

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

**USE OF MATHEMATICAL MODELING
FOR THE OPTIMIZATION OF
PORTFOLIO SELECTION PROCESS**

UĞUR HANAY
A THESIS SUBMITTED FOR THE DEGREE OF
MASTER OF SCIENCE
DEPARTMENT OF MATHEMATICS

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**THESIS SUPERVISOR
ASSIST. PROF. DR. UĞUR KAPLAN**

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T.C.
GEBZE TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

OPTİMİZE PORTFÖY SEÇİM
SÜRECİNDE MATEMATİKSEL
MODELLEME KULLANIMI

UĞUR HANAY
YÜKSEK LİSANS TEZİ
MATEMATİK ANABİLİM DALI

DANIŞMANI
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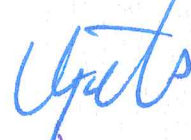
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İMZA/MÜHÜR

SUMMARY

In this thesis, the portfolio construction methods are discussed. Minimum variance, equally-weighted portfolio and equally-weighted risk contributions (ERC) portfolios are the methods that we studied. We specifically focused on the equally-weighted risk contributions (ERC) portfolios. The aim of this method is to find the risk weighted coefficients which makes the risk contribution of stocks constituting the portfolio equal. For analysing, the companies publicly traded on the exchange and in the BIST-30 Index are considered. Daily closing prices are used and the time window is from January 2015 to December 2015 (1 year). Daily stock returns are used in analysis. In general literature, referring also to optimize the portfolio selection process, we are looking at equal risk contribution with the required statical analysis. In the last part, we emphasize that these analysis require the use of specific criterias for target model. In the portfolio, the proportional of stocks risk contribution is required to be equal, this is one of our constraints. Our problem is to find weights of stocks for an equal risk contribution. We showed that the total weight of all the equities equal to 1, this restriction helps us to find the results of analysis. We consider the most important resource that as a reference an article : “On the properties of equally-weighted risk contributions portfolios”, the first version published by Sebastian Maillard, Thierry Roncalli and Jerome Teiletche, in 2008.

Key Words: Equally Weighted Risk Contribution, Mininum Variance, Risk Contribution, Portfolio Construction.

ÖZET

Bu tezde, portföy oluşturma yöntemleri ele alınmıştır. İncelenen yöntemler; minimum varyans, eşit ağırlıklı portföy ve eşit ağırlıklı risk katkısı olarak sıralanabilir. Özellikle eşit ağırlıklı risk katkısı üzerinde çalışma yoğunlaştırılmıştır. Bu yöntemde amaç, portföyümüzü oluşturan hisse senetlerinin risk katkılarını eşit yapacak olan ağırlık katsayılarını bulmaktır. Borsada işlem gören ve BIST-30 endeksi içerisinde yer alan şirketlerin hisse senedi fiyatları dikkate alınmıştır. Kullanılan fiyatlar gün sonu kapanışlarıdır ve Ocak 2015 – Aralık 2015 (1 yıllık) tarih aralığını kapsamaktadır. Analizlerde hisse senetlerinin günlük getirileri kullanılmıştır. Genel literatürde kullanılan, optimize portföy oluşturma yöntemlerine de değinerek eşit risk dağılımları yapısını gerekli istatistiki analizlerle inceliyoruz. Son durumda oluşturulması hedeflenen modelde bu analizler sonucu belirli kriterlerin kullanılması gerektiğini vurguluyoruz. Portföyde yer alan hisse senedinin oransal olarak risk katkısı eşit olması istenmektedir. Bu kullandığımız kısıtlardan bir tanesidir. Problemimiz de eşit risk katkısını veren hisse senedi ağırlıklarını bulmaktır. Burada tüm hisse ağırlıklarının toplamı 1' e eşit olacaktır, bu kısıtlama analiz sonucunu bulmamızda bize yardımcı olmaktadır. Referans olarak dikkate aldığımız en önemli kaynak, ilk versiyonu 2008 yılında Sebastian Maillard, Thierry Roncalli ve Jerome Teiletche tarafından yayınlanan “On the properties of equally-weighted risk contributions portfolios (Eşit risk ağırlıklı dağılımın özellikleri)” makalesidir.

Anahtar Kelimeler: Eşit Ağırlıklı Risk Katkısı, Minimum Varyans, Risk Getirisi, Portföy Oluşturma.

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LIST of ABBREVIATIONS and ACRONYMS

<u>Abbreviations</u>	<u>Explanations</u>
<u>and Acronyms</u>	
BIST	: Borsa Istanbul
BIST30	: Borsa Istanbul – 30 companies
CPF	: Corner Point Feasible
CTA	: Commodity Trading Advisor
ERC	: Equally Risk Weighted Contribution
GAMS	: General Algebraic Modeling System
GTU	: Gebze Technical University
LP	: Linear Programming
Max	: Maximum
MINLP	: Mixed-Integer Non-Linear Programming
MIP	: Mixed-Integer Programming
Min	: Minimum
MV	: Minimum Variance
NLP	: Non-Linear Programming
Stdev	: Standard Deviation

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1. INTRODUCTION

1.1. Aim of the Study

This MS thesis addresses modeling with equally-weighted risk contributions portfolios and optimization of portfolio selection process.

In literature, the studies is referred to as the portfolio optimization. The definition of portfolio optimization has many types, options. Firstly, we can start with a question that; “ What is an optimal portfolio? ”. The aim is to understand the definition of the optimal portfolio. In the next paragraphs, we are trying to find the answer to this question with an example.

The case is about probability of winning and losing which will be associated with investor’s risk perception and appetite. Let buy a lottery ticket, there are two options. The price of tickets is 1 TL. In the first one, there is a probability of 10% winning 100 TL and probability of 90% chance of winning nothing. For the other ticket, there is a probability of 10% of winning 500 TL and 90% chance of winning nothing, but if you lose you will pay extra 20 TL to lottery tickets seller. You have the right to purchase tickets once. Which lottery ticket will you buy?

In the beginning, for our example, the price is same, but in the end of game you may have to pay an extra fee. That we encountered in this case, the last results or lottery trends can determine which tickets give more chance. The statistical values can help in finding the answer. Let try to answer which ticket will we get with the results of expected returns and standard deviations.

First of all, we will discuss on expected returns of lottery tickets. In the following table, we can see the details for lottery tickets;

Table 1.1: Details of lottery tickets.

	Price (x)	Probability of winning (w)	Prize (y)	Probability of losing (l)	Penalty, if lose (p)
<i>Ticket 1</i>	1 TL	10%	100 TL	90%	0 TL
<i>Ticket 2</i>	1 TL	10%	500 TL	90%	20 TL

x: price, w: probability of winning, y: prize, l: probability of losing, p: penalty

The expected returns for the lottery tickets are calculated.

Let expected returns be equal to $f(x)$;

$$f(x) = y \times w + l \times p \quad (1.1)$$

is expressed in that formula. When the values are substituted in the formula (1.1), the achievement of the values in the table below;

Table 1.2: Expected return of lottery tickets.

	$f(x)$: expected return
<i>Ticket 1</i>	10
<i>Ticket 2</i>	32

When we analyse the result of expected returns of lottery tickets, it seems more logical to buy the Ticket 2. Now, let's share more details the properties of the tickets. There is the possibility of cancellation of the second lottery ticket. In addition, a refund is out of the question, so the second ticket is a risky and it has also high returns. If you are a "risk loving" investor, then you can purchase that tickets. This is completely related to your risk appetite. The positive relationship exists between the risk and expected return which means that as risk increases, the expected return increases. When the investors are choosing between products that have the same expected return, will choose a less risky. However, in cases of expected return varied, is engaged in the risk appetite of the investor. In accordance with these informations, we can say that are two types of investors which are named "risk loving" and "risk averse".

Let's go back to our example in which are going to purchase the ticket and continue to look for the answer the question. We will continue with the calculation of the standard deviation to study the properties of the tickets. The distribution can be measured by the standard deviation method.

We first take a look at the formulation of standard deviation in the following formulation;

$$\text{stdev}(x) = \sqrt{w \times (y - f(x))^2 + l \times (p - f(x))^2} \quad (1.2)$$

is expressed in the formula.

In Table 1.1, the details of lottery tickets exist which are by using the information in the table calculate the standard deviation. When the values are substituted in the formula (1.2), the achievement of the values in the table below;

Table 1.3: Standard deviation of lottery tickets.

	$stdev(x)$
<i>Ticket 1</i>	30,0
<i>Ticket 2</i>	157,9

The standard deviation method is generally used for portfolio optimization. In our example, we see that the Ticket 1 has smaller expected return but, in parallel, smaller standard deviation which means less risky. This method is used to examine the risk measure that is common and popular. Harry Markowitz is also used the standard deviation method in the portfolio optimization theorem [1]. These are the generally accepted methods.

Harry Markowitz is considered the inventor of the modern portfolio theory. Nearly 70 years ago, in an article published in 1952, "Portfolio Selection", The Journal of Finance, March 1952, 7,77-91, the investors who purchase securities of various how to achieve the maximum return with certain level of risk at the end of a time window or certain period.

2. LITERATURE REVIEW

2.1. Portfolio Optimization

Securities can be found in different types. With fixed interest rates, variable interest rates, indexed to a value or a portfolio etc. Eventually they are always assets that have a certain risk, that can be bought or sold. So what is this risk? What is the risk for a portfolio that we will construct with at least two securities? In literature, we can say as a general definition of risk : Risk is the potential of gaining or losing something of value.

Generally, risk is a term that expresses negativities. It is taking the chance of being exposed to certain situations for an expected profit. We might encounter many kinds of risks throughout our daily lives. A risky business life, a risky working field, risky transactions, a risky route, a risky route etc. Risk is a term that has permeated into our entire lives. It is encountered in many areas: finance, insurance, securities trading, workplace etc.

Risk is ubiquitous in all areas of life so, risk management is a must for every system. To manage risk, the risk must always be defined and diversified. Sometimes the risk is a fire, a flood, a volcanic eruption, but sometimes it is money.

As the products present in the finance sector are in the study, moving on through financial risk identification will be appropriate. Risk in the financial sector is the chance that the return achieved on an investment will be different from expected and also takes into account the size of difference. This includes the possibility of losing some or all of the original investment. Financial risk identification is expanding more each day in our time. Because the technology is developing, new investment products are arising and with the new terms, risk definition is expanding more and more. In financial risk, “credit risk, operational risk, market risk, liquidity risk, reputational risk, concentration risk, interest rate risk, currency risk, equity risk, commodity risk, country risk, legal risk, political risk, profit risk, systemic risk, volatility risk, model risk, etc.” can be listed. Financial risk definition to include all of these risk categories; “meaning the uncertainty of a return and the potential for financial loss” will be more common [2].

In 1952, The Journal of Finance published the article called “Portfolio Selection” authored by Harry Markowitz. Thanks to this article, the term Modern Portfolio Theory was established and developed. So, the terms financial risk and market management have too developed in within this frame. In modern portfolio theory, the standard deviation (or variance) of a portfolio is used as the definition of risk. At the same time, in Markowitz’s article it is defined how to find the probable maximum profit percentage that is possible at certain levels of risk regarding the securities in the portfolios created by investors.

We stated the definition of risk above. So, what is profit? Let’s move on with the answer to this question.

As for the definition of profit we can say that “it is the earnings that are expected by investing on a product in a given period of time and taking a certain risk during this investment”. By minimizing these risks, trying to maximize the earnings, the profit is the subject of Markowitz’s modern portfolio theorem. By diversification and inverse correlation between securities, the risk can be reduced significantly. According to the theorem, investors can choose alternative portfolios that have the same risk percentage but different profits. Generally in studies, Markowitz described with limitations, creating a portfolio while maximizing the average profit and minimizing the standard deviation, to put in a different way the construction of the portfolio in a way that it will provide the requested expected minimum profit. The limitations in the average profit maximization for the items that form the portfolio should have these qualities;

- i) Their weight should greater than 0 and equal to each other,
- ii) The total of their weights should equal to 1,
- iii)The standard deviation of the portfolio should smaller that a certain value.

This value, will symbolize the standard deviation that we find when we construct the portfolio with equal weights. For the minimum expected profit limitations the items that form the portfolio;

- i) Their weight should greater than 0 and equal to each other,
- ii) The total of their weights should equal to 1,
- iii)There will be a limit down for the average portfolio profit.

The limit down will symbolize the average profit we find when we construct the portfolio with items that have the same weight.

The modern portfolio theorem and many studies in the wake of the theorem are used commonly in the financial markets. The aim is to always minimize the risk to present the maximum possible income to the investor. The minimization of the portfolio's risk and the maximization of profit at an acceptable risk level under a stable profit are methods that are used in the average risk approach.

In the portfolios that are built by choosing from hundreds of items, there are the items with the best risk-profit relationships, this is also made real by Markowitz's Modern Portfolio Theorem. It is possible to see the efficient frontier, the efficient frontier that the active portfolios are on, on the chart that was built according to the risk profit relationship.

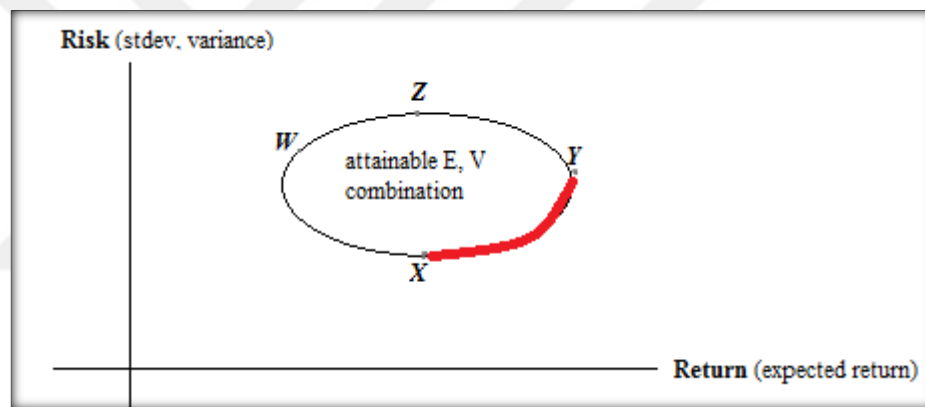


Figure 2.1: The attainable expected return (E) and variance (V).

As shown in the chart, the portfolio Z has the highest risk and the portfolio X has the lowest risk. However, the return of both portfolio are same. So, which are will choose? On the other hand, it is possible to say that the portfolio Y has the highest portfolio return. In this case, the best portfolios that the maximum return and minimum risk, we can say that among the X – Y. If the investors choose a portfolio or the portfolios with respect to this range, the expected return in certain risk level said it would be effectively.

It is possible to see the studies concerning portfolio construction and optimization as two periods in literature. “The Traditional Approach” and arose from Markowitz's article “The Modern Approach”. We can say that the start of the traditional approach dates back to the article written by Hicks in 1935. In the article,

the theory of money is explained while a portfolio approach is also mentioned. Here, only a simple process in mathematics, averaging is used. Another scientist who contributed to traditional approach is Marschak (1938). Like Hicks, he focused too on the theory of money more in portfolio construction. In Marschak's explanations as statistical calculations, he stated that the standard deviation and correlation relationships between different factors too can be used for measuring risk. This study also has contributions in the construction of other models. In 1938, Williams wrote a book about risk factors in securities. In the book, he stated that the expected profit or the cost of capital to the presenter of the investment product can not be estimated precisely. We can say that the most appropriate and the last study of the first period is done by Leavens in 1945. He pointed out that investigating and diversifying the risk factors is a way of reducing risk. Also, an investment on a security of a business that is active in only one sector is riskier in the case of a bad scenario in that sector. Leavens emphasized that the diversification, the risk's dependence on the sector will be reduced and higher profit will be possible.

In the traditional approach, the product that was invested on was always one. So we can say that there is a reverse situation in the risk-profit relationship. The modern portfolio approach, the approach which started with Markowitz, says that the risk can not be reduced with only portfolio diversification, the course and the volume of the relationships in the securities too have an effect. With the mean-variance model this was put forward. Also, many scientists contributed to the modern portfolio approach. In 1958, Tobin in his study, established a portfolio with "n amounts of risky and a riskless financial assets and as a result he said that the risk of risky financial assets are detached from each other. Here, he analyzed the covariance matrix. With the same method, Sharpe (1963), Cohen-Pogue (1967), Elton-Gruber (1973), Rosenberg (1974) have carried out analyses using real market data. The contribution of many scientists to the modern portfolio approach have continued in the recent past and different methods, ways were proposed. For some years now, another route has been explored by considering some inquiring methods like the minimum variance, equal risk contribution or equally-weighted portfolios which are special cases of a more general allocation approach based on risk parity [3], [4].

2.2. Equally Weighted Risk Contributions (ERC)

Over several decades ago, Markowitz (1952, 1956) suggested a practical implementation called mean – variance framework which is very well-known in optimal portfolio construction [1]. This method assumes that investors aim to maximize the expected return for a given volatility level. However, it has a few critical drawbacks which are tendence to be overly concentrated in the limited subset in the whole assets or securities and excessive sensitivity to the input parameters.

In the literature, there are alternative methods to deal with this problems, like portfolio resampling (Michaud, 1989) or robust asset allocation (Tütüncü and Koenig, 2004). Nevertheless, they have their own shortcomings such as additional computational encumbrance for investors, shrinkage estimator problems (Jorion, 1986) and inferior out-of-sample performance (Scherer, 2007a, 2007b). Whereas, most of investors prefer simple solutions which are computationally easy to carry out and are assumed robust since they are not subjected to expected returns [5].

There are two well-known techniques; the minimum variance portfolio which is special case on the mean – variance efficient frontier and the equally – weighted portfolios which are also known as “1/n” portfolios widely used in practice (Bernartzi and Thaler, 2001, Windcliff and Boyle, 2004) and are shown to be competent out-of-sample (DeMiguel, Garlappi and Uppal, 2009) [6]. Although, mean – variance portfolios are simple to implement due to the unique solution and is known as robust, they suffer from the asset density or concentration. A simple way for this drawback is to assign the same weight to all the assets included in the portfolio, which means equally weighted portfolios. However, they have also drawback of very limited diversification of risks where the individual risks seriously differ from each other.

We consider the most important resource that as a reference an article : “On the properties of equally-weighted risk contributions portfolios”, the first version published by Sebastian Maillard, Thierry Roncalli and Jerome Teiletche, in 2008 [5]. In this paper, another heuristic approach is analyzed, which is in the middle – ground stemming between minimum variance and equally – weighted portfolios. In this approach, the main idea is to make equal risk contributions from the disparate ingredients of the portfolio. Institutional investors are already practising on risk

contributions under risk budgeting which is more about risk contributions than about portfolio weights. Equalizing risk contributions implicitly presumes the homogeneity of the correlation structure. Thus, it rules out the correlation impact among strategies. However, it has become a standard practice for multistrategy hedge funds like CTAs.

Equally – weighted risk contributions (ERC) reproduce the effect of diversification in equally – weighted portfolios for single and joint risk contributions of the assets. On the other hand, the minimum – variance portfolios also make equal risk contributions, but with respect to a marginal basis. However, investors often focus on its risks in a limited number of positions by dropping out the benefits of diversification in practice. ERC is also based on optimality arguments which are shown as a solution to Markowitz's continuous time portfolio issue in terms of equalization of quantities related to risk contributions by assuming positive drift rates as Brownian motions governing stock prices.

The fundamental of ERC strategy is that the risk contributions of assets of the portfolio are equal to each other, which means a risk – balanced portfolio. Since most of investors cannot take short positions and it is better to keep similar conditions in comparison of ERC among other approaches, short selling is ignored in cases analyzed in the paper.

Minimum – variance (MV) and $1/n$ portfolios are widely used methods as pointed out previously [7]. ERC portfolios, on the other hand, are naturally located between these two approaches. Under a unique correlation assumption, $1/n$ portfolio becomes a special case where all volatilities are equal. Furthermore, it is straightforward to show that MV portfolio coincides with the ERC one for the lowest possible values of correlation matrix, that is the highest cross – diversification. In order to get the ERC portfolio, one minimizes the volatility of the portfolio subject to an additional constraint that is based on the minimum level of diversification among the factors which is essential due to getting the ERC portfolio. According to values of this constraint, it is possible to turn the portfolio into MV or $1/n$ one. This reinforces the interpretation of the ERC portfolio as an intermediary portfolio between both. Therefore, there is a natural order of the volatilities of the portfolios with the MV being the less volatile, the $1/n$ being the more volatile and the ERC stated between both [8].

Additionally, the ERC portfolio can be invested in all assets compared to the minimum variance one. Although its volatility is higher than the volatility of MV, it

is lower than the $1/n$ approach. In general, weights are ranked in the same order for both the ERC and MV portfolios. Nevertheless, ERC portfolio is obviously more balanced according to risk contributions. Moreover, the ERC portfolio is dominated on a risk adjusted basis because of the low volatility of MV. However, the main difference of ERC lies in the diversification since MV portfolios have excessive concentration and a much lower turnover.

On the other hand, the ERC portfolios transcend with respect to returns and risk compared to $1/n$ portfolios. Although the $1/n$ portfolio is balanced in weights, it is actually not balanced in risk contributions. Further, the MV portfolio condenses its weights and risk by the less volatile goods. Since the ERC approach focus on balance in risk and not too much concentrated on weights. Therefore, it seems apperantly more robust solution.

Investors tend to be more skeptical of traditional asset allocation methodologies comprising expected returns since lack of robustness and discomfort with emprical results. Thus, they focus on minimum variance which is independent of return expectations and equally – weighted ($1/n$) portfolios. Although they present robust solutions, both have their own limitations such as a lack of risk diversification for $1/n$ portfolios and striking asset density for MV ones. Therefore, ERC portfolios are an alternative approach based on equalizing risk contributions from different constituents of the portfolio. In this approach, the distribution of risks is maximized with a kind of “ $1/n$ ” filter with regard to risk so that asset allocator is dividing the risk budget equally to each component. Thus, none is dominating. While looking at the hierarchy of volatilities, this middle – ground positioning becomes more clear. Nevertheless, there is something to keep in mind that ERC approach may be computationally intensive since numerical optimization is necessary for determining the ERC solution for a large portfolio although the closed – form solutions derived in the analysis.

Even though empirical applications might be followed in different ways like in comparing the behaviour of equally – weighted risk contributions portfolios with other weighting techniques such as capitalization – weighted, equally – weighted, fundamentally – weighted (Arnott et al., 2005) and minimum – variance weighted (Clarke et al., 2002) portfolios, the methodology of ERC portfolios comparing with these approach is still an fascinating question.

2.3. Simplex, Branch and Bounds

Generally stated, the simplex method is an algebraic method with mathematical operations. However, geometry is also the basis of the method. First applications of this method are seen in works Dantzing in 1947. In details and principles of this method, optimal solution is seen to be achieved by algebraic repeating, in other words iteration method [9].

The simplex method applied in linear programming problem solving is the best method for the optimal solution of applications with many variables. For example, graphical method is also used in linear programming problems. However it is sufficient only for resolution of problems with up to three variables. When the number of variables increases, optimal solution is put away. For this reason resolution of real linear programming problems is ensured by simplex method [10].

The simplex method firstly creates the simplex table. Repetitive operations and rules as well as the structure reach developing solutions by application of a certain calculation method and algebraic iteration. Operations are repeated until the optimal solution is reached [11].

The simplex method is used in problems of industrial blends, multiple correlation problems' application, work and wage assessment problems, problems with profit maximization and cost minimization targets, optimization of structure models and many other fields. In this method, while trying to reach the best solution, a problem is brought to a standard form and resolution is reached by following the rules and determined by iteration method. Different points are tried within the solution area. Every new resolution is simplex method gives the closest result to the optimized solution at least as the previous one. Thus, after a certain number of iteration operation, the resolution is reached. So, what a standard form is?

In order for the linear programming problem to be in a standard form, all constraints must be equal, all variables must be positive and target function should be of minimization or maximization structure. For ensuring equality of constraints in standard form, new variables must be applied in accordance with the inequality. We can ensure this by adding or subtracting slack variables. The coefficients for slack variables are equal to one in constraints and to zero in objective function. The slack variables take place in the resolution too, but their values show the raw material

volumes or capacities which are not used in problems. Also, in the standard form, the rightside of the equation must always be positive, namely, greater than zero. If there is a negative value, the equation can be reversed by multiplication standard method.

We will see all points described above reaching an optimal solution over a single example, both graphically and algebraically, with certain iteration constraints. Our example will be an example which takes place in the book of Hillier and Lieberman, “Introduction to Operations Research”, 2002, 7th Edition and explains the simplex method in details. With details of example;

$$\begin{aligned}
 &\text{maximize } Z = 3x_1 + 5x_2 \\
 &x_1 \leq 4 \\
 &2x_2 \leq 12 \\
 &3x_1 + 2x_2 \leq 18 \\
 &x_1 \geq 0, x_2 \geq 0
 \end{aligned} \tag{2.1}$$

First we will try to find optimal solution with graph and geometric method. In the following graph, the corners and edges has been showed;

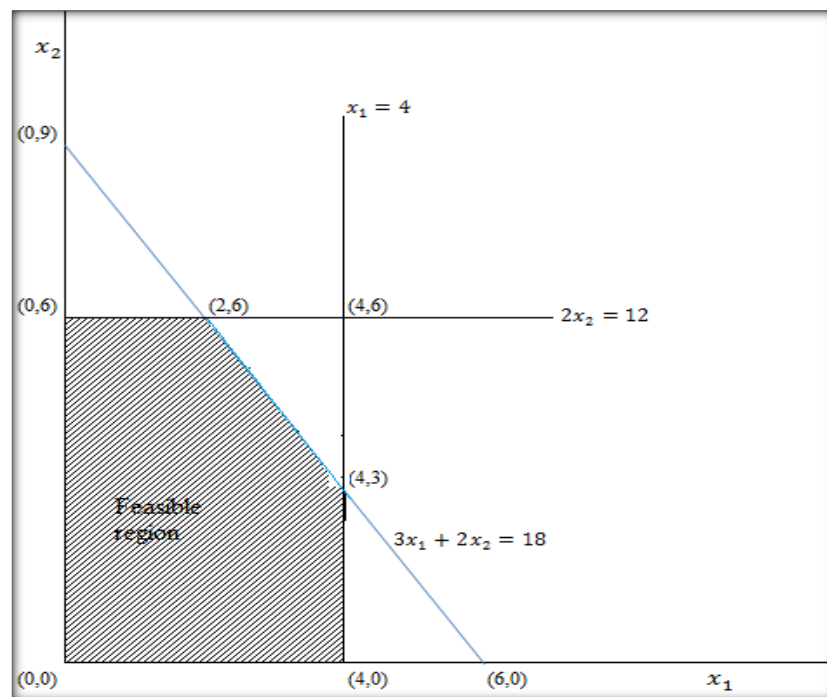


Figure 2.2: Feasible Region for optimal solutions of example.

The five constraint boundaries and their points of intersection are highlighted in that figure, they will play the key roles for analysis. They lie on the corners of the feasible region are the corner point feasible (CPF) solutions. They are; (0,0), (0,6), (2,6), (4,3), (4,0). The corner points infeasible solutions are; (0,9), (4,6), (6,0).

In our example, each corner point solution lies at the intersection of two constraint boundaries. The following general definitions say that; For any linear programming problem with n decision variables, two CPF solutions are adjacent to each other if they share $n-1$ constraint boundaries. The two adjacent CPF solutions are connected by a line segment that lies on these same shared constraint boundaries. Such a line segment is referred to as an edge of the feasible region.

In example, $n = 2$ so two of its CPF solutions are adjacent if they share one constraint boundary. We can say that (0,0) and (0,6) are adjacent because they share the $x_1 = 0$ constraint boundary. For all CPF solutions, we can say their adjacent CPF solutions with respect to that definition.

Table 2.1: The CPF Solutions and adjacents.

CPF	Their adjacents
Solutions	CPF solutions
(0,0)	(0,6) and (4,0)
(0,6)	(2,6) and (0,0)
(2,6)	(4,3) and (0,6)
(4,3)	(4,0) and (2,6)
(4,0)	(0,0) and (4,3)

Then we have to check whether a CPF solution is an optimal solution. So, what is the optimality test? We mean that optimality test have to apply for finding optimal solution. Consider any linear programming problem that possesses at least one optimal solution. If a CPF solution has no adjacent CPF solutions that are better, then it must be an optimal solution [11].

With (0,6), $Z = 30$; (4,3), $Z = 27$; (2,6), $Z = 36$ is larger, so it must be optimal solution. With same method and steps, we can solve the example for all of CPF solutions. Final, we will find (2,6) is optimal solution, since we want to maximize Z .

In the following sentences, the solution concept defines the simplex method with diagram;

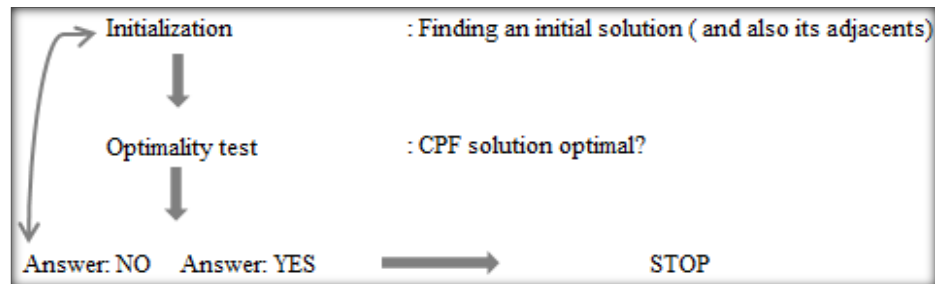


Figure 2.3: Iteration Diagram.

We can develop the solution method a little bit more by using the same iteration logic. You wonder how? As it was mentioned above, we can bring our model to an equaiton by using slack values. In the latter case, we can write our model as follows;

$$\begin{aligned}
 & \text{maximize } Z = 3x_1 + 5x_2 \\
 & x_1 + x_3 = 4 \\
 & 2x_2 + x_4 = 12 \\
 & 3x_1 + 2x_2 + x_5 = 18 \\
 & x_j \geq 0, \quad (j = 1, \dots, 5)
 \end{aligned} \tag{2.2}$$

Our model has the same meaning in both forms. The new created model is called as “an augmented form”. We can see the summary of all steps in geometrical and algebraic breakdown in the table below. The interpretations are the geometric and algebraic with the simplex method.

Table 2.2: The geometric and algebraic interpretations of simplex method.

Method Sequence	Geometric Interpretation	Algebraic Interpretation
Initialization	Choose (0,0) to be the CPF solution	Choose x_1 and x_2 , for the initial solution (0,0,4,12,18).
Optimality test	No, because moving along edge from (0,0) increases Z.	No, because increasing x_1 or x_2 increases Z.
Iteration 1		
Step 1	Move up on the x_2 axis	Increase x_2
Step 2	Stop when the $2x_2 - 12$ is reached.	Stop when x_3, x_4 or x_5 drops to 0, (x_4).
Step 3	Find the new pair : (0,6) is the new CPF.	x_2 basic, x_4 non-basic solve equations: (0,6,4,0,6) is new BF solution.
Optimality test	No, because moving along edge from (0,6) increases Z.	No, because increasing x_1 increases Z.
Iteration 2		
Step 1	Move along this edge to the right.	Increase x_1
Step 2	Stop when the $3x_1 + 2x_2 - 18$ is reached.	Stop when x_2, x_3 or x_5 drops to 0, (x_5).
Step 3	Find the new pair : (2,6) is the new CPF.	x_1 basic, x_5 non-basic solve equations: (2,6,2,0,0) is new BF solution.
Optimality test	(2,6) is optimal, because moving along edge from (2,6) decreases Z.	(2,6,2,0,0) is optimal, because increasing x_4 or x_5 decreases Z.

For sure the result is the same and the optimal solution is on $x_1 = 2$ and $x_2 = 6$, so $Z = 3x_1 + 5x_2 = 36$, will be found.

The algebraic form of the simplex method may be the best one for learning underlying logic of the algorithm, and also the table method can be used in simplex method as a different solution method. It is possible to apply methods with algorithmic logic by means of a computer. Technological products are used for the solutions as well. But if you want to resolve the problem manually, by hand, table is the best and most appropriate method. The iteration logic is the same in this form too. It is just about the unique structure of solution steps and some new definitions. Needless to say that the result will be the same, if you made no mistake in solution steps.

The branch and bound method is a method that helps us to classify algorithms in mathematical models that we will produce in the linear programming problems. The objective here is to find the best optimal solution as usual, if it exists. First of all, we need to show step by step what the problem is and what is the meaning of data in the search for the solution. The road to follow is definite after we have identified the entries. The entrances are broken up and the progress is made by the iteration method. More simply, we can call it the branching method. The next step is to determine the upper and lower bounds for these parts. In the latter case, the optimal solution is obtained [11].

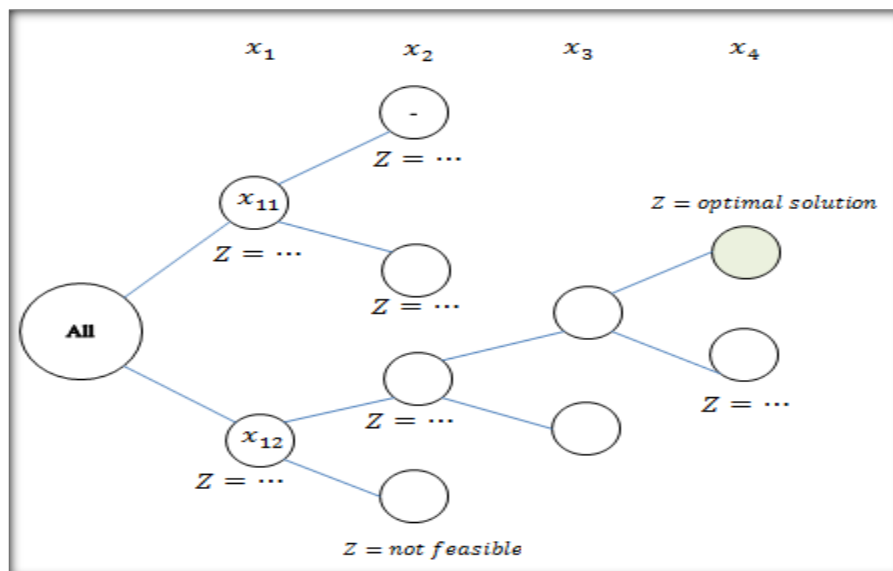


Figure 2.4: Branch and Bound Method – Iteration Graph.

As shown in the graphic above, it is possible to move forward with possible restrictions. In the latter case, there is an optimal solution that gives the best results and provides all the constraints.

The branch and bound methods refer to something else, general enumerating all solutions of an integer program, to solve it more efficiently. This is the divide method, on the other hand meaning conquer method. In the branch method and steps, we divide the problem into a few smaller ones. In the conquering part, we try to get for each smaller problems to do this, we have to divide the problem further until we get a problem that we can handle, that is the bound part. In the example, we have shown these steps better through. With details of example;

$$\begin{aligned}
 & \text{maximize } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4 \\
 & 6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10 \\
 & x_3 + x_4 \leq 1 \\
 & -x_1 + x_3 \leq 0 \\
 & -x_2 + x_4 \leq 0 \\
 & x_i \text{ are binary, for } i = 1, 2, 3, 4.
 \end{aligned} \tag{2.3}$$

Let start to branching; Branch 1 : $x_1 = 0$ or $x_1 = 1$. On the next steps, we have to apply bounding method for optimal solution and also with simplex method we can say the linear programming relaxation yields its optimal solution $(x_1, x_2, x_3, x_4) = (\frac{5}{6}, 1, 0, 1)$ and $Z = 16.5$. The iteration methods will continue for all steps and the optimal solution searching is also exist. With these results, if we continue to Branch 1:

If $x_1 = 0$, the problem;

$$\begin{aligned}
 & \text{maximize } Z = 5x_2 + 6x_3 + 4x_4 \\
 & 3x_2 + 5x_3 + 2x_4 \leq 10 \\
 & x_3 + x_4 \leq 1 \\
 & x_3 \leq 0 \\
 & -x_2 + x_4 \leq 0 \\
 & x_i \text{ are binary, for } i = 2, 3, 4.
 \end{aligned} \tag{2.4}$$

The optimal solution is in (1,1) and $Z = 9$. If $x_1 = 1$, the problem;

$$\begin{aligned}
& \text{maximize } Z = 9 + 5x_2 + 6x_3 + 4x_4 \\
& 3x_2 + 5x_3 + 2x_4 \leq 4 \\
& x_3 + x_4 \leq 1 \\
& x_3 \leq 1 \\
& -x_2 + x_4 \leq 0 \\
& x_i \text{ are binary, for } i = 2, 3, 4.
\end{aligned} \tag{2.5}$$

The optimal solution is in (1,0.8,0,0.8) and $Z = 16.2$.

In that example, if we keep going to all branches and find the all possible values of Z , in final we find the optimal solution. With simplex method and iteration method, in final, we find the optimal solution and current best solution. We have step by step analyzed the all possibilities and feasible solution. It is in the $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0$. The current best (1,1,0,0) and $Z = 14$ is the optimal solution. In addition, all subproblems (branches) were fathomed and the relaxation of the subproblem has an optimal solution with $Z < Z^*$ is the current best solution [11]. The tree and all steps and results are shown in the following graph.

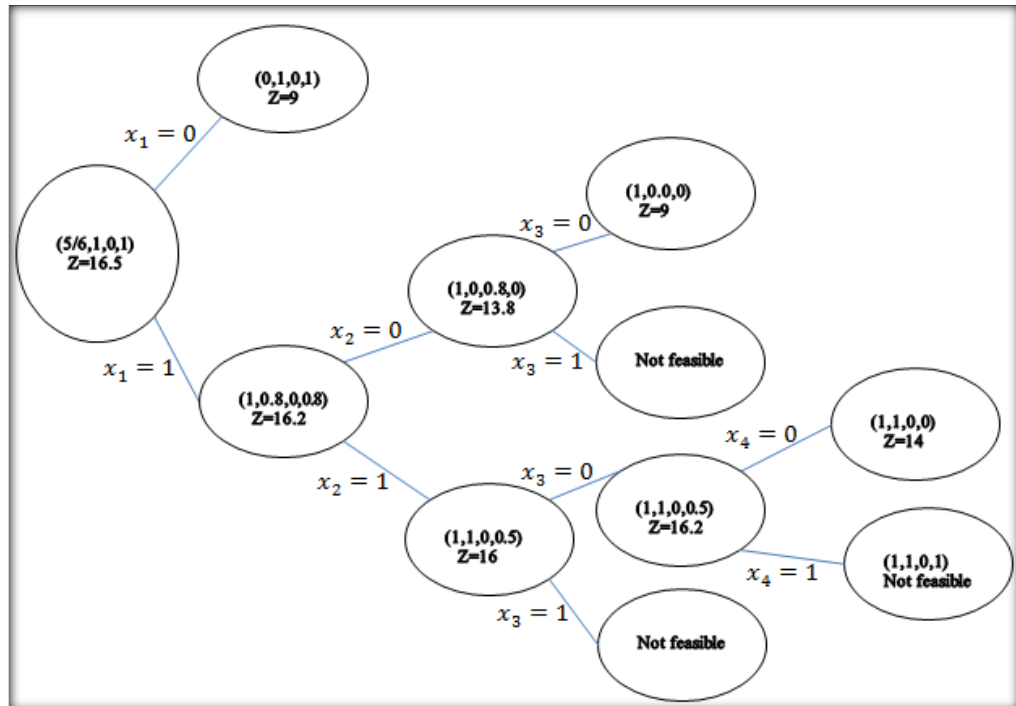


Figure 2.5: The results of branch and bound for all iterations.

Consequently, the optimality tests and iteration steps indicate that $(x_1, x_2, x_3, x_4) = (1, 1, 0, 0)$ is optimal solution with $Z = 14$.

2.3.1. Linear Programming (LP)

The linear programming is an important part of the literature when we consider the mathematical model for the optimization of portfolio selection process. Modelling is an expression that stated by words, symbols or mathematical formulas which analyzes the changes of a system or a process and examines its behaviours under spesific conditions; as result of this helps controlling and making predictions about future. In our work, we will use mathematical modelling. Well, what is mathematical modelling? The general name of system that connects words, symbols or formulas used in model by functions is mathematical modelling. The mathematical model has three main components which are decision variables, objective function and constraints [12].

The decision variable, in mathematical model the calculation value is decision variable within the analyze process. Before model set up phase these values are unknown values, symbols. As result of model, they will be calculated.

Objective fuction is a linear function that provides to obtain the best solution (maximum or minimum) consisting of decision variables.

$$Z_{max/min} = \sum_{j=1}^n c_j x_j \quad (2.6)$$

Constraints are called as the essential relationship between in decision variables and mathematical model of decision variables. Being greater or less than each other, greater than zero, totally less than a parameter can be exemplified.

$$\sum_{j=1}^n c_j x_j \leq b_i \quad (i = 1, \dots, m) \quad (2.7)$$

$$x_j \geq 0 \quad (j = 1, \dots, n)$$

A linear programming problem may be defined as the problem of maximizing or minimizing a linear function subject to linear constraints. The constraints may be equalities or inequalities. In linear programming, all of the mathematical expressions for the objective function and the constraints are linear. For solving mathematical models, the numerical values have to find for decision variables to minimize or maximize an objective function in the presence of constraints. In the example, we can see that a linear programming problem which is with 2x2 dimension :

$$\begin{aligned} Z_{max} &= c_1x_1 + c_2x_2 \\ a_{11}x_1 + a_{12}x_2 &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 &\leq b_2 \\ x_1, x_2 &\geq 0 \end{aligned} \tag{2.8}$$

The some linear programming application fields which are production program, feeding program, capital budgeting program, distribution program, stock control program, production line balance program can be considered.

The linear programming has six assumptions. We can see them in the following sentences;

- Certainty
- Linearity
- Divisibility
- Additivity
- Proportionality
- Non-negativity

A linear programming model consists of a linear objective function which is optimized within the scope of linear equality or inequality restricted equations. In general form expressed as;

$$Z_{max/min} = \sum_{j=1}^n c_j x_j \tag{2.9}$$

The solution is usually more than one in a linear programming problem. However, there is one optimum solution (usually, because it could be an alternative solution). The resources to achieve the goal is not infinite amount. In parallel, it plays an active role in determining the solution with the least resources. It should also be expressed, the decision variables in linear programming problem must not be negative [12].

2.3.2. Mixed Integer Program (MIP)

For many science interested people, modern linear programming problem comes to light after work of George Dentzing published in 1947, yet this issue is still debatable. In this paper, mixed integer programming will be discussed. Development of mixed integer programming is successor of linear programming as well as integer programming. Thus, in order to comprehend mixed integer programming, integer programming should be discussed.

The mixed integer linear program is of the form;

$$\begin{aligned}
 \max/\min &= c^T x_j \\
 Ax &= b \\
 x_j &\geq 0 \quad (j = 1, \dots, n) \\
 x_j &\text{ integer} \quad (\text{for some or all } j = 1, \dots, n)
 \end{aligned}
 \tag{2.10}$$

If all variables need to be integer it is called a integer linear program. On the other hand, if all variables need to be 0 or 1, it is called a 0-1 linear program.

Mathematically, the Mixed Integer Linear Programming (MIP) Problem looks like;

- *maximize or minimize* $c_1 t + c_2 u + c_3 v + c_4 w + c_5 x + c_6 y + c_7 z$
- *subject to* $A_1 t + A_2 u + A_3 v + A_4 w + A_5 x + A_6 y + A_7 z$

The linear programming models have been argued, far all have been continuous, it has the same meaning that decision variables are allowed to be

fractional. This is a often realistic assumption. However, fractional solutions are not realistic, so we have to suppose the optimization problem;

$$\begin{aligned}
 & \text{maximize } Z = \sum_{j=1}^n c_j x_j \\
 & \sum_{j=1}^n a_{ij} x_j = b_i \quad (i = 1, \dots, m) \\
 & x_j \geq 0 \quad (j = 1, \dots, n) \\
 & x_j \text{ integer} \quad (\text{for some or all } j = 1, \dots, n)
 \end{aligned} \tag{2.11}$$

is called the integer programming problem. It is said to be a mixed integer program when several variables are restricted to be integer.

An integer programming problem is a mathematical optimization problem or feasibility problem that contains all variables as integer. Integer programming arises from need of daily life and industry. In other words, integer programming models are developed as practically every area of application of mathematical programming. Capital budgeting, warehouse location and scheduling problems can be given as an integer programming [11], [12].

In a typical capital-budgeting problem, decision involve the selection of a number of potential investments. The investment decisions might be to choose among possible plant locations, to select a configuration of capital equipment, or to settle upon a set of research and development projects. Often it makes no sense to consider partial investments in these activities, and so the problem becomes a go-no-go integer program, where the decision variables are taken to be $x_j = 0$ or 1 , indicating that the j th investment is rejected or accepted. Assuming that c_j is the contribution resulting from the j th investment and that a_{ij} is the amount of resource i , such as cash or manpower, used on the j th investment, we can state the problem formally as;

$$\begin{aligned}
 & \text{maximize } Z = \sum_{j=1}^n c_j x_j \\
 & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad (i = 1, \dots, m) \\
 & x_j = 0 \text{ or } 1 \quad (j = 1, \dots, n).
 \end{aligned} \tag{2.12}$$

The objective is to maximize total contribution from all investments without exceeding the limited availability b_i of any resource.

One important special scenario for the capital-budgeting problem involves cash flow constraints. In this case, the constraints

$$\sum_{j=1}^n a_{ij} x_j \leq b_i \quad (2.13)$$

reflect incremental cash balance in each period. The coefficients a_{ij} represent the net cash flow from investment j in period i . If the investment requires additional cash in period i , then $a_{ij} > 0$, while if the investment generates cash in period i , then $a_{ij} < 0$. The righthand-side coefficients b_i represent the incremental exogenous cash flows. If additional funds are made available in period i , then $b_i > 0$, while if funds are withdrawn in period i , then $b_i < 0$. These constraints state that the funds required for investment must be less than or equal to funds generated from prior investments plus external funds made available.

When it comes to mixed integer programming, firstly it can be said that definition has a crucial distinction between integer programming and mixed integer programming which is condition of variables. In this programming, all variables are not integer. The mixed integer programming problem contains some of decision variables are constrained to be integer valued.

Mixed integer programming is vastly used to provide an explanation and achieve a solution of optimization problems. For instance, travelling salesmen problem, sudoku modelling problem, warehouse sales allocation and portfolio optimization problem are some of mixed integer problem.

To conclude that mixed integer programming play significant role in mathematical problems. After developing super computers and advacing technologies, integer programming as well as mixed once is hot topic in mathematics and computer science in order to solve people's daily life problems.

2.3.3. GAMS

The Gams –General Algebraic Modeling System- is a high level programming language which used to modeling and solving the optimization problems (which are linear, non-linear and mixed-integer). The program has been developed for the World Bank by first optimization specialist Dr. Anthony Brook and Dr. Alexander Meeraus. GAMS began to worldwide recognition in 1987 after receiving awards from “The Computer Science Technical Section of the Operations Research Society of America”. One of the best feature of the program is to be strong as well as ease of use and moreover flexibility. GAMS contains the solver algorithms such as CPLEX, lingo, dicopt in order to solve linear, non-linear and mixed integer problems. Furthermore, GAMS can easily change the methods while solve the problems.

The GAMS can be used not only for mathematical models with algebraic equations, but also, acquainted with methods of solving differential equations in partial derivatives. The mathematical problems are solved with some iterations. The using of GAMS is really comprehensive.

In GAMS, each model have the structure which are “SET, DATA, VARIABLES, EQUATIONS, MODEL, SOLVERS, OUTPUT”. In the example, Consider the problem of assigning process streams to heat exchangers as described in pp.409-410 of the book "Optimization of Chemical Process" by Edgar and Himmelblau. The optimization problem is given by:

$$\begin{aligned} \text{minimize } z &= \sum_i \sum_j c_{ij} x_{ij} \\ \sum_i x_{ij} &= 1 \quad (j = 1, \dots, n) \\ \sum_j x_{ij} &= 1 \quad (i = 1, \dots, n), \quad \sum_i x_{ij} = 1 \quad (j = 1, \dots, n) \\ x_{ij} &= 0, 1 \quad (i, j = 1, \dots, n) \end{aligned} \quad (2.14)$$

which corresponds to the well knowm assignment problem. Here i represents the index for the n streams and j the index for the n exchangers. The binary variable $x_{ij} = 1$ if stream i is assigned to exchanger j , and $x_{ij} = 0$ if it is not. The two equations simply state that every exchanger j must be assigned to one stream. and every stream i must be assigned to one exchanger.

The cost C_{ij} of assigning stream i to exchanger j is as follows:

Table 2.3: The cost of assigning stream to exchangers.

				Exchangers
Streams	1	2	3	4
A	94	1	54	68
B	74	10	88	82
C	73	88	8	76
D	11	74	81	21

We can formulate the above problem in GAMS in the form of the model in (4) using index sets. As shown in the output except for the use of indices. Note that the data are given in terms of SETS and the TABLE. The elements of a set can be names (A, B, C, D) or numbers. In the latter case we can use the * sign to denote the range (1*4, means 1, 2, 3, 4). For the TABLE there is no need to place the numbers in precise positions; they only have to be consistent, data can also be entered using the keywords SCALAR and PARAMETERS. Note that we can specify x_{ij} as a variable with indices, $X(I,J)$, and similarly the two equations: $ASSI(J)$ means for $j = 1, 2, 3, 4$; $ASSJ(I)$ means for $i = A, B, C, D$. Also, in this case since x_{ij} is restricted to 0-1 values, we use the keyword BINARY VARIABLES. The summation is expressed in the form of SUM (index of summation, terms in sum).

Last, for the input, we have used the OPTION SOLPRINT=OFF statement, so as to only print the values of the variables x_{ij} and the objective Z. This is done with the DISPLAY keyword where we list the level value of the x_{ij} (X.L.– no indices are needed) and of Z (Z.L.). Finally, note that in the SOLVE statement we specify the solver Mixed Integer Program (MIP) due to the fact that the x_{ij} are binary variables. The solve summary indicates that the optimum objective function is $Z=97$. We also note that the solution was obtained from the relaxed Linear Programming (LP). This is not surprising since it is well known that the assignment problem has a “unimodular” matrix and therefore the solutions for the x_{ij} are guaranteed to be 0-1 if we solve the problem as an LP.

Finally, due to the use of the DISPLAY statement the requested variables are printed. Note that stream A is assigned to exchanger 4, B to 2, C to 3 and D to 1 with

a minimum cost of 97. The GAMS program codes, details –for sample model- , inputs and outputs are located in the appendices.



3. PROBLEM DEFINITION, SOLUTION and RESULTS

3.1. Formulation as a GAMS

In the research, the mathematical modeling will be used for the optimization of portfolio selection process. The data used are the equity values of the companies included in the BIST30 Index. For analysing, the companies publicly traded on the exchange and BIST30 Index are considered. The graphics, trend values, matrix and statistical results are prepared in excel.

The BIST30 Index is a capitalization weighted index composed of National Market companies except investment trusts and it can also be used for trading in Derivatives Market.

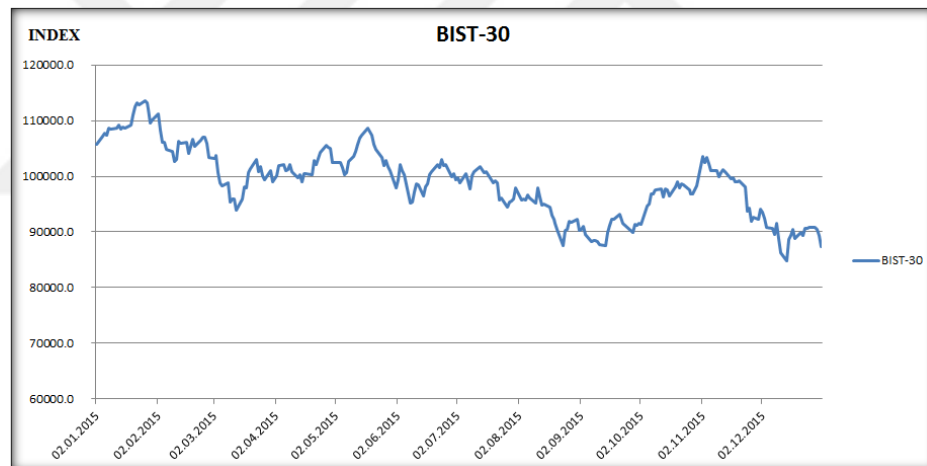


Figure 3.1: The time window 01.01.2015 - 31.12.2015, BIST30 Index trend.

The graphic above shows the trend of BIST30 Index for the time window 01.01.2015 and 31.12.2015. As can be seen from the graph, the BIST30 Index values reached a minimum of 84.893 and a maximum of 113.615 in 2015. The average annual value of the index is 98.899.

The equity price figures for BIST30 companies in the year 2015, are given in the graphic below. The daily closing prices have used and shown for the time window January 2015 and December 2015.

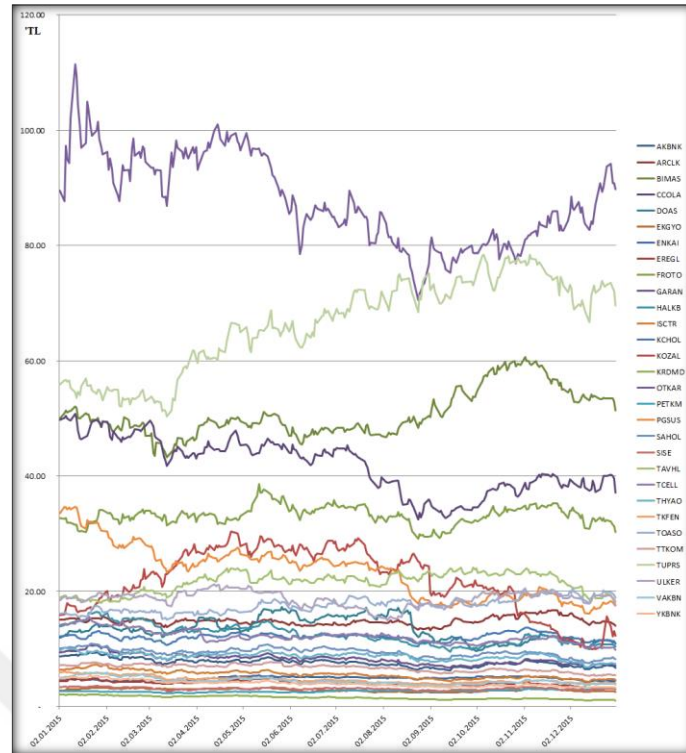


Figure 3.2: In 2015, the equity price trend of BIST30 companies.

It is possible to give the following summary statistical information for the daily closing prices of some related companies after 2015.

	BIST-30	AKBNK	BIMAS	CCOLA	DOAS	EKGYO	FROTO	GARAN	HALKB	ISCTR	KCHOL	OTKAR	PETKM	SISE	TCELL	ULKER	VAKBN	YKBNK
MAX	113,614,870	9,500	60,600	50,750	17,250	3,280	38,600	10,700	16,450	7,250	13,700	111,500	3,133	3,624	15,600	21,250	5,940	5,310
MIN	84,893,120	6,290	43,200	32,450	9,670	2,380	29,050	6,700	9,560	4,330	10,700	70,500	2,273	2,650	9,900	14,950	3,530	3,210
MEDIAN	99,170,660	7,670	49,500	43,100	13,150	2,830	33,150	8,290	12,350	5,570	12,050	86,350	2,653	3,105	12,000	18,900	4,340	3,910
STDEV	6,142,764	0,692	4,030	4,694	2,003	0,251	1,638	0,874	1,669	0,695	0,588	7,871	0,205	0,206	1,204	1,289	0,573	0,527

Figure 3.3: The statical results of equity price.

The chart below shows the correlation matrix of equity prices and BIST30 Index. The general meaning of correlation is the relationship between something else. The correlation indicates whether there is linear relationship between the two variables, that is, whether the changes in the variables affect each other. This is represented by the correlation coefficient that is between -1 and 1. While the negative values indicate a negative relationship, the positive values are positive relationship. If the values are -1 or 1, then that exist a perfect relationship between variables. In additon, it can be said that as the values approach zero, the relationship between the variables decreases. For example, as can be clearly seen in the following chart, AKBNK has a high positive correlation with CCOLA but negatively correlated and near-zero with ENKAI.

PRICE	AKBNK	ARCLK	BIMAS	CCOLA	DOAS	EKGYO	ENKAI	EREG	FROTO	GARAN	HALKS	ISCTR	KCHOL	KOZAL	KROMD	OTKAR	PETKM	PGSUS	SAHOL	SISE	TAVHL	TECEL	THYAO	TKFEN	TOASO	TTKOM	TUPRS	ULKER	VAKBN	YKBNK	BIST-30
AKBNK	1.000	0.266	-0.249	0.889	0.337	0.806	-0.093	0.788	0.328	0.970	0.936	0.941	0.472	0.167	0.902	0.752	-0.351	0.897	0.872	0.837	-0.432	0.849	0.820	0.917	-0.455	0.827	-0.677	0.348	0.944	0.948	0.976
ARCLK	0.266	1.000	0.632	0.092	-0.285	0.433	0.106	0.024	0.433	0.137	0.138	0.062	0.689	-0.435	0.035	0.138	0.451	-0.025	0.098	0.435	0.101	0.087	0.217	0.167	0.412	-0.041	0.194	0.440	0.284	0.101	0.306
BIMAS	-0.249	0.632	1.000	-0.478	-0.646	-0.085	0.289	-0.337	0.080	-0.412	-0.421	-0.487	0.447	-0.612	-0.501	-0.373	0.731	-0.545	-0.326	0.660	0.322	-0.417	-0.357	-0.327	0.615	-0.372	0.671	0.321	-0.238	-0.399	-0.112
CCOLA	0.889	0.092	-0.478	1.000	0.416	0.767	-0.218	0.709	0.331	0.809	0.916	0.915	0.237	0.269	0.905	0.834	-0.434	0.910	0.767	0.704	-0.309	0.734	0.706	0.893	-0.519	0.830	-0.796	0.317	0.861	0.896	0.855
DOAS	0.337	-0.285	-0.646	0.416	1.000	0.122	0.258	0.555	0.406	0.394	0.385	0.405	0.125	0.708	0.407	0.220	-0.419	0.506	0.579	0.251	0.044	0.337	0.572	0.184	0.609	-0.241	-0.307	0.198	0.302	0.390	
EKGYO	0.806	0.433	-0.085	0.767	0.122	1.000	-0.174	0.588	0.405	0.782	0.854	0.781	0.441	0.041	0.745	0.754	-0.266	0.628	0.605	0.660	-0.331	0.613	0.932	0.789	-0.114	0.667	-0.599	0.612	0.771	0.819	
ENKAI	-0.093	0.106	0.289	-0.218	0.258	-0.174	1.000	0.337	0.422	-0.211	-0.222	-0.211	0.529	0.446	-0.143	-0.318	0.049	-0.142	0.283	0.077	0.797	0.279	-0.294	0.247	0.085	0.556	0.025	-0.235	-0.252	0.052	
EREG	0.788	0.024	-0.337	0.709	0.555	0.588	0.337	1.000	0.349	0.775	0.768	0.798	0.453	0.581	0.838	0.548	-0.581	0.809	0.910	0.624	0.001	0.672	0.854	0.700	-0.533	0.898	-0.515	0.244	0.709	0.766	0.856
FROTO	0.328	0.433	0.080	0.331	0.406	0.405	0.422	0.349	1.000	0.226	0.263	0.200	0.635	0.246	0.160	0.143	0.055	0.159	0.465	0.409	0.209	0.083	0.413	0.127	0.317	0.358	0.090	0.254	0.223	0.134	0.401
GARAN	0.970	0.137	-0.412	0.909	0.394	0.782	0.211	0.775	0.226	1.000	0.965	0.981	0.323	0.214	0.943	0.785	-0.443	0.954	0.832	0.756	-0.510	0.801	0.789	0.931	-0.551	0.844	-0.776	0.275	0.941	0.975	0.945
HALKS	0.936	0.138	-0.421	0.916	0.385	0.854	0.222	0.768	0.263	0.965	1.000	0.970	0.290	0.232	0.938	0.805	-0.481	0.903	0.793	0.728	-0.490	0.844	0.760	0.940	-0.545	0.857	-0.803	0.347	0.961	0.967	0.927
ISCTR	0.941	0.062	-0.487	0.915	0.405	0.781	0.211	0.798	0.200	0.961	0.970	1.000	0.260	0.309	0.969	0.795	-0.550	0.844	0.830	0.698	-0.480	0.888	0.785	0.941	-0.635	0.890	-0.828	0.283	0.933	0.979	0.927
KCHOL	0.472	0.689	0.447	0.257	0.125	0.441	0.529	0.453	0.635	0.328	0.290	0.260	1.000	0.062	0.252	0.135	0.234	0.216	0.533	0.317	0.207	0.559	0.235	0.268	0.308	0.219	0.379	0.347	0.246	0.567	
KOZAL	0.167	-0.435	-0.612	0.263	0.708	0.041	0.446	0.581	0.246	0.214	0.232	0.309	0.062	1.000	0.375	0.126	-0.696	0.351	0.507	-0.010	0.388	0.151	0.452	0.107	-0.437	0.617	-0.199	-0.071	0.042	0.183	0.259
KROMD	0.902	0.035	-0.326	0.905	0.407	0.743	0.143	0.838	0.160	0.943	0.938	0.969	0.323	0.375	1.000	0.802	-0.606	0.955	0.814	0.650	-0.400	0.844	0.796	0.929	-0.711	0.897	-0.827	0.318	0.884	0.954	0.900
OTKAR	0.752	0.138	-0.373	0.894	0.220	0.754	0.138	0.548	0.143	0.785	0.805	0.795	0.135	0.126	0.802	1.000	-0.363	0.745	0.543	0.535	-0.498	0.582	0.502	0.780	-0.506	0.636	-0.737	0.491	0.763	0.774	0.705
PETKM	-0.351	0.451	0.701	0.454	-0.419	-0.266	0.040	-0.161	0.055	-0.443	-0.481	-0.510	0.234	0.309	-0.468	-0.365	1.000	-0.556	-0.447	0.053	0.063	-0.411	-0.379	-0.438	0.835	-0.689	0.680	-0.669	-0.332	-0.489	-0.355
PGSUS	0.897	-0.025	-0.365	0.932	0.506	0.628	0.142	0.809	0.159	0.934	0.903	0.944	0.317	0.351	0.915	0.743	-0.536	1.000	0.843	0.694	-0.464	0.866	0.832	0.905	-0.642	0.876	-0.794	0.117	0.854	0.939	0.878
SAHOL	0.872	0.098	-0.326	0.767	0.579	0.605	0.283	0.910	0.485	0.832	0.793	0.830	0.533	0.507	0.814	0.533	-0.447	0.843	1.000	0.763	-0.105	0.732	0.907	0.736	-0.411	0.894	-0.481	0.167	0.761	0.800	0.914
SISE	0.837	0.435	0.608	0.704	0.251	0.660	0.077	0.624	0.409	0.754	0.728	0.688	0.650	0.010	0.450	0.135	0.055	0.694	0.763	1.000	-0.303	0.641	0.740	-0.121	-0.066	0.580	-0.337	0.308	0.794	0.730	0.850
TAVHL	-0.432	0.101	0.332	-0.309	0.044	-0.331	0.797	0.001	0.209	-0.101	-0.490	-0.480	0.317	0.388	-0.400	-0.498	0.063	-0.464	-0.105	-0.303	1.000	-0.514	-0.037	0.433	0.299	-0.186	0.675	0.017	-0.525	-0.534	-0.282
TECEL	0.849	0.087	-0.417	0.754	0.337	0.613	0.264	0.672	0.083	0.891	0.844	0.888	0.207	0.151	0.844	0.582	-0.431	0.866	0.732	0.641	-0.514	1.000	0.743	0.852	-0.550	0.779	-0.734	0.035	0.842	0.895	0.884
THYAO	0.820	0.217	-0.387	0.794	0.370	0.510	0.179	0.464	0.113	0.789	0.760	0.781	0.398	0.452	0.798	0.532	-0.379	0.831	0.907	0.740	0.697	0.743	1.000	0.718	-0.365	0.832	-0.403	0.135	0.719	0.763	0.887
TKFEN	0.817	0.167	-0.327	0.893	0.189	0.788	0.294	0.700	0.127	0.851	0.840	0.941	0.235	0.107	0.829	0.780	-0.458	0.905	0.736	0.721	-0.315	0.852	0.718	1.000	-0.609	0.776	-0.822	0.374	0.955	0.975	0.886
TOASO	-0.455	0.412	0.625	-0.184	-0.314	0.247	-0.133	0.317	-0.151	-0.143	-0.151	0.268	-0.437	-0.111	-0.108	0.833	0.642	-0.411	-0.066	0.299	-0.130	-0.365	-0.409	1.000	-0.620	-0.788	-0.110	-0.471	-0.624	-0.415	
TTKOM	0.822	-0.041	-0.172	0.830	0.609	0.667	0.085	0.898	0.358	0.848	0.857	0.890	0.308	0.617	0.897	0.436	-0.681	0.876	0.894	0.580	-0.186	0.759	0.832	0.776	-0.620	1.000	-0.698	0.189	0.763	0.833	0.858
TUPRS	0.627	0.194	0.871	0.796	0.241	-0.109	0.556	-0.111	0.090	0.774	0.803	0.828	0.219	-0.199	0.837	0.737	0.680	-0.794	-0.481	-0.337	0.875	-0.734	-0.488	1.000	-0.174	-0.739	-0.827	-0.196	0.827	0.887	0.966
ULKER	0.348	0.440	0.321	0.327	-0.307	0.637	0.025	0.244	0.254	0.275	0.347	0.283	0.379	-0.071	0.318	0.491	-0.069	0.117	0.167	0.308	0.037	0.035	0.135	0.374	-0.110	0.189	-0.174	1.000	0.382	0.289	0.369
VAKBN	0.944	0.264	-0.228	0.861	0.198	0.852	-0.235	0.739	0.223	0.941	0.961	0.931	0.347	0.042	0.884	0.743	-0.352	0.804	0.761	0.794	-0.513	0.842	0.719	0.935	-0.471	0.763	-0.739	0.382	1.000	0.963	0.926
YKBNK	0.948	0.101	-0.399	0.896	0.301	0.771	-0.252	0.746	0.134	0.771	0.767	0.797	0.246	0.164	0.771	0.489	-0.581	0.891	0.876	0.797	0.730	0.816	0.743	0.931	-0.581	0.811	-0.740	0.940	1.000	0.921	
BIST-30	0.976	0.306	-0.212	0.853	0.390	0.813	0.052	0.876	0.401	0.945	0.927	0.927	0.467	0.259	0.930	0.725	-0.355	0.878	0.914	0.830	-0.282	0.874	0.987	0.886	-0.415	0.858	-0.506	0.369	0.972	0.921	1.000

Figure 3.4: The correlation matrix of equity prices.

When preparing the correlation matrix of equity, the following general formula was used :

$$r = \frac{\Sigma(xy) - (\Sigma x)(\Sigma y)/n}{\sqrt{(\Sigma x^2 - (\Sigma x)^2/n)(\Sigma y^2 - (\Sigma y)^2/n)}} \quad (3.1)$$

The following graphic shows the covariance matrix of BIST30 companies. The covariance shows the relationship of two variables to each other. The positive values indicate that there is a positive link between the variables, that they move in the same direction. The covariance matrix is calculated with the daily stock returns. This method is used in many optimize portfolio formation, and also we will use in equal risk contribution [13].

	AKBNK	ARCLK	BIMAS	CCOLA	DOAS	EKGYO	ENKAI	EREG	FROTO	GARAN	HALKS	ISCTR	KCHOL	KOZAL	KROMD	OTKAR	PETKM	PGSUS	SAHOL	SISE	TAVHL	TECEL	THYAO	TKFEN	TOASO	TTKOM	TUPRS	ULKER	VAKBN	YKBNK
AKBNK	0.04%	0.02%	0.01%	0.02%	0.03%	0.03%	0.01%	0.02%	0.02%	0.04%	0.04%	0.02%	0.02%	0.03%	0.02%	0.02%	0.03%	0.02%	0.03%	0.02%	0.01%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.01%	0.04%	0.03%
ARCLK	0.02%	0.01%	0.01%	0.01%	0.02%	0.01%	0.01%	0.01%	0.02%	0.02%	0.02%	0.01%	0.02%	0.01%	0.01%	0.02%	0.01%	0.01%	0.02%	0.01%	0.01%	0.02%	0.01%	0.02%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%
BIMAS	0.01%	0.01%	0.02%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%
CCOLA	0.02%	0.01%	0.01%	0.01%	0.01%	0.02%	0.01%	0.01%	0.01%	0.02%	0.02%	0.02%	0.01%	0.02%	0.02%	0.01%	0.02%	0.01%	0.01%	0.02%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%
DOAS	0.03%	0.02%	0.01%	0.01%	0.03%	0.01%	0.01%	0.02%	0.02%	0.03%	0.03%	0.02%	0.02%	0.02%	0.02%	0.03%	0.02%	0.02%	0.03%	0.02%	0.01%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.03%	0.03%
EKGYO	0.05%	0.01%	0.01%	0.02%	0.03%	0.05%	0.01%	0.02%	0.02%	0.03%	0.04%	0.03%	0.02%	0.01%	0.02%	0.	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.03%	0.03%
ENKAI	0.01%	0.01%	0.01%	0.01%	0.01%	0.02%	0.01%	0.02%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%
EREG	0.02%	0.01%	0.01%	0.01%	0.02%	0.02%	0.01%	0.02%	0.02%	0.02%	0.02%	0.02%	0.01%	0.02%	0.02%	0.01%	0.02%	0.01%	0.01%	0.02%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%
FROTO	0.02%	0.02%	0.01%	0.01%	0.02%	0.02%	0.01%	0.01%	0.04%	0.02%	0.02%	0.02%	0.01%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.01%	0.02%	0.01%	0.02%	0.01%	0.02%	0.01%	0.01%	0.02%	0.02%
GARAN	0.04%	0.02%	0.01%	0.02%	0.03%	0.01%	0.01%	0.02%	0.02%	0.04%	0.05%	0.04%	0.02%	0.02%	0.03%	0.02%	0.02%	0.02%	0.03%	0.02%	0.01%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.01%	0.04%	0.04%
HALKS	0.04%	0.02%	0.01%	0.02%	0.03%	0.01%	0.01%	0.02%	0.02%	0.04%	0.05%	0.04%	0.02%	0.02%	0.03%	0.02%	0.02%	0.02%	0.03%	0.02%	0.01%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.01%	0.04%	0.04%
ISCTR	0.02%	0.02%	0.01%	0.02%	0.03%	0.01%	0.01%	0.02%	0.02%	0.04%	0.04%	0.02%	0.02%	0.03%	0.02%	0.02%	0.02%	0.02%	0.03%	0.02%	0.01%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.01%	0.04%	0.04%
KCHOL	0.02%	0.01%	0.01%	0.02%	0.02%	0.01%	0.01%	0.02%	0.02%	0.03%	0.02%	0.02%	0.02%	0.01%	0.01%	0.02%	0.01%	0.01%	0.02%	0.02%	0.01%	0.01%	0.02%	0.02%	0.02%	0.01%	0.01%	0.01%	0.02%	0.02%
KOZAL	0.02%	0.01%	0.01%	0.02%	0.02%	0.01%	0.01%	0.02%	0.02%	0.03%	0.02%	0.02%	0.02%	0.01%	0.01%	0.02%	0.01%	0.01%	0.02%	0.02%	0.01%	0.01%	0.02%	0.02%	0.02%	0.01%	0.01%	0.01%	0.02%	0.02%
KROMD	0.01%	0.01%	0.02%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%
OTKAR	0.02%	0.01%	0.01%	0.02%	0.02%	0.01%	0.01%	0.02%	0.02%	0.03%	0.02%	0.02%	0.01%	0.02%	0.02%	0.01%	0.02%	0.01%	0.01%	0.02%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%
PETKM	0.02%	0.01%	0.01%	0.02%	0.02%	0.01%	0.01%	0.02%	0.02%	0.03%	0.02%	0.02%	0.01%	0.02%	0.02%	0.01%	0.02%	0.01%	0.01%	0.02%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%
PGSUS	0.02%	0.01%	0.02%	0.02%	0.03%	0.02%	0.01%	0.02%	0.02%	0.03%	0.03%	0.02%	0.01%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.01%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.03%	0.03%
SAHOL	0.03%	0.01%	0.01%	0.02%	0.02%	0.01%	0.02%	0.02%	0.02%	0.03%	0.03%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.01%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.01%	0.04%	0.04%
SISE	0.01%	0.01%	0.01%	0.01%	0.01%	0.02%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%
TAVHL	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%
TECEL	0.02%	0.01%	0.01%	0.02%	0.02%	0.01%	0.01%	0.02%	0.02%	0.02%	0.02%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%
THYAO	0.02%	0.01%	0.01%	0.02%	0.02%	0.01%	0.01%	0.02%	0.02%	0.02%	0.02%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%
TKFEN	0.02%	0.01%	0.01%	0.02%	0.02%	0.01%	0.01%	0.02%	0.02%	0.02%	0.02%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%
TOASO	0.02%	0.01%	0.01%	0.02%	0.02%	0.01%	0.01%	0.02%	0.02%	0.02%	0.02%	0.02%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%
TTKOM	0.02%	0.01%	0.01%	0.02%	0.02%	0.01%	0.01%	0.02%	0.02%	0.02%	0.02%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%
TUPRS	0.01%	0.01%	0.01%	0.01%	0.01%	0.02%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%
ULKER	0.01%	0.01%	0.01%	0.02%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%	0.01%	0.01%	0.02%	0.01%	0.02%	0.01%	0.01%	0.01%	0.01%	0.02%	0.01%	0.02%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%
VAKBN	0.04%	0.02%	0.01%	0.02%	0.03%	0.01%	0.01%	0.02%	0.02%	0.04%	0.05%	0.04%	0.02%	0.02%	0.03%	0.02%	0.02%	0.02%	0.03%	0.02%	0.01%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.01%	0.04%	0.04%
YKBNK	0.04%	0.02%	0.01%	0.02%	0.03%	0.01%	0.01%	0.02%	0.02%	0.04%	0.05%	0.04%	0.02%	0.02%	0.03%	0.02%	0.02%	0.02%	0.03%	0.02%	0.01%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.01%	0.04%	0.04%

$$COV(x, y) = \frac{\sum_{i=1}^n (x_i - x^*)(y_i - y^*)}{n - 1} \quad (3.2)$$

where x^* is average of x_i and y^* is average of y_i .

In all analysis, the daily stock returns were used and in general literature, referring also to optimize the portfolio selection process, we are looking to equal risk contribution with required statistical analysis.

We will use these studies and results for the our mathematical model which will be data set. The model will produce results for the optimize portfolio.

3.2. Solutions

For the mathematical modeling, we will try to 2 different solution methods to optimize the results. These solutions methods can be respectively called Non-Linear Programming (NLP) and Mixed Integer Non-Linear Programming (MINLP). That will be possible to decide on the optimize portfolio selection by seeing and comparing which model produces more suitable results. The comparison of the results for the portfolio optimization increases the selection and investment possibilities.

We can give a mathematical representation of the model as follows:

$$\begin{aligned} \text{minimize } z &= \sum_i (\vartheta_{x_i} \sigma(x) - r_i)^2 \\ \forall r_i &= \frac{1}{30} \text{ (equally risk contributions - ERC)} \\ \sum x_i &= 1 \text{ (} i = 1, \dots, n \text{),} \end{aligned} \quad (3.3)$$

and $\vartheta_{x_i} \sigma(x)$ are defined:

$$\vartheta_{x_i} \sigma(x) = \frac{x_i \sigma_i^2 + \sum x_j \sigma_{ij}}{\sigma(x)} \quad (3.4)$$

In here, we are looking for value of $\sigma(x)$ and it is known that this value is not equal to $0,0\overline{3333}$ and we are trying to get as close as possible to that value. The

fundamental of ERC strategy is that the risk contributions of assets of the portfolio are equal to each other, which means a risk – balanced portfolio.

For the GAMS codes given in the appendices with the details, it can be summarized the relevant part as follows:

Equations

- objectives
- obj.. $\sum(i, \text{resSqr}(i)) = e = z;$
- TR .. $\sum(i, \text{TotalRC}(i)) = e = \text{TotalR};$
- TRC(i) .. $(\sum(j, x(i)*x(j)*\text{cov}(i,j)) + (x(i)*x(i)*\text{sigma}(i)*\text{sigma}(i))) * x(i) / (\text{TotalR} + 0.00001) = e = \text{TotalRC}(i);$
- TRCResidual(i) .. $\text{TotalRC}(i) - r(i)*\text{TotalR} = e = \text{res}(i);$
- remSqr(i) .. $\text{resSqr}(i) = e = \text{res}(i)*\text{res}(i);$
- weight .. $\sum(i, x(i)) = e = 1;$
- risk .. $\sum(i, r(i)) = l = 1;$

In accordance with the formulas, relevant part can be given as above. Some changes in the code, e.g. x_i values have a certain lower bound (MINLP solution method), will allow to improve the solution method to be NLP or MINLP.

3.3. Results and Trial for the Real Data

After the GAMS codes running, the result that we have found say to us how much weight -and also the sum of all weights will be equal to 1- should have from the equity in the portfolio. The code shared by the appendices and given the related part above produced the result with the NLP solution method. According to this result, the weight of equities in the portfolio is as follows:

Equity	AKBNK	ARCLK	BIMAS	COLA	DOAS	EKGYO	ENKAI	ERGL	FROTO	GARAN
Weighted	0.05	0.07	0.02	0.03	0.04	0.04	0.02	0.05	0.03	0.07
Equity	HALKB	ISCTR	KCHOL	KOZAL	KRDMD	OTKAR	PETKM	PGSUS	SAHOL	SISE
Weighted	0.01	0.01	0.03	0.05	0.01	0.03	0.05	0.02	0.01	0.01
Equity	TAVHL	TCELL	THYAO	TKFEN	TOASO	TTKOM	TUPRS	ULKER	VAKBN	YKBNK
Weighted	0.08	0.05	0.02	0.01	0.01	0.05	0.08	0.02	0.01	0.02

Figure 3.6: The equity weighted of BIST30 companies in portfolio.

On the other hand, we can create a portfolio in order to test whether the optimized portfolio can also be found. In the analysis, we used BIST30 Index companies and closing equity prices at the end of 2015. For the backtesting, 04.01.2016 and 30.06.2016 closing equity prices were taken into consideration.

Assume that we have TL 30.000 on January 4, 2016 and we buy BIST30 Index companies' equity from each as a total of TL 1.000 for our portfolio. This portfolio has been re-evaluated at the end of day closing prices on June 30, 2016. The value of the portfolio within this timeframe consists of the details shared below.

<i>equally</i>	Equity	Weighted	04.01.2016 Closing Price	30.06.2016 Closing Price	04.01.2016-pcs	30.06.2016-Values
1	AKBNK	0.0333333	6.58	8.23	152	1,251
2	ARCLK	0.0333333	13.7	18.93	73	1,382
3	BIMAS	0.0333333	50.8	56.05	20	1,103
4	CCOLA	0.0333333	37.9	35.12	26	927
5	DOAS	0.0333333	11	10.6	91	964
6	EKGYO	0.0333333	2.55	2.86	392	1,122
7	ENKAI	0.0333333	4.25	4.39	235	1,033
8	EREGL	0.0333333	2.96	4.06	338	1,372
9	FROTO	0.0333333	30.34	30.6	33	1,009
10	GARAN	0.0333333	6.98	7.58	143	1,086
11	HALKB	0.0333333	10.19	8.54	98	838
12	ISCTR	0.0333333	4.53	4.56	221	1,007
13	KCHOL	0.0333333	10.71	13.12	93	1,225
14	KOZAL	0.0333333	11.54	12.73	87	1,103
15	KRDMD	0.0333333	1.06	1.28	943	1,208
16	OTKAR	0.0333333	92.8	97	11	1,045
17	PETKM	0.0333333	4.49	3.84	223	855
18	PGSUS	0.0333333	16.95	13.57	59	801
19	SAHOL	0.0333333	7.96	9.42	126	1,183
20	SISE	0.0333333	3.09	3.55	324	1,149
21	TAVHL	0.0333333	17.69	12.33	57	697
22	TCELL	0.0333333	9.79	10.57	102	1,080
23	THYAO	0.0333333	7.14	5.72	140	801
24	TKFEN	0.0333333	3.92	7.37	255	1,880
25	TOASO	0.0333333	18.73	23.64	53	1,262
26	TTKOM	0.0333333	5.34	6.05	187	1,133
27	TUPRS	0.0333333	71	63.75	14	898
28	ULKER	0.0333333	17.94	21	56	1,171
29	VAKBN	0.0333333	3.75	4.51	267	1,203
30	YKBNK	0.0333333	3.22	3.98	311	1,236
		1.0000			TOTAL :	33,021

Figure 3.7: The equity prices and values of BIST30 companies in portfolio.

While the value of the portfolio was TL 30.000 on January 4, 2016, it was TL 33.021 as of June 30, 2016.

The values of the portfolio created using equity weighted according to mathematical modeling are given in the table below. For non linear programming, we proceeded two different criterias and the first one is in the below.

NLP-1	Equity	Weighted	04.01.2016 Closing Price	30.06.2016 Closing Price	04.01.2016-pcs	30.06.2016-Values
1	AKBNK	0,05	6,58	8,23	228	1.876
2	ARCLK	0,07	13,7	18,93	153	2.902
3	BIMAS	0,02	50,8	56,05	12	662
4	CCOLA	0,03	37,9	35,12	24	834
5	DOAS	0,04	11	10,6	109	1.156
6	EKGYO	0,04	2,55	2,86	471	1.346
7	ENKAI	0,02	4,25	4,39	141	620
8	EREGL	0,05	2,96	4,06	507	2.057
9	FROTO	0,03	30,34	30,6	30	908
10	GARAN	0,07	6,98	7,58	301	2.281
11	HALKB	0,01	10,19	8,54	29	251
12	ISCTR	0,01	4,53	4,56	66	302
13	KCHOL	0,03	10,71	13,12	84	1.103
14	KOZAL	0,05	11,54	12,73	130	1.655
15	KRDMD	0,01	1,06	1,28	283	362
16	OTKAR	0,03	92,8	97	10	941
17	PETKM	0,05	4,49	3,84	334	1.283
18	PGSUS	0,02	16,95	13,57	35	480
19	SAHOL	0,01	7,96	9,42	38	355
20	SISE	0,01	3,09	3,55	97	345
21	TAVHL	0,08	17,69	12,33	136	1.673
22	TCELL	0,05	9,79	10,57	153	1.620
23	THYAO	0,02	7,14	5,72	84	481
24	TKFEN	0,01	3,92	7,37	77	564
25	TOASO	0,01	18,73	23,64	16	379
26	TTKOM	0,05	5,34	6,05	281	1.699
27	TUPRS	0,08	71	63,75	34	2.155
28	ULKER	0,02	17,94	21	33	702
29	VAKBN	0,01	3,75	4,51	80	361
30	YKBNK	0,02	3,22	3,98	186	742
		1,0000			TOTAL :	32.093

Figure 3.8: The equity weighted and values.

While the value of the portfolio was TL 30.000 on January 4, 2016, it was TL 32.093 as of June 30, 2016.

The results of the second criteria has used for non linear programming are also given below. The method used here did not put a limit on the total error margin, it has been released. The equity weight and the portfolio values for each equity are shown in detail. It is easy to say that the total portfolio value is slightly lower than the date considered.

NLP-2	Equity	Weighted	04.01.2016 Closing Price	30.06.2016 Closing Price	04.01.2016-pcs	30.06.2016-Values
1	AKBNK	0,03	6,58	8,23	137	1.126
2	ARCLK	0,06	13,7	18,93	131	2.487
3	BIMAS	0,01	50,8	56,05	6	331
4	CCOLA	0,03	37,9	35,12	24	834
5	DOAS	0,06	11	10,6	164	1.735
6	EKGYO	0,05	2,55	2,86	588	1.682
7	ENKAI	0,03	4,25	4,39	212	930
8	EREGL	0,04	2,96	4,06	405	1.646
9	FROTO	0,03	30,34	30,6	30	908
10	GARAN	0,06	6,98	7,58	258	1.955
11	HALKB	0,02	10,19	8,54	59	503
12	ISCTR	0,02	4,53	4,56	132	604
13	KCHOL	0,02	10,71	13,12	56	735
14	KOZAL	0,06	11,54	12,73	156	1.986
15	KRDMD	0,02	1,06	1,28	566	725
16	OTKAR	0,04	92,8	97	13	1.254
17	PETKM	0,04	4,49	3,84	267	1.026
18	PGSUS	0,04	16,95	13,57	71	961
19	SAHOL	0,01	7,96	9,42	38	355
20	SISE	0,02	3,09	3,55	194	689
21	TAVHL	0,06	17,69	12,33	102	1.255
22	TCELL	0,04	9,79	10,57	123	1.296
23	THYAO	0,03	7,14	5,72	126	721
24	TKFEN	0,01	3,92	7,37	77	564
25	TOASO	0,03	18,73	23,64	48	1.136
26	TTKOM	0,04	5,34	6,05	225	1.360
27	TUPRS	0,06	71	63,75	25	1.616
28	ULKER	0,01	17,94	21	17	351
29	VAKBN	0,02	3,75	4,51	160	722
30	YKBNK	0,01	3,22	3,98	93	371
		1,0000			TOTAL :	31.861

Figure 3.9: NLP - The equity weighted and values of BIST30 companies in portfolio.

While the value of the portfolio was TL 30.000 on January 4, 2016, it was TL 31.861 as of June 30, 2016.

In addition to these results, different equity weights were calculated by developing the GAMS code with linear programming and mixed integer non-linear programming solution methods.

In all analysis in the above, we observed the equity weights, portfolio values and the end-of day values of June 30, 2016 based on the solution methods used in the model. The non-linear programming (NLP) solution methods were used for the calculation. Two different calculations were used for NLP. We released the error margin in the second calculation, so we have not put a limit on the total error margin. We have seen that the calculated portfolio values at each step have decreased compared to the initial equally weighted portfolio result.

In addition to these, the final solution method will be mixed integer non-linear programming, so the MINLP.

We are still looking for the answer to the question of which optimal portfolio can build how equally weighted which has a potential with a low risk and high return. The solution method which gives the highest portfolio value on the base date – June 30, 2016 – will be the optimal solution method for us and the optimal portfolio will be created with these equity weights.

We have seen the portfolio value formed when the equity weighted is in equally and the portfolio values according to the non-linear programming solution methods. For the moment, it can be said that the highest portfolio value is in equal weighted. However, this is not the evidence of a generalization that this will be the same in every created portfolio. It will not be possible to perform this generalization according to the market conditions and historical characteristics of the data. The volatility is also important in the period taken here as a section. Depending on the market conditions, the model or optimal solution method that will provide the optimal portfolio may vary [14].

Now, with all of this in mind, we can proceed the final solution method. The results of the mixed integer non-linear programming solution method which are equity weights and portfolio values are listed below.

MINLP	Equity	Weighted	04.01.2016 Closing Price	30.06.2016 Closing Price	04.01.2016-pcs	30.06.2016-Values
1	AKBNK	0,04	6,58	8,23	182	1.501
2	ARCLK	0,07	13,7	18,93	153	2.902
3	BIMAS	0,03	50,8	56,05	18	993
4	CCOLA	0,01	37,9	35,12	8	278
5	DOAS	0,01	11	10,6	27	289
6	EKGYO	0,02	2,55	2,86	235	673
7	ENKAI	0,03	4,25	4,39	212	930
8	EREGL	0,05	2,96	4,06	507	2.057
9	FROTO	0,01	30,34	30,6	10	303
10	GARAN	0,03	6,98	7,58	129	977
11	HALKB	0,01	10,19	8,54	29	251
12	ISCTR	0,01	4,53	4,56	66	302
13	KCHOL	0,06	10,71	13,12	168	2.205
14	KOZAL	0,05	11,54	12,73	130	1.655
15	KRDMD	0,04	1,06	1,28	1.132	1.449
16	OTKAR	0,04	92,8	97	13	1.254
17	PETKM	0,01	4,49	3,84	67	257
18	PGSUS	0,01	16,95	13,57	18	240
19	SAHOL	0,03	7,96	9,42	113	1.065
20	SISE	0,06	3,09	3,55	583	2.068
21	TAVHL	0,01	17,69	12,33	17	209
22	TCELL	0,03	9,79	10,57	92	972
23	THYAO	0,01	7,14	5,72	42	240
24	TKFEN	0,08	3,92	7,37	612	4.512
25	TOASO	0,04	18,73	23,64	64	1.515
26	TTKOM	0,04	5,34	6,05	225	1.360
27	TUPRS	0,01	71	63,75	4	269
28	ULKER	0,03	17,94	21	50	1.054
29	VAKBN	0,06	3,75	4,51	480	2.165
30	YKBNK	0,07	3,22	3,98	652	2.596
		1,0000			TOTAL :	36.540

Figure 3.10: The results of MINLP solution method: equity weighted and values.

While the value of the portfolio was TL 30.000 on January 4, 2016, it was TL 36.540 as of June 30, 2016.

As a result of the mathematical model, our portfolio values, which were formed on June 30, 2016 are respectively; when we have the BIST30 companies in the portfolio with equally weighted is TL 33.021, NLP solution method first result is TL 32.093, NLP solution method second result -here did not put a limit on the total error margin- is TL 31.861 and the results of MINLP solution method is TL 36.540. In the

light of these results, we will have reached the portfolio optimize by forming the portfolios according to the risk weights formed in the result of MINLP solution method. The solution method which gives the highest portfolio value on the base date – June 30, 2016 – will be the optimal solution method for us and the optimal portfolio will be created with mixed integer non-linear programming solution method. In that solution method, we put a minimum threshold for lower bound of the equity weighted of BIST30 companies in the portfolio which is 0.01.

In the mathematical model, for the optimize portfolio, we run all possible logical methods with the results of the solution methods. As a result, the best result at the based date could be reached with the MINLP solution method. The optimize portfolio will consist of BIST30 companies in line with the weights seen on the table below.

Equity	AKBNK	ARCLK	BIMAS	CCOLA	DOAS	EKGYO	ENKAI	EREGL	FROTO	GARAN
Weighted	0.04	0.07	0.03	0.01	0.01	0.02	0.03	0.05	0.01	0.03
Equity	HALKB	ISCTR	KCHOL	KOZAL	KRDMD	OTKAR	PETKM	PGSUS	SAHOL	SISE
Weighted	0.01	0.01	0.06	0.05	0.04	0.04	0.01	0.01	0.03	0.06
Equity	TAVHL	TCELL	THYAO	TKFEN	TOASO	TTKOM	TUPRS	ULKER	VAKBN	YKBNK
Weighted	0.01	0.03	0.01	0.08	0.04	0.04	0.01	0.03	0.06	0.07

Figure 3.11: The equity weighted of BIST30 companies in portfolio for portfolio optimized.

4. CONCLUSIONS

In all analysis, we address modeling with equally-weighted risk contributions portfolios and optimization of portfolio selection process.

In the general literature, there is always a question for the investor. Which asset is obtained with the highest return and lowest risk ? The question has been sought for the years by different methods. Harry Markowitz, one of the most important senior research scientist who sought to the question, is considered the inventor of the modern portfolio theory. In literature, the portfolio which has high return and low risk is an optimize portfolio. We searched for the possibility of using mathematical modeling for portfolio optimization and that many results could be obtained with different solution methods. In model approach, we consider the most important resource that as a reference an article : “On the properties of equally-weighted risk contributions portfolios”, the first version published by Sebastian Maillard, Thierry Roncalli and Jerome Teiletche, in 2008.

The structure of equal risk contribution, which are used in the general literature, is also examined by analyzing the necessary statistics. The theoretical information and applications have been tried to find the answer of the investor’s question.

In literature, the studies is referred to as the portfolio optimization. The definition of portfolio optimization has many types, options. First, we started with a question that; “ What is an optimal portfolio? ”. The answer to this question is that many theoretical knowledge and statistics are subjectively given as a result of the analysis. The definition of equally risk contribution (ERC) is given in the application field and a formula is written for the proposed mathematical formula. In the mathematical modelling, non-linear programming, linear programming and mixed integer programming solution methods are used respectively.

In all analysis , we observed the equity weights, portfolio values and the end-of day values of June 30, 2016 based on the solution methods used in the model. The non-linear programming (NLP) solution methods were used for the calculation. Two different calculations were used for NLP. We released the error margin in the second calculation, so we have not put a limit on the total error margin for NLP. We have seen that the calculated portfolio values at each step have decreased compared to the

initial equally weighted portfolio result. The final solution method was mixed integer non-linear programming. The solution method which gives the highest portfolio value on the base date – June 30, 2016 – will be the optimal solution method for us and the optimal portfolio will be created with mixed integer non-linear programming solution method. However, this is not the evidence of a generalization that this will be the same in every created portfolio. It will not be possible to perform this generalization according to the market conditions and historical characteristics of the data. The volatility is also important in the period taken here as a section. Depending on the market conditions, the model or optimal solution method that will provide the optimal portfolio may vary.

In conclusion, we could generate the our mathematical modelling to a specific criterion for optimizing portfolio formation and the optimized portfolio shown what solution method we can achieve.

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BIOGRAPHY

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APPENDICES

Appendix A: The Details of GAMS Example

Consider the problem of assigning process streams to heat exchangers as described in pp.409-410 of the book "Optimization of Chemical Process" by Edgar and Himmelblau.

In the beginnig, we have started to program with these codes, a sample model;

Table A1.1: The GAMS sample model on "Optimization of Chemical Process".

\$TITLE Test Problem
\$OFFSYMXREF
\$OFFSYMLIST
* Assignment problem for heat exchangers from pp.409-410 in
* Optimization of Chemical Processes" by Edgar and Himmelblau
SETS
I streams / A, B, C, D /
J exchangers / 1*4 / ;
TABLE C(I,J) Cost of assigning stream i to exchanger j
1 2 3 4
A 94 1 54 68
B 74 10 88 82
C 73 88 8 76
D 11 74 81 21 ;
VARIABLES X(I,J), Z;
BINARY VARIABLES X(I,J);
EQUATIONS ASSI(J), ASSJ(I), OBJ;
ASSI(J).. SUM(I, X(I,J)) =E= 1;
ASSJ(I).. SUM(J, X(I,J)) =E= 1;
OBJ.. Z =E= SUM ((I,J), C(I,J)*X(I,J)) ;
MODEL HEAT / ALL /;
OPTION LIMROW = 0;
OPTION LIMCOL = 0;
OPTION SOLPRINT = OFF; SOLVE HEAT USING MIP MINIMIZING Z;

Table A1.1: Continuation of the table.

DISPLAY X.L, Z.L ;

After program running, the results was in the screen with details;

GAMS 2.25 PC AT/XT

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Test Problem

4

5 *

6 * Assignment problem for heat exchangers from pp.409-410
in

7 * Optimization of Chemical Processes" by Edgar and
Himmelblau

8 *

9

10 SETS

11 I streams / A, B, C, D /

12 J exchangers / 1*4 / ;

13

14 TABLE C(I,J) Cost of assigning stream i to exchanger j

15

16 1 2 3 4

17 A 94 1 54 68

18 B 74 10 88 82

19 C 73 88 8 76

20 D 11 74 81 21 ;

21

22

23 VARIABLES X(I,J), Z;

24 BINARY VARIABLES X(I,J);

25

26 EQUATIONS ASSI(J), ASSJ(I), OBJ;

27

28 ASSI(J).. SUM(I, X(I,J)) =E= 1;

29 ASSJ(I).. SUM(J, X(I,J)) =E= 1;

30 OBJ.. Z =E= SUM ((I,J), C(I,J)*X(I,J)) ;

31

32 MODEL HEAT / ALL /;

33

34 OPTION LIMROW = 0;

35 OPTION LIMCOL = 0;

36 OPTION SOLPRINT = OFF;

37

38 SOLVE HEAT USING MIP MINIMIZING Z;

39 40 DISPLAY X.L, Z.L ;

Table A1.1: Continuation of the table.

COMPILATION TIME	=	0.060 SECONDS	VERID TP5-00-038
GAMS 2.25 PC AT/XT		09/16/97	
23:53:18 PAGE	2		
Test Problem			
Model Statistics	SOLVE HEAT USING MIP FROM LINE 38		
MODEL STATISTICS			
BLOCKS OF EQUATIONS	3	SINGLE EQUATIONS	9
BLOCKS OF VARIABLES	2	SINGLE VARIABLES	17
NON ZERO ELEMENTS	49	DISCRETE VARIABLES	16
GENERATION TIME	=	0.050 SECONDS	
EXECUTION TIME	=	0.160 SECONDS	VERID TP5-00-038
GAMS 2.25 PC AT/XT		09/16/97	
23:53:18 PAGE	3		
Test Problem			
Solution Report	SOLVE HEAT USING MIP FROM LINE 38		
S O L V E S U M M A R Y			
MODEL HEAT	OBJECTIVE Z		
TYPE MIP	DIRECTION MINIMIZE		
SOLVER ZOOM	FROM LINE 38		
**** SOLVER STATUS	1	NORMAL COMPLETION	
**** MODEL STATUS	1	OPTIMAL	
**** OBJECTIVE VALUE	97.0000		
RESOURCE USAGE, LIMIT	0.047	1000.000	
ITERATION COUNT, LIMIT	16	1000	
Z O O M / X M P --- PC Version 2.2 Nov 1990			
Dr Roy E. Marsten and Dr Jaya Singhal, XMP Optimization Software Inc. Tucson, Arizona			
D E M O N S T R A T I O N M O D E			
You do not have a full license for this program.			
The following size restrictions apply:			
Total nonzero elements: 1000			
Total discrete variables: 20			
Estimate work space needed	--	16 Kb	
Work space allocated	--	273 Kb	

Table A1.1: Continuation of the table.

Iterations	Time				
Initial LP	16.00	Heuristic	0.00	Branch and bound	0.00
Final LP	0.00	*****			
REPORT SUMMARY : 0 NONOPT					
0 INFEASIBLE 0 UNBOUNDED					
GAMS 2.25	PC AT/XT			09/16/97	
23:53:18	PAGE	4			
Test Problem					
Execution					
----	40	VARIABLE	X.L		
	1	2	3	4	
A				1.000	
B		1.000			
C			1.000		
D	1.000				
----	40	VARIABLE	Z.L	=	97.000
EXECUTION TIME	=	0.060	SECONDS	VERID	TP5-00-038
USER:	CACHE DESIGN CASE STUDIES SERIES			G911007-	
1447AX-TP5					
GAMS DEMONSTRATION VERSION					
***** FILE SUMMARY					
INPUT	C:\GAMS\HEAT.GMS				
OUTPUT	C:\GAMS\HEAT.LST				

After running, it is easy to say that stream A is assigned to exchanger 4, B to 2, C to 3 and D to 1 with a minimum cost of 97.

Appendix B: The Details of Mathematical Models' GAMS entries

Table B1.1: The Mathematical Models' GAMS entries.

Sets	
i	assets /1*30/;alias(i,j);parametersr(i) risk contribution /
1	0.033333333
2	0.033333333
3	0.033333333
4	...

Table B1.1: Continuation of the table.

30	0.033333333/ sigma(i) standard deviation/							
1	0.073223931, 2	0.079319318, 3	0.102617329, 4	0.055206159, 5	0.06788669, 6	0.070110586, 7	0.075748286, 8	0.08505397, 9
	0.10795073, 10	0.089740406, 11	0.096671639, 12	0.034505652, 13	0.10755005, 14	0.062478156, 15	0.069264132, 16	0.101107844, 17
	0.126370431, 18	0.054387131, 19	0.098868218, 20	0.121105668, 21	0.086720833, 22	0.09261982, 23	0.068830537, 24	0.066660419, 25
	0.076128113, 26	0.10179308, 27	0.08893666, 28	0.123841454, 29	0.123581138, 30	0.119587171/;		
Table cor(i,j) the correlation matrix								
1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30						
1	1.0000	0.5483	0.7088	0.6479	0.7163	0.8990	0.8770	
	0.9090	0.8251	0.8778	0.8036	0.0825	0.9126	0.7328	
	0.8788	0.8062	0.8704	0.7398	0.7961	0.8293	0.8838	
	0.6380	0.7873	0.6431	0.7697	0.7547	0.7043	0.6977	
	0.8843	0.8837						
2	0.5483	1.0000	0.3801	0.5377	0.4143	0.5753	0.5492	
	0.4375	0.6621	0.4545	0.5959	0.0217	0.4329	0.6356	
	0.4131	0.6155	0.5586	0.6418	0.5572	0.3188	0.6385	
	0.7348	0.3570	0.5173	0.7049	0.2657	0.3084	0.1247	
	0.4459	0.4443						
3	0.7088	0.3801	1.0000	0.5649	0.6442	0.7605	0.6822	
	0.7959	0.8113	0.7126	0.7234	0.4160	0.7982	0.6705	
	0.8041	0.7346	0.7243	0.4887	0.7083	0.7953	0.6549	
	0.6881	0.5833	0.6653	0.6258	0.7578	0.5915	0.6884	
	0.7605	0.7922						
4	0.6479	0.5377	0.5649	1.0000	0.4669	0.7210	0.6660	
	0.6572	0.7503	0.7836	0.7498	0.2584	0.6623	0.7358	
	0.5449	0.8742	0.7828	0.6355	0.7621	0.5182	0.7318	

Table B1.1: Continuation of the table.

0.5961	0.4093	0.7011	0.5786	0.6405	0.3694	0.3593	
0.7391	0.7426						
5	0.7163	0.4143	0.6442	0.4669	1.0000	0.8222	0.8027
0.7099	0.7041	0.7270	0.8148	0.3587	0.7722	0.5685	
0.6459	0.6729	0.7193	0.5224	0.7123	0.8233	0.7838	
0.5936	0.7198	0.4258	0.7677	0.6050	0.8929	0.7018	
0.7285	0.7576						
6	0.8990	0.5753	0.7605	0.7210	0.8222	1.0000	0.8598
0.8549	0.8468	0.8881	0.8919	0.2599	0.8923	0.7761	
0.8289	0.8410	0.8483	0.6672	0.8732	0.8081	0.8534	
0.6937	0.7860	0.7080	0.8209	0.7809	0.7456	0.6766	
0.9010	0.8958						
7	0.8770	0.5492	0.6822	0.6660	0.8027	0.8598	1.0000
0.8864	0.8877	0.8403	0.8534	0.1385	0.8818	0.7481	
0.8255	0.8352	0.8751	0.7004	0.8439	0.8556	0.9093	
0.7268	0.7796	0.6559	0.8492	0.6470	0.7958	0.7387	
0.8871	0.9030						
8	0.9090	0.4375	0.7959	0.6572	0.7099	0.8549	0.8864
1.0000	0.8901	0.9191	0.8048	0.1527	0.9436	0.7606	
0.8709	0.8191	0.8536	0.7550	0.8434	0.9175	0.8478	
0.6265	0.7810	0.7359	0.8221	0.8219	0.7063	0.7225	
0.9277	0.9677						
9	0.8251	0.6621	0.8113	0.7503	0.7041	0.8468	0.8877
0.8901	1.0000	0.8309	0.8710	0.2154	0.8726	0.8300	
0.7921	0.8691	0.8392	0.7310	0.8499	0.8329	0.8429	
0.8577	0.7236	0.7691	0.8587	0.6921	0.6254	0.6033	
0.8713	0.8928						
10	0.8778	0.4545	0.7126	0.7836	0.7270	0.8881	0.8403
0.9191	0.8309	1.0000	0.9003	0.1194	0.9414	0.8240	
0.8114	0.8550	0.9171	0.8120	0.9405	0.8485	0.8935	
0.5460	0.7238	0.7916	0.8157	0.8345	0.7162	0.6631	
0.9480	0.9703						

Table B1.1: Continuation of the table.

11	0.8036	0.5959	0.7234	0.7498	0.8148	0.8919	0.8534
0.8048	0.8710	0.9003	1.0000	0.2832	0.9047	0.8495	
0.7244	0.8781	0.9185	0.7167	0.9246	0.7871	0.9106	
0.6712	0.6982	0.8079	0.8862	0.7733	0.7172	0.6189	
0.8761	0.8931						
12	0.0825	0.0217	0.4160	0.2584	0.3587	0.2599	0.1385
0.1527	0.2154	0.1194	0.2832	1.0000	0.1666	0.1001	
0.1000	0.3666	0.1617	-0.0834	0.1311	0.2429	0.0978	
0.1759	0.0047	0.1706	0.1132	0.3876	0.0860	0.1407	
0.1403	0.1942						
13	0.9126	0.4329	0.7982	0.6623	0.7722	0.8923	0.8818
0.9436	0.8726	0.9414	0.9047	0.1666	1.0000	0.8154	
0.8472	0.8455	0.9297	0.7169	0.9047	0.9037	0.9130	
0.6413	0.8152	0.7758	0.8304	0.8648	0.7213	0.7068	
0.9523	0.9677						
14	0.7328	0.6356	0.6705	0.7358	0.5685	0.7761	0.7481
0.7606	0.8300	0.8240	0.8495	0.1001	0.8154	1.0000	
0.6254	0.8109	0.8622	0.6850	0.8878	0.7024	0.8307	
0.6872	0.5690	0.7036	0.7759	0.7747	0.4825	0.4461	
0.8440	0.8126						
15	0.8788	0.4131	0.8041	0.5449	0.6459	0.8289	0.8255
0.8709	0.7921	0.8114	0.7244	0.1000	0.8472	0.6254	
1.0000	0.6851	0.7769	0.6156	0.7677	0.8440	0.7452	
0.5965	0.7217	0.6746	0.7128	0.6667	0.7214	0.7812	
0.8160	0.8546						
16	0.8062	0.6155	0.7346	0.8742	0.6729	0.8410	0.8352
0.8191	0.8691	0.8550	0.8781	0.3666	0.8455	0.8109	
0.6851	1.0000	0.9097	0.7201	0.8767	0.7109	0.8887	
0.6977	0.5945	0.7821	0.7533	0.7825	0.5287	0.4779	
0.8567	0.8764						
17	0.8704	0.5586	0.7243	0.7828	0.7193	0.8483	0.8751
0.8536	0.8392	0.9171	0.9185	0.1617	0.9297	0.8622	

Table B1.1: Continuation of the table.

0.7769	0.9097	1.0000	0.7177	0.9100	0.7732	0.9735
0.6266	0.6458	0.7730	0.8046	0.8212	0.6704	0.6015
0.9007	0.9160					
18	0.7398	0.6418	0.4887	0.6355	0.5224	0.6672
0.7550	0.7310	0.8120	0.7167	-0.0834	0.7169	0.6850
0.6156	0.7201	0.7177	1.0000	0.7999	0.6327	0.7546
0.5349	0.5095	0.6913	0.7283	0.4816	0.5613	0.4629
0.7451	0.7761					
19	0.7961	0.5572	0.7083	0.7621	0.7123	0.8732
0.8434	0.8499	0.9405	0.9246	0.1311	0.9047	0.8878
0.7677	0.8767	0.9100	0.7999	1.0000	0.8010	0.8902
0.6500	0.6802	0.8200	0.8335	0.7611	0.6591	0.5783
0.9211	0.9334					
20	0.8293	0.3188	0.7953	0.5182	0.8233	0.8081
0.9175	0.8329	0.8485	0.7871	0.2429	0.9037	0.7024
0.8440	0.7109	0.7732	0.6327	0.8010	1.0000	0.7790
0.6347	0.8155	0.5655	0.7733	0.7510	0.7811	0.7753
0.8746	0.9034					
21	0.8838	0.6385	0.6549	0.7318	0.7838	0.8534
0.8478	0.8429	0.8935	0.9106	0.0978	0.9130	0.8307
0.7452	0.8887	0.9735	0.7546	0.8902	0.7790	1.0000
0.6770	0.7092	0.7120	0.8640	0.7441	0.7185	0.5900
0.8775	0.8943					
22	0.6380	0.7348	0.6881	0.5961	0.5936	0.6937
0.6265	0.8577	0.5460	0.6712	0.1759	0.6413	0.6872
0.5965	0.6977	0.6266	0.5349	0.6500	0.6347	0.6770
1.0000	0.6303	0.5194	0.6875	0.4051	0.4333	0.4330
0.6396	0.6250					
23	0.7873	0.3570	0.5833	0.4093	0.7198	0.7860
0.7810	0.7236	0.7238	0.6982	0.0047	0.8152	0.5690
0.7217	0.5945	0.6458	0.5095	0.6802	0.8155	0.7092
0.6303	1.0000	0.5485	0.7629	0.6018	0.6583	0.5808

Table B1.1: Continuation of the table.

0.8034	0.7640						
24	0.6431	0.5173	0.6653	0.7011	0.4258	0.7080	0.6559
0.7359	0.7691	0.7916	0.8079	0.1706	0.7758	0.7036	
0.6746	0.7821	0.7730	0.6913	0.8200	0.5655	0.7120	
0.5194	0.5485	1.0000	0.7603	0.6875	0.4048	0.4010	
0.7654	0.8059						
25	0.7697	0.7049	0.6258	0.5786	0.7677	0.8209	0.8492
0.8221	0.8587	0.8157	0.8862	0.1132	0.8304	0.7759	
0.7128	0.7533	0.8046	0.7283	0.8335	0.7733	0.8640	
0.6875	0.7629	0.7603	1.0000	0.6535	0.7048	0.5620	
0.7901	0.8359						
26	0.7547	0.2657	0.7578	0.6405	0.6050	0.7809	0.6470
0.8219	0.6921	0.8345	0.7733	0.3876	0.8648	0.7747	
0.6667	0.7825	0.8212	0.4816	0.7611	0.7510	0.7441	
0.4051	0.6018	0.6875	0.6535	1.0000	0.4832	0.5229	
0.8229	0.8384						
27	0.7043	0.3084	0.5915	0.3694	0.8929	0.7456	0.7958
0.7063	0.6254	0.7162	0.7172	0.0860	0.7213	0.4825	
0.7214	0.5287	0.6704	0.5613	0.6591	0.7811	0.7185	
0.4333	0.6583	0.4048	0.7048	0.4832	1.0000	0.8272	
0.7119	0.7397						
28	0.6977	0.1247	0.6884	0.3593	0.7018	0.6766	0.7387
0.7225	0.6033	0.6631	0.6189	0.1407	0.7068	0.4461	
0.7812	0.4779	0.6015	0.4629	0.5783	0.7753	0.5900	
0.4330	0.5808	0.4010	0.5620	0.5229	0.8272	1.0000	
0.6452	0.7054						
29	0.8843	0.4459	0.7605	0.7391	0.7285	0.9010	0.8871
0.9277	0.8713	0.9480	0.8761	0.1403	0.9523	0.8440	
0.8160	0.8567	0.9007	0.7451	0.9211	0.8746	0.8775	
0.6396	0.8034	0.7654	0.7901	0.8229	0.7119	0.6452	
1.0000	0.9712						

Table B1.1: Continuation of the table.

30	0.8837	0.4443	0.7922	0.7426	0.7576	0.8958	0.9030
0.9677	0.8928	0.9703	0.8931	0.1942	0.9677	0.8126	
0.8546	0.8764	0.9160	0.7761	0.9334	0.9034	0.8943	
0.6250	0.7640	0.8059	0.8359	0.8384	0.7397	0.7054	
0.9712	1.0000;						
Parameter cov(i,j) the covariance matrix;							
cov(i,j)=cor(i,j)*sigma(i)*sigma(j);							
Variables							
z							
res(i) residual remaining							
resSqr(i) square of remaining							
TotalR total risk							
TotalRC(i);							
Positive variables							
x(i) total weight contribution Total=1;							
Equations							
obj							
TR total risk							
TRC(i) total risk contribution							
TRCResidual(i) residual total risk contribution							
remSqr(i) square of the difference remaining							
weight sum of weight							
risk sum of the risk;obj.. sum(i,resSqr(i))=e=z;TR .. sum(i>TotalRC(i))=e=TotalR;							
TRC(i) .. (sum(j, x(i)*x(j)*cov(i,j))+(x(i)*x(i)*sigma(i)*sigma(i))) * x(i) /							
(TotalR+0.00001) =e= TotalRC(i);TRCResidual(i) .. TotalRC(i) - r(i)*TotalR =e=							
res(i);remSqr(i) .. resSqr(i)=e=res(i)*res(i);weight .. sum(i,x(i))=e=1;risk ..							
sum(i,r(i))=l=1;Model portfolio /all/;							
Solve portfolio using NLP minimizing z;							
Option NLP=BARON;Display z.l,x.l, TotalR.l, TotalRC.l;							
file uh /'ugur.txt'/;put uh;loop((i)\$x.l(i)>0), put i.tl ;)*file uh /'ugur.txt'/;*put							
uh;*loop((i,j,k)\$x.l(i,j,k)=1),** put k.tl, ',', i.tl, ',', j.tl /;*)*;							