'|) \right) \right] = - \frac{1}{4\pi} \left[\frac{x^{(i)}(\phi) x^{(ii)}(\phi) + y^{(i)}(\phi) y^{(ii)}(\phi)}{l(\phi)^3} \right] \quad (2.88)$$

$$\lim_{\rho \rightarrow \rho'} \left[\frac{\partial}{\partial l} G_i(\rho, \rho') - \frac{1}{2\pi l(\phi)} \frac{\partial}{\partial \phi} \left(C l_1(\phi - \phi') J_0(k_{\rho,i} |\rho - \rho'|) \right) \right] = - \frac{1}{4\pi} \left[\frac{x^{(i)}(\phi) x^{(ii)}(\phi) + y^{(i)}(\phi) y^{(ii)}(\phi)}{l(\phi)^3} \right] \quad (2.89)$$

$$\lim_{\rho \rightarrow \rho'} \left[\frac{\partial}{\partial l} G_i(\rho, \rho') - \frac{1}{2\pi l(\phi)} \frac{\partial}{\partial \phi} C l_1(\phi - \phi') \right] = - \frac{1}{4\pi} \left[\frac{x^{(i)}(\phi) x^{(ii)}(\phi) + y^{(i)}(\phi) y^{(ii)}(\phi)}{l(\phi)^3} \right] \quad (2.90)$$

where γ is Euler-Mascheroni constant. For the second-order normal derivative with respect to n and n' , the corresponding singular functions and limiting forms are given below:

$$\lim_{\boldsymbol{\rho} \rightarrow \boldsymbol{\rho}'} \left[\frac{\partial^2}{\partial n \partial n'} G_i(\boldsymbol{\rho}, \boldsymbol{\rho}') + \frac{1}{2\pi l(\phi)l(\phi')} \frac{\partial^2}{\partial \phi \partial \phi'} \left(C l_1(\phi - \phi') J_0(k_{\rho,i} |\boldsymbol{\rho} - \boldsymbol{\rho}'|) \right) \right] = A^{lim}(\phi) \quad (2.91)$$

$$\lim_{\boldsymbol{\rho} \rightarrow \boldsymbol{\rho}'} \left[\frac{\partial^2}{\partial n \partial n'} G_i(\boldsymbol{\rho}, \boldsymbol{\rho}') + \frac{1}{2\pi l(\phi)l(\phi')} \frac{\partial^2}{\partial \phi \partial \phi'} (C l_1(\phi - \phi')) \right] = A^{lim}(\phi) \quad (2.92)$$

$$\lim_{\boldsymbol{\rho} \rightarrow \boldsymbol{\rho}'} \left[\frac{\partial^2}{\partial n \partial n'} G_i(\boldsymbol{\rho}, \boldsymbol{\rho}') + \frac{1}{2\pi l(\phi)l(\phi')} \frac{\partial^2}{\partial \phi \partial \phi'} (C l_1(\phi - \phi')) - \frac{k_{\rho,i}}{2\pi} C l_1(\phi - \phi') \frac{J_1(k_{\rho,i} |\boldsymbol{\rho} - \boldsymbol{\rho}'|)}{|\boldsymbol{\rho} - \boldsymbol{\rho}'|} (\hat{\mathbf{a}}_n \cdot \hat{\mathbf{a}}_{n'}) \right] = A^{lim}(\phi) + B^{lim}(\phi) \quad (2.93)$$

where

$$A^{lim}(\phi) = \frac{1}{4\pi} \left[\frac{\left(x^{(n)}(\phi) y^{(n)}(\phi) - x^{(n)}(\phi) y^{(n)}(\phi) \right)^2}{l(\phi)^6} \right] \quad (2.94)$$

$$B^{lim}(\phi) = \left[\left(-\frac{k_{\rho,i}^2}{2} \right) \left(\frac{j}{4} + \frac{\gamma}{2\pi} + \frac{1}{2\pi} \ln \left(\frac{k_{\rho,i} l(\phi)}{2} \right) - \frac{1}{4\pi} \right) + \left[\left(\frac{1}{12} \right) \left(\frac{1}{2\pi} \right) \left(\frac{-T^{(1,1)}(\phi) + 2T^{(1,3)}(\phi) - 3T^{(2,2)}(\phi)}{l(\phi)^4} \right) \right] \right] \quad (2.95)$$

$$T^{(i,j)}(\phi) = x^{(i)}(\phi) x^{(j)}(\phi) + y^{(i)}(\phi) y^{(j)}(\phi) \quad (2.96)$$

In the limits given above, $x^{(i)}$, $x^{(j)}$, $y^{(i)}$, and $y^{(j)}$ denote the i^{th} and j^{th} order derivatives of x and y with respect to ϕ , respectively. Owing to the additional terms introduced into the equation kernels in Equations (2.85) – (2.93), the corresponding limits exist as the observation point p approaches the source point q , i.e., as $p \rightarrow q$ and $\boldsymbol{\rho} \rightarrow \boldsymbol{\rho}'$.

To recover the original equation kernel, the same term initially added to regularize the kernel must be subtracted afterward. The Fourier series coefficients of this subtracted term can be evaluated by convolving the numerically computed Fourier coefficients of the accompanying function, if present, with the analytically available Fourier coefficients given in (2.71), (2.73), and (2.75), despite the presence of singular behavior.

The limits that must be considered when evaluating the integrals of kernels that do not contain singularities as the observation point approaches the source point are given below:

$$\lim_{\rho \rightarrow \rho'} \left[\frac{\partial}{\partial n'} G_i(\rho, \rho') \right] = -\frac{1}{4\pi} \left[\frac{x^{(')}(\phi) y^{(')}(\phi) - x^{(')}(\phi) y^{(')}(\phi)}{l(\phi)^3} \right] \quad (2.97)$$

$$\lim_{\rho \rightarrow \rho'} \left[\frac{\partial}{\partial n} G_i(\rho, \rho') \right] = -\frac{1}{4\pi} \left[\frac{x^{(')}(\phi) y^{(')}(\phi) - x^{(')}(\phi) y^{(')}(\phi)}{l(\phi)^3} \right] \quad (2.98)$$

$$\begin{aligned} \lim_{\rho \rightarrow \rho'} \left[\left(\frac{\partial^2}{\partial n \partial l'} - \frac{\partial^2}{\partial l \partial n'} \right) G_i(\rho, \rho') \right] &= \lim_{\rho \rightarrow \rho'} [k_{\rho,i}^2 (\hat{\mathbf{a}}_n \cdot \hat{\mathbf{a}}_{l'}) G_i(\rho, \rho')] \\ &= \lim_{\rho \rightarrow \rho'} [-k_{\rho,i}^2 (\hat{\mathbf{a}}_l \cdot \hat{\mathbf{a}}_{n'}) G_i(\rho, \rho')] = 0 \end{aligned} \quad (2.99)$$

The definitions of the unit vectors that appear in the integral equations in the form of scalar products are given below:

$$\hat{\mathbf{a}}_n = \hat{\mathbf{a}}_x \frac{y^{(')}(\phi)}{l(\phi)} - \hat{\mathbf{a}}_y \frac{x^{(')}(\phi)}{l(\phi)}; \quad \hat{\mathbf{a}}_{n'} = \hat{\mathbf{a}}_x \frac{y^{(')}(\phi')}{l(\phi')} - \hat{\mathbf{a}}_y \frac{x^{(')}(\phi')}{l(\phi')} \quad (2.100)$$

$$\hat{\mathbf{a}}_l = \hat{\mathbf{a}}_x \frac{x^{(')}(\phi)}{l(\phi)} + \hat{\mathbf{a}}_y \frac{y^{(')}(\phi)}{l(\phi)}; \quad \hat{\mathbf{a}}_{l'} = \hat{\mathbf{a}}_x \frac{x^{(')}(\phi')}{l(\phi')} + \hat{\mathbf{a}}_y \frac{y^{(')}(\phi')}{l(\phi')} \quad (2.101)$$

Using the above definitions, limits and the procedure described, the boundary integral equations are discretized by applying the method presented in the next section, and the corresponding solutions are sought.

2.2. Solution of the 2.5D Boundary Integral Equations

The component in one direction of the 2.5D boundary integral equations of a vector electromagnetic problem involving a cylindrical scatterer excited by a source from its exterior region usually appears in the following form:

$$\frac{1}{2}U(\boldsymbol{\rho}) + \int_C U(\boldsymbol{\rho}') K(\boldsymbol{\rho}, \boldsymbol{\rho}') dl' = S^{\text{inc}}(\boldsymbol{\rho}) \quad (2.102)$$

where U is the unknown total field, K is the kernel of the integral equation and S is the excitation source. The integral equation (2.102) valid at the boundary can be expressed in terms of the source and observation ϕ and ϕ' angle respectively as follows:

$$\frac{1}{2}U(\boldsymbol{\rho}(\phi)) + \int_C U(\boldsymbol{\rho}'(\phi')) K(\boldsymbol{\rho}(\phi), \boldsymbol{\rho}'(\phi')) l(\phi') d\phi' = S^{\text{inc}}(\boldsymbol{\rho}(\phi)) \quad (2.103)$$

where $l(\phi')$ represents the arclength function as previously defined in (2.77) and is given by:

$$l(\phi') = \sqrt{\left(\frac{dx'(\phi')}{d\phi'}\right)^2 + \left(\frac{dy'(\phi')}{d\phi'}\right)^2} \quad (2.104)$$

Equation (2.103) can be discretized by expanding the unknown into a series with a basis function and testing both sides of the equation on the contour with a test function. Since the basis and test functions will be chosen as complex exponentials for the solution, let us perform a smart scaling of equation (2.103) to work with the Fourier coefficients of the unknown field outside the integral due to it depending on the observation point and write all terms dependent on ϕ and ϕ' :

$$\frac{1}{2} \frac{U(\phi)}{l(\phi)} + \frac{1}{l(\phi)} \int_C U(\phi') K(\phi, \phi') l(\phi') d\phi' = \frac{S^{\text{inc}}(\phi)}{l(\phi)} \quad (2.105)$$

The resulting equation (2.105) allows for discretization into a matrix system of equations consisting of Fourier series coefficients, with the basis and test functions chosen as complex exponentials. The series expansion of $U(\phi)$ in terms of complex exponential basis function (Fourier series expansion) is as follows:

$$U(\phi') = \sum_{m=-\infty}^{\infty} u_m e^{jm\phi'} \quad (2.106)$$

Substituting (2.106) in (2.105):

$$\frac{1}{2} \frac{U(\phi)}{l(\phi)} + \frac{1}{l(\phi)} \int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} u_m e^{jm\phi'} K(\phi, \phi') l(\phi') d\phi' = \frac{S^{\text{inc}}(\phi)}{l(\phi)} \quad (2.107)$$

Now we can test the equation (2.107) on the contour with $e^{-jn\phi}$ on both sides:

$$\begin{aligned} \int_{-\pi}^{\pi} \left\{ \frac{1}{2} \frac{U(\phi)}{l(\phi)} + \frac{1}{l(\phi)} \int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} u_m e^{jm\phi'} K(\phi, \phi') l(\phi') d\phi' \right\} e^{-jn\phi} l(\phi) d\phi \\ = \int_{-\pi}^{\pi} \frac{S^{\text{inc}}(\phi)}{l(\phi)} e^{-jn\phi} l(\phi) d\phi \end{aligned} \quad (2.108)$$

The smart scaling done in obtaining (2.105) allows equation (2.108) to be discretized by Fourier integrals. Accordingly, if (2.108) is rearranged,

$$\begin{aligned} \frac{1}{2} \int_{-\pi}^{\pi} U(\phi) e^{-jn\phi} d\phi + \sum_{m=-\infty}^{\infty} u_m \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K(\phi, \phi') l(\phi') e^{jm\phi'} e^{-jn\phi} d\phi' d\phi \\ = \int_{-\pi}^{\pi} S^{\text{inc}}(\phi) e^{-jn\phi} d\phi \end{aligned} \quad (2.109)$$

Multiplying both sides of (2.109) by $\frac{1}{2\pi}$ and adjusting the coefficients to convert the integral on the left-hand side into a Fourier integral gives the following equation:

$$\begin{aligned}
& \frac{1}{2} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} U(\phi) e^{-jn\phi} d\phi \right) \\
& + \sum_{m=-\infty}^{\infty} u_m 2\pi \left(\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K(\phi, \phi') l(\phi') e^{jm\phi'} e^{-jn\phi} d\phi' d\phi \right) \quad (2.110) \\
& = \frac{1}{2\pi} \int_{-\pi}^{\pi} S^{\text{inc}}(\phi) e^{-jn\phi} d\phi
\end{aligned}$$

and note that:

$$U(\phi) = \sum_{n=-\infty}^{\infty} u_n e^{jn\phi} \quad (2.111)$$

$$S^{\text{inc}}(\phi) = \sum_{n=-\infty}^{\infty} s_n e^{jn\phi} \quad (2.112)$$

$$u_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} U(\phi) e^{-jm\phi} d\phi \quad (2.113)$$

$$s_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} S^{\text{inc}}(\phi) e^{-jn\phi} d\phi \quad (2.114)$$

$$k_{m,n} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (K(\phi, \phi') l(\phi')) e^{-jm\phi'} e^{-jn\phi} d\phi' d\phi \quad (2.115)$$

Using expressions (2.111) to (2.115), (2.110) can be written in discrete form in terms of Fourier series coefficients:

$$\frac{1}{2} u_m + 2\pi \sum_{n=-\infty}^{\infty} u_m k_{-m,n} = s_n \quad (2.116)$$

The $k_{-m,n}$ in (2.116) is the Fourier series coefficients calculated with (2.115) flipped along the column (from left to right). These Fourier series coefficients in (2.116) can be easily obtained numerically by FFT operation. This finally leads to the following matrix-equation system:

$$\left(\frac{1}{2}\bar{\mathbf{I}} + \bar{\mathbf{Z}}\right)\mathbf{f} = \mathbf{b} \quad (2.117)$$

Here $\mathbf{f}$ is the unknown coefficient vector of the total field at the boundary and its elements are u_m and coefficient vector of incident field $\mathbf{b}$'s elements are also s_n . The elements of $\bar{\mathbf{Z}}$ are $2\pi k_{-m,n}$. A system of the form (2.117) constitutes the matrix blocks of a scattering problem to be solved by boundary integral equations. For a perfectly electrically conducting scatterer, the $\frac{1}{2}\bar{\mathbf{I}}$ term in (2.117) vanishes and forms the solution system by substituting the appropriate kernel into the system.

2.3. Combination of Interior and Exterior Field Representations

The super-algebraic convergent solutions of the boundary integral equations [93] are important for validating the field solutions obtained using the FMM accelerated and well-conditioned TMM proposed in this thesis study. However, the solution of a set of fundamental integral equations may not be unique at certain frequencies. Although the occurrence of non-uniqueness is rare, and thus typically does not cause significant errors in numerical computations, it remains a potential source of instability. Nevertheless, when the frequency coincides with a resonance, the solution can become seriously inaccurate. At these resonance frequencies, the solution becomes contaminated with eigen solutions that radiate a nonphysical field within the interior region [92]. Although these eigen solutions occur only at discrete frequency points, the accuracy of the numerical solution degrades not only at these points but also within a finite range surrounding them.

Therefore, as a means of eliminating resonance solutions, the integral equations formulated in the interior and exterior regions are combined with a specific weighting factor. The unknowns are then determined by solving a combined set of electric field integral equations and magnetic field integral equations.

Let us revisit the four integral equations (2.57) – (2.60) formulated on the boundary of the scatterer involving four different kernel operators $\mathcal{S}, \mathcal{D}, \mathcal{H}$ and $\mathcal{R}$. Based on these equations, the electric and magnetic field integral equation operators are defined as follows:

$$EFIE_i^z\{\boldsymbol{\rho}, \boldsymbol{\rho}', \mathbf{K}(k_z), \mathbf{K}_m(k_z), k_z\} = \frac{(-1)^i}{2} K_{ml}(k_z) + \frac{j}{\omega \varepsilon_i} \mathcal{S}_i\{\mathbf{K}(k_z), k_z\} - \mathcal{D}_i\{\mathbf{K}_m(k_z), k_z\} \quad (2.118)$$

$$MFIE_i^z\{\boldsymbol{\rho}, \boldsymbol{\rho}', \mathbf{K}(k_z), \mathbf{K}_m(k_z), k_z\} = \frac{(-1)^{i+1}}{2} K_l(k_z) + \frac{j}{\omega \mu_i} \mathcal{S}_i\{\mathbf{K}_m(k_z), k_z\} + \mathcal{D}_i\{\mathbf{K}(k_z), k_z\} \quad (2.119)$$

$$EFIE_i^l\{\boldsymbol{\rho}, \boldsymbol{\rho}', \mathbf{K}(k_z), \mathbf{K}_m(k_z), k_z\} = \frac{(-1)^{i+1}}{2} K_{mz}(k_z) + \frac{j}{\omega \varepsilon_i} \mathcal{H}_i\{\mathbf{K}(k_z), k_z\} - \mathcal{R}_i\{\mathbf{K}_m(k_z), k_z\} \quad (2.120)$$

$$MFIE_i^l\{\boldsymbol{\rho}, \boldsymbol{\rho}', \mathbf{K}(k_z), \mathbf{K}_m(k_z), k_z\} = \frac{(-1)^i}{2} K_z(k_z) + \frac{j}{\omega \mu_i} \mathcal{H}_i\{\mathbf{K}_m(k_z), k_z\} + \mathcal{R}_i\{\mathbf{K}(k_z), k_z\} \quad (2.121)$$

In addition, the integral equations for i^{th} region can be expressed in vector form as follows:

$$\mathbf{EFIE}_i = \begin{bmatrix} EFIE_i^l \\ EFIE_i^z \end{bmatrix}; \quad \mathbf{MFIE}_i = \begin{bmatrix} MFIE_i^l \\ MFIE_i^z \end{bmatrix} \quad (2.122)$$

$\mathbf{EFIE}_i$ and $\mathbf{MFIE}_i$ in (2.124) can be combined in different ways. Among these combinations, the classical combined-region formulation in which region 1 denotes the interior and region 2 the exterior domain, is given as follows:

$$p_1 \mathbf{EFIE}_1 + p_2 \mathbf{EFIE}_2 \quad (2.123)$$

$$q_1 \mathbf{MFIE}_1 + q_2 \mathbf{MFIE}_2 \quad (2.124)$$

If in equations (2.123) and (2.124), the parameters are chosen as $p_1 = -p_2 = 1$ and $q_1 = -q_2 = 1$, the resulting equation corresponds to the PMCHWT formulation. On the other hand, if $p_1 = 1$, $p_2 = -\frac{\varepsilon_1}{\varepsilon_2}$, $q_1 = 1$, $q_2 = -\frac{\mu_1}{\mu_2}$ [92], the resulting equation becomes the Müller integral equation. In this type of formulation, the singularities are eliminated, the kernels are smooth [97].

Combined region formulations may also be obtained from field-combined equations. Among these, a JMCFIE formulation is one of the widely used approaches and is expressed as follows [109]:

$$c_1 \mathbf{JCFIE}_1 + c_2 \mathbf{JCFIE}_2 \quad (2.125)$$

$$d_1 \mathbf{MCFIE}_1 + d_2 \mathbf{MCFIE}_2 \quad (2.126)$$

where c_i and d_i are non-zero implementation coefficients. In (2.125), $\mathbf{JCFIE}$ is referred to as the electric current combined field integral equation and in (2.126) $\mathbf{MCFIE}$ called as magnetic current combined field integral equations, similarly. These equations are defined as follows:

$$\mathbf{JCFIE}_i = a_i \mathbf{EFIE}_i + b_i \hat{\mathbf{a}}_{n,i} \times \mathbf{MFIE}_i \quad (2.127)$$

$$\mathbf{MCFIE}_i = -a_i \hat{\mathbf{a}}_{n,i} \times \mathbf{EFIE}_i + b_i \mathbf{MFIE}_i \quad (2.128)$$

where $\hat{\mathbf{a}}_n = \hat{\mathbf{a}}_{n,2} = -\hat{\mathbf{a}}_{n,1}$, $a_i = \alpha$ and $b_i = (1 - \alpha)\eta_i$ for $0 < \alpha < 1$ and $\eta_i = \sqrt{\frac{\mu_i}{\epsilon_i}}$.

As discussed above, the most prominent and widely recognized combined integral equations are considered in this study. These formulations are analyzed to ensure the reliable validation of the proposed well-conditioned and accelerated T-matrix approach and to illustrate the methodology adopted for the 2.5D scattering problem. Therefore, alternative combinations will not be elaborated, and the focus will be limited to classical combined-region integral equations.

2.4. 2D Representation of 3D Excitation Sources: Electric Dipole Fields for 2.5D Cylindrical Scattering Problems

In the previous sections, boundary integral equations were derived from the most general three-dimensional integral representations to solve the problem of an obliquely incident wave exciting a cylindrical scatterer at various incidence angles (i.e., for different k_z values), and the evaluation of these equations was discussed. This section focuses on the treatment of the three-dimensional source excitation term appearing on the right-hand side of these equations (e.g., an electric dipole source) and on the procedure for reconstructing the full three-dimensional fields from the fields obtained by solving the integral equations.

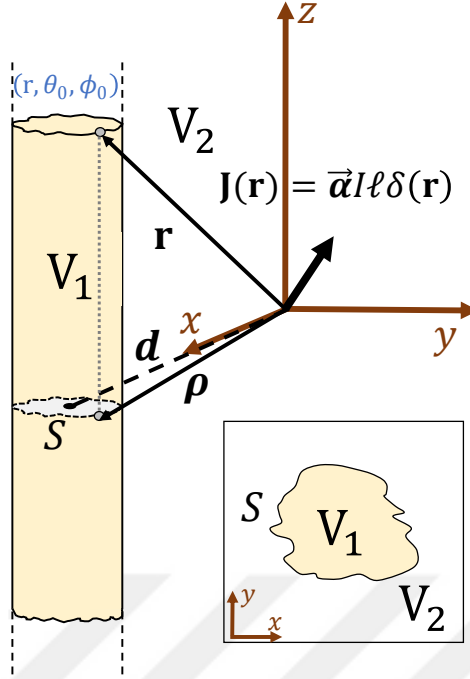


Figure 2.4: The scattering scenario of a dipole field from an infinite cylinder of arbitrary cross-section.

Consider a point source or a Hertzian dipole oriented in the $\vec{\alpha}$ direction and with electrical current density $\mathbf{J} = \vec{\alpha} l \delta(\mathbf{r})$ in a homogeneous medium, as shown in **Figure 2.4**. The electric and magnetic fields radiated by this source can be derived using either the vector potential method or the dyadic Green's function approach. Accordingly, the field produced by a Hertzian dipole is given as follows:

$$\mathbf{E}(\mathbf{r}) = j\omega\mu \left(\bar{\mathbf{I}} + \frac{\nabla\nabla}{k^2} \right) \cdot \vec{\alpha} l \frac{e^{-jkr}}{4\pi r} \quad (2.129)$$

$$\mathbf{H}(\mathbf{r}) = \nabla \times \vec{\alpha} l \frac{e^{-jkr}}{4\pi r} \quad (2.130)$$

where l is the current moment and k is the wavenumber of the homogeneous medium. With $\vec{\alpha}$ defined as $\vec{\alpha} = \hat{\mathbf{a}}_x \alpha_x + \hat{\mathbf{a}}_y \alpha_y + \hat{\mathbf{a}}_z \alpha_z$, the electric and magnetic field components, as derived from equations (2.129) and (2.130), are given below:

$$\mathbf{E}(\mathbf{r}) = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \frac{j\omega\mu ll}{4\pi k^2} \begin{bmatrix} \alpha_x \left(k^2 + \frac{\partial^2}{\partial x^2} \right) + \alpha_y \left(\frac{\partial^2}{\partial x \partial y} \right) + \alpha_z \left(\frac{\partial^2}{\partial x \partial z} \right) \\ \alpha_x \left(\frac{\partial^2}{\partial y \partial x} \right) + \alpha_y \left(k^2 + \frac{\partial^2}{\partial y^2} \right) + \alpha_z \left(\frac{\partial^2}{\partial y \partial z} \right) \\ \alpha_x \left(\frac{\partial^2}{\partial z \partial x} \right) + \alpha_y \left(\frac{\partial^2}{\partial z \partial y} \right) + \alpha_z \left(k^2 + \frac{\partial^2}{\partial z^2} \right) \end{bmatrix} \frac{e^{-jkr}}{4\pi r} \quad (2.131)$$

$$\mathbf{H}(\mathbf{r}) = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \frac{Il}{4\pi} \begin{bmatrix} \alpha_z \left(\frac{\partial}{\partial y} \right) - \alpha_y \left(\frac{\partial}{\partial z} \right) \\ \alpha_x \left(\frac{\partial}{\partial z} \right) - \alpha_z \left(\frac{\partial}{\partial x} \right) \\ \alpha_y \left(\frac{\partial}{\partial x} \right) - \alpha_x \left(\frac{\partial}{\partial y} \right) \end{bmatrix} \frac{e^{-jkr}}{4\pi r} \quad (2.132)$$

The expressions given in (2.131) and (2.132) can be written as a superposition of line sources with different k_z components, from which the TM and TE components with respect to the z -direction can be obtained.

(2.131) and (2.132) fields can be represented with path integrals of two-dimensional fields, which are functions of the complex k_z varying in the continuous spectrum and whose z -dependence is a plane wave. For electric or magnetic dipole source excitation oriented in any direction, the cylindrical scattering problem reduces to a two-dimensional coupled problem with excitation of the electric or magnetic line sources, whose radial spectral dependence is cylindrical and obliquely incident with varying complex angles.

The exposition of the integration procedure for a point source gains great importance since it forms the path for integration of the 2-D fields in the longitudinal plane wave (k_z) spectrum. Therefore, let us begin by writing the integral equivalent of the term

$\frac{e^{-jkr}}{4\pi r}$ with substituting $|\mathbf{r} - \mathbf{r}'|$ instead of r for source located at $\mathbf{r}'$:

$$\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_z \left(-\frac{j}{4} H_0^{(2)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}'|) \right) e^{-jk_z(z-z')} \quad (2.133)$$

Furthermore, integrand in (2.133) represents the 2.5-D Green's function $g_0(k_z, \boldsymbol{\rho}, \mathbf{r}') = G(\boldsymbol{\rho}, \mathbf{r}') e^{jk_z z'}$ involves the Green's function of homogeneous 2-D space. The infinite integral in (2.133) can be calculated as a saddle point integration with a given observation position angle θ_0 stated to be the best in [108] as it minimizes

oscillations and provides exponential convergence over the integration path ($\tilde{\Gamma}_\theta$ in **Figure 2.5**) defined on the complex plane.

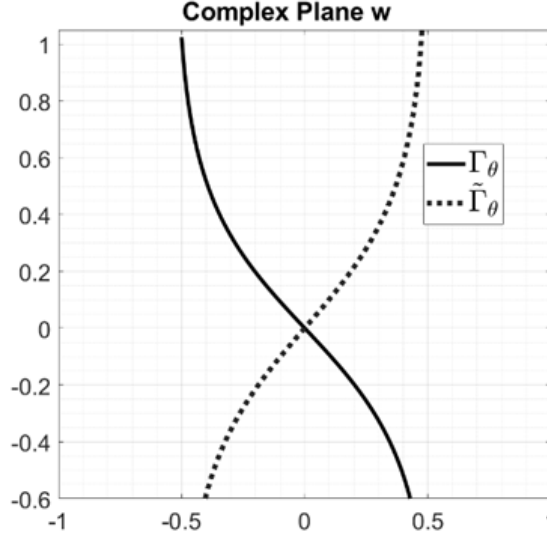


Figure 2.5: The integration contour Γ_θ and $\tilde{\Gamma}_\theta$ on complex plane $w=\theta/\pi$ that are exploited during integration of the longitudinal plane wave spectra.

The source point residing at the origin of the 3-D space defines $g_0(r) = g_0(\mathbf{r}, \mathbf{r}')|_{\mathbf{r}'=(0,0,0)}$ and $\theta \in \mathbb{C}$ as the elevation angle from the transverse plane, three back-to-back changes of variables assuming $k_\rho(\theta) = k_0 \cos \theta$, $k_z(\theta) = k_0 \sin \theta$ leads us firstly to $\theta = \theta_0 + \theta' + j\theta''(\theta') \in \mathbb{C}$, then, to $\theta' \in \mathbb{R}$, and finally, to $t \in [-1,1] \in \mathbb{R}$ with rewriting (2.133) [108]:

$$\frac{e^{-jk|\mathbf{r}|}}{4\pi|\mathbf{r}|} = \frac{1}{2\pi} \int_{\tilde{\Gamma}_\theta} F_0(\theta) d\theta = \frac{1}{2\pi} \int_{-\frac{\pi}{2}+\alpha}^{\frac{\pi}{2}-\alpha} d\theta' c(\theta') F_0(\theta) \quad (2.134)$$

$$\frac{e^{-jk|\mathbf{r}|}}{4\pi|\mathbf{r}|} = \frac{\pi - 2\alpha}{4\pi} \int_{-1}^1 dt c(\theta'(t)) F_0(\theta(t)) \quad (2.135)$$

where

$$F_0(\theta) = \frac{-jk}{4} \cos \theta H_0^{(2)}(k_\rho(\theta)\rho) e^{-jk_z(\theta)z} \quad (2.136)$$

$$c(\theta') = 1 + j \left[\frac{\cos(\theta')}{(\cos \alpha - \sin \theta')} + \tan(\alpha - \theta') \right] \quad (2.137)$$

$$\theta''(\theta') = \ln \left(\frac{\cos(\alpha) - \sin(\theta')}{\cos(\alpha - \theta')} \right) \quad (2.138)$$

$$\theta'(t) = \left(\frac{\pi}{2} - \alpha \right) t \quad (2.139)$$

Steps in (2.135), for the approximate integral calculation, lead to sampling at a finite number of points on the appropriately selected integration path $\tilde{\Gamma}_\theta$ that suppresses the logarithmic singularity of $F_0(\theta)$. In (2.135), the rightmost integral approximation can be sampled at Gauss-Legendre integration points of number N and calculated after their weighted sum with the mentioned variables being computed in the order of their dependencies as in [108].

As an example, a vertical electric dipole located at ρ_s can be considered. In this case, $\alpha_x = \alpha_y = 0, \alpha_z = 1$, and the resulting E_z (TM^z) and H_z (TE^z) field components are expressed as follows:

$$E_z^{inc} = \frac{j\omega\mu l}{4\pi k^2} \left(k^2 + \frac{\partial^2}{\partial z^2} \right) \frac{e^{jk|\mathbf{r}-\mathbf{r}_s|}}{4\pi|\mathbf{r}-\mathbf{r}_s|} \quad (2.140)$$

$$H_z^{inc} = 0 \quad (2.141)$$

Using equation (2.133), (2.140) can be expressed as follows:

$$E_z^{inc} = \frac{j\omega\mu l}{4\pi k^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_z (k_\rho^2) \left(-\frac{j}{4} H_0^{(2)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_s|) e^{-jk_z(z-z_s)} \right) \quad (2.142)$$

where $k^2 - k_z^2 = k_\rho^2$. (2.142) can be numerically integrated using Gauss-Legendre integration points as in (2.135) along the integration path in **Figure 2.5**. However, when solving the scattering problem, this integral is not evaluated in advance. Instead, the integrand is placed on the right-hand side of the 2.5D integral equations, and solutions are obtained for each value of θ (or $k_z(\theta)$) along the chosen integration path. The final three-dimensional solution is then constructed by performing an inverse Fourier transform, that is, by integrating the resulting solutions over k_z . This means:

$$EFIE_i^z\{\boldsymbol{\rho}, \boldsymbol{\rho}', \mathbf{K}(k_z), \mathbf{K}_m(k_z), k_z\} = \hat{\delta}_{i,2} E_z^{inc}(\boldsymbol{\rho}, k_z) \quad (2.143)$$

$$MFIE_i^z\{\boldsymbol{\rho}, \boldsymbol{\rho}', \mathbf{K}(k_z), \mathbf{K}_m(k_z), k_z\} = \hat{\delta}_{i,2} H_z^{inc}(\boldsymbol{\rho}, k_z) \quad (2.144)$$

$$EFIE_i^l\{\boldsymbol{\rho}, \boldsymbol{\rho}', \mathbf{K}(k_z), \mathbf{K}_m(k_z), k_z\} = \hat{\delta}_{i,2} E_l^{inc}(\boldsymbol{\rho}, k_z) \quad (2.145)$$

$$MFIE_i^l\{\boldsymbol{\rho}, \boldsymbol{\rho}', \mathbf{K}(k_z), \mathbf{K}_m(k_z), k_z\} = \hat{\delta}_{i,2} H_l^{inc}(\boldsymbol{\rho}, k_z) \quad (2.146)$$

where

$$\tilde{E}_z^{inc}(\boldsymbol{\rho}, k_z) = \left(\frac{j\omega\mu l}{4\pi k^2} k_\rho^2 \right) \left(-\frac{j}{4} H_0^{(2)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_s|) \right) \quad (2.147)$$

$$E_z^{inc} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_z \tilde{E}_z^{inc}(\boldsymbol{\rho}, k_z) e^{-jk_z z} \quad (2.148)$$

and the following inverse Fourier transform relationship applies:

$$\mathbf{F}(\mathbf{r}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_z \tilde{\mathbf{F}}(\boldsymbol{\rho}, k_z) e^{-jk_z z} \quad (2.149)$$

In (2.149), $\mathbf{F}(\boldsymbol{\rho}, k_z)$ may present any of the 2.5D field or current quantities, $\mathbf{E}(\boldsymbol{\rho}, k_z)$, $\mathbf{H}(\boldsymbol{\rho}, k_z)$ or $\mathbf{K}(\boldsymbol{\rho}, k_z)$, $\mathbf{K}_m(\boldsymbol{\rho}, k_z)$, whereas $\mathbf{F}(\mathbf{r})$ may represent any of the corresponding 3D field or current quantities: $\mathbf{E}(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$ or $\mathbf{J}(\mathbf{r})$, $\mathbf{M}(\mathbf{r})$. In this way, the transition from 2.5D fields to 3D fields is accomplished. If the steady-state solution in the time domain is of interest, then—as is well known—the following relation can be used:

$$\mathbf{F}(\mathbf{r}, t) = \text{Re}\{\mathbf{F}(\mathbf{r})e^{j\omega t}\} \quad (2.150)$$

Thus, the three-dimensional scattering problem of cylindrical scatterers excited by dipole sources (or more generally, fields radiated by arbitrary antennas) is addressed and formulated.

3. T MATRIX METHOD

The T matrix method (TMM) is an indispensable instrument in the multiple scattering calculations within a wide range of scientific research from nanotechnology to astrophysics. Its applications to the novel scattering problems [110-112] are disseminated so frequently that the comprehensive thematic T-matrix reference database was regularly updated [113] as a tradition kept by Michael I. Mishchenko. This tradition may have sadly come to an end after the last update [85] as he passed away according to [113].

The theoretical ground of the method is based on the extinction theorem which is formulated with one of the two outcomes of the surface equivalence principle [114, 36]. This principle formulates the fields satisfying homogeneous Helmholtz equations in the homogeneous regions of the free space separated with a smooth boundary whereas these fields also satisfy certain boundary conditions. While the equivalent surface current densities on the boundary are the source of the scattering field in the region on one side of the boundary, these densities extinguish the fields on the other side causing null fields in this region.

It would not be unfair to state that the engineering with the latter outcome of the surface equivalence principle i.e. the null-field methods applied to scattering and radiation of not only electromagnetic waves but also waves of various origins such as elastic or acoustic waves [115, 116] are performed with a more specific tuning and variations than the former outcome of the surface equivalence principle: The boundary element methods (BEM) or boundary integral equation (BIE) methods [93, 117] which are more universal than the latter. For the same variety of applications, establishing a wide variety of direct solutions on the object boundary in applications drives the modern software on electromagnetic engineering rendering the BEM or BIE methods to be more widespread currently [118]. But the null-field methods are in continuous demand in the important scope of applications. Even more, they formed the establishment of multiple scattering mechanisms consisting of open boundaries, isolated T-matrices of which exploited concepts in BIE formulations [29].

The TMM, which is also called the EBCM as one of the null field methods, extends the boundary to a surface in the null-field region that can be one of the eigen-surfaces of the proper separable coordinates with respect to the homogenous Helmholtz equation. Beyond these eigen-surfaces in the null-field region, the eigenfunction expansions of the unknown surface currents on the arbitrarily shaped scatterer boundary are valid if it is not elongated or corrugated. The generalized multipole method [119] and the method of auxiliary sources [120] are the other two methods that do not pronounce such a limitation which are also based on the extinction theorem. They trace the unknown coefficients of the eigenfunction expansions of the point sources in the null field interspersed along the boundary. They may arrange the types and amplitudes of the expansions as well as the locations of the point sources in due process of satisfying the boundary condition. The null field integral representations can further be exploited to form relevant BIE formulations [36, 121]. The main concern of these applications is efficient scattering formulations when the boundary is quite arbitrary. The TMM on the other hand is functional for scattering waves from a multitude of objects with one type of eigen-function expansion for all [4].

In this section, the ill-conditioning of this functional method is examined, and a well-conditioned formulation is presented for multiple dielectric scatterers with arbitrary cylindrical cross sections. There are two aspects of the ill-conditioning regarding the TMM first of which is geometrical. The less the deviation of the scatterer surface from the eigen-surface that the eigen-expansions used in calculations, the less the ill-conditioning [36]. The second aspect of the ill-conditioning occurs even when the surface is an eigen-surface due to the eigenfunction expansions used i.e., a circle/sphere in 2/3 dimensions. Especially when the radial eigenfunctions e.g., Bessel and Hankel functions of integer/half-integer indices for 2/3 dimensional problems in case the cylindrical/spherical harmonic expansions are used, the unknown coefficients possess fast decay and growth with respect to the truncation numbers [115]. This feature of the unknowns is a threat known as “numerical catastrophe” to the accuracy of the computer with finite precision where doubling the precision can only shift it to higher truncation numbers [125]. It is observed to prevent the expression of the resonant content of the obtained solutions in [122]. The regularization of this feature has been of critical importance especially when the interaction between more than one boundary must be formulated with linear algebraic equation systems (LAES) in the

computer with an algorithm that avoids the round-off errors during matrix inversions [123]. Although awareness of this second aspect has a history of about four decades, we witness that the advent of computer hardware allowing more precision in calculations is leaving it behind. Nevertheless, we can find works on resolving the stability and condition of specific problems every now and then [124].

In this section, regularization is applied to the TMM for scattering by arbitrarily shaped multiple dielectric boundaries. The formulation is based on the Analytical Regularization Method (ARM) [125], which establishes pairs of left- and right-hand regularizers for operators of integral, integro-differential, or matrix type that lead to ill-conditioned LAES, thereby transforming them into well-conditioned LAES. Although ARM is typically defined for BIE formulations, a version consistent with null-field methods presented in [76] involves eigenfunction solutions of dielectric scatterers with circular boundaries. In that work, the use of left- and right-hand regularizer pairs for the eigen-surface scattering problem is shown to significantly improve numerical stability and conditioning.

Here, the regularization developed for eigen-surface scatterers is extended to T-matrix formulations involving extended boundaries. With the incorporation of this regularization, the system is formulated in an FMM accelerated manner, resulting in a well-conditioned and accelerated T-matrix.

3.1. 2D T Matrix Method

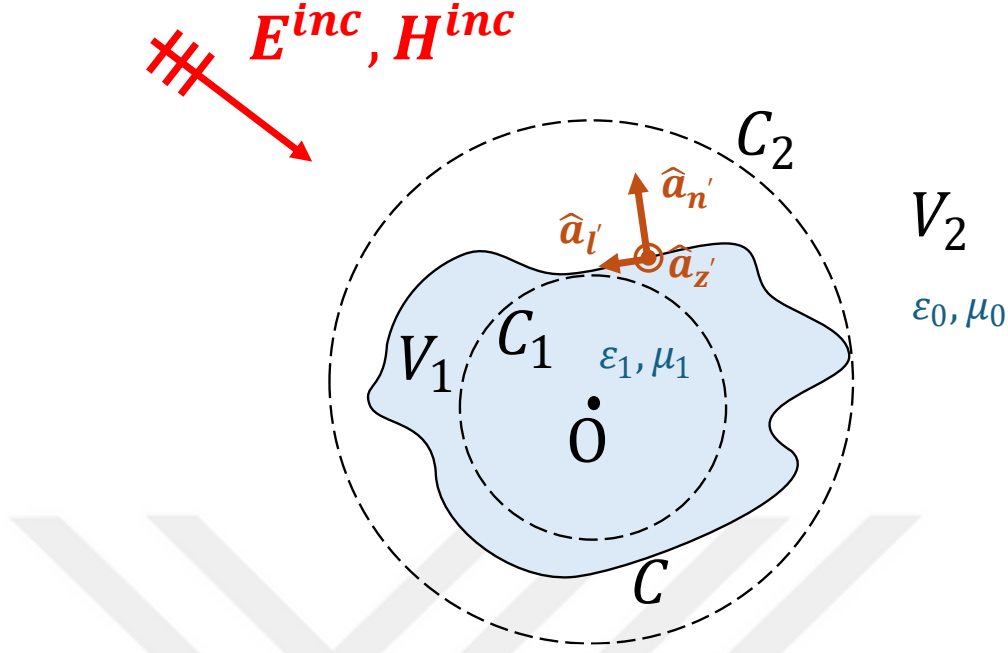


Figure 3.1: Arbitrary contour of a scatterer with its extended boundary.

For multiple dielectric cylindrical scatterers, the TMM is first formulated for a single scatterer, as illustrated in **Figure 3.1**, and is then generalized to the multiple scatterer case. The expression for the incident field on an arbitrary shaped cylindrical scatterer with wave number k_1 in a medium with wave number k_0 can be given as follows:

$$\phi_i(\rho) = \Re g \psi^t(k_0, \rho_0) \cdot \mathbf{a} \quad (3.1)$$

The scattered field expression is as follows:

$$\phi_s(\rho) = \psi^t(k_0, \rho_1) \cdot \mathbf{f}_1 \quad (3.2)$$

Here, the superscript t denotes the transpose of the vector and $\psi^t(k_0, \rho_1)$ is the row vector for the scattered fields in terms of outgoing cylindrical harmonics. $\mathbf{a}$ and $\mathbf{f}_1$ are column vectors containing the coefficients of the incident field and the scattered field, respectively. Let n be an integer and $H_n^{(2)}(x)$ is n^{th} order Hankel function of the second kind,

$$[\psi(k_0, \rho_1)]_n = H_n^{(2)}(k_0 \rho_1) e^{jn\phi_1}, \quad n \in [-M, M] \quad (3.3)$$

where ρ_1 and ϕ_1 are polar coordinates with respect to the local coordinate system of the first scatterer considering the center of the scatterer as the origin. Similarly, let $J_n(x)$ is the n^{th} order Bessel function,

$$[\Re g\psi(k_0, \rho_0)]_n = J_n(k_0\rho_0)e^{jn\phi_0}, \quad n \in [-M, M] \quad (3.4)$$

“ $\Re g$ ” stands for the regular part of the function in front of it, which transforms the Hankel function, which is singular at $\rho_i = 0$ in the two-dimensional case, into a regular Bessel function at this point. Also, ρ_0 and ϕ_0 in (3.4) are polar coordinates written in the global coordinate system. Total field expression in the medium with wave number k_0 outside the scatterer is:

$$\phi(\rho) = \Re g\psi^t(k_0, \rho_0) \cdot \mathbf{a} + \psi^t(k_0, \rho_1) \cdot \mathbf{f}_1 \quad (3.5)$$

In (3.5), the incident field is written in the global coordinate system, and the scattered field is written in the local coordinate system of the scatterer. It is possible to express the incident field on the local coordinate of the scatterer by the Graf Addition Theorem. According to this theorem,

$$\Re g\psi^t(k_0, \rho_0) = \Re g\psi^t(k_0, \rho_1) \cdot \bar{\boldsymbol{\beta}}_{10} \quad (3.6)$$

In (3.6), $\bar{\boldsymbol{\beta}}_{10}$ is the "transformation matrix" that transforms the standing wave function into a standing wave function, which provides the translation between the 1st coordinate system and the 0th coordinate system, and,

$$[\bar{\boldsymbol{\beta}}_{10}]_{np} = J_{p-n}(k_0 d_{10})e^{-j(p-n)\phi_{10}} \quad (3.7)$$

where d_{10} is the distance between the origin of coordinate systems 0 and 1 and ϕ_{10} is the angle between the line segment of length d_{10} and the x_1 axis. If (3.6) is written in the local coordinate of the scatterer using (3.5) to express the total field, we obtain:

$$\phi(\rho) = \Re g\psi^t(k_0, \rho_1) \cdot \bar{\boldsymbol{\beta}}_{10} \cdot \mathbf{a} + \psi^t(k_0, \rho_1) \cdot \mathbf{f}_1 \quad (3.8)$$

The first term of the above equation is the incident field on the scatterer, and the second term is the scattered field. Thus, a relationship between $\mathbf{a}$ and $\mathbf{f}_1$ can be constructed as follows,

$$\mathbf{f}_1 = \bar{\mathbf{T}}_{1(1)} \cdot \bar{\boldsymbol{\beta}}_{10} \cdot \mathbf{a} \quad (3.9)$$

In (3.9), $\bar{\mathbf{T}}_{1(1)}$ is the T matrix of the scatterer for the case where there is one scatterer and is called a single-scatterer T matrix or an isolated-scatterer T matrix. The sub-index (1) indicates that the T matrix is an isolated-scatterer T matrix and the number in front of the parenthesis indicates the number of scatterers. In the case of multiple scatterers, isolated scatterer T matrices can also be written for the i^{th} scatterer. The isolated T Matrix is defined as follows:

$$\bar{\mathbf{T}}_{1(1)} = -(\Re g \bar{\mathbf{Q}}_{10}) \cdot \bar{\mathbf{Q}}_{10}^{-1} \quad (3.10)$$

and,

$$Q_{nm}^{10} = \int_C \{ \psi_n(k_0, \boldsymbol{\rho}'_1) \hat{\mathbf{a}}_{n'} \cdot \nabla' \Re g \psi_m(k_1, \boldsymbol{\rho}'_1) \frac{p_1}{p_0} - \Re g \psi_m(k_1, \boldsymbol{\rho}'_1) \hat{\mathbf{a}}_{n'} \cdot \nabla' \psi_n(k_0, \boldsymbol{\rho}'_1) \} dl' \quad (3.11)$$

where p_1 and p_0 correspond to $\frac{1}{\mu_1}$ and $\frac{1}{\mu_0}$ respectively for perpendicularly incident TM case. For TE case, p_1 and p_0 are $\frac{1}{\varepsilon_1}$ and $\frac{1}{\varepsilon_0}$ respectively. (3.11) can be written with the following series expansion of the homogeneous medium Green's function. This expansion is as follows:

$$\begin{aligned} G_1(\boldsymbol{\rho}_1, \boldsymbol{\rho}'_1) &= -\frac{j}{4} H_0^{(2)}(k_0 |\boldsymbol{\rho}_1 - \boldsymbol{\rho}'_1|) \\ &= -\frac{j}{4} \sum_{n=-\infty}^{\infty} H_n^{(2)}(k_0 \rho'_1) J_n(k_0 \rho_1) e^{jn(\phi_1 - \phi'_1)}, \quad \rho_1 < \rho'_1 \end{aligned} \quad (3.12)$$

With the expansion (3.12), the cylindrical harmonics given as follows:

$$\psi_n(k_0, \boldsymbol{\rho}'_1) = H_n^{(2)}(k_0 \rho'_1) e^{-jn\phi'_1}, \quad \rho_1 < \rho'_1 \quad (3.13)$$

$$\Re g \psi_n(k_0, \boldsymbol{\rho}_1) = J_n(k_0 \rho_1) e^{jn\phi_1}, \quad \rho_1 < \rho'_1 \quad (3.14)$$

$$\Re g \psi_m(k_1, \boldsymbol{\rho}'_1) = J_m(k_1 \rho'_1) e^{jm\phi'_1}, \quad \rho_1 < \rho'_1 \quad (3.15)$$

The $\overline{\mathbf{Q}}$ matrix assembled using (3.11) within the isolated T matrix given by (3.10) becomes ill-conditioned for elongated and corrugated scatterers and therefore requires regularization to yield a well-conditioned formulation.

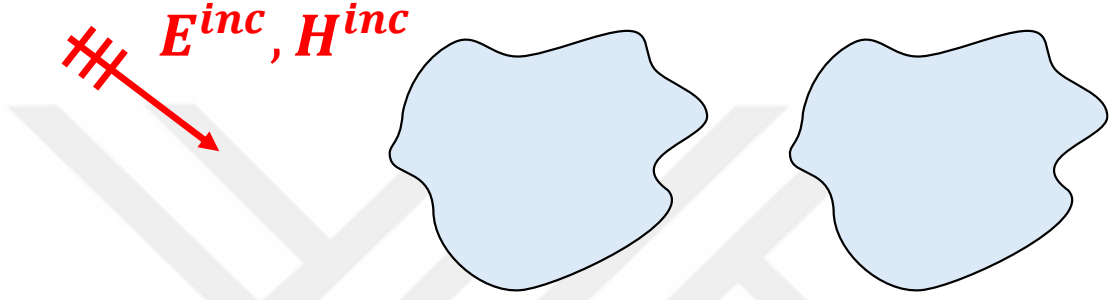


Figure 3.2: Two scatterer case.

If the T Matrix method for a single scatterer is extended to the case with two scatterers, as shown in **Figure 3.2**, the total field expression given in (3.5) becomes as follows:

$$\phi(\boldsymbol{\rho}) = \Re g \boldsymbol{\psi}^t(k_0, \boldsymbol{\rho}_0) \cdot \mathbf{a} + \boldsymbol{\psi}^t(k_0, \boldsymbol{\rho}_1) \cdot \mathbf{f}_1 + \boldsymbol{\psi}^t(k_0, \boldsymbol{\rho}_2) \cdot \mathbf{f}_2 \quad (3.16)$$

In the above equation, the scattered waves from each scatterer are expressed in terms of outgoing waves in their respective coordinate systems, while the incident field is written in terms of standing waves in the global coordinate system. To find the coefficients of the scattered waves from a scatterer of interest, the total field expression (3.16) must be written in the local coordinate system of that scatterer. According to the Graf Addition Theorem:

$$\Re g \boldsymbol{\psi}^t(k_0, \boldsymbol{\rho}_i) = \Re g \boldsymbol{\psi}^t(k_0, \boldsymbol{\rho}_j) \cdot \overline{\boldsymbol{\beta}}_{ji}, \quad \forall |\boldsymbol{\rho}_j| \quad (3.17)$$

$$\boldsymbol{\psi}^t(k_0, \boldsymbol{\rho}_i) = \boldsymbol{\psi}^t(k_0, \boldsymbol{\rho}_j) \cdot \overline{\boldsymbol{\beta}}_{ji}, \quad |\boldsymbol{\rho}_j| > d_{ji} \quad (3.18)$$

$$\boldsymbol{\psi}^t(k_0, \boldsymbol{\rho}_i) = \Re g \boldsymbol{\psi}^t(k_0, \boldsymbol{\rho}_j) \cdot \overline{\boldsymbol{\alpha}}_{ji}, \quad |\boldsymbol{\rho}_j| < d_{ji} \quad (3.19)$$

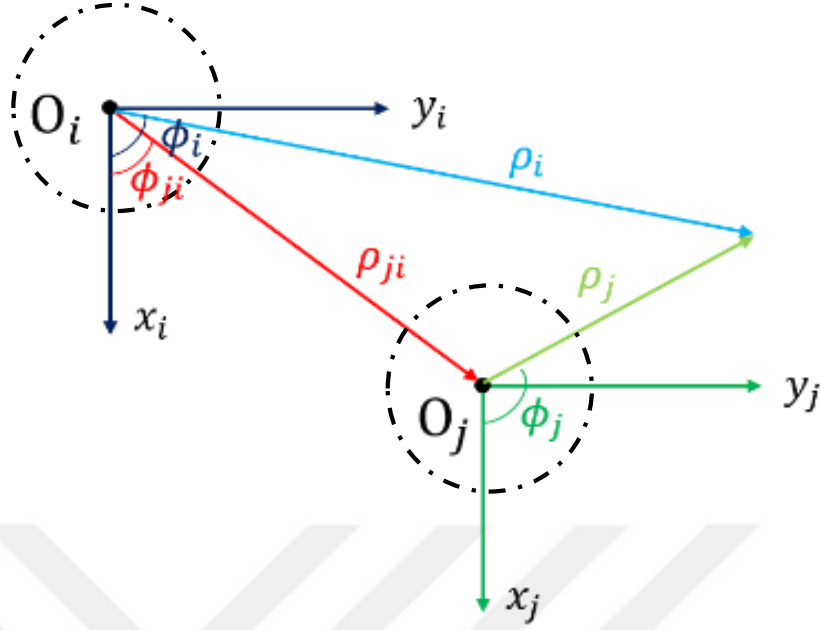


Figure 3.3: Transformation between the j^{th} coordinate system and the i^{th} coordinate system.

In above, $\bar{\beta}_{ji}$ is the “transformation matrix” that make transformation between the j^{th} coordinate system and the i^{th} coordinate system, converting standing waves to standing waves in the multiple scatterer case, and converting outgoing waves to outgoing waves again in the multilayer case. Similarly, $\bar{\alpha}_{ji}$ is the “transformation matrix” that converts the outgoing wave function into a standing wave function. The elements of these transformation matrices are as follows:

$$[\bar{\beta}_{ji}]_{np} = J_{p-n}(k_0 d_{ji}) e^{-j(p-n)\phi_{ji}} \quad (3.20)$$

$$[\bar{\alpha}_{ji}]_{np} = H_{p-n}^{(2)}(k_0 d_{ji}) e^{-j(p-n)\phi_{ji}} \quad (3.21)$$

Also, the following relationship can be established between $\bar{\beta}_{ji}$ and $\bar{\alpha}_{ji}$:

$$\bar{\beta}_{ji} = \Re g \bar{\alpha}_{ji} \quad (3.22)$$

Using the above relations, the following expressions can be written for the unknown coefficients of the scattered fields from each scatterer:

$$\mathbf{f}_1 - \bar{\mathbf{T}}_{1(1)} \cdot \bar{\boldsymbol{\alpha}}_{12} \cdot \mathbf{f}_2 = \bar{\mathbf{T}}_{1(1)} \cdot \bar{\boldsymbol{\beta}}_{10} \cdot \mathbf{a} \quad (3.23)$$

$$\mathbf{f}_2 - \bar{\mathbf{T}}_{2(1)} \cdot \bar{\boldsymbol{\alpha}}_{21} \cdot \mathbf{f}_1 = \bar{\mathbf{T}}_{2(1)} \cdot \bar{\boldsymbol{\beta}}_{20} \cdot \mathbf{a} \quad (3.24)$$

where $\bar{\mathbf{T}}_{i(1)}$ is the isolated T matrix for the i^{th} scatterer. Using (3.23) and (3.24), one can construct $\bar{\mathbf{A}} \cdot \mathbf{f} = \mathbf{b}$ matrix equation system and then search for a solution to the unknown coefficients of the scattered field. For such a system, the blocks of the matrix $\bar{\mathbf{A}}$ and the vector $\mathbf{b}$ are as follows:

$$A_{ij} = \begin{cases} \bar{\mathbf{I}} & i = j \\ -\bar{\mathbf{T}}_{i(1)} \cdot \bar{\boldsymbol{\alpha}}_{ij} & i \neq j \end{cases} \quad (3.25)$$

$$b_i = \bar{\mathbf{T}}_{i(1)} \cdot \bar{\boldsymbol{\beta}}_{i0} \cdot \mathbf{a} \quad (3.26)$$

(3.25) and (3.26) also are valid for the case with N scatterers and the matrix A is ill-conditioned regardless of whether it has an arbitrary cross-section or a circular cross-section.

To treat a scatterer consisting of multiple layers, we first consider the scatterer depicted in **Figure 3.1** as a single cylindrical layer. The total field expression for a cylindrical layer excited from outside in a k_0 wavenumber medium is as follows:

$$\phi(\boldsymbol{\rho}) = \Re g \boldsymbol{\psi}^t(k_0, \boldsymbol{\rho}_1) \cdot \bar{\boldsymbol{\beta}}_{10} \cdot \mathbf{a}_0 + \boldsymbol{\psi}^t(k_0, \boldsymbol{\rho}_1) \cdot \mathbf{b}_0 \quad (3.27)$$

and $\mathbf{b}_0$ is

$$\mathbf{b}_0 = \bar{\mathbf{R}}_{01} \cdot \bar{\boldsymbol{\beta}}_{10} \cdot \mathbf{a}_0 \quad (3.28)$$

The $\bar{\mathbf{R}}_{01}$ matrix and $\mathbf{b}_0$ vector in (3.28) are equivalent to the $\bar{\mathbf{T}}_{1(1)}$ and $\mathbf{f}_1$ in (3.9) respectively and the matrix $\bar{\mathbf{R}}_{01}$ is the scattering matrix describing the outgoing waves

scattered from the 1st scatterer towards the 0th medium. In addition, the field expression transmitted to the inner region of the scatterer:

$$\phi_i(\rho) = \Re g \psi^t(k_1, \rho_1) \cdot \mathbf{a}_1 \quad (3.29)$$

Where the vector $\mathbf{a}_1$ is the coefficient vector of the field transmitted into the 1st layer. The expression relating the transmitted field to the coefficients of the incident field is as follows:

$$\mathbf{a}_1 = 4j\bar{\mathbf{Q}}_{10}^{-1} \cdot \mathbf{a}_0 \quad (3.30)$$

And in (3.30),

$$\bar{\mathbf{T}}_{01} = 4j\bar{\mathbf{Q}}_{10}^{-1} \quad (3.31)$$

For the solution of the multilayer scattering problem by the T matrix method, the $\bar{\mathbf{Q}}_{10}$ matrix in (3.31) must first be well-conditioned for well-conditioning of the resulting matrix. The regularization for a single layer or a single scatterer is extended to the multilayer or multiple scatterer case and solutions are sought numerically in a stable system. However, even if the system is numerically stable, regularization does not guarantee an accurate solution for elongated or corrugated geometries due to the expansion of the fields into cylindrical harmonics. However, the regularization process guarantees that errors in the solution of unknown fields for geometries with arbitrary cross-sections are not due to system conditioning and round-off errors. Therefore, the constraints of the TMM and possible improvements for arbitrary cross sections should be investigated after the regularization process.

If the scatterer is excited by a field coming from its internal region and expressed in outgoing waves, then the expression for the incident field in the medium with wave number k_1 is as below:

$$\phi_i(\rho) = \psi^t(k_1, \rho_1) \cdot \mathbf{b}_1 \quad (3.32)$$

In this case, the scattered field expression is:

$$\phi_s(\boldsymbol{\rho}) = \Re g \boldsymbol{\psi}^t(k_1, \boldsymbol{\rho}_1) \cdot \mathbf{a}_1 \quad (3.33)$$

Thus, the total field expression inside the scatterer

$$\phi(\boldsymbol{\rho}) = \boldsymbol{\psi}^t(k_1, \boldsymbol{\rho}_1) \cdot \mathbf{b}_1 + \Re g \boldsymbol{\psi}^t(k_1, \boldsymbol{\rho}_1) \cdot \mathbf{a}_1 \quad (3.34)$$

The vectors $\mathbf{a}_1$ and $\mathbf{b}_1$ in equation (3.34) are the coefficient vector of the scattered field and the coefficient vector of the incident field, respectively. Therefore, a relationship can be established between vectors $\mathbf{a}_1$ and $\mathbf{b}_1$:

$$\mathbf{a}_1 = \bar{\mathbf{R}}_{10} \cdot \mathbf{b}_1 \quad (3.35)$$

The $\bar{\mathbf{R}}_{10}$ matrix in (3.35) is the scattering matrix describing the scattered fields in the standing waves with direction from the boundary with medium 0 to medium 1 and is defined as follows:

$$\bar{\mathbf{R}}_{10} = -(\Re g^{-1} \bar{\mathbf{P}}_{10}) \cdot \bar{\mathbf{P}}_{10}^{-1} \quad (3.36)$$

and,

$$P_{nm}^{10} = \int_C \{ \Re g \psi_n(k_1, \boldsymbol{\rho}'_1) \hat{\mathbf{a}}_{n'} \cdot \nabla' \psi_m(k_0, \boldsymbol{\rho}'_1) \frac{p_0}{p_1} - \psi_m(k_0, \boldsymbol{\rho}'_1) \hat{\mathbf{a}}_{n'} \cdot \nabla' \Re g \psi_n(k_1, \boldsymbol{\rho}'_1) \} dl' \quad (3.37)$$

where the elements of the matrix $\bar{\mathbf{P}}_{10}$ are P_{nm}^{10} . Similar to the relationship between $\mathbf{a}_1$ and $\mathbf{b}_1$, a relationship can be established between the coefficient vector of the field transmitted to medium 0 and $\mathbf{b}_1$. The field transmitted to medium 0,

$$\phi_t(\boldsymbol{\rho}) = \boldsymbol{\psi}^t(k_0, \boldsymbol{\rho}_1) \cdot \mathbf{b}_0 \quad (3.38)$$

and,

$$\mathbf{b}_0 = \bar{\mathbf{T}}_{10} \cdot \mathbf{b}_1 \quad (3.39)$$

where $\mathbf{b}_0$ is the coefficient vector of the field transmitted to the 0th medium.

The $\bar{\mathbf{T}}_{10}$ matrix in (3.39) is the transition matrix describing the outgoing waves transmitted from medium 1 to medium 0 and is given by:

$$\bar{\mathbf{T}}_{10} = 4j\bar{\mathbf{P}}_{10}^{-1} \quad (3.40)$$

Thus, using the matrix expressions $\bar{\mathbf{R}}_{01}$, $\bar{\mathbf{T}}_{01}$, $\bar{\mathbf{R}}_{10}$ and $\bar{\mathbf{T}}_{10}$ the solution of the single interface problem can be extended to multi-interface problems. It should be noted that $\bar{\mathbf{T}}_{01}$ and $\bar{\mathbf{T}}_{10}$ are distinct from $\bar{\mathbf{T}}_{1(1)}$, the T-matrix of an isolated scatterer introduced in the previous section.

For the multi-interface/layered problem, the expressions for the multilayered problem can be extended considering the discussion above. For this, cylindrical harmonics are as follows:

$$\psi_n(k_i, \rho'_{i+1}) = H_n^{(2)}(k_i \rho'_{i+1}) e^{-jn\phi'_{i+1}}, \quad \rho_{i+1} < \rho'_{i+1} \quad (3.41)$$

$$\Re g \psi_m(k_{i+1}, \rho'_{i+1}) = J_m(k_{i+1} \rho'_{i+1}) e^{jm\phi'_{i+1}}, \quad \rho_{i+1} < \rho'_{i+1} \quad (3.42)$$

$$\Re g \psi_n(k_{i+1}, \rho_{i+1}') = J_n(k_{i+1} \rho_{i+1}') e^{-jn\phi'_{i+1}}, \quad \rho_{i+1} > \rho_{i+1}' \quad (3.43)$$

$$\psi_m(k_i, \rho'_{i+1}) = H_n^{(2)}(k_i \rho'_{i+1}) e^{jm\phi'_{i+1}}, \quad \rho_{i+1} > \rho_{i+1}' \quad (3.44)$$

and the equations for a two-layer problem with these cylindrical harmonics become as below:

$$\mathbf{b}_0 - \bar{\mathbf{T}}_{10} \cdot \bar{\boldsymbol{\beta}}_{12} \cdot \mathbf{b}_1 = \bar{\mathbf{R}}_{01} \cdot \bar{\boldsymbol{\beta}}_{10} \cdot \mathbf{a}_0 \quad (3.45)$$

$$\mathbf{a}_1 - \bar{\mathbf{R}}_{10} \cdot \bar{\boldsymbol{\beta}}_{12} \cdot \mathbf{b}_1 = \bar{\mathbf{T}}_{01} \cdot \bar{\boldsymbol{\beta}}_{10} \cdot \mathbf{a}_0 \quad (3.46)$$

$$\mathbf{b}_1 - \bar{\mathbf{R}}_{12} \cdot \bar{\boldsymbol{\beta}}_{21} \cdot \mathbf{a}_1 = \mathbf{0} \quad (3.47)$$

$$\mathbf{a}_2 - \bar{\mathbf{T}}_{12} \cdot \bar{\boldsymbol{\beta}}_{21} \cdot \mathbf{a}_1 = \mathbf{0} \quad (3.48)$$

To compute $\bar{\mathbf{R}}_{12}$ in Equation (3.47), the expression for $\bar{\mathbf{R}}_{01}$ given in (3.10) can be used by replacing the index 0 with 1 and 1 with 2. Similarly, $\bar{\mathbf{T}}_{12}$ in (3.48) can be obtained from (3.31) by applying the same index substitution. In the case of more than two layers, the corresponding matrices can be computed for each pair (i, j) by performing this substitution accordingly. Thus, a system of matrix equations such as $\bar{\mathbf{A}} \cdot \mathbf{x} = \mathbf{y}$ is created and solutions are sought.



3.2. Construction of the 2.5D T Matrix Method

In this section, the 2.5D TMM will be constructed based on the null field integral equations (2.45) and (2.46) where the fields on boundary are unknown instead of the currents. Accordingly, we updated the T Matrix methodology discussed in the previous section for 2.5D by maintaining the same methodology. Maintaining the exposition of the previous section, the approach will be reformulated by updating and restating Equations (3.1) – (3.48) in the following.

The expression for the incident field on an arbitrary shaped cylindrical scatterer with wave number k_1 in a medium with wave number k_0 given as follows:

$$\boldsymbol{\phi}_i(\boldsymbol{\rho}) = \Re g \bar{\Psi}(k_{\rho,0}, \boldsymbol{\rho}_0) \cdot \mathbf{a} \quad (3.49)$$

The scattered field expression is as follows:

$$\boldsymbol{\phi}_s(\boldsymbol{\rho}) = \bar{\Psi}(k_{\rho,0}, \boldsymbol{\rho}_1) \cdot \mathbf{f}_1 \quad (3.50)$$

Here, $\mathbf{a}$ and $\mathbf{f}_1$ are column vectors containing the coefficients of the z-components of the electric and magnetic fields as blocks, unlike in the 2D case. $\mathbf{a}$ and $\mathbf{f}_1$ vectors are related to the coefficients of the incident field and the scattered field, respectively. $\bar{\Psi}$ is a 2x2 block-diagonal matrix whose two diagonal blocks are given by $\boldsymbol{\psi}^t$. Let n be an integer and $H_n^{(2)}(x)$ is n^{th} order Hankel function of the second kind,

$$[\boldsymbol{\psi}(k_{\rho,0}, \boldsymbol{\rho}_1)]_n = H_n^{(2)}(k_{\rho,0}\rho_1)e^{jn\phi_1}, \quad n \in [-M, M] \quad (3.51)$$

where ρ_1 and ϕ_1 are polar coordinates with respect to the local coordinate system of the first scatterer considering the center of the scatterer as the origin. Similarly, let $J_n(x)$ is the n^{th} order Bessel function,

$$[\Re g \boldsymbol{\psi}(k_{\rho,0}, \boldsymbol{\rho}_0)]_n = J_n(k_{\rho,0}\rho_0)e^{jn\phi_0}, \quad n \in [-M, M] \quad (3.52)$$

“ $\Re g$ ” stands for the regular part of the function in front of it, which transforms the Hankel function, which is singular at $\boldsymbol{\rho}_i = 0$ in the 2.5D case, into a regular Bessel

function at this point. Also, ρ_0 and ϕ_0 in (3.52) are polar coordinates written in the global coordinate system. Total field expression in the medium with wave number k_0 outside the scatterer is:

$$\phi(\rho) = \Re g \bar{\Psi}(k_{\rho,0}, \rho_0) \cdot \mathbf{a} + \bar{\Psi}(k_{\rho,0}, \rho_1) \cdot \mathbf{f}_1 \quad (3.53)$$

In (3.53), the incident field is written in the global coordinate system, and the scattered field is written in the local coordinate system of the scatterer. It is possible to express the incident field on the local coordinate of the scatterer by the Graf Addition Theorem. According to this theorem,

$$\Re g \psi^t(k_{\rho,0}, \rho_0) = \Re g \psi^t(k_{\rho,0}, \rho_1) \cdot \bar{\beta}_{10} \quad (3.54)$$

In (3.54), $\bar{\beta}_{10}$ is the "transformation matrix" that transforms the standing wave function into a standing wave function, which provides the translation between the 1st coordinate system and the 0th coordinate system, and,

$$[\bar{\beta}_{10}]_{np} = J_{p-n}(k_{\rho,0} d_{10}) e^{-j(p-n)\phi_{10}} \quad (3.55)$$

where d_{10} is the distance between the origin of coordinate systems 0 and 1 and ϕ_{10} is the angle between the line segment of length d_{10} and the x_1 axis. If (3.53) is written in the local coordinate of the scatterer using (3.54) to express the total field, we obtain:

$$\phi(\rho) = \Re g \bar{\Psi}(k_{\rho,0}, \rho_1) \cdot \bar{\mathbf{B}}_{10} \cdot \mathbf{a} + \bar{\Psi}(k_{\rho,0}, \rho_1) \cdot \mathbf{f}_1 \quad (3.56)$$

The first term of the above equation is the incident field on the scatterer, and the second term is the scattered field. Thus, a relationship between $\mathbf{a}$ and $\mathbf{f}_1$ can be written as follows,

$$\mathbf{f}_1 = \bar{\mathbf{T}}_{1(1)} \cdot \bar{\mathbf{B}}_{10} \cdot \mathbf{a} \quad (3.57)$$

In (3.57), $\bar{\mathbf{T}}_{1(1)}$ is the 2.5D T matrix of the scatterer for the case where there is one scatterer and is called a single-scatterer 2.5D T matrix or an isolated-scatterer 2.5D T matrix. Note that in equations (3.49) to (3.56), $k_{\rho,0}$ is written instead of k_0 in the 2D case. Also, the isolated scatterer T matrices here consist of 4 matrix blocks and include coupling terms. The isolated T Matrix is defined as follows:

$$\bar{\mathbf{T}}_{1(1)} = -(\Re g \bar{\mathbf{Q}}_{10}) \cdot \bar{\mathbf{Q}}_{10}^{-1} \quad (3.58)$$

$$\bar{\mathbf{Q}}_{10} = \begin{bmatrix} \bar{\mathbf{Q}}_{10}^{(TM \rightarrow TM)} & \mathbf{Q}_{10}^{(TM \rightarrow TE)} \\ \bar{\mathbf{Q}}_{10}^{(TE \rightarrow TM)} & \bar{\mathbf{Q}}_{10}^{(TE \rightarrow TE)} \end{bmatrix} \quad (3.59)$$

The matrix $\bar{\mathbf{Q}}_{10}$ given by (3.59) consists of four blocks and it can be easily understood that these matrix blocks arise due to the coupling terms in the null field equations (2.45) and (2.46). $Q_{nm}^{10 (\mathcal{N} \rightarrow \mathcal{M})}$ is being the elements of the corresponding matrix block coupling the N mode to the M mode,

$$Q_{nm}^{10 (\mathcal{N} \rightarrow \mathcal{N})} = \int_C \{ \psi_n(k_{\rho,0}, \boldsymbol{\rho}'_1) \hat{\mathbf{a}}_{n'} \cdot \nabla' \Re g \psi_m(k_{\rho,1}, \boldsymbol{\rho}'_1) \frac{p_1^{(\mathcal{N} \rightarrow \mathcal{N})}}{p_0^{(\mathcal{N} \rightarrow \mathcal{N})}} - \Re g \psi_m(k_{\rho,1}, \boldsymbol{\rho}'_1) \hat{\mathbf{a}}_{n'} \cdot \nabla' \psi_n(k_{\rho,0}, \boldsymbol{\rho}'_1) \} dl' \quad (3.60)$$

$$Q_{nm}^{10 (\mathcal{N} \rightarrow \mathcal{M})} = \int_C \{ \Re g \psi_m(k_{\rho,1}, \boldsymbol{\rho}'_1) \hat{\mathbf{a}}_{l'} \cdot \nabla' \psi_n(k_{\rho,0}, \boldsymbol{\rho}'_1) \frac{q_1^{(\mathcal{N} \rightarrow \mathcal{M})}}{q_0^{(\mathcal{N} \rightarrow \mathcal{M})}} \} dl' \quad (3.61)$$

where $p_1^{(\mathcal{N} \rightarrow \mathcal{N})}$ and $p_0^{(\mathcal{N} \rightarrow \mathcal{N})}$ correspond to $(k_{\rho,2})^2 \varepsilon_1$ and $(k_{\rho,1})^2 \varepsilon_2$ respectively when $(\mathcal{N} \rightarrow \mathcal{N})$ is $(TM \rightarrow TM)$. For $(TE \rightarrow TE)$ case, $p_1^{(\mathcal{N} \rightarrow \mathcal{N})}$ and $p_0^{(\mathcal{N} \rightarrow \mathcal{N})}$ are $(k_{\rho,2})^2 \mu_1$ and $(k_{\rho,1})^2 \mu_2$ respectively.

In addition, $q_1^{(\mathcal{N} \rightarrow \mathcal{M})}$ and $q_0^{(\mathcal{N} \rightarrow \mathcal{M})}$ in (3.55) are $-(k_1^2 - k_2^2)k_z$ and $\omega(k_{\rho,1})^2 \varepsilon_2$ respectively, if $(\mathcal{N} \rightarrow \mathcal{M})$ is $(TM \rightarrow TE)$. Otherwise, i.e. for $(TE \rightarrow TM)$, $q_1^{(\mathcal{N} \rightarrow \mathcal{M})}$ and $q_0^{(\mathcal{N} \rightarrow \mathcal{M})}$ are $(k_1^2 - k_2^2)k_z$ and $\omega(k_{\rho,1})^2 \mu_2$ respectively.

(3.60) and (3.61) can be written with the following series expansion of the homogeneous medium 2.5D Green's function. This expansion is as follows:

$$\begin{aligned}
G_1(\boldsymbol{\rho}_1, \boldsymbol{\rho}'_1) &= -\frac{j}{4} H_0^{(2)}(k_{\rho,0} |\boldsymbol{\rho}_1 - \boldsymbol{\rho}'_1|) \\
&= -\frac{j}{4} \sum_{n=-\infty}^{\infty} H_n^{(2)}(k_{\rho,0} \rho'_1) J_n(k_{\rho,0} \rho_1) e^{jn(\phi_1 - \phi'_1)}, \quad \rho_1 < \rho'_1
\end{aligned} \tag{3.62}$$

With the expansion (3.62), cylindrical harmonics are given as follows:

$$\psi_n(k_{\rho,0}, \boldsymbol{\rho}'_1) = H_n^{(2)}(k_{\rho,0} \rho'_1) e^{-jn\phi'_1}, \quad \rho_1 < \rho'_1 \tag{3.63}$$

$$\Re g \psi_n(k_{\rho,0}, \boldsymbol{\rho}_1) = J_n(k_{\rho,0} \rho_1) e^{jn\phi_1}, \quad \rho_1 < \rho'_1 \tag{3.64}$$

$$\Re g \psi_m(k_{\rho,1}, \boldsymbol{\rho}'_1) = J_m(k_{\rho,1} \rho'_1) e^{jm\phi'_1}, \quad \rho_1 < \rho'_1 \tag{3.65}$$

The $\bar{\mathbf{Q}}_{10}$ matrix filled with the help of (3.62) in the isolated T matrix given by (3.58) shows ill-conditioned properties and needs to be regularized for well conditioning.

If the T Matrix method for a single scatterer is extended to the case with two scatterers, the total field expression becomes as follows:

$$\boldsymbol{\phi}(\boldsymbol{\rho}) = \Re g \bar{\boldsymbol{\Psi}}(k_{\rho,0}, \boldsymbol{\rho}_0) \cdot \mathbf{a} + \bar{\boldsymbol{\Psi}}(k_{\rho,0}, \boldsymbol{\rho}_1) \cdot \mathbf{f}_1 + \bar{\boldsymbol{\Psi}}(k_{\rho,0}, \boldsymbol{\rho}_2) \cdot \mathbf{f}_2 \tag{3.66}$$

According to the Graf Addition Theorem:

$$\Re g \psi^t(k_{\rho,0}, \boldsymbol{\rho}_i) = \Re g \psi^t(k_{\rho,0}, \boldsymbol{\rho}_j) \cdot \bar{\boldsymbol{\beta}}_{ji}, \quad \forall |\boldsymbol{\rho}_j| \tag{3.67}$$

$$\psi^t(k_{\rho,0}, \boldsymbol{\rho}_i) = \psi^t(k_{\rho,0}, \boldsymbol{\rho}_j) \cdot \bar{\boldsymbol{\beta}}_{ji}, \quad |\boldsymbol{\rho}_j| > d_{ji} \tag{3.68}$$

$$\psi^t(k_{\rho,0}, \boldsymbol{\rho}_i) = \Re g \psi^t(k_{\rho,0}, \boldsymbol{\rho}_j) \cdot \bar{\boldsymbol{\alpha}}_{ji}, \quad |\boldsymbol{\rho}_j| < d_{ji} \tag{3.69}$$

In above, $\bar{\boldsymbol{\beta}}_{ji}$ is the “transformation matrix” that make transformation between the j^{th} coordinate system and the i^{th} coordinate system, converting standing waves to standing waves in the multiple scatterer case, and converting outgoing waves to outgoing waves again in the multilayer case. Similarly, $\bar{\boldsymbol{\alpha}}_{ji}$ is the “transformation matrix” that converts the outgoing wave function into a standing wave function. The elements of these transformation matrices are as follows:

$$[\bar{\boldsymbol{\beta}}_{ji}]_{np} = J_{p-n}(k_0 d_{ji}) e^{-j(p-n)\phi_{ji}} \quad (3.70)$$

$$[\bar{\boldsymbol{\alpha}}_{ji}]_{np} = H_{p-n}^{(2)}(k_0 d_{ji}) e^{-j(p-n)\phi_{ji}} \quad (3.71)$$

Also, the following relationship can be established between $\bar{\boldsymbol{\beta}}_{ji}$ and $\bar{\boldsymbol{\alpha}}_{ji}$:

$$\bar{\boldsymbol{\beta}}_{ji} = \Re g \bar{\boldsymbol{\alpha}}_{ji} \quad (3.72)$$

Using the above relations, the following expressions can be written for the unknown coefficients of the scattered fields from each scatterer:

$$\mathbf{f}_1 - \bar{\mathbf{T}}_{1(1)} \cdot \bar{\mathbf{A}}_{12} \cdot \mathbf{f}_2 = \bar{\mathbf{T}}_{1(1)} \cdot \bar{\mathbf{B}}_{10} \cdot \mathbf{a} \quad (3.73)$$

$$\mathbf{f}_2 - \bar{\mathbf{T}}_{2(1)} \cdot \bar{\mathbf{A}}_{21} \cdot \mathbf{f}_1 = \bar{\mathbf{T}}_{2(1)} \cdot \bar{\mathbf{B}}_{20} \cdot \mathbf{a} \quad (3.74)$$

where $\bar{\mathbf{T}}_{i(1)}$ is the isolated T matrix for the i^{th} scatterer. Using (3.73) and (3.74), one can construct $\bar{\mathbf{Z}} \cdot \mathbf{f} = \mathbf{b}$ matrix equation system and then search for a solution to the unknown coefficients of the scattered field. For such a system, the blocks of the matrix $\bar{\mathbf{Z}}$ and the vector $\mathbf{b}$ are as follows:

$$\bar{\mathbf{Z}}_{ij} = \begin{cases} \bar{\mathbf{I}} & i = j \\ -\bar{\mathbf{T}}_{i(1)} \cdot \bar{\mathbf{A}}_{ij} & i \neq j \end{cases} \quad (3.75)$$

$$\mathbf{b}_i = \bar{\mathbf{T}}_{i(1)} \cdot \bar{\mathbf{B}}_{i0} \cdot \mathbf{a} \quad (3.76)$$

Here $\bar{\mathbf{T}}_{i(1)}$ is the 2.5D isolated-scatterer T matrix as given in (3.58). $\bar{\mathbf{A}}_{ij}$, $\bar{\mathbf{B}}_{i0}$, $\mathbf{f}$ and $\mathbf{b}$ appear in the coupled case in the following form:

$$\bar{\mathbf{A}}_{ij} = \begin{bmatrix} \bar{\boldsymbol{\alpha}}_{ij} & 0 \\ 0 & \bar{\boldsymbol{\alpha}}_{ij} \end{bmatrix} \quad (3.77)$$

$$\bar{\mathbf{B}}_{i0} = \begin{bmatrix} \bar{\boldsymbol{\beta}}_{i0} & 0 \\ 0 & \bar{\boldsymbol{\beta}}_{i0} \end{bmatrix} \quad (3.78)$$

$$\mathbf{b}_i = \bar{\mathbf{T}}_{i(1)} \cdot \bar{\mathbf{B}}_{i0} \cdot \begin{bmatrix} \mathbf{a}_E \\ \mathbf{a}_H \end{bmatrix} \quad (3.79)$$

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_E \\ \mathbf{f}_H \end{bmatrix} \quad (3.80)$$

For the cylindrical layered case, a 2.5D formulation can be made in a similar way to the above. In this case, the resulting matrices will again consist of four blocks for the same reason mentioned above.

The total field expression for a cylindrical layer excited from outside in a k_0 wavenumber medium is as follows:

$$\boldsymbol{\phi}(\boldsymbol{\rho}) = \Re g \bar{\Psi}(k_{\rho,0}, \boldsymbol{\rho}_1) \cdot \bar{\mathbf{B}}_{10} \cdot \mathbf{a}_0 + \bar{\Psi}(k_{\rho,0}, \boldsymbol{\rho}_1) \cdot \mathbf{b}_0 \quad (3.81)$$

and $\mathbf{b}_0$ is

$$\mathbf{b}_0 = \bar{\mathbf{R}}_{01} \cdot \bar{\mathbf{B}}_{10} \cdot \mathbf{a}_0 \quad (3.82)$$

The $\bar{\mathbf{R}}_{01}$ matrix and $\mathbf{b}_0$ vector in (3.76) are equivalent to the $\bar{\mathbf{T}}_{1(1)}$ and $\mathbf{f}_1$ in (3.57) respectively and the matrix $\bar{\mathbf{R}}_{01}$ is the scattering matrix describing the outgoing waves scattered from the 1st scatterer towards the 0th medium. In addition, the field expression transmitted to the inner region of the scatterer:

$$\boldsymbol{\phi}_i(\boldsymbol{\rho}) = \Re g \bar{\Psi}(k_{\rho,1}, \boldsymbol{\rho}_1) \cdot \mathbf{a}_1 \quad (3.83)$$

where the vector $\mathbf{a}_1$ is the coefficient vector of the field transmitted into the 1st layer. The expression relating the transmitted field to the coefficients of the incident field is as follows:

$$\mathbf{a}_1 = \bar{\mathbf{T}}_{01} \cdot \mathbf{a}_0 \quad (3.84)$$

And in (3.84),

$$\bar{T}_{01} = 4j\bar{Q}_{10}^{-1} \quad (3.85)$$

If the scatterer is excited by a field coming from its internal region and expressed in outgoing waves, then the expression for the incident field in the medium with wave number k_1 is as below:

$$\phi_i(\rho) = \bar{\Psi}(k_{\rho,1}, \rho_1) \cdot \mathbf{b}_1 \quad (3.86)$$

In this case, the scattered field expression is:

$$\phi_s(\rho) = \Re g \bar{\Psi}(k_{\rho,1}, \rho_1) \cdot \mathbf{a}_1 \quad (3.87)$$

Thus, the total field expression inside the scatterer:

$$\phi(\rho) = \bar{\Psi}(k_{\rho,1}, \rho_1) \cdot \mathbf{b}_1 + \Re g \bar{\Psi}(k_{\rho,1}, \rho_1) \cdot \mathbf{a}_1 \quad (3.88)$$

The vectors $\mathbf{a}_1$ and $\mathbf{b}_1$ in equation (3.88) are the coefficient vector of the scattered field and the coefficient vector of the incident field, respectively. Therefore, a relationship can be established between vectors $\mathbf{a}_1$ and $\mathbf{b}_1$:

$$\mathbf{a}_1 = \bar{\mathbf{R}}_{10} \cdot \mathbf{b}_1 \quad (3.89)$$

The $\bar{\mathbf{R}}_{10}$ matrix in (3.89) is the scattering matrix describing the scattered fields in the standing waves with direction from the boundary with medium 0 to medium 1 and is defined as follows:

$$\bar{\mathbf{R}}_{10} = -(\Re g^{-1} \bar{\mathbf{P}}_{10}) \cdot \bar{\mathbf{P}}_{10}^{-1} \quad (3.90)$$

$$\bar{\mathbf{P}}_{10} = \begin{bmatrix} \bar{\mathbf{P}}_{10}^{(TM \rightarrow TM)} & \bar{\mathbf{P}}_{10}^{(TM \rightarrow TE)} \\ \bar{\mathbf{P}}_{10}^{(TE \rightarrow TM)} & \bar{\mathbf{P}}_{10}^{(TE \rightarrow TE)} \end{bmatrix} \quad (3.91)$$

and,

$$P_{nm}^{10(\mathcal{N} \rightarrow \mathcal{N})} = \int_C \{ \Re g \psi_n(k_{\rho,1}, \boldsymbol{\rho}'_1) \hat{\mathbf{a}}_{n'} \cdot \nabla' \psi_m(k_{\rho,0}, \boldsymbol{\rho}'_1) \frac{p_0^{(\mathcal{N} \rightarrow \mathcal{N})}}{p_1^{(\mathcal{N} \rightarrow \mathcal{N})}} - \psi_m(k_{\rho,0}, \boldsymbol{\rho}'_1) \hat{\mathbf{a}}_{n'} \cdot \nabla' \Re g \psi_n(k_{\rho,1}, \boldsymbol{\rho}'_1) \} dl' \quad (3.92)$$

$$P_{nm}^{10(\mathcal{N} \rightarrow \mathcal{M})} = \int_C \{ \psi_m(k_{\rho,0}, \boldsymbol{\rho}'_1) \hat{\mathbf{a}}_{l'} \cdot \nabla' \Re g \psi_n(k_{\rho,1}, \boldsymbol{\rho}'_1) \frac{s_0^{(\mathcal{N} \rightarrow \mathcal{M})}}{s_1^{(\mathcal{N} \rightarrow \mathcal{M})}} \} dl' \quad (3.93)$$

where the elements of the matrix $\bar{\mathbf{P}}_{10}$ are $P_{nm}^{10(\mathcal{N} \rightarrow \mathcal{M})}$. The definition of $p_i^{(\mathcal{N} \rightarrow \mathcal{N})}$ in (3.92) is as given earlier. However, the definition of $s_i^{(\mathcal{N} \rightarrow \mathcal{M})}$ is slightly different from $q_i^{(\mathcal{N} \rightarrow \mathcal{M})}$. In addition, $s_0^{(\mathcal{N} \rightarrow \mathcal{M})}$ and $s_1^{(\mathcal{N} \rightarrow \mathcal{M})}$ in (3.93) are $(k_1^2 - k_2^2)k_z$ and $\omega(k_{\rho,0})^2 \varepsilon_1$ respectively, if $(\mathcal{N} \rightarrow \mathcal{M})$ is $(TM \rightarrow TE)$. Otherwise, i.e. for $(TE \rightarrow TM)$, $s_0^{(\mathcal{N} \rightarrow \mathcal{M})}$ and $s_1^{(\mathcal{N} \rightarrow \mathcal{M})}$ are $-(k_1^2 - k_2^2)k_z$ and $\omega(k_{\rho,0})^2 \mu_1$ respectively.

Like the relationship between $\mathbf{a}_1$ and $\mathbf{b}_1$, a relationship can be established between the coefficient vector of the field transmitted to medium 0 and $\mathbf{b}_1$. The field transmitted to medium 0,

$$\boldsymbol{\phi}_t(\boldsymbol{\rho}) = \bar{\boldsymbol{\Psi}}(k_{\rho,0}, \boldsymbol{\rho}_1) \cdot \mathbf{b}_0 \quad (3.94)$$

$$\mathbf{b}_0 = \bar{\mathbf{T}}_{10} \cdot \mathbf{b}_1 \quad (3.95)$$

where $\mathbf{b}_0$ is the coefficient vector of the field transmitted to the 0th medium.

The $\bar{\mathbf{T}}_{10}$ matrix in (3.95) is the transition matrix describing the outgoing waves transmitted from medium 1 to medium 0 and is given by:

$$\bar{\mathbf{T}}_{10} = 4j\bar{\mathbf{P}}_{10}^{-1} \quad (3.96)$$

Thus, using the matrix expressions $\bar{\mathbf{R}}_{01}$, $\bar{\mathbf{T}}_{01}$, $\bar{\mathbf{R}}_{10}$ and $\bar{\mathbf{T}}_{10}$ the solution of the single interface problem can be extended to multi-interface problems. For the multi-interface/layered problem, the cylindrical harmonics are as follows:

$$\psi_n(k_{\rho,i}, \boldsymbol{\rho}'_{i+1}) = H_n^{(2)}(k_{\rho,i} \rho'_{i+1}) e^{-jn\phi'_{i+1}}, \quad \rho_{i+1} < \rho'_{i+1} \quad (3.97)$$

$$\Re g \psi_m(k_{\rho,i+1}, \boldsymbol{\rho}'_{i+1}) = J_m(k_{\rho,i+1} \rho'_{i+1}) e^{jm\phi'_{i+1}}, \quad \rho_{i+1} < \rho'_{i+1} \quad (3.98)$$

$$\Re g \psi_n(k_{\rho,i+1}, \boldsymbol{\rho}_{i+1}') = J_n(k_{\rho,i+1} \rho_{i+1}') e^{-jn\phi'_{i+1}}, \quad \rho_{i+1} > \rho_{i+1}' \quad (3.99)$$

$$\psi_m(k_{\rho,i}, \boldsymbol{\rho}'_{i+1}) = H_m^{(2)}(k_{\rho,i} \rho'_{i+1}) e^{jm\phi'_{i+1}}, \quad \rho_{i+1} > \rho_{i+1}' \quad (3.100)$$

and the equations for a two-layer problem with these cylindrical harmonics become as below:

$$\mathbf{b}_0 - \bar{\mathbf{T}}_{10} \cdot \bar{\mathbf{B}}_{12} \cdot \mathbf{b}_1 = \bar{\mathbf{R}}_{01} \cdot \bar{\mathbf{B}}_{10} \cdot \mathbf{a}_0 \quad (3.101)$$

$$\mathbf{a}_1 - \bar{\mathbf{R}}_{10} \cdot \bar{\mathbf{B}}_{12} \cdot \mathbf{b}_1 = \bar{\mathbf{T}}_{01} \cdot \bar{\mathbf{B}}_{10} \cdot \mathbf{a}_0 \quad (3.102)$$

$$\mathbf{b}_1 - \bar{\mathbf{R}}_{12} \cdot \bar{\mathbf{B}}_{21} \cdot \mathbf{a}_1 = \mathbf{0} \quad (3.103)$$

$$\mathbf{a}_2 - \bar{\mathbf{T}}_{12} \cdot \bar{\mathbf{B}}_{21} \cdot \mathbf{a}_1 = \mathbf{0} \quad (3.104)$$

Thus, a system of matrix equations such as $\bar{\mathbf{Z}} \cdot \mathbf{x} = \mathbf{y}$ is created and solutions are sought. In here, $\bar{\mathbf{B}}_{ij}$, $\mathbf{b}_i$ and $\mathbf{a}_i$ appear in the coupled case in the following form:

$$\bar{\mathbf{B}}_{i0} = \begin{bmatrix} \bar{\boldsymbol{\beta}}_{ij} & 0 \\ 0 & \bar{\boldsymbol{\beta}}_{ij} \end{bmatrix} \quad (3.105)$$

$$\mathbf{b}_i = \begin{bmatrix} \mathbf{b}_{i,E} \\ \mathbf{b}_{i,H} \end{bmatrix} \quad (3.106)$$

$$\mathbf{a}_i = \begin{bmatrix} \mathbf{a}_{i,E} \\ \mathbf{a}_{i,H} \end{bmatrix} \quad (3.107)$$

3.3. Slow Convergence and Improvement of the Addition Theorem for Near Interactions and Proximate Sources

An outgoing cylindrical wave can be expressed in terms of standing waves using the Graf's addition theorem. This expression is given again below as in [36]:

$$\psi^t(k_\rho, \boldsymbol{\rho}_p) = \Re g \psi^t(k_\rho, \boldsymbol{\rho}_q) \cdot \bar{\alpha}_{qp}, \quad |\boldsymbol{\rho}_q| < d_{pq} \quad (3.108)$$

The more expressive representation of (3.108) is as follows:

$$H_n^{(2)}(k_\rho \rho_p) e^{jn\phi_p} = \sum_{m=-\infty}^{\infty} \left[H_{m-n}^{(2)}(k_\rho d_{pq}) e^{-j(m-n)\phi_{pq}} \right] J_m(k_\rho \rho_q) e^{jm\phi_q}, \quad \rho_q < d_{pq} \quad (3.109)$$

$$[\bar{\alpha}_{pq}]_{m,n} = H_{m-n}^{(2)}(k_\rho d_{pq}) e^{-j(m-n)\phi_{pq}} \quad (3.110)$$

where $d_{pq} = |\boldsymbol{\rho}_p - \boldsymbol{\rho}_q|$ and $\phi_{pq} = \arg\{\boldsymbol{\rho}_p - \boldsymbol{\rho}_q\}$. As can be seen from (3.109), as the distance d_{pq} decreases and ρ_q approaches ρ_p , the function $H_{m-n}^{(2)}(k_\rho d_{pq})$ approaches its singularity. Therefore, the sum in (3.109) converges slowly for small d_{pq} distances, i.e. for very close scatterers each other. This leads to numerical errors and inaccuracies when calculating the scattered field from scatterers close to each other, or when the excitation source is placed too close to the scatterers.

Due to this constraint, scattering problems from circular cross-sectional bodies require a large number of cylindrical harmonics, so that the infinite series in (3.109) is truncated by a large truncation number M. For such problems, the accuracy of the solution increases as M is increased, but this is not the same for arbitrary cross-sectional bodies. For scattering problems involving arbitrary cross-sectional bodies located proximate to each other, or excitation sources such as dipoles or line sources located very close to these bodies, a modification of the method (and Graf's addition theorem summation) is necessary in order to search for the solution with the T matrix method in a salutary way. Such a modification was achieved in [91] for elliptical scatterers whose centers lie within the circumscribing circle of another scatterer by replacing the Hankel function in (3.109) by its integral equivalent and truncating the path along which this spectral integral is taken. Nevertheless, the present approach

needs to be generalized for objects with lossy medium parameters and for each longitudinal plane-wave spectrum (l-PWS) component arising from a dipole source located in the near vicinity of the scatterer.

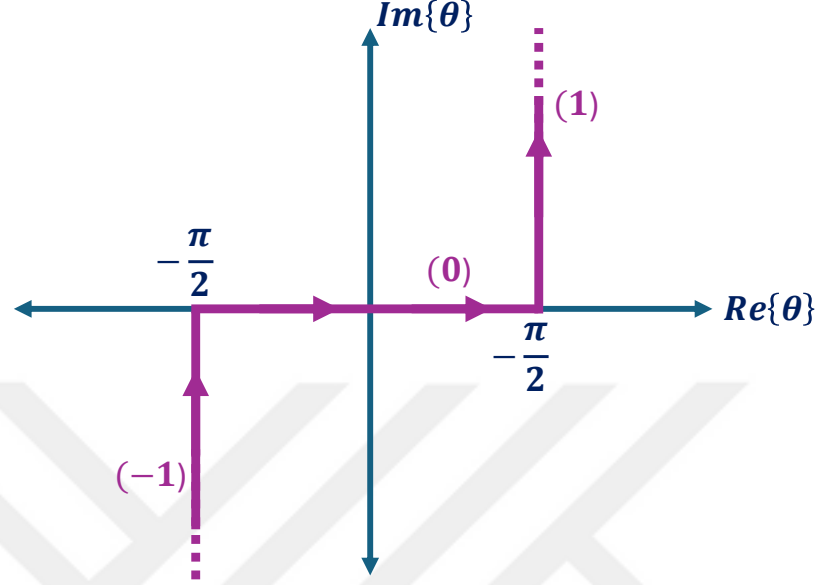


Figure 3.4: Integration path for Schlöfli-Sommerfeld integral.

With this motivation, we shall write the Schlöfli-Sommerfeld Integral for the n^{th} order Hankel function of the second kind as in [126]:

$$H_{m-n}^{(2)}(z) = \frac{1}{\pi} \int_{-j\infty-\frac{\pi}{2}}^{j\infty+\frac{\pi}{2}} (-j)^{-(m-n)} e^{j(z \cos \theta + (m-n)\theta)} d\theta \quad (3.111)$$

It is also possible to express it more compactly as follows:

$$H_{m-n}^{(2)}(z) = W_{m-n}(z) = \frac{1}{\pi} \int_{-j\infty-\frac{\pi}{2}}^{j\infty+\frac{\pi}{2}} F_{m-n}(z, \theta) d\theta \quad (3.112)$$

where

$$F_{m-n}(z, \theta) = (-j)^{-(m-n)} e^{j(z \cos \theta + (m-n)\theta)} \quad (3.113)$$

To analyze the $W_{m-n}(z)$ integral and take it in a proper form, let us divide the path of integration as shown in **Figure 3.4**, and hence the integral, into three parts:

$$\begin{aligned} W_{m-n}(z) &= \frac{1}{\pi} \left[\int_{-j\infty - \frac{\pi}{2}}^{-\frac{\pi}{2}} d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta + \int_{\frac{\pi}{2}}^{j\infty + \frac{\pi}{2}} d\theta \right] \{F_{m-n}(z, \theta)\} \\ &= W_{m-n}^{(-1)}(z) + W_{m-n}^{(0)}(z) + W_{m-n}^{(1)}(z) \end{aligned} \quad (3.114)$$

with definitions below:

$$W_{m-n}^{(-1)}(z) = \frac{1}{\pi} \int_{-j\infty - \frac{\pi}{2}}^{-\frac{\pi}{2}} F_{m-n}(z, \theta) d\theta \quad (3.115)$$

$$W_{m-n}^{(0)}(z) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F_{m-n}(z, \theta) d\theta \quad (3.116)$$

$$W_{m-n}^{(1)}(z) = \frac{1}{\pi} \int_{\frac{\pi}{2}}^{j\infty + \frac{\pi}{2}} F_{m-n}(z, \theta) d\theta \quad (3.117)$$

Now consider the $W_{m-n}^{(-1)}(z)$ integral with $\theta = \theta' - j\theta''$ and $\theta' = -\frac{\pi}{2}$. Hence, $\frac{d\theta}{d\theta''} = -j$ and $d\theta = -jd\theta''$. Moreover, $\theta'' = \frac{\theta + \frac{\pi}{2}}{-j}$ and the integral can be written as follows:

$$W_{m-n}^{(-1)}(z) = \frac{1}{\pi} \int_{\infty}^0 -j F_{m-n}(z, \theta(\theta'')) d\theta'' \quad (3.118)$$

In order to avoid the singularity of the Hankel function, the spectrum of evanescent plane waves should be truncated. Thus,

$$W_{m-n}^{(-1)}(z) \approx \tilde{W}_{m-n}^{(-1)}(z, L) = \frac{1}{\pi} \int_L^0 -j F_{m-n}(z, \theta(\theta'')) d\theta'' \quad (3.119)$$

Note that we converted the infinite integral to finite integral. For numerical implementation of the (3.119) with Gaussian-Legendre (G-L) quadrature, an additional transform $\theta'' = -\frac{L}{2}t + \frac{L}{2}$; $d\theta'' = -\frac{L}{2}dt$ was applied to standardize the expression. As a result, we can use legacy table of G-L quadrature points and weights to integrate:

$$\tilde{W}_{m-n}^{(-1)}(z, L) = \frac{1}{\pi} \int_{-1}^1 \frac{jL}{2} F_{m-n}(z, \theta(t)) dt \quad (3.120)$$

with $t = -\frac{2\theta''}{L} + 1$; $t \in [-1, 1]$. The following integrals are also obtained by a similar procedure:

$$\tilde{W}_{m-n}^{(-1)}(z, L) = \frac{1}{\pi} \int_{-1}^1 \frac{jL}{2} F_{m-n}(z, \theta(t)) dt \quad (3.121)$$

where $\theta = \theta' - j\theta''$, $\theta' = \frac{\pi}{2}$ and $\theta'' = -\frac{L}{2}t - \frac{L}{2}$; $t \in [-1, 1]$. The $\tilde{W}_{m-n}^{(0)}(z, L)$ integral is:

$$\tilde{W}_{m-n}^{(0)}(z, L) = \frac{1}{\pi} \int_{-1}^1 \frac{\pi}{2} F_{m-n}(z, \theta(t)) dt \quad (3.122)$$

with $\theta = \theta' - j\theta''$, $\theta'' = 0$ and $\theta' = \frac{\pi}{2}$. The integrals (3.120), (3.121) and (3.122) can take a general form as follows:

$$\tilde{W}_{m-n}^{(i)}(z, L) = \frac{1}{\pi} \int_{-1}^1 C_i F_{m-n}(z, \theta_i(t)) dt \quad (3.123)$$

$$C_i = \begin{cases} \frac{jL}{2}, & |i| = 1 \\ \frac{\pi}{2}, & |i| = 0 \end{cases} \quad (3.124)$$

$$\theta_i = \begin{cases} -\frac{\pi}{2} - \frac{j(-Lt + L)}{2}, & i = -1 \\ \frac{\pi t}{2}, & |i| = 0 \\ \frac{\pi}{2} - \frac{j(-Lt - L)}{2}, & i = 1 \end{cases} \quad (3.125)$$

and,

$$W_{m-n}(z) \approx \tilde{W}_{m-n}(z, L) = \sum_{i=-1}^1 \tilde{W}_{m-n}^{(i)}(z, L) \quad (3.126)$$

$$\lim_{L \rightarrow \infty} \tilde{W}_{m-n}(z, L) = W_{m-n}(z) = H_{m-n}^{(2)}(z) \quad (3.127)$$

Note that (3.112) and (3.123) integrals valid for $|\arg(z)| < \frac{\pi}{2}$ analytically but used for $|\arg(z)| < \frac{\pi}{4}$ numerically as in [127]. This means that these formulas will work if the medium outside the scatterer is lossless or low-lossy. For $|\arg(z)| > \frac{\pi}{4}$ we will use the Mehler Sonine integrals in [126] which are analytically valid for $-\pi < \arg(z) < 0$:

$$H_{m-n}^{(2)}(z) = M_{m-n}(z) = \frac{je^{\frac{j(m-n)\pi}{2}}}{\pi} \int_{-\infty}^{\infty} G_{m-n}(z, \theta) d\theta \quad (3.128)$$

$$G_{m-n}(z, \theta) = e^{-jz \cosh \theta - (m-n)\theta} \quad (3.129)$$

Truncated version of (3.128) is:

$$M_{m-n}(z) \approx \tilde{M}_{m-n}(z, L) = \frac{je^{\frac{j(m-n)\pi}{2}}}{\pi} \int_{-L}^L G_{m-n}(z, \theta) d\theta \quad (3.130)$$

In a manner similar to what was done to the previous integrals, we apply an additional transformation for numerical implementation of the (3.130) with G-L quadrature. As a result, with $\theta = Lt$, $t = \frac{\theta}{L}$; $t \in [-1,1]$ and $d\theta = Ldt$ we obtain:

$$\tilde{M}_{m-n}(z, L) = \frac{je^{\frac{j(m-n)\pi}{2}}}{\pi} \int_{-L}^L G_{m-n}(z, \theta(t)) dt \quad (3.131)$$

$$\lim_{L \rightarrow \infty} \tilde{M}_{m-n}(z, L) = M_{m-n}(z) = H_{m-n}^{(2)}(z) \quad (3.132)$$

(3.126) and (3.131) are complementary spectrum truncated functions for different $|arg(z)|$ values. Hence, let us define the “(m-n)th order Modified - Spectrum Truncated Hankel Function of second kind” $\tilde{\mathcal{H}}_{m-n}^{(2)}(z)$:

$$\tilde{\mathcal{H}}_{m-n}^{(2)}(z, L) = \begin{cases} \tilde{W}_{m-n}(z, L), & |arg(z)| \leq \frac{\pi}{4} \\ \tilde{M}_{m-n}(z, L), & |arg(z)| > \frac{\pi}{4} \end{cases} \quad (3.133)$$

$$\lim_{L \rightarrow \infty} \tilde{\mathcal{H}}_{m-n}^{(2)}(z, L) = H_{m-n}^{(2)}(z) \quad (3.134)$$

Therefore, translation matrices by using the Graf’s addition theorem can be updated for near proximate/neighbor targets:

$$[\bar{\alpha}_{qp}]_{m,n} = \begin{cases} \tilde{\mathcal{H}}_{m-n}^{(2)}(k_\rho d_{pq}, L) e^{-j(m-n)\phi_{pq}}, & d_{pq} < 0.5\lambda \text{ (small } d_{pq}) \\ H_{m-n}^{(2)}(k_\rho d_{pq}) e^{-j(m-n)\phi_{pq}}, & d_{pq} \geq 0.5\lambda \end{cases} \quad (3.135)$$

Alternatively, for a line source placed near the scatterer of interest, the following modification can be applied:

$$\begin{aligned} & -\frac{j}{4} H_0^{(2)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_s|) \\ & = -\frac{j}{4} \sum_{n=-\infty}^{\infty} \tilde{\mathcal{H}}_n^{(2)}(k_\rho \rho_s, L) J_n(k_\rho \rho) e^{jn(\phi - \phi_s)}, \quad \rho < \rho_s \end{aligned} \quad (3.136)$$

For (3.135) to work properly in the presence of closely spaced scatterers, and similarly for (3.136) when dealing with nearby sources, the choice of L is critical. Although some details are given in [91], since the way the integrals of interest are computed is different in this work, genetic algorithm optimizations as in [127] will guide the determination of the truncation number L in a way that minimizes errors. With a proper choice of the truncation number L , the proposed “Spectrum Truncated Hankel Function” $\tilde{\mathcal{H}}_n^{(2)}(k_\rho \rho, L)$ leads to a significant improvement in the accuracy of the solutions. With an appropriate cylindrical harmonic truncation number, the method improves significantly.



3.4. Analytical Regularization of the T Matrix Method for Scatterers with Arbitrary Cross-Section

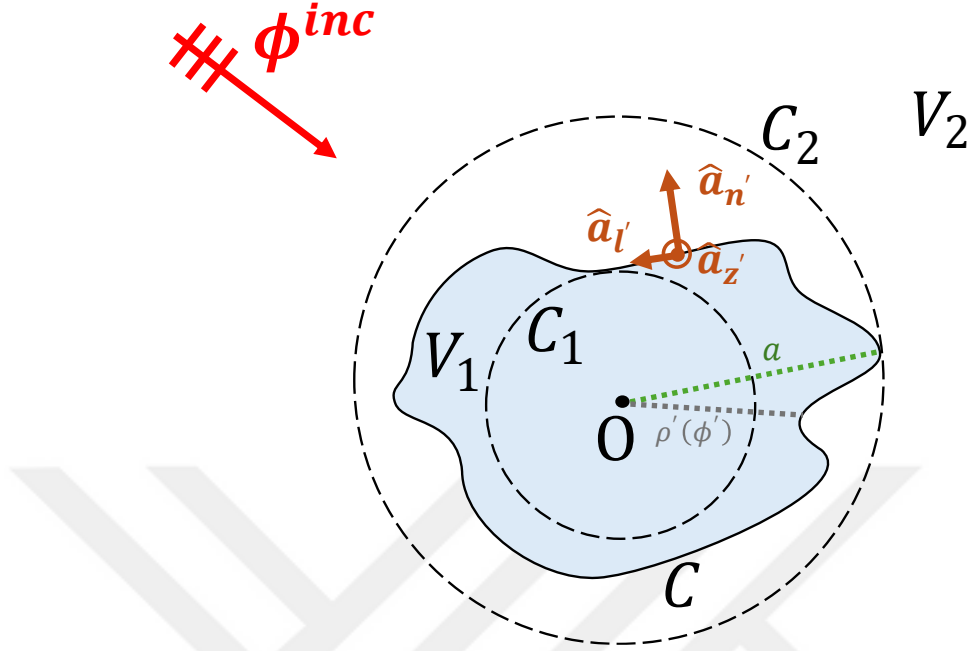


Figure 3.5: Arbitrary contour of a scatterer with its circumscribing and inscribed circles for analytical regularization.

Let us consider an arbitrary contour C as in **Figure 3.5**, which is a boundary of some obstacle. For this boundary, let us apply the following procedure. First, (2.35) and (2.36) are evaluated by setting $i = 1$, (2.37) and (2.38) are used, and the resulting expressions are substituted into (2.31) and (2.32). The same procedure is then repeated for $i = 2$. As a result, the following integral equations hold both outside and inside such a contour:

$$\phi_2(\rho) = \phi^{inc}(\rho) - \int_C \left\{ \frac{\partial \phi_2(\rho')}{\partial n'} G_2(\rho, \rho') - \phi_2(\rho') \frac{\partial G_2(\rho, \rho')}{\partial n'} \right\} dl' ; \rho \in V_2 \quad (3.137)$$

$$\phi^{inc}(\rho) = \int_C \left\{ \frac{\partial \phi_2(\rho')}{\partial n'} G_2(\rho, \rho') - \phi_2(\rho') \frac{\partial G_2(\rho, \rho')}{\partial n'} \right\} dl' ; \rho \in V_1 \quad (3.138)$$

$$0 = \int_C \left\{ \frac{\partial \phi_1(\rho')}{\partial n'} G_1(\rho, \rho') - \phi_1(\rho') \frac{\partial G_1(\rho, \rho')}{\partial n'} \right\} dl' ; \rho \in V_2 \quad (3.139)$$

$$\phi_1(\boldsymbol{\rho}) = \int_c \left\{ \frac{\partial \phi_1(\boldsymbol{\rho}')}{\partial n'} G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') - \phi_1(\boldsymbol{\rho}') \frac{\partial G_1(\boldsymbol{\rho}, \boldsymbol{\rho}')}{\partial n'} \right\} dl' ; \boldsymbol{\rho} \in V_1 \quad (3.140)$$

Here the $\phi(\boldsymbol{\rho})$ field is E_z and H_z and equation (3.137) – (3.140) expresses a z component of the equations involving coupling in the case of excitation with a three-dimensional arbitrary source. It is also known that in the T Matrix method, the fields are expressed in terms of cylindrical harmonics as follows:

$$\phi_2(\boldsymbol{\rho}) = \phi_{inc}(\boldsymbol{\rho}) + \phi_{sca}(\boldsymbol{\rho}) \quad (3.141)$$

$$\phi_{inc}(\boldsymbol{\rho}) = \sum_{n=-\infty}^{\infty} a_n J_n(k_{\rho,2}\rho) e^{jn\phi} \quad (3.142)$$

$$\phi_{sca}(\boldsymbol{\rho}) = \sum_{n=-\infty}^{\infty} f_n H_n^{(2)}(k_{\rho,2}\rho) e^{jn\phi} \quad (3.143)$$

$$\phi_1(\boldsymbol{\rho}) = \sum_{m=-\infty}^{\infty} \alpha_m J_m(k_{\rho,1}\rho) e^{jm\phi} \quad (3.144)$$

The first step of regularization is to determine the nature of the unknown in the matrix system of equations and to regularize this nature into a bounded vector in l_2 space. For this purpose, let us recall the content of the isolated scatterer T matrix and how it is obtained:

$$\bar{\mathbf{T}}_{1(1)} = -(\Re g \bar{\mathbf{Q}}) \cdot \bar{\mathbf{Q}}^{-1} \quad (3.145)$$

$$\bar{\mathbf{Q}} \cdot \boldsymbol{\alpha} = \mathbf{a} \quad (3.146)$$

$$\Re g \bar{\mathbf{Q}} \cdot \boldsymbol{\alpha} = \mathbf{f} \quad (3.147)$$

To obtain the isolated scatterer T matrix accurately, without error, the inverse of the $\bar{\mathbf{Q}}$ matrix must be taken reliably. Therefore, let us consider the matrix system of equations (3.146) and analyze the nature of the $\boldsymbol{\alpha}$, the coefficient of the field transmitted to the scatterer. Firstly, let us perform a Fourier series expansion to the fields $\phi_1(\boldsymbol{\rho})$ and

$\frac{\partial \phi_1(\rho')}{\partial n'}$. Then, let us write the series expression of the Green's function in terms of cylindrical harmonics with the help of the Graf addition theorem:

$$\phi_1(\rho) = \sum_{n=-\infty}^{\infty} u_n^{(1)} e^{jn\phi'} \quad (3.148)$$

$$\frac{\partial \phi_1(\rho')}{\partial n'} l(\phi') = \sum_{n=-\infty}^{\infty} v_n^{(1)} e^{jn\phi'} \quad (3.149)$$

$$\begin{aligned} G_1(\rho, \rho') &= -\frac{j}{4} H_0^{(2)}(k_{\rho,1} |\rho - \rho'|) \\ &= -\frac{j}{4} \sum_{m=-\infty}^{\infty} H_m^{(2)}(k_{\rho,1} \rho') e^{-jm\phi} J_m(k_{\rho,1} \rho) e^{jm\phi}, \quad \rho < \rho' \end{aligned} \quad (3.150)$$

$$\frac{\partial G_1(\rho, \rho')}{\partial n'} = -\frac{j}{4} \sum_{m=-\infty}^{\infty} \frac{\partial}{\partial n'} \{H_m^{(2)}(k_{\rho,1} \rho') e^{-jm\phi}\} J_m(k_{\rho,1} \rho) e^{jm\phi}, \quad \rho < \rho' \quad (3.151)$$

Now let us substitute equations (3.148) – (3.151) in (3.140) to analyze the content of the $\phi_1(\rho)$ field, i.e. the content of the α_m coefficient:

$$\phi_1(\rho) = \sum_{m=-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} \frac{j}{4} u_n^{(1)} I_{m,n}^{u(1)} - \frac{j}{4} v_n^{(1)} I_{m,n}^{v(1)} \right\} J_m(k_{\rho,1} \rho) e^{jm\phi} \quad (3.152)$$

Here the integrals $I_{m,n}^{u(1)}$ and $I_{m,n}^{v(1)}$ are defined as follows:

$$\begin{aligned} I_{m,n}^{u(1)} &= \int_{-\pi}^{\pi} \left\{ \left[k_{\rho,1} H_m'^{(2)}(k_{\rho,1} \rho'(\phi')) \frac{\vec{\rho} \cdot \hat{\mathbf{a}}_{n'}}{\rho'(\phi')} \right. \right. \\ &\quad \left. \left. + jm H_m^{(2)}(k_{\rho,1} \rho'(\phi')) \frac{\vec{\rho} \cdot \hat{\mathbf{a}}_{l'}}{(\rho'(\phi'))^2} \right] e^{-jm\phi} l(\phi') \right\} e^{jn\phi'} d\phi' \end{aligned} \quad (3.153)$$

$$I_{m,n}^{v(1)} = \int_{-\pi}^{\pi} \left\{ H_m^{(2)}(k_{\rho,1} \rho'(\phi')) e^{-jm\phi} \right\} e^{jn\phi'} d\phi' \quad (3.154)$$

From these, it is understood that α_m is as follows:

$$\alpha_m = \sum_{n=-\infty}^{\infty} \frac{j}{4} u_n^{(1)} I_{m,n}^{u(1)} - \frac{j}{4} v_n^{(1)} I_{m,n}^{v(1)} \quad (3.155)$$

Moreover, the following conditions apply for the Fourier coefficients $u_n^{(1)}$ and $v_n^{(1)}$:

$$\sum_{n=-\infty}^{\infty} |u_n^{(1)}|^2 (1 + |n|) < \infty \quad (3.156)$$

$$\sum_{n=-\infty}^{\infty} |v_n^{(1)}|^2 (1 + |n|)^{-1} < \infty \quad (3.157)$$

Conditions in (3.156) and (3.157) mean that values $u_n^{(1)}$ and $v_n^{(1)}$ have the following representation:

$$u_n^{(1)} = \tau_n^{-1} \tilde{u}_n^{(1)} \quad (3.158)$$

$$v_n^{(1)} = \tau_n \tilde{v}_n^{(1)} \quad (3.159)$$

where $\tilde{u}_n^{(1)}$ and $\tilde{v}_n^{(1)}$ are some values which belong to l_2 space:

$$\sum_{n=-\infty}^{\infty} |\tilde{u}_n^{(1)}|^2 < \infty \quad (3.160)$$

$$\sum_{n=-\infty}^{\infty} |\tilde{v}_n^{(1)}|^2 < \infty \quad (3.161)$$

where

$$\tau_n = \max(1, \sqrt{|n|}) \quad (3.162)$$

Therefore, expression (3.155) can be rewritten as follows:

$$\alpha_m = \sum_{n=-\infty}^{\infty} \frac{j}{4} \tau_n^{-1} \tilde{u}_n^{(1)} I_{m,n}^{u(1)} - \frac{j}{4} \tau_n \tilde{v}_n^{(1)} I_{m,n}^{v(1)} \quad (3.163)$$

If the scatterer contour is a circle, (3.163) becomes as follows:

$$\alpha_m = \frac{j\pi}{2} \tau_m^{-1} \tilde{u}_m^{(1)} H_m'^{(2)}(k_{\rho,2}\rho') - \frac{j\pi}{2} \tau_m \tilde{v}_m^{(1)} H_m^{(2)}(k_{\rho,2}\rho') \quad (3.164)$$

The asymptotic expression of the function $\tau_m^{-1} H_m'^{(2)}$ and $\tau_m H_m^{(2)}$ as n goes to infinity is as follows:

$$\tau_m^{-1} H_m'^{(2)}(z) = -\frac{j}{\pi z} \frac{m!}{\sqrt{m}} \left(\frac{z}{2}\right)^{-m} [1 + O(m^{-1})] \quad (3.165)$$

$$\tau_m H_m^{(2)}(z) = \frac{j}{\pi} \frac{m!}{\sqrt{m}} \left(\frac{z}{2}\right)^{-m} [1 + O(m^{-1})] \quad (3.166)$$

The asymptotic expressions of (3.165) and (3.166) are approximately equal for a fixed z and large values of n . Therefore, the following expression can be written:

$$\alpha_m = \tau_m H_m^{(2)}(k_{\rho,2}\rho') \tilde{\alpha}_m = R_{\alpha,m} \tilde{\alpha}_m \quad (3.167)$$

$$\sum_{n=-\infty}^{\infty} |\tilde{\alpha}_m|^2 < \infty \quad (3.168)$$

The expression (3.167) shows the content of the α_m coefficients. If an unknown coefficient (essentially the transmitted field) of finite energy in l_2 space satisfying condition (3.168) is sought, the right-hand regularization operator $R_{\alpha,m}$ is the one that makes this possible. However, (3.167) and (3.168) are valid when the scatterer has a circular cross section. For arbitrary cross-sectional bodies, it is necessary to obtain a finite energy coefficient from the expression in the sum of (3.155) by a similar procedure. For this, let us express the sum (3.155) as follows:

$$\alpha_m = \sum_{n=-\infty}^{\infty} A_{m,n} \quad (3.169)$$

If an asymptotic analysis like that in (3.165) and (3.166) is applied for $\tau_n^{-1} I_{m,n}^{u(2)}$ and $\tau_n I_{m,n}^{v(2)}$, using (3.165) and (3.166), the following result is obtained:

$$\tau_n^{-1} I_{m,n}^{u(1)} = \tau_n I_{m,n}^{v(1)} \circledast \frac{m}{n} L_n^{u(1)} \quad (3.170)$$

where $\circledast$ is discrete linear convolution operator and $L_m^{u(1)}$ is defined as follows:

$$L_n^{u(1)} = \int_{-\pi}^{\pi} \left\{ \left[\frac{\vec{\rho} \cdot \hat{\mathbf{a}}_{n'}}{\rho'(\phi')} + j \frac{\vec{\rho} \cdot \hat{\mathbf{a}}_{l'}}{\rho'(\phi')} \right] \frac{l(\phi')}{\rho'(\phi')} \right\} e^{jn\phi'} d\phi' \quad (3.171)$$

After some mathematical manipulations on (3.171), the expression for $L_n^{u(1)}$ is obtained as follows:

$$L_n^{u(1)} = \hat{\delta}_n - jn \int_{-\pi}^{\pi} \ln(\rho'(\phi')) e^{jn\phi'} d\phi' \quad (3.172)$$

where $\hat{\delta}_n$ is Kronecker delta. Consequently, after substitution of (3.172) to (3.163), the $A_{m,n}$ coefficients in (3.169) are as follows:

$$A_{m,n} = \frac{j}{4} \left(\frac{m}{n} \tilde{u}_n^{(1)} - \tilde{v}_n^{(1)} \right) \tau_n I_{m,n}^{v(1)} - \frac{m}{4} \tilde{u}_n^{(1)} \left(\tau_n I_{m,n}^{v(1)} \circledast b_n \right) \quad (3.173)$$

The expression (3.173) shows that, for bodies of arbitrary cross-section, it is not possible to obtain a coefficient which satisfies condition (3.168) completely by merely multiplying by an operator. In addition, this expression gives an indication that the T matrix method is not valid when the scatterer body has cross-sections that deviate too much from the circle. The approximate expression of (3.173), assuming that the cross section of the body does not deviate too much from the circle:

$$A_{m,n} \cong (\tilde{C}_1 \tilde{u}_n^{(1)} - \tilde{C}_2 \tilde{v}_n^{(1)}) \tau_n I_{m,n}^{v(1)} = R_{\alpha,m,n} \tilde{a}_n \quad (3.174)$$

$$\alpha_m = \sum_{n=-\infty}^{\infty} \tau_n I_{m,n}^{v(1)} \tilde{a}_n \quad (3.175)$$

Using (3.174) and (3.175), the alpha coefficient in matrix-vector product form becomes as follows:

$$\alpha = \bar{R}_\alpha \cdot \tilde{a} \quad (3.176)$$

Here $\bar{R}_\alpha$ is a full matrix in contrast to the right-hand regularization operator for the circular scatterer. Applying the same analysis to (3.138) to obtain the left-hand regularization operator $\bar{L}_\alpha$, the following expression is obtained:

$$a_n = \sum_{m=-\infty}^{\infty} \Lambda_{n,m} \quad (3.177)$$

$$\Lambda_{n,m} \cong (\tilde{C}_1 \tilde{u}_m^{(2)} - \tilde{C}_2 \tilde{v}_m^{(2)}) \tau_m I_{n,m}^{v(2)} = L_{\alpha,n,m}^{-1} \tilde{a}_m \quad (3.178)$$

$$a_n = \sum_{m=-\infty}^{\infty} \tau_m I_{n,m}^{v(2)} \tilde{a}_m \quad (3.179)$$

$$a = \bar{L}_\alpha^{-1} \cdot \tilde{a} \quad (3.180)$$

$$\tilde{a} = \bar{L}_\alpha \cdot a \quad (3.181)$$

The matrix $\bar{L}_\alpha$ in (3.181) is a full matrix as is $\bar{R}_\alpha$. Using (3.176) and (3.181), expression (3.146) can be updated as follows:

$$\bar{L}_\alpha \cdot \bar{Q} \cdot \bar{R}_\alpha \cdot \tilde{a} = \bar{R}_\alpha \cdot a \quad (3.182)$$

$$\bar{L}_\alpha \cdot \bar{Q} \cdot \bar{R}_\alpha \cdot \tilde{a} = \tilde{a} \quad (3.183)$$

Here the $I_{n,m}^{v(2)}$ forming the $\bar{L}_\alpha^{-1}$ matrix is given as follows:

$$I_{n,m}^{v(2)} = \int_{-\pi}^{\pi} \left\{ H_n^{(2)} \left(k_{\rho,2} \rho'(\phi') \right) e^{-jn\phi} \right\} e^{jm\phi'} d\phi' \quad (3.184)$$

$\bar{\mathbf{L}}_\alpha$ and $\bar{\mathbf{R}}_\alpha$ given above can be chosen as regularization operators. However, the difficulty in this choice is that the $\bar{\mathbf{L}}_\alpha$ matrix must be calculated by taking the inverse of the $\bar{\mathbf{L}}_\alpha^{-1}$ matrix. Since the $\bar{\mathbf{L}}_\alpha^{-1}$ matrix is a full matrix, even when taking the inverse of the regularization operator, the inverse may not be taken accurately due to the ill-conditionality of the matrix. Instead, diagonal regularization matrices are more useful, and the inverses of these matrices can be easily calculated. Such a right-hand regularization matrix can be calculated by the component of the integrand of $I_{m,n}^{v(2)}$ where rho takes its maximum value i.e. radius of the circumscribing circle C_1 . In this case, the left-hand regularization matrix is the matrix product of the inverse of the right-hand regularization matrix $\bar{\mathbf{R}}_\alpha$ and the inverse of the diagonal of the matrix $\bar{\mathbf{Q}}$. This means:

$$\bar{\mathbf{R}}_\alpha = \text{diag} \left[H_n^{(1)} \left(k_{\rho,1} \max(\rho'(\phi')) \right) \right] \quad (3.185)$$

$$\bar{\mathbf{L}}_\alpha = \bar{\mathbf{L}}_\alpha^{(1)} \cdot \bar{\mathbf{L}}_\alpha^{(2)} \quad (3.186)$$

$$\bar{\mathbf{L}}_\alpha^{(1)} = \left[\text{diag}[\bar{\mathbf{Q}}] \right]^{-1} \quad (3.187)$$

$$\bar{\mathbf{L}}_\alpha^{(2)} = \bar{\mathbf{R}}_\alpha^{-1} \quad (3.188)$$

For the $\bar{\mathbf{L}}_\alpha$ and $\bar{\mathbf{R}}_\alpha$ regularization operators defined above refer to the first choice of regularization operator, while $\bar{\mathbf{L}}_\alpha$ and $\bar{\mathbf{R}}_\alpha$ above denote the second choice. For a scatterer with a circular cross section, (3.185) and (3.187) reduce to the following form, while (3.186) and (3.188) remain valid:

$$\bar{\mathbf{R}}_\alpha = \text{diag} \left[H_n^{(1)}(k_{\rho,1}a) \right] \quad (3.189)$$

$$\bar{\mathbf{L}}_\alpha^{(1)} = \bar{\mathbf{P}}^{-1} \quad (3.190)$$

the diagonal entries of the matrix $\bar{\mathbf{Q}}$ for circle of radius a being defined as the diagonal matrix $\bar{\mathbf{P}}$:

$$\bar{\mathcal{P}} = \text{diag} \left\{ -\frac{j\pi a}{2} \left(H_n^{(2)}(k_{\rho,2}a) \hat{\mathbf{a}}_{n'} \cdot \nabla' J_n(k_{\rho,1}a) \frac{p_1}{p_2} - J_n(k_{\rho,1}a) \hat{\mathbf{a}}_{n'} \cdot \nabla' H_n^{(2)}(k_{\rho,2}a) \right) \right\} \quad (3.191)$$

Equations (3.186), (3.188), (3.189), and (3.190) define the regularization operators for a circular cross section and can also be applied as a third choice for scatterer with arbitrary cross section by taking $a = \max(\rho'(\phi'))$, i.e., the extended circular boundary of the non-null field region [128]. As in [36,76], for the set of equations that leads to the same T-matrix via the Rayleigh formulation, regularization operators based on traveling waves are defined as follows:

$$\bar{\mathcal{R}}_f = \text{diag} \left[H_n^{(1)}(k_{\rho,2}a) \right]^{-1} \quad (3.192)$$

$$\bar{\mathcal{L}}_f = \bar{\mathcal{L}}_f^{(1)} \cdot \bar{\mathcal{L}}_f^{(2)} \quad (3.193)$$

$$\bar{\mathcal{L}}_f^{(1)} = \bar{\mathcal{P}}^{-1} \quad (3.194)$$

$$\bar{\mathcal{L}}_f^{(2)} = \bar{\mathcal{R}}_f^{-1} \quad (3.195)$$

Thus, two well-conditioned systems based on the extended boundary condition method and the Rayleigh hypothesis are obtained respectively, both leading to the same T-matrix [128]:

$$\bar{\mathcal{L}}_\alpha \cdot \bar{\mathcal{Q}} \cdot \bar{\mathcal{R}}_\alpha \cdot \tilde{\alpha} = \bar{\mathcal{L}}_\alpha \cdot \mathbf{a} \quad (3.196)$$

$$\bar{\mathcal{L}}_f \cdot \bar{\mathcal{Q}}^t \cdot \bar{\mathcal{R}}_f \cdot \tilde{\mathbf{f}} = \bar{\mathcal{L}}_f \cdot \Re g \bar{\mathcal{Q}}^t \cdot \mathbf{a} \quad (3.197)$$

Note that the $H_n^{(1)}$ or its reciprocal is used for the factorization of the asymptotic behaviours for coefficients α and $\mathbf{f}$ instead of $H_n^{(2)}$ and J_n respectively. Because $H_n^{(1)}$ imposes its zeros on the phasors which are in the Riemann sheet that is impossible to meet within the time-harmonic regime of the problem [76]. In the most general case, $k_{\rho,i}$ may be a complex number. This can arise when the excitation angle θ becomes complex, as in formulations where the response to a dipole source is obtained by solving the problem separately for each k_z component, or it may result from the lossy

nature of the scatterers. Therefore, to regularize the behavior that renders the system ill-conditioned and to ensure the correct asymptotic behavior, the complex conjugate of k_ρ should be used in the regularization operators instead of k_ρ . With this choice, the zeros of the Hankel function $H_n^{(1)}$ do not coincide with those of $H_n^{(2)}$, thereby preserving the physically correct asymptotic behavior. As a result, while preserving the validity of the regularization operators given in (3.186) and (3.193), the most general forms of $\bar{\mathcal{R}}_\alpha$ and $\bar{\mathcal{R}}_f$ are obtained for p^{th} scatterer as follows [127]:

$$\bar{\mathcal{R}}_{f,p} = \text{diag}\left[H_n^{(1)}(\text{conj}(k_{\rho,p^-})a_p)\right]^{-1} \quad (3.198)$$

$$\bar{\mathcal{R}}_{\alpha,p} = \text{diag}\left[H_n^{(1)}(\text{conj}(k_{\rho,p})a_p)\right] \quad (3.199)$$

where $\text{conj}(k_{\rho,i})$ is complex conjugate of $k_{\rho,p}$ and p^- denotes exterior region of p^{th} scatterer. For the multiple scatterer case, consider the two-scatterer matrix equation $\bar{\mathbf{Z}} \cdot \mathbf{f} = \mathbf{b}$ constructed from (3.73) and (3.74). In this case, the regularization operators must take the form of a 2×2 block-diagonal matrix to act on the blocks associated with both the E_z and H_z equations. Accordingly, the coupled 2.5D regularization operators are given as follows:

$$\bar{\mathcal{L}}_{f,p}^c = \begin{bmatrix} \bar{\mathcal{L}}_{f,p}^{(2)} & 0 \\ 0 & \bar{\mathcal{L}}_{f,p}^{(2)} \end{bmatrix} \bar{\mathcal{L}}_\alpha^c \quad (3.200)$$

$$\bar{\mathcal{R}}_{f,p}^c = \begin{bmatrix} \bar{\mathcal{R}}_{f,p} & 0 \\ 0 & \bar{\mathcal{R}}_{f,p} \end{bmatrix} \quad (3.201)$$

where the superscript $\mathcal{C}$ denotes the coupling case. Using (3.200) and (3.201), the regularization operators that incorporate the regularization for each scatterer block in the system and are applied to the entire equation system are given as follows:

$$\bar{\mathcal{R}} = \begin{bmatrix} \bar{\mathcal{R}}_{f,1}^c & \mathbf{0} \\ \mathbf{0} & \bar{\mathcal{R}}_{f,2}^c \end{bmatrix} \quad (3.202)$$

$$\bar{\mathcal{L}} = \begin{bmatrix} \bar{\mathcal{L}}_{f,1}^c & \mathbf{0} \\ \mathbf{0} & \bar{\mathcal{L}}_{f,2}^c \end{bmatrix} \quad (3.203)$$

(3.202) and (3.203) apply to the resulting system $\bar{\mathcal{Z}} \cdot \mathbf{f} = \mathbf{b}$ for the 2.5D multiple scattering problem. If right-hand and left-hand regularization operators are applied to such a system,

$$\bar{\mathcal{L}} \cdot \bar{\mathcal{Z}} \cdot \bar{\mathcal{R}} = \bar{\mathbf{I}} + \bar{\mathbf{K}} \quad (3.204)$$

$$\mathbf{z} = \bar{\mathcal{R}}^{-1} \cdot \mathbf{f} \quad (3.205)$$

$$\mathbf{g} = \bar{\mathcal{L}} \cdot \mathbf{b} \quad (3.206)$$

will be appeared and thus the well-conditioned $(\bar{\mathbf{I}} + \bar{\mathbf{K}}) \cdot \mathbf{z} = \mathbf{g}$ system is solved to find the unknowns. In the case where the E_z and H_z fields are decoupled, as discussed in Chapter 2, the matrices $\bar{\mathcal{R}}$ and $\bar{\mathcal{L}}$ are replaced by $\bar{\mathcal{L}}_{f,p}^{(2)}$ and $\bar{\mathcal{R}}_{f,p}$ instead of 2×2 block diagonal matrices. Above, the equation system and the corresponding regularization operators have been formulated for a two-scatterer configuration; the extension to a system comprising N scatterers follows straightforwardly.

3.5. Regularized and FMM Accelerated T Matrix Method

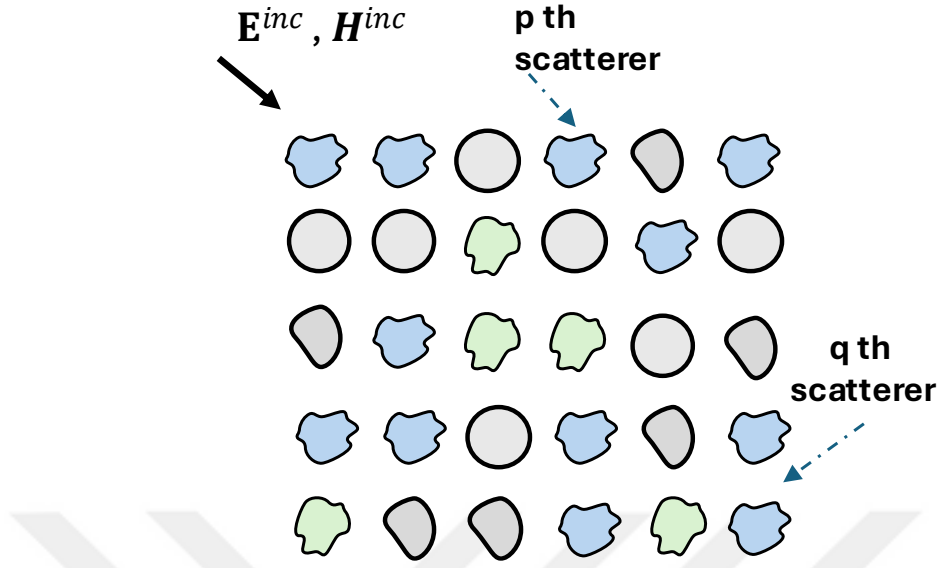


Figure 3.6: N cylindrical rods with different cross sections.

In this section, the TMM for a scattering problem involving N scatterers is considered together with the regularization procedure introduced in the previous section, and its acceleration for far interactions is achieved using the FMM. In the fast multipole approach reported in [61], the truncation number M of the cylindrical harmonics in the T-matrix method must be selected at the outset of the solution, and the box sizes are determined accordingly.

With this choice, the resulting accelerated scheme reduces the computational complexity from $O(N^2)$ to $O(N^{1.5})$. However, this approach does not guarantee the accuracy of the solution. Although it accelerates the computation, the resulting system remains ill-conditioned in nature. If the cylindrical harmonic number M is increased, the condition number of the system grows significantly. Moreover, larger box sizes are required, which may cause far scatterers to be grouped into near interactions.

Conversely, when a small value of M is employed, smaller boxes are formed, and solutions can be obtained since the condition number of the ill-conditioned system does not become excessively large. This situation forces the choice of M to depend on the electrical sizes of the scatterers, making the approach inherently M dependent.

If the system is made well-conditioned through regularization, the accuracy of the solutions no longer exhibits sensitivity to the choice of M . In this case, M can be

selected arbitrarily without incurring accuracy loss due to ill-conditioning, while the classical error bounds of the FMM remain applicable.

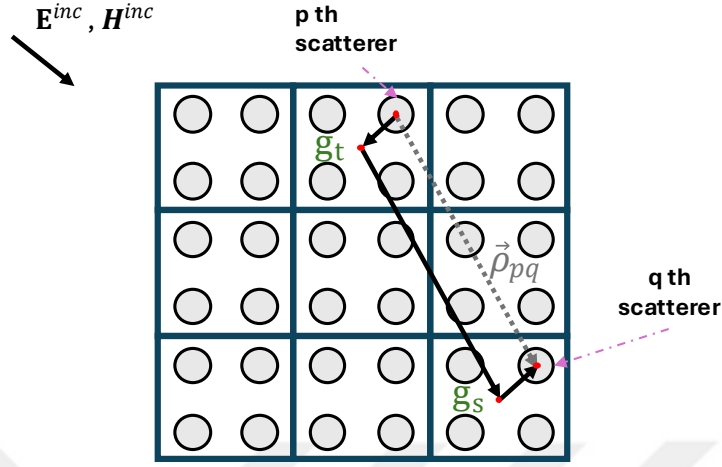


Figure 3.7: Groups of cylindrical rods for fast multipole approximation.

To construct the regularized and accelerated T-matrix method, consider a set of cylindrical rods shown in **Figure 3.6**, where the rods may have different cross sections (shapes) and distinct electromagnetic properties, such as dielectric permittivity and magnetic permeability. In a manner like (3.73) and (3.74), for a system of N scatterers, the unknown coefficients of the fields scattered by each rod can be expressed in terms of the incident field and the contributions from the scattered field coefficients of the other rods as follows:

$$\mathbf{f}_p = \bar{\mathbf{T}}_{p(1)} \cdot \left(\bar{\mathbf{B}}_{p0} \cdot \mathbf{a} + \sum_{\substack{q=1 \\ q \neq p}}^N \bar{\mathbf{A}}_{pq} \cdot \mathbf{f}_q \right) \quad (3.207)$$

where $\bar{\mathbf{T}}_{p(1)}$ denotes the isolated scatterer T matrix for p th scatterer, $\mathbf{f}_p$ represents the coefficients of the fields scattered by the p th scatterer, and $\mathbf{a}$ denotes the coefficients of the incident fields, which are given in the following form:

$$\bar{\mathbf{T}}_{p(1)} = -\Re g \bar{\mathbf{Q}}_{p0} \cdot \bar{\mathbf{Q}}_{p0}^{-1} \quad (3.208)$$

$$\mathbf{f}_p = \begin{bmatrix} \mathbf{f}_p^{E_z} \\ \mathbf{f}_p^{H_z} \end{bmatrix} \quad (3.209)$$

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}^{E_z} \\ \mathbf{a}^{H_z} \end{bmatrix} \quad (3.210)$$

and the translation matrices $\bar{\mathbf{A}}_{pq}$ and $\bar{\mathbf{B}}_{p0}$ have the form given below:

$$\bar{\mathbf{B}}_{p0} = \begin{bmatrix} \bar{\boldsymbol{\beta}}_{p0} & 0 \\ 0 & \bar{\boldsymbol{\beta}}_{p0} \end{bmatrix} \quad (3.211)$$

$$\bar{\mathbf{A}}_{pq} = \begin{bmatrix} \bar{\boldsymbol{\alpha}}_{pq} & 0 \\ 0 & \bar{\boldsymbol{\alpha}}_{pq} \end{bmatrix} \quad (3.212)$$

and from (3.110), we know that

$$[\bar{\boldsymbol{\alpha}}_{pq}]_{m,n} = H_{m-n}^{(2)}(k_\rho d_{pq}) e^{-j(m-n)\phi_{pq}} \quad (3.213)$$

Before applying regularization to the system, a fast multipole expansion is constructed for (3.213). For the grouping shown in **Figure 3.7**, instead of applying translation directly from the q^{th} scatterer to the p^{th} scatterer, the expansion is carried out via a three-stage translation: from the q^{th} scatterer to the source group g_s , from the source group g_s to the target group g_t , and from the target group g_t to the p^{th} scatterer. By applying the raising and lowering operators for Bessel and Hankel functions given in [61] and [36] to the fast multipole expression valid for $H_0^{(2)}(k_\rho d_{pq})$, the following expansion is obtained [61]:

$$[\bar{\boldsymbol{\alpha}}_{pq}]_{m,n} \cong \frac{1}{2\pi} \int_0^{2\pi} \beta_n^{qs}(\varphi) T^{st}(\varphi, P) \beta_m^{*pt}(\varphi) d\varphi \quad (3.214)$$

$$\beta_n^{qs}(\varphi) = e^{jk_\rho(x_{qs} \cos \varphi + y_{qs} \sin \varphi) + jn(\varphi + \frac{\pi}{2})} \quad (3.215)$$

$$\beta_m^{pt}(\varphi) = e^{-jk_\rho(x_{pt} \cos \varphi + y_{pt} \sin \varphi) - jm(\varphi + \frac{\pi}{2})} \quad (3.216)$$

$$T^{st}(\varphi, P) = \sum_{r=-P}^P H_r^{(2)}(k_\rho d_{st}) e^{-jr(\phi_{st}-\varphi-\frac{\pi}{2})} \quad (3.217)$$

where superscript “*” is complex conjugate, $x_{pt} = x_p - x_t$, $y_{pt} = y_p - y_t$, $x_{qs} = x_q - x_s$ and $y_{qs} = y_q - y_s$. To decompose the $\bar{\alpha}_{pq}$ matrix into a product of three sparse matrices for far interaction, the integral (3.214) is discretized with uniform theta samples and the following sum is obtained:

$$\left[\bar{\alpha}_{pq}^{far} \right]_{m,n} \cong \frac{1}{Q} \sum_{v=1}^Q \beta_n^{qs}(\varphi_v) T^{st}(\varphi_v, P) \beta_m^{*pt}(\varphi_v) = \frac{1}{Q} \bar{\mathbf{V}}_p^+ \cdot \bar{\mathbf{M}}_{pq} \cdot \bar{\mathbf{V}}_q \quad (3.218)$$

where $\varphi_v = \frac{2\pi v}{Q-1}$, $\bar{\mathbf{V}}_p^+$ and $\bar{\mathbf{V}}_q$ are $M \times Q$ disaggregation and $Q \times N$ aggregation matrix, respectively. Also, $\bar{\mathbf{M}}_{pq}$ is $Q \times Q$ diagonal translation matrix. The superscript “+” denotes the conjugate transpose matrix. By partitioning the scatterers into near and far interactions and using (3.218) and (3.207), the following matrix equation system is obtained:

$$\left[\bar{\mathbf{I}} + \bar{\mathbf{T}} \cdot \left(\bar{\mathbf{A}}^{near} + \frac{1}{Q} \bar{\mathbf{V}}^+ \cdot \bar{\mathbf{M}} \cdot \bar{\mathbf{V}} \right) \right] \cdot \mathbf{f} = \bar{\mathbf{T}} \cdot \bar{\mathbf{B}} \cdot \mathbf{a} \quad (3.219)$$

Equation (3.219) represents the fast multipole accelerated T-matrix formulation, where $\bar{\mathbf{A}}^{near}$ accounts for near group interactions. The sparse matrices $\bar{\mathbf{V}}^+$, $\bar{\mathbf{M}}$ and $\bar{\mathbf{V}}$ correspond to the disaggregation, translation, and aggregation matrix, respectively. This equation system remains ill-conditioned, and such ill-conditioning delays convergence even when matrix–vector products and iterative solution methods are employed. To make the system well-conditioned, the regularization operators given in (3.202) and (3.203), derived in the previous section, are applied as in (3.204) – (3.206). Consequently, (3.219) is transformed into the following form:

$$\left[\bar{\mathbf{I}} + \bar{\mathcal{L}} \cdot \bar{\mathbf{T}} \cdot \left(\bar{\mathbf{A}}^{near} + \frac{1}{Q} \bar{\mathbf{V}}^+ \cdot \bar{\mathbf{M}} \cdot \bar{\mathbf{V}} \right) \cdot \bar{\mathcal{R}} \right] \cdot \tilde{\mathbf{f}} = \bar{\mathcal{L}} \cdot \bar{\mathbf{T}} \cdot \bar{\mathbf{B}} \cdot \mathbf{a} \quad (3.220)$$

Equation (3.220) represents the well-conditioned fast multipole accelerated T-matrix formulation and can be decomposed into the following equations:

$$\begin{aligned}
\tilde{\mathbf{f}}_p^{E_z} = & \bar{\mathcal{L}}_{f,p}^{(2)} \cdot \bar{\mathbf{T}}_{p(1)}^{(TM \rightarrow TM)} \cdot \left(\bar{\boldsymbol{\beta}}_{p0} \cdot \mathbf{a}^{E_z} + \sum_{q \in \mathcal{G}_S; \mathcal{G}_S \in \mathcal{N}\{\mathcal{G}_t\}}^N \bar{\boldsymbol{\alpha}}_{pq}^{near} \cdot \bar{\mathbf{R}}_{f,q} \cdot \tilde{\mathbf{f}}_q^{E_z} \right. \\
& \left. + \sum_{q \in \mathcal{G}_S; \mathcal{G}_S \in \mathcal{F}\{\mathcal{G}_t\}}^N \bar{\boldsymbol{\alpha}}_{pq}^{far} \cdot \bar{\mathbf{R}}_{f,q} \cdot \tilde{\mathbf{f}}_q^{E_z} \right) \\
& + \bar{\mathcal{L}}_{f,p}^{(2)} \cdot \bar{\mathbf{T}}_{p(1)}^{(TM \rightarrow TE)} \cdot \left(\bar{\boldsymbol{\beta}}_{p0} \cdot \mathbf{a}^{H_z} + \sum_{q \in \mathcal{G}_S; \mathcal{G}_S \in \mathcal{N}\{\mathcal{G}_t\}}^N \bar{\boldsymbol{\alpha}}_{pq}^{near} \cdot \bar{\mathbf{R}}_{f,q} \cdot \tilde{\mathbf{f}}_q^{H_z} \right. \\
& \left. + \sum_{q \in \mathcal{G}_S; \mathcal{G}_S \in \mathcal{F}\{\mathcal{G}_t\}}^N \bar{\boldsymbol{\alpha}}_{pq}^{far} \cdot \bar{\mathbf{R}}_{f,q} \cdot \tilde{\mathbf{f}}_q^{H_z} \right)
\end{aligned} \tag{3.221}$$

and

$$\begin{aligned}
\tilde{\mathbf{f}}_p^{H_z} = & \bar{\mathcal{L}}_{f,p}^{(2)} \cdot \bar{\mathbf{T}}_{p(1)}^{(TE \rightarrow TE)} \cdot \left(\bar{\boldsymbol{\beta}}_{p0} \cdot \mathbf{a}^{H_z} + \sum_{q \in \mathcal{G}_S; \mathcal{G}_S \in \mathcal{N}\{\mathcal{G}_t\}}^N \bar{\boldsymbol{\alpha}}_{pq}^{near} \cdot \bar{\mathbf{R}}_{f,q} \cdot \tilde{\mathbf{f}}_q^{H_z} \right. \\
& \left. + \sum_{q \in \mathcal{G}_S; \mathcal{G}_S \in \mathcal{F}\{\mathcal{G}_t\}}^N \bar{\boldsymbol{\alpha}}_{pq}^{far} \cdot \bar{\mathbf{R}}_{f,q} \cdot \tilde{\mathbf{f}}_q^{H_z} \right) \\
& + \bar{\mathcal{L}}_{f,p}^{(2)} \cdot \bar{\mathbf{T}}_{p(1)}^{(TE \rightarrow TM)} \cdot \left(\bar{\boldsymbol{\beta}}_{p0} \cdot \mathbf{a}^{E_z} + \sum_{q \in \mathcal{G}_S; \mathcal{G}_S \in \mathcal{N}\{\mathcal{G}_t\}}^N \bar{\boldsymbol{\alpha}}_{pq}^{near} \cdot \bar{\mathbf{R}}_{f,q} \cdot \tilde{\mathbf{f}}_q^{E_z} \right. \\
& \left. + \sum_{q \in \mathcal{G}_S; \mathcal{G}_S \in \mathcal{F}\{\mathcal{G}_t\}}^N \bar{\boldsymbol{\alpha}}_{pq}^{far} \cdot \bar{\mathbf{R}}_{f,q} \cdot \tilde{\mathbf{f}}_q^{E_z} \right)
\end{aligned} \tag{3.222}$$

It should be noted that in the above formulas, $\mathcal{N}$ indicates near groups, $\mathcal{F}$ indicates far groups and N is the number of scatterers. In the implementation of the well-conditioned FMM accelerated T matrix method, the isolated scatterer T matrices are first calculated by means of the regularized $\bar{\mathbf{Q}}_{p0}$ matrices in the form shown below.

$$\bar{\mathbf{T}}_{p(1)} = - \left(\Re g \bar{\mathbf{Q}}_{p0} \right) \cdot \bar{\mathbf{R}}_{\alpha}^c \cdot \left(\bar{\mathcal{L}}_{\alpha}^c \cdot \bar{\mathbf{Q}}_{p0} \cdot \bar{\mathbf{R}}_{\alpha}^c \right)^{-1} \cdot \bar{\mathcal{L}}_{\alpha}^c \tag{3.223}$$

$$\overline{\mathbf{T}}_{p(1)} = \begin{bmatrix} \overline{\mathbf{T}}_{p(1)}^{(TM \rightarrow TM)} & \overline{\mathbf{T}}_{p(1)}^{(TM \rightarrow TE)} \\ \overline{\mathbf{T}}_{p(1)}^{(TE \rightarrow TM)} & \overline{\mathbf{T}}_{p(1)}^{(TE \rightarrow TE)} \end{bmatrix} \quad (3.224)$$

Then the scatterers are grouped and the matrix vector products (3.221) and (3.222) are performed and the unknown regularized coefficient $\tilde{\mathbf{f}}$ of the scattered field is calculated using an iterative solver like generalized minimal residual method (GMRES) or biconjugate gradient stabilized method (BiCGStab). Finally, using the following expression,

$$\overline{\mathbf{R}} \cdot \tilde{\mathbf{f}} = \mathbf{f} \quad (3.225)$$

the unknown coefficient of the scattered field is found. With the FMM approach, a faster solution is expected to be obtained than storing the scattering matrix and solving it by taking its inverse. In addition, the nature of this system is well-conditioned and therefore, with increasing number of truncations, the iterative solution of the system is expected to converge in fewer iterations than before regularization.

4. NUMERICAL RESULTS

This chapter presents a comprehensive set of numerical results corresponding to the problems detailed and formulated in Chapters 2 and 3, together with the proposed improvements, and is organized into four subsections.

4.1. Numerical Results for Analytical Regularization: Well-Conditioned T Matrix Method

In this subsection, a well-conditioning strategy for the T-Matrix algorithm exemplified on cylindrical dielectric scatterers and layers of arbitrary cross section was introduced.

Figure 4.1 illustrates the scatterer types and corrugation/elongation levels for contours generated via The Modified Superformula.

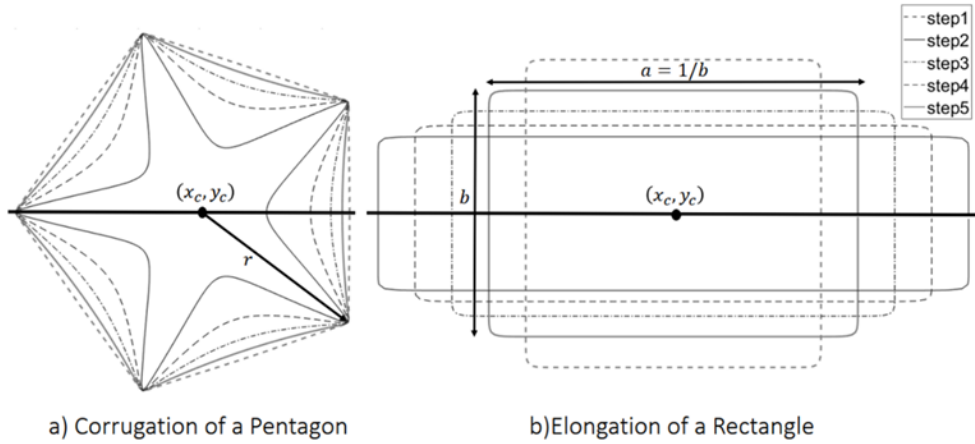


Figure 4.1: Scatterer contours generated with modified superformula and their corrugations and elongations.

The Modified Superformula Contour C with the parametrizations in [129] is given as input to the calculation of the integrals of the $\bar{Q}$ matrices in the T Matrix method and $\rho(\phi)$ is as below:

$$\rho(\phi) = \left[\left(\frac{\sqrt{\cos^2\left(\frac{m\phi}{4}\right) + \epsilon_1^2}}{a} \right)^{n_2} + \left(\frac{\sqrt{\sin^2\left(\frac{m\phi}{4}\right) + \epsilon_2^2}}{b} \right)^{n_3} \right]^{\frac{1}{n_1}} \quad (4.1)$$

For numerical results, the line source emitting waves of length λ_0 is either situated to the central coordinates to each contour i.e., (x_c, y_c) , or in the center of the global coordinates $0.5\lambda_0$ far from the leftmost nearest point of the scatterer along the horizontal axis. The electrical sizes of scatterers are scaled with the usage of the parameter r exploited below defined as $r_i = t_i\lambda_0$ where $t_i = [0.01, 0.05, 0.1, 0.5, 1]$.

With $a = b = 1$, $m = 5$, $n_2 = n_3 = 60$, choosing $n_1 \in \{100, 80, 60, 40, 20\}$ (4.1) corrugates the pentagon to a star in five steps. Here (4.1) is normalized by its maximum before scaling it with r to electrical dimension preserving the circumscribing circle radius to be $r: r \times \frac{\rho(\phi)}{\max(\rho(\phi))}, \phi \in (-\pi, \pi]$.

Setting $a \in \{1, 1.25, 1.5, 1.75, 2\} = \frac{1}{b}$, $m = 5$, $n_1 = n_2 = n_3 = 30$, (4.1) elongates the rectangle in five steps where its scaling to the electrical dimension is performed with $r \times \rho(\phi), \phi \in (-\pi, \pi]$ preserving its λ^2 area.

The infinitely smooth boundary of 2D isolated scatterer of relative permittivity $\epsilon_r = 4$ situated in free space ($\epsilon_r = 1$) is parameterized using (4.1), choosing $\epsilon_{1,2} = 0$. Such arrangements for (4.1) are critical not only working out the conditioning of the T Matrix Method weakening with the corrugation and elongation of the boundaries but also for the super algebraic convergence of the BIE solver in [93] used for benchmark.

Predicting the truncation number of the infinite linear algebraic equation systems of the first kind which are ill-conditioned (LAES_i) solving for an isolated scatterer always requires a preprocess depending on the satisfaction of the boundary conditions with a certain error. This error depends on the size of this scatterer and must be controlled either by confirmation of boundary condition for each truncation or via usage of the empirical formula [130]. The complexity of the latter during the multiple scattering calculations makes the satisfaction of the former critical. Even for eigen-surfaces as scatterers, the excess of the critical truncation number in the corresponding ill-conditioned LAES_i (that a double precision environment can only shift it to its larger values rather than eliminating it) leads to noisy solutions introduced by severe round-off errors. But the well-conditioned LAES_w provides all the precision that a computer can deliver as truncation numbers keep on increasing [76].

Here we elaborate on similar aspects during calculation of the scattering using T-matrix algorithm, from an arbitrary shaped object accounting its corrugation and

elongation which erode the condition of the system. Even before detailing the numerical results setup, we can apply to the help of **Table 4.1** and **Table 4.2** in order to develop the understanding of the characteristics of the conditioning before and after the regularization we employ, where we pick two excitation cases and illuminate pentagonal/tetragonal dielectric cylindrical boundary with low and high corrugation/elongation both at different electrical sizes, respectively. We witness single occurrence of regularization outcome as in eigen-surfaces as in [76] i.e. for the low level of corrugation of a pentagon, where the condition number of $\bar{\mathbf{Q}} - \bar{\mathbf{L}}_\alpha \cdot \bar{\mathbf{Q}} \cdot \bar{\mathbf{R}}_\alpha$, i.e. before - after regularization is growing with acceleration - uniformly bounded respectively. When the pentagon is highly corrugated, even though we lose the uniform bound after regularization we have a better conditioned system than before its regularization. Keeping in mind that, roughly in a 16 decimal digit precision environment, the condition number of the levels 10^γ , provide $(16 - \gamma)$ correct figure precision as $\gamma \geq 16$ would mean total loss of accuracy. Thus, after the applied regularization we not only can extend the correct precision range for the truncation numbers but also can trace the direct effect of corrugation. We can draw exactly the same conclusions for highly elongated tetragon example much stronger and observe that we cannot exceed 15 as a truncation number for some correct decimal figures in the calculation where this number was 30 in previous example. Even for the low elongation case we witness that the regularization works for well-conditioning leading to a slow increase rather than uniform bound for the condition numbers. There are three reasons we can point out for this behavior: (i) the polarization, (ii) the electrical size being larger than previous example, and (iii) exciting with line source inside meets concave boundaries, i.e. effective corrugation accompanies the low level of elongation. The third reason is more generalizable than the other two as a rule of thumb.

As has been built within the context of **Table 4.1** and **Table 4.2**, the regularizers described in subsection 3.4 established for a circularly cylindrical boundary [76] are specified here to the obstacle of arbitrary shape by setting its constant radius to that of the closest extended circular boundary in the non-null (source) field region with the constitutive parameters of the null-field region. We discuss the validity of the suggested regularization and its well-conditioning the T-matrices of the excited with line source that can be situated outside or inside the isolated cylindrical object under 5 different corrugation and elongation levels modeled by an infinitely smooth

parametrization in polar coordinates [129] of 5 different electrical sizes (for brevity, 3 of TM polarized and 2 of TE polarized are used among 5) upon the results in **Figure 4.2-Figure 4.5**.

Table 4.1: Typical condition number behavior of the nonregularized and regularized T-matrix algebraic systems for low and high-level corrugated scatterers under exterior TE line source excitation.

Truncation Number M	TE line source outside exciting pentagon with $r_3 = 0.1\lambda_0$ size			
	No corrugation – level 0		High corrugation – level 4	
	LAES _i	LAES _w	LAES _i	LAES _w
5	4.520726E+04	1.463209E+00	2.374010E+07	1.476931E+02
10	3.688527E+15	1.597232E+00	1.730765E+23	6.491989E+05
15	1.623003E+29	1.647045E+00	3.723354E+39	2.093149E+07
20	6.794381E+46	1.673941E+00	3.276547E+61	1.441523E+09
25	3.339624E+62	1.690682E+00	2.388575E+79	1.081758E+12
30	1.291599E+78	1.701773E+00	8.276802E+95	1.699688E+14
35	5.342410E+92	1.709357E+00	1.353189E+113	2.121259E+16
40	1.706067E+106	1.714623E+00	2.107318E+128	7.331803E+18
45	2.141027E+121	1.718299E+00	1.909644E+146	1.824876E+23
50	3.256766E+136	1.720863E+00	1.033283E+164	6.051799E+28

Table 4.2: Typical condition number behavior of the nonregularized and regularized T-matrix algebraic systems for low and high-level elongated scatterers under interior TM line source excitation.

Truncation Number M	TM line source inside exciting tetragon with $r_4 = 0.5\lambda_0$ size			
	No elongation – level 0		High elongation – level 4	
	LAES _i	LAES _w	LAES _i	LAES _w
5	2.342328E+00	4.822262E+00	9.360798E+01	1.182854E+03
10	2.036235E+03	4.974743E+01	1.453967E+04	9.821883E+08
15	4.363847E+06	6.147461E+01	1.038366E+07	9.052588E+13
20	9.429246E+11	8.366477E+01	3.586011E+10	9.231106E+18
25	1.855299E+19	2.197917E+02	1.127861E+17	5.459557E+24
30	3.868848E+27	9.369490E+02	1.120411E+30	1.965060E+30
35	1.652012E+37	8.341208E+03	1.763890E+42	3.346918E+34
40	1.350175E+47	8.486358E+04	1.113437E+57	6.786387E+36
45	1.445377E+59	1.135594E+06	8.136292E+69	4.285280E+40
50	1.345394E+70	1.342509E+07	3.113532E+82	3.222684E+43

The definitions of source locations and size formations are below. **Table 4.1** and **Table 4.2** exploits other cases than those of **Figure 4.2-Figure 4.5**. The well-conditioned

version of this algorithm with the formula given in subsection 3.4 is used throughout these processes.

The abscissa, ordinate pair of local frames for **Figure 4.2**, **Figure 4.3** and **Figure 4.4**, **Figure 4.5** are the truncation numbers of the algebraic systems and from none to high, 5 levels of corrugation and elongation, respectively.

The ordinates of global frames for **Figure 4.2**, **Figure 4.3** and **Figure 4.4**, **Figure 4.5** are samples from the used line source excitation where the 5 row bottom-up line up has $r_{2,4}$ for TE and $r_{1,3,5}$ for TM cases picked for brevity among 5 levels of obstacle sizes and two different polarizations. Each consecutive pair of the abscissas of 4 columns of global frames present data when the line source is placed outside and inside, respectively.

Let the relative l_2 norm of Y with respect to X be abbreviated as follows: $[Y, X]_2 = \|X - Y\|_2 / \|X\|_2$. This is equivalent to relative version of root mean square (RMS) deviation of Y from X . All the relative errors of Y with respect to X are calculated using this formula in the definition of figures below.

The abscissas of each consecutive pair of 4 columns of global frames for **Figure 4.2** and **Figure 4.3** stand for relative errors for the scattered field values observed on the extended cylindrical boundary C_{1-2} (like in **Figure 3.5**), respectively. First and second of these pairs present a comparison of values obtained via LAES_w and LAES_i w.r.t. the BIE method in [93] and LAES_w explained in this work. The physical relevance of the results that LAES_w provides is clearly traceable in the comparisons made with BIE at a large enough truncation number especially when corrugation or elongation is not increased to a certain level. These results exhibit large deviations from that of LAES_i as truncation number increases as well as the effect of elongation and corrugation does.

The abscissas of 4 columns of global frames for **Figure 4.4** and **Figure 4.5** bear data of the condition number of the regularized matrix $\bar{\mathcal{L}}_\alpha \cdot \bar{\mathbf{Q}} \cdot \bar{\mathcal{R}}_\alpha$ and relative error of the coefficients α_i obtained via the relation $\alpha_i = \bar{\mathbf{Q}}^{-1} \cdot \mathbf{a}$ w.r.t coefficients α_w where $\alpha_w = \bar{\mathcal{R}}_\alpha \cdot \tilde{\mathbf{a}}$. Working only with these transmission coefficients (α) to save space contents us since their characteristics before and after regularization reflect parallel features with that of the reflection coefficients ($\mathbf{f} = -\Re g \bar{\mathbf{Q}} \cdot \bar{\mathbf{Q}}^{-1} \cdot \mathbf{a} = -[\bar{\mathbf{Q}}^t]^{-1} \cdot \Re g \bar{\mathbf{Q}}^t \cdot \mathbf{a} = \bar{\mathcal{R}}_f \cdot \tilde{\mathbf{f}}$) based on the expressions in subsection 3.4. These transmission

coefficients also exhibit large deviations before and after regularization agreeing on a few initial coefficients only, which are due to the ill-conditioned nature of $LAES_i$ tabulated with accelerated increase of their condition numbers in **Table 4.1** and **Table 4.2**. On the contrary the condition numbers of $LAES_w$ here exhibit much predictably increasing character owing to the increasing elongation and corrugation. When effects of these are coming in sight the condition number exhibits a regulated version of the increase before the regularization. Otherwise, a uniform bound for the condition number behavior with increasing truncation is observed. We understand that the applied regularization based on the extended circular boundary is sufficient for these latter cases while for the former they do not fully and there is still room for further improvement. We can conclude that after applying the suggested regularization we can reach solutions with much greater number of significant digits than before its application provided an error controlling mechanism.

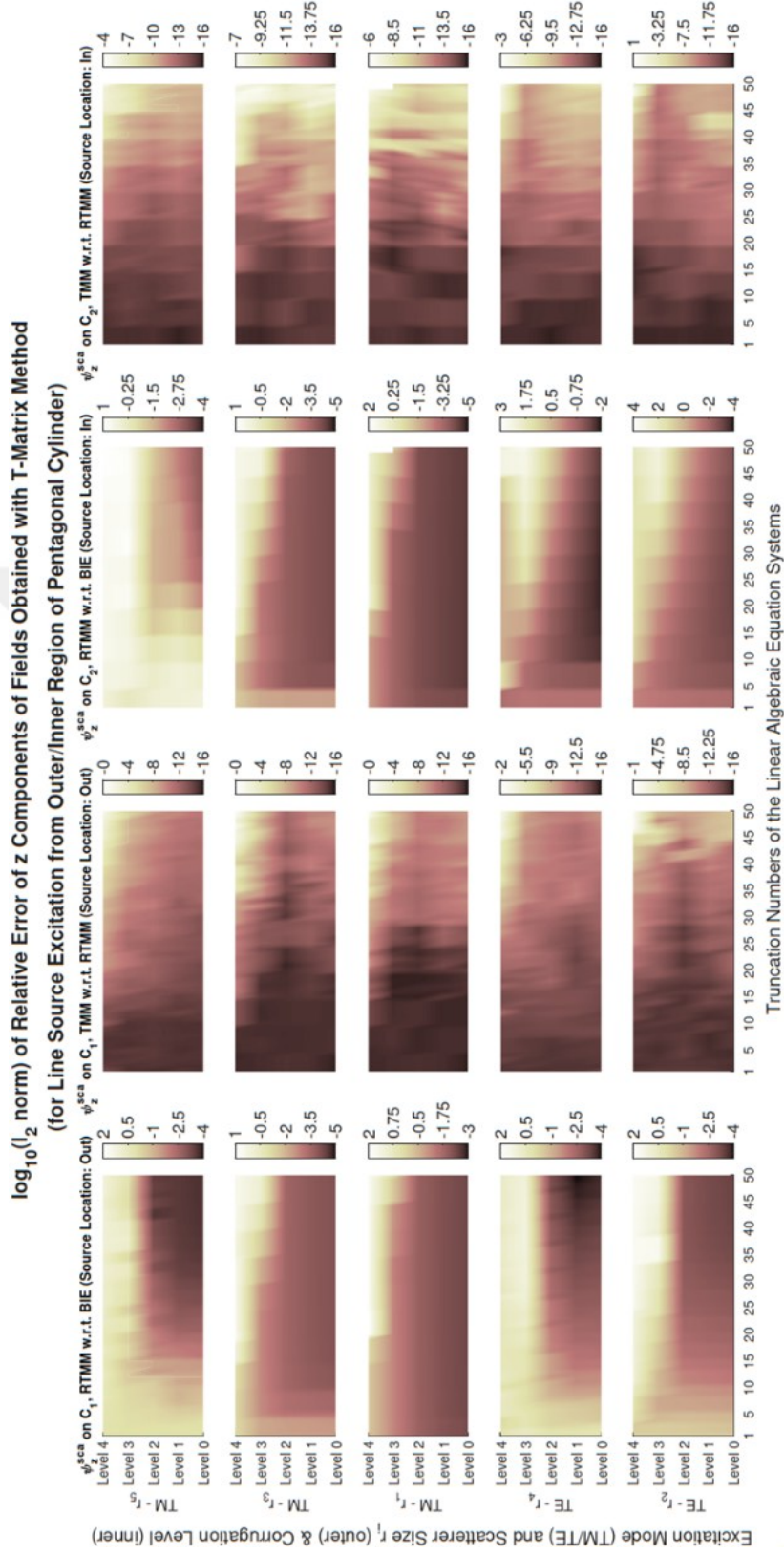


Figure 4.2: Plots with respect to increasing truncation number of the algebraic systems regarding the relative error of the z component of electric (TM)- magnetic(TE) scattered fields from the dielectric pentagon of electrical sizes $\mathbf{r}_{1,3,5}$ and $\mathbf{r}_{2,4}$ at rows (1-3) - (4-5) measured on the extended boundary. The excitation with a line source outside - inside are worked out at columns (1-2) - (3-4) with 5 steps of corrugation from levels 0 to 4. The relative errors in columns (1,3) and (2,4) belong to (the regularized T-Matrix method w.r.t the BIE and (traditional T-Matrix method w.r.t the regularized one) respectively.

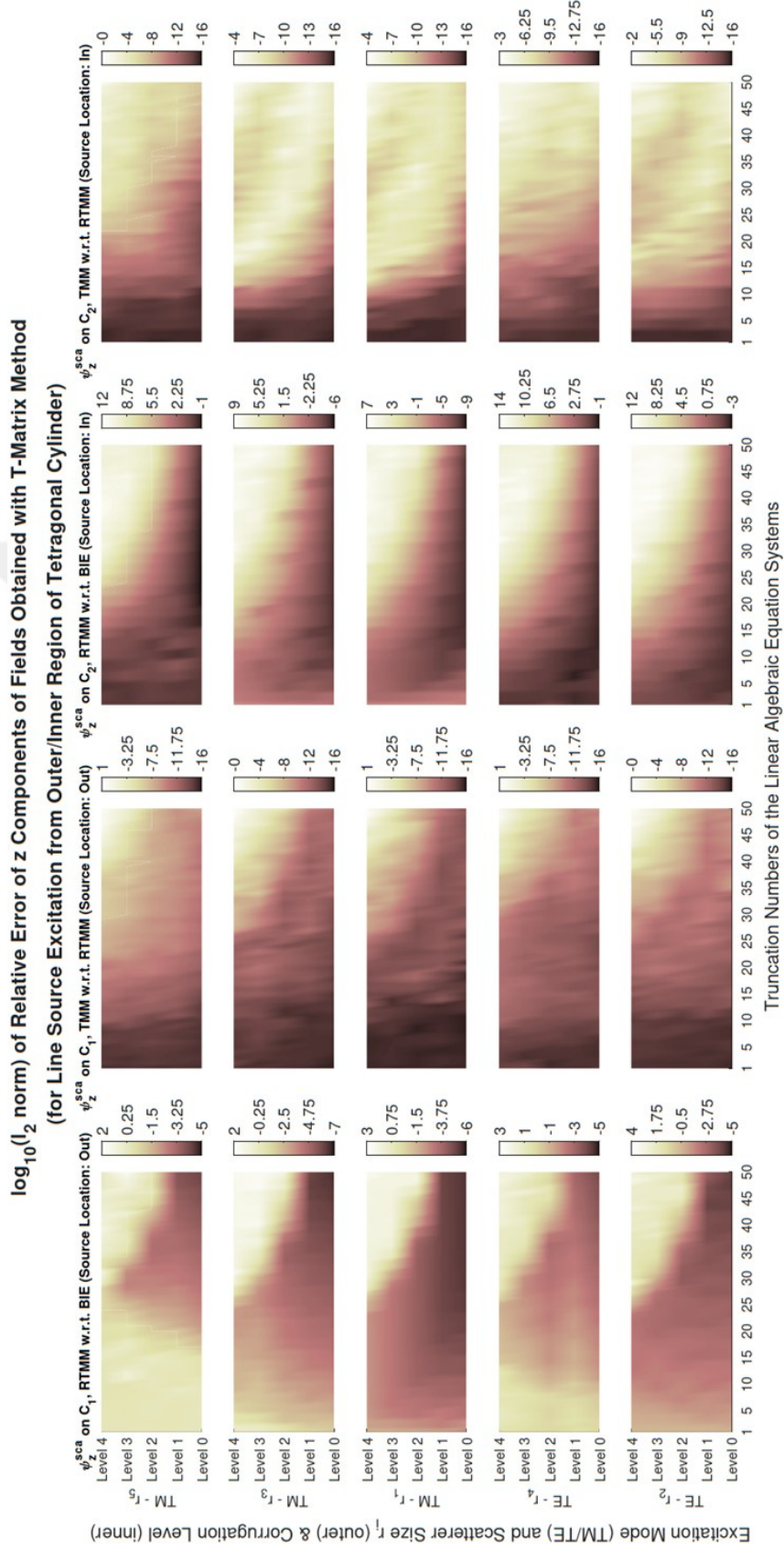


Figure 4.3: Plots with respect to increasing truncation number of the algebraic systems regarding the relative error of the z component of electric (TM)- magnetic(TE) scattered fields from the dielectric tetragon of electrical sizes $r_{1,3,5}$ and $r_{2,4}$ at rows (1-3) - (4-5) measured on the extended boundary. The excitation with a line source outside - inside are worked out at columns (1-2) - (3-4) with 5 steps of elongation from levels 0 to 4. The relative errors in columns (1,3) and (2,4) belong to (the regularized T-Matrix method w.r.t the BIE) and (traditional T-Matrix method w.r.t the regularized one) respectively.

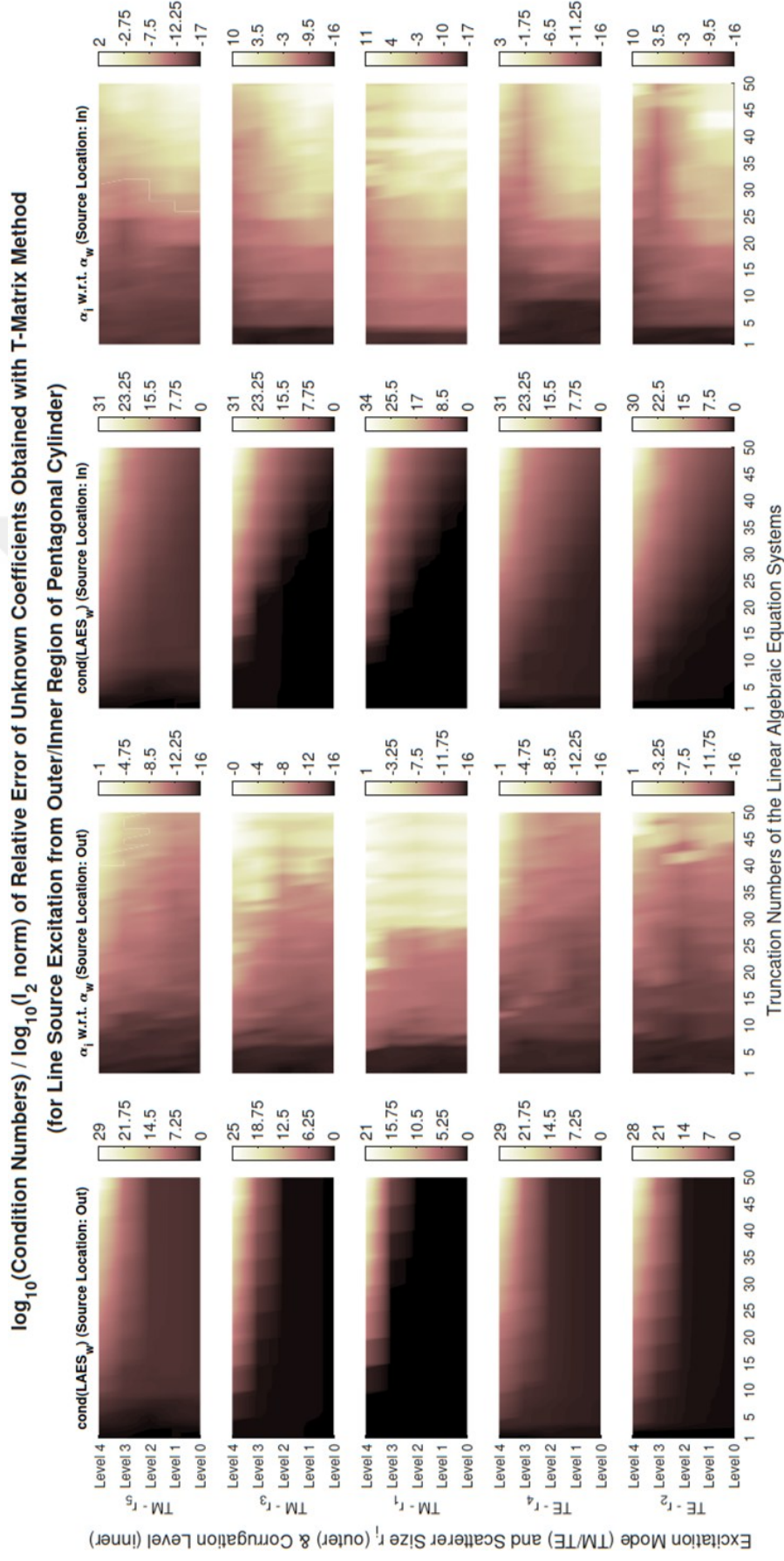


Figure 4.4: Plots with respect to increasing truncation number of the algebraic systems regarding the condition numbers of the well-conditioned LAES (LAES_w) at columns (1,3) and relative errors of the solution vectors LAES_i provide w.r.t solution vectors LAES_w provide at columns (2,4) used for the excitation scenarios in **Figure 4.2** with 5 steps of corrugation from levels 0 to 4.

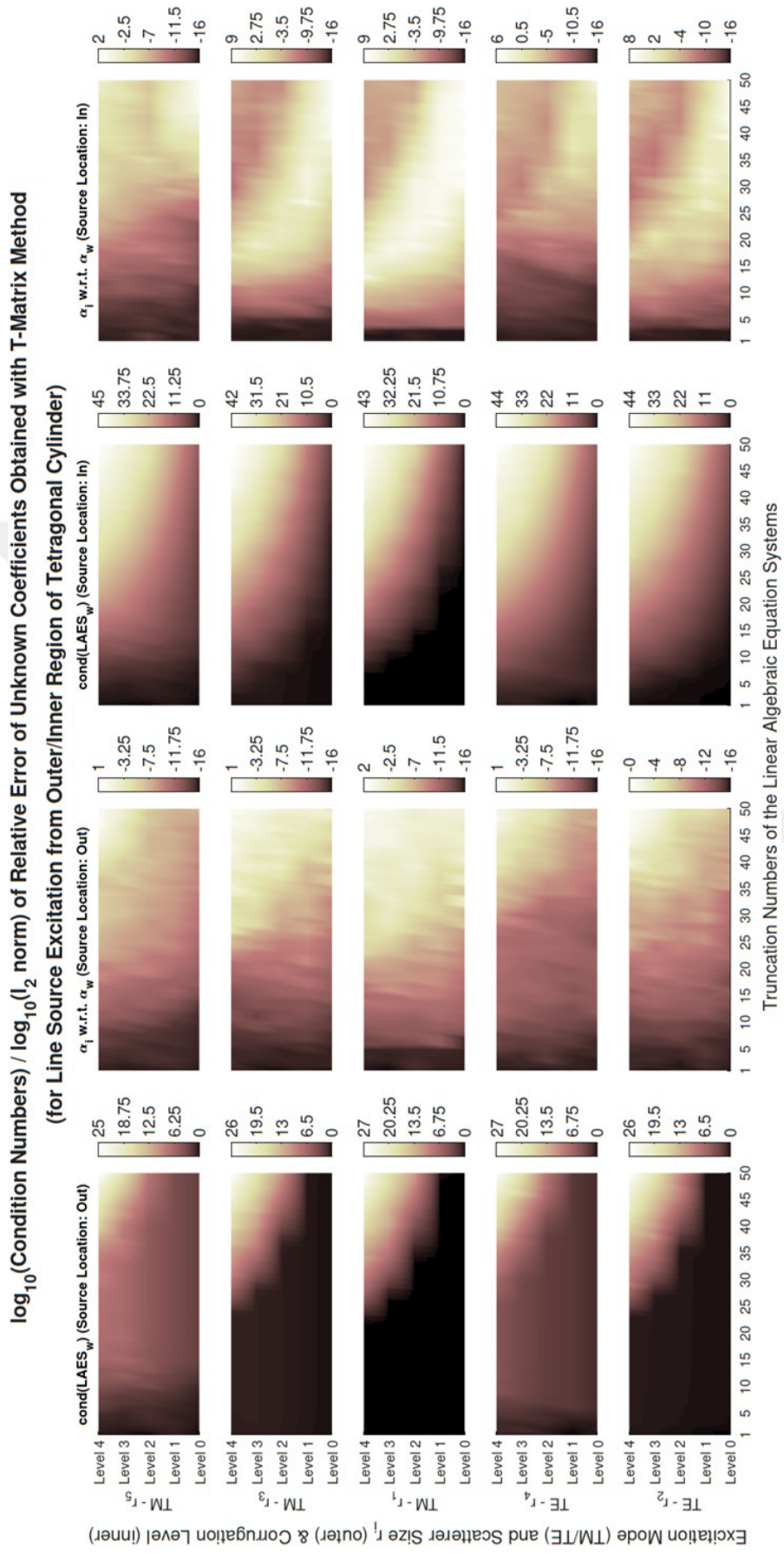


Figure 4.5: Plots with respect to increasing truncation number of the algebraic systems regarding the condition numbers of the well-conditioned LAES (LAES_w) at columns (1,3) and relative errors of the solution vectors LAES_i provide w.r.t solution vectors LAES_w provide at columns (2,4) used for the excitation scenarios **Figure 4.3** with 5 steps of elongation from levels 0 to 4.

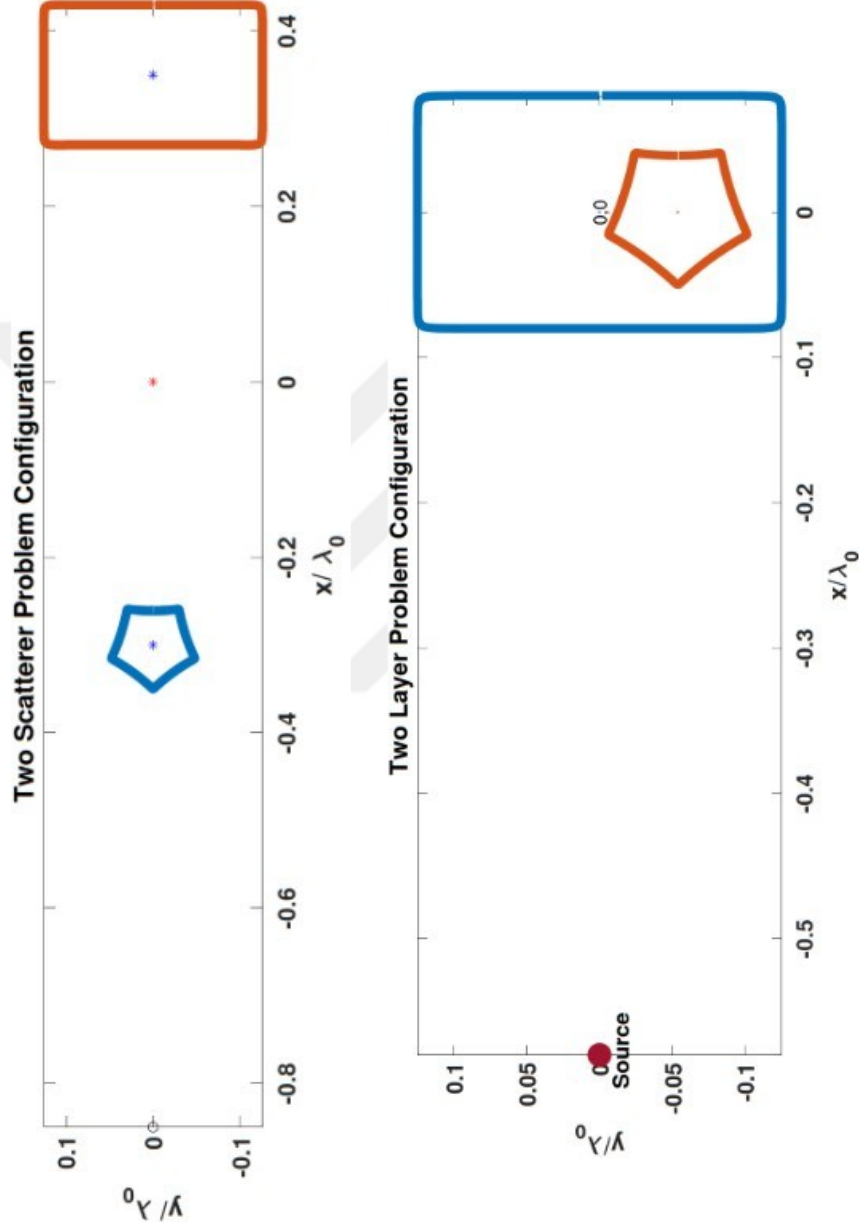


Figure 4.6: Multiple and multilayer scatterer problem configuration. (top) A nonmagnetic pentagonal scatterer with $\mathbf{r}_2$ electrical size, corrugation level 2 and $\epsilon_r = 4$ dielectric permittivity and a nonmagnetic tetragonal scatterer with $\mathbf{r}_3$ electrical size, elongation level 2 and $\epsilon_r = 16$ dielectric permittivity. (bottom) A nonmagnetic pentagonal scatterer with $\mathbf{r}_2$ electrical size, corrugation level 2 and $\epsilon_r = 4$ dielectric permittivity and a nonmagnetic tetragonal scatterer with $\mathbf{r}_3$ electrical size, elongation level 2 and $\epsilon_r = 4$ dielectric permittivity.

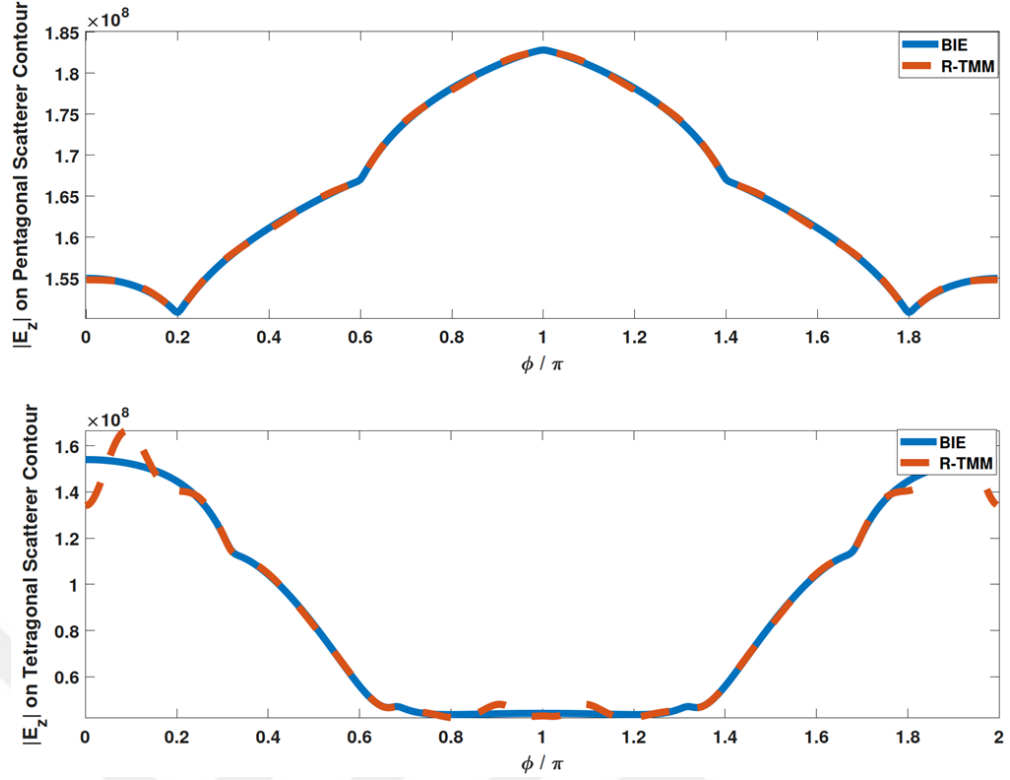


Figure 4.7: Absolute value of z component of electric field (l component of magnetic current density) on scatterer contours. (top) On pentagonal scatterer contour. (bottom) On tetragonal scatterer contour.

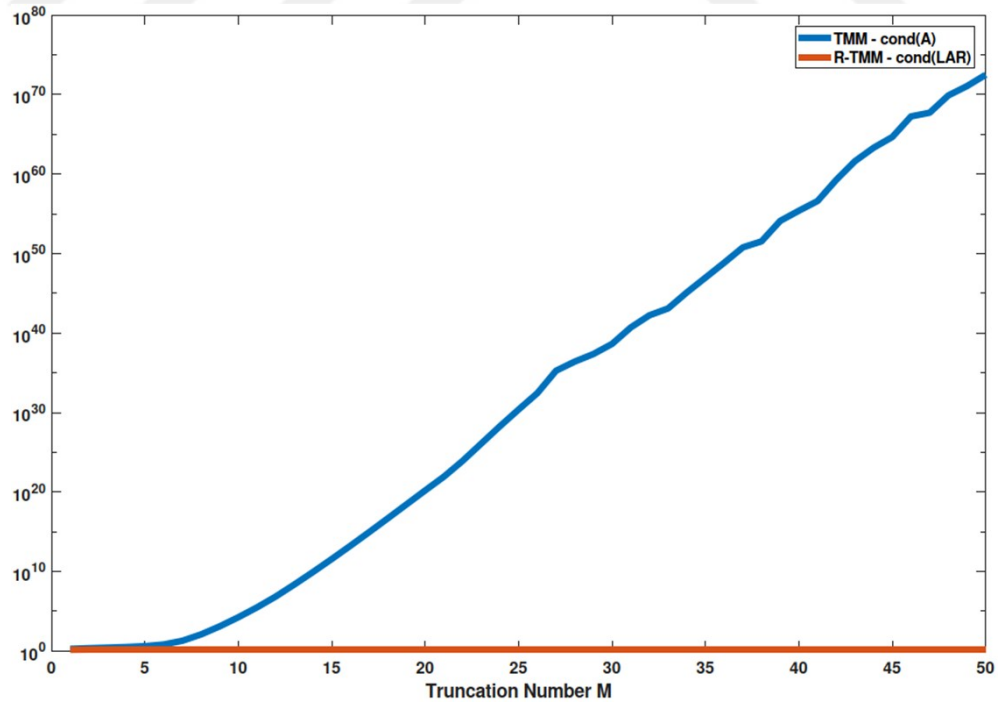


Figure 4.8: Condition numbers of resulting LAES from T Matrix and Regularized T Matrix Method for two scatterer case. (LAR : LAES_w, A: LAES_i). For $M=9$, $\text{cond}(A) = 1244.29$, and $\text{cond}(\text{LAR}) = 1.415$.

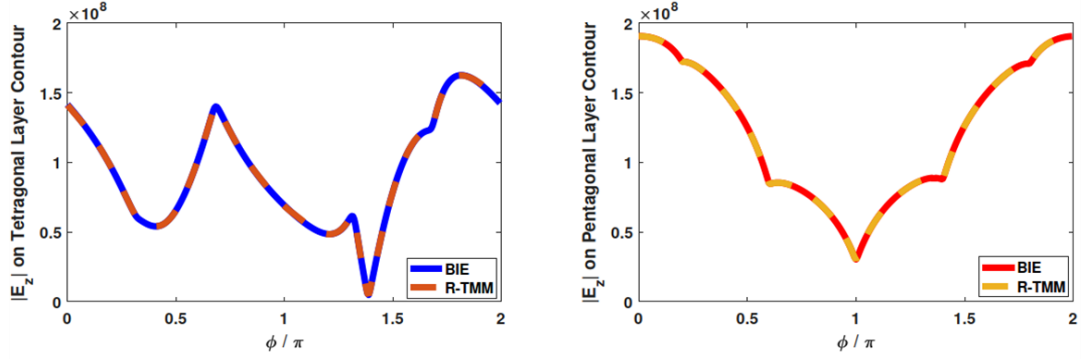


Figure 4.9: Absolute value of z component of electric field (l component of magnetic current density) on layer contours. On tetragonal layer contour (left). On pentagonal layer contour (right).

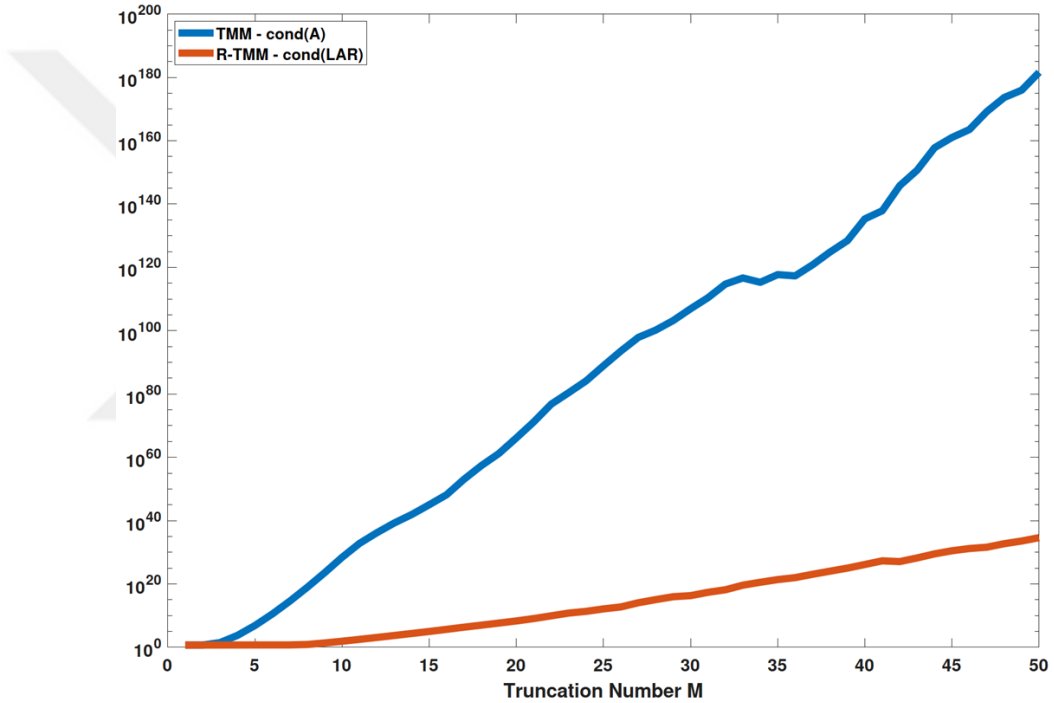


Figure 4.10: Condition numbers of resulting LAES from T Matrix and Regularized T Matrix Method for two scatterer case. (LAR : LAES_w, A: LAES_i). For M=9, cond(A) = 1244.29, and cond(LAR) = 1.415.

From an overall perspective, **Figure 4.2** and **Figure 4.3** show the relative errors of the solutions obtained by the Regularized (Well-Conditioned) TMM for pentagonal and tetragonal cylinders with boundary integral equation solutions with respect to the electrical size and deformation level of the object contour. In addition, **Figure 4.4** and **Figure 4.5** show the relative errors of the unknown coefficients obtained by the Regularized T Matrix Method (R-TMM) with solutions of the unknown coefficients obtained by the traditional TMM for the pentagonal and tetragonal cylinders separately

and the conditioning of the linear system of equations with respect to the electrical size and deformation level of the contours.

Besides, **Figure 4.7** and **Figure 4.9** illustrate the validity of the R-TMM in the light of the results in **Figure 4.2** to **Figure 4.5** for a multiple and multiple layer problem illustrated in **Figure 4.6**, respectively. Moreover, **Figure 4.8** and **Figure 4.10** illustrate the improving effect of regularization on the TMM for the same problems, respectively.

Consequently, **Figure 4.2** to **Figure 4.5** and **Figure 4.7** to **Figure 4.10** contains a comprehensive analysis of the T Matrix method for the effect of analytical regularization on the method. The results show that the regularization of the TMM makes it possible to obtain a well-conditioned numerically stable system that is insensitive to the increasing number of cylindrical harmonic truncations. However, the expansion of the fields on the scatterer into cylindrical harmonics is not suitable for arbitrary sections, especially for very long or very corrugated arbitrary sections, which leads to errors in the numerical results.

Inevitable native weakness of this method, i.e., the geometrical ill-conditioning effects of increasing corrugation and elongation of these cross sections became numerically traceable with this novel strategy that reduces the capacity for potential round-off errors through the applied regularization on the circular extended boundaries in the counter-null-fields. A super algebraically convergent boundary integral equation formulation is taken as benchmark solver under excitation of a line source both from inside and outside the chosen variety of boundaries demonstrating the physical relevance of the new results obtained. Regarding the regularization of multiple scattering of arbitrary cross-section boundaries, the multiple scatterer and multilayer case is considered, and it is shown that the solutions are consistent with the solutions obtained by direct solution of the integral equations and that the regularization is an operation that improves the conditioning of LAES for problems involving arbitrary cross-sections.

4.2. Numerical Results for Arbitrary Shape Analytical Regularization Operators and Their Use in an FMM-Accelerated System

In this subsection, the arbitrary cross section regularization operators are tested for various cases. The scatterers of interest for the regularization are given in **Figure 4.11**. Consider cylindrical rods with arbitrary cross section and different sizes. The parameters of rods are $c = 0.5\lambda_0$ and $0.05\lambda_0$, $\varepsilon_1 = 4\varepsilon_0$ and, $\mu = 0.25\mu_0$. The condition numbers of the $\bar{\mathbf{Q}}$ matrices calculated for these scatterers and the condition number of regularized matrices by the regularization operators explained in subsection 3.4 are as shown in **Figure 4.12** - **Figure 4.15**.

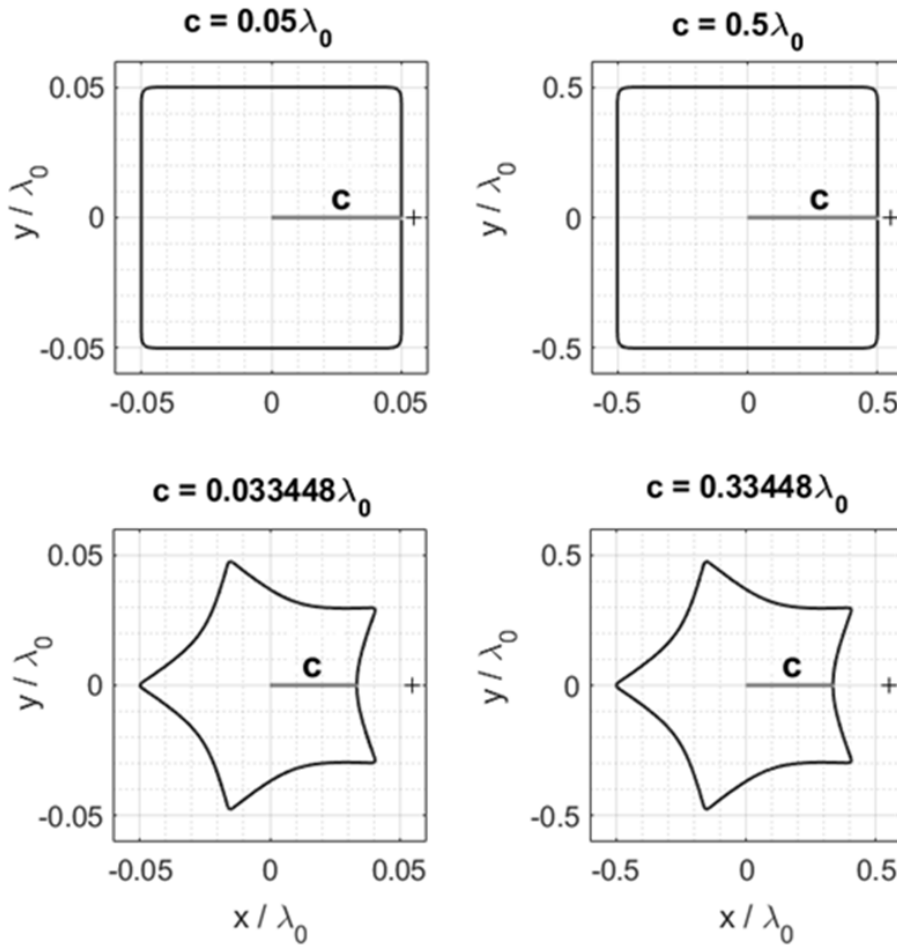


Figure 4.11: The scatterers of interest with different cross section. Top-left: Square(1), top-right: Square(2), bottom-left: Star(1) and bottom-right: Star(2).

Figure 4.12 - Figure 4.15 show that the $\bar{\mathbf{Q}}$ matrix is clearly ill-conditioned and that the regularization operators improve the conditioning of the $\bar{\mathbf{Q}}$ matrix. However, the second regularization operator obtained for the arbitrary cross section makes the matrix more stable compared to the first operator and improves its conditioning compared to the circular cross-section regularization (third regularization operator). In addition, the second regularization operator makes the condition number of the matrix uniformly bounded for Square(1), Square(2) and Star(1) scatterers, while for Star(2) it makes the condition number closer to uniformly bounded. Therefore, the second pair of arbitrary cross-sectional regularization operators ($\bar{\mathbf{R}}_\alpha$ and $\bar{\mathbf{L}}_\alpha$ in subsection 3.4) should be used as the regularization operator.

Making the $\bar{\mathbf{Q}}$ matrix well-conditioned essentially relies on applying a regularization procedure that drives it toward the identity matrix. Let us assume that the left and right regularization operators, $\bar{\mathbf{L}}_\alpha$ and $\bar{\mathbf{R}}_\alpha$, are ideally chosen such that the regularized $\bar{\mathbf{Q}}$ matrix approaches the identity matrix. Suppose further that such regularization operators are obtained through a matrix decomposition applied to $\bar{\mathbf{Q}}$, for instance, via QR decomposition. In this case, the operators $\bar{\mathbf{L}}_\alpha$ and $\bar{\mathbf{R}}_\alpha$ would correspond to the inverse of the orthogonal Q matrix and the inverse of the upper triangular R matrix, respectively. Although the inverse of the orthogonal Q matrix can be readily computed by taking its complex conjugate transpose, the inverse of the ill-conditioned R matrix cannot be obtained reliably. Consequently, deriving the regularization operators through such a decomposition-based procedure is not practical, and the analysis presented in subsection 3.4 becomes necessary. For objects with arbitrary cross-sections, **Figure 4.12 - Figure 4.15** demonstrate that all three proposed regularization operators successfully make the system well-conditioned, with the second operator providing the best overall performance in terms of conditioning.

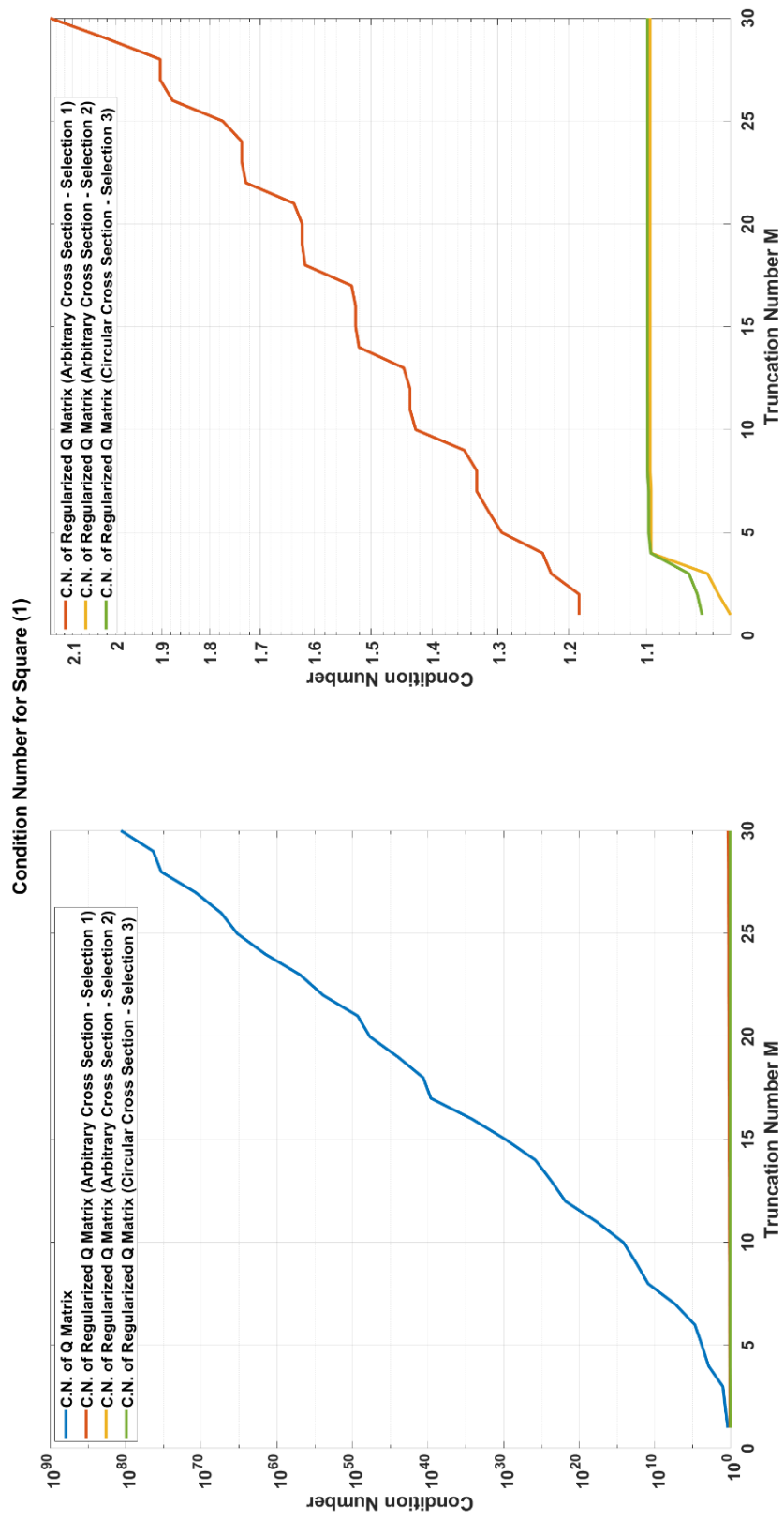


Figure 4.12: Condition numbers of the $\bar{Q}$ matrices regularized by the three regularization operators introduced in subsection 3.4 and of the non-regularized $\bar{Q}$ matrix for the Square (1) case.

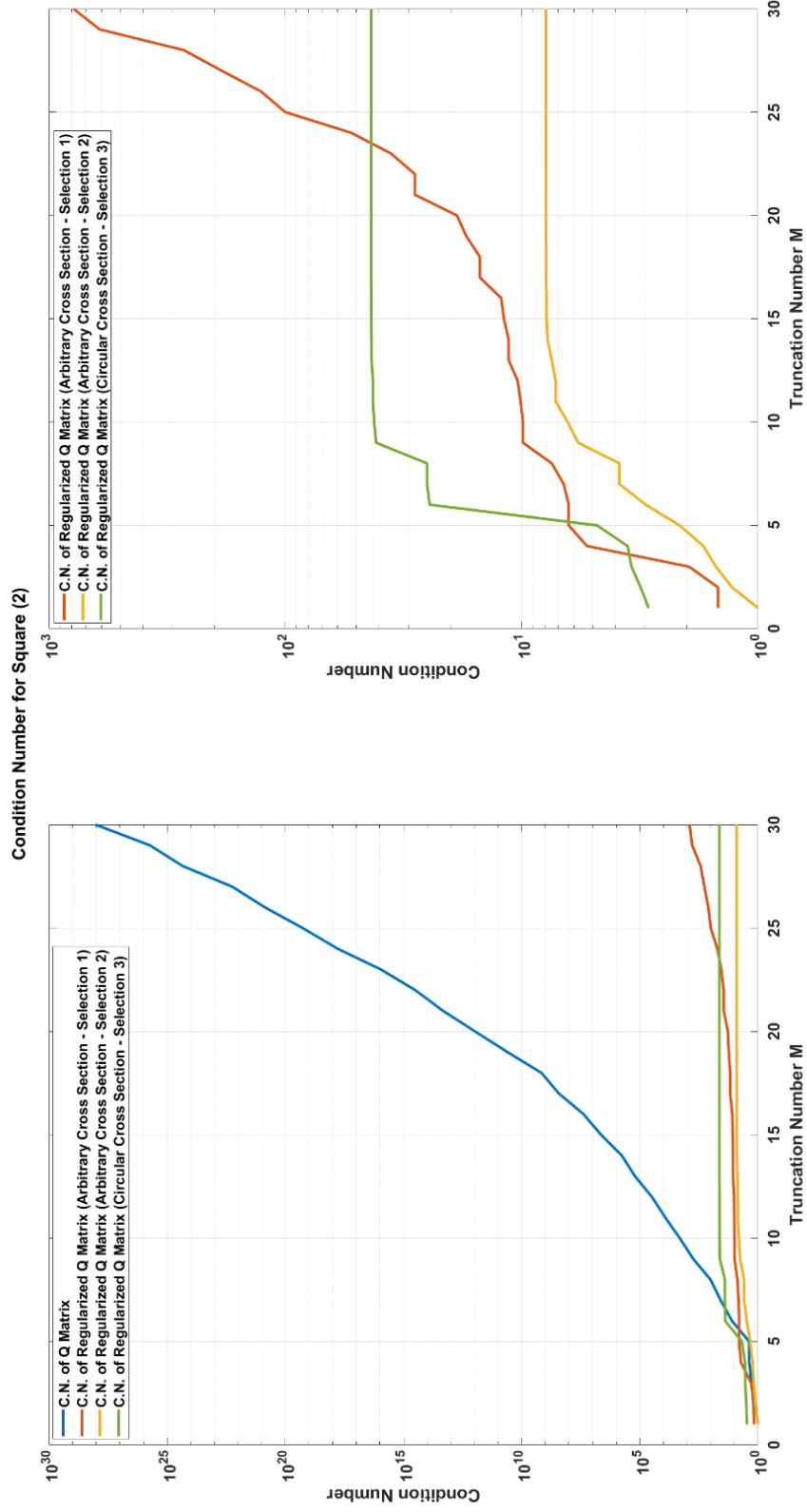


Figure 4.13: Condition numbers of the $\bar{Q}$ matrices regularized by the three regularization operators introduced in subsection 3.4 and of the non-regularized $\bar{Q}$ matrix for the Square (2) case.

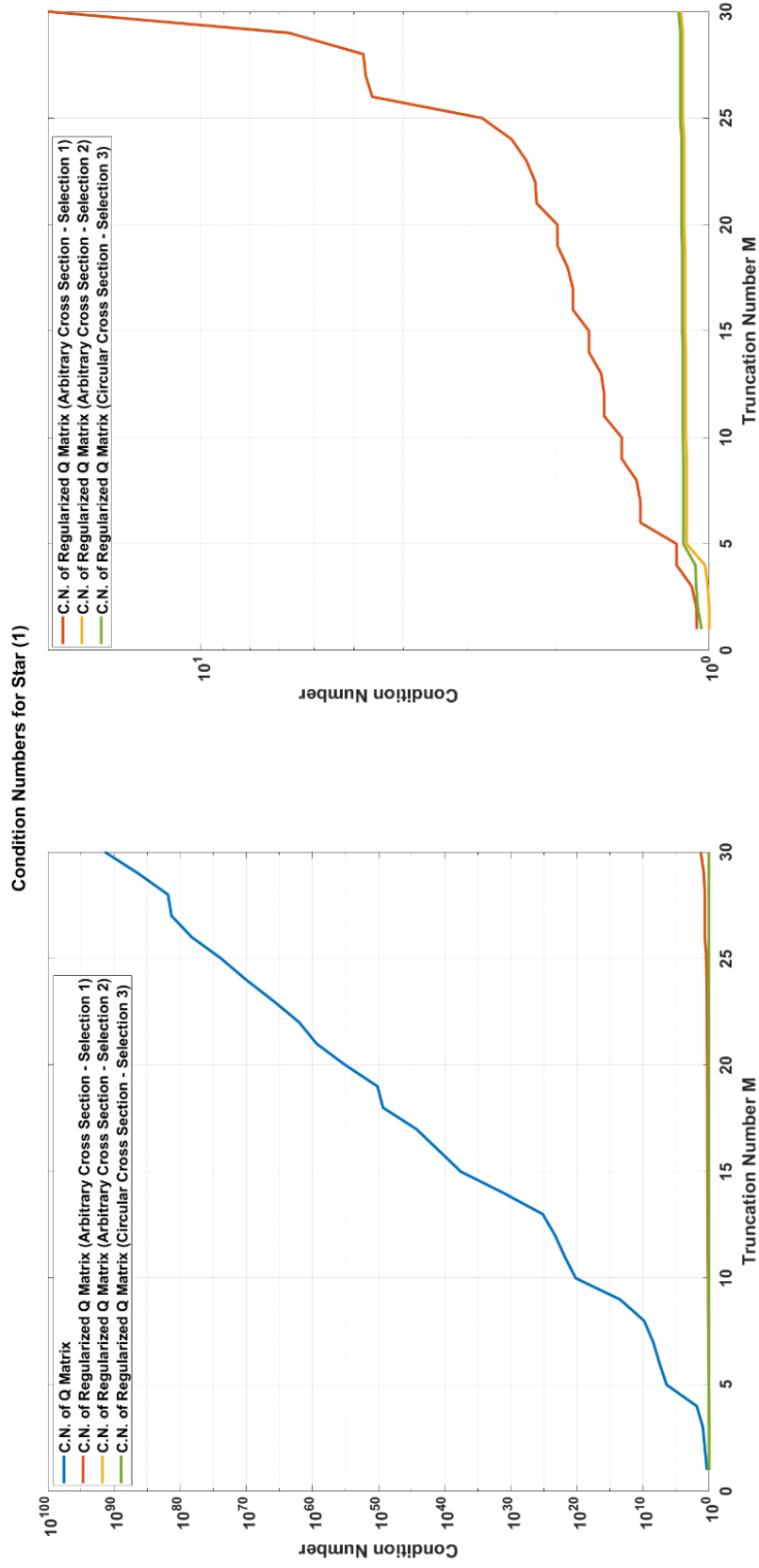


Figure 4.14: Condition numbers of the $\bar{Q}$ matrices regularized by the three regularization operators introduced in subsection 3.4 and of the non-regularized $\bar{Q}$ matrix for the Star (1) case.

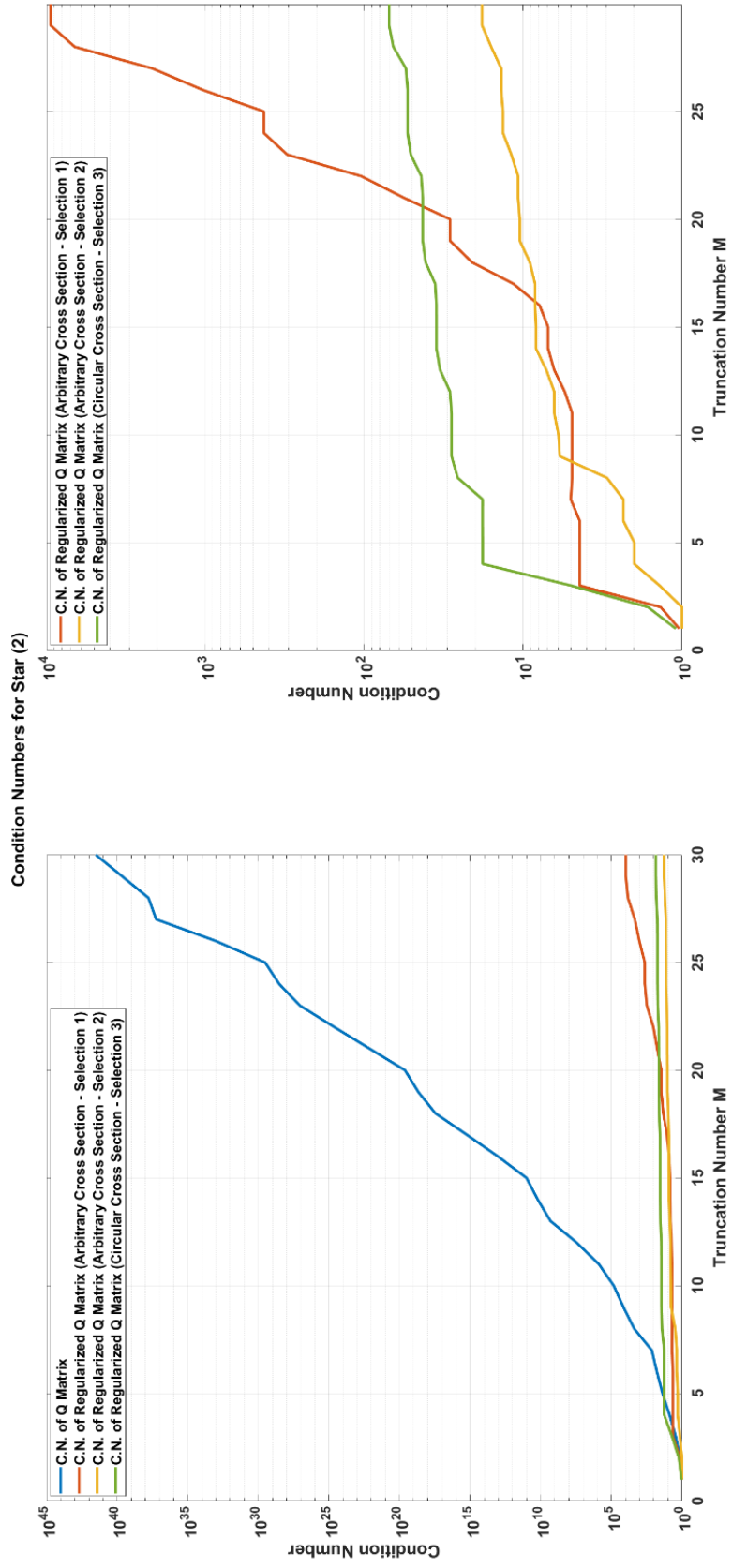


Figure 4.15: Condition numbers of the $\bar{Q}$ matrices regularized by the three regularization operators introduced in subsection 3.4 and of the non-regularized $\bar{Q}$ matrix for the Star (2) case.

Following the regularization of the $\bar{\mathbf{Q}}$ matrix within the isolated-scatterer T-matrix formulation for a single scatterer, and in order to extend this regularization to the entire system accelerated by FMM, we start with casting some of the well-conditioned features of the numerical process applied through the principles in subsection 3.4 during the interaction of two anti-isorefractive (AIR) cylinders with different cross-sections in **Figure 4.16** illuminated by a plane wave with a varying propagation direction in the y - z plane during a polar-angle sweep.

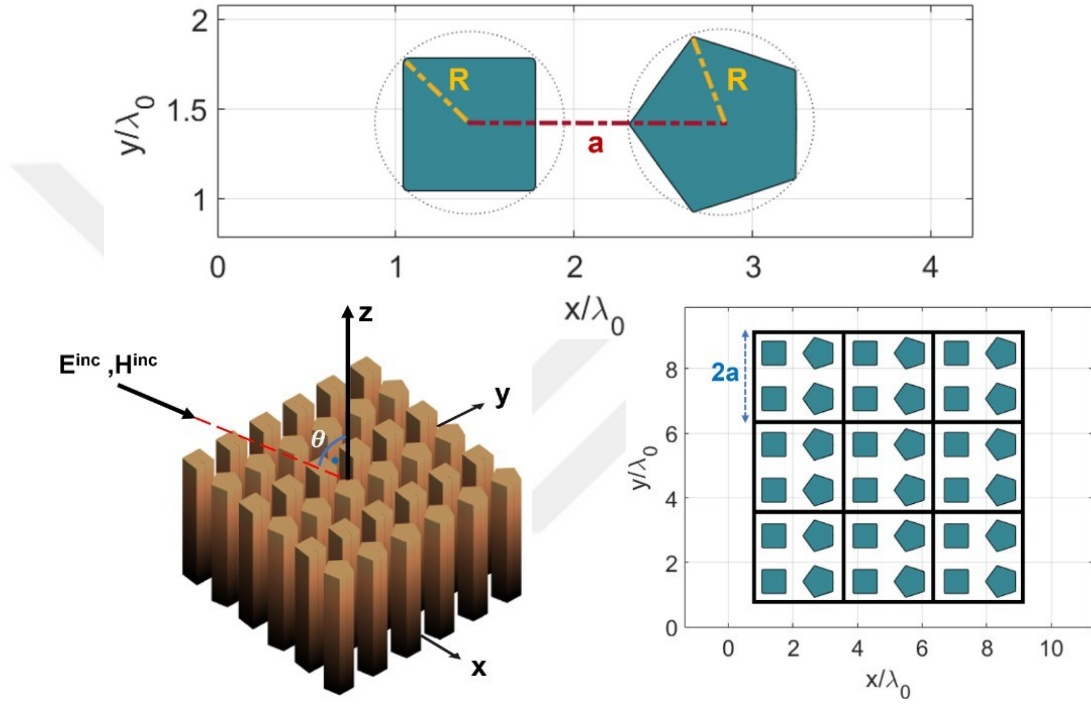


Figure 4.16: 2D view of two DNG AIR scatterers with $\epsilon_r = -4.2$, $\mu_r = 1/\epsilon$, $a = 5.5$ mm, $R = 2$ mm (top). A unit amplitude $f = 77$ GHz oblique TM plane wave excites the infinitely long cylinders with square and pentagonal cross-section. 3D view of oblique plane wave excitation and 6x6 AIR rod array constructed with the same scatterers (bottom-left). 2D view of 6x6 DNG AIR scatterer array with FMM groups with $2a = 11$ mm box size (bottom-right).

Then we will point out the benefits that this well conditioning can offer for an FMM algorithm grouping of a multitude of similar objects. **Figure 4.17** demonstrates that the algebraic systems obtained after regularization remain full rank and possess uniformly bounded condition numbers for the oblique scattering interaction between the two objects considered. Without regularization, the resulting systems become rank-deficient and ill-conditioned, leading to numerical instability.

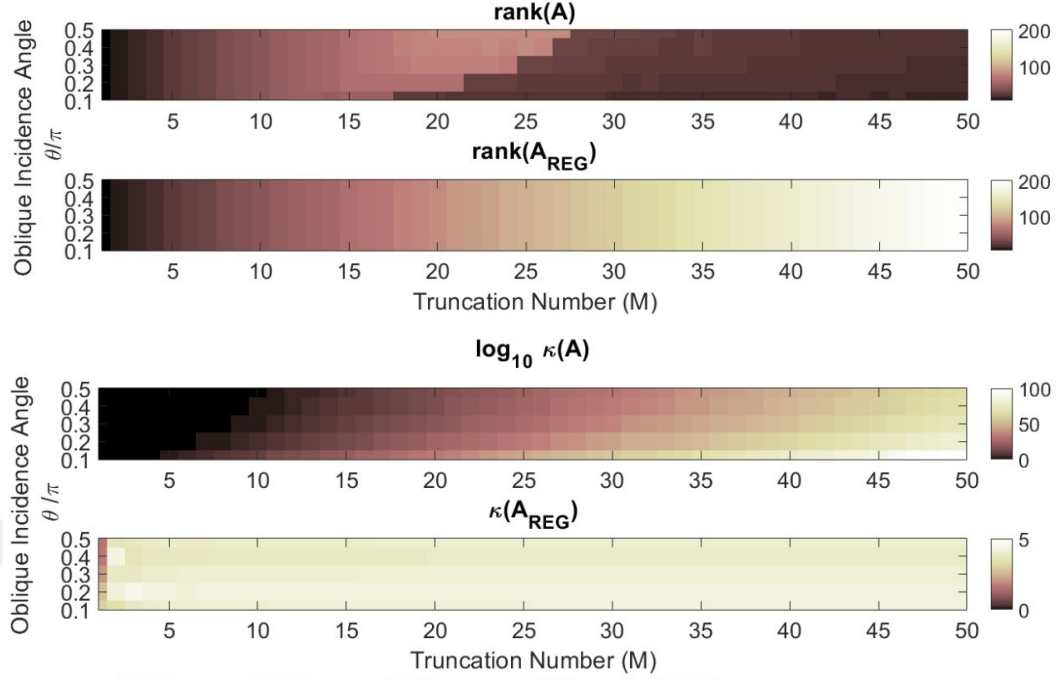


Figure 4.17: Condition features of the regularized ($\mathbf{A}_{\text{REG}}$) and non-regularized ($\mathbf{A}$) T-Matrix algebraic systems modeling the interaction between two objects in **Figure 4.16** w.r.t obliquity and the truncation number. The ranks (top), the condition numbers κ (bottom).

The BIE solver provides the lineal current density values, amplitudes of which are equal to values of the tangential electric and magnetic fields on the boundary contours. The physical relevance of this model is now to be exhibited with a measure relative to outputs of another solver via BIE using the following relative error formulation via the root mean square of a value over positions on the parametrized boundary contour at the parameters:

$$RMSRE(X_{RTMM}) = \sqrt{\frac{1}{N} \sum_{i=1}^N \left(\frac{X_{RTMM}(\phi_i) - X_{BIE}(\phi_i)}{X_{BIE}(\phi_i)} \right)^2} \quad (4.2)$$

The values obtained by the RTMM in (4.2) correspond to the amplitudes of the tangential field components on the object boundaries, computed from the solved cylindrical harmonic coefficients, whereas the values obtained from the BIE correspond to the associated equivalent magnetic line current densities determined

through the integral-equation formulation. The resulting comparisons on both boundaries under consideration are illustrated in **Figure 4.18**. Furthermore, the relevance of the R-TMM and BIE-based magnetic current density distributions is demonstrated in **Figure 4.19**, where the two magnetic current densities are plotted together for the case corresponding to the worst error profile observed in **Figure 4.18**.

The presented results confirm the numerical stability and physical relevance of the proposed model for the interaction of two scattering bodies. These findings motivate their extension to FMM-accelerated scenarios involving many objects. Following the FMM based T-matrix formulation in subsection 3.5, the configuration in **Figure 4.15** is analyzed using a single-buffer interaction scheme. The resulting system is solved via GMRES with a maximum of 200 iterations and a tolerance of 10^{-15} under oblique plane-wave excitation with $\theta_{inc} = 0.3\pi$.

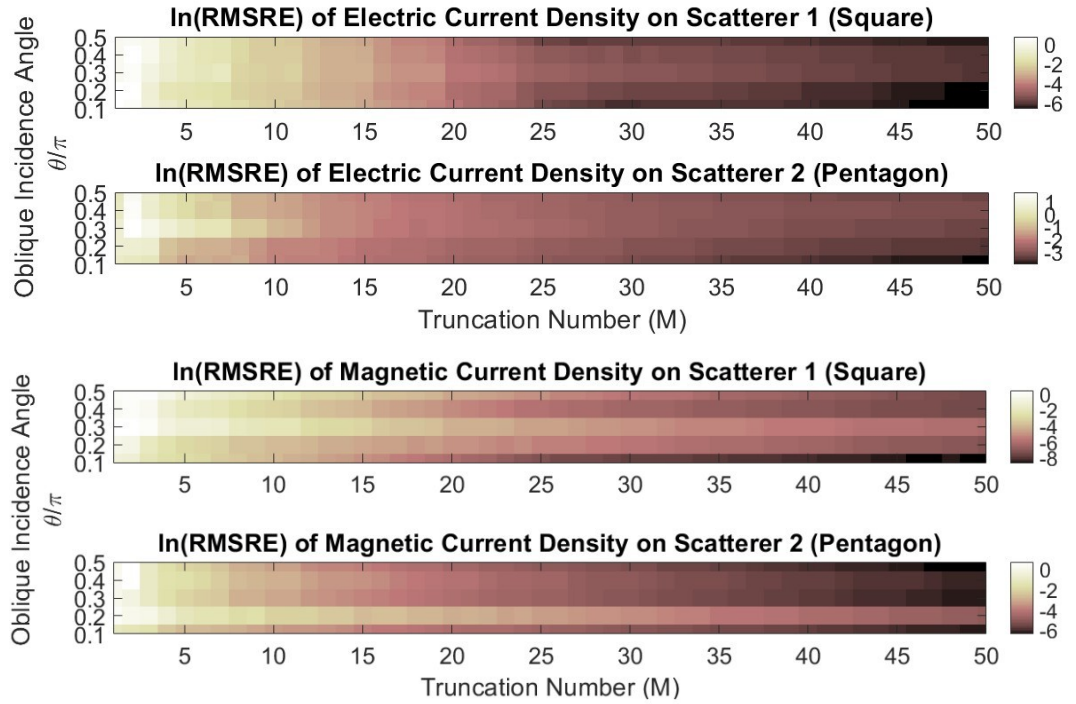


Figure 4.18: Natural logarithms of the RMSRE of physical features of the R-TMM modeling the interaction between two objects in **Figure 4.16** w.r.t obliquity and the truncation number. Electric Current Density (top). Magnetic Current Density (bottom).

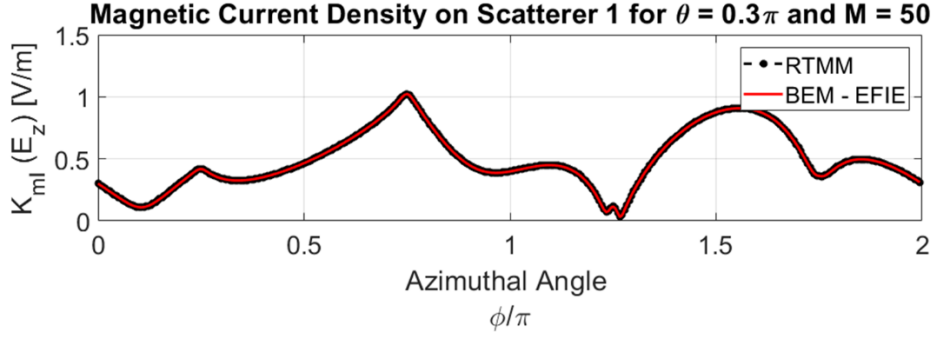


Figure 4.19: Depiction of the physical relevance of the obtained results at the best precision by means of the case which belongs to the maximally erroneous RMSRE profile in **Figure 4.18**.

As shown in **Figure 4.20** and **Figure 4.21**, the non-regularized system exhibits a truncation threshold that limits the effective use of the FMM algorithm. This limitation vanishes after regularization, leading to a well-conditioned system suitable for large-scale N-body scattering problems.

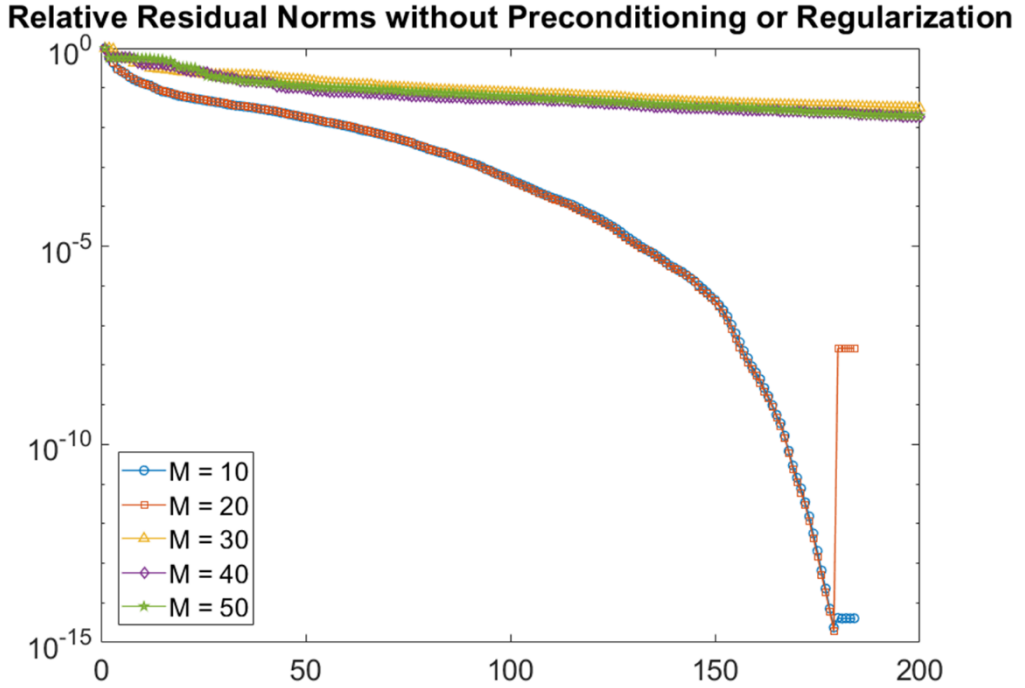


Figure 4.20: The relative residual norms of the GMRES iterations for algebraic systems of different sizes without regularization, which are ill-conditioned.

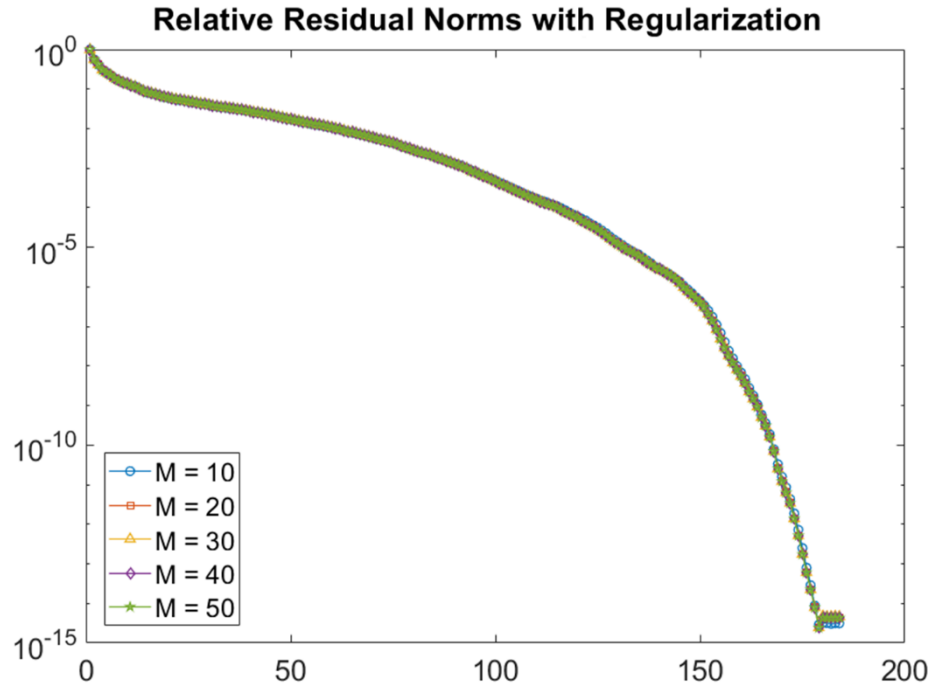


Figure 4.21: The relative residual norms of the GMRES iterations for regularized algebraic systems of different sizes, demonstrating well-conditioned behavior.

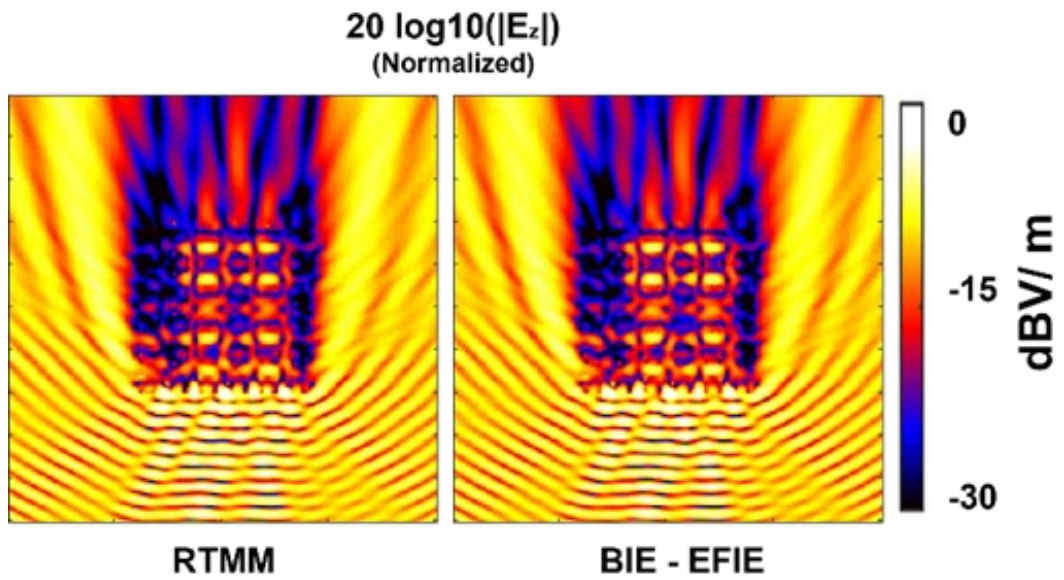


Figure 4.22: Depiction of the physical relevance of the multiple scattering problem under consideration.

Figure 4.22 depicts physical relevance through the near fields with R-TMM and the BIE calculations. The near-field plot shows that FMM accelerated R-TMM produces solutions that are validated by the super-algebraically convergent solution of EFIE using the method in [93].

Retaining the configuration shown in **Figure 4.16**, the scatterer material is replaced with zirconium oxide, and the corresponding scattering scenario is investigated. For this case, both the well-conditioned nature of the regularized matrix equation system and the relative residual norms of the regularized and FMM-accelerated matrix–vector product formulation are examined throughout the GMRES iterations. The scatterers are composed of zirconium oxide with a dielectric permittivity of $\varepsilon_1 = (23.52 - 0.2285j)\varepsilon_0$ and a magnetic permeability of $\mu_1 = 0.9999862$ at 77 GHz. The array is excited by TM-polarized plane waves with $\theta = \pi/2$ and $\phi = \pi$.

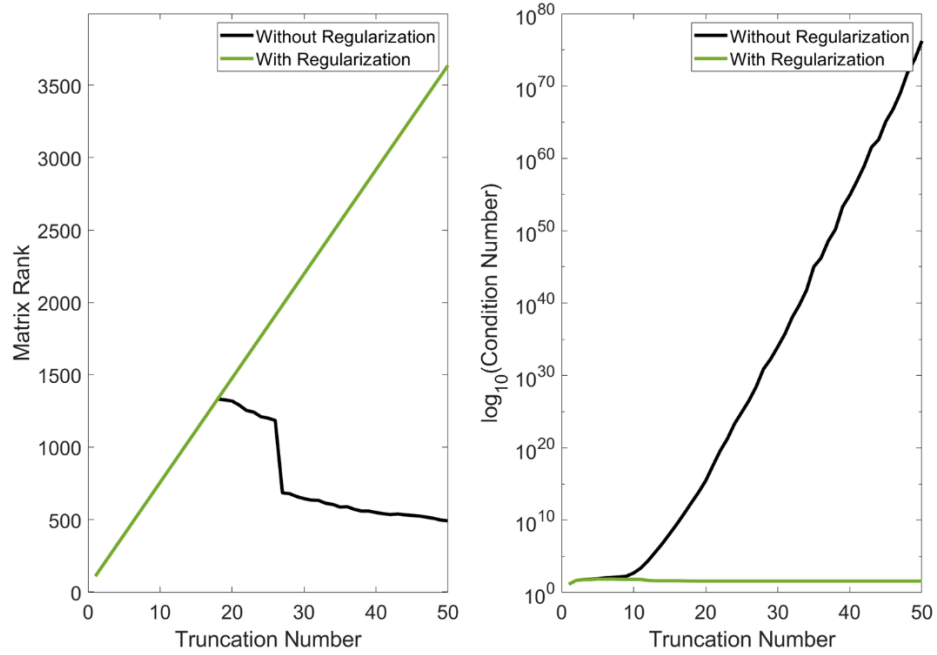


Figure 4.23: Rank and condition number of the system matrix with respect to truncation number for the regularized and non-regularized cases.

For this case, **Figure 4.23** presents the rank and condition number of the system matrix $\bar{\mathbf{A}}$ in the scattering-matrix formulation $\bar{\mathbf{A}}\mathbf{x} = \mathbf{b}$, shown as functions of the increasing truncation number, both with and without regularization. It is clearly observed that the application of regularization leads to a well-conditioned system.

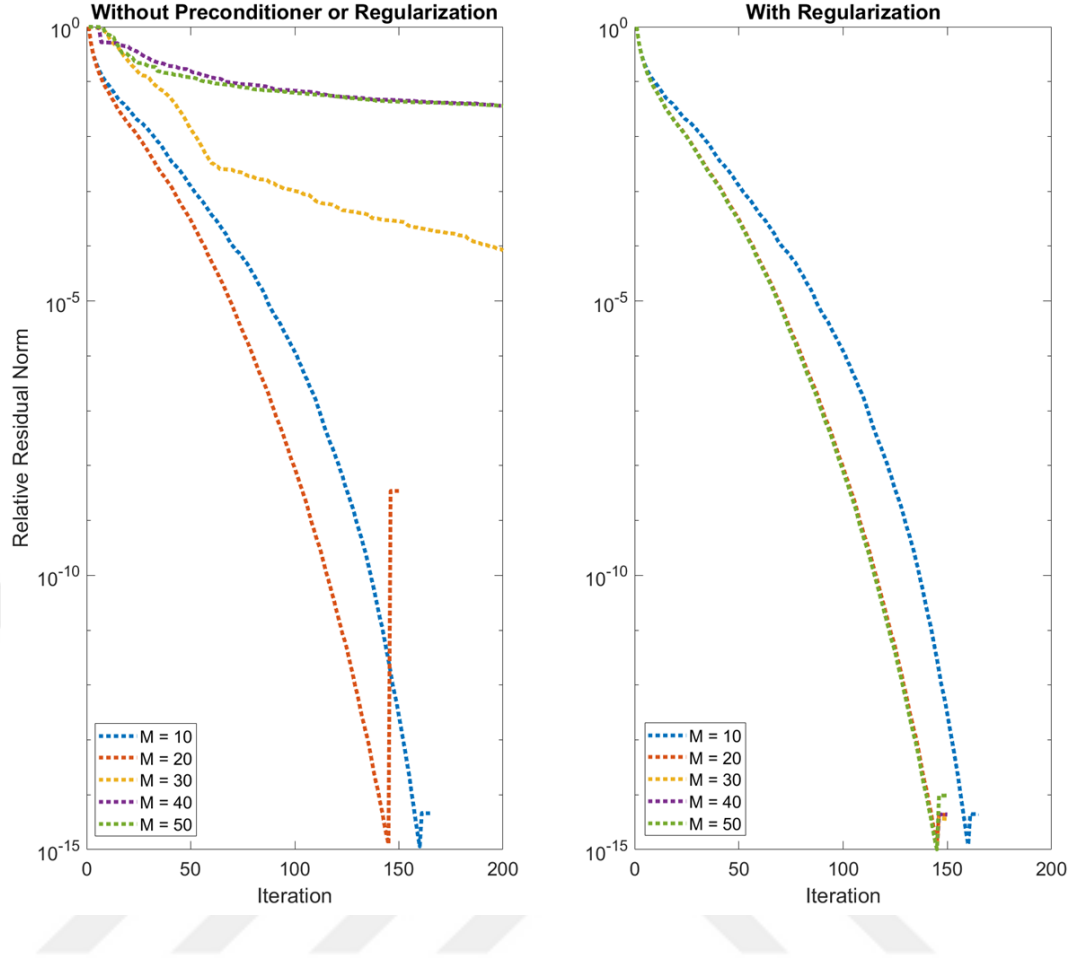


Figure 4.24: Relative residual norms of the GMRES iterations for the FMM-accelerated system with and without regularization for zirconium oxide scatterers.

Figure 4.24 presents the relative residual norms over GMRES iterations for the FMM-accelerated system formulated in matrix–vector product form. The results show that, for truncation numbers beyond $M = 30$, achieving lower solution errors requires a significantly increased number of iterations. It is clearly observed that the well-conditioned (regularized) system substantially reduces the required iteration count, thereby improving the efficiency of the iterative solution process.

In **Figure 4.25**, the resulting electric fields are compared over an observation frame in units of dBV/m². When examining the relative errors between the FMM-accelerated well-conditioned T-matrix method and the boundary integral equation solution, it is observed that although the null-field method naturally yields relatively higher error near the scatterer boundaries, it provides accurate enough in the regions outside.

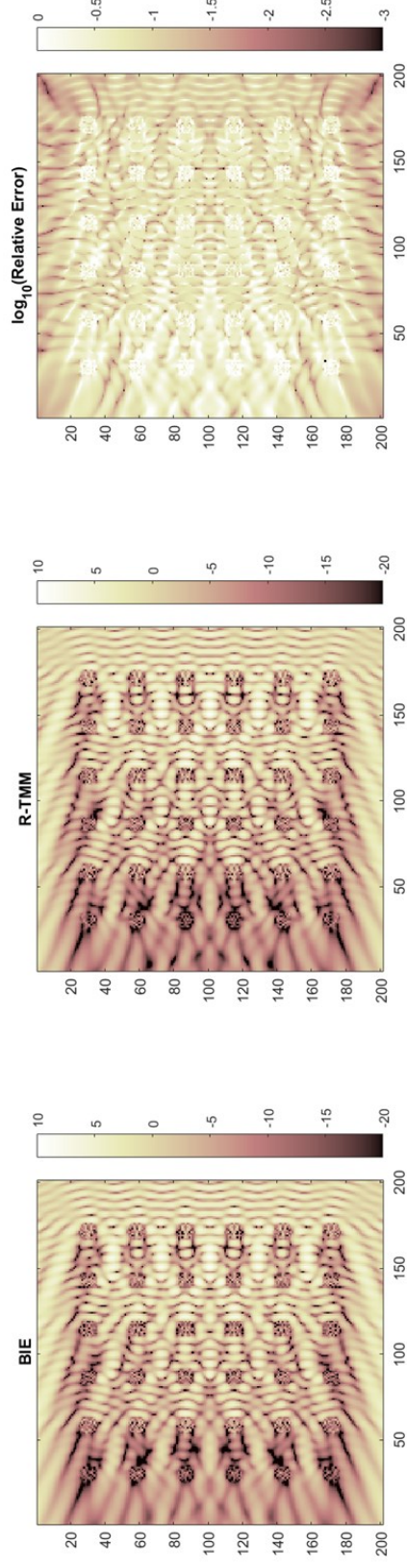


Figure 4.25: $20\log_{10}|E_z|$ on $20\lambda_0 \times 20\lambda_0$ frame [dBV/m²] obtained with BIE (left) and FMM Accelerated – Well Conditioned TMM (middle) and $\log_{10}(\text{Relative Error})$ (right) for zirconium oxide scatterers.

4.3. Numerical Results for Oblique Plane Wave and 3D Dipole Excitations

In this study, the boundary integral equations and null field integral equations are reformulated in 2.5D and for the case where the z components of the electric and magnetic fields are coupled, and the boundary integral equation and T matrix method based on it are reconstructed.

To demonstrate this, firstly, we consider the scenario shown in **Figure 2.3** where a non-magnetic circular cylinder of radius λ_0 with $\varepsilon_1 = 4\varepsilon_0$ dielectric permittivity is illuminated by a TM wave incident obliquely at $\theta = \pi/4$ from the second medium with free space parameter. At $z=0$, the tangential (l) and z components of the fields in steps of $\Delta\phi = 0.1^\circ$ on the cylinder cross-section are obtained in 2.5D with the infinite length of the cylinder by the two methods mentioned above (BIE and TMM) by taking 100 and 360 samples respectively. The numerical results are then obtained for the infinitely long cylinder by applying a periodic boundary condition in the z -direction to a cylinder of length $0.1\lambda_0$ by dividing it into triangular elements with one side of length $0.11\lambda_0$ using the Altair® FEKO® student version, using the MoM/MLFMM based solver according to the surface equivalence principle. Each simulation was performed on a machine with a clock speed of 2.33 GHz, 2 Intel Xeon E5410 processors with a total of 8 cores and 64 GB random access memory.

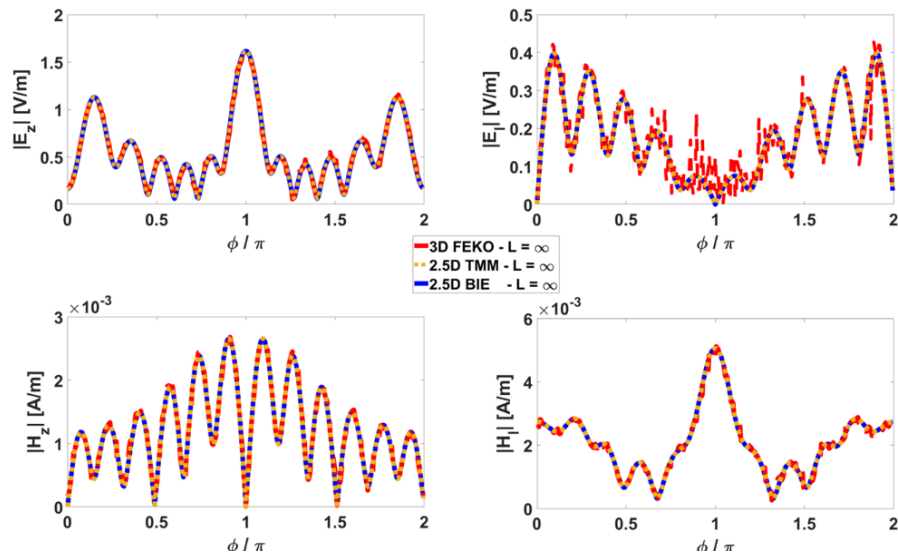


Figure 4.26: Scattering of TM Waves incident obliquely on a single circular cylinder - fields on the scatterer boundary.

Figure 4.26 shows that the fields on the scatterer boundary obtained from the 2.5D solution of the boundary integral equation, the T Matrix method and the FEKO simulation validate each other. The fact that the results obtained by three different methods confirm each other proves that the formulation and analysis made in this study are correct, accurate and consistent.

Secondly, the boundary integral equations and null field integral equations are solved in 2.5D for various cases. The scatterers of interest for the solution are given in **Figure 4.11**. We consider the scenario where a non-magnetic arbitrary shaped cylinders with $c = 0.5\lambda_0$ and $0.05\lambda_0$ and $\varepsilon_1 = 4\varepsilon_0$ dielectric permittivity is illuminated by a TM wave incident obliquely at $\theta^{inc} = \pi/3$ and $\theta^{inc} = \pi/6$ from the second medium with free space parameter. The z components of the fields in steps of $\Delta\phi = 0.1^\circ$ on the cylinder cross-section are obtained in 2.5D for the infinite cylinder by the two methods mentioned in previous sections (BIE and TMM) by taking 200 and 360 samples respectively.

Figure 4.27 demonstrates that the fields on the scatterer boundary obtained from the 2.5D solutions of the BIE and R-TMM validate each other for the coupled scattering case at different oblique incidence angles θ^{inc} . The solutions obtained by the R-TMM are evaluated directly on the scatterer boundary. While cylindrical wave function expansions are complete everywhere for circular cross-sections, their validity on the boundary becomes questionable for scatterers with arbitrary cross-sections. Since the T-matrix method is based on a null-field formulation, it may fail to represent the boundary fields accurately for corrugated geometries, such as star-shaped scatterers, although it remains valid both inside and outside the scatterer. Despite this limitation, the R-TMM yields boundary field solutions that are consistent with and in good agreement with the corresponding BIE results. This agreement confirms that the formulation and analysis presented in this study are accurate, and consistent for coupled scattering cases.

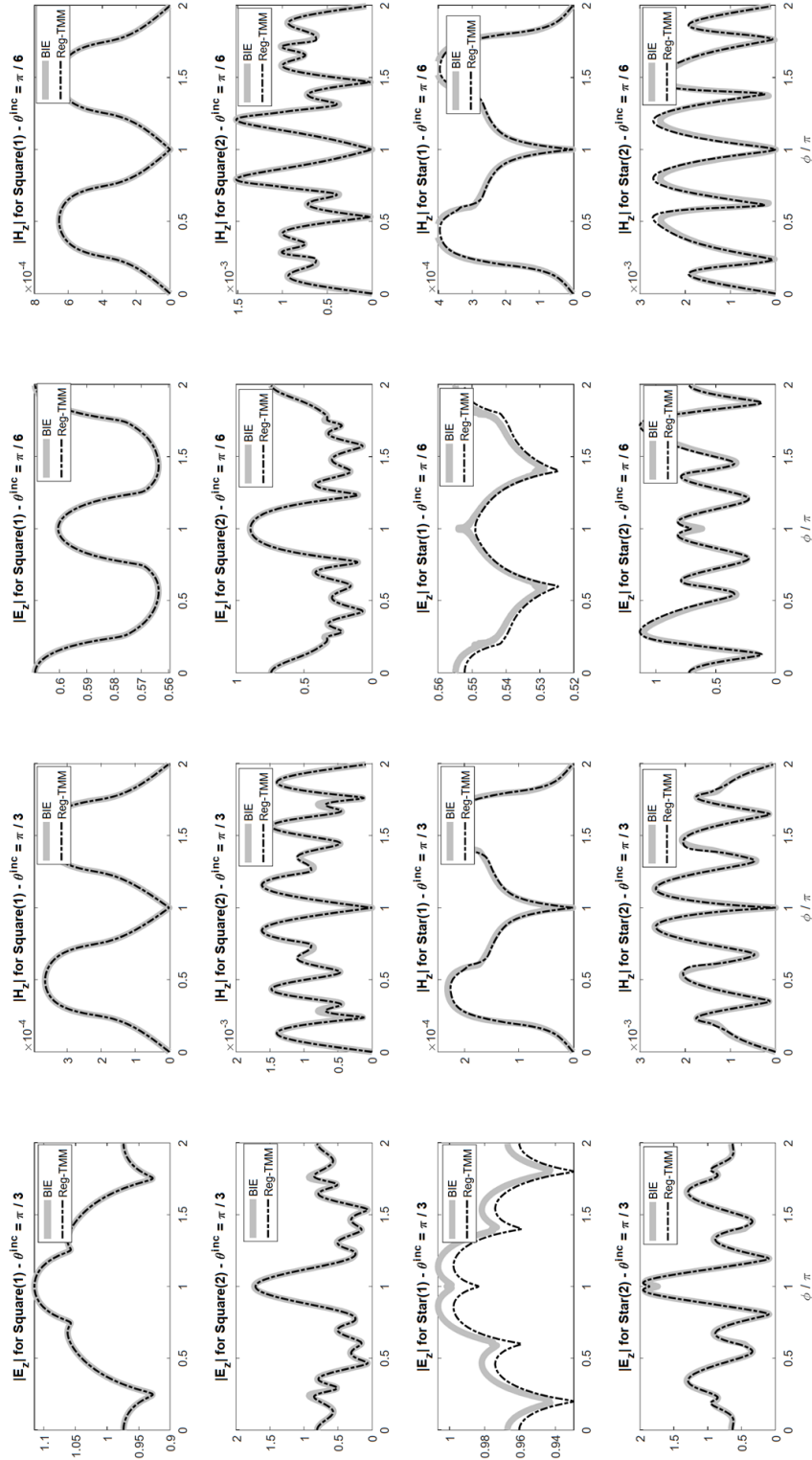


Figure 4.27: Scattering of TM Waves incident obliquely on a dielectric cylinder with arbitrary cross-section. Fields on the scatterer boundary for different oblique incident angles $\theta^{inc} = \pi/3$ and $\theta^{inc} = \pi/6$.

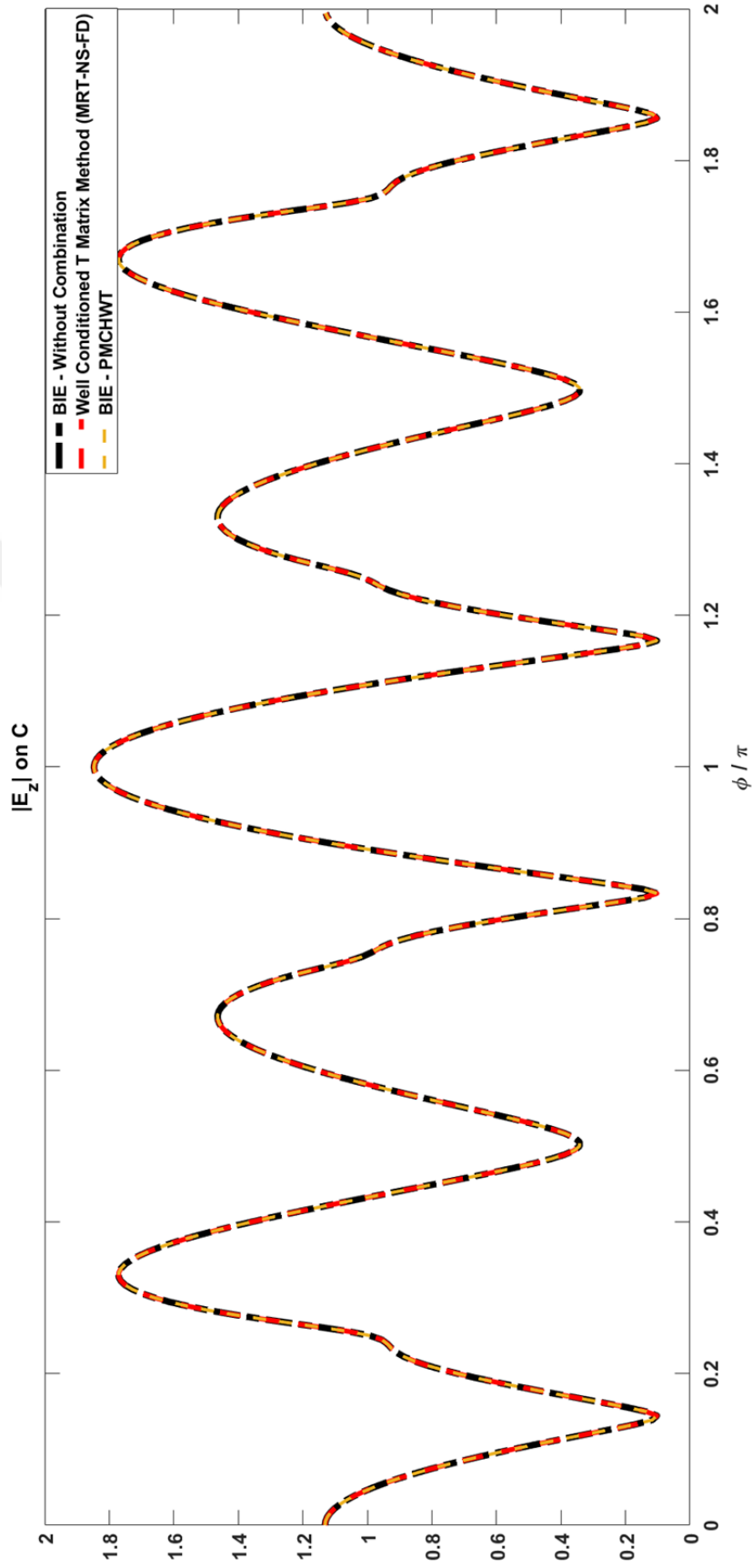


Figure 4.28: $|E_z|$ (i.e. l component of magnetic current density- $|\mathbf{K}_{ml}|$) values on boundary of the scatterer with respect to azimuthal angle.

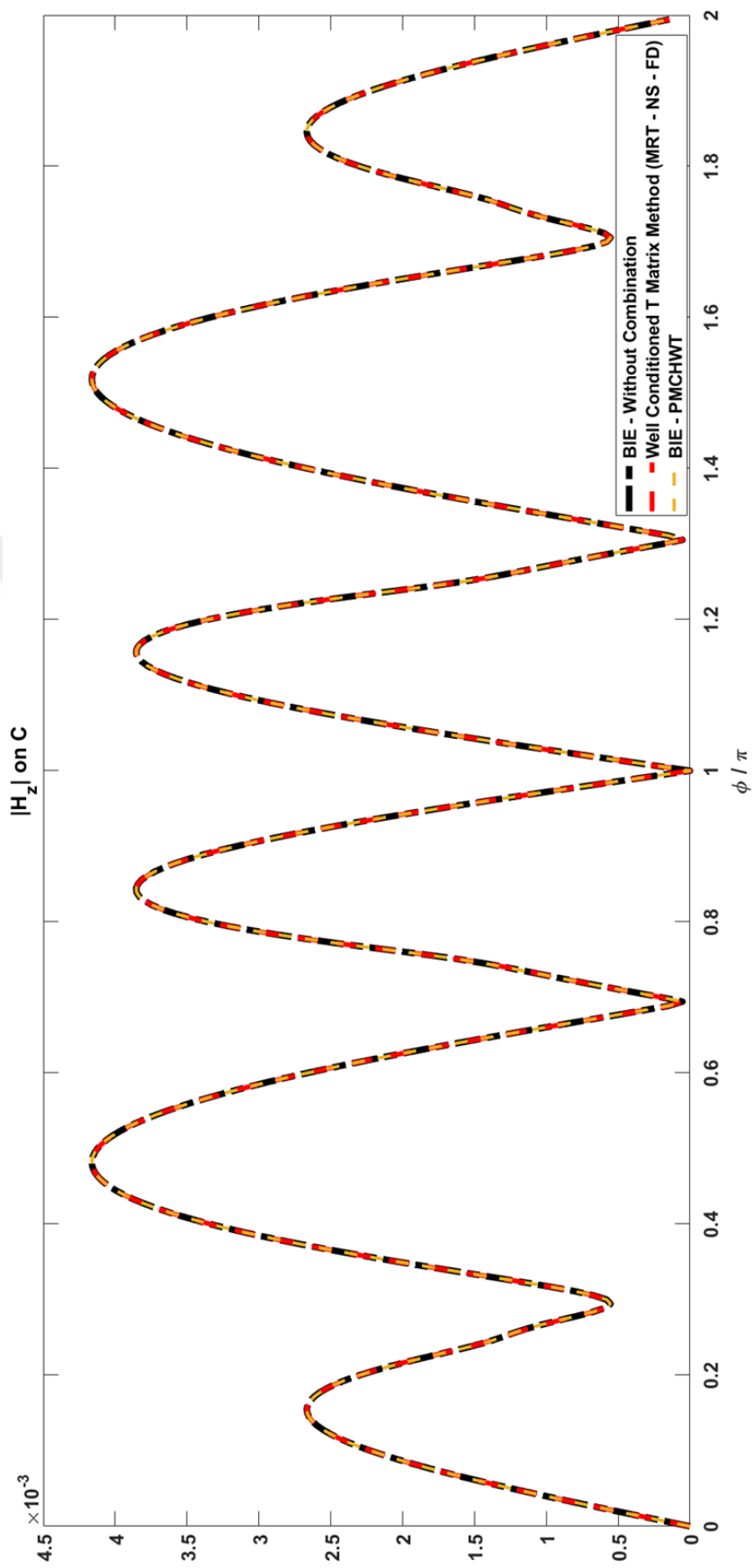


Figure 4.29: $|H_z|$ (i.e. 1 component of electric current density - $|\mathbf{K}_1|$) values on boundary of the scatterer with respect to azimuthal angle.

Thirdly, integral equations without combination, PMCHWT formulation, proposed T Matrix Method are tested. The scatterer of interest for the PMCHWT solution is Square as in **Figure 4.11**. The side length of the non-magnetic lossy scatterer is $0.5\lambda_0$ ($c = 0.25\lambda_0$), and its dielectric permittivity $\epsilon_1 = (8 - 0.2j)\epsilon_0$ for frequency $f = 1$ GHz. The scatterer is illuminated by a TM-polarized plane wave incident at oblique angles $\theta_{inc} = \frac{\pi}{4}$, $\phi_{inc} = \pi$. **Figure 4.28** and **Figure 4.29** show the field distributions on the boundary of the scatterer for three different methods and formulations. Based on these plots, it can be concluded that all three methods are in good agreement, and that the proposed T-matrix approach produces numerically accurate results.

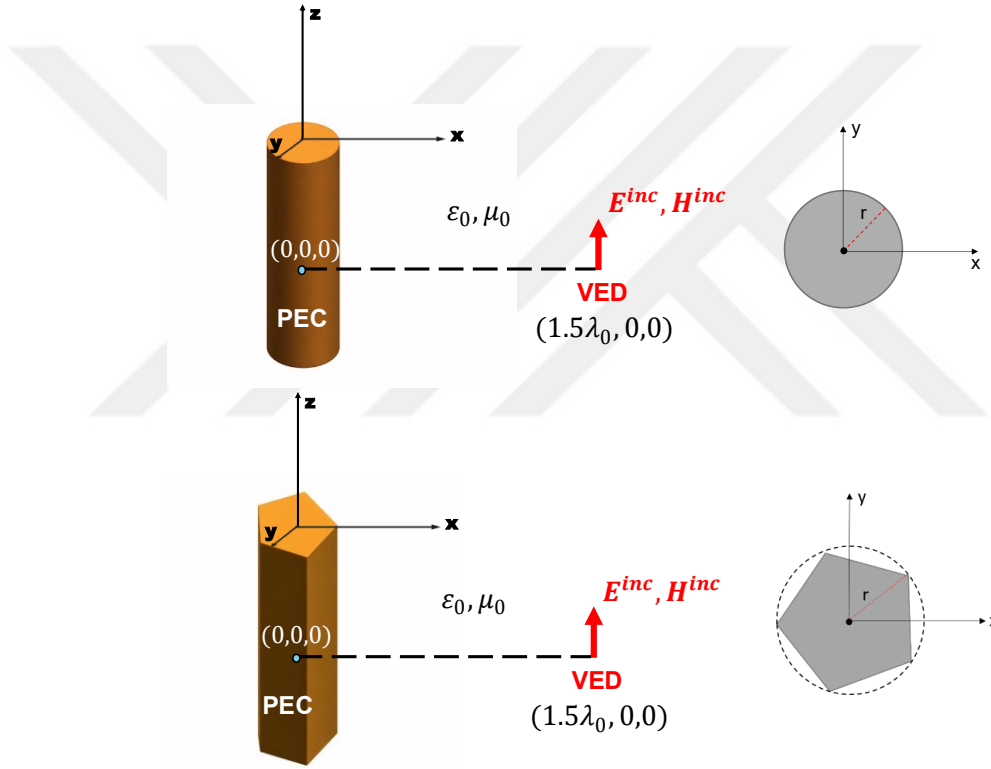


Figure 4.30: Scattering problem of a perfectly electrically conducting (PEC) cylinder illuminated by a vertical electric dipole (VED).

As the final case considered in this subsection, an infinitely long perfect electric conductor (PEC) circular and pentagonal cylinder of radius λ_0 (for the pentagonal cross-section, radius of the circumscribed circle λ_0), centered at $(0,0,0)$ in Cartesian coordinates as shown in **Figure 4.30**, is excited by a vertical electric dipole (VED) located at $(1.5\lambda_0, 0, 0)$ in free space. For this configuration, the incident field has a nonzero E_z component while its H_z component vanishes, as given by (2.140) and

(2.141). Consequently, only an electric current oriented along the z -direction is induced on the scatterer boundary, corresponding to the tangential component of the magnetic field. The solutions for this unknown current density are obtained by discretizing the boundary at $N = 100$ points using (2.135) and solving the EFIE following the techniques described in [93] and [125], with 500 sampling points taken along the cross-section. Subsequently, for the same VED excitation positioned at the identical location, reference solutions are computed using the Altair® FEKO® student version by employing a Method of Moments/Multilevel Fast Multipole Algorithm (MoM/MLFMA)-based solver under the surface equivalence principle for a circular cylinder of length $10\lambda_0$, discretized into triangular elements with an edge length of $0.2\lambda_0$. In addition, taking $\mathbf{r}' = (1.5\lambda_0, 0, 0)$, relative errors are evaluated for each θ value in the range $\theta \in [0, \pi/2]$ at $r = \lambda_0$ and $\phi = 0$ in spherical coordinates, in accordance with (2.142), by comparing the results obtained from (2.135).

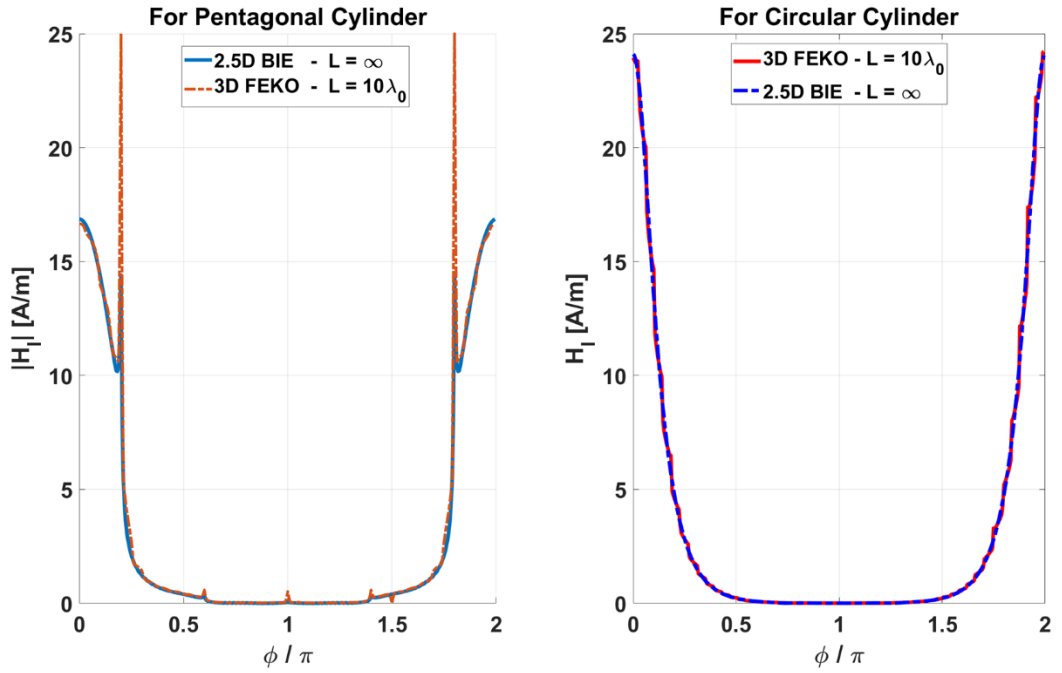


Figure 4.31: Magnitude of the tangential component of the magnetic field on the boundaries of the PEC pentagonal and circular cylinders at $z = 0$.

Since an infinitely long cylinder cannot be modeled in FEKO by imposing periodic boundary conditions along the z -direction for dipole excitation, finite-length cylinders with a length of $10\lambda_0$ were considered in the FEKO simulations and used for comparison. Under the infinite-length approximation, the VED source is represented as a superposition of k_z dependent two-dimensional line sources. For each spectral

component, the corresponding boundary integral equations are solved, and the induced magnetic fields on the scatterer boundaries are obtained, as illustrated in **Figure 4.31**. Despite these modeling differences, the results are in good agreement.

Figure 4.32 shows the relative error of the 2.5D computation with respect to the 3D solution for various number of integration points, for a vertical electric dipole located at $\mathbf{r}' = (1.5\lambda_0, 0, 0)$, observed at $r = \lambda_0$ and $\phi = 0$, as a function of different θ values. In **Figure 4.32**, it is observed that the relative error decreases to the order of 10^{-12} as the number of Gauss–Legendre integration points N increases. Based on this observation, it can be concluded that the 2.5D formulation accurately represents the 3D formulation and can reliably be used as the right-hand side in 2.5D scattering problems.

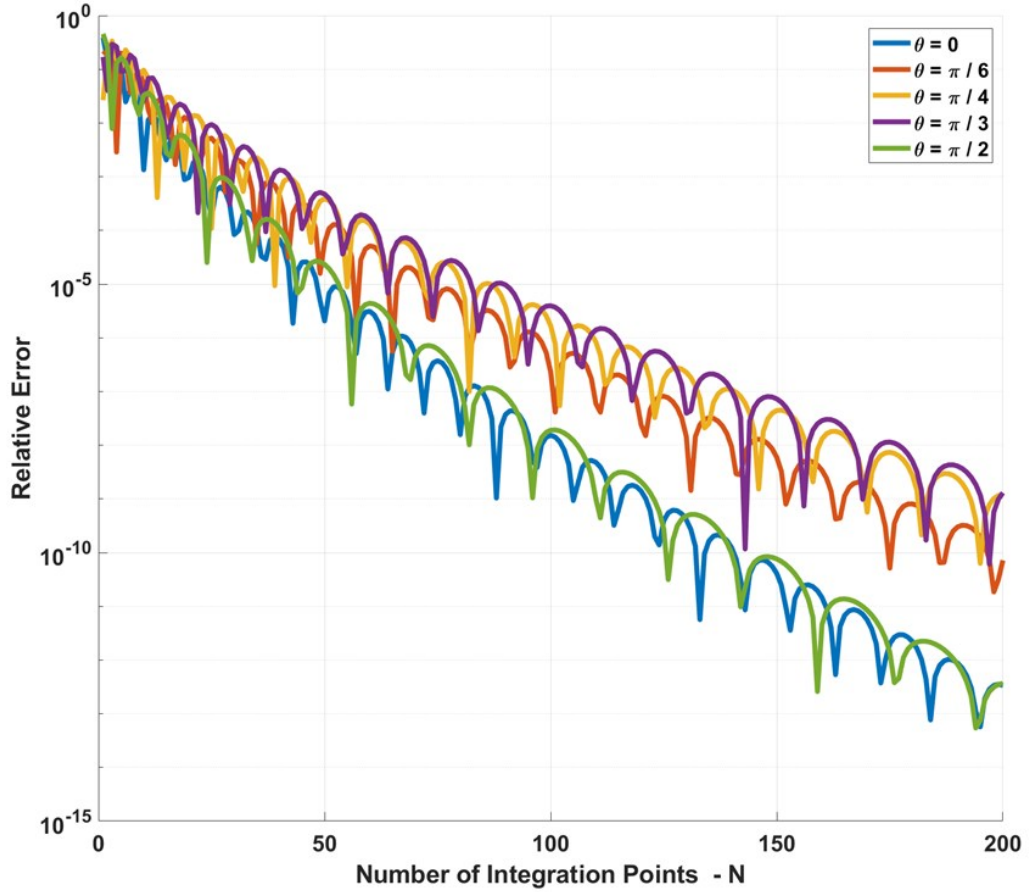


Figure 4.32: Relative error of the 2.5D integral calculation for a vertical electric dipole compared to the 3D calculation, as a function of the number of integration points for different θ values.

In conclusion, the numerical findings demonstrate that the T-matrix method formulated for oblique plane wave incidence and the BIE method yield consistent results. Moreover, for scatterers excited by dipole fields, the BIE solutions confirm that the 2.5D representation of the dipole excitation is accurately accounted for. The solutions obtained using the T-matrix method for dipole source excitation, as well as the scattering problem involving sources located in the vicinity of the scatterer, will be addressed in the following subsection.



4.4. Numerical Results for Well-Conditioned T-Matrix Method for a Polarization-Decoupling Cylinder with Arbitrary Cross-Section Near a Dipole Source

In this subsection, the stages of a T-matrix method (TMM) implementation are examined throughout the modeling process of a Hertzian dipole excitation of a polarization-decoupled (PD) isolated cylindrical scatterer with an arbitrary cross-section [127]. Iso-refractive scatterers, which also constitute special cases beyond the analytical scope of classical formulations such as those in [131], allow the decoupling of fields into TM and TE components in 2.5-D scattering problems. This property enables the use of classical 2-D TMM algorithms for infinitely long cylindrical bodies (uniform along the z -direction) excited by polarization-coupling sources with longitudinal dependence, such as an obliquely incident plane wave (PW).

A dipole field is the longitudinal inverse Fourier transform of a line source type field. Its spectrum is the longitudinal continuous PW spectrum (l-PWS). A novel algorithmic procedure for the l-PWS integration of dipole excitation is readily available as underlined in subsection 2.4. The transverse 2-D solutions can also be integrated along the same l-PWS to obtain the 3-D fields. The complex-valued transverse wavenumber on each member of this spectrum leads each 2-D solution possess a numerical stability issue during truncation of the infinite cylindrical wave spectrum (CWS) for the TMM other than that of the case with real-valued wavenumber. This problem is considered following the procedure presented in subsections 3.3 and 3.4. Finally, the adopted approach allows the analysis of dipole excitations in close proximity to the scatterer boundaries.

Every cylindrical field excited by each line source in the l-PWS of a dipole field is a superposed form of a standing CWS. Their amplitudes are the traveling cylindrical wave harmonics (CWH) containing the n th order Hankel function. Each of these functions is an infinite integration of plane waves forming the transverse continuous plane wave spectrum (t-PWS) with complex arguments, following a proper path in the complex plane.

The alleviation of the problem addressed in the TMM implementation through novel extension presented hereby, and the adaptation of a modern technique which is applied for surface integrated waveguides structures and handles the addition theorem for bodies close to each other, requires the truncation of this infinite integration of the t -

PWS. After a regularization operation to make the system of equations well-conditioned, this version of the TMM becomes numerically stable for the problem of scattering from objects with arbitrary cross-sections located close to the 3-D excitation source due to the appropriate truncation numbers of the discrete CWS and the continuous t-PWS. To shed light on the nature of the new approach, these truncation numbers are determined with the aid of Genetic Algorithms (GA), and this version of the method is optimized for both truncation numbers. To validate the results, a super-algebraically convergent BIE solver for dielectric cylinders and reference solutions obtained using a commercial 3-D solver are employed.

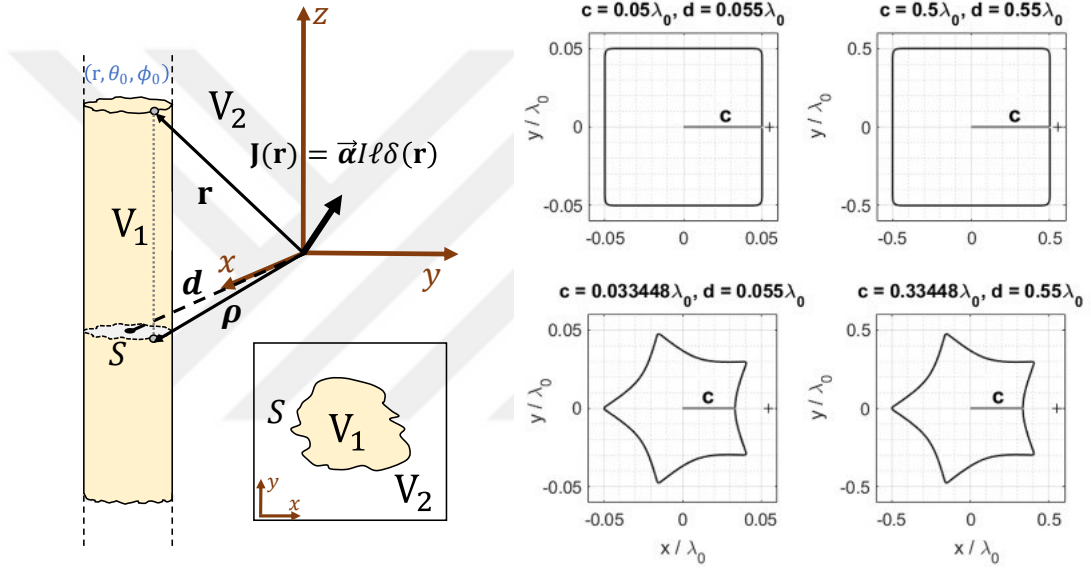


Figure 4.33: Polarization-decoupling isorefractive scatterer excited by a dipole source located at a distance d , defined by $\alpha_x = 0$, $\alpha_y = \alpha_z = 1$ and $l\ell = 1$ (left). Excited scatterers with different c values and cross-sectional geometries (right).

With the scatterer centered at the origin of the coordinate system, the excitation is provided by the longitudinal plane-wave spectrum (l-PWS) components of a dipole source characterized by the current density $\mathbf{J}(\mathbf{r}) = \delta(x - x_s)\delta(y)\delta(z)\vec{\alpha}$ where $\alpha_x = 0$ and $\alpha_y = \alpha_z = 1$. Each l-PWS component is represented by an equivalent line source located at a distance d from the scatterer center, such that $x_s = d$, as illustrated in **Figure 4.33**. To incorporate these excitations into the T-matrix formulation, the line-source fields associated with each l-PWS component are expanded into cylindrical harmonics using the addition theorem. Instead of employing the resulting expansion coefficients expressed in terms of n th-order Hankel functions directly, a truncated

spectrum is constructed by limiting the corresponding transverse plane-wave spectrum (t-PWS), and this truncated spectrum is subsequently used in computations. In this context, both the truncation number of t-PWS and the number of cylindrical harmonics retained play a crucial role in the accuracy of the solutions. To analyze and determine appropriate truncation parameters, genetic algorithm optimizations are performed for various scatterer cross-sections.

To gain insight into the parameters to be applied in the algorithms, a genetic algorithm (GA) from the MATLAB is employed. Using the tournament selection function with default GA parameters, including the mutation rate, the algorithm is executed on a modern desktop computer for two optimization runs:

- i) To minimize the relative error between the left and right-hand sides of the equation (2.133) by tuning the parameters α and N . δ in [108] could have been optimized for a general case. Still, it need not be optimized here, since throughout the calculations, the observation position angle is $\theta_0 = 0$, on the $z = 0$ plane. Usage of GA is recommended here since its duration does not exceed several seconds for reasonable population (~ 100) and generation (maximum value < 1000) sizes with 10^{-10} desired tolerance as a stopping criterion. Averaging of the optimized values for each observation point on the boundary contour ends up with the values $\alpha = 0$, and $N = 101$ under the constraints $\alpha \in [0, 1]$, $N \in [1, 501]$.
- ii) To minimize the cumulative pointwise relative error of the fields tangential to the boundary between the outcomes of the methods in section 3 and in [93] w.r.t. the latter. This tunes the truncation number values L for t-PWS in (3.126) and (3.131), M - for CWHs, and the switch between (3.126) and (3.131) as in (3.133).

It is noted that this optimization constitutes an exploratory study carried out solely for the E_z equation in for polarization-decoupled (iso-refractive) cylindrical scatterers. Completing even fully CPU-parallelized versions of this optimization may require excessive computational time, on the order of several days. For this reason, the outcomes obtained from this study are also utilized for the H_z equation. The reliability of this approach is consistently observed throughout numerical calculations. The population size and the maximum number of generations are fixed to 20 and 50,

respectively, with a desired tolerance of 10^{-6} adopted as the stopping criterion. The variation of the truncation parameters L and M with respect to the sampling points on $\tilde{\Gamma}_\theta$ in **Figure 2.5**, subject to the constraints $L \in [0,10]$ and $M \in [1,51]$, is presented in **Figure 4.34**. The switch parameter takes the integer value 1 or 2 depending on whether (3.126) and (3.131) is employed. The resulting criterion of the switch value is given in (3.133). Therefore, the variation of the switch parameter itself is not presented here.

Figure 4.35 and **Figure 4.36** illustrate the accuracy and success in solving the near target-source problem by using the Modified - Spectrum Truncated Hankel Function of second kind” $\tilde{\mathcal{H}}_n^{(2)}(z)$. The Circle (1) and (2) in **Figure 4.35** are circular scatterers with radius c . The relative error formula is:

$$Relative\ Error = \frac{|\Psi^{TMM} - \Psi^{BIE}|}{|\Psi^{BIE}|} \quad (4.3)$$

In the proximate target/source problem, solutions can be improved by increasing the number of cylindrical harmonics due to the slow convergence of the addition theorem for the circular scatterer. However, this is not possible for a scatterer with an arbitrary cross section such as a square. However, an increase in the accuracy of the solutions is possible with the function $\tilde{\mathcal{H}}_{m-n}^{(2)}(z)$ and a suitable spectrum truncation number L chosen by genetic algorithm optimizations. The optimized L and M parameters that give reliable and accurate solutions for Square (1) are 4.9740 and 50 respectively. For Square (2), they are 2.6776 and 50, respectively.

Figure 4.36 illustrates similar results to **Figure 4.35** for complex k_ρ and different $\arg(z)$ values (for lossy medium) for star-shaped iso-refractive (iso-refractive to free space) scatterers with $z = k_\rho \rho_s$. The optimized L and M parameters that give reliable and accurate solutions for Star (1) – $|\arg(z)| = 0.31\pi$ case are 5.9313 and 31 respectively. for Star (1) – $|\arg(z)| = 0.17\pi$ they are 5.9330 and 31, respectively.

The optimized L and M parameters that give accurate solutions for Star (2) – $|\arg(z)| = 0.31\pi$ case are 4.7745 and 33 respectively. for Star (2) – $|\arg(z)| = 0.17\pi$ they are 3.2941 and 22, respectively.

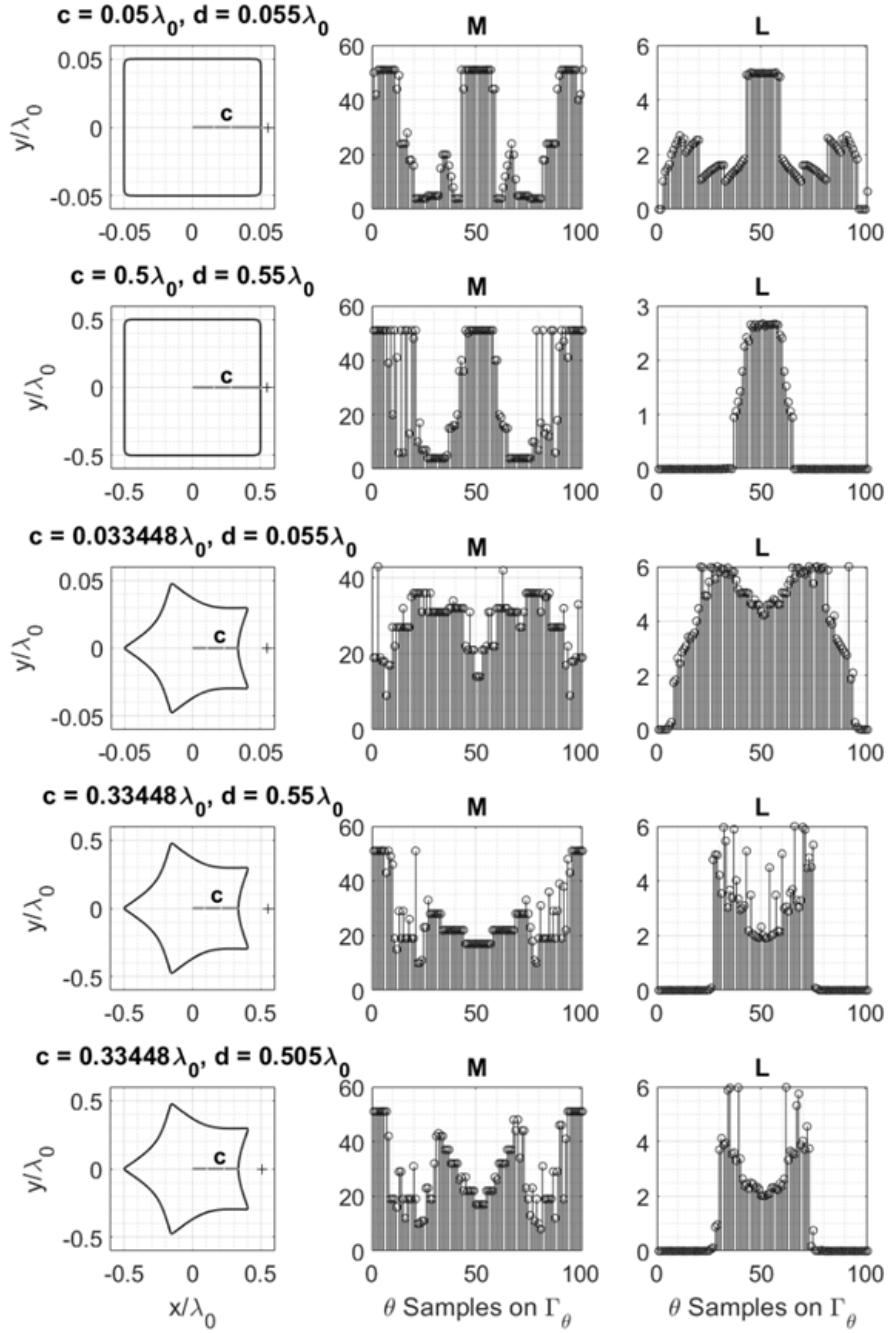


Figure 4.34: Variance of the GA-optimized truncation parameters of cylindrical harmonics (M) and the transverse plane wave spectrum of the incident proximate dipole field of free-space wavelength λ_0 (L) for each member of the longitudinal plane wave spectrum integrated on the integration curve $\tilde{\Gamma}_\theta$. Cross-section of the cylinder (row-wise: Square (1,2), Star (1,2,3)) parametrized according to formula given in [128] and the location of the dipole on the transverse plane are depicted on the left. Distance to the center of the scattering geometry of dipole - d - and the closest point on the boundary - c - are given.

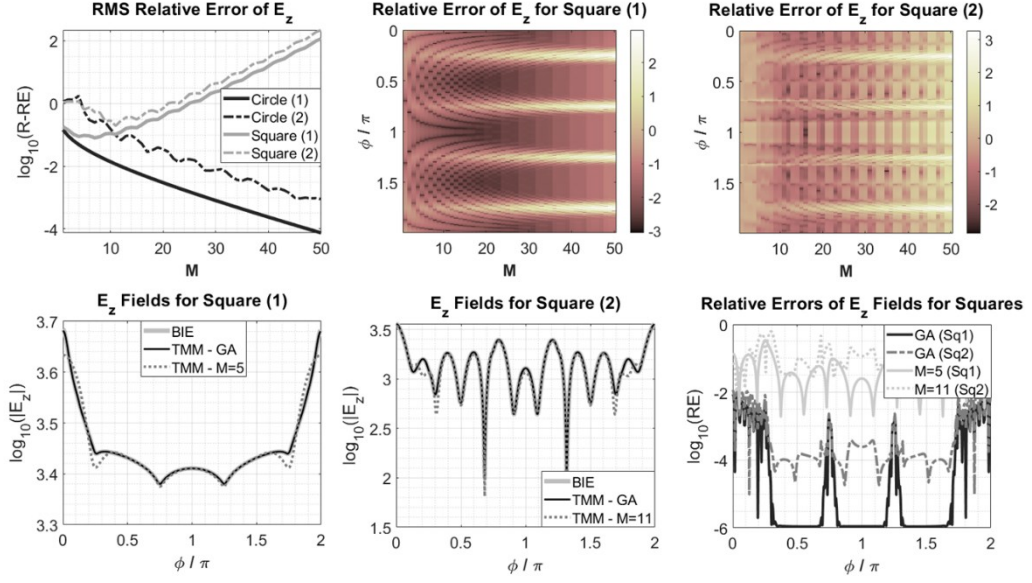


Figure 4.35: E_z fields and relative errors for scatterers with square cross section (with improved convergence of Graf's addition theorem using $\tilde{\mathcal{H}}_n^{(2)}(z)$ function for $|\arg(z)| = 0$).

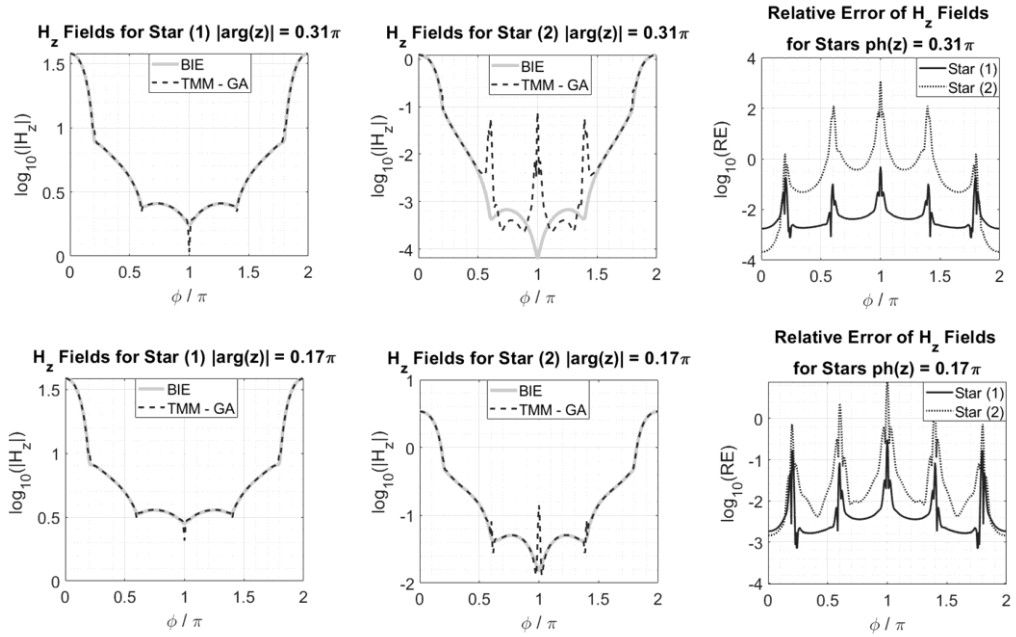


Figure 4.36: H_z fields and relative errors for scatterers with star cross section (with improved convergence of Graf's addition theorem using $\tilde{\mathcal{H}}_n^{(2)}(z)$ function for various $|\arg(z)|$ values).

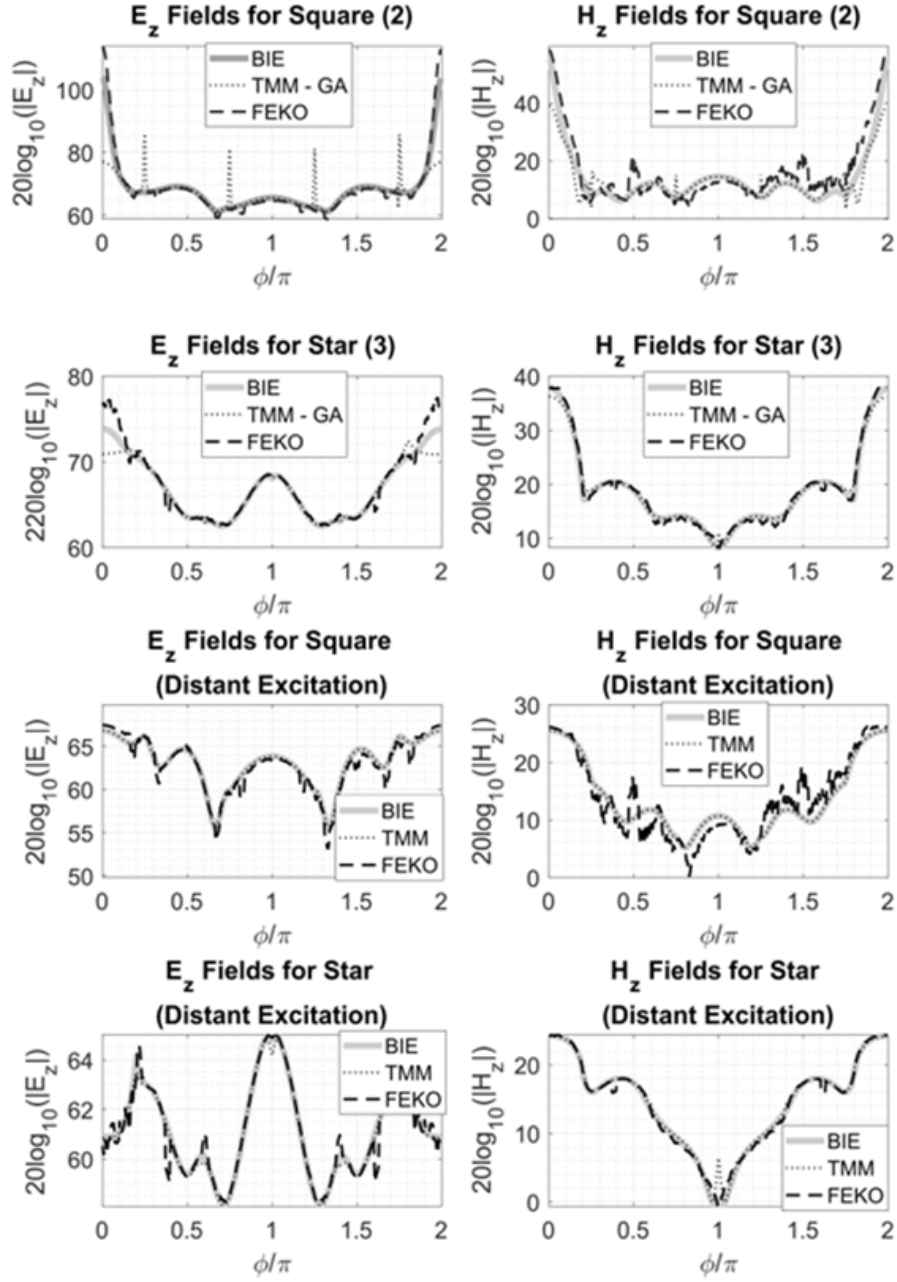


Figure 4.37: E_z and H_z on S solved by TMM for the scenario in **Figure 4.33** for bodies from **Figure 4.34** illuminated at depicted proximity and at a distance $d=\lambda$. Comparisons with respect to BIE and MLFMM-SEP from FEKO of -respectively- longitudinally infinite and finite cylinders iso-refractive to free space can be followed.

Figure 4.37 presents a pattern of deviation connected only to the features of the considered 3-D and 2-D models. Thus, the relevance of the solutions obtained with our proposed procedure to the ones of the 3-D models and their accuracy-improving effect is evident. From the presented cases, where the dipole is located inside and outside the circumscribing circle of Square(2) and Star(3) respectively, we can conclude the

following: Firstly, mild corner spikes of the electric field of Square(2) stand out as a deviation. Secondly, the results of Star(3) with 5 corners exemplify the coverage of the circular cylindrical harmonics in TMM. Thus, we can see the successful agreement between the 2-D TMM solutions and those of 2-D BIE and 3-D solver within the relevance scope presented. is evident.

Undoubtedly, TMM without well-conditioning could not be a candidate for optimization, run with results as in **Figure 4.34**. The underlying reason is illustrated in **Figure 4.38** through the condition numbers (CNs) of the LAES before and after the regularization proposed in subsection 3.4. This regularization is performed for each quadrature entry of the l-PWS integral. This guarantees the minimization of the behavior of the CNs with increasing M . The levels of these minimal values remain much below the value that may threaten the accuracy of calculations that the significant figures on a modern desktop computer can represent. **Figure 4.38** displays not only this but also the vast range of values that the CNs could get without the suggested regularization applied. The CNs of the LAES of a TMM implementation would be uniformly bounded after regularization if the object's cross-section were circular since the applied regularization was adopted from that of the circles. The reason behind the non-uniform course of the minimalized CNs is the presence of a deviation of the corresponding boundary contour's electrical shape from a circle for a particular frequency. Indeed, the Square(1) in **Figure 4.38** leads to CNs that tend to a uniformly bounded w.r.t. M .

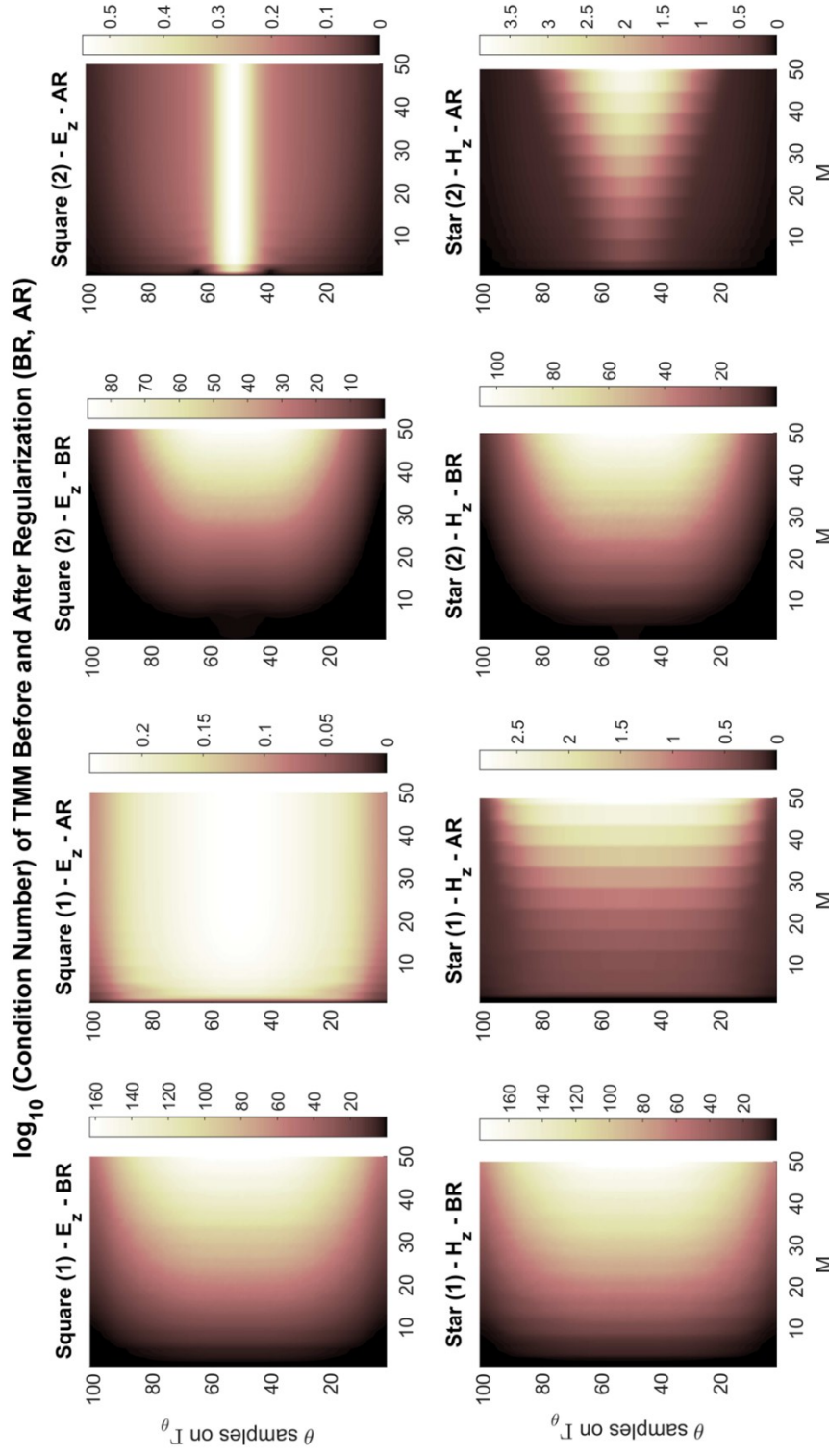


Figure 4.38: The condition numbers for each θ on on $\tilde{\Gamma}_\theta$ in the parametrized interval $t \in [-1, 1]$ for the cases in Figure 4.34. The depictions H_z of the first row and E_z of the second row are omitted for brevity due to similarity in their courses.

5. CONCLUSION AND FUTURE WORK

In this thesis, the TMM is enhanced for 2-D and 2.5-D scattering problems by achieving well-conditioning under a variety of excitation scenarios, including distant and near dipole excitations. Starting from the three-dimensional integral-equation formulation, the well-conditioned TMM and its acceleration via the fast multipole approach are investigated. By regularizing the inherently ill-conditioned accelerated system, a well-conditioned formulation is obtained, resulting in a fast and numerically stable algorithm for electromagnetic scattering from multiple dielectric cylinders relevant to microwave, light scattering and photonic crystal applications. It is demonstrated that, when the regularized system is solved using iterative techniques, stable solutions are achieved with a significantly reduced number of iterations.

With this improvement, the TMM becomes not only numerically robust but also practically efficient, rendering it well suited for design and optimization tasks in photonic and microwave beam-shaping problems, where a desired field distribution or power density is prescribed at a specific point or over a given region. Beyond frequency selective surfaces (FSS) and meta surfaces, the proposed and improved methodology provides a reliable and practical computational tool for microwave and optical photonic device design and optimization in communication applications. Moreover, owing to its numerical stability and efficiency, the resulting formulation is particularly attractive as a data generation engine for machine learning models, supplying reliable training data for both forward and inverse electromagnetic problems.

As future work, the extension of the proposed approach to planar layered media is envisioned, along with its application to large scale photonic devices and reconfigurable intelligent surface (RIS) applications in modern communication systems. Further directions include the integration of improved T-matrix framework with learning-based surrogate and inverse-design models, enabling scalable optimization and rapid design-space exploration for complex microwave and photonic systems.

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BIOGRAPHY

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PUBLICATIONS AND PRESENTATIONS FROM THE THESIS

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