

T.R.
GEBZE INSTITUTE OF TECHNOLOGY
INSTITUTE OF SCIENCES AND ENGINEERING

REAL-VALUED TIME-DOMAIN MODES IN
HOLLOW WAVEGUIDES

Özlem AKGÜN
PH. D. DISSERTATION
ELECTRONICS ENGINEERING DEPARTMENT

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DOKTORA TEZİ JÜRİ ONAY SAYFASI

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SUMMARY

TITLE : REAL-VALUED TIME-DOMAIN MODES IN HOLLOW
WAVEGUIDES

AUTHOR : Özlem AKGÜN

In this doctoral dissertation, an initial-boundary value problem for the system of Maxwell's equations with time derivative is formulated and solved rigorously for transient modes in a hollow waveguide with perfectly conducting surface. The cross section is bounded by a closed singly-connected contour of arbitrary but smooth enough shape. Hence, the complete sets of TE and TM modes are obtained and studied.

Every modal field component is obtained as a product of a vector function of transverse coordinates, $(\mathbf{r})$, and a scalar amplitude dependent on axial coordinate, z , and time, t . It has been established that the study is reduced to solving two autonomous problems. 1) *A modal basis problem*. Final result of this step is the definition of complete set of functions (in Hilbert space, $\mathcal{L}_2(\mathcal{S})$) dependent on the transverse coordinates that originates a basis. 2) *A modal amplitude problem*. It is proved that the amplitudes are generated by the solutions to the Klein-Gordon Equation (KGE) with z and t as independent variables. The KGE has been derived from Maxwell's equations directly. Solutions to the KGE are invariant under the relativistic Lorentz transformations and subjected to the causality principle. Special attention is paid to various ways leading to analytical solutions to the KGE. As an example, one case among eleven others is considered in detail. The modal amplitudes are found explicitly and expressed via products of the Airy functions with arguments dependent on z and t .

Keywords: Maxwell's equations, waveguide theory, time-domain modes, law of conservation of energy, Klein-Gordon equation, Airy functions.

ÖZET

TEZİN BAŞLIĞI : İÇİ BOŞ DALGA KILAVUZLARINDA REEL
DEĞERLİ ZAMAN UZAYI MODLARI
YAZAR ADI : Özlem AKGÜN

Bu tezde, zamana göre türev ihtiva eden Maxwell denklemler sistemi için bir başlangıç-sınır değer problemi ortaya konmuştur. Bu problem içi boş bir dalga kılavuzunda geçici modlar için analitik olarak çözülmüştür. Dalga kılavuzunun mükemmel iletken yüzeye sahip olduğu varsayılmıştır. Kılavuzun dikey kesitinin tek parça kapalı bir konturdan oluştuğu ve bu konturun yeterince düzgün olduğu kabul edilmiştir. Tek iletkenli dalga kılavuzları söz konusu olduğundan TE ve TM modları çalışılmıştır.

Her bir modal alan bileşeni iki çarpandan oluşacak şekilde elde edilmiştir. Bunlardan biri dalga kılavuzunun enine koordinatlarının, $(\mathbf{r})$, vektör fonksiyonu, diğeri ise eksenel koordinat olan z ve zaman t 'ye bağımlı skaler bir fonksiyondur. Bu çalışmada, zaman uzayı dalga kılavuzu modlarını belirlemek için birbirinden neredeyse bağımsız iki problem çalışılmıştır. 1) *Modal baz problemi*. Bu problemin çözümü Hilbert uzayı $\mathcal{L}_2(\mathcal{S})$ 'de tamamlanmış fonksiyon setini verir. Bu fonksiyonlar kılavuzun enine koordinatlarına bağılırlar ve modal baz oluştururlar. 2) *Modal genlik problemi*. Modal genliklerin z ve t bağımsız değişkenlerine sahip Klein-Gordon Denklemi (KGD) 'nin çözümleri tarafından üretildiği ispatlanmıştır. KGD doğrudan Maxwell denklemlerinden türetilmiştir. KGD 'nin çözümleri rölativistik Lorentz dönüşümleri altında değişimsizdir ve nedensellik ilkesine uyarlar. KGD 'yi değişkenlerine ayırarak analitik çözümüne imkan sağlayacak dönüşüm çiftleri verilmiştir. Bu tezde, çözüm önerilerinden biri ayrıntılı bir şekilde çalışılmıştır. Modal genliklerin analitik çözümü yapılmış ve bu genlikler z ve t bağımsız değişkenlerine sahip Airy fonksiyonlarının çarpımı olarak elde edilmiştir.

Anahtar kelimeler: Maxwell denklemleri, dalga kılavuzu teorisi, zaman-uzayı modları, enerjinin korunumu kanunu, Klein-Gordon denklemi, Airy fonksiyonları.

Dedicated to my dear mother Nejla Akgün.

Without you, I could never succeed.

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NOTATIONS AND ABBREVIATIONS

$\mathcal{E}$	: Electric field strength vector (V / m)
$\mathcal{H}$	: Magnetic field strength vector (A / m)
$\mathbf{E}$	: Transverse components of electric field vector (V / m)
$\mathbf{H}$	: Transverse components of magnetic field vector (A / m)
E_z	: z – component of electric field (V / m)
H_z	: z – component of magnetic field (A / m)
W	: Energy density of a field (J / m^3)
P_z	: z – component of the Poynting vector (W / m^2)
ε_0	: Dielectric constant of free space (F / m)
μ_0	: Magnetic permeability of free space (H / m)
c	: Velocity of light in vacuum (m / s)
∇	: Nabla operator
$\nabla_{\perp}$	: Nabla transverse
$\nabla_{\perp}^2$	: Transverse Laplacian
rot	: Rotation
div	: Divergence
Oz	: Waveguide axis
S	: Waveguide cross section
L	: Contour of the waveguide cross section, S
$\mathbf{z}$	: Unit vector along the waveguide axis, Oz
$\mathbf{l}$	: Unit vector tangential to the contour, L
$\mathbf{n}$	: Outer normal unit vector to the domain S
$\mathbf{R}$	: Three-component position vector (m)
$\mathbf{r}$	: Projection of the position vector $\mathbf{R}$ onto the domain S (m)
t	: Observation time (s)
∂_t	: Partial derivative with respect to t ($1 / s$)
∂_z	: Partial derivative with respect to z ($1 / m$)

$\partial_{\mathbf{n}}$	: Partial derivative with respect to the outer normal unit vector, $\mathbf{n}$
TE	: Transverse Electric
TM	: Transverse Magnetic
ψ_n	: Eigensolutions to the Neumann boundary-eigenvalue problem
ν_n^2	: Eigenvalues of the Neumann boundary-eigenvalue problem ($1 / m^2$)
ϕ_n	: Eigensolutions to the Dirichlet boundary-eigenvalue problem
κ_n^2	: Eigenvalues of the Dirichlet boundary-eigenvalue problem ($1 / m^2$)
KGE	: Klein-Gordon Equation
$h_n(z, t)$	: Potential function of the KGE for the TE modes
$e_n(z, t)$	: Potential function of the KGE for the TM modes
τ	: Dimensionless time
ξ	: Dimensionless coordinate
α	: Constant of separation of variables
$Ai(\cdot)$	: Airy function of the first kind
$Bi(\cdot)$	: Airy function of the second kind
F	: Inertial frame in a state of rest
$\bar{F}$	: Inertial frame moving with constant velocity
v_z	: Constant velocity along the Oz axis (m / s)
$\bar{\gamma}$	: Lorentz factor

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1. INTRODUCTION

1.1. Subject of the Study

From a general point of view, the time-domain studies in electromagnetic field theory can be categorized into two groups.

The first one is development and application of powerful numerical methods. The most popular ones among these are the Finite Difference Time Domain (FDTD) methods. The main advantages, in our opinion, as stated in [Taflove and Hagness, 2005] are: *i)* There is no limitation on the number of unknown coefficients up to a limit of 10^9 . *ii)* They are capable of calculating the response of a nonlinear electromagnetic system. However, it is appropriate to mention here that a physical explanation of the numerical data can be achieved under the presence of analytical results. Besides, the accuracy of the solution of a problem solved via said methods is limited by memory volume and efficient computer use.

The group of analytical studies can also be divided into two subcategories. The first consists of frequency domain studies where the Fourier integral transform is needed in order to find time-domain solutions in the final step. Nevertheless, in these studies, there are some drawbacks with regard to the causality principle. However, in case of using Laplace transform, causal solutions can be obtained [Anderson and Barnes, 1953], [Coppens, 1966]. On the other hand, in the second subcategory, the solutions are obtained in the time domain *directly* without recourse to any integral transform. The Evolutionary Approach to Electromagnetics (EAE) also used in this dissertation can be given as an example to this subcategory. Inspired by the works suggested in the 1980's to solve cavity and waveguide problems [Tretyakov, 1986], [Tretyakov, 1989], and motivated by the desire to solve the problems using rather simple mathematical techniques, the EAE is proposed here for the study of electromagnetic field propagation in hollow waveguides accompanied with transient processes in the time domain.

Development of the waveguide theory has been pioneered by Lord Rayleigh's

work [Rayleigh, 1897]. Since then, extension of the waveguide theory has occurred mostly within the framework of the classical time-harmonic field theory where $-\infty < t < \infty$. However, in a real physical problem an electromagnetic field must be causal and should start with a source applied at a fixed initial time, say, $t = 0$. Moreover, the continuous superpositions of the time-harmonic waves can synthesize the transient fields in principle, however, it is hard to achieve that in practice. In [Dvorak and Dudley, 1995], comprehensive information on this topic can be found. But, the use of EAE proposed here gives us a great facility on not only obtaining transient fields but steady state solutions, as well. Surely, the evolutionary approach satisfies the correspondence principle¹ which is accepted in the theoretical studies. The classical time-harmonic field theory appears as the simplest particular case among the others (see Appendix C) that open up new lines of analytical studies in time-domain electromagnetics.

1.2. Purpose and Contents

There are two coupled objectives of this dissertation. The primary is to solve an initial-boundary value problem for the system of Maxwell's equations with time derivative, ∂_t , in order to obtain transient modes in a hollow waveguide. It is known that Maxwell's equations with ∂_t are invariant under the relativistic Lorentz transformations which are inapplicable to the time harmonic waveguide waves in the frequency domain studied more than a century. We separate the initial-boundary value problem into a pair of autonomous problems: *i*) a modal basis problem, and *ii*) a modal amplitude problem. The secondary objective is to study the modal amplitudes generated by the solutions to the Klein-Gordon Equation (KGE), derived directly from Maxwell's equations, with axial waveguide coordinate, z , and time, t , as independent variables. Obtaining the KGE from Maxwell's equations with ∂_t is a common objective of all analytical time-domain methods since the KGE is a relativistic equation, as well. A new approach is proposed here to express

¹ Every new theory should reproduce the results of previously well-established theories as a limiting case in the domains of previous theories.

the modal amplitudes via various special functions of mathematical physics having z and t in their arguments. To illustrate the feasibility of the proposed method, one case, where the modal amplitudes are found explicitly and expressed via products of the Airy functions, is considered. It is worth noting that the Airy functions also appear in the study of Airy light beams in optics. Comprehensive information on this topic can be found in recent article [Torre, 2009]. In the light beam theory, Airy equation can be obtained as a result of paraxial *approximation* of a scalar wave equation. In our theory, however, Airy equation is obtained as an exact form of the KGE derived *rigorously* from Maxwell's equations with ∂_t .

The chapters are organized as follows. Chapter 2 provides the formulation of the problem for finding solutions in Hilbert space, $\mathcal{L}_2(\mathcal{S})$. Maxwell's equations in the form of transverse-longitudinal decompositions, obtained without any restriction, are given also in this chapter. In Chapter 3, we utilize the results of Chapter 2 for the study of complete sets of the *TE* and *TM* time-domain waveguide modes, where the field components are obtained in a class of real-valued functions of coordinates and time. These complete sets of modes are reformulated in a common form which enables studying the modal amplitudes of the field components *in parallel*. The time-harmonic waveguide waves appear as the simplest particular case. In Chapter 4, various ways in order to extent the analytical solutions for the modal amplitudes are discussed. Non-sinusoidal modal amplitudes, expressed via products of Airy functions, are obtained, and numerical results are illustrated. Chapter 5 is devoted to relativistic effects on the wave propagation in a hollow waveguide. Finally, concluding remarks are given in Chapter 6.

1.3. Historical Background

1.3.1. Historical Background of the Method

The research in this dissertation is carried out in the framework of the EAE proposed for the study of electromagnetic field propagation in hollow waveguides. The key point of the method is to preserve time as an independent variable in the

process of solving Maxwell's equations with time derivative, ∂_t . This contradicts to the time-harmonic field concept where t is lost in the solving procedure due to the choice of the time dependency of the field as $\exp(\pm i\omega t)$. Since it is accepted that the electromagnetic theory is founded on Maxwell's equations, most of the research has been done in Fourier space. The use of Fourier transform eliminates the time as a parameter. Thus, the problem can be described as obtaining the effective frequencies to calculate the amplitudes of the fields at these frequencies. In such problems, time-domain solutions can be found by applying inverse Fourier transform as follows

$$f(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\mathbf{r}, \omega) e^{i\omega t} d\omega. \quad (1.1)$$

In contrast to the approach mentioned above, the EAE does not accept the time dependency of the field as $\exp(\pm i\omega t)$. Using the EAE, it is possible to derive modal decompositions as functions of coordinate and time for electromagnetic signals capable of propagating in a waveguide. The modal amplitudes in these decompositions are the solutions to the systems of time-dependent partial differential equations with appropriate initial conditions and are obtained by taking projections on Maxwell's equations. The differential equations with time derivative are known as evolution equations. A solution to an evolution equation supplemented with the initial conditions exhibits how a physical event evolves from an initial state up to the state of observation. For this reason, the method is called Evolutionary Approach (EA). We have adapted this approach for the study of electromagnetic signal propagation in a hollow waveguide. In addition to requiring rather simple mathematical technique, in our opinion, as compared to other time-domain methods, the proposed approach also leads to various ways to express the modal amplitudes via non-sinusoidal functions. Non-sinusoidal signals can also be found in the comprehensive time-domain studies performed by Harmuth². To this purpose, Maxwell's equations are amended by introducing magnetic current density as a real

² Dr. Henning F. Harmuth is the author of more than 150 journal publications and of 15 books. Complete information about his contribution to science can be found in the special issue of the journal *Electromagnetic Phenomena*, which is devoted to his 80th anniversary, by the link, see bibliography.

physical quantity, and also by adding magnetic equivalent of Ohm's law to the set of constitutive relations. However, the propagation of non-sinusoidal signals in waveguides can still be studied remaining within the scope of generally accepted theory [Aksoy and Tretyakov, 2003], [Tretyakov and Akgun, 2010].

Publications that are given below can be used to understand the EAE since the method presented in them is ideologically close to ours. The method was introduced first in the 1980's, and early studies were published in Russian [Tretyakov, 1986] and [Tretyakov, 1989]. The main advantage of the method over the classical time-harmonic field concept is that it is free of restrictions on the constitutive relations [Tretyakov, 1993], [Tretyakov, 1994]. In [Aksoy and Tretyakov, 2004], oscillations in a cavity excited by the Walsh function signals are studied. In [Erden and Tretyakov, 2008], exact solutions for the modal amplitudes of the real-valued electromagnetic fields in a cavity are obtained as a convolution integral.

1.3.2. Historical Background of the Problem

Lord Rayleigh is known as the pioneer who helped to open up a new line of research with the paper on the propagation of electromagnetic signals through “tubes” [Rayleigh, 1897]. Since then, the theory of waveguide has been studied as a part of classical time-harmonic field concept where all the field quantities are replaced by their phasor forms. In this concept, constitutive relations are forced to be linear, and the state of medium in a waveguide or in a cavity should be stationary. In addition to the restrictions, there are also some drawbacks with regard to the causality principle. These are the reasons to search for new methods directly in the time domain.

To the best of our knowledge, Van Bladel was the first who derived the evolution equations for the waveguide modes, although he considered only time harmonic waves [Van Bladel, 1964]. Then, Felsen and Marcuvitz used another way to solve these equations [Felsen and Marcuvitz, 1973]. A new theorem on a class of four-dimensional skew-symmetric tensors was proved by Gabriel. In [Gabriel, 1980], linear Maxwell's equations in the relativistic covariant form are studied. The work was concerned with the analysis of a network formalism within the concepts of

electromagnetism. Borisov considered a two-dimensional boundary value time-domain problem and solved it via Riemann method [Borisov, 1987]. Another time-domain study on the propagation of transient electromagnetic waves in waveguides was performed in [Kristensson, 1995]. In this work, an elegant method relying on wave splitting technique has been proposed. The field in the waveguide is represented exactly as a time convolution of a Green function and a source function. Exact numerical data can be obtained effectively at small distances between source point and the point of observation. However, when t and z become relatively large, the convolution integral fails in calculations due to fast oscillations of the kernel in the integrand. We solved this problem in a new way (the details are presented in section “4.1. Time-Domain Modes Expressible via Airy Functions”), without the need for the convolution integral. We succeeded in an analytical expansion of the initial data, given on time semi-axis, $t \geq 0$, over the whole domain of propagation, $0 \leq z \leq ct$, where c is the velocity of light in vacuum. This way is equivalent to analytical calculation of the convolution integral without approximations of its kernel [Tretyakov and Akgun, 2010].

2. STATEMENT OF THE PROBLEM

In this work, an ideal hollow waveguide having a cross-section domain, S , invariable along the waveguide axis, Oz , is considered. We assume that S is bounded by a closed singly connected contour, L . We also assume that the contour has an arbitrary but smooth enough shape implying that none of its inner angles (i.e., measured within S) exceed π . In the rest of the work, we will make use of a right-handed triplet $(\mathbf{z}, \mathbf{l}, \mathbf{n})$ of mutually orthogonal unit vectors (i.e., $\mathbf{z} \times \mathbf{l} = \mathbf{n}$) where $\mathbf{n}$ is the outer normal to S , $\mathbf{z}$ is oriented along the Oz axis, and $\mathbf{l}$ is tangential to L . A point of observation within the waveguide by a three-component vector, $\mathbf{R}$, and an observation time, t , are also denoted.

2.1. Formulation of the Problem

We solve the system of vectorial Maxwell's equations for the electric and magnetic field strength vectors, $\mathcal{E}$ and $\mathcal{H}$,

$$\begin{aligned}\nabla \times \mathcal{E}(\mathbf{R}, t) &= -\mu_0 \partial_t \mathcal{H}(\mathbf{R}, t) \\ \nabla \times \mathcal{H}(\mathbf{R}, t) &= \varepsilon_0 \partial_t \mathcal{E}(\mathbf{R}, t)\end{aligned}\tag{2.1}$$

where μ_0 and ε_0 denote the magnetic permeability and the dielectric constant in the free space, respectively. The scalar Maxwell's equations

$$\begin{aligned}\nabla \cdot \mathcal{E}(\mathbf{R}, t) &= 0 \\ \nabla \cdot \mathcal{H}(\mathbf{R}, t) &= 0\end{aligned}\tag{2.2}$$

will also be used, though they are the corollaries of (2.1). The systems given in (2.1) and (2.2) hold within the waveguide volume, except for its surface. Assume that the surface has the physical properties of the perfect electric conductor, the field components should then satisfy the following boundary conditions

$$\begin{aligned}
(\mathbf{n} \cdot \mathcal{H}) \Big|_L &= 0 \\
(\mathbf{l} \cdot \mathcal{E}) \Big|_L &= 0 \\
(\mathbf{z} \cdot \mathcal{E}) \Big|_L &= 0
\end{aligned} \tag{2.3}$$

where $(\cdot)$ denotes the scalar product. Due to the hyperbolic nature of Maxwell's equations, a solution to (2.1) should be subjected to the following initial conditions

$$\begin{aligned}
\mathcal{E}(\mathbf{R}, 0) &= 0, \\
\mathcal{H}(\mathbf{R}, 0) &= 0.
\end{aligned} \tag{2.4}$$

Thus, we solve the initial-boundary value problem (2.1)-(2.4) in a class of the real-valued functions, provided that the expected solution is square-integrable in Hilbert space, $\mathcal{L}_2(\mathcal{S})$. We now introduce quadratic characteristics for the real-valued vector fields $\mathcal{E}$, and $\mathcal{H}$

$$\begin{aligned}
W(z, t) &= \frac{1}{2S} \int_S \{ \varepsilon_0 (\mathcal{E} \cdot \mathcal{E}) + \mu_0 (\mathcal{H} \cdot \mathcal{H}) \} ds \\
P_z(z, t) &= \frac{1}{S} \int_S \mathbf{z} \cdot [\mathcal{E} \times \mathcal{H}] ds
\end{aligned} \tag{2.5}$$

where $(\times)$ denotes the vector product. The integration is performed at a fixed instant of time, t , over the waveguide cross section, S , located at a fixed coordinate, z . $W(z, t)$ defined in (2.5) denotes the energy density of the field averaged over S . And, the z -component of the Poynting vector averaged over S , $P_z(z, t)$ also defined in (2.5), is interpreted as the intensity of energy flow at the point z in the field. Poynting's theorem, applied to (2.1), yields the following differential equation

$$\partial_z P_z(z, t) + \partial_t W(z, t) \approx 0. \tag{2.6}$$

The differential form for the law of conservation of energy given in (2.6) is valid for the time-domain solutions. For the proof of (2.6), the readers are referred to Appendix A. Square integrability in (2.5) specifies the global property of the expected solution, whereas the differentiability with respect to the variables z and t

in (2.6) specifies the local property. Maxwell's equations in (2.1) are invariant under the relativistic Lorentz transformations. Hence, a solution to the initial-boundary value problem should be found in compliance with the causality principle that will be discussed in the section "4.1. Time-Domain Modes Expressible via Airy Functions."

2.2. Transverse-Longitudinal Decompositions

The three-component position vector, $\mathbf{R}$, can be presented as

$$\mathbf{R} = \mathbf{r} + \mathbf{z}z \quad (2.7)$$

where $\mathbf{r}$ is a projection of the vector $\mathbf{R}$ onto the domain S indicating a point of observation within S . And, the nabla operator, ∇ , can be written as

$$\nabla = \nabla_{\perp} + \mathbf{z}\partial_z. \quad (2.8)$$

The differential procedure $\nabla_{\perp}$ acts on the transverse coordinates ($\mathbf{r}$). In a similar fashion, we present the field vectors $\mathcal{E}$ and $\mathcal{H}$ as the sums of their transverse and longitudinal components

$$\begin{aligned} \mathcal{E}(\mathbf{R}, t) &= \mathbf{E}(\mathbf{r}, z, t) + \mathbf{z}E_z(\mathbf{r}, z, t) \\ \mathcal{H}(\mathbf{R}, t) &= \mathbf{H}(\mathbf{r}, z, t) + \mathbf{z}H_z(\mathbf{r}, z, t) \end{aligned} \quad (2.9)$$

where $\mathbf{E}$ and $\mathbf{H}$ are the projections of the electric and magnetic field vectors onto the domain S , respectively, E_z and H_z are the magnitudes of z – components of the same vector fields.

By making use of (2.8) and (2.9), we rewrite the vectorial Maxwell's equations as follows

$$\begin{aligned} \text{rot}\mathcal{E} &\equiv [(\nabla_{\perp} + \mathbf{z}\partial_z) \times (\mathbf{E} + \mathbf{z}E_z)] = -\mu_0\partial_t\mathbf{H} - \mathbf{z}\mu_0\partial_tH_z, \\ \text{rot}\mathcal{H} &\equiv [(\nabla_{\perp} + \mathbf{z}\partial_z) \times (\mathbf{H} + \mathbf{z}H_z)] = \varepsilon_0\partial_t\mathbf{E} + \mathbf{z}\varepsilon_0\partial_tE_z. \end{aligned} \quad (2.10)$$

In order to find left-hand sides of (2.10), we introduce a three-component vector field, $\mathcal{A}$, and calculate the curl of the vector as shown below

$$\begin{aligned} \text{rot}\mathcal{A} &\equiv [(\nabla_{\perp} + \mathbf{z}\partial_z) \times (\mathbf{A} + \mathbf{z}A_z)] \\ &= [\nabla_{\perp} \times \mathbf{A}] + [\nabla_{\perp} \times \mathbf{z}A_z] + [\mathbf{z}\partial_z \times \mathbf{A}] + [\mathbf{z}\partial_z \times \mathbf{z}A_z]. \end{aligned} \quad (2.11)$$

It is evident that $[\mathbf{z}\partial_z \times \mathbf{z}A_z] = 0$, $[\nabla_{\perp} \times \mathbf{z}A_z] = [\nabla_{\perp} A_z \times \mathbf{z}] = [\nabla_{\perp} \times \mathbf{z}]A_z$, and $[\mathbf{z}\partial_z \times \mathbf{A}] = \partial_z[\mathbf{z} \times \mathbf{A}]$, thus, (2.11) can be simplified to the following form

$$\begin{aligned} \text{rot}\mathcal{A} &\equiv [(\nabla_{\perp} + \mathbf{z}\partial_z) \times (\mathbf{A} + \mathbf{z}A_z)] \\ &= [\nabla_{\perp} \times \mathbf{A}] + [\nabla_{\perp} \times \mathbf{z}]A_z + \partial_z[\mathbf{z} \times \mathbf{A}]. \end{aligned} \quad (2.12)$$

We now can apply the result of (2.12) to the left-hand sides of (2.10) as given below

$$\begin{aligned} \text{rot}\mathcal{E} &\equiv [\nabla_{\perp} \times \mathbf{E}] + [\nabla_{\perp} \times \mathbf{z}]E_z + \partial_z[\mathbf{z} \times \mathbf{E}] = -\mu_0\partial_t\mathbf{H} - \mathbf{z}\mu_0\partial_tH_z, \\ \text{rot}\mathcal{H} &\equiv [\nabla_{\perp} \times \mathbf{H}] + [\nabla_{\perp} \times \mathbf{z}]H_z + \partial_z[\mathbf{z} \times \mathbf{H}] = \varepsilon_0\partial_t\mathbf{E} + \mathbf{z}\varepsilon_0\partial_tE_z. \end{aligned} \quad (2.13)$$

By making use of a well-known vector identity $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$, the scalar multiplication of (2.13) by the unit vector $\mathbf{z}$ yields

$$\begin{aligned} \mathbf{z} \cdot \text{rot}\mathcal{E} &\equiv \mathbf{z} \cdot [\nabla_{\perp} \times \mathbf{E}] = -\nabla_{\perp} \cdot [\mathbf{z} \times \mathbf{E}] = -\mu_0\partial_tH_z, \\ \mathbf{z} \cdot \text{rot}\mathcal{H} &\equiv \mathbf{z} \cdot [\nabla_{\perp} \times \mathbf{H}] = \nabla_{\perp} \cdot [\mathbf{H} \times \mathbf{z}] = \varepsilon_0\partial_tE_z. \end{aligned} \quad (2.14)$$

The projection of (2.13) onto S can be written as follows

$$\begin{aligned} [\nabla_{\perp} \times \mathbf{z}]E_z + \partial_z[\mathbf{z} \times \mathbf{E}] &= -\mu_0\partial_t\mathbf{H}, \\ [\nabla_{\perp} \times \mathbf{z}]H_z + \partial_z[\mathbf{z} \times \mathbf{H}] &= \varepsilon_0\partial_t\mathbf{E}. \end{aligned} \quad (2.15)$$

Utilizing the identity $\mathbf{A} \times \mathbf{B} \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ known as triple product expansion or Lagrange's Formula, the vector multiplication of (2.15) by the unit vector $\mathbf{z}$ yields

$$\begin{aligned} \nabla_{\perp}E_z &= \mu_0\partial_t[\mathbf{H} \times \mathbf{z}] + \partial_z\mathbf{E} \\ \nabla_{\perp}H_z &= \varepsilon_0\partial_t[\mathbf{z} \times \mathbf{E}] + \partial_z\mathbf{H} \end{aligned} \quad (2.16)$$

In a similar fashion, the scalar Maxwell's equations can also be rewritten by utilizing (2.8) and (2.9) as shown below

$$\begin{aligned} \text{div}\mathcal{E} &\equiv (\nabla_{\perp} + \mathbf{z}\partial_z) \cdot (\mathbf{E} + \mathbf{z}E_z) = \nabla_{\perp} \cdot \mathbf{E} + \partial_z E_z = 0, \\ \text{div}\mathcal{H} &\equiv (\nabla_{\perp} + \mathbf{z}\partial_z) \cdot (\mathbf{H} + \mathbf{z}H_z) = \nabla_{\perp} \cdot \mathbf{H} + \partial_z H_z = 0. \end{aligned} \quad (2.17)$$

Finally, the equations given in (2.14), (2.16), and (2.17) can be grouped into two systems of equations: The first includes only $E_z(\mathbf{r}, z, t)$

$$\begin{aligned} \nabla_{\perp} E_z &= \mu_0 \partial_t [\mathbf{H} \times \mathbf{z}] + \partial_z \mathbf{E}, & (a) \\ \varepsilon_0 \partial_t E_z &= \nabla_{\perp} \cdot [\mathbf{H} \times \mathbf{z}], & (b) \\ \partial_z E_z &= -\nabla_{\perp} \cdot \mathbf{E}, & (c) \end{aligned} \quad (2.18)$$

whereas the other system includes only $H_z(\mathbf{r}, z, t)$

$$\begin{aligned} \nabla_{\perp} H_z &= \varepsilon_0 \partial_t [\mathbf{z} \times \mathbf{E}] + \partial_z \mathbf{H}, & (a) \\ \mu_0 \partial_t H_z &= \nabla_{\perp} \cdot [\mathbf{z} \times \mathbf{E}], & (b) \\ \partial_z H_z &= -\nabla_{\perp} \cdot \mathbf{H}. & (c) \end{aligned} \quad (2.19)$$

(2.18) and (2.19) are Maxwell's equations, given in (2.1) and (2.2), in the form of transverse-longitudinal decompositions obtained without any restriction.

2.3. Boundary Conditions

After the decomposition of the vector fields into their transverse and longitudinal parts, the first two of boundary conditions in the system of (2.3) can be rewritten as follows

$$\begin{aligned} (\mathbf{n} \cdot \mathbf{H})|_L &= 0, \\ (\mathbf{l} \cdot \mathbf{E})|_L &= 0. \end{aligned} \quad (2.20)$$

After substituting the last condition of (2.3), $E_z|_L = 0$, into the equations (b) and (c) of (2.18), we obtain the following boundary conditions

$$\begin{aligned}(\nabla_{\perp} \cdot [\mathbf{H} \times \mathbf{z}])\big|_L &= 0, \\ (\nabla_{\perp} \cdot \mathbf{E})\big|_L &= 0,\end{aligned}\tag{2.21}$$

respectively. The boundary conditions of (2.21) are the corollaries of the conditions given in (2.20). For the details, the readers are referred to Appendix B.

3. SOLVING THE PROBLEM

In this chapter, we shall solve the problem given in (2.18)-(2.21) with a method known as *incomplete separation of variables*³ [Polyanin and Manzhirov, 2006], specifically, every field component of (2.18) and (2.19) can be presented as a product of two factors: One is a function of transverse coordinates, $(\mathbf{r})$, and the other is dependent on the variables (z, t) . We shall also search for the complete sets of transverse electric and transverse magnetic time-domain modes. Transverse modes appear due to the boundary conditions enforced on the wave because of the physical properties of the waveguide. As an example, a radio wave in a hollow waveguide with perfectly conducting surface must have zero-valued tangential electric field component at the walls of the waveguide. Hence, the transverse pattern of electric field is restricted to those which fit between the walls. The allowed modes can be found via solving Maxwell's equations under the boundary conditions over a waveguide surface. These solutions can be divided into two basic sets of modes in the interior of a waveguide. For the one set of modes, there is no axial electric field component. However, these modes have an axial magnetic field and are called “magnetic type” that is H modes or “transverse electric” that is TE modes. The other basic set of modes has no magnetic field along the waveguide axis but has an axial electric field instead, hence, they are referred to as “electric type” that is E modes or “transverse magnetic” that is TM modes.

3.1. Complete Set of the Transverse Electric (TE) Time-Domain Modes

The system of (2.18) should be solved, provided that $E_z(\mathbf{r}, z, t) = 0$. The lines (b) and (c) of (2.18)

$$\begin{aligned}\nabla_{\perp} \cdot [\mathbf{H} \times \mathbf{z}] &= -\nabla_{\perp} \cdot [\mathbf{z} \times \mathbf{H}] = -[\nabla_{\perp} \times \mathbf{z}] \cdot \mathbf{H} = 0 \\ \nabla_{\perp} \cdot \mathbf{E} &= 0\end{aligned}\tag{3.1}$$

³ The method differs from that of complete separation as compared to the number of functions to be found. In case of using complete separation for this study, one should find two more distinct functions since every factor has only one independent variable.

should give the identity $0 = 0$. In the first line of (3.1), vector identities $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$, $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$, and $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$ were used, respectively. In order to obtain the identity $[\nabla_{\perp} \times \nabla_{\perp} \psi(\mathbf{r})] \equiv 0$ in (3.1), the transverse components of electric and magnetic field strength vectors should be in the following forms

$$\begin{aligned} \mathbf{E}(\mathbf{r}, z, t) &= V(z, t) [\nabla_{\perp} \psi(\mathbf{r}) \times \mathbf{z}] \\ \mathbf{H}(\mathbf{r}, z, t) &= I(z, t) \nabla_{\perp} \psi(\mathbf{r}) \end{aligned} \quad (3.2)$$

where $\psi(\mathbf{r})$, $V(z, t)$, and $I(z, t)$ should be found.

Consider now (2.19), and choose the z -component of magnetic field as shown below

$$H_z = A(z, t) \psi(\mathbf{r}) \quad (3.3)$$

where $A(z, t)$ is an unknown function to be found. Substitution of the right-hand side of (3.3) for H_z , and the vectors $\mathbf{E}$ and $\mathbf{H}$, given in (3.2), into the lines (b) and (c) of (2.19) yields

$$\begin{aligned} -\mu_0 \psi(\mathbf{r}) \partial_t A(z, t) &= -V(z, t) \nabla_{\perp} \cdot [\mathbf{z} \times [\nabla_{\perp} \psi(\mathbf{r}) \times \mathbf{z}]], \\ \psi(\mathbf{r}) \partial_z A(z, t) &= -I(z, t) \nabla_{\perp} \cdot \nabla_{\perp} \psi(\mathbf{r}), \end{aligned} \quad (3.4)$$

respectively. After applying the triple product expansion to the right-hand side of the first line, and making use of $\nabla_{\perp} \cdot \nabla_{\perp} \psi(\mathbf{r}) = \nabla_{\perp}^2 \psi(\mathbf{r})$, we obtain

$$\begin{aligned} -\mu_0 \psi(\mathbf{r}) \partial_t A(z, t) &= [-\nabla_{\perp}^2 \psi(\mathbf{r})] V(z, t) \\ \psi(\mathbf{r}) \partial_z A(z, t) &= [-\nabla_{\perp}^2 \psi(\mathbf{r})] I(z, t) \end{aligned} \quad (3.5)$$

which indicates that $\psi(\mathbf{r})$ is twice differentiable with respect to the transverse coordinates, $(\mathbf{r})$. After substituting (3.2) and (3.3) into the line (a) of (2.19), utilizing $\mathbf{z} \times [\nabla_{\perp} \psi(\mathbf{r}) \times \mathbf{z}] = \nabla_{\perp} \psi(\mathbf{r})$, and eliminating the term $\nabla_{\perp} \psi(\mathbf{r})$ from both

sides of the equation, we obtain

$$A(z, t) = \varepsilon_0 \partial_t V(z, t) + \partial_z I(z, t). \quad (3.6)$$

Now, consider the boundary conditions, given in (2.20), for the transverse components of magnetic and electric field vectors. Substitution of (3.2) into (2.20) yields the following conditions

$$\begin{aligned} I(z, t) [\mathbf{n} \cdot \nabla_{\perp} \psi(\mathbf{r})] \Big|_L &= 0, \\ V(z, t) (\mathbf{l} \cdot [\nabla_{\perp} \psi(\mathbf{r}) \times \mathbf{z}]) \Big|_L &= 0. \end{aligned} \quad (3.7)$$

Notice that $I(z, t)$ and $V(z, t)$ must not be zero, otherwise, the transverse components of magnetic field and electric field vectors in (3.2) will be zero. Since nontrivial solutions are searched for, (3.7) should be interpreted as follows

$$\begin{aligned} [\mathbf{n} \cdot \nabla_{\perp} \psi(\mathbf{r})] \Big|_L &= 0, \\ (\mathbf{l} \cdot [\nabla_{\perp} \psi(\mathbf{r}) \times \mathbf{z}]) \Big|_L &= 0. \end{aligned} \quad (3.8)$$

The boundary conditions of (3.8) for the field vectors $\mathbf{E}$ and $\mathbf{H}$ yield the same condition for the potential $\psi(\mathbf{r})$ as follows

$$\partial_{\mathbf{n}} \psi(\mathbf{r}) \Big|_L = 0 \quad (3.9)$$

stating that the normal derivative of $\psi(\mathbf{r})$ on the contour L equals to zero. Substitution of the definitions, given in (3.2), for the vectors into the boundary conditions of (2.21) both yields the identity $0 = 0$. It is clear from (3.5) that the potential $\psi(\mathbf{r})$ is twice differentiable with respect to the transverse coordinates, $(\mathbf{r})$. Following the Sturm-Liouville Theorems, the boundary condition of (3.9) suggests using the Neumann boundary-eigenvalue problem for the transverse Laplacian, $\nabla_{\perp}^2$,

$$\begin{aligned} \nabla_{\perp}^2 \psi_n(\mathbf{r}) + \nu_n^2 \psi_n(\mathbf{r}) &= 0 \\ \partial_{\mathbf{n}} \psi_n(\mathbf{r}) \Big|_L &= 0 \end{aligned} \quad (3.10)$$

where $\mathbf{r} \in S$, the eigenvalues $\nu_n^2 \geq 0$ ($n = 0, 1, 2, \dots$) are the discrete values of a spectral parameter ν . Discreteness of the spectrum is caused by the boundedness of the domain S . The elements of spectrum $\{\nu_n^2\}_{n=0}^{\infty}$ are real numbers. The spectrum has a single point of condensation located at infinity. The subscript, n , regulates the distribution of the eigenvalues on a real axis in the increasing order of their numerical values. The set $\{\psi_n(\mathbf{r})\}_{n=0}^{\infty}$ of the eigensolutions to the problem (3.10), corresponding to all the eigenvalues from the set $\{\nu_n^2\}_{n=0}^{\infty}$, is complete in Hilbert space $\mathcal{L}_2(\mathcal{S})$ (for a proof of completeness, the readers are referred to [Tretyakov, 1993]). Hence, any potential $\psi(\mathbf{r})$ satisfying (3.9) can be presented as a decomposition by making use of the functions $\psi_n(\mathbf{r})$ as the basis elements. To this aim, the functions $\psi_n(\mathbf{r})$ should be normalized thereafter. The complete set of functions, ψ_n , generates a complete set of TE modes in the time domain, see [Tretyakov, 1993]. In (3.2) and (3.5), the potential ψ can be taken as an eigenfunction ψ_n . After introducing the following

$$A(z, t) = \nu_n^2 h_n(z, t), \quad (3.11)$$

then substituting the term $\nu_n^2 \psi_n$ for $[-\nabla_{\perp}^2 \psi_n]$ and the term $\nu_n^2 h_n(z, t)$ for $A(z, t)$ into (3.5), and eliminating $\nu_n^2 \psi_n$ from both sides of the equation, we obtain

$$\begin{aligned} V(z, t) &= -\mu_0 \partial_t h_n(z, t), \\ I(z, t) &= \partial_z h_n(z, t). \end{aligned} \quad (3.12)$$

After utilizing (3.11), (3.12), and applying $\varepsilon_0 \mu_0 = 1/c^2$, (3.6) results in the following differential equation for the function $h_n(z, t)$

$$\partial_{ct}^2 h_n(z, t) - \partial_z^2 h_n(z, t) + \nu_n^2 h_n(z, t) = 0 \quad (3.13)$$

where $\partial_{ct} = \partial / c\partial t$. (3.13) is known as the Klein-Gordon Equation (KGE) in mathematical physics [Kragh, 1984] and [Polyanin, 2002].

After these efforts, all the field components of TE modes can be listed as follows

$$\begin{aligned} E'_{zn}(\mathbf{r}, z, t) &= 0 \\ \mathbf{E}'_n(\mathbf{r}, z, t) &= [-\partial_t h_n(z, t)] \mu_0 \nabla_{\perp} \psi_n(\mathbf{r}) \times \mathbf{z} \\ \mathbf{H}'_n(\mathbf{r}, z, t) &= [\partial_z h_n(z, t)] \nabla_{\perp} \psi_n(\mathbf{r}) \\ H'_{zn}(\mathbf{r}, z, t) &= [v_n h_n(z, t)] v_n \psi_n(\mathbf{r}) \end{aligned} \quad (3.14)$$

where $n = 1, 2, \dots$ and prime $(\cdot)'$ is used as a notation for the TE modes. Note that the terms in the square brackets of (3.14) are the modal amplitudes in physical sense. And, the terms, which are the vector functions of transverse coordinates, $(\mathbf{r})$, originate the modal basis. In order to provide the field components with appropriate physical dimensions, it is necessary to normalize $\psi_n(\mathbf{r})$ properly. By making use of (2.5), $\psi_n(\mathbf{r})$ for $n \neq 0$ can be normalized as follows

$$\frac{\mu_0 v_n^2}{S} \int_S \psi_n^2(\mathbf{r}) ds = 1 \quad (3.15)$$

which enables us to rewrite the quadratic characteristics of the field (3.14) in a simpler form as shown below (for the detailed information, the readers are referred to Appendix A)

$$\begin{aligned} W'_n(z, t) &= \frac{1}{2} \left[(\partial_{ct} h_n)^2 + (\partial_z h_n)^2 + (v_n h_n)^2 \right], \\ P'_{zn}(z, t) &= (-\partial_t h_n)(\partial_z h_n). \end{aligned} \quad (3.16)$$

This result coincides with the previous one obtained by a more general method, see [Aksoy and Tretyakov, 2003]. The infinite set of TE modes given in (3.14) is generated by the eigensolutions to the problem (3.10), corresponding to all the

eigenvalues, except for $v_n^2 = 0$. However, $\psi_0(\mathbf{r})$, corresponding to the eigenvalue $v_0^2 = 0$, is also an eigenfunction. The minimum-maximum theorem for the harmonic functions asserts that it is a constant: $\psi_0(\mathbf{r}) = C$ for $\mathbf{r} \in L, S$ that generates a one-component modal field as follows

$$\begin{aligned}\mathcal{E}'_0(\mathbf{r}, z, t) &= \mathbf{0}, \\ \mathcal{H}'_0(\mathbf{r}, z, t) &= \mathbf{z}h_0(z, t)C.\end{aligned}\tag{3.17}$$

In case of the hollow waveguide, the function $h_0(z, t)$ is constant, as well. This fact follows from the lines (b) and (c) of (2.19). Without loss of generality, we may assume that $C = h_0 = 1$.

The potentials $\psi_n(\mathbf{r})$ generating the modal field patterns in the waveguide cross section coincide with those known for the time-harmonic waves. This can be illustrated with the following example.

Example 3.1: Consider a waveguide with a rectangular cross section as illustrated in Fig. 3.1, and assume that its contour is smooth enough; none of its inner angles (equal to $\pi/2$ each) exceed π . The eigensolutions to the Neumann problem (3.10) can be obtained by the method of complete separation of variables as given below

$$\psi_n(\mathbf{r}) \equiv \psi_n(x, y) = A'_n \cos\left(\frac{\pi p x}{a}\right) \cos\left(\frac{\pi q y}{b}\right)\tag{3.18}$$

where A'_n is the normalization constant, p and q are the integers. The eigenfunctions of (3.18) correspond to the following eigenvalues

$$v_n^2 = \left(\frac{\pi p}{a}\right)^2 + \left(\frac{\pi q}{b}\right)^2.\tag{3.19}$$

After substituting (3.18) into the normalization condition of (3.15), the constants A'_n can be obtained as

$$A'_n = \frac{\sqrt{(2 - \delta_{p,0})(2 - \delta_{q,0})}}{v_n \sqrt{\mu_0}} \quad (3.20)$$

where $\delta_{p,0}$ and $\delta_{q,0}$ are the Kronecker deltas. Any combination of the integers $p = 0, 1, 2, \dots$ and $q = 0, 1, 2, \dots$ is available, provided that $p + q \neq 0$. The subscript n , which is introduced above as a regulator of positions of the eigenvalues on the real axis, can be interpreted as the doublet: $n \rightarrow (p, q)$.

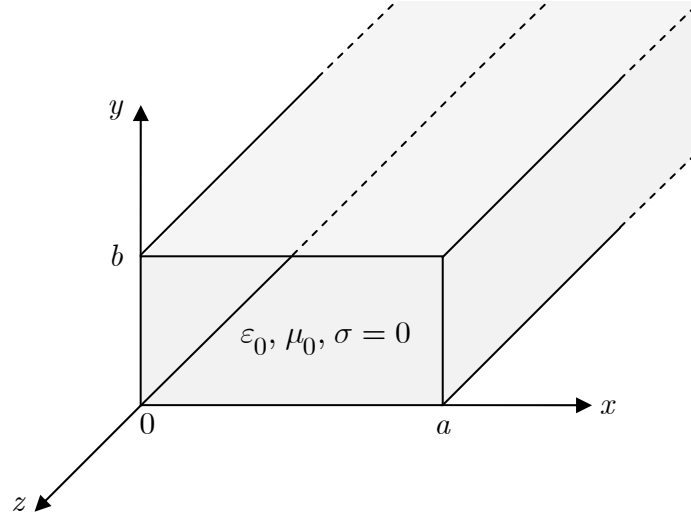


Figure 3.1. A rectangular hollow waveguide with perfectly conducting surface.

Thus, we have obtained the complete set of TE waveguide modes in the time domain. Their field components are given in (3.14) and (3.17). The set of TE modes is complete in $\mathcal{L}_2(\mathcal{S})$. This fact has been proved also resting on the Weyl Theorem on orthogonal detachments of $\mathcal{L}_2(\mathcal{S})$ [Tretyakov, 1993]. The modal amplitudes depending on the axial coordinate, z , and time, t , can be found via solving a one-dimensional KGE of (3.13). This equation is invariant under the relativistic Lorentz transformations. Hence, all the TE time-domain modes obtained above have the same relativistic properties, as well.

3.2. Complete Set of the Transverse Magnetic (TM) Time-Domain Modes

Similar to the analysis of TE modes, (2.18) and (2.19) can be solved, provided that $H_z(\mathbf{r}, z, t) = 0$. The set of TM modes is generated by the complete set of eigensolutions to the Dirichlet boundary-eigenvalue problem

$$\begin{aligned} \nabla_{\perp}^2 \phi_n(\mathbf{r}) + \kappa_n^2 \phi_n(\mathbf{r}) &= 0 & (a) \\ \phi_n(\mathbf{r})|_L &= 0 & (b) \\ \frac{\varepsilon_0 \kappa_n^2}{S} \int_S \phi_n^2(\mathbf{r}) ds &= 1 & (c) \end{aligned} \quad (3.21)$$

where $\mathbf{r} \in S$, the eigenvalues $\kappa_n^2 \geq 0$ ($n = 0, 1, 2, \dots$) are discrete values of a spectral parameter κ , and $\phi_n(\mathbf{r})$ are the eigenfunctions corresponding to these eigenvalues. For $n = 0$, $\kappa_0^2 = 0$ is also an eigenvalue, and $\phi_0(\mathbf{r})$ is the corresponding eigenfunction. When the contour L is singly connected, then $\phi_0(\mathbf{r}) = 0$. This follows from the minimum-maximum theorem for the harmonic functions.

The complete set $\{\phi_n(\mathbf{r})\}_{n=1}^{\infty}$ of the eigenfunctions ϕ_n generates the complete set of TM modes. The field components are

$$\begin{aligned} H''_{zn}(\mathbf{r}, z, t) &= 0 \\ \mathbf{H}''_n(\mathbf{r}, z, t) &= [-\partial_t e_n(z, t)] \mathbf{z} \times \varepsilon_0 \nabla_{\perp} \phi_n(\mathbf{r}) \\ \mathbf{E}''_n(\mathbf{r}, z, t) &= [\partial_z e_n(z, t)] \nabla_{\perp} \phi_n(\mathbf{r}) \\ E''_{zn}(\mathbf{r}, z, t) &= [\kappa_n e_n(z, t)] \kappa_n \phi_n(\mathbf{r}) \end{aligned} \quad (3.22)$$

where $n = 0, 1, 2, \dots$ and double prime $(\cdot)''$ is used as a notation for the TM modes. The time dependent modal amplitudes in the square brackets are specified by $e_n(z, t)$, which is the solution to the following KGE

$$\partial_{ct}^2 e_n(z, t) - \partial_z^2 e_n(z, t) + \kappa_n^2 e_n(z, t) = 0. \quad (3.23)$$

The quadratic field characteristics for the TM modes can be obtained in a similar manner used for the TE modes as shown below

$$\begin{aligned} W_n''(z, t) &= \frac{1}{2} \left[(\partial_{ct} e_n)^2 + (\partial_z e_n)^2 + (\kappa_n e_n)^2 \right], \\ P_{zn}''(z, t) &= (-\partial_t e_n)(\partial_z e_n). \end{aligned} \quad (3.24)$$

In addition to **Ex. 3.1**, the potentials $\phi_n(\mathbf{r})$, generating the modal field patterns of the TM modes in the waveguide cross section, can be found as follows:

Example 3.2: Solving the Dirichlet problem of (3.21) for the rectangular waveguide illustrated in Fig.3.1 results in

$$\phi_n(\mathbf{r}) \equiv \phi_n(x, y) = A_n'' \sin\left(\frac{\pi p x}{a}\right) \sin\left(\frac{\pi q y}{b}\right) \quad (3.25)$$

where $p = 1, 2, \dots$ and $q = 1, 2, \dots$. These eigensolutions correspond to the following eigenvalues

$$\kappa_n^2 = \left(\frac{\pi p}{a}\right)^2 + \left(\frac{\pi q}{b}\right)^2 \quad (3.26)$$

where the subscript n can be interpreted as $n \rightarrow (p, q)$. The normalization constants A_n'' can be given by

$$A_n'' = \frac{2}{\kappa_n \sqrt{\varepsilon_0}}. \quad (3.27)$$

Thus, we have obtained the complete set of TM modes. Completeness in $\mathcal{L}_2(\mathcal{S})$ has been proved in [Tretyakov, 1993]. The TM modes possess all the appropriate relativistic properties in the time domain similarly to the TE modes.

3.3. General Remarks

1. In order to provide the electric and magnetic field strength vectors defined in (3.14) with the required dimensions of Vm^{-1} and Am^{-1} , respectively, the dimension of *force* (i.e., newton, N) should be assigned to the constant at the right-hand side of normalization condition given in (3.15). This assignment provides the normalization constant for the TE modes defined in (3.20) with the following dimension

$$[A'_n] = \frac{N^{1/2}m}{(Hm^{-1})^{1/2}} \quad (3.28)$$

where Hm^{-1} is the dimension of magnetic permeability of free space, μ_0 . After utilizing (3.18) and (3.28) for the dimension of $\psi_n(\mathbf{r})$, and substituting the dimensions of every terms into the $\mathbf{E}'_n(\mathbf{r}, z, t)$ of TE modes, the dimensional analysis can be made as follows

$$\begin{aligned} \mathbf{E}'_n(\mathbf{r}, z, t) &= \underbrace{[-\partial_t h_n(z, t)]}_{s^{-1}} \underbrace{\mu_0}_{Hm^{-1}} \underbrace{\nabla_{\perp} \psi_n(\mathbf{r})}_{m^{-1} \frac{N^{1/2}m}{(Hm^{-1})^{1/2}}} \times \mathbf{z}, \\ [\mathbf{E}'_n] &= kg \frac{m}{s^2} \frac{1}{As} = Vm^{-1}. \end{aligned} \quad (3.29)$$

The dimension of $\mathbf{H}'_n(\mathbf{r}, z, t)$ for the TE modes can be given by

$$\mathbf{H}'_n(\mathbf{r}, z, t) = \underbrace{[\partial_z h_n(z, t)]}_{m^{-1}} \underbrace{\nabla_{\perp} \psi_n(\mathbf{r})}_{m^{-1} \frac{N^{1/2}m}{(Hm^{-1})^{1/2}}} \Rightarrow [\mathbf{H}'_n] = Am^{-1}. \quad (3.30)$$

And, the dimensional analysis of $H'_{zn}(\mathbf{r}, z, t)$ for the TE modes can be made as shown below

$$H'_{zn}(\mathbf{r}, z, t) = \underbrace{[v_n h_n(z, t)]}_{m^{-1}} \underbrace{v_n}_{m^{-1}} \underbrace{\psi_n(\mathbf{r})}_{\frac{N^{1/2}m}{(Hm^{-1})^{1/2}}} \Rightarrow [H'_{zn}] = Am^{-1}. \quad (3.31)$$

In a fashion similar to the *TE* modes, the dimensional analysis of the *TM* modes can also be made. For the *TM* modes, however, the normalization constant defined in (3.27) has the following dimension

$$[A''_n] = \frac{N^{1/2}m}{(Fm^{-1})^{1/2}} \quad (3.32)$$

where Fm^{-1} is the dimension of free-space dielectric constant, ε_0 . After utilizing (3.25) and (3.32) for the dimension of $\phi_n(\mathbf{r})$, and substituting the dimensions of every terms into the field components of (3.22), the electric and magnetic strength vectors have the dimensions of Vm^{-1} and Am^{-1} , respectively.

2. As an example, in case of the time-harmonic fields propagating along the Oz axis, the KGE has the following particular solution

$$g_n(z, t) = a_n \sin(\omega t - \alpha_n z + \varphi) = \begin{cases} h_n(z, t), & TE \\ e_n(z, t), & TM \end{cases} \quad (3.33)$$

where a_n , φ are real constants, and $\alpha_n = \sqrt{\omega^2 \varepsilon_0 \mu_0 - k_n^2}$ with the parameters ω (a given frequency) and k_n^2 . For the *TM* modes, k_n^2 should be replaced with the eigenvalue κ_n^2 , and for the *TE* modes, k_n^2 is v_n^2 . The condition of $\alpha_n = 0$ defines the cut-off frequencies ω_n of the time-harmonic modes, specifically, for the *TE* modes, $\omega'_n = cv_n$, and for the *TM* modes, $\omega''_n = c\kappa_n$.

3. The KGE can also be given in the following canonical form

$$\partial_{\tau}^2 f(\xi, \tau) - \partial_{\xi}^2 f(\xi, \tau) + f(\xi, \tau) = 0. \quad (3.34)$$

The KGE's, both (3.13) and (3.23), have this form after scaling the time, t , and the coordinate, z , as follows

$$\begin{aligned} \tau = v_n ct = 2\pi t / T_n', \quad \xi = v_n z = 2\pi z / \lambda_n', & \Leftarrow TE \\ \tau = \kappa_n ct = 2\pi t / T_n'', \quad \xi = \kappa_n z = 2\pi z / \lambda_n'', & \Leftarrow TM \end{aligned} \quad (3.35)$$

where T_n' and T_n'' are the periods corresponding to the cut-off frequencies ω_n' and ω_n'' , respectively. Similarly, λ_n' and λ_n'' are the corresponding cut-off wavelengths. When the scaled variables are chosen as $\tau = v_n ct$ and $\xi = v_n z$, the solution $f(\xi, \tau)$ to (3.34) is $h_n(\xi, \tau)$; otherwise (i.e., $\tau = \kappa_n ct$ and $\xi = \kappa_n z$), $f(\xi, \tau)$ is $e_n(\xi, \tau)$. The KGE should be supplemented with a pair of initial conditions as

$$\begin{aligned} f(\xi, \tau) \Big|_{\xi=0} &= \varsigma(\tau) \\ -\partial_{\tau} f(\xi, \tau) \Big|_{\xi=0} &= \nu(\tau) \end{aligned} \quad (3.36)$$

where the functions $\varsigma(\tau)$ and $\nu(\tau)$, playing role of the source functions, are given on the semi-axis $\tau \geq 0$, but $\varsigma(\tau) = \nu(\tau) = 0$ when $\tau < 0$. Though the variables τ and ξ have different numerical scaling for the different modal fields, the TE and TM modes can be rewritten in a common form which enables studying the amplitudes of the field components *in parallel*. The TE time-domain modes of (3.14) can be reformulated by means of the dimensionless modal amplitudes as

$$\begin{aligned} E_{zn}'(\mathbf{r}, \xi, \tau) &= 0, \\ v_n^{-1} \sqrt{\varepsilon_0 / \mu_0} \mathbf{E}_n'(\mathbf{r}, \xi, \tau) &= \mathcal{A}(\xi, \tau) \nabla_{\perp} \psi_n(\mathbf{r}) \times \mathbf{z}, \\ v_n^{-1} \mathbf{H}_n'(\mathbf{r}, \xi, \tau) &= \mathcal{B}(\xi, \tau) \nabla_{\perp} \psi_n(\mathbf{r}), \\ v_n^{-1} H_{zn}'(\mathbf{r}, \xi, \tau) &= f(\xi, \tau) v_n \psi_n(\mathbf{r}). \end{aligned} \quad (3.37)$$

And, simple manipulations with (3.22) yield the following TM time-domain modes via the dimensionless modal amplitudes

$$\begin{aligned}
H''_{zn}(\mathbf{r}, \xi, \tau) &= 0 \\
\kappa_n^{-1} \sqrt{\mu_0 / \varepsilon_0} \mathbf{H}''_n(\mathbf{r}, \xi, \tau) &= \mathcal{A}(\xi, \tau) \mathbf{z} \times \nabla_{\perp} \phi_n(\mathbf{r}) \\
\kappa_n^{-1} \mathbf{E}''_n(\mathbf{r}, \xi, \tau) &= \mathcal{B}(\xi, \tau) \nabla_{\perp} \phi_n(\mathbf{r}) \\
\kappa_n^{-1} E''_{zn}(\mathbf{r}, \xi, \tau) &= f(\xi, \tau) \kappa_n \phi_n(\mathbf{r})
\end{aligned} \tag{3.38}$$

where the modal amplitudes $\mathcal{A}(\xi, \tau)$ and $\mathcal{B}(\xi, \tau)$ of the transverse field components are generated by the modal amplitude $f(\xi, \tau)$ of the longitudinal field component, *identically*, $\mathcal{A}(\xi, \tau) = -\partial_{\tau} f(\xi, \tau)$ and $\mathcal{B}(\xi, \tau) = \partial_{\xi} f(\xi, \tau)$.

4. Based on (3), the quadratic characteristics of (3.16) and (3.24) both have the following forms in terms of the new notation

$$\begin{aligned}
k_n^{-2} W(\xi, \tau) &= \frac{1}{2} (\mathcal{A}^2 + \mathcal{B}^2 + f^2) \\
k_n^{-2} P_z(\xi, \tau) &= c \mathcal{A} \mathcal{B}
\end{aligned} \tag{3.39}$$

where k_n^{-2} is v_n^{-2} for the *TE* modes, or it is κ_n^{-2} for the *TM* modes. The conservation of energy law given in (2.6) should be interpreted as

$$\partial_{\xi} P_z(\xi, \tau) + c \partial_{\tau} W(\xi, \tau) \approx 0. \tag{3.40}$$

The energy density and the z -component of Poynting vector defined in (3.39) are valid for both *TE* and *TM* time-domain waveguide modes. As an example, the quadratic characteristics of the time-harmonic fields propagating along the Oz axis will be calculated by means of (3.39). In particular case of (3.33), the function $f(\xi, \tau)$ can be given as follows

$$f(\xi, \tau) = \sin(\eta\tau - \vartheta\xi + \varphi) \tag{3.41}$$

where $\eta = \omega / \omega_n^c \geq 1$, ω_n^c is a cut-off frequency, and $\vartheta = \sqrt{\eta^2 - 1} \geq 0$. The real constant a_n present in (3.33) is assumed to be 1 in (3.41). Substitution of (3.41) into (3.39) yields

$$\begin{aligned} k_n^{-2}W(\xi, \tau) &= \vartheta^2 \cos^2(\eta\tau - \vartheta\xi + \varphi) + \frac{1}{2}, \\ k_n^{-2}P_z(\xi, \tau) &= c\eta\vartheta \cos^2(\eta\tau - \vartheta\xi + \varphi). \end{aligned} \quad (3.42)$$

Fig. 3.2 illustrates the energy density and the z -component of Poynting vector (i.e., the intensity of energy flow along the waveguide axis Oz) given in (3.42) both normalized by k_n^2 . The quantity $P_z(\xi, \tau)$ is also normalized by the velocity of light, c , standing at the right-hand side of the second line of (3.42). Fig. 3.2 is obtained, provided that $\eta = 1.2$, $\varphi = 0$, and $\xi = 0$. Referring to Fig. 3.2, we can state that the energy should be accumulated when the z -component of Poynting vector, specifying the flow of energy along the waveguide, is zero. Utilizing the second line of (3.39), we can conclude that the energy flow is specified by $\mathcal{A}(\xi, \tau)$ and $\mathcal{B}(\xi, \tau)$, which are the modal amplitudes of the transverse components of the field, see (3.37) and (3.38). Hence, the energy accumulated should be specified by $f(\xi, \tau)$, the modal amplitude of the longitudinal component of the field, see (3.37) and (3.38), which has no contribution to the energy flow.

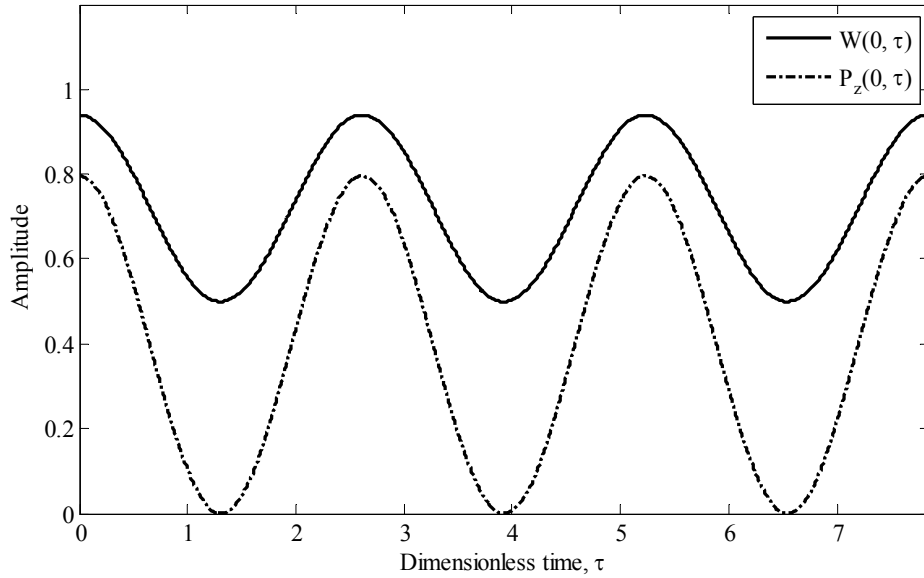


Figure 3.2. The energy density and the z -component of Poynting vector.

In order to find the accumulated energy density and the transferred energy density along the waveguide, the total energy density, the first line of (3.42), can be expressed as the sums of the energy density stored in the transverse components of

the field; $k_n^{-2}W_{tr}(\xi, \tau) = (\mathcal{A}^2 + \mathcal{B}^2) / 2$, and the energy density stored in the longitudinal component of the field; $k_n^{-2}W_z(\xi, \tau) = f^2 / 2$, as shown below

$$\begin{aligned} k_n^{-2}W_{tr}(\xi, \tau) &= \frac{1}{2}(\eta^2 + \vartheta^2) \cos^2(\eta\tau - \vartheta\xi + \varphi), \\ k_n^{-2}W_z(\xi, \tau) &= \frac{1}{2} \sin^2(\eta\tau - \vartheta\xi + \varphi). \end{aligned} \quad (3.43)$$

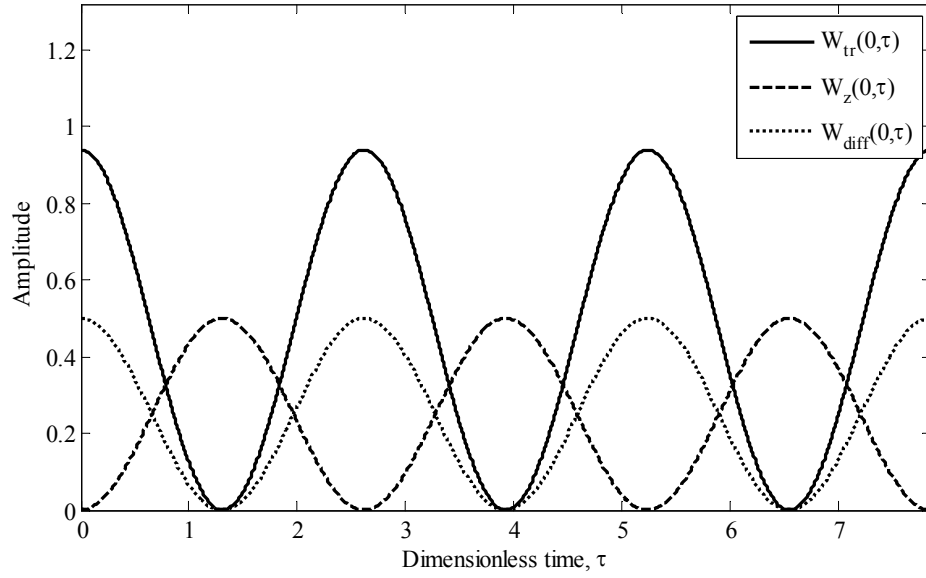


Figure 3.3. The energy densities.

In Fig. 3.3, W_{tr} and W_z normalized by k_n^2 are illustrated, provided that $\eta = 1.2$, $\varphi = 0$, and $\xi = 0$. Notice that when the intensity of energy flow in Fig. 3.2, P_z , decreases to zero, W_{tr} in Fig. 3.3 also decreases to zero, whereas W_z increases up to its maximum value of 0.5. Similarly, when W_{tr} increases, W_z decreases due to the conservation of energy law. From these facts, we can conclude that $k_n^{-2}W_{tr}(\xi, \tau)$ is the amount of energy density transferred along the waveguide, and $k_n^{-2}W_z(\xi, \tau)$ is the amount of energy density accumulated. Fig. 3.3 also illustrates the difference of the energy densities stored in the modal amplitudes of transverse components of the electric and magnetic field; $k_n^{-2}W_{diff}(\xi, \tau) = (\mathcal{A}^2 - \mathcal{B}^2) / 2$. The quantity W_{diff} has the same amplitude as W_z and has a phase shift of $\pi / 2$ as compared to W_z . Based

on the latter, we can conclude that W_{diff} is the amount of energy density participating in the periodical exchange with W_z , which illustrates that the energetic process in the waveguide is also a wave process.

The relation $P_z(\xi, \tau) / W(\xi, \tau)$, specifying a dynamic velocity, v_{dyn} , of the wave energy transferred along the waveguide, can be calculated by utilizing (3.42)

$$v_{dyn} = \frac{c \eta \vartheta \cos^2(\eta\tau - \vartheta\xi + \varphi)}{\vartheta^2 \cos^2(\eta\tau - \vartheta\xi + \varphi) + 1/2}. \quad (3.44)$$

The calculations of W^{avg} (the energy density averaged over the period T corresponding to the frequency ω) and P_z^{avg} (the Poynting vector averaged in the same way) yield

$$\begin{aligned} k_n^{-2} W^{avg}(\xi, \tau) &= k_n^{-2} \frac{1}{T} \int_{\theta}^{\theta+T} W(\xi, \tau) d\tau = \frac{\eta^2}{2} \\ k_n^{-2} P_z^{avg}(\xi, \tau) &= k_n^{-2} \frac{1}{T} \int_{\theta}^{\theta+T} P_z(\xi, \tau) d\tau = \frac{c \eta \vartheta}{2} \end{aligned} \quad (3.45)$$

where the coordinate ξ is arbitrary, θ is a constant, the period T in time τ is equal to $2\pi / \eta$. The relation P_z^{avg} / W^{avg} , specifying an averaged velocity, v_{avg} , of the wave energy transferred along the waveguide, can be calculated by making use of (3.45), and the definitions of η and ϑ as follows

$$v_{avg} = c \frac{\sqrt{\omega^2 - (\omega_n^c)^2}}{\omega}. \quad (3.46)$$

Fig. 3.4 illustrates the dynamic velocity of (3.44) and the averaged velocity of (3.46), provided that $\eta = 1.2$, $\varphi = 0$, and $\xi = 0$. Both velocities in the figure are normalized by the velocity of light, c .

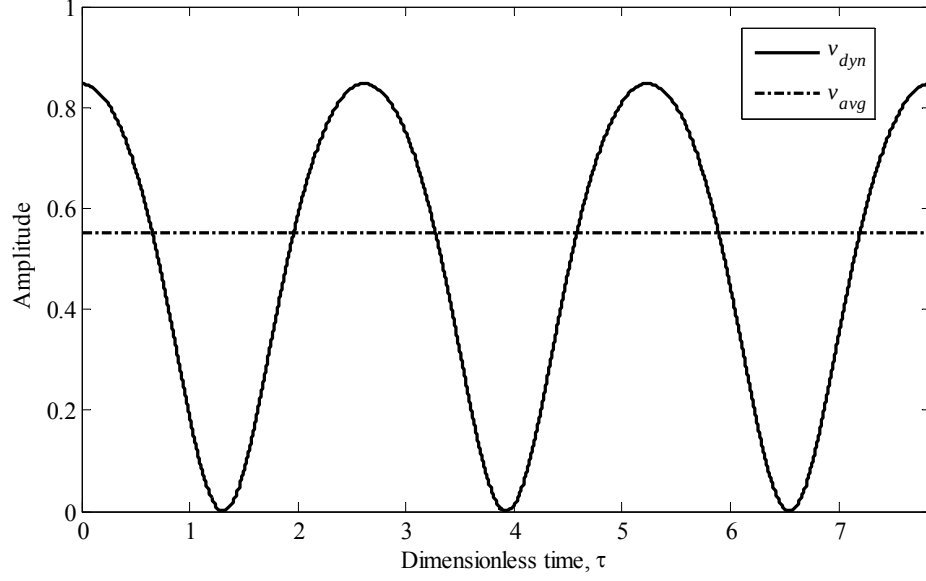


Figure 3.4. The dynamic velocity and the averaged velocity.

The phase velocity, v_{ph} , can also be calculated in the standard manner as follows

$$v_{ph} = c \frac{\omega}{\sqrt{\omega^2 - (\omega_n^c)^2}}. \quad (3.47)$$

To the best of our knowledge, though the phase velocity is well-studied in the framework of time-harmonic fields, the dynamic velocity, and consequently the energy density of the field and the z -component of Poynting vector, $v_{dyn} = P_z(z, t) / W(z, t)$, have not been considered. The reason is that the quadratic characteristics mentioned above are specified as complex-valued quantities

$$\begin{aligned} W(z) &= \frac{1}{2S} \int_S \left\{ \epsilon_0 (\mathcal{E} \cdot \mathcal{E}^*) + \mu_0 (\mathcal{H} \cdot \mathcal{H}^*) \right\} ds \\ P_z(z) &= \frac{1}{S} \int_S \mathbf{z} \cdot [\mathcal{E} \times \mathcal{H}^*] ds \end{aligned} \quad (3.48)$$

where $(\cdot)^*$ denotes complex conjugation. Since the time dependency of field vectors are expressed via $\exp(\pm i\omega t)$ for the time-harmonic fields, the parameter t in the integrands of (3.48) is eliminated due to the complex conjugation. In contrast to the

time-harmonic field theory, the proposed method operates with the real-valued field, the quadratic characteristics of which are the functions of t under the definition of (2.5). This fact gives us the opportunity to study dynamic processes for the quadratic characteristics.

5. The KGE obeys the fundamental principle of relativistic physics: The form of KGE is invariant in any inertial reference frame, specifically, the KGE must maintain its form under the action of Poincaré group within the framework of the Group Theory. This fact enables us to use remarkable properties of symmetry, which the solutions to the equation possess. These properties were published in [Miller, 1977].

In this chapter, the field components of TE and TM time-domain waveguide modes, defined in (3.14) and (3.22), have been obtained together with their quadratic characteristics of (3.16) and (3.24). It is proved that the modal amplitudes of the field components are generated by the solutions to the KGE 's of (3.13) and (3.23). In order to study the modal amplitudes of the fields for TE and TM modes *in parallel*, the dimensionless variables of (3.35) have been introduced. As an example, the quadratic characteristics of time-harmonic fields in terms of the dimensionless variables have been obtained in (3.42). In the next chapter, based on the properties of symmetry, which the solution to the KGE possess and mentioned in **(5)**, new analytical solutions to non-sinusoidal modal amplitudes will be derived.

4. NON-SINUSOIDAL ANALYTICAL SOLUTIONS

In [Miller, 1977], it has been proposed that the solution $f(\xi, \tau)$ to (3.34) can be interpreted as a function of new independent variables u and v yet to be defined. It is supposed, however, that u and v are twice differentiable functions of the old variables (ξ, τ) . Substitution of $f[u(\xi, \tau); v(\xi, \tau)]$ into (3.34) yields

$$\begin{aligned} & \left\{ [(\partial_\tau u)^2 - (\partial_\xi u)^2] \partial_u^2 + [(\partial_\tau v)^2 - (\partial_\xi v)^2] \partial_v^2 \right. \\ & \quad \left. + 2[(\partial_\tau u)(\partial_\tau v) - (\partial_\xi u)(\partial_\xi v)] \partial_{uv}^2 \right. \\ & \quad \left. + [\partial_\tau^2 u - \partial_\xi^2 u] \partial_u + [\partial_\tau^2 v - \partial_\xi^2 v] \partial_v + 1 \right\} f(u, v) = 0. \end{aligned} \quad (4.1)$$

The crucial idea in [Miller, 1977] is as follows: *i)* The functions $u(\xi, \tau)$ and $v(\xi, \tau)$ should be found using the properties of symmetry of the KGE. This enables us to express the coefficients in the square brackets as some given functions of the variables u and v . *ii)* Substitution of these functions of (u, v) into (4.1) results in the factorization of the solution as $f(u, v) = U(u)V(v)$. Eventually, $U(u)$ and $V(v)$ appear as some special functions of mathematical physics with the arguments dependent on (ξ, τ) . This enables us to obtain new analytical solutions for the time-domain modes, which are yet to be known, to the best of our knowledge.

A complete set consisting of eleven pairs of the inverse functions $\tau(u, v)$ and $\xi(u, v)$ needed for derivation of $u(\xi, \tau)$ and $v(\xi, \tau)$ has been established in [Miller, 1977]. For the completeness of the presentation, the set is also given in Appendix C. The first pair leads to the time-harmonic solutions. The second pair was used for the studies of digital signal propagation in waveguides [Aksoy and Tretyakov, 2003]. The seventh and the eighth pairs are still under study. We consider here which solution can be supplied with the fifth pair.

4.1. Time-Domain Modes Expressible via Airy Functions

Inversion of the formulas given in the fifth pair yields

$$\begin{aligned} u + v &= \frac{(\tau + \xi)}{2}, \\ u - v &= \pm \sqrt{\tau - \xi}. \end{aligned} \quad (4.2)$$

When the sign ($\pm$) is chosen as minus ($-$), then

$$\begin{aligned} u &= \frac{(\tau + \xi)}{4} - \frac{\sqrt{\tau - \xi}}{2}, \\ v &= \frac{(\tau + \xi)}{4} + \frac{\sqrt{\tau - \xi}}{2}. \end{aligned} \quad (4.3)$$

Otherwise, i.e., choosing the plus ($+$), the definitions of u and v will be replaced with each other. After substituting (4.3) into (4.1), the coefficients in the square brackets of (4.1) are obtained as follows

$$\left[(\partial_\tau u)^2 - (\partial_\xi u)^2 \right] = - \left[(\partial_\tau v)^2 - (\partial_\xi v)^2 \right] = \frac{1}{4(u - v)}, \quad (4.4)$$

and all the other coefficients are equal to zero. Substitution of these coefficients into (4.1) yields

$$\frac{\partial^2}{\partial u^2} f(u, v) + 4u f(u, v) = \frac{\partial^2}{\partial v^2} f(u, v) + 4v f(u, v). \quad (4.5)$$

Now, it is evident that $f(u, v)$ is a product of two functions. Substitution of the product, $U(u)V(v)$, into (4.5) results in

$$\frac{1}{U(u)} \frac{d^2}{du^2} U(u) + 4u = \frac{1}{V(v)} \frac{d^2}{dv^2} V(v) + 4v = 4\alpha \quad (4.6)$$

where α is a constant of separation of the variables. At this point, we introduce two new auxiliary variables $\bar{u}$ and $\bar{v}$ as

$$\begin{aligned} \bar{u} &= \sqrt[3]{4} (\alpha - u), \\ \bar{v} &= \sqrt[3]{4} (\alpha - v), \end{aligned} \quad (4.7)$$

then (4.6) yields the Airy differential equations [Vallée and Soares, 2004]

$$\begin{aligned} \frac{d^2}{d\bar{u}^2} U(\bar{u}) - \bar{u} U(\bar{u}) &= 0, \\ \frac{d^2}{d\bar{v}^2} V(\bar{v}) - \bar{v} V(\bar{v}) &= 0. \end{aligned} \quad (4.8)$$

The Airy functions as the solutions to (4.8) are real-valued, provided that the constant parameter α in their arguments given in (4.7) is chosen to be real valued.

Suppose that the solution should have a *beginning in time*: say, at $t = 0$. Physically, this implies that the field sources do not act before the instant $t = 0$. In compliance with the causality principle, a final solution to the KGE should be presented as the following piecewise function

$$f(\xi, \tau) = \begin{cases} 0 & \text{if } \tau < 0, \\ U(\bar{u})V(\bar{v}) & \text{if } 0 \leq \xi \leq \tau, \\ 0 & \text{if } \xi > \tau. \end{cases} \quad (4.9)$$

The first line of (4.9) is written in accordance with the weak causality condition. The condition states that all the fields are zero for $t < 0$, when the sources of the field are zero at that time. The lower lines of (4.9) correspond to the strong causality condition [Hansen and Yaghjian, 1999]. This follows from the axiom of the special theory of relativity asserting that any electromagnetic field transfers a signal with the velocity of light, c . Specifically, the modal fields are zero beyond the distance $z = ct$ along the Oz axis from a source located at $z = 0$ that turns on at instant $t = 0$. The solution $f(\xi, \tau) = U(\bar{u})V(\bar{v})$ assumes implicitly that the initial conditions of (3.36) are chosen as

$$\begin{aligned} \varsigma(\tau) &= U(\bar{u})V(\bar{v})\Big|_{\xi=0}, \\ \nu(\tau) &= -\partial_{\tau}[U(\bar{u})V(\bar{v})]\Big|_{\xi=0}. \end{aligned} \quad (4.10)$$

Hence, our procedure has sense of the expansion of initial values, given for $\xi = 0$, over the whole domain of propagation, $0 \leq \xi \leq \tau < +\infty$. This procedure is

equivalent to the exact calculation of convolution integral mentioned in the section titled “1.3.2. Historical Background of the Problem.”

4.2. Numerical Examples

Each equation in (4.8) has two linearly independent solutions

$$\begin{aligned} U(\bar{u}) &= a_1 Ai(\bar{u}) + b_1 Bi(\bar{u}) \\ V(\bar{v}) &= a_2 Ai(\bar{v}) + b_2 Bi(\bar{v}) \end{aligned} \quad (4.11)$$

where $a_{1,2}$ and $b_{1,2}$ are arbitrary constants, $Ai(\cdot)$ is the Airy function of the first kind, and $Bi(\cdot)$ is the Airy function of the second kind. Their arguments as the functions of dimensionless time, τ , and dimensionless coordinate, ξ , are

$$\begin{aligned} \bar{u} &= \sqrt[3]{4} \left[\alpha - \frac{(\tau + \xi)}{4} + \frac{\sqrt{\tau - \xi}}{2} \right], \\ \bar{v} &= \sqrt[3]{4} \left[\alpha - \frac{(\tau + \xi)}{4} - \frac{\sqrt{\tau - \xi}}{2} \right]. \end{aligned} \quad (4.12)$$

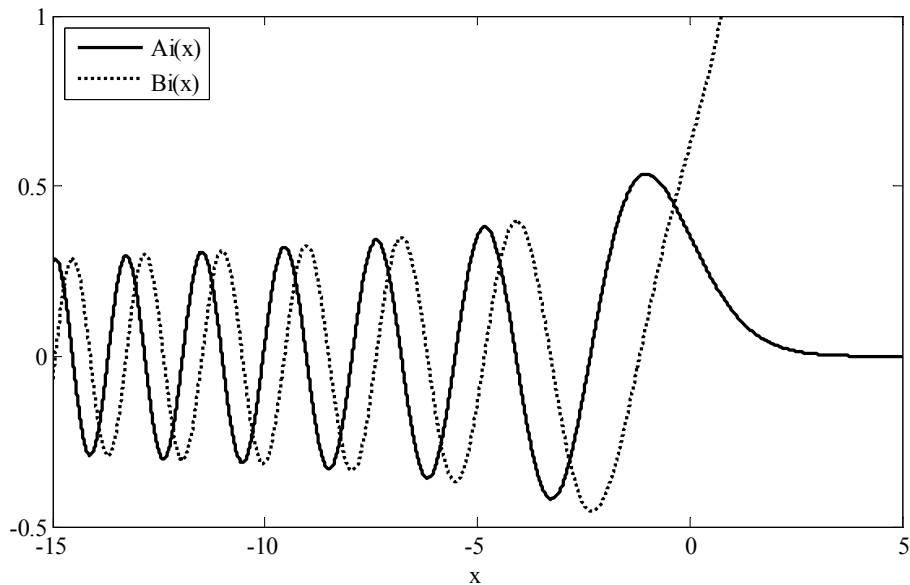


Figure 4.1. The Airy functions of the first and second kinds.

In Fig. 4.1, in order to illustrate the Airy functions, we denote their arguments as x . When x is positive, $Ai(x)$ is positive, convex and decreasing exponentially to

zero. The function $Bi(x)$ for positive x is also positive and convex, but contrary to the $Ai(x)$ it is increasing exponentially. When x is negative, $Ai(x)$ and $Bi(x)$ oscillate around zero with ever-increasing frequency and ever-decreasing amplitude. The decrease in the amplitude of oscillations can be given by $x^{-1/4}$ in average. The function $Bi(x)$ is defined as the solution with the same amplitude of oscillation as the $Ai(x)$ (as x goes to $-\infty$), which differs in phase by $\pi/2$, see the asymptotic formula given below in (4.15). It is clear from Fig. 4.1 that negative values of the argument of Airy functions provide oscillations with decreasing amplitude. Based on (4.9) and (4.11), in order to have finite-valued modal amplitude for the longitudinal component of the field, the arguments of Airy functions in (4.12) should be negative. This fact gives us the opportunity to define the following criteria

$$\begin{aligned} \bar{u} : \quad \alpha &< \frac{(\tau + \xi)}{4} - \frac{\sqrt{\tau - \xi}}{2} \\ \bar{v} : \quad \alpha &< \frac{(\tau + \xi)}{4} + \frac{\sqrt{\tau - \xi}}{2} \end{aligned} \quad (4.13)$$

for choosing the possible values of α for the arguments $\bar{u}$ and $\bar{v}$ that are given in (4.12). Note that the criterion written for $\bar{u}$ automatically satisfies the one for $\bar{v}$. When the value of α is chosen from the first criterion of (4.13), the positive values of τ and ξ provide negative values of the arguments of Airy functions leading to oscillations with decreasing amplitude. Considering (4.9) and (4.11) together in the domain of propagation, $0 \leq \xi \leq \tau < +\infty$, we can conclude that all the following combinations of the products of Airy functions

$$\begin{aligned} f_1(\xi, \tau) &= Ai(\bar{u})Ai(\bar{v}) & f_2(\xi, \tau) &= Ai(\bar{u})Bi(\bar{v}) \\ f_3(\xi, \tau) &= Bi(\bar{u})Ai(\bar{v}) & f_4(\xi, \tau) &= Bi(\bar{u})Bi(\bar{v}) \end{aligned} \quad (4.14)$$

can have physical sense as the solution $f(\xi, \tau)$, provided that the value of α is chosen properly.

An interesting case occurs when time, τ , and coordinate, ξ , both are in vicinity of the characteristic line of (3.34), $\tau = \xi$. The variables τ and ξ are

coupled by the condition $\tau = \xi + \delta$ where $\delta < 1$ is a small parameter, specifically, $\delta = 0.05$. For the minimum values of τ and ξ in Figures 4.2-4.9, the inequality $\alpha < -0.0993$ is calculated by virtue of the first criterion of (4.13) and the figures are obtained for $\alpha = -1$. Fig. 4.2 illustrates the initial values of (4.10) as the source functions for $f_1(\xi, \tau)$ given in (4.14).

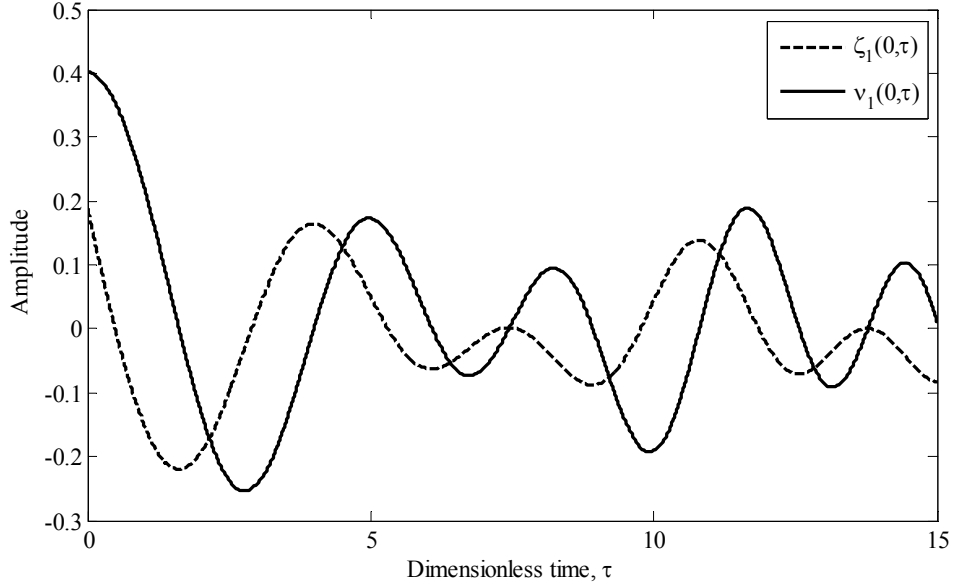


Figure 4.2. The initial values as the source functions for $f_1(\xi, \tau) = Ai(\bar{u})Ai(\bar{v})$ when $\alpha = -1$.

In Fig. 4.3, the modal amplitudes specified by the initial values shown in Fig. 4.2 are illustrated. The dashed line corresponds to the function $f_1(\xi, \tau)$ specifying the longitudinal component of a three-component modal field. In (3.37), it is the modal amplitude of the field $v_n^{-1}H'_{zn}(\mathbf{r}, \xi, \tau)$ for the TE modes. In (3.38), it is the modal amplitude of the field $\kappa_n^{-1}E''_{zn}(\mathbf{r}, \xi, \tau)$ for the TM modes. The solid line corresponds to the function $\mathcal{A}_1(\xi, \tau) = -\partial_\tau f_1(\xi, \tau)|_{\xi=\tau-\delta}$ specifying the transverse components of a two-component modal field. In (3.37) and (3.38), it is either the modal amplitude of the field $v_n^{-1}\sqrt{\varepsilon_0/\mu_0}\mathbf{E}'_n(\mathbf{r}, \xi, \tau)$ for the TE modes, or it is the modal amplitude of the field $\kappa_n^{-1}\sqrt{\mu_0/\varepsilon_0}\mathbf{H}''_n(\mathbf{r}, \xi, \tau)$ for the TM modes. The dotted line corresponds to the function $\mathcal{B}_1(\xi, \tau) = \partial_\xi f_1(\xi, \tau)|_{\xi=\tau-\delta}$ specifying the

transverse components of a three-component modal field. In (3.37) and (3.38), it is either the modal amplitude of the field $v_n^{-1}\mathbf{H}'_n(\mathbf{r}, \xi, \tau)$ for the TE modes, or it is the modal amplitude of the field $\kappa_n^{-1}\mathbf{E}''_n(\mathbf{r}, \xi, \tau)$ for the TM modes.

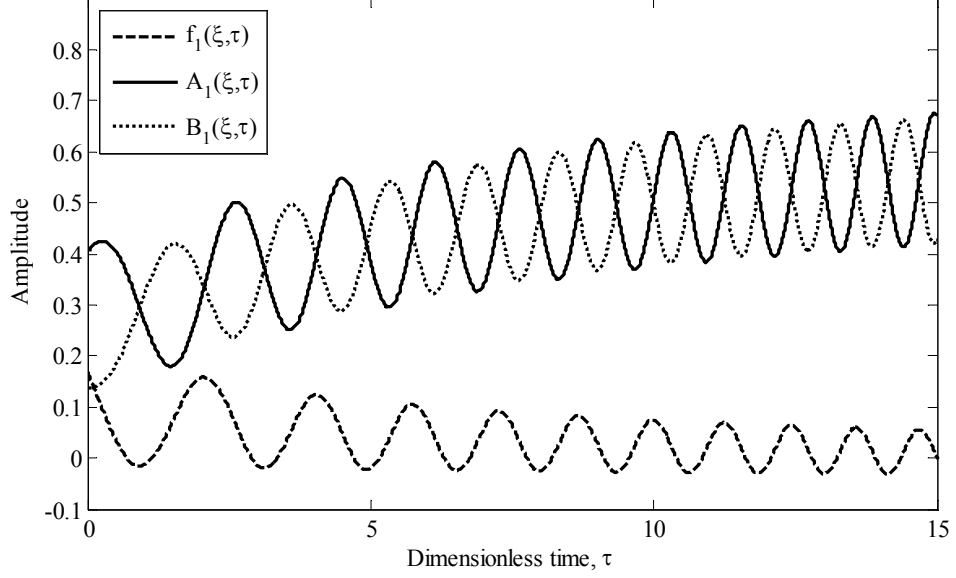


Figure 4.3. The modal amplitudes specified by the initial values shown in Fig. 4.2 when $\alpha = -1$.

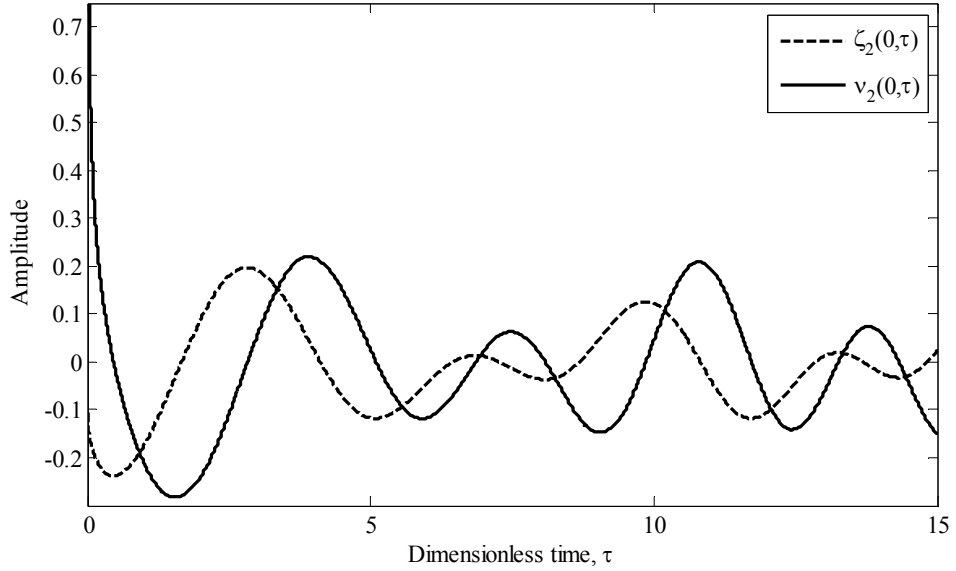


Figure 4.4. The initial values as the source functions for $f_2(\xi, \tau) = Ai(\bar{u})Bi(\bar{v})$ when $\alpha = -1$.

Fig. 4.4 illustrates the initial values of (4.10) as the source functions for $f_2(\xi, \tau)$ given in (4.14), whereas Fig. 4.5 illustrates the modal amplitudes generated

by the solution $f_2(\xi, \tau)$ to the *KGE*. In Fig. 4.5, the system of line notation is similar to that of Fig. 4.3.

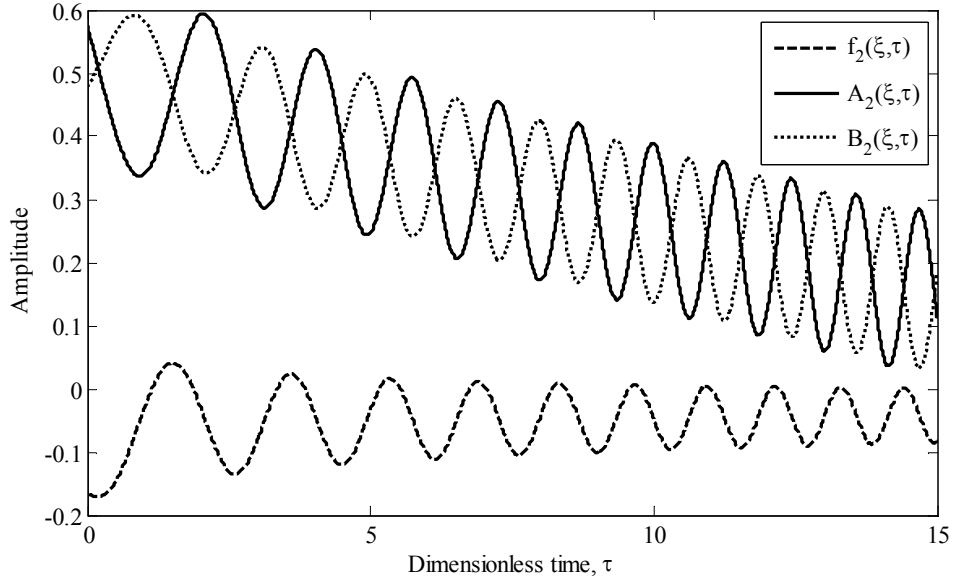


Figure 4.5. The modal amplitudes specified by the initial values shown in Fig. 4.4 when $\alpha = -1$.

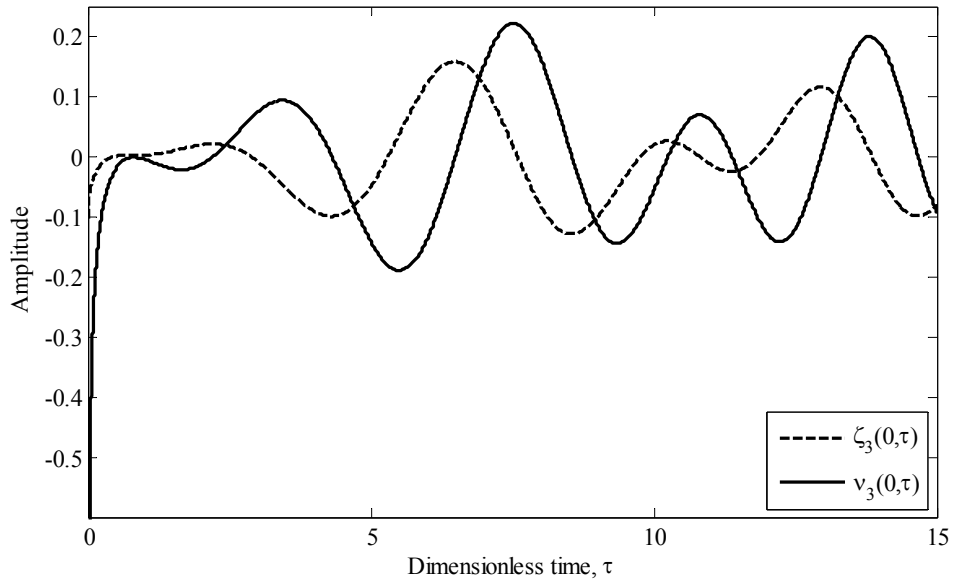


Figure 4.6. The initial values as the source functions for $f_3(\xi, \tau) = Bi(\bar{u})Ai(\bar{v})$ when $\alpha = -1$.

Fig. 4.6 illustrates the initial values of (4.10) as the source functions for $f_3(\xi, \tau)$ given in (4.14), whereas Fig. 4.7 illustrates the modal amplitudes generated by the solution $f_3(\xi, \tau)$ to the *KGE*. In Fig. 4.7, the system of line notation is

similar to that of Fig. 4.3.

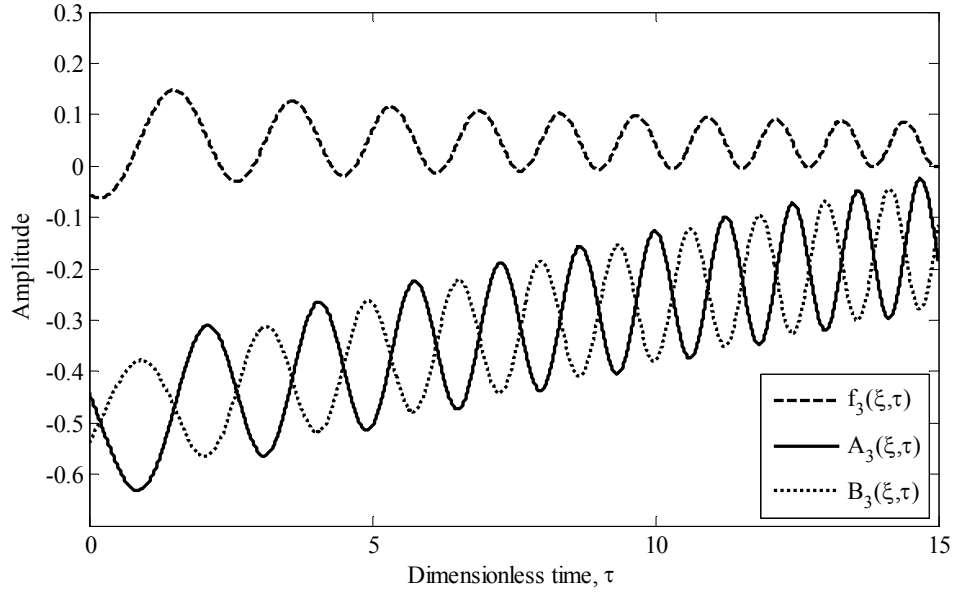


Figure 4.7. The modal amplitudes specified by the initial values shown in Fig. 4.6 when $\alpha = -1$.

Fig. 4.8 illustrates the initial values of (4.10) as the source functions for $f_4(\xi, \tau)$ given in (4.14), whereas Fig. 4.9 illustrates the modal amplitudes generated by the solution $f_4(\xi, \tau)$ to the *KGE*. In Fig. 4.9, the system of line notation is similar to that of Fig. 4.3.

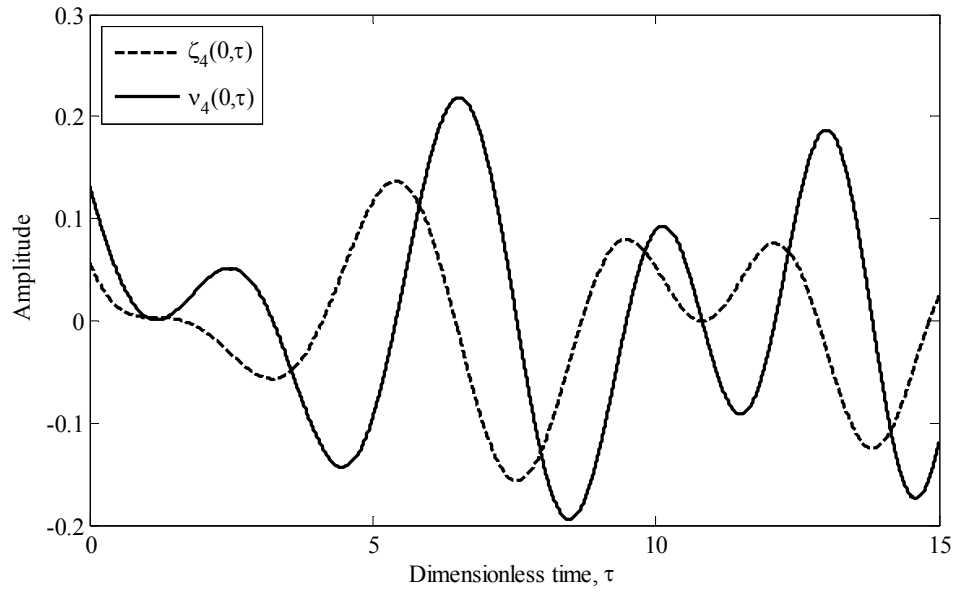


Figure 4.8. The initial values as the source functions for $f_4(\xi, \tau) = Bi(\bar{u})Bi(\bar{v})$ when $\alpha = -1$.

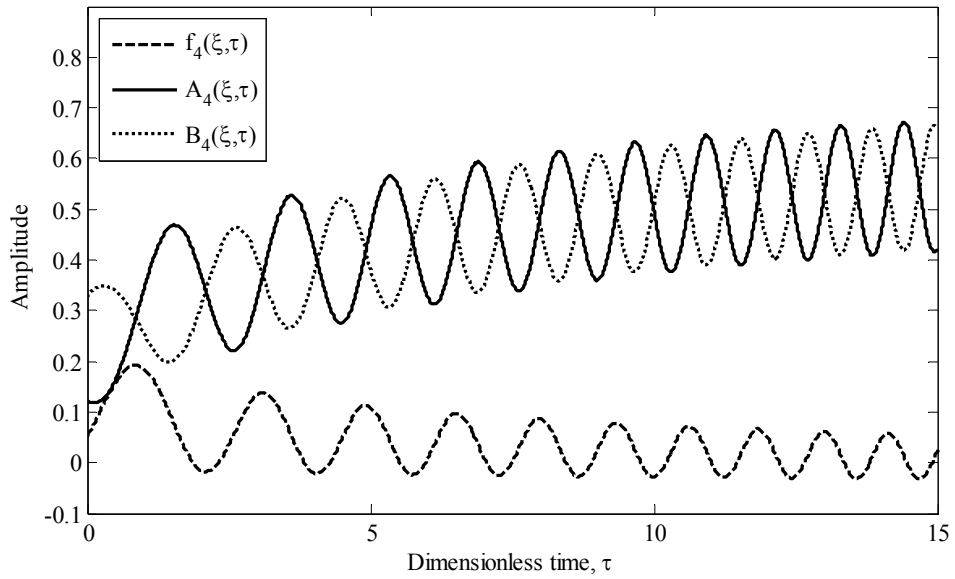


Figure 4.9. The modal amplitudes specified by the initial values shown in Fig. 4.8 when $\alpha = -1$.

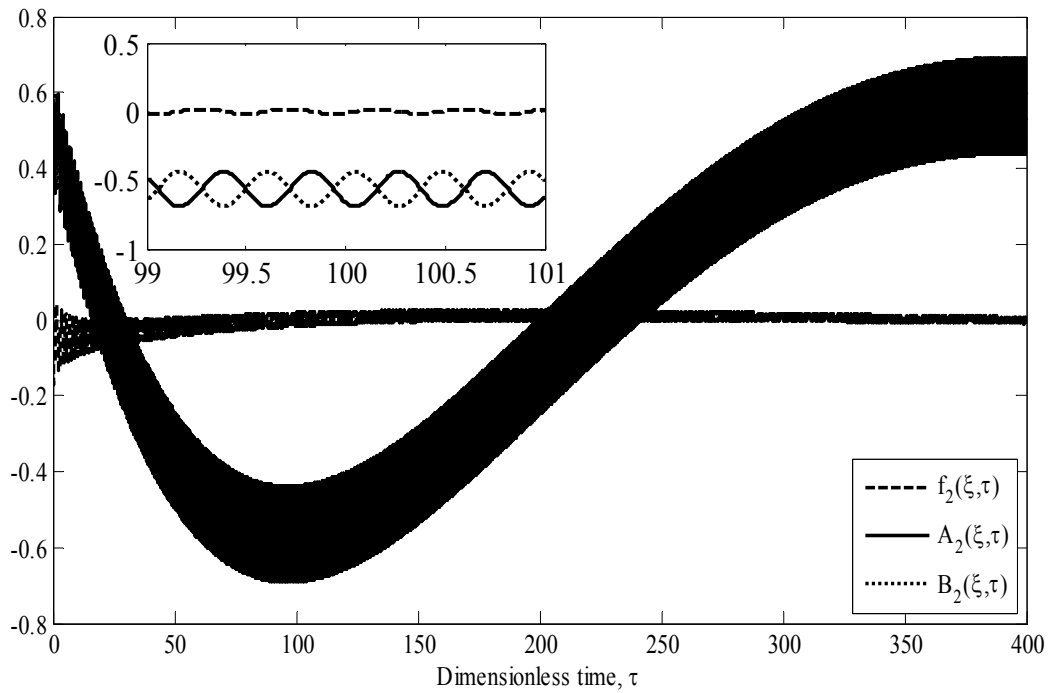


Figure 4.10. The modal amplitudes specified by the initial values shown in Fig. 4.4 for large values of τ when $\alpha = -1$.

The asymptotic behaviour of the Airy functions as the argument (for example, x) goes to $+\infty$ is given by the following asymptotic formulas

$$\begin{aligned}
Ai(-x) &\sim \frac{1}{\sqrt{\pi}} x^{-1/4} \sin\left(\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right), \\
Bi(-x) &\sim \frac{1}{\sqrt{\pi}} x^{-1/4} \cos\left(\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right).
\end{aligned}
\tag{4.15}$$

This behaviour can also be seen from Fig. 4.10 obtained for $f_2(\xi, \tau) = Ai(\bar{u})Bi(\bar{v})$ for large values of τ . The only difference between Figures 4.5 and 4.10 is the maximum value of τ -axis, hence, $\alpha = -1$, $\tau = \xi + \delta$, and $\delta = 0.05$ in both figures. As explained before, the positive values of τ and ξ provide negative values of the arguments of Airy functions. That is, the asymptotic formula of (4.15) is given for the limit in the negative direction. Due to the slow oscillations with respect to τ steps, the interval between $\tau = 99$ and $\tau = 101$ is zoomed to see the modal amplitudes in detail. As the value of τ increases further, the amplitudes of the oscillations decreases.

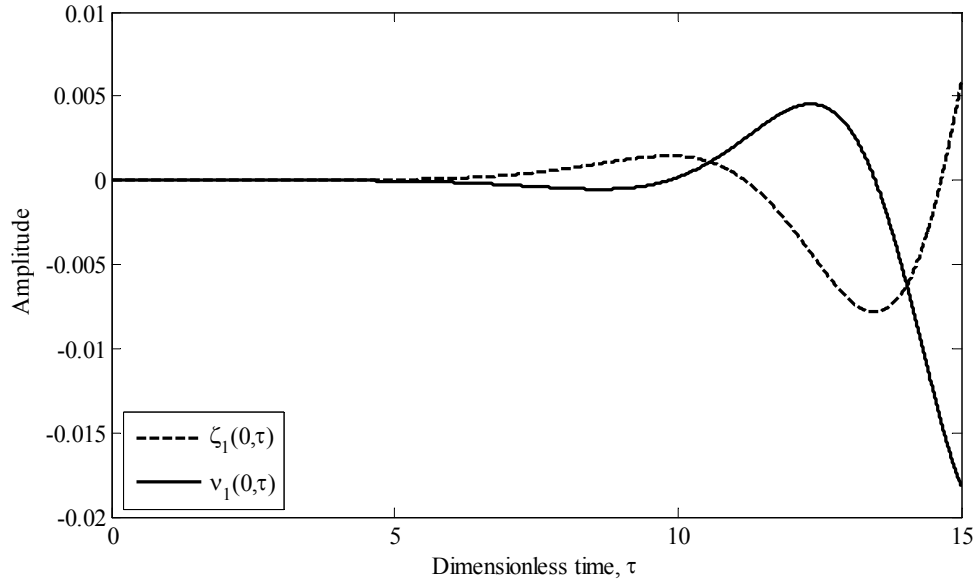


Figure 4.11. The initial values as the source functions for $f_1(\xi, \tau)$ when $\alpha = 3$.

In Figures 4.11-4.18, the effect of α on the modal amplitudes is examined by choosing a value which is not satisfying the proposed condition, $\alpha < -0.0993$. All the other parameters are the same as in Figures 4.2-4.9. All the combinations of the products of Airy functions given in (4.14) are illustrated together with their initial values since they are also dependent on the value of α . In Fig. 4.11, the initial values as the source functions for the modal amplitude $f_1(\xi, \tau)$ is illustrated for $\alpha = 3$. In

Fig. 4.12, the modal amplitudes specified by the initial values shown in Fig. 4.11 are illustrated for $\alpha = 3$. From the first criterion of (4.13), it can be concluded that the arguments of the Airy functions given in (4.14) have positive values until $\tau > 6.2486$. For the values $\tau < 6.2486$, the $Ai(\bar{u})$ and $Ai(\bar{v})$, the factors of $f_1(\xi, \tau)$, are decreasing exponentially to zero causing the zero valued transients of modal amplitudes. For the values $\tau > 6.2486$, the arguments $\bar{u}$ and $\bar{v}$ are negative valued leading to the retarded oscillations.

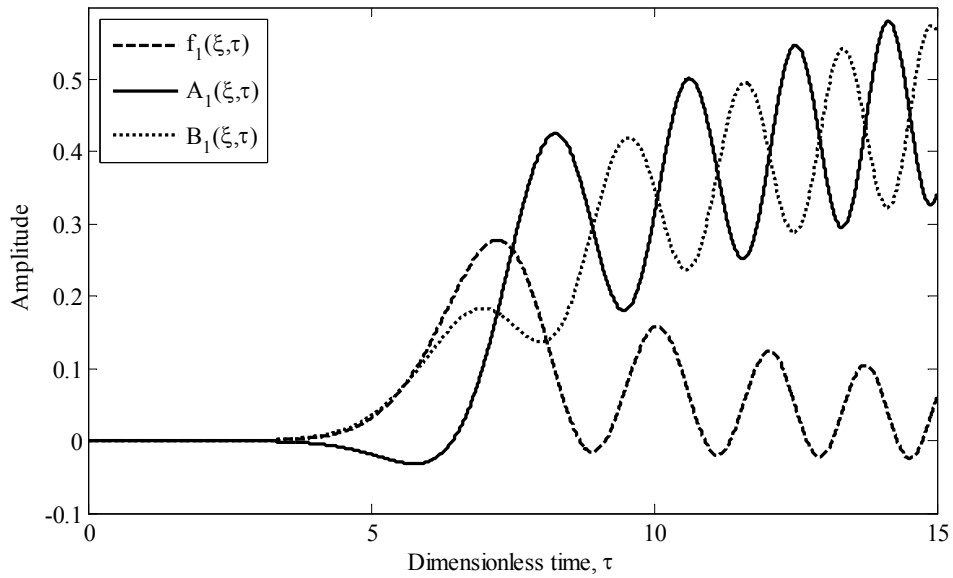


Figure 4.12. The modal amplitudes specified by the initial values shown in Fig. 4.11 when $\alpha = 3$.

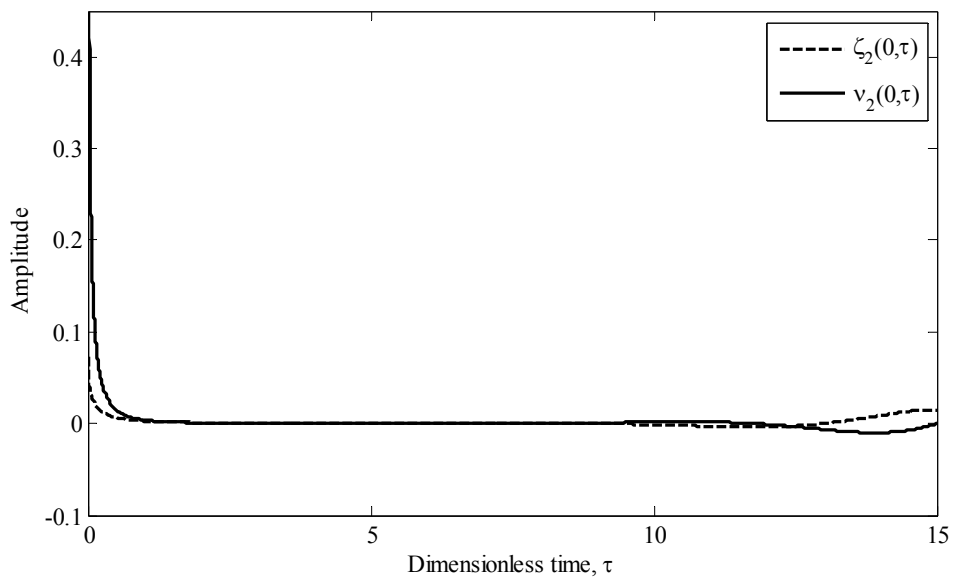


Figure 4.13. The initial values as the source functions for $f_2(\xi, \tau)$ when $\alpha = 3$.

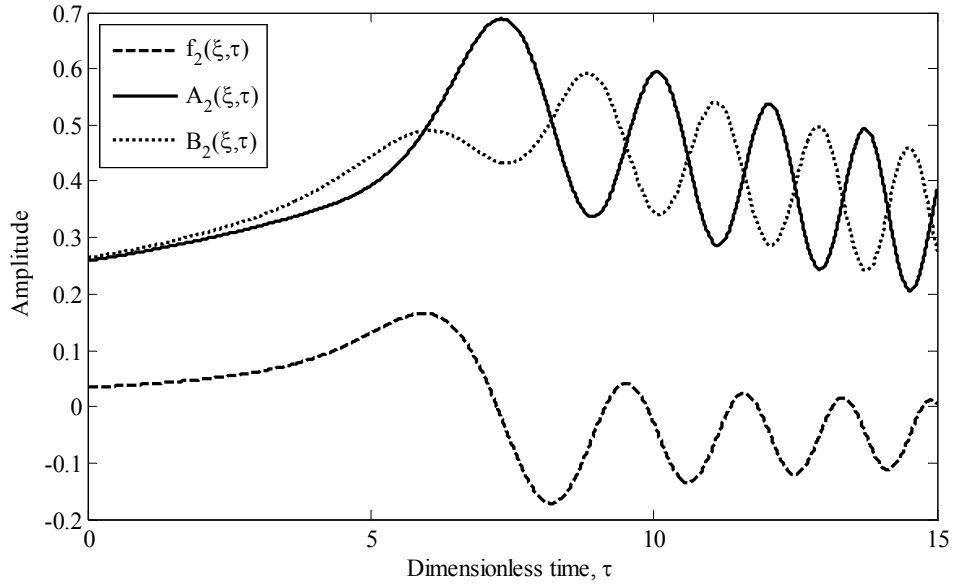


Figure 4.14. The modal amplitudes specified by the initial values shown in Fig. 4.13 when $\alpha = 3$.

Fig. 4.13 illustrates the initial values of (4.10) for $\alpha = 3$ as the source functions for $f_2(\xi, \tau)$ given in (4.14). Whereas Fig. 4.14 illustrates the modal amplitudes generated by the solution $f_2(\xi, \tau)$ to the *KGE*.

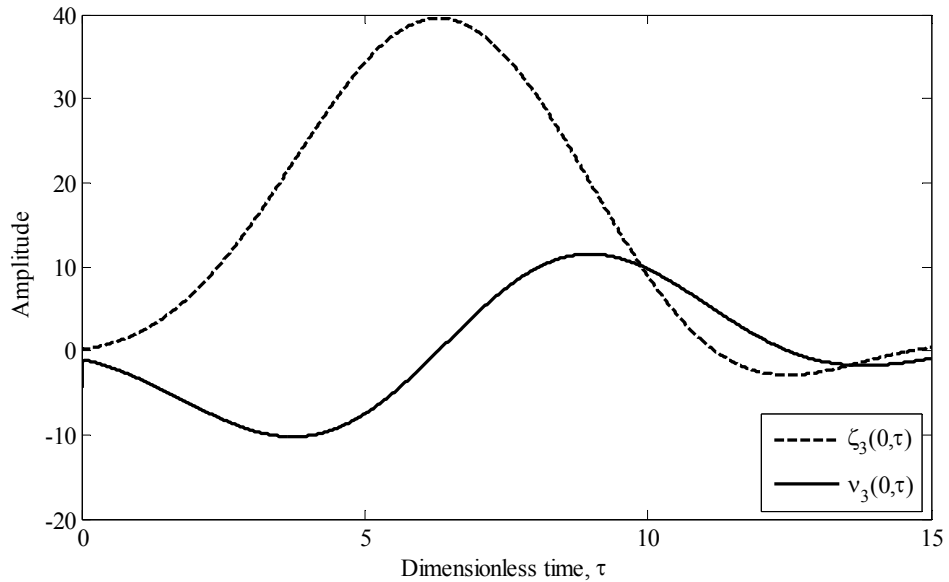


Figure 4.15. The initial values as the source functions for $f_3(\xi, \tau)$ when $\alpha = 3$.

Fig. 4.15 illustrates the initial values of (4.10) for $\alpha = 3$ as the source functions for $f_3(\xi, \tau)$ given in (4.14). Whereas Fig. 4.16 illustrates the modal amplitudes generated by the solution $f_3(\xi, \tau)$ to the *KGE*.

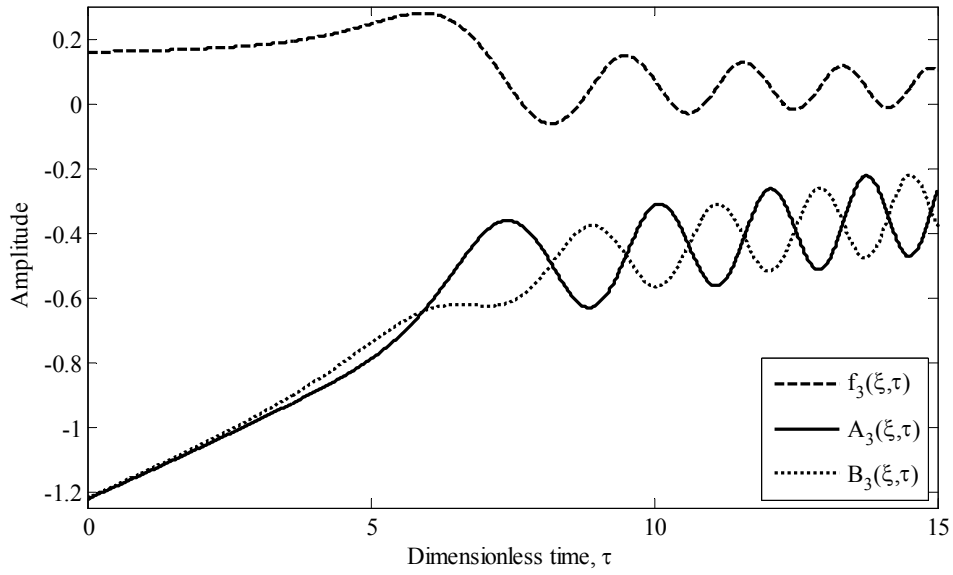


Figure 4.16. The modal amplitudes specified by the initial values shown in Fig. 4.15 when $\alpha = 3$.

In Fig. 4.17, the initial values as the source functions for the modal amplitude $f_4(\xi, \tau)$ is illustrated for $\alpha = 3$. In Fig. 4.18, the modal amplitudes specified by the initial values shown in Fig. 4.17 are illustrated for $\alpha = 3$. For the values of $\tau < 6.2486$, $Bi(\bar{u})$ and $Bi(\bar{v})$, the factors of $f_4(\xi, \tau)$, are increasing exponentially due to the positive valued arguments. For the values of $\tau > 6.2486$, the arguments $\bar{u}$ and $\bar{v}$ are negative valued leading to the oscillations.

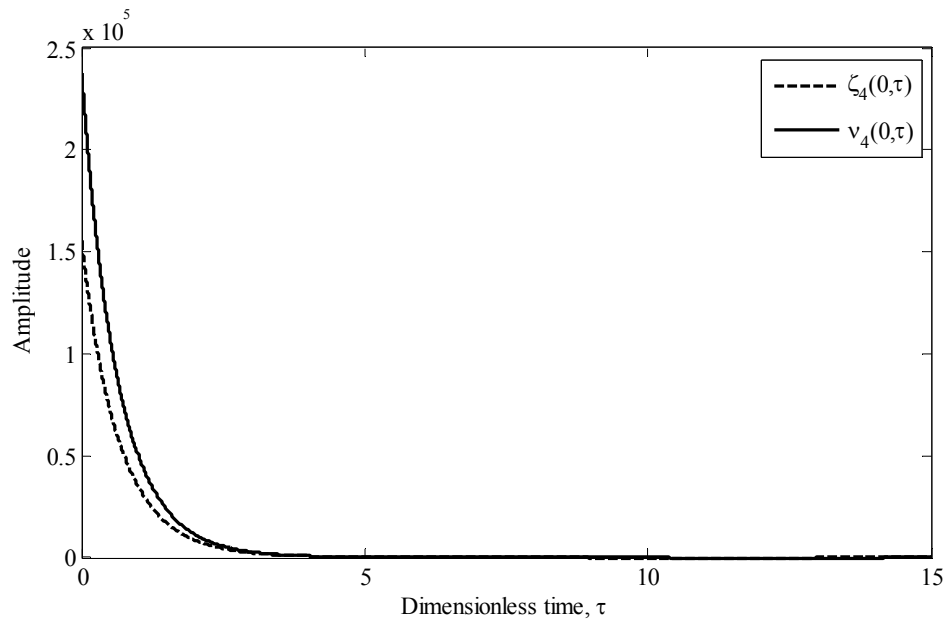


Figure 4.17. The initial values as the source functions for $f_4(\xi, \tau)$ when $\alpha = 3$.

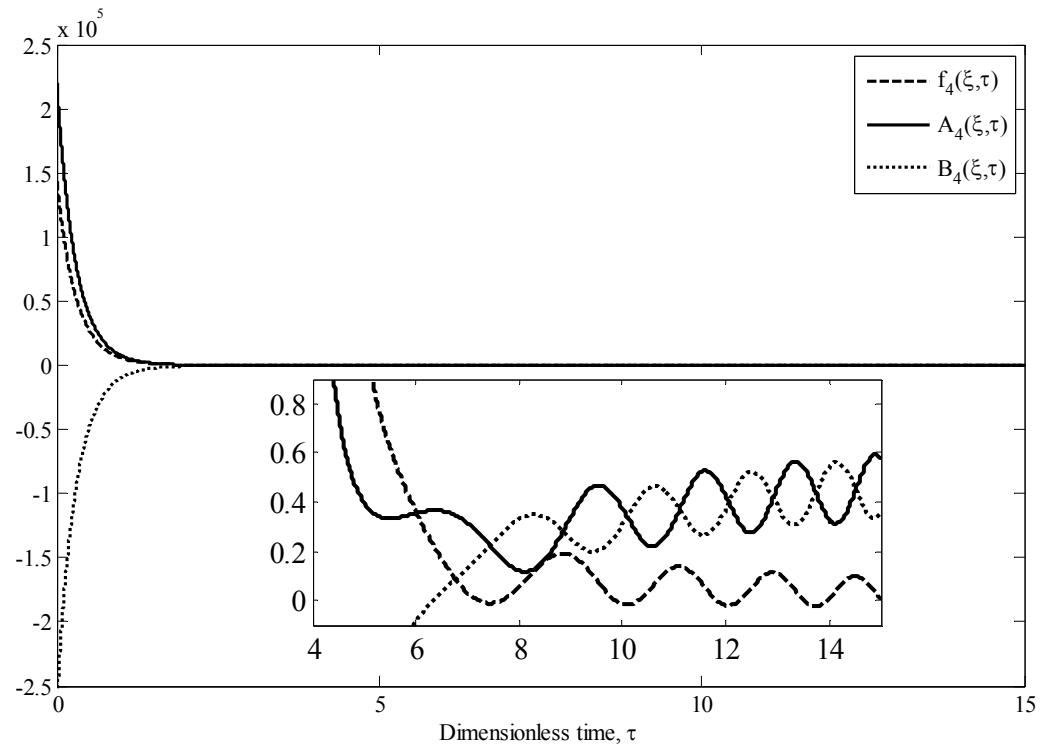


Figure 4.18. The modal amplitudes specified by the initial values shown in Fig. 4.17 when $\alpha = 3$.

5. RELATIVISTIC EFFECTS ON THE WAVE PROPAGATION IN A HOLLOW WAVEGUIDE

5.1. Introduction to the Special Theory of Relativity

Galilei and Newton have introduced a concept of a family of inertial reference frames that are assumed to be in a state of constant motion along a straight line (i.e., uniform rectilinear motion) with respect to one another. In the Galilean relativity also known as the Newtonian relativity as applied to the Newtonian mechanics [Newton, 1848], specifically, the following two arguments are assumed to be true:

- i)** There exists an absolute space, in which Newton's laws of motion hold. An inertial frame is a reference frame in uniform rectilinear motion relatively to the absolute space.
- ii)** All inertial frames of reference share an absolute time.

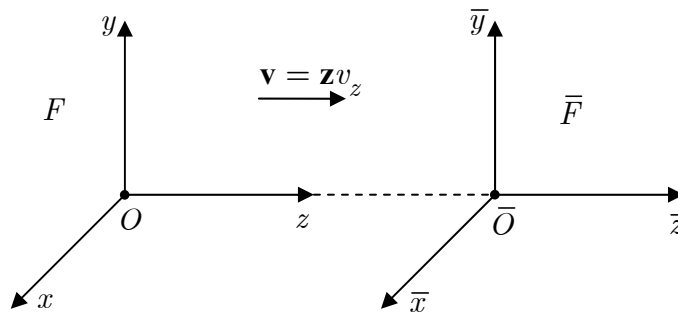


Figure 5.1. Two inertial frames of reference in relative motion.

The Galilean transformation, converting measurements in one inertial frame to another, can be derived as follows. We consider two inertial frames F and $\bar{F}$, as illustrated in Fig. 5.1, assuming that their origins coincide when time equals to zero. A system of values $(x, y, z; t)$ defines the place and time of a material point in the frame F , whereas $(\bar{x}, \bar{y}, \bar{z}; \bar{t})$ represents the same point relatively to the system $\bar{F}$. Assume that the coordinate systems are oriented so that the Ox axis is parallel to the $\bar{O}\bar{x}$ axis, as are Oy and $\bar{O}\bar{y}$ axes, while the Oz axis and the $\bar{O}\bar{z}$ axis are collinear. Also assume that $\bar{F}$ starts to a uniform rectilinear motion relatively to the frame F at rest with constant velocity, v_z , along the common Oz axis, meanwhile the other

two axes still remain parallel to each other. The latter corresponds to a boost in which the entire system is shifted without any rotation of its axes. Hereinafter, we will refer to the reference frame in a state of rest as F , and to the moving frame as $\bar{F}$.

The total displacement z in F along the direction of the constant velocity is the sum of the relative displacement $\bar{z}$ in $\bar{F}$ and of the distance between the two origins, $z - \bar{z}$. Meanwhile, the other space measurements remain unchanged. Thus, the Galilean transformation can be given by

$$\begin{aligned} x &= \bar{x}, & y &= \bar{y}, \\ z &= \bar{z} + v_z \bar{t}, & t &= \bar{t}. \end{aligned} \tag{5.1}$$

Note that $t = \bar{t}$ is the result of the second axiom above expressing that the universal (or absolute) time is independent of the relative motion of different observers.

Though the Galilean relativity, stating that the laws of motion are the same in all the inertial reference frames, is valid for nearly all mechanical systems, the field equations of electrodynamics, formulated by James Clerk Maxwell, are not invariant under this transformation. The special theory of relativity proposed by Albert Einstein [Einstein, 1905] removes this lack of precision. The fundamentals of special theory of relativity are founded on the following principles:

- i)** The principle of relativity: The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of coordinates in uniform translatory motion.
- ii)** The principle of the constancy of the velocity of light: Any ray of light moves in the “stationary” system of coordinates with the determined velocity c , whether the ray be emitted by a stationary or by a moving body.

In the special theory of relativity as a modification of the Galilean relativity, varying measurements of space and time for the two different observers can be converted into each other's frames of reference by making use of the Lorentz transformation

$$\begin{aligned} x &= \bar{x}, & y &= \bar{y}, \\ z &= \bar{\gamma}(\bar{z} + v_z \bar{t}), & t &= \bar{\gamma}[\bar{t} + (v_z / c^2)\bar{z}], \end{aligned} \quad (5.2)$$

where $\bar{\gamma} = 1 / \sqrt{1 - \beta^2}$ is called the Lorentz factor, and $\beta = v_z / c$. For the relative velocities much less than the velocity of light, or mathematically, when $\beta \ll 1$, the Lorentz transformation reduces to the Galilean transformation in accordance with the correspondence principle defined in the first footnote of the section “1.1. Subject of the Study.” Accordingly, the Galilean transformation can be regarded as a low-velocity approximation to the Lorentz's. The inverse Lorentz transformation can be obtained by transposing the coordinates, and replacing the relative velocity v_z with $-v_z$ in (5.2) as follows

$$\begin{aligned} \bar{x} &= x, & \bar{y} &= y, \\ \bar{z} &= \bar{\gamma}(z - v_z t), & \bar{t} &= \bar{\gamma}[t - (v_z / c^2)z]. \end{aligned} \quad (5.3)$$

In the special theory of relativity, time is not absolute as in the Galilean relativity, specifically, time passes at different rates for different observers: A moving clock ticks more slowly than when it is at rest with respect to the observer, which is known as *time dilation*. The other astounding consequence of this theory is *length contraction*: A moving rod may be found to be shorter than when it is at rest with respect to the observer. Therefore, as a result of time dilation and length contraction, the manifolds, space and time are combined into a single one, a unified four-dimensional continuum called space-time, rather than as separate entities.

In Chapter 3, the time-domain modes, which are valid for various hollow waveguides with perfectly conducting surface, have been derived via solving the initial-boundary value problem for the system of Maxwell's equations with time derivative, ∂_t . It is known that these equations are invariant under the relativistic Lorentz transformations. Proposed technique has led to the complete sets of *TE* and *TM* waveguide modes, each of which has similar relativistic properties, as well. Notice that the relativistic Lorentz transformations are inapplicable to the time-harmonic waveguide waves in the frequency domain studied more than a century. In

Chapter 4, it is proved that the modal amplitudes are generated by the KGE , derived from Maxwell's equations directly, with t and z as independent variables. The KGE obeys the fundamental principle of relativistic physics, that is, the form of KGE is invariant in any inertial reference frame. This implies that the KGE must maintain its form under the action of the Poincaré group within the framework of the Group Theory.

The organization of the chapter is as follows: The proof of relativistic invariance of the KGE is given in the next section. Section 5.3 provides the relativistic effects on the wave propagation in a hollow waveguide for the TE and TM modes. In Section 5.4, the relativistic analysis of the waveguide modes via the dimensionless modal amplitudes is made, and the quadratic characteristics of the modes are studied. As an example, the quadratic characteristics of time-harmonic fields are given.

5.2. Relativistic Invariance of the KGE

To prove that (3.34) has the same form both in the frame at rest and in the moving frame, we first adapt the second line of (5.2) to the dimensionless coordinate and time, (ξ, τ) in F and $(\bar{\xi}, \bar{\tau})$ in $\bar{F}$, by virtue of (3.35)

$$\begin{aligned}\xi &= \bar{\gamma}(\bar{\xi} + \beta\bar{\tau}), \\ \tau &= \bar{\gamma}(\bar{\tau} + \beta\bar{\xi}).\end{aligned}\tag{5.4}$$

In a similar manner, the second line of (5.3) can be rewritten as follows

$$\begin{aligned}\bar{\xi} &= \bar{\gamma}(\xi - \beta\tau), \\ \bar{\tau} &= \bar{\gamma}(\tau - \beta\xi).\end{aligned}\tag{5.5}$$

The solution $f(\xi, \tau)$ to (3.34) can be interpreted as $f[\bar{\xi}(\xi, \tau); \bar{\tau}(\xi, \tau)]$. Utilizing (5.5) for the first partial derivatives of $f(\bar{\xi}, \bar{\tau})$, we obtain

$$\begin{aligned}\partial_{\bar{\tau}} f(\bar{\xi}, \bar{\tau}) &= \bar{\gamma}(-\beta \partial_{\bar{\xi}} + \partial_{\bar{\tau}}) f(\bar{\xi}, \bar{\tau}), \\ \partial_{\bar{\xi}} f(\bar{\xi}, \bar{\tau}) &= \bar{\gamma}(\partial_{\bar{\xi}} - \beta \partial_{\bar{\tau}}) f(\bar{\xi}, \bar{\tau}).\end{aligned}\tag{5.6}$$

After repeating the differentiation procedure to calculate the second-order derivatives, we get

$$\begin{aligned}\partial_{\bar{\tau}}^2 f(\bar{\xi}, \bar{\tau}) &= (\beta^2 \bar{\gamma}^2 \partial_{\bar{\xi}}^2 - 2\beta \bar{\gamma}^2 \partial_{\bar{\xi}} \partial_{\bar{\tau}} + \bar{\gamma}^2 \partial_{\bar{\tau}}^2) f(\bar{\xi}, \bar{\tau}), \\ \partial_{\bar{\xi}}^2 f(\bar{\xi}, \bar{\tau}) &= (\bar{\gamma}^2 \partial_{\bar{\xi}}^2 - 2\beta \bar{\gamma}^2 \partial_{\bar{\xi}} \partial_{\bar{\tau}} + \beta^2 \bar{\gamma}^2 \partial_{\bar{\tau}}^2) f(\bar{\xi}, \bar{\tau}).\end{aligned}\tag{5.7}$$

Substituting (5.7) into (3.34), and recalling that $\bar{\gamma}^2(1 - \beta^2) = 1$, the *KGE* in the moving frame $\bar{F}$ can be obtained as

$$\partial_{\bar{\tau}}^2 f(\bar{\xi}, \bar{\tau}) - \partial_{\bar{\xi}}^2 f(\bar{\xi}, \bar{\tau}) + f(\bar{\xi}, \bar{\tau}) = 0\tag{5.8}$$

which completes the proof. $\square$

5.3. Relativistic Effects on the Time-Domain Waveguide Modes

5.3.1. *TE* time-domain waveguide modes

The field components of *TE* modes in the rest frame, F , have been defined in (3.14). To rewrite (3.14) in a suitable form for the ease of the analysis, we now introduce auxiliary functions for the modal amplitudes of $\mathbf{E}'_n(\mathbf{r}, z, t)$, and $\mathbf{H}'_n(\mathbf{r}, z, t)$ as follows

$$\begin{aligned}\mathcal{A}_n^h(z, t) &= [-\partial_t h_n(z, t)], \\ \mathcal{B}_n^h(z, t) &= [\partial_z h_n(z, t)].\end{aligned}\tag{5.9}$$

After providing (5.9) for the modal amplitudes, the field in the rest frame F , defined in (3.14), can be reformulated as

$$\begin{aligned}
E'_{zn}(\mathbf{r}, z, t) &= 0, \\
\mathbf{E}'_n(\mathbf{r}, z, t) &= \mathcal{A}_n^h(z, t) \mu_0 \nabla_{\perp} \psi_n(\mathbf{r}) \times \mathbf{z}, \\
\mathbf{H}'_n(\mathbf{r}, z, t) &= \mathcal{B}_n^h(z, t) \nabla_{\perp} \psi_n(\mathbf{r}), \\
H'_{zn}(\mathbf{r}, z, t) &= [v_n h_n(z, t)] v_n \psi_n(\mathbf{r}).
\end{aligned} \tag{5.10}$$

In order to find the field in the frame $\bar{F}$, we will make use of the following Lorentz transformation for the field components [Van Bladel, 2007]

$$\begin{aligned}
\bar{E}_{\bar{z}}(\bar{\mathbf{r}}, \bar{z}, \bar{t}) &= E_z(\mathbf{r}, z, t), & (a) \\
\bar{\mathbf{E}}(\bar{\mathbf{r}}, \bar{z}, \bar{t}) &= \bar{\gamma}[\mathbf{E}(\mathbf{r}, z, t) + \mu_0 \mathbf{v} \times \mathbf{H}(\mathbf{r}, z, t)], & (b) \\
\mu_0 \bar{\mathbf{H}}(\bar{\mathbf{r}}, \bar{z}, \bar{t}) &= \bar{\gamma} \left[\mu_0 \mathbf{H}(\mathbf{r}, z, t) - \frac{1}{c^2} \mathbf{v} \times \mathbf{E}(\mathbf{r}, z, t) \right], & (c) \\
\bar{H}_{\bar{z}}(\bar{\mathbf{r}}, \bar{z}, \bar{t}) &= H_z(\mathbf{r}, z, t). & (d)
\end{aligned} \tag{5.11}$$

Hereinafter, the transverse coordinates $\bar{\mathbf{r}}$ of the frame $\bar{F}$ will be taken as $\mathbf{r}$ due to the first line of (5.3), and the unit vector $\bar{\mathbf{z}}$ along $\bar{O}\bar{z}$ axis will be taken as $\mathbf{z}$, since $\bar{\mathbf{z}} \parallel \mathbf{z}$. For the sake of clarity in the presentation, we study the field components in the frame $\bar{F}$ independently.

Since the TE time-domain waveguide modes are under study, the z – component of electric field in $\bar{F}$ can be given by

$$\bar{E}_{\bar{z}}(\mathbf{r}, \bar{z}, \bar{t}) = 0. \tag{5.12}$$

After substituting $\mathbf{E}'_n(\mathbf{r}, z, t)$ and $\mathbf{H}'_n(\mathbf{r}, z, t)$ defined in (5.10) into the right-hand side of the line (b) of (5.11), and also the term $\mathbf{z}v_z$ for the velocity vector $\mathbf{v}$, the transverse component of electric field in $\bar{F}$ can be obtained as

$$\bar{\mathbf{E}}'_n(\mathbf{r}, \bar{z}, \bar{t}) = \bar{\gamma} [\mathcal{A}_n^h(z, t) - \beta c \mathcal{B}_n^h(z, t)] \Big|_{z(\bar{z}, \bar{t}); t(\bar{z}, \bar{t})} \mu_0 \nabla_{\perp} \psi_n(\mathbf{r}) \times \mathbf{z} \tag{5.13}$$

where the functions $z(\bar{z}, \bar{t})$ and $t(\bar{z}, \bar{t})$, the second line of the direct Lorentz transformation defined in (5.2), should be substituted into the arguments of the

modal amplitudes in the right-hand side of (5.13).

To calculate the transverse components of magnetic field in $\bar{F}$, we substitute $\mathbf{E}'_n(\mathbf{r}, z, t)$ and $\mathbf{H}'_n(\mathbf{r}, z, t)$ defined in (5.10) into the right-hand side of the line (c) of (5.11), the term $\mathbf{z}v_z$ for the velocity vector $\mathbf{v}$, then eliminate μ_0 from both side of the equation, and make use of the triple product expansion for $\mathbf{z} \times [\nabla_{\perp} \psi_n(\mathbf{r}) \times \mathbf{z}] = \nabla_{\perp} \psi_n(\mathbf{r})$ to obtain

$$\bar{\mathbf{H}}'_n(\mathbf{r}, \bar{z}, \bar{t}) = \bar{\gamma} \left[\mathcal{B}_n^h(z, t) - \frac{\beta}{c} \mathcal{A}_n^h(z, t) \right] \Big|_{z(\bar{z}, \bar{t}); t(\bar{z}, \bar{t})} \nabla_{\perp} \psi_n(\mathbf{r}) \quad (5.14)$$

where the functions $z(\bar{z}, \bar{t})$ and $t(\bar{z}, \bar{t})$, the second line of the direct Lorentz transformation defined in (5.2), should be substituted into the arguments of the modal amplitudes in the right-hand side of (5.14).

After substituting $H'_{zn}(\mathbf{r}, z, t)$ defined in (5.10) into the right-hand side of the line (d) of (5.11), the z -component of magnetic field in the frame $\bar{F}$ can be obtained as

$$\bar{H}'_{zn}(\mathbf{r}, \bar{z}, \bar{t}) = [v_n h_n(z, t)] \Big|_{z(\bar{z}, \bar{t}); t(\bar{z}, \bar{t})} v_n \psi_n(\mathbf{r}) \quad (5.15)$$

where the second line of the direct Lorentz transformation defined in (5.2) should be substituted into the argument of the function $h_n(z, t)$ thereafter. To sum up the field components of TE time-domain waveguide modes in $\bar{F}$, (5.12), (5.13), (5.14) and (5.15) can be grouped as follows

$$\begin{aligned} \bar{E}'_{zn}(\mathbf{r}, \bar{z}, \bar{t}) &= 0 \\ \bar{\mathbf{E}}'_n(\mathbf{r}, \bar{z}, \bar{t}) &= \bar{\gamma} [\mathcal{A}_n^h(z, t) - \beta c \mathcal{B}_n^h(z, t)] \Big|_{z(\bar{z}, \bar{t}); t(\bar{z}, \bar{t})} \mu_0 \nabla_{\perp} \psi_n(\mathbf{r}) \times \mathbf{z} \\ \bar{\mathbf{H}}'_n(\mathbf{r}, \bar{z}, \bar{t}) &= \bar{\gamma} \left[\mathcal{B}_n^h(z, t) - \frac{\beta}{c} \mathcal{A}_n^h(z, t) \right] \Big|_{z(\bar{z}, \bar{t}); t(\bar{z}, \bar{t})} \nabla_{\perp} \psi_n(\mathbf{r}) \\ \bar{H}'_{zn}(\mathbf{r}, \bar{z}, \bar{t}) &= [v_n h_n(z, t)] \Big|_{z(\bar{z}, \bar{t}); t(\bar{z}, \bar{t})} v_n \psi_n(\mathbf{r}) \end{aligned} \quad (5.16)$$

where the functions $z(\bar{z}, \bar{t})$ and $t(\bar{z}, \bar{t})$, the second line of the direct Lorentz transformation defined in (5.2), should be substituted into the arguments of the modal amplitudes in the right-hand side of (5.16). Comparing (5.16) with (5.10), we can state that the modal bases, which are the vector functions of transverse coordinates, $(\mathbf{r})$, are *absolute* for the frames F and $\bar{F}$. In other words, due to $\bar{\mathbf{r}} = \mathbf{r}$, the eigenvalues ν_n^2 of the problem (3.10) as well as the corresponding eigensolutions $\psi_n(\mathbf{r})$ remain unchanged in the frame $\bar{F}$. Consequently, the waveguide modal basis is independent of the relative motion of the reference frames. From (5.16), we can also conclude that the modal amplitudes, which are the functions of $(\bar{z}, \bar{t})$, are *not* absolute for both frames.

To make a similar presentation as in (5.10), we introduce the following modal amplitudes

$$\begin{aligned}\bar{\mathcal{A}}_n^h(\bar{z}, \bar{t}) &= \bar{\gamma} [\mathcal{A}_n^h(z, t) - \beta c \mathcal{B}_n^h(z, t)] \Big|_{z(\bar{z}, \bar{t}); t(\bar{z}, \bar{t})} \\ \bar{\mathcal{B}}_n^h(\bar{z}, \bar{t}) &= \bar{\gamma} \left[\mathcal{B}_n^h(z, t) - \frac{\beta}{c} \mathcal{A}_n^h(z, t) \right] \Big|_{z(\bar{z}, \bar{t}); t(\bar{z}, \bar{t})}\end{aligned}\quad (5.17)$$

where the functions $z(\bar{z}, \bar{t})$ and $t(\bar{z}, \bar{t})$, the second line of the direct Lorentz transformation defined in (5.2), should be substituted into the arguments of the modal amplitudes in the right-hand side of (5.17). After utilizing (5.17) for the modal amplitudes, the field components of TE time-domain waveguide modes in $\bar{F}$ given in (5.16) can be rewritten as

$$\begin{aligned}\bar{E}'_{zn}(\mathbf{r}, \bar{z}, \bar{t}) &= 0, \\ \bar{\mathbf{E}}'_n(\mathbf{r}, \bar{z}, \bar{t}) &= \bar{\mathcal{A}}_n^h(\bar{z}, \bar{t}) \mu_0 \nabla_{\perp} \psi_n(\mathbf{r}) \times \mathbf{z}, \\ \bar{\mathbf{H}}'_n(\mathbf{r}, \bar{z}, \bar{t}) &= \bar{\mathcal{B}}_n^h(\bar{z}, \bar{t}) \nabla_{\perp} \psi_n(\mathbf{r}), \\ \bar{H}'_{zn}(\mathbf{r}, \bar{z}, \bar{t}) &= [v_n h_n(\bar{z}, \bar{t})] v_n \psi_n(\mathbf{r}).\end{aligned}\quad (5.18)$$

To illustrate the change in the modal amplitudes, the following example is considered.

Example 5.1: In case of the time-harmonic fields propagating along the Oz axis, the solution to the KGE of (3.13) can be given by

$$h_n(z, t) = a_n \sin(\omega t - \alpha_n z + \varphi) \quad (5.19)$$

where the parameters a_n , φ are real constants, and $\alpha_n = \sqrt{\omega^2 \varepsilon_0 \mu_0 - \nu_n^2}$ with ω being a given frequency, and ν_n^2 previously defined in (3.19). Without loss of generality, we may assume that $a_n = 1$ and $\varphi = 0$. After substituting (5.19) into (5.9), the modal amplitudes of $\mathbf{E}'_n(\mathbf{r}, z, t)$, and $\mathbf{H}'_n(\mathbf{r}, z, t)$ in the frame F can be obtained respectively as follows

$$\begin{aligned} \mathcal{A}_n^h(z, t) &= [-\omega \cos(\omega t - \alpha_n z)], \\ \mathcal{B}_n^h(z, t) &= [-\alpha_n \cos(\omega t - \alpha_n z)]. \end{aligned} \quad (5.20)$$

Substitution of (5.19), and (5.20) into (5.10) yields the following field components of TE modes in F

$$\begin{aligned} E'_{zn}(\mathbf{r}, z, t) &= 0, \\ \mathbf{E}'_n(\mathbf{r}, z, t) &= [-\omega \cos(\omega t - \alpha_n z)] \mu_0 \nabla_{\perp} \psi_n(\mathbf{r}) \times \mathbf{z}, \\ \mathbf{H}'_n(\mathbf{r}, z, t) &= [-\alpha_n \cos(\omega t - \alpha_n z)] \nabla_{\perp} \psi_n(\mathbf{r}), \\ H'_{zn}(\mathbf{r}, z, t) &= [v_n \sin(\omega t - \alpha_n z)] v_n \psi_n(\mathbf{r}). \end{aligned} \quad (5.21)$$

In order to find the field components of TE modes in $\bar{F}$, we substitute (5.19) and (5.20) into (5.16) to obtain

$$\begin{aligned} \bar{E}'_{zn}(\mathbf{r}, \bar{z}, \bar{t}) &= 0, \\ \bar{\mathbf{E}}'_n(\mathbf{r}, \bar{z}, \bar{t}) &= \{-\bar{\gamma}(\omega - \beta c \alpha_n) \cos[\omega t(\bar{z}, \bar{t}) - \alpha_n z(\bar{z}, \bar{t})]\} \mu_0 \nabla_{\perp} \psi_n(\mathbf{r}) \times \mathbf{z}, \\ \bar{\mathbf{H}}'_n(\mathbf{r}, \bar{z}, \bar{t}) &= \left\{ -\bar{\gamma} \left(\alpha_n - \frac{\beta}{c} \omega \right) \cos[\omega t(\bar{z}, \bar{t}) - \alpha_n z(\bar{z}, \bar{t})] \right\} \nabla_{\perp} \psi_n(\mathbf{r}), \\ \bar{H}'_{zn}(\mathbf{r}, \bar{z}, \bar{t}) &= \{v_n \sin[\omega t(\bar{z}, \bar{t}) - \alpha_n z(\bar{z}, \bar{t})]\} v_n \psi_n(\mathbf{r}). \end{aligned} \quad (5.22)$$

After utilizing the second line of (5.2) for the functions $z(\bar{z}, \bar{t})$ and $t(\bar{z}, \bar{t})$ in the

right-hand side of (5.22), the components of the time-harmonic fields for the TE modes in the frame $\bar{F}$ can be obtained as

$$\begin{aligned}
 \bar{E}'_{zn}(\mathbf{r}, \bar{z}, \bar{t}) &= 0, \\
 \bar{\mathbf{E}}'_n(\mathbf{r}, \bar{z}, \bar{t}) &= \left\{ -\bar{\gamma}(\omega - \beta c \alpha_n) \cos \left[\bar{\gamma}(\omega - \beta c \alpha_n) \bar{t} - \bar{\gamma} \left(\alpha_n - \frac{\beta}{c} \omega \right) \bar{z} \right] \right\} \mu_0 \nabla_{\perp} \psi_n(\mathbf{r}) \times \mathbf{z}, \\
 \bar{\mathbf{H}}'_n(\mathbf{r}, \bar{z}, \bar{t}) &= \left\{ -\bar{\gamma} \left(\alpha_n - \frac{\beta}{c} \omega \right) \cos \left[\bar{\gamma}(\omega - \beta c \alpha_n) \bar{t} - \bar{\gamma} \left(\alpha_n - \frac{\beta}{c} \omega \right) \bar{z} \right] \right\} \nabla_{\perp} \psi_n(\mathbf{r}), \\
 \bar{H}'_{zn}(\mathbf{r}, \bar{z}, \bar{t}) &= \left\{ v_n \sin \left[\bar{\gamma}(\omega - \beta c \alpha_n) \bar{t} - \bar{\gamma} \left(\alpha_n - \frac{\beta}{c} \omega \right) \bar{z} \right] \right\} v_n \psi_n(\mathbf{r}).
 \end{aligned} \tag{5.23}$$

To make a similar presentation as in (5.21), we introduce two new auxiliary parameters

$$\begin{aligned}
 \bar{\omega} &= \bar{\gamma}(\omega - \beta c \alpha_n) \\
 \bar{\alpha}_n &= \bar{\gamma} \left(\alpha_n - \frac{\beta}{c} \omega \right)
 \end{aligned} \tag{5.24}$$

where $\bar{\omega}$ is the angular frequency, and $\bar{\alpha}_n$ is the angular wavenumber both in $\bar{F}$. Notice that when the velocity of the moving frame $\bar{F}$ is zero, or mathematically, when $\beta = 0$, $\bar{\omega}$ and $\bar{\alpha}_n$ reduce to ω and α_n , respectively. Substitution of (5.24) back into (5.23) yields the components of the time-harmonic fields for the TE modes in the frame $\bar{F}$ as follows

$$\begin{aligned}
 \bar{E}'_{zn}(\mathbf{r}, \bar{z}, \bar{t}) &= 0, \\
 \bar{\mathbf{E}}'_n(\mathbf{r}, \bar{z}, \bar{t}) &= [-\bar{\omega} \cos(\bar{\omega} \bar{t} - \bar{\alpha}_n \bar{z})] \mu_0 \nabla_{\perp} \psi_n(\mathbf{r}) \times \mathbf{z}, \\
 \bar{\mathbf{H}}'_n(\mathbf{r}, \bar{z}, \bar{t}) &= [-\bar{\alpha}_n \cos(\bar{\omega} \bar{t} - \bar{\alpha}_n \bar{z})] \nabla_{\perp} \psi_n(\mathbf{r}), \\
 \bar{H}'_{zn}(\mathbf{r}, \bar{z}, \bar{t}) &= [v_n \sin(\bar{\omega} \bar{t} - \bar{\alpha}_n \bar{z})] v_n \psi_n(\mathbf{r}).
 \end{aligned} \tag{5.25}$$

Recalling that $\bar{\omega} \bar{t} - \bar{\alpha}_n \bar{z} = \text{const.}$, taking the derivative of the latter with respect to $\bar{t}$, and making use of (5.24), the phase velocity in $\bar{F}$, $\bar{v}_{ph}$, can be obtained as

$$\bar{v}_{ph} = \frac{d\bar{z}}{d\bar{t}} = \frac{(\omega - \beta c \alpha_n)}{\left(\alpha_n - \frac{\beta}{c} \omega \right)}. \quad (5.26)$$

After dividing each term in (5.26) by α_n , and utilizing $v_{ph} = \omega / \alpha_n$, the following relation between the phase velocities in the frames F and $\bar{F}$ can be given by

$$\bar{v}_{ph} = c \frac{(v_{ph} - \beta c)}{(c - \beta v_{ph})}. \quad (5.27)$$

Based on (5.27), when $\beta = 0$, which is the case that the velocity of $\bar{F}$ equals to zero, then $\bar{v}_{ph} = v_{ph}$.

5.3.2. TM time-domain waveguide modes

The field components of TM modes in the rest frame, F , have been defined in (3.22). To rewrite (3.22) in a suitable form for the relativistic analysis, we introduce auxiliary functions for the modal amplitudes of $\mathbf{H}_n''(\mathbf{r}, z, t)$ and $\mathbf{E}_n''(\mathbf{r}, z, t)$ as follows

$$\begin{aligned} \mathcal{A}_n^e(z, t) &= [-\partial_t e_n(z, t)], \\ \mathcal{B}_n^e(z, t) &= [\partial_z e_n(z, t)]. \end{aligned} \quad (5.28)$$

After providing (5.28) for the modal amplitudes, the field in F , defined in (3.22), can be reformulated as

$$\begin{aligned} H_{zn}''(\mathbf{r}, z, t) &= 0 \\ \mathbf{H}_n''(\mathbf{r}, z, t) &= \mathcal{A}_n^e(z, t) \mathbf{z} \times \varepsilon_0 \nabla_{\perp} \phi_n(\mathbf{r}) \\ \mathbf{E}_n''(\mathbf{r}, z, t) &= \mathcal{B}_n^e(z, t) \nabla_{\perp} \phi_n(\mathbf{r}) \\ E_{zn}''(\mathbf{r}, z, t) &= [\kappa_n e_n(z, t)] \kappa_n \phi_n(\mathbf{r}) \end{aligned} \quad (5.29)$$

In order to find the field in the frame $\bar{F}$, we will make use of the Lorentz transformation for the field components given in (5.11). After utilizing (5.29) for the

field components in F at the right-hand side of (5.11), substituting the term $\mathbf{z}v_z$ for the velocity vector $\mathbf{v}$, and making use of the triple product expansion for $\mathbf{z} \times [\mathbf{z} \times \nabla_{\perp} \phi_n(\mathbf{r})] = -\nabla_{\perp} \phi_n(\mathbf{r})$, the TM waveguide modes in $\bar{F}$ can be obtained as follows

$$\begin{aligned}
\bar{H}_{zn}''(\mathbf{r}, \bar{z}, \bar{t}) &= 0 \\
\bar{\mathbf{H}}_n''(\mathbf{r}, \bar{z}, \bar{t}) &= \bar{\gamma} [\mathcal{A}_n^e(z, t) - \beta c \mathcal{B}_n^e(z, t)] \Big|_{z(\bar{z}, \bar{t}); t(\bar{z}, \bar{t})} \mathbf{z} \times \varepsilon_0 \nabla_{\perp} \phi_n(\mathbf{r}) \\
\bar{\mathbf{E}}_n''(\mathbf{r}, \bar{z}, \bar{t}) &= \bar{\gamma} \left[\mathcal{B}_n^e(z, t) - \frac{\beta}{c} \mathcal{A}_n^e(z, t) \right] \Big|_{z(\bar{z}, \bar{t}); t(\bar{z}, \bar{t})} \nabla_{\perp} \phi_n(\mathbf{r}) \\
\bar{E}_{zn}''(\mathbf{r}, \bar{z}, \bar{t}) &= [\kappa_n e_n(z, t)] \Big|_{z(\bar{z}, \bar{t}); t(\bar{z}, \bar{t})} \kappa_n \phi_n(\mathbf{r})
\end{aligned} \tag{5.30}$$

where the functions $z(\bar{z}, \bar{t})$ and $t(\bar{z}, \bar{t})$, the second line of the direct Lorentz transformation defined in (5.2), should be substituted into the arguments of the modal amplitudes in the right-hand side of (5.30). Comparing (5.30) with (5.29), we can state that the modal bases, which are the vector functions of transverse coordinates, $(\mathbf{r})$, are *absolute* for the frames F and $\bar{F}$. In other words, due to $\bar{\mathbf{r}} = \mathbf{r}$, the eigenvalues κ_n^2 of the problem (3.21) as well as the corresponding eigensolutions $\phi_n(\mathbf{r})$ remain unchanged in the frame $\bar{F}$. Consequently, the waveguide modal basis is independent of the relative motion of the reference frames. From (5.30), we can also conclude that the modal amplitudes, which are the functions of $(\bar{z}, \bar{t})$, are *not* absolute for both frames.

To make a similar presentation as in (5.29), we introduce the following modal amplitudes

$$\begin{aligned}
\bar{\mathcal{A}}_n^e(\bar{z}, \bar{t}) &= \bar{\gamma} [\mathcal{A}_n^e(z, t) - \beta c \mathcal{B}_n^e(z, t)] \Big|_{z(\bar{z}, \bar{t}); t(\bar{z}, \bar{t})} \\
\bar{\mathcal{B}}_n^e(\bar{z}, \bar{t}) &= \bar{\gamma} \left[\mathcal{B}_n^e(z, t) - \frac{\beta}{c} \mathcal{A}_n^e(z, t) \right] \Big|_{z(\bar{z}, \bar{t}); t(\bar{z}, \bar{t})}
\end{aligned} \tag{5.31}$$

where the second line of the direct Lorentz transformation of (5.2) should be substituted into the arguments of modal amplitudes in the right-hand side of (5.31).

After utilizing (5.31) for the modal amplitudes, the field components of TM time-domain waveguide modes in $\bar{F}$ given in (5.30) can be rewritten as

$$\begin{aligned}\bar{H}_{zn}''(\mathbf{r}, \bar{z}, \bar{t}) &= 0, \\ \bar{\mathbf{H}}_n''(\mathbf{r}, \bar{z}, \bar{t}) &= \bar{\mathcal{A}}_n^e(\bar{z}, \bar{t}) \mathbf{z} \times \varepsilon_0 \nabla_{\perp} \phi_n(\mathbf{r}), \\ \bar{\mathbf{E}}_n''(\mathbf{r}, \bar{z}, \bar{t}) &= \bar{\mathcal{B}}_n^e(\bar{z}, \bar{t}) \nabla_{\perp} \phi_n(\mathbf{r}), \\ \bar{E}_{zn}''(\mathbf{r}, \bar{z}, \bar{t}) &= [\kappa_n e_n(\bar{z}, \bar{t})] \kappa_n \phi_n(\mathbf{r}).\end{aligned}\tag{5.32}$$

To illustrate the change in the modal amplitudes, the following example is considered.

Example 5.2: In case of the time-harmonic fields propagating along the Oz axis, the solution to the KGE of (3.23) can be given by

$$e_n(z, t) = a_n \sin(\omega t - \alpha_n z + \varphi) \tag{5.33}$$

where the parameters a_n , φ are real constants, and $\alpha_n = \sqrt{\omega^2 \varepsilon_0 \mu_0 - \kappa_n^2}$ with ω being a given frequency, and κ_n^2 previously defined in (3.26). Without loss of generality, we may assume that $a_n = 1$ and $\varphi = 0$. After substituting (5.33) into (5.28), the modal amplitudes of $\mathbf{H}_n''(\mathbf{r}, z, t)$ and $\mathbf{E}_n''(\mathbf{r}, z, t)$ in the frame F can be obtained respectively as follows

$$\begin{aligned}\mathcal{A}_n^e(z, t) &= [-\omega \cos(\omega t - \alpha_n z)], \\ \mathcal{B}_n^e(z, t) &= [-\alpha_n \cos(\omega t - \alpha_n z)].\end{aligned}\tag{5.34}$$

Substitution of (5.33) and (5.34) into (5.29) yields the following field components of TM modes in F

$$\begin{aligned}H_{zn}''(\mathbf{r}, z, t) &= 0, \\ \mathbf{H}_n''(\mathbf{r}, z, t) &= [-\omega \cos(\omega t - \alpha_n z)] \mathbf{z} \times \varepsilon_0 \nabla_{\perp} \phi_n(\mathbf{r}), \\ \mathbf{E}_n''(\mathbf{r}, z, t) &= [-\alpha_n \cos(\omega t - \alpha_n z)] \nabla_{\perp} \phi_n(\mathbf{r}), \\ E_{zn}''(\mathbf{r}, z, t) &= [\kappa_n \sin(\omega t - \alpha_n z)] \kappa_n \phi_n(\mathbf{r}).\end{aligned}\tag{5.35}$$

In order to find the field components of *TM* modes in $\bar{F}$, we substitute (5.33) and (5.34) into (5.30) to obtain

$$\begin{aligned}\bar{H}_{zn}''(\mathbf{r}, \bar{z}, \bar{t}) &= 0, \\ \bar{\mathbf{H}}_n''(\mathbf{r}, \bar{z}, \bar{t}) &= \{-\bar{\gamma}(\omega - \beta c \alpha_n) \cos[\omega t(\bar{z}, \bar{t}) - \alpha_n z(\bar{z}, \bar{t})]\} \mathbf{z} \times \varepsilon_0 \nabla_{\perp} \phi_n(\mathbf{r}), \\ \bar{\mathbf{E}}_n''(\mathbf{r}, \bar{z}, \bar{t}) &= \left\{ -\bar{\gamma} \left(\alpha_n - \frac{\beta}{c} \omega \right) \cos[\omega t(\bar{z}, \bar{t}) - \alpha_n z(\bar{z}, \bar{t})] \right\} \nabla_{\perp} \phi_n(\mathbf{r}), \\ \bar{E}_{zn}''(\mathbf{r}, \bar{z}, \bar{t}) &= \{\kappa_n \sin[\omega t(\bar{z}, \bar{t}) - \alpha_n z(\bar{z}, \bar{t})]\} \kappa_n \phi_n(\mathbf{r}).\end{aligned}\tag{5.36}$$

After utilizing the second line of (5.2) for the functions $z(\bar{z}, \bar{t})$ and $t(\bar{z}, \bar{t})$ in the right-hand side of (5.36), the components of the time-harmonic fields for the *TM* modes in the frame $\bar{F}$ can be obtained as

$$\begin{aligned}\bar{H}_{zn}''(\mathbf{r}, \bar{z}, \bar{t}) &= 0, \\ \bar{\mathbf{H}}_n''(\mathbf{r}, \bar{z}, \bar{t}) &= \left\{ -\bar{\gamma}(\omega - \beta c \alpha_n) \cos \left[\bar{\gamma}(\omega - \beta c \alpha_n) \bar{t} - \bar{\gamma} \left(\alpha_n - \frac{\beta}{c} \omega \right) \bar{z} \right] \right\} \mathbf{z} \times \varepsilon_0 \nabla_{\perp} \phi_n(\mathbf{r}), \\ \bar{\mathbf{E}}_n''(\mathbf{r}, \bar{z}, \bar{t}) &= \left\{ -\bar{\gamma} \left(\alpha_n - \frac{\beta}{c} \omega \right) \cos \left[\bar{\gamma}(\omega - \beta c \alpha_n) \bar{t} - \bar{\gamma} \left(\alpha_n - \frac{\beta}{c} \omega \right) \bar{z} \right] \right\} \nabla_{\perp} \phi_n(\mathbf{r}), \\ \bar{E}_{zn}''(\mathbf{r}, \bar{z}, \bar{t}) &= \left\{ \kappa_n \sin \left[\bar{\gamma}(\omega - \beta c \alpha_n) \bar{t} - \bar{\gamma} \left(\alpha_n - \frac{\beta}{c} \omega \right) \bar{z} \right] \right\} \kappa_n \phi_n(\mathbf{r}).\end{aligned}\tag{5.37}$$

To make a similar presentation as in (5.35), we utilize $\bar{\omega}$ and $\bar{\alpha}_n$ defined in (5.24) for reformulating (5.37) as follows

$$\begin{aligned}\bar{H}_{zn}''(\mathbf{r}, \bar{z}, \bar{t}) &= 0 \\ \bar{\mathbf{H}}_n''(\mathbf{r}, \bar{z}, \bar{t}) &= \{-\bar{\omega} \cos(\bar{\omega} \bar{t} - \bar{\alpha}_n \bar{z})\} \mathbf{z} \times \varepsilon_0 \nabla_{\perp} \phi_n(\mathbf{r}) \\ \bar{\mathbf{E}}_n''(\mathbf{r}, \bar{z}, \bar{t}) &= \{-\bar{\alpha}_n \cos(\bar{\omega} \bar{t} - \bar{\alpha}_n \bar{z})\} \nabla_{\perp} \phi_n(\mathbf{r}) \\ \bar{E}_{zn}''(\mathbf{r}, \bar{z}, \bar{t}) &= \{\kappa_n \sin(\bar{\omega} \bar{t} - \bar{\alpha}_n \bar{z})\} \kappa_n \phi_n(\mathbf{r})\end{aligned}\tag{5.38}$$

which yields the *TM* waveguide modes for the time-harmonic fields in the frame $\bar{F}$.

5.4. Relativistic Waveguide Modes via the Dimensionless Modal Amplitudes

5.4.1. *TE* time-domain waveguide modes

The field components of *TE* modes by means of the dimensionless modal amplitudes in the rest frame, F , were defined in (3.37). In order to find the field in the frame $\bar{F}$, we will make use of the Lorentz transformation for the field components given in (5.11). After utilizing (3.37) for the field components in F at the right-hand side of (5.11), substituting the term $\mathbf{z}v_z$ for the velocity vector $\mathbf{v}$, and making use of the triple product expansion for $\mathbf{z} \times [\nabla_{\perp} \psi_n(\mathbf{r}) \times \mathbf{z}] = \nabla_{\perp} \psi_n(\mathbf{r})$, the *TE* waveguide modes by means of the dimensionless modal amplitudes in $\bar{F}$ can be obtained as follows

$$\begin{aligned} \bar{E}'_{zn}(\mathbf{r}, \bar{\xi}, \bar{\tau}) &= 0 \\ v_n^{-1} \sqrt{\varepsilon_0 / \mu_0} \bar{\mathbf{E}}'_n(\mathbf{r}, \bar{\xi}, \bar{\tau}) &= \bar{\gamma} [\mathcal{A}(\xi, \tau) - \beta \mathcal{B}(\xi, \tau)] \Big|_{\xi(\bar{\xi}, \bar{\tau}); \tau(\bar{\xi}, \bar{\tau})} \nabla_{\perp} \psi_n(\mathbf{r}) \times \mathbf{z} \\ v_n^{-1} \bar{\mathbf{H}}'_n(\mathbf{r}, \bar{\xi}, \bar{\tau}) &= \bar{\gamma} [\mathcal{B}(\xi, \tau) - \beta \mathcal{A}(\xi, \tau)] \Big|_{\xi(\bar{\xi}, \bar{\tau}); \tau(\bar{\xi}, \bar{\tau})} \nabla_{\perp} \psi_n(\mathbf{r}) \\ v_n^{-1} \bar{H}'_{zn}(\mathbf{r}, \bar{\xi}, \bar{\tau}) &= [f(\xi, \tau)] \Big|_{\xi(\bar{\xi}, \bar{\tau}); \tau(\bar{\xi}, \bar{\tau})} v_n \psi_n(\mathbf{r}) \end{aligned} \quad (5.39)$$

where the functions $\xi(\bar{\xi}, \bar{\tau})$ and $\tau(\bar{\xi}, \bar{\tau})$ has been defined in (5.4). To make a similar presentation as in (3.37), we introduce the following dimensionless modal amplitudes in $\bar{F}$

$$\begin{aligned} \bar{\mathcal{A}}(\bar{\xi}, \bar{\tau}) &= \bar{\gamma} [\mathcal{A}(\xi, \tau) - \beta \mathcal{B}(\xi, \tau)] \Big|_{\xi(\bar{\xi}, \bar{\tau}); \tau(\bar{\xi}, \bar{\tau})}, \quad (a) \\ \bar{\mathcal{B}}(\bar{\xi}, \bar{\tau}) &= \bar{\gamma} [\mathcal{B}(\xi, \tau) - \beta \mathcal{A}(\xi, \tau)] \Big|_{\xi(\bar{\xi}, \bar{\tau}); \tau(\bar{\xi}, \bar{\tau})}, \quad (b) \\ \bar{f}(\bar{\xi}, \bar{\tau}) &= [f(\xi, \tau)] \Big|_{\xi(\bar{\xi}, \bar{\tau}); \tau(\bar{\xi}, \bar{\tau})}. \quad (c) \end{aligned} \quad (5.40)$$

Recall that the dimensionless modal amplitudes have been introduced first in section “3.3. General Remarks” to study the amplitudes of the field components for the *TE* and *TM* modes *in parallel*. Hence, (5.40) is also valid for the *TM* modes. After utilizing (5.40) for the modal amplitudes, the field components of *TE* time-domain

waveguide modes in $\bar{F}$ given in (5.39) can be rewritten as

$$\begin{aligned}
 \bar{E}'_{zn}(\mathbf{r}, \bar{\xi}, \bar{\tau}) &= 0, \\
 v_n^{-1} \sqrt{\varepsilon_0 / \mu_0} \bar{\mathbf{E}}'_n(\mathbf{r}, \bar{\xi}, \bar{\tau}) &= \bar{\mathcal{A}}(\bar{\xi}, \bar{\tau}) \nabla_{\perp} \psi_n(\mathbf{r}) \times \mathbf{z}, \\
 v_n^{-1} \bar{\mathbf{H}}'_n(\mathbf{r}, \bar{\xi}, \bar{\tau}) &= \bar{\mathcal{B}}(\bar{\xi}, \bar{\tau}) \nabla_{\perp} \psi_n(\mathbf{r}), \\
 v_n^{-1} \bar{H}'_{zn}(\mathbf{r}, \bar{\xi}, \bar{\tau}) &= \bar{f}(\bar{\xi}, \bar{\tau}) v_n \psi_n(\mathbf{r}).
 \end{aligned} \tag{5.41}$$

5.4.2. *TM* time-domain waveguide modes

In a fashion similar to the *TE* modes, the field components of *TM* time-domain waveguide modes by means of the dimensionless modal amplitudes in the frame $\bar{F}$ can be obtained as

$$\begin{aligned}
 \bar{H}''_{zn}(\mathbf{r}, \bar{\xi}, \bar{\tau}) &= 0 \\
 \kappa_n^{-1} \sqrt{\mu_0 / \varepsilon_0} \bar{\mathbf{H}}''_n(\mathbf{r}, \bar{\xi}, \bar{\tau}) &= \bar{\mathcal{A}}(\bar{\xi}, \bar{\tau}) \mathbf{z} \times \nabla_{\perp} \phi_n(\mathbf{r}) \\
 \kappa_n^{-1} \bar{\mathbf{E}}''_n(\mathbf{r}, \bar{\xi}, \bar{\tau}) &= \bar{\mathcal{B}}(\bar{\xi}, \bar{\tau}) \nabla_{\perp} \phi_n(\mathbf{r}) \\
 \kappa_n^{-1} \bar{E}''_{zn}(\mathbf{r}, \bar{\xi}, \bar{\tau}) &= \bar{f}(\bar{\xi}, \bar{\tau}) \kappa_n \phi_n(\mathbf{r})
 \end{aligned} \tag{5.42}$$

where the dimensionless modal amplitudes, which are the functions of $(\bar{\xi}, \bar{\tau})$, has been defined in (5.40).

5.4.3. Quadratic Characteristics of the Time-Domain Modes

The quadratic characteristics of the time-domain modes in the rest frame F were defined in (3.39). In the frame $\bar{F}$, these characteristics can be given by

$$\begin{aligned}
 k_n^{-2} \bar{W}(\bar{\xi}, \bar{\tau}) &= \frac{1}{2} (\bar{\mathcal{A}}^2 + \bar{\mathcal{B}}^2 + \bar{f}^2) \\
 k_n^{-2} \bar{P}_z(\bar{\xi}, \bar{\tau}) &= c \bar{\mathcal{A}} \bar{\mathcal{B}}
 \end{aligned} \tag{5.43}$$

where k_n^{-2} is v_n^{-2} for the *TE* modes and κ_n^{-2} for the *TM* modes, and the dimensionless modal amplitudes in the right-hand side of (5.43) has been defined in (5.40). As an example, the quadratic characteristics of the time-harmonic fields will

be calculated next by means of (5.43), which is valid for the time-domain modes.

Example 5.3: The solution, $f(\xi, \tau)$, to the KGE of (3.34) in F , which is the modal amplitude of the longitudinal field component, was defined in (3.41). In the frame $\bar{F}$, $\bar{f}(\bar{\xi}, \bar{\tau})$ can be obtained after substituting (3.41) into the line (c) of (5.40), and utilizing (5.4) for the functions $\xi(\bar{\xi}, \bar{\tau})$ and $\tau(\bar{\xi}, \bar{\tau})$ as follows

$$\bar{f}(\bar{\xi}, \bar{\tau}) = \sin[\bar{\gamma}(\eta - \beta\vartheta)\bar{\tau} - \bar{\gamma}(\vartheta - \beta\eta)\bar{\xi} + \varphi]. \quad (5.44)$$

Recalling that $\mathcal{A}(\xi, \tau) = -\partial_\tau f(\xi, \tau)$ and $\mathcal{B}(\xi, \tau) = \partial_\xi f(\xi, \tau)$, the modal amplitudes of transverse components of the field in the frame F are obtained as

$$\begin{aligned} \mathcal{A}(\xi, \tau) &= -\eta \cos(\eta\tau - \vartheta\xi + \varphi), \\ \mathcal{B}(\xi, \tau) &= -\vartheta \cos(\eta\tau - \vartheta\xi + \varphi). \end{aligned} \quad (5.45)$$

After substituting (5.45) into the lines (a) and (b) of (5.40), and utilizing (5.4) for the functions $\xi(\bar{\xi}, \bar{\tau})$ and $\tau(\bar{\xi}, \bar{\tau})$, we obtain the modal amplitudes of transverse components of the field in the frame $\bar{F}$ as

$$\begin{aligned} \bar{\mathcal{A}}(\bar{\xi}, \bar{\tau}) &= -\bar{\gamma}(\eta - \beta\vartheta) \cos[\bar{\gamma}(\eta - \beta\vartheta)\bar{\tau} - \bar{\gamma}(\vartheta - \beta\eta)\bar{\xi} + \varphi], \\ \bar{\mathcal{B}}(\bar{\xi}, \bar{\tau}) &= -\bar{\gamma}(\vartheta - \beta\eta) \cos[\bar{\gamma}(\eta - \beta\vartheta)\bar{\tau} - \bar{\gamma}(\vartheta - \beta\eta)\bar{\xi} + \varphi]. \end{aligned} \quad (5.46)$$

After substituting (5.44) and (5.46) into (5.43), the quadratic characteristics of the time-harmonic fields can be obtained as follows

$$\begin{aligned} \arg. &= \bar{\gamma}(\eta - \beta\vartheta)\bar{\tau} - \bar{\gamma}(\vartheta - \beta\eta)\bar{\xi} + \varphi \\ k_n^{-2} \bar{W}(\bar{\xi}, \bar{\tau}) &= \frac{1}{2} \left\{ \bar{\gamma}^2 [(\eta - \beta\vartheta)^2 + (\vartheta - \beta\eta)^2] \cos^2(\arg.) + \sin^2(\arg.) \right\} \\ k_n^{-2} \bar{P}_z(\bar{\xi}, \bar{\tau}) &= c \bar{\gamma}^2 (\eta - \beta\vartheta)(\vartheta - \beta\eta) \cos^2(\arg.) \end{aligned} \quad (5.47)$$

In Fig. 5.2 and 5.3, the energy density and the z -component of Poynting vector, both normalized by k_n^2 and given in (5.47), are illustrated for the different values of β . The quantity $\bar{P}_z(\bar{\xi}, \bar{\tau})$ is also normalized by the velocity of light, c . The figures are obtained, provided that $\eta = 1.2$, $\varphi = 0$, $\xi = 0$, and $\vartheta = \sqrt{\eta^2 - 1} = 0.6633$. These

parameters are the same as the ones used to obtain Fig. 3.2 exhibited in section “3.3. General Remarks.” Hence, Fig. 3.2 can be considered as the case in which $\bar{W}(\bar{\xi}, \bar{\tau})$ and $\bar{P}_z(\bar{\xi}, \bar{\tau})$ defined in (5.47) are obtained for $\beta = 0$, after replacing the horizontal axis τ with $\bar{\tau}$. In Fig. 5.3, $\bar{P}_z(\bar{\xi}, \bar{\tau})$ is negative valued due to the negativeness of the term $(\vartheta - \beta\eta)$ when $\beta = 0.9$. For the above values of η and ϑ , $(\vartheta - \beta\eta)$ is negative valued when $\beta > 0.5528$.

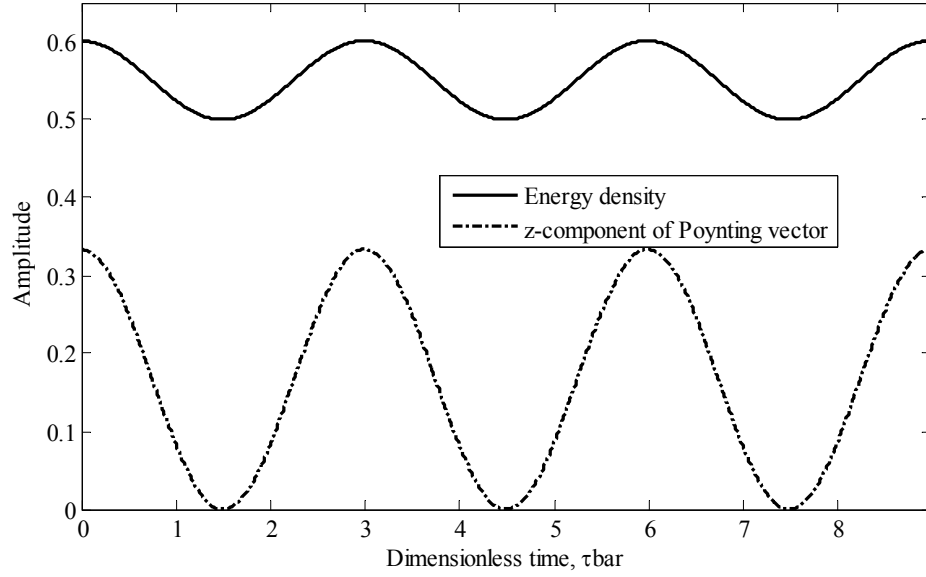


Figure 5.2. The energy density and the z -component of Poynting vector in $\bar{F}$ when $\beta = 0.3$.

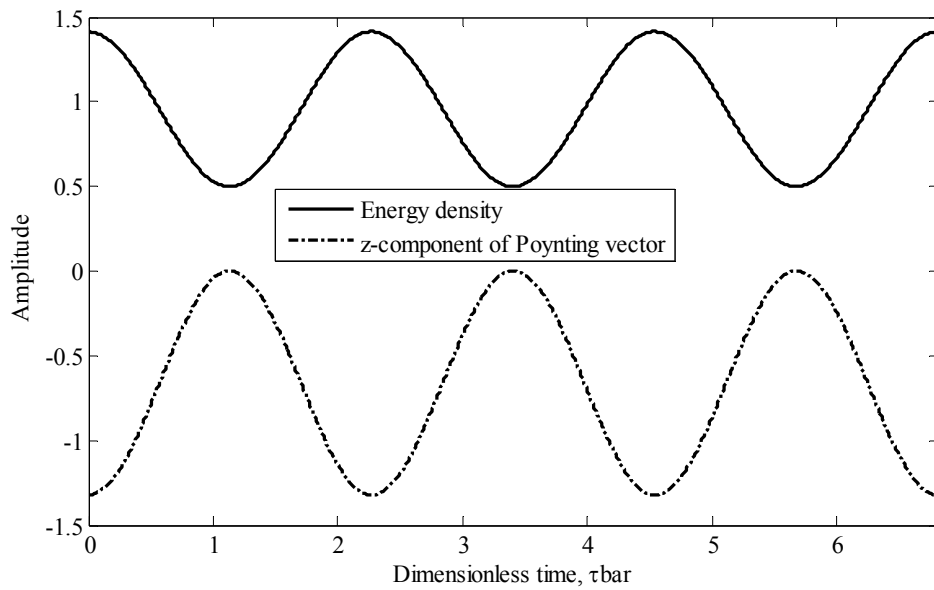


Figure 5.3. The energy density and the z -component of Poynting vector in $\bar{F}$ when $\beta = 0.9$.

By utilizing (5.44) and (5.46), we calculate the energy density stored in the transverse components of the field; $k_n^{-2}\bar{W}_{tr} = (\bar{\mathcal{A}}^2 + \bar{\mathcal{B}}^2) / 2$, the energy density stored in the longitudinal component of the field; $k_n^{-2}\bar{W}_z = \bar{f}^2 / 2$, and the difference of the energy densities specified by the modal amplitudes of the transverse components of the field; $k_n^{-2}\bar{W}_{diff} = (\bar{\mathcal{A}}^2 - \bar{\mathcal{B}}^2) / 2$, in the frame $\bar{F}$ as follows

$$\begin{aligned} \arg. &= \bar{\gamma}(\eta - \beta\vartheta)\bar{\tau} - \bar{\gamma}(\vartheta - \beta\eta)\bar{\xi} + \varphi, \\ k_n^{-2}\bar{W}_{tr}(\bar{\xi}, \bar{\tau}) &= \frac{\bar{\gamma}^2}{2}[(\eta - \beta\vartheta)^2 + (\vartheta - \beta\eta)^2]\cos^2(\arg.), \quad (a) \\ k_n^{-2}\bar{W}_z(\bar{\xi}, \bar{\tau}) &= \frac{1}{2}\sin^2(\arg.), \quad (b) \\ k_n^{-2}\bar{W}_{diff}(\bar{\xi}, \bar{\tau}) &= \frac{\bar{\gamma}^2}{2}[(\eta - \beta\vartheta)^2 - (\vartheta - \beta\eta)^2]\cos^2(\arg.). \quad (c) \end{aligned} \quad (5.48)$$

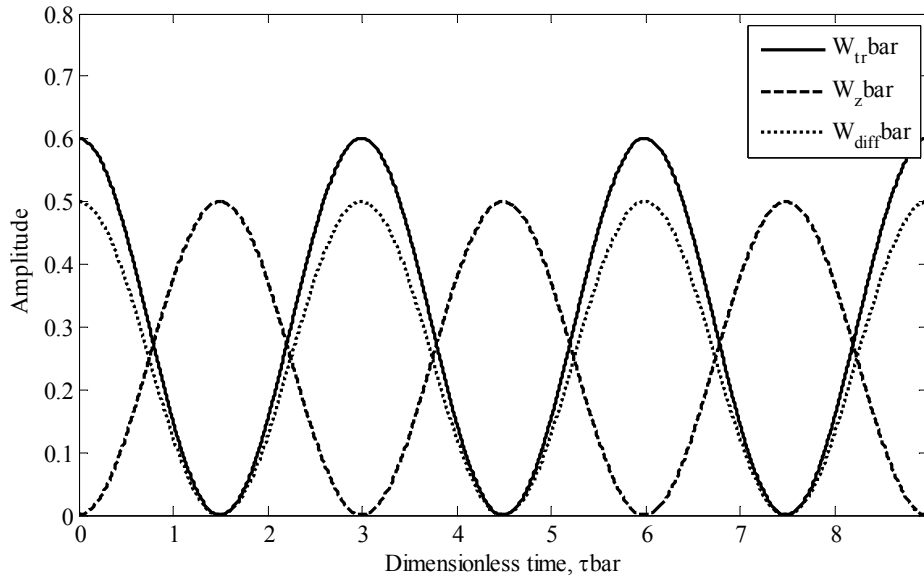


Figure 5.4. The energy densities in $\bar{F}$ when $\beta = 0.3$.

In Figures 5.4 and 5.5, $\bar{W}_{tr}(\bar{\xi}, \bar{\tau})$, $\bar{W}_z(\bar{\xi}, \bar{\tau})$, and $\bar{W}_{diff}(\bar{\xi}, \bar{\tau})$ both normalized by k_n^2 are illustrated for the different values of β . Notice that the peak-to-peak amplitude of $\bar{W}_{diff}(\bar{\xi}, \bar{\tau})$ is the same for the two figures, though the value of β is changed. This fact can be valid only when $\bar{\gamma}^2[(\eta - \beta\vartheta)^2 - (\vartheta - \beta\eta)^2] = 1$. The latter can be proved by substituting (5.44) into the KGE of (5.8). The parameters used to obtain

the figures ($\eta = 1.2$, $\varphi = 0$, $\xi = 0$, and $\vartheta = \sqrt{\eta^2 - 1} = 0.6633$) are the same as the ones in Fig. 3.3 exhibited in section “3.3. General Remarks.” Hence, Fig. 3.3 can be considered as the case in which the equations defined in (5.47) are obtained for $\beta = 0$, after replacing the horizontal axis τ with $\bar{\tau}$.

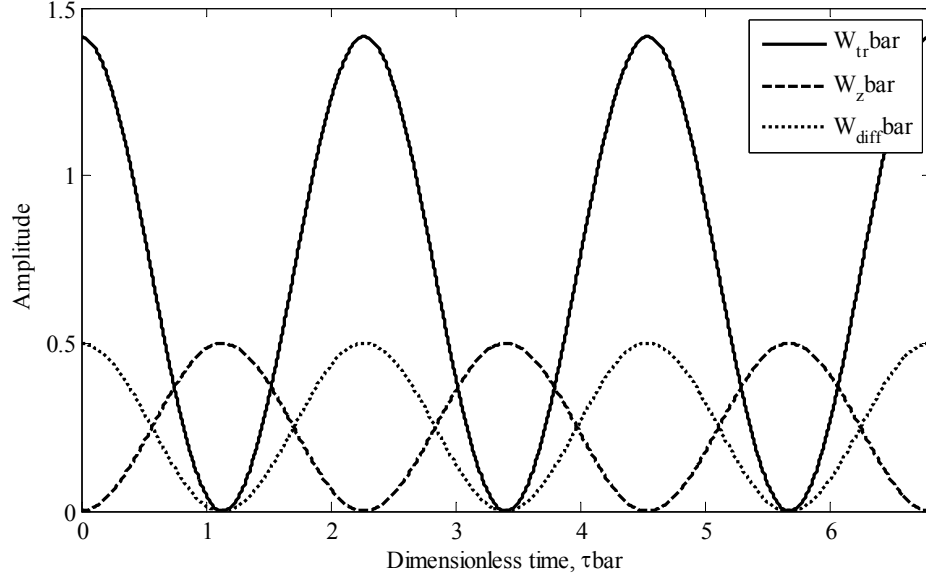


Figure 5.5. The energy densities in $\bar{F}$ when $\beta = 0.9$.

By utilizing (5.47), the dynamic velocity, $\bar{v}_{dyn} = \bar{P}_z(\bar{\xi}, \bar{\tau}) / \bar{W}(\bar{\xi}, \bar{\tau})$, of the wave energy transferred along the waveguide in the frame $\bar{F}$, can be calculated as

$$\begin{aligned} \arg. &= \bar{\gamma}(\eta - \beta\vartheta)\bar{\tau} - \bar{\gamma}(\vartheta - \beta\eta)\bar{\xi} + \varphi, \\ \bar{v}_{dyn} &= \frac{2c\bar{\gamma}^2(\eta - \beta\vartheta)(\vartheta - \beta\eta) \cos^2(\arg.)}{\bar{\gamma}^2[(\eta - \beta\vartheta)^2 + (\vartheta - \beta\eta)^2] \cos^2(\arg.) + \sin^2(\arg.)}. \end{aligned} \quad (5.49)$$

The calculations of $\bar{W}^{avg}$ (energy density averaged over the period $\bar{T}$ corresponding to the frequency $\bar{\omega}$) and $\bar{P}_z^{avg}$ (Poynting vector averaged in the same way) yield

$$\begin{aligned} k_n^{-2} \bar{W}^{avg}(\bar{\xi}, \bar{\tau}) &= k_n^{-2} \frac{1}{\bar{T}} \int_{\theta}^{\theta+\bar{T}} \bar{W}(\bar{\xi}, \bar{\tau}) d\bar{\tau} = \frac{1}{4} \bar{\gamma}^2 [(\eta - \beta\vartheta)^2 + (\vartheta - \beta\eta)^2] + \frac{1}{4} \\ k_n^{-2} \bar{P}_z^{avg}(\bar{\xi}, \bar{\tau}) &= k_n^{-2} \frac{1}{\bar{T}} \int_{\theta}^{\theta+\bar{T}} \bar{P}_z(\bar{\xi}, \bar{\tau}) d\bar{\tau} = \frac{1}{2} c\bar{\gamma}^2 (\eta - \beta\vartheta)(\vartheta - \beta\eta) \end{aligned} \quad (5.50)$$

where the coordinate $\bar{\xi}$ is arbitrary, θ is a constant, the period $\bar{T}$ in time $\bar{\tau}$ is equal to $2\pi / [\bar{\gamma}(\eta - \beta\vartheta)]$. The relation $\bar{P}_z^{avg} / \bar{W}^{avg}$, specifying the averaged velocity, $\bar{v}_{avg}$, of the wave energy transferred along the waveguide in the frame $\bar{F}$, can be calculated by making use of (5.50) as

$$\bar{v}_{avg} = \frac{c\bar{\gamma}^2(\eta - \beta\vartheta)(\vartheta - \beta\eta)}{\frac{1}{2}\bar{\gamma}^2[(\eta - \beta\vartheta)^2 + (\vartheta - \beta\eta)^2] + \frac{1}{2}}. \quad (5.51)$$

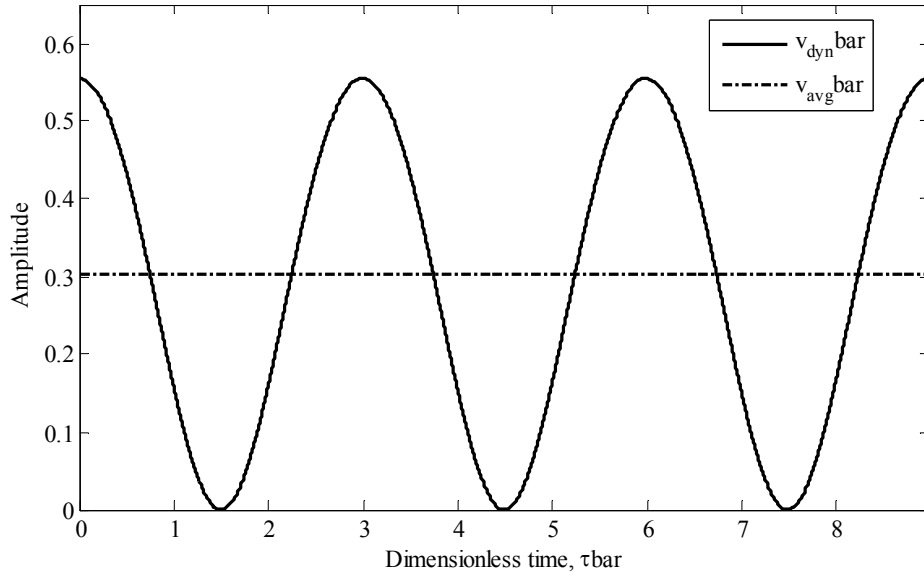


Figure 5.6. The dynamic velocity and the averaged velocity in $\bar{F}$ when $\beta = 0.3$.

Figures 5.6 and 5.7 illustrate the dynamic velocity of (5.49) and the averaged velocity of (5.51) in the frame $\bar{F}$. Both velocities in the figures are normalized by the velocity of light, c . The parameters used to obtain the figures ($\eta = 1.2$, $\varphi = 0$, $\xi = 0$, and $\vartheta = \sqrt{\eta^2 - 1} = 0.6633$) are the same as the ones in Fig. 3.4 exhibited in section “3.3. General Remarks.” Hence, Fig. 3.4 can be considered as the case in which (5.49) and (5.51) are obtained for $\beta = 0$, after replacing the horizontal axis τ with $\bar{\tau}$. In Fig. 5.7, the velocities are negative valued due to the negativeness of the common term of (5.49) and (5.51), $(\vartheta - \beta\eta)$, when $\beta = 0.9$. As explained before, for the above values of η and ϑ , $(\vartheta - \beta\eta)$ is negative valued when $\beta > 0.5528$. From this negativeness, we can state that the velocities in Fig. 5.7 are

directed opposite to the power flow in the waveguide.

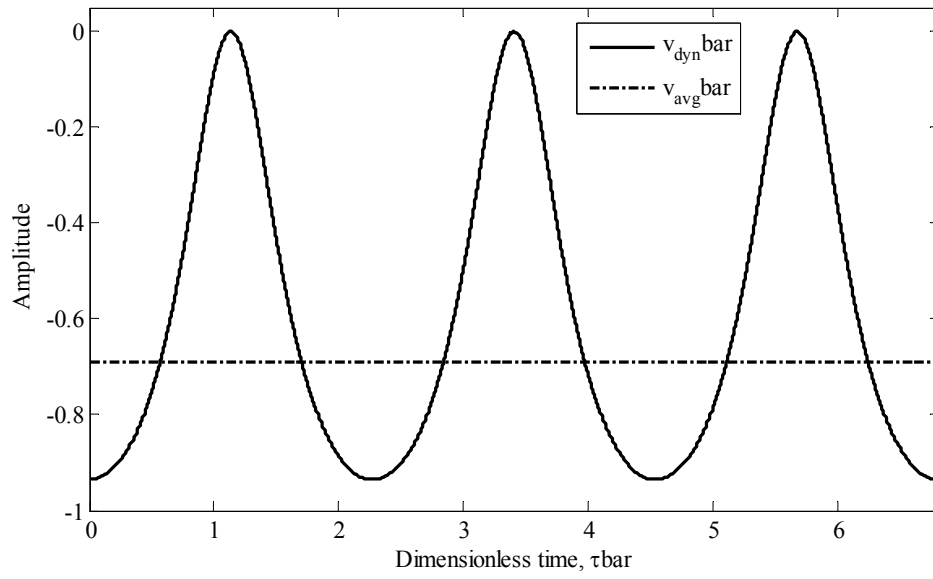


Figure 5.7. The dynamic velocity and the averaged velocity in $\bar{F}$ when $\beta = 0.9$.

6. CONCLUSIONS

In this dissertation, we propose an alternative technique for the study of electromagnetic field propagation in a hollow waveguide accompanied with transient processes in the time domain. The proposed technique has been developed for solving the initial-boundary value problem for the system of Maxwell's equations *directly* in the time domain. The motivation is to obtain new analytical solutions to the waveguide modes without recourse to the time-harmonic fields with non-causal solutions. With the method described here, the solutions are found in compliance with the causality principle.

To sum up, the study of the time-domain modes can be separated into two autonomous problems.

i) The modal basis problem: Determination of the complete set of modal field patterns in the waveguide cross section is the same, in the final analysis, as for the classical time-harmonic modes. The two-dimensional Dirichlet and Neumann boundary eigenvalue problems for the transverse Laplacian should be solved. This results in two complete sets of eigensolutions. The eigensolutions generate the basis elements and exhibit the field pattern configurations of the modal field components. Practically, one has a wide choice of methods for solving these problems. The already known results for typical waveguides can also be used after providing them with appropriate normalizations. Additionally, the dimension of force (i.e., newton N) should be assigned to the constants at the right-hand sides of the normalization conditions. This provides the electric and the magnetic field strength vector with required dimensions Vm^{-1} and Am^{-1} , respectively.

ii) The modal amplitude problem: We have derived a pair of potentials that generates explicit formulas for the modal amplitudes in the time domain. These potentials satisfy the KGE with independent variables. The modal amplitudes of the TE and TM modes can be studied *in parallel* after replacing the independent variables by the dimensionless ones.

The technique also includes a novel approach for expressing the modal amplitudes, the solutions to the KGE , via various special functions of mathematical

physics. The standard explicit solution to the KGE as a convolution integral can be found in handbooks, e.g., [Polyanin and Manzhirov, 2006]. However, exact calculations of the integral are often troublesome in realistic situations. Exact numerical data can be obtained effectively at small distances between source point and the point of observation. But when t and z become relatively large, the convolution integral fails in calculations due to fast oscillations of the kernel in the integrand [Kristensson, 1995]. Therefore, a new technique has been provided for solving the KGE analytically without recourse to the convolution integral. This technique is based on the properties of symmetry of the KGE which were disclosed within the framework of Group Theory [Miller, 1977]. It turns out that the solution to the KGE can be presented as a product of two functions if and only if the functions are specified in pairs in compliance with the properties of symmetry. A complete set of possible pairs is listed in Appendix C. The simplest pair is given in the first pair of the list which leads to the time-harmonic fields. From this, we can conclude that the time-harmonic field is only a particular case. Hence, the other ten cases open up new lines of analytical studies in the time-domain electromagnetics. In this dissertation, the fifth pair of the list is studied in detail, where the modal amplitude of the longitudinal component of the field is expressed as a product of two Airy functions. In fact, this solution is an exact expansion of the initial data over the whole domain of propagation which is equivalent to exact analytical calculation of the convolution integral without approximations of its kernel [Tretyakov and Akgun, 2010].

We have also made the relativistic analysis of the time domain waveguide modes. It is known that the system of Maxwell's equations with time derivative are invariant under the relativistic Lorentz transformations. Hence, the complete sets of TE and TM waveguide modes, which are derived via solving the initial-boundary value problem for the mentioned system, have similar relativistic properties. Notice that the relativistic Lorentz transformations are inapplicable to the time-harmonic waveguide waves in the frequency domain. We have proved that the modal bases of the time domain waveguide modes are *absolute* for the inertial frames of reference. In other words, the waveguide modal basis is independent of the relative motion of the reference frames. However, the modal amplitudes of the time domain waveguide modes are *not* absolute for the inertial reference frames.

APPENDICES

APPENDIX A: Proof of the Law of Conservation of Energy

A relation between the rate of change of the energy stored in the field and the energy flow can be deduced as a general integral of the field equations. We note that $\mathcal{H} \cdot (\nabla \times \mathcal{E})$ and $\mathcal{E} \cdot (\nabla \times \mathcal{H})$ have the dimensions of power expended per unit volume (watts per cubic meter) suggesting the scalar multiplication of the magnetic and electric field strength vectors by the system of vectorial Maxwell's equations

$$\begin{aligned}\mathcal{H} \cdot (\nabla \times \mathcal{E}) &= -\mu_0 \mathcal{H} \cdot \partial_t \mathcal{H}, \\ \mathcal{E} \cdot (\nabla \times \mathcal{H}) &= \varepsilon_0 \mathcal{E} \cdot \partial_t \mathcal{E}.\end{aligned}\tag{6.1}$$

Subtracting the second line from the first of (6.1), and applying the followings

$$\mathcal{A} \cdot \partial_t \mathcal{A} = \frac{1}{2} \partial_t (\mathcal{A} \cdot \mathcal{A}) \quad \text{and} \quad \mathcal{H} \cdot (\nabla \times \mathcal{E}) - \mathcal{E} \cdot (\nabla \times \mathcal{H}) = \nabla \cdot [\mathcal{E} \times \mathcal{H}], \quad \text{we}$$

obtain

$$\nabla \cdot [\mathcal{E} \times \mathcal{H}] = -\frac{1}{2} \partial_t [\varepsilon_0 (\mathcal{E} \cdot \mathcal{E}) + \mu_0 (\mathcal{H} \cdot \mathcal{H})]\tag{6.2}$$

which is the Poynting theorem in its differential form. After choosing a finite volume V within the waveguide between two cross sections located at z_1 and z_2 (where $z_2 = z_1 + \delta z$, $\delta z \ll 1$), and taking the integral of both sides of (6.2) over V , we obtain

$$\frac{1}{V} \int_V \nabla \cdot [\mathcal{E} \times \mathcal{H}] dv = -\frac{1}{2V} \partial_t \int_V [\varepsilon_0 (\mathcal{E} \cdot \mathcal{E}) + \mu_0 (\mathcal{H} \cdot \mathcal{H})] dv.\tag{6.3}$$

By applying Gauss' theorem to the left-hand side, we obtain

$$\frac{1}{V} \int_V \nabla \cdot [\mathcal{E} \times \mathcal{H}] dv = \frac{1}{V} \oint_S \mathbf{N} \cdot \mathbf{P} ds\tag{6.4}$$

where $\mathbf{P} = [\mathcal{E} \times \mathcal{H}]$ is the Poynting vector, and $\mathbf{N}$ is the outward normal to the closed surface S bounding the volume V . The surface S consists of three

constituents: $\mathbf{S} = S_1 + S_2 + S_w$, where S_1 and S_2 are the waveguide cross sections located at $z = z_1$ and $z = z_2$, respectively, and S_w coincides with the waveguide surface. The outward normals to the surfaces S_1 , S_2 , and S_w can be denoted as $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_w$, respectively. Therefore, the right-hand side of (6.4) can be rewritten as follows

$$\frac{1}{V} \oint_{\mathbf{S}} \mathbf{N} \cdot \mathbf{P} \, ds = \frac{1}{V} \int_{S_1} \mathbf{n}_1 \cdot \mathbf{P} \, ds + \frac{1}{V} \int_{S_2} \mathbf{n}_2 \cdot \mathbf{P} \, ds + \frac{1}{V} \int_{S_w} \mathbf{n}_w \cdot \mathbf{P} \, ds \quad (6.5)$$

where $\mathbf{n}_1 = -\mathbf{z}$, $\mathbf{n}_2 = \mathbf{z}$, and $\mathbf{n}_w$ coincides with the unit vector $\mathbf{n}$ normal to the contour L of the waveguide cross section. In view of the identity $\mathbf{n}_w \cdot \mathbf{P} = \mathbf{n} \cdot [\mathcal{E} \times \mathcal{H}] = [\mathbf{n} \times \mathcal{E}] \cdot \mathcal{H}$, it is clear that $[\mathbf{n} \times \mathcal{E}]|_{S_w}$ vanishes as the tangential components of the electric field vanish over the perfectly conducting waveguide surface. By taking into account that the waveguide cross section is invariable along the Oz axis, S_1 and S_2 can be taken as S , thus, (6.5) can be reformulated as

$$\frac{1}{V} \oint_{\mathbf{S}} \mathbf{N} \cdot \mathbf{P} \, ds = \frac{1}{V} \int_S (\mathbf{n}_1 \cdot \mathbf{P})|_{z=z_1} \, ds + \frac{1}{V} \int_S (\mathbf{n}_2 \cdot \mathbf{P})|_{z=z_2} \, ds. \quad (6.6)$$

At this point, we need the explicit formula of the Poynting vector, $\mathbf{P} = [\mathcal{E} \times \mathcal{H}]$. Consider the case of TE modes, the field vectors of which are given in (3.14), and calculate the Poynting vector as shown below

$$\mathbf{P} = \mathbf{z} \mu_0 P_z(z, t) \nabla_{\perp} \psi_n(\mathbf{r}) \cdot \nabla_{\perp} \psi_n(\mathbf{r}) \quad (6.7)$$

where $P_z(z, t)$ is defined as follows

$$P_z(z, t) = [-\partial_t h_n(z, t)][\partial_z h_n(z, t)]. \quad (6.8)$$

Substitution of (6.7) into (6.6) yields

$$\frac{1}{V} \oint_S \mathbf{N} \cdot \mathbf{P} ds = \frac{1}{(z_2 - z_1)} (P_z|_{z=z_2} - P_z|_{z=z_1}) \frac{\mu_0}{S} \int_S \nabla_{\perp} \psi_n \cdot \nabla_{\perp} \psi_n ds \quad (6.9)$$

where the term $S(z_2 - z_1)$ has been substituted for V . After utilizing the vector identity $\nabla_{\perp} \cdot (\psi_n \nabla_{\perp} \psi_n) = \nabla_{\perp} \psi_n \cdot \nabla_{\perp} \psi_n + \psi_n \nabla_{\perp}^2 \psi_n$ for the integrand, taking the integral over the surface S , and making use of the two-dimensional Gauss' theorem and also the Neumann boundary-eigenvalue problem given in (3.10), we obtain

$$\frac{\mu_0}{S} \int_S \nabla_{\perp} \psi_n \cdot \nabla_{\perp} \psi_n ds = \frac{\mu_0 v_n^2}{S} \int_S \psi_n^2 ds + \frac{\mu_0}{S} \oint_L \psi_n \partial_{\mathbf{n}} \psi_n dl. \quad (6.10)$$

By making use of the normalization condition for the TE modes given in (3.15) and the Neumann boundary condition, (6.10) can be reduced to

$$\frac{\mu_0}{S} \int_S \nabla_{\perp} \psi_n \cdot \nabla_{\perp} \psi_n ds = 1. \quad (6.11)$$

Substitution of (6.11) back into (6.9) yields

$$\frac{1}{V} \oint_S \mathbf{N} \cdot \mathbf{P} ds = \frac{1}{(z_2 - z_1)} (P_z|_{z=z_2} - P_z|_{z=z_1}). \quad (6.12)$$

Reminding that $z_2 = z_1 + \delta z$, and assuming that $\delta z \ll 1$, the Taylor series of

$P_z(z, t)|_{z=z_2}$ in the neighborhood of z_1 can be given by

$$P_z(z, t)|_{z=z_2} = P_z(z, t)|_{z=z_1} + \delta z \partial_z P_z(z, t)|_{z=z_1} + O((\delta z)^2). \quad (6.13)$$

After substituting (6.13) into (6.12), and recalling that the choice of z_1 can be arbitrary, we obtain

$$\frac{1}{V} \oint_S \mathbf{N} \cdot \mathbf{P} ds = \partial_z P_z(z, t) + O((\delta z)^2). \quad (6.14)$$

Utilizing (6.4), (6.14) gives the left-hand side of (6.3). Consider now the right-hand side of (6.3). We first present the explicit formula of the integrand, surely, for the TE modes as given below

$$\begin{aligned} \varepsilon_0(\mathcal{E} \cdot \mathcal{E}) + \mu_0(\mathcal{H} \cdot \mathcal{H}) = & \mu_0[(\partial_{ct} h_n)^2 + (\partial_z h_n)^2] v_n^2 \psi_n^2 \\ & + \mu_0[(\partial_{ct} h_n)^2 + (\partial_z h_n)^2] \nabla_\perp \cdot (\psi_n \nabla_\perp \psi_n) \quad (6.15) \\ & + \mu_0 h_n^2 v_n^4 \psi_n^2. \end{aligned}$$

Substitution of (6.15) into the integrand in the right-hand side of (6.3) yields

$$\begin{aligned} & -\frac{1}{2V} \partial_t \int_V [\varepsilon_0(\mathcal{E} \cdot \mathcal{E}) + \mu_0(\mathcal{H} \cdot \mathcal{H})] dv = \\ & -\frac{1}{2\delta z} \frac{\mu_0 v_n^2}{S} \partial_t \int_V [(\partial_{ct} h_n)^2 + (\partial_z h_n)^2] \psi_n^2 dv \\ & -\frac{1}{2} \frac{1}{S \delta z} \mu_0 \partial_t \int_V [(\partial_{ct} h_n)^2 + (\partial_z h_n)^2] \nabla_\perp \cdot (\psi_n \nabla_\perp \psi_n) dv \\ & -\frac{1}{2\delta z} \frac{\mu_0 v_n^2}{S} \partial_t \int_V v_n^2 h_n^2 \psi_n^2 dv. \end{aligned} \quad (6.16)$$

Consider the volume integral as a decomposition into a surface integral over the waveguide cross section, S , and a line integral along the waveguide axis, Oz . By making use of the normalization condition for the TE modes given in (3.15) and the Neumann boundary condition for the surface integral, and providing the first mean value theorem for the line integral, we obtain

$$-\frac{1}{2V} \partial_t \int_{z_1}^{z_2} \left\{ \int_S [\varepsilon_0(\mathcal{E} \cdot \mathcal{E}) + \mu_0(\mathcal{H} \cdot \mathcal{H})] ds \right\} dz = -\partial_t W(z, t) \frac{1}{(z_2 - z_1)} \int_{z_1}^{z_2} dz \quad (6.17)$$

where $W(z, t)$ is defined as follows

$$W(z, t) = \frac{1}{2} \left[(\partial_{ct} h_n(z, t))^2 + (\partial_z h_n(z, t))^2 + (v_n h_n(z, t))^2 \right]. \quad (6.18)$$

Thus, (6.17) can be rewritten in the following form

$$-\frac{1}{2V}\partial_t\int_{z_1}^{z_2}\left\{\int_S[\varepsilon_0(\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\mathcal{E}})+\mu_0(\boldsymbol{\mathcal{H}}\cdot\boldsymbol{\mathcal{H}})]ds\right\}dz=-\partial_tW(z,t). \quad (6.19)$$

(6.19) gives the right-hand side of (6.3) that can be written in the final form as shown below

$$\partial_zP_z(z,t)+\partial_tW(z,t)\approx 0 \quad (6.20)$$

which completes the proof. The definitions of $P_z(z,t)$ and $W(z,t)$ for the TE modes are given together in (3.16). $\square$

APPENDIX B: Boundary Conditions

Boundary conditions of (2.21) follow from that of (2.20) as the potentials, ϕ_n , for the TM modes satisfy the Helmholtz equation presented in (3.21). This fact can be illustrated as follows:

Substitution of the transverse components of electric and magnetic field vectors for the TM modes

$$\begin{aligned}\mathbf{E}(\mathbf{r}, z, t) &= V(z, t) \nabla_{\perp} \phi(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}, z, t) &= I(z, t) [\nabla_{\perp} \phi(\mathbf{r}) \times \mathbf{z}]\end{aligned}\tag{7.1}$$

into (2.21) yields the following conditions

$$\begin{aligned}-I(z, t) \{ \nabla_{\perp} \cdot (\mathbf{z} \times [\nabla_{\perp} \phi(\mathbf{r}) \times \mathbf{z}]) \} \Big|_L &= 0, \\ V(z, t) [\nabla_{\perp} \cdot \nabla_{\perp} \phi(\mathbf{r})] \Big|_L &= 0.\end{aligned}\tag{7.2}$$

After utilizing the triple product expansion for the first line of (7.2), and recalling that $I(z, t)$ and $V(z, t)$ must not be zero, we obtain the same boundary condition for $\phi(\mathbf{r})$ as

$$\nabla_{\perp}^2 \phi(\mathbf{r}) \Big|_L = 0.\tag{7.3}$$

The unknown potential $\phi(\mathbf{r})$ can be presented as Fourier series in terms of the basis $\phi_n(\mathbf{r})$

$$\phi(\mathbf{r}) = \sum_n c_n \phi_n(\mathbf{r})\tag{7.4}$$

where $r \in L, S$ and $\{\phi_n(\mathbf{r})\}_{n=1}^{\infty}$ is the complete set of solutions to the Dirichlet boundary-eigenvalue problem of (3.21). Now, substituting (7.4) back into (7.3), and changing the order of the sigma operator and the transverse Laplacian operator, we get

$$\sum_n c_n [\nabla_{\perp}^2 \phi_n(\mathbf{r})] \Big|_L = 0. \quad (7.5)$$

Utilizing the Helmholtz equation, the term $-\kappa_n^2 \phi_n(\mathbf{r})$ can be substituted for $\nabla_{\perp}^2 \phi_n(\mathbf{r})$, finally, we obtain

$$\phi_n(\mathbf{r}) \Big|_L = 0 \quad (7.6)$$

which satisfies the boundary condition of the *TM* modes on the contour *L* individually for the values of *n*, where $n = 1, 2, \dots$.

APPENDIX C: Complete List of Substitutions

A complete list of substitutions for $u(\xi, \tau)$ and $v(\xi, \tau)$ that factorize the solution to (4.1) as $f(u, v) = U(u)V(v)$ is given below.

- 1) $\tau = u$ and $\xi = v$, where $-\infty < u, v < \infty$ yield $f(u, v)$ as a product of exponential functions.
- 2) $\tau = u \cosh v$ and $\xi = u \sinh v$, where $0 \leq u < \infty$, $-\infty < v < \infty$ yield $f(u, v)$ as a product of an exponential and the Bessel functions.
- 3) $\tau = (u^2 + v^2) / 2$ and $\xi = uv$, where $0 \leq u < \infty$, $-\infty < v < \infty$ yield $f(u, v)$ as a product of parabolic cylinder functions.
- 4) $\tau = uv$ and $\xi = (u^2 + v^2) / 2$, where $0 \leq u < \infty$, $-\infty < v < \infty$ yield $f(u, v)$ as a product of parabolic cylinder functions.
- 5) $\tau + \xi = 2(u + v)$ and $\tau - \xi = (u - v)^2$, where $-\infty < u, v < \infty$ yield $f(u, v)$ as a product of Airy functions.
- 6) $\tau + \xi = \cosh[(u - v) / 2]$ and $\tau - \xi = \sinh[(u + v) / 2]$, where $-\infty < u < \infty$ and $-\infty < v < \infty$ yield $f(u, v)$ as a product of Mathieu functions.
- 7) $\tau + \xi = 2 \sinh(u - v)$ and $\tau - \xi = \exp(u + v)$, where $-\infty < u, v < \infty$ yield $f(u, v)$ as a product of Bessel functions.
- 8) $\tau + \xi = 2 \cosh(u - v)$ and $\tau - \xi = \exp(u + v)$, where $-\infty < u, v < \infty$ yield $f(u, v)$ as a product of Bessel functions.
- 9) $\tau = \sinh u \cosh v$ and $\xi = \cosh u \sinh v$, where $-\infty < u, v < \infty$ yield $f(u, v)$ as a product of Mathieu functions.
- 10) $\tau = \cosh u \cosh v$ and $\xi = \sinh u \sinh v$, where $-\infty < u < \infty$, and $0 \leq v < \infty$ yield $f(u, v)$ as a product of Mathieu functions.
- 11) $\tau = \cos u \cos v$ and $\xi = \sin u \sin v$, where $0 < u < 2\pi$, and $0 \leq v < \pi$ yield $f(u, v)$ as a product of Mathieu functions.

The substitutions given above specify orthogonal systems of coordinates (u, v) . Besides, there are non-orthogonal systems which enable us to separate the variables u and v in the KGE , see [Miller, 1977].

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