

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

**INITIAL TIME DIFFERENCE STABILITY ANALYSIS
OF SET DIFFERENTIAL EQUATIONS
INVOLVING CAUSAL OPERATORS**

HAZM TALAB
**A THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
DEPARTMENT OF MATHEMATICS**

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T.C.
GEBZE TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

NEDENSEL OPERATÖRLERİ İÇEREN
KÜME DİFERANSİYEL DENKLEMLERİN
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SUMMARY

This work examines initial time difference (ITD) stability analysis of set differential equations (SDEs) with causal operators. In particular, the stability properties of the solutions of perturbed forms of SDEs with causal operators in correspondence to the properties of the solutions of un-perturbed forms have been considered.

Through analogous approaches to Lyapunov second method in stability analysis, we study the stability properties taking into account the differences in the initial conditions (time and position) between the solutions of perturbed and un-perturbed forms of SDEs with causal operators.

Additionally, we establish generalized variational comparison results aimed to examine the stability properties of the solutions of SDEs with causal operators in terms of two measures and take into consideration the differences in the starting parameters in the initial states.

Later, we employ those generalized variational comparison results in proving theorems that give sufficient criteria for diverse qualitative properties in terms of two measures of the solutions of perturbed forms of SDEs with causal operators in correspondence to the properties of the solutions of un-perturbed forms, with considering the differences in the initial conditions.

Key Words: Causal Operators, Comparison Results, Initial Time Difference (ITD), Lyapunov, Perturbed and Un-perturbed Set Differential Equations, Set Differential Equations (SDEs), Stability, Two Measures.

ÖZET

Bu çalışma, nedensel operatörleri içeren küme diferansiyel denklemlerinin başlangıç zaman farklı kararlılık analizini incelemektedir. Özellikle, pertürbe edilmiş nedensel operatörleri içeren küme diferansiyel denklemlerinin çözümlerinin karşılık gelen pertürbe edilmemiş olanlara göre başlangıç zaman farklı kararlılık özellikleri araştırılmıştır.

Nedensel operatörleri içeren küme diferansiyel denklemlerin pertürbe edilmiş ve pertürbe edilmemiş formlarının çözümleri arasındaki başlangıç koşullarındaki (zaman ve pozisyon) farklılıkları düşünülerek, kararlılık analizinde Lyapunov'un ikinci yöntemine benzer yaklaşımlar yoluyla, kararlılık özellikleri incelenmektedir.

Ek olarak başlangıç koşullarındaki farkı dikkate alınarak nedensel operatörleri içeren küme diferansiyel denklemlerinin çözümlerinin kararlılık özelliklerini iki ölçü açısından incelemeyi amaçlayan genelleştirilmiş varyasyonel karşılaştırma sonuçları araştırılmıştır.

Bu genelleştirilmiş varyasyonel karşılaştırma sonuçları pertürbe edilmiş nedensel operatörleri içeren diferansiyel denklemlerinin çözümlerinin karşılık gelen pertürbe edilmemiş olanlara göre başlangıç zaman farklı olan iki ölçü cinsinden çeşitli niteliksel özellikler için yeterli koşulları sağlayan teoremleri kanıtlamada kullanılmıştır.

Anahtar Kelimeler: Nedensel Operatörler, Karşılaştırma Sonuçları, Başlangıç zaman farkı, Lyapunov, Pertürbe edilmiş ve pertürbe edilmemiş küme diferansiyel denklemleri, Küme diferansiyel denklemleri, İstikrar, Kararlılık, İki Ölçü.

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LIST OF ABBREVIATIONS AND ACRONYMS

Abbreviations Explanations and Acronyms

$\mathbb{N}$	: The set of all natural numbers
$\mathbb{R}$	: The set of all real numbers
$\mathbb{R}_+$	: The set of all positive real numbers
$\mathbb{R}^n$	: The set of all n-tuples of real numbers
IVP	: Initial Value Problem
ODE	: Ordinary Differential Equation
ODEs	: Ordinary Differential Equations
SDE	: Set Differential Equation
SDEs	: Set Differential Equations
ITD	: Initial Time Difference
w. r. t.	: with respect to
a. e.	: almost everywhere
$D^+f(t)$	: Right-upper Dini derivative of $f \in C[[t_0, +\infty), \mathbb{R}]$ $D^+f(t) = \overline{\lim}_{h \rightarrow 0^+} \frac{f(t+h) - f(t)}{h} = \lim_{h \rightarrow 0^+} \sup \frac{f(t+h) - f(t)}{h},$
$D_+f(t)$	: Right-lower Dini derivative of $f \in C[[t_0, +\infty), \mathbb{R}]$ $D_+f(t) = \underline{\lim}_{h \rightarrow 0^+} \frac{f(t+h) - f(t)}{h} = \lim_{h \rightarrow 0^+} \inf \frac{f(t+h) - f(t)}{h}$
$D^-f(t)$	: Left-upper Dini derivative of $f \in C[[t_0, +\infty), \mathbb{R}]$ $D^-f(t) = \overline{\lim}_{h \rightarrow 0^-} \frac{f(t+h) - f(t)}{h} = \lim_{h \rightarrow 0^-} \sup \frac{f(t+h) - f(t)}{h}$
$D_-f(t)$	: Left-lower Dini derivative of $f \in C[[t_0, +\infty), \mathbb{R}]$ $D_-f(t) = \underline{\lim}_{h \rightarrow 0^-} \frac{f(t+h) - f(t)}{h} = \lim_{h \rightarrow 0^-} \inf \frac{f(t+h) - f(t)}{h}$
$\ x\ $	: $\ x\ = \sqrt{\sum_{i=1}^n x_i^2}, x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$
$d(x, y)$	: $d(x, y) = \ x - y\ = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}, x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$
$B_r(x)$	: $B_r(x) = \{y \in \mathbb{R}^n : \ x - y\ < r\}, x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$

$K_c(\mathbb{R}^n)$	: The set of all non-empty compact convex subsets of $\mathbb{R}^n$
$K(\mathbb{R}^n)$	: The set of all non-empty compact subsets of $\mathbb{R}^n$
$\mathcal{C}(\mathbb{R}^n)$	: The set of all non-empty closed subsets of $\mathbb{R}^n$
E_1	: $E_1 = \{\Omega \in K_c(\mathbb{R}^n) : L(s, \Omega(s)) \leq L(t, \Omega(t)), \quad t_0 \leq s \leq t\}$
E_A	: $E_A = \{\Omega \in K_c(\mathbb{R}^n) : L(s, \Omega(s)) A(s) \leq L(t, \Omega(t)) A(t),$ $t_0 \leq s \leq t\}$
$S(\rho)$	: $S(\rho) = S_\rho = B(\theta, \rho) = \{U \in K_c(\mathbb{R}^n) : D[U, \tilde{0}] < \rho < \infty\}$
$\mathcal{K}$	: $\mathcal{K} = \left\{ a \in C[\mathbb{R}_+, \mathbb{R}_+] : \begin{array}{l} a(u) \text{ is strictly increasing in } u \\ \text{and } a(0) = 0 \end{array} \right\}$
$\mathcal{L}$	: $\mathcal{L} = \left\{ \sigma \in C[\mathbb{R}_+, \mathbb{R}_+] : \begin{array}{l} \sigma(u) \text{ is strictly decreasing in } u \\ \text{and } \lim_{u \rightarrow \infty} \sigma(u) = 0 \end{array} \right\}$
$\mathcal{CK}$	: $\mathcal{CK} = \left\{ a \in C[\mathbb{R}_+^2, \mathbb{R}_+] : \begin{array}{l} a(t, s) \in \mathcal{K} \text{ for each } t \\ \text{and } a(t, s) \text{ is continuous for each } s \end{array} \right\}$
Γ	: $\Gamma = \left\{ h \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+] : \inf_{(t, U)} h(t, U) = 0 \right\}$
Γ_0	: $\Gamma_0 = \left\{ h \in \Gamma : \inf_U h(t, U) = 0, \text{ for each } t \in \mathbb{R}_+ \right\}$
$S(h, \rho)$	: $S(h, \rho) = \{(t, U) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n), \ h(t, U) < \rho, \ \rho > 0\}$

1. INTRODUCTION

Studying set differential equations (SDEs) in an appropriate metric space and their stability features was first presented by Lakshmikantham, Leela and Vatsala in 2002 [1]. Later, many researches have been done studying SDEs in the recent decades due to their unifying properties.

Lakshmikantham, Leela and Devi provided an approach to eliminate a disadvantage in the formulation of SDEs that the diameter of solutions of SDEs being nondecreasing over time, which causes sometimes a mismatch in the behavioral properties of solutions of the ordinary differential equations (ODEs) from which SDEs can be generated [2].

The stability criteria for SDEs, analogous to those for ODEs in the original Lyapunov results and their boundedness results, were presented by Bhaskar and Devi in [3], [4].

Lakshmikantham, Bhaskar and Devi, highlighted the unifying properties of SDEs and many other significant findings in one of the most important resources on SDEs [5]. Authors endeavored to treat SDEs as a distinct field of study, discussing the underlying theory, the fundamental comparison results, and the stability concerns for their dynamical systems, employing smooth and non-smooth analytical approaches in their pursuit and investigated the relationships between SDEs and fuzzy differential equations. In addition to providing a contemporary introduction to the topic of SDEs, authors drew attention to their exquisite applications in other fields, laying the groundwork for future research in metric spaces.

The comprehensiveness of the SDEs is driven by the fact that they encompass the conventional differential and integral equations when the Hukuhara difference and integrals defined on the SDEs are restricted to $\mathbb{R}$; whereas they give us vector differential equations when the restriction is done to $\mathbb{R}^n$ [6]–[8]. Furthermore, understanding SDEs gives us a more precise overview when considering multivalued differential equations and inclusions [5] and when considering fuzzy differential equations [9].

Afterwards, SDEs got more attention to study them in depth with their applications. Tu and Tung introduced some results on stability of sheaf-solutions,

exponential stability of solutions and sheaf-solutions of SDEs and their applications [10].

Causal operators, on the other hand, were introduced by Tonneli [11]. Corduneanu presented a monograph that overviews the general theory of the functional differential equations involving causal operators [12]. And since then there has been a growing interest in causal operators and their differential equations [13]–[15].

Using the Lyapunov function technique, Drici, Mcrae, and Devi developed stability conclusions in the broad framework of SDEs involving causal operators integrating the unifying concepts of both SDEs and causal operators' theories [16].

Devi initiated a generalization that considers SDEs involving causal operators with memory [17], and obtained existence and uniqueness results of their IVP [18]. Further, Devi developed generalized monotone iterative technique for SDEs involving causal operators with memory [19].

Devi and Naidu concluded stability and boundedness results for SDEs involving causal operators with memory and impulsive SDEs involving causal operators with memory, obtaining results parallel to ODEs with delay [20]–[22].

SDEs with causal operators were tempting to researchers considering that many well-known forms of differential equations, including integro-differential equations [23], impulsive differential equations [24], and differential equations with delay [25], are examples of equations involving causal operators. Consequently, SDEs with causal operators provide a thorough overview of several types of differential equations.

Furthermore, SDEs with causal operators unify the fundamental theory of SDEs, including various corresponding dynamical systems.

Although it is never feasible to know the exact solutions of all dynamical systems in practice, their attributes may be determined through a variety of qualitative studies such as stability analysis [26], [27], initial time difference (ITD) stability analysis [28]–[39], practical stability analysis [40]–[42], boundedness [43], [44], etc.

Many techniques have been used in this process, including the Lyapunov second method, variation of parameters [45], “in terms of two measures” methodology [46], and so on.

ITD stability analysis, which can be considered as an extension of the conventional idea of stability, examines the behavioral attributes of the solutions of differential equations considering the differences in the starting parameters in the initial state since, in realistic circumstances, the initial place and time of the initial value problems might vary as well as their solutions.

In the qualitative theory of differential equations, such a generalized perspective to stability helps assess their solutions' stability and other behavioral features in a more practical style that considers the deviations in the baseline parameters.

The relevance of this technique and others in the qualitative theory of differential equations stems from the fact that they focus on the behavior of solutions rather than the solutions themselves. In many circumstances, knowing the precise solution for a system is not feasible; yet, it is far simpler to describe these solutions than to calculate them.

Stability in terms of two measures, on the other hand, is a notion that unites multiple forms of stability and may be expressed in a variety of ways. Devi, Naidu and Dhaigude published a research paper in which they attempted to establish stability findings for SDEs involving causal operators using this “two measures” methodology [47].

We conducted an ITD stability analysis of SDEs involving causal operators. In this regard, we studied the fundamental concepts and conclusions relating to ITD stability and stability of the null solution of SDEs involving causal operators. We also provided adequate ITD stability and ITD asymptotic stability criteria for SDEs involving causal operators.

It is shown that attempting to derive ITD stability features from those of the null solution is challenging, and this is demonstrated by contrasting classical and ITD perspectives of the stability of solutions of SDEs involving causal operators.

We cleared up any lingering ambiguities between ITD stability and the traditional concept of stability by establishing ITD comparison theorems which allowed us to overcome the problems associated with the conventional concept of stability in the first place.

Using an appropriate comparison system, we described a developed method that overcomes the challenges we faced and allows us to deduce stability features for

solutions of perturbed forms of SDEs with causal operators compared to their unperturbed forms.

Additionally, we developed generalized variational comparison results to assess the stability and other qualitative aspects in terms of two measures for solutions of SDEs involving causal operators, considering the variations in the starting parameters.

Next, we employed these comparison results in proving theorems that give adequate criteria for practical stability, equi-boundedness, equi-attractiveness in the large, and Lagrange stability in terms of two measures with ITD for the solutions of perturbed SDEs with causal operators in comparison to their unperturbed counterparts.

Our concluded results have been prepared in three research papers. Two of them have been published [48], [49]. The third one [50] has been accepted and it is currently in press.

2. PRELIMINARIES FOR SDES AND CAUSAL OPERATORS

2.1. Introduction

In this chapter, we present the preliminaries for SDEs and Causal Operators. We laid the foundation for studying SDEs with causal operators by presenting the basic definitions and fundamental comparison results, local and global existence results, and the most relevant theorems regarding the stability of the solutions of SDEs with Causal Operators.

2.2. Basic Definitions

2.2.1. $\mathbb{R}^n$

$\mathbb{R}^n$ denotes the set of all n -tuples of the form $(x_1, x_2, \dots, x_n)$ where each x_i is a real number for $1 \leq i \leq n$. $\mathbb{R}^n$ forms an n -dimensional vector space over the field of real numbers $\mathbb{R}$, with the following operations defined on $\mathbb{R}^n$ by

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n), \\ \alpha x &= (\alpha x_1, \alpha x_2, \dots, \alpha x_n) \end{aligned} \tag{2.1}$$

for any $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, and any $\alpha \in \mathbb{R}$.

The zero vector of $\mathbb{R}^n$ is given by $(0, 0, \dots, 0)$, and the additive inverse of the vector

$x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is given by $(-x_1, -x_2, \dots, -x_n)$.

The relevance of this structure arises from the fact that any n -dimensional real vector space is isomorphic to the vector space $\mathbb{R}^n$.

The Euclidean norm of $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is defined by

$$\|x\| = \sqrt{\sum_{i=1}^n x_i^2} \tag{2.2}$$

The Euclidean distance between any $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ is defined by

$$d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \quad (2.3)$$

Each Euclidean norm and the Euclidean distance satisfy the norm and distance's propositions, respectively.

2.2.2. Open, Closed, Bounded, Compact, and Convex Subsets of $\mathbb{R}^n$

For any $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, the open ball centered at x with the radius $r > 0$ is defined by

$$B_r(x) = \{y \in \mathbb{R}^n : \|x - y\| < r\} \quad (2.4)$$

A set $U \subset \mathbb{R}^n$ is said to be open if and only if for any $x \in U$ there exists $r > 0$, such the open ball $B_r(x)$ is a subset of U .

$U \subset \mathbb{R}^n$ is said to be closed if and only if its complement U^c is open.

Any open ball is an open set; and any set is open if it is a union of open balls; $\emptyset$ and $\mathbb{R}^n$ are both open and closed sets.

$U \subset \mathbb{R}^n$ is said to be bounded if and only if there exists a $x \in \mathbb{R}^n$ and $r > 0$ such that the open ball $B_r(x)$ contains U .

$U \subset \mathbb{R}^n$ is said to be compact if and only if it is closed and bounded.

$U \subset \mathbb{R}^n$ is said to be convex if for any $a_1, a_2 \in U$ and any $\lambda \in [0, 1]$, we have

$$\lambda a_1 + (1 - \lambda) a_2 \in U \quad (2.5)$$

We consider the following subspaces of $\mathbb{R}^n$:

- i) $K_c(\mathbb{R}^n)$ the set of all non-empty compact convex subsets of $\mathbb{R}^n$;
- ii) $K(\mathbb{R}^n)$ the set of all non-empty compact subsets of $\mathbb{R}^n$;
- iii) $C(\mathbb{R}^n)$ the set of all non-empty closed subsets of $\mathbb{R}^n$.

2.2.3. Continuity in $\mathbb{R}^n$

A function f defined on $X \subset \mathbb{R}^n$ and its codomain is $\mathbb{R}^m$ is said to be continuous at $x_0 \in \mathbb{R}^n$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\forall x \in X, \|x - x_0\| < \delta \text{ implies } \|f(x) - f(x_0)\| < \varepsilon \quad (2.6)$$

$f: X \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous if and only if f^{-1} of any open set $U \subset \mathbb{R}^m$ is an open set in $\mathbb{R}^n$.

2.2.4. The Hausdorff Metric

The Hausdorff metric between any bounded sets A and B in $\mathbb{R}^n$ is defined by

$$D(A, B) = \max \left\{ \sup_{x \in B} d(x, A), \sup_{y \in A} d(y, B) \right\} \quad (2.7)$$

where $d(x, A) = \inf \{d(x, y) : y \in A\}$.

Each of $(K(\mathbb{R}^n), D)$ and $(K_c(\mathbb{R}^n), D)$ forms a complete metric space. As a semi-linear metric space with natural addition and non-negative scalar multiplication, the space $K_c(\mathbb{R}^n)$ can be embedded as a cone into a corresponding Banach space.

As an example of calculating the Hausdorff metric between two bounded subsets in $\mathbb{R}$, let $A = [0, 10], B = [20, 25] \in K_c(\mathbb{R})$.

Let us calculate $\sup_{x \in B} d(x, A), \sup_{y \in A} d(y, B)$.

For $x \in B = [20, 25], \{d(x, y) : y \in [0, 10]\}$ reaches its infimum value when $y = 10$; Thus,

$$\begin{aligned} \sup_{x \in B} d(x, A) &= \sup_{x \in B} \inf_{y \in A} \{d(x, y) : y \in A\} \\ &= \sup_{x \in [20, 25]} \inf_{y \in [0, 10]} \{d(x, y) : y \in [0, 10]\} = \sup_{x \in [20, 25]} d(x, 10) \\ &= d(25, 10) = |25 - 10| = 15 \end{aligned} \quad (2.8)$$

On the other hand, for $y \in A = [0, 10], \{d(y, x) : x \in [20, 25]\}$ reaches its infimum value when $x = 20$; Thus,

$$\begin{aligned}
\sup_{y \in A} d(y, B) &= \sup_{y \in A} \inf \{ d(y, x) : x \in B \} \\
&= \sup_{y \in [0,10]} \inf \{ d(y, x) : x \in [20, 25] \} = \sup_{y \in [0,10]} d(y, 20) \\
&= d(0, 20) = |0 - 20| = 20
\end{aligned} \tag{2.9}$$

Hence,

$$D(A, B) = \max \left\{ \sup_{x \in B} d(x, A), \sup_{y \in A} d(y, B) \right\} = \max\{15, 20\} = 20 \tag{2.10}$$

The Hausdorff metric satisfies the following properties:

$$\begin{aligned}
\text{i)} \quad & D(A, B) = D(B, A) \\
\text{ii)} \quad & D(A + C, B + C) = D(A, B) \\
\text{iii)} \quad & D(kA, kB) = k D(A, B) \\
\text{iv)} \quad & D(A, B) \leq D(A, C) + D(C, B)
\end{aligned} \tag{2.11}$$

for any $A, B, C \in K_c(\mathbb{R}^n)$ and $k \in \mathbb{R}_+$, where Minkowski addition of any two non-empty subsets A and B of $\mathbb{R}^n$ is defined by

$$A + B = \{a + b : a \in A, b \in B\} \tag{2.12}$$

and where scalar multiplication of a value $k \in \mathbb{R}$ and a non-empty subset A of $\mathbb{R}^n$ is defined by

$$kA = \{ka : a \in A\} \tag{2.13}$$

If $k = -1$, we get $-A = (-1)A = \{-a : a \in A\}$.

2.2.5. Hukuhara Difference

In general, $A + (-A) \neq \{0\}$ (unless $A = \{a\}$ is a singleton), which in its turn does not constitute a natural operation of subtraction. For example, if $A = [0, 5]$ then $-A = (-1)A = [-5, 0]$, and therefore $A + (-A) = [0, 5] + [-5, 0] = [-5, 5] \neq \{0\}$.

To overcome with such implication of Minkowski difference, i. e.

$$A + (-1)B = \{a - b : a \in A, \quad b \in B\} \quad (2.14)$$

Hukuhara difference between two sets $A, B \in K_c(\mathbb{R}^n)$ is defined as follows:

If there exists a set $C \in K_c(\mathbb{R}^n)$ such that $C + B = A$, then Hukuhara difference exists, and we denote it by $A \ominus B$, or simply, $A - B$, where there is no misunderstanding with Minkowski difference. i. e.

$$A \ominus B = C \Leftrightarrow C + B = A \quad (2.15)$$

An essential characteristic of Hukuhara difference is $A - A = \{0\}$ for $A \in K_c(\mathbb{R}^n)$.

2.2.6. Hukuhara Differentiability and Integrality

Let $U : I \rightarrow K_c(\mathbb{R}^n)$ be a given multifunction, where I is an interval of real numbers. U is Hukuhara differentiable at a point $t_0 \in I$, if there exists $D_H U(t_0) \in K_c(\mathbb{R}^n)$ such that the limits

$$\lim_{h \rightarrow 0^+} \frac{U(t_0 + h) - U(t_0)}{h} \quad \text{and} \quad \lim_{h \rightarrow 0^+} \frac{U(t_0) - U(t_0 - h)}{h} \quad (2.16)$$

both exist in the topology of $K_c(\mathbb{R}^n)$ and are equivalent to $D_H U(t_0)$.

It is implicit in the definition of $D_H U(t_0)$ the existence of the differences $U(t_0 + h) - U(t_0)$ and $U(t_0) - U(t_0 - h)$, for sufficiently small $h > 0$.

By embedding $K_c(\mathbb{R}^n)$ as a complete cone in a corresponding Banach space and taking into consideration the conclusion on differentiation of Bochner integral, we get that if

$$G(t) = G(t_0) + \int_{t_0}^t F(s) ds, \quad t \in I \quad (2.17)$$

where $F : I \rightarrow K_c(\mathbb{R}^n)$ is integrable in the sense of Bochner, then G is Hukuhara differentiable, i. e. $D_H G(t)$ exists, and the equality $D_H G(t) = F(t)$, a. e. on I , holds.

Also, the Hukuhara integral

$$\int_I F(s) ds = \left[\int_I f(s) ds : f \text{ is a continuous selector of } F \right] \quad (2.18)$$

for any compact set $I \subset \mathbb{R}_+$.

2.2.7. Causal Operator

Let $E = C[[t_0, T], K_c(\mathbb{R}^n)]$ with norm

$$\|U\| = D_0[U, \theta] = \sup_{t_0 \leq t \leq T} D[U(t), \theta] \quad (2.19)$$

where $U \in E, \theta$ is the zero element of $\mathbb{R}^n$, which is regarded as a point set. E equipped with such a norm is a Banach space.

Let $Q \in C[E, E]$. Q is said to be a causal operator or non-anticipative operator if $U(s) = V(s)$, $t_0 \leq s \leq t \leq T$, and $U, V \in E$, then

$$(QU)(s) = (QV)(s), \quad t_0 \leq s \leq t. \quad (2.20)$$

2.3. SDEs with Casual Operators

In this section, we restate Lakshmikanthan's foundational findings from "Theory of Set Differential Equations in Metric Spaces" [5], addressing the existence and uniqueness results for SDEs with casual operators and investigating the qualitative behaviors of their solutions using comparison results in terms of Lyapunov-like functions, which under adequate conditions are then utilized to prove stability features of SDEs with casual operators. Additionally, we restate the comparison results presented by Drici Z., Mcrae F. A., and Devi J. V. in [16].

In this context, the Lyapunov-like function acts as a mechanism that translates the SDEs into corresponding comparison systems, making it sufficient to derive the stability features of their solutions straightforwardly by evaluating the qualitative aspects of the simpler corresponding comparison systems.

We consider the IVP of the following SDE involving a causal operator, which is given using the Hukuhara derivative as:

$$D_H U(t) = (QU)(t), \quad U(t_0) = U_0 \in K_c(\mathbb{R}^n) \quad (2.21)$$

and consider the following scalar IVP

$$w' = g(t, w), \quad w(t_0) = w_0 \geq 0 \quad (2.22)$$

where $g \in C[[t_0, T] \times \mathbb{R}_+, \mathbb{R}_+]$

2.3.1. Comparison Results

2.3.1.1. Theorem (Comparison Theorem 1)

For $m \in C[J, \mathbb{R}_+]$, $g \in C[J \times \mathbb{R}_+, \mathbb{R}_+]$ where $J = [t_0, T]$, assume that

$$D_- m(t) \leq g(t, |m|_0(t)), \quad t \in J \quad (2.23)$$

where $D_- m(t)$ is a Dini derivative of $m(t)$

$$D_- m(t) = \liminf_{h \rightarrow 0^-} \frac{m(t+h) - m(t)}{h} \quad (2.24)$$

and

$$|m|_0(t) = \sup_{0 \leq s \leq t} |m(s)| \quad (2.25)$$

Additionally, suppose that there exists a maximal solution $r(t) = r(t, t_0, w_0)$ on J of the following equation

$$w' = g(t, w), \quad w(t_0) = w_0 \geq 0 \quad (2.26)$$

Then, $m(t_0) \leq w_0$ implies

$$m(t) \leq r(t), \quad t \in J \quad (2.27)$$

Proof:

For a sufficiently small enough $\varepsilon > 0$, let us consider

$$w' = g(t, w) + \varepsilon, \quad w(t_0) = w_0 + \varepsilon \geq 0 \quad (2.28)$$

whose any solution $w(t, t_0, w_0, \varepsilon)$ of it satisfies that

$$\lim_{\varepsilon \rightarrow 0^+} w(t, t_0, w_0, \varepsilon) = r(t, t_0, w_0) \quad (2.29)$$

Then, in order to prove (2.27), all that is needed to be done is to demonstrate that

$$m(t) \leq w(t, t_0, w_0, \varepsilon), \quad t \in J \quad (2.30)$$

If we suppose that (2.30) is not true, then considering that $m(t_0) < w_0 + \varepsilon$ (as $m(t_0) \leq w_0$), there exists a $t_1 > t_0$ such that

$$\begin{aligned} m(t) &< w(t, t_0, w_0, \varepsilon), \quad t_0 \leq t < t_1, \\ m(t_1) &= w(t_1, t_0, w_0, \varepsilon) \end{aligned} \quad (2.31)$$

Thus,

$$\begin{aligned} D_- m(t_1) &= D_- w(t_1, t_0, w_0, \varepsilon) \geq w'(t_1, t_0, w_0, \varepsilon) \\ &= g(t_1, w(t_1, t_0, w_0, \varepsilon)) + \varepsilon > g(t_1, w(t_1, t_0, w_0, \varepsilon)) \end{aligned} \quad (2.32)$$

On the other hand, since $g(t, w) \geq 0$ this indicates that $w(t, t_0, w_0, \varepsilon)$ is non-decreasing in t , and given (2.31), this gives us that

$$|m|_0(t_1) = \sup_{0 \leq s \leq t_1} |m(s)| = w(t_1, t_0, w_0, \varepsilon) = m(t_1) \quad (2.33)$$

Hence, by (2.23), (2.33) and (2.31), we have

$$D_- m(t_1) \leq g(t_1, |m|_0(t_1)) = g(t_1, m(t_1)) = g(t_1, w(t_1, t_0, w_0, \varepsilon)) \quad (2.34)$$

which contradicts (2.32) completing the proof. ■

Following that, we consider the estimation of any two solutions $U(t), V(t)$ of the SDE with a causal operator (2.21) by means of the maximal solution $r(t) = r(t, t_0, w_0)$ of the scalar differential equation (2.22) using the Comparison Theorem 1 in 2.3.1.1.

We define D_0 on $E = C[[t_0, T], K_c(\mathbb{R}^n)]$ as follows

$$D_0[U, V](t) = \max_{t_0 \leq s \leq t} D[U(s), V(s)] \quad (2.35)$$

2.3.1.2. Theorem (Comparison Theorem 2)

Let $Q \in C[E, E]$ be a causal operator where $E = C[[t_0, T], K_c(\mathbb{R}^n)]$, and assume that

$$D[(QU)(t), (QV)(t)] \leq g(t, D_0[U, V](t)), \quad t \in J = [t_0, T] \quad (2.36)$$

where $g \in C[J \times \mathbb{R}_+, \mathbb{R}_+]$.

Additionally, suppose the existence of a maximal solution $r(t) = r(t, t_0, w_0)$ on J of the scalar differential equation (2.22).

Then, for any two solutions $U(t), V(t)$ of the SDE with a causal operator (2.21) on J with the initial conditions $U(t_0) = U_0, V(t_0) = V_0$, respectively, satisfying that $D[U_0, V_0] \leq w_0$; we have

$$D[U(t), V(t)] \leq r(t, t_0, w_0), \quad t \in J \quad (2.37)$$

Proof:

Setting $m(t) = D[U(t), V(t)]$, gives us

$$m(t_0) = D[U(t_0), V(t_0)] = D[U_0, V_0] \leq w_0 \quad (2.38)$$

and for a sufficiently small enough $h > 0$, we can write

$$m(t + h) = D[U(t + h), V(t + h)] \quad (2.39)$$

Using the following properties sequentially of the Hausdorff metric D

$$\begin{aligned} \text{i) } D[A, B] &\leq D(A, C) + D(C, B) \\ \text{ii) } D(A + C, B + C) &= D(A, B) \end{aligned} \quad (2.40)$$

for any $A, B, C \in K_c(\mathbb{R}^n)$, and in view of the fact that the Hukuhara differences $U(t + h) - U(t)$ and $V(t + h) - V(t)$ exist for any sufficiently small enough $h > 0$, and it leads us to the following

$$\begin{aligned}
m(t+h) &= D[U(t+h), V(t+h)] \\
&\leq D[U(t+h), U(t) + h(QU)(t)] \\
&\quad + D[U(t) + h(QU)(t), V(t+h)] \\
&\leq D[U(t+h), U(t) + h(QU)(t)] \\
&\quad + D[U(t) + h(QU)(t), V(t) + h(QV)(t)] \\
&\quad + D[V(t) + h(QV)(t), V(t+h)] \tag{2.41} \\
&\leq D[U(t+h), U(t) + h(QU)(t)] \\
&\quad + D[U(t) + h(QU)(t), U(t) + h(QV)(t)] \\
&\quad + D[U(t) + h(QV)(t), V(t) + h(QV)(t)] \\
&\quad + D[V(t) + h(QV)(t), V(t+h)]
\end{aligned}$$

Hence,

$$\begin{aligned}
m(t+h) &\leq D[U(t) + Z(t, h), U(t) + h(QU)(t)] + D[h(QU)(t), h(QV)(t)] \\
&\quad + D[U(t), V(t)] + D[V(t) + h(QV)(t), V(t) + Y(t, h)] \tag{2.42}
\end{aligned}$$

where

$$U(t+h) = U(t) + Z(t, h) \quad \text{and} \quad V(t+h) = V(t) + Y(t, h) \tag{2.43}$$

Also, using the property (ii) in (2.40) of the Hausdorff metric D gives us

$$\begin{aligned}
m(t+h) &\leq D[Z(t, h), h(QU)(t)] + D[h(QU)(t), h(QV)(t)] \\
&\quad + D[U(t), V(t)] + D[h(QV)(t), Y(t, h)] \tag{2.44}
\end{aligned}$$

Considering that the Hukuhara differences exist, and replacing $Z(t, h)$ and $Y(t, h)$ by $U(t+h) - U(t)$ and $V(t+h) - V(t)$ respectively, we have

$$\begin{aligned}
m(t+h) &\leq D[U(t+h) - U(t), h(QU)(t)] + D[h(QU)(t), h(QV)(t)] \\
&\quad + D[U(t), V(t)] + D[h(QV)(t), V(t+h) - V(t)] \tag{2.45}
\end{aligned}$$

Subtracting $m(t)$ from both sides of the inequality in (2.45) gives us

$$\begin{aligned}
m(t+h) - m(t) &\leq D[U(t+h) - U(t), h(QU)(t)] + D[h(QU)(t), h(QV)(t)] \\
&\quad + D[h(QV)(t), V(t+h) - V(t)] \tag{2.46}
\end{aligned}$$

By dividing both sides in the (2.46) over $h > 0$, we get

$$\begin{aligned} & \frac{m(t+h) - m(t)}{h} \\ & \leq D \left[\frac{U(t+h) - U(t)}{h}, (QU)(t) \right] + D[(QU)(t), (QV)(t)] \\ & + D \left[(QV)(t), \frac{V(t+h) - V(t)}{h} \right] \end{aligned} \quad (2.47)$$

To obtain $D^+m(t)$ on the left side in (2.47), we take the limit supremum as $h \rightarrow 0^+$ in both sides of the inequality in (2.47). Then, by making use of the fact that both $U(t)$ and $V(t)$ are solutions of the SDE with a causal operator (2.21) on J with the initial conditions $U(t_0) = U_0$, $V(t_0) = V_0$, respectively; together with the assumption (2.36) gives us for $t \in J$

$$D^+m(t) \leq D[(QU)(t), (QV)(t)] \leq g(t, D_0[U, V](t)) = g(t, |m|_0(t)), \quad (2.48)$$

Hence, by the previous Comparison Theorem 1 in 2.3.1.1. we conclude the desired estimation. ■

We can directly conclude from the previous Comparison Theorem 2 in 2.3.1.2. the following corollary, simply by substituting $V(t)$ by the null solution for (2.21).

2.3.1.3. Corollary

Let $Q \in C[E, E]$ be a causal operator where $E = C[[t_0, T], K_c(\mathbb{R}^n)]$, and assume that

$$D[(QU)(t), \theta] \leq g(t, D_0[U, \theta](t)) \quad (2.49)$$

where $g \in C[[t_0, T] \times \mathbb{R}_+, \mathbb{R}_+]$.

Additionally, suppose the existence of a maximal solution $r(t) = r(t, t_0, w_0)$ on J of the scalar differential equation (2.22).

Then, for any solution $U(t)$ of the SDE with a causal operator (2.21) on J with the initial conditions $U(t_0) = U_0$, satisfying that $D[U_0, \theta] \leq w_0$; we have

$$D[U(t), \theta] \leq r(t, t_0, w_0), \quad t \in J \quad (2.50)$$

Now, we can state and prove a local existence result of the solution $U(t)$ of the SDE involving causal operator (2.21), using successive approximations.

2.3.1.4. Theorem (Local Existence Result)

Assume that

- i) $Q \in C[B, E]$ is a causal operator, where $B = B(U_0, b) = \{U \in E : D_0[U, U_0] \leq b\}$ and $E = C[[t_0, T], K_c(\mathbb{R}^n)]$, and it satisfies on B the following inequality

$$D_0[(QU), \theta](t) \leq M_1 \quad (2.51)$$

- ii) $g(t, w) \in C[J \times [0, 2b], \mathbb{R}_+]$ is non-decreasing in w for each $t \in J = [t_0, T]$, $g(t, 0) \equiv 0$, and satisfies the inequality

$$g(t, w) \leq M_2 \quad (2.52)$$

for $(t, w) \in J \times [0, 2b]$.

Additionally, $w(t) = 0$ is the only solution of

$$w' = g(t, w), \quad w(t_0) = 0, \quad t \in J \quad (2.53)$$

- iii)

$$D[(QU)(t), (QV)(t)] \leq g(t, D_0[U, V](t)) \quad (2.54)$$

on B .

Then, the successive approximations defined by

$$U_{n+1}(t) = U_0 + \int_{t_0}^t (QU_n)(s) ds, \quad n = 0, 1, 2, \dots \quad (2.55)$$

exist on $J_0 = [t_0, t_0 + \eta)$ where $\eta = \min\left\{T - t_0, \frac{b}{M}\right\}$ and $M = \max\{M_1, M_2\}$, and uniformly converge to the only solution $U(t)$ of (2.21).

Proof:

For $t \in J_0 = [t_0, t_0 + \eta)$, we have, by induction, used the following features of the Hausdorff metric D

$$D(A + C, B + C) = D(A, B) \quad (2.56)$$

for $A, B \in K_c(\mathbb{R}^n)$ and $C \in K(\mathbb{R}^n)$; and that

$$D \left[\int_{t_0}^t F(s) ds, \int_{t_0}^t G(s) ds \right] \leq \int_{t_0}^t D[F(s), G(s)] ds \quad (2.57)$$

for any integrable set-valued mappings $F, G: [t_0, t] \rightarrow K_c(\mathbb{R}^n)$,

$$\begin{aligned} D[U_{n+1}(t), U_0] &= D \left[U_0 + \int_{t_0}^t (QU_n)(s) ds, U_0 \right] = D \left[\int_{t_0}^t (QU_n)(s) ds, \theta \right] \\ &\leq \int_{t_0}^t D[(QU_n)(s), \theta] ds \leq \int_{t_0}^t D_0[QU_n, \theta](t) ds \leq M_1(t - t_0) \\ &\leq M(t - t_0) \leq b \end{aligned} \quad (2.58)$$

which demonstrates the well-definedness of the successive approximations on J_0 .

Following that, we then define the following successive approximations

$$\begin{aligned} w_0(t) &= M(t - t_0) \\ w_{n+1}(t) &= \int_{t_0}^t g(s, w_n(s)) ds, \quad t \in J_0 = [t_0, t_0 + \eta), \quad n = 0, 1, 2, \dots \end{aligned} \quad (2.59)$$

We aim to prove the uniform convergence of the sequence $\{w_k(t)\}_{k \geq 1}$ to the zero solution $w(t) = 0$ of the scalar differential equation (2.53).

First, let us demonstrate by mathematical induction the monotonicity of the sequence $\{w_k(t)\}_{k \geq 1}$.

Starting from (2.59), we get the following estimation regarding $w_1(t)$ using (2.52)

$$\begin{aligned} w_1(t) &= \int_{t_0}^t g(s, w_0(s)) ds = \int_{t_0}^t g(s, M(t - t_0)) ds \leq M_2(t - t_0) \\ &\leq M(t - t_0) = w_0(t), \quad t \in J_0 = [t_0, t_0 + \eta) \end{aligned} \quad (2.60)$$

For $k > 1$, let us assume that

$$w_k(t) \leq w_{k-1}(t), \quad t \in J_0 = [t_0, t_0 + \eta) \quad (2.61)$$

Then, by the monotonicity of g , for $w_k(t) \leq w_{k-1}(t)$, we obtain the following inequality

$$g(t, w_k(t)) \leq g(t, w_{k-1}(t)), \quad t \in J_0 = [t_0, t_0 + \eta) \quad (2.62)$$

So, for $t \in J_0 = [t_0, t_0 + \eta)$,

$$w_{k+1}(t) = \int_{t_0}^t g(s, w_k(s)) ds \leq \int_{t_0}^t g(s, w_{k-1}(s)) ds = w_k(t), \quad (2.63)$$

Thus, the sequence $\{w_k(t)\}_{k \geq 1}$ is monotone decreasing. Moreover,

$$w'_k(t) = g(t, w_{k-1}(t)) \leq M_2, \quad t \in J_0, \quad (2.64)$$

Hence, by Arzelà–Ascoli theorem and the monotonicity of $\{w_k(t)\}_{k \geq 1}$, we conclude the uniform convergence of the sequence $\{w_k(t)\}_{k \geq 1}$ to $w(t)$ on J_0 .

$$\lim_{k \rightarrow \infty} w_k(t) = w(t), \quad t \in J_0, \quad (2.65)$$

Since $w(t)$ satisfies equation (2.53), we get from (ii) that $w(t) \equiv 0$ on J_0 .

Now, let us show by mathematical induction that

$$D_0[U_{n+1}, U_n](t) \leq w_n(t) \quad (2.66)$$

for $t \in J_0$ and for all $n = 0, 1, 2, \dots$

First, for $t \in J_0$, $t_0 \leq s \leq t$, and using the features of the Hausdorff metric D and the assumption in (i), we have

$$\begin{aligned} D[U_1(s), U_0] &= D \left[U_0 + \int_{t_0}^s (QU_0)(\xi) d\xi, U_0 \right] = D \left[\int_{t_0}^s (QU_0)(\xi) d\xi, \theta \right] \\ &\leq \int_{t_0}^s D[(QU_0)(\xi), \theta] d\xi \leq D_0[(QU_0), \theta](\xi)(s - t_0) \\ &\leq D_0[(QU_0), \theta](\xi)(t - t_0) \leq M_1(t - t_0) \leq M(t - t_0) \\ &= w_0(t) \end{aligned} \quad (2.67)$$

Thus,

$$D_0[U_1, U_0](t) \leq w_0(t), \quad t \in J_0 \quad (2.68)$$

Next, we assume the following statement holds for some $k > 1$

$$D_0[U_k, U_{k-1}](t) \leq w_{k-1}(t), \quad t \in J_0 \quad (2.69)$$

We shall show that

$$D_0[U_{k+1}, U_k](t) \leq w_k(t), \quad t \in J_0 \quad (2.70)$$

For $t \in J_0$, $t_0 \leq s \leq t$, and by making use of the attributes of D , the assumption in (iii), and the induction hypothesis together with the monotonicity of g , we reach the following conclusion

$$\begin{aligned} D[U_{k+1}(s), U_k(s)] &= D \left[U_0 + \int_{t_0}^s (QU_k)(\xi) d\xi, U_0 + \int_{t_0}^s (QU_{k-1})(\xi) d\xi \right] \\ &= D \left[\int_{t_0}^s (QU_k)(\xi) d\xi, \int_{t_0}^s (QU_{k-1})(\xi) d\xi \right] \\ &\leq \int_{t_0}^s D[(QU_k)(\xi), (QU_{k-1})(\xi)] d\xi \\ &\leq \int_{t_0}^s g(\xi, D_0[U_k, U_{k-1}](\xi)) d\xi \leq \int_{t_0}^s g(\xi, w_{k-1}(\xi)) d\xi \\ &\leq \int_{t_0}^t g(\xi, w_{k-1}(\xi)) d\xi = w_k(t) \end{aligned} \quad (2.71)$$

Thus,

$$D_0[U_{k+1}, U_k](t) \leq w_k(t), \quad t \in J_0 \quad (2.72)$$

Hence,

$$D_0[U_{n+1}, U_n](t) \leq w_n(t), \quad t \in J_0 \quad (2.73)$$

for all $n = 0, 1, 2, \dots$

Now, we aim to show that $\{U_n(t)\}_{n \geq 1}$ is a Cauchy sequence. For this, let us set

$$v(t) = D[U_n(t), U_m(t)], \quad t \in J_0 \quad (2.74)$$

for $n \leq m$.

Using the successive approximations defined by (2.55), the properties of the Hausdorff metric D , the assumption in (iii), (2.73), together with the monotonicity of g , we have

$$\begin{aligned} D^+v(t) &\leq D[D_H U_n(t), D_H U_m(t)] \\ &= D \left[D_H \left(U_0 + \int_{t_0}^t (QU_{n-1})(s) ds \right), D_H \left(U_0 + \int_{t_0}^t (QU_{m-1})(s) ds \right) \right] \\ &= D[(QU_{n-1})(t), (QU_{m-1})(t)] \\ &\leq D[(QU_{n-1})(t), (QU_n)(t)] + D[(QU_n)(t), (QU_m)(t)] \\ &\quad + D[(QU_m)(t), (QU_{m-1})(t)] \\ &\leq g(t, D_0[U_{n-1}, U_n](t)) + g(t, D_0[U_n, U_m](t)) \\ &\quad + g(t, D_0[U_m, U_{m-1}](t)) \\ &\leq g(t, D_0[U_n, U_{n-1}](t)) + g\left(t, \max_{t_0 \leq s \leq t} D[U_n(s), U_m(s)]\right) \\ &\quad + g(t, D_0[U_m, U_{m-1}](t)) \\ &\leq g(t, w_{n-1}(t)) + g(t, |v|_0(t)) + g(t, w_{n-1}(t)) \\ &= g(t, |v|_0(t)) + 2g(t, w_{n-1}(t)) \end{aligned} \quad (2.75)$$

Then, applying the Comparison Theorem 1 in 2.3.1.1., the previous inequalities in (2.75) give us that

$$v(t) \leq r_n(t), \quad t \in J_0 \quad (2.76)$$

where $r_n(t)$ is the maximal solution of

$$r'_n(t) = g(t, r_n) + 2g(t, w_{n-1}(t)), \quad r_n(t_0) = 0 \quad (2.77)$$

for each n , on J_0 .

Since $2g(t, w_{n-1}(t)) \rightarrow 0$ uniformly on J_0 , as $n \rightarrow \infty$; it follows by Lemma 1.3.1 in [6] that $r_n(t) \rightarrow 0$ uniformly on J_0 , as $n \rightarrow \infty$.

This implies from (2.73) that $U_n(t)$ converges uniformly to $U(t)$ on J_0 , as $n \rightarrow \infty$. And clearly $U(t)$ is a solution of (2.21).

To prove that the solution $U(t)$ of (2.21) is unique, let us suppose that there exists another solution such $V(t)$ of (2.21) on J_0 ; and let us set

$$m(t) = D[U(t), V(t)] \quad (2.78)$$

Then, we have

$$m(t_0) = D[U(t_0), V(t_0)] = D[U_0, U_0] = 0 \quad (2.79)$$

and by a similar manner to that in (2.75), it is concluded that

$$D^+m(t) \leq g(t, |m|_0(t)), \quad t \in J_0 \quad (2.80)$$

Thus, applying the Comparison Theorem 1 in 2.3.1.1. gives us

$$m(t) \leq r(t, t_0, 0), \quad t \in J_0 \quad (2.81)$$

where $r(t, t_0, 0)$ is the maximal solution of (2.53).

The assumption in (ii) that $w(t) = 0$ is the only solution of (2.53) gives us that $m(t) = 0$, or in other words $U(t) = V(t)$, $t \in J_0$,

This proves the uniqueness and completes the proof. ■

Let us now present a global existence result for the solution $U(t)$ of the SDE with a causal operator (2.21), following the local existence result.

2.3.1.5. Theorem (Global Existence Result)

Assume that

- i) $Q \in C[E, E]$ is a causal operator, where $E = C[[t_0, T], K_c(\mathbb{R}^n)]$,
- ii) $g(t, w) \in C[\mathbb{R}_+^2, \mathbb{R}_+]$ is non-decreasing in w for all $t \in \mathbb{R}_+$, and the following inequality holds

$$D[(QU)(t), \theta] \leq g(t, D_0[U, \theta](t)) \quad (2.82)$$

iii) There exists a maximal solution $r(t) = r(t, t_0, w_0)$ of

$$w' = g(t, w), \quad w(t_0) = 0, \quad t \in [t_0, \infty) \quad (2.83)$$

iv) Q is sufficiently smooth to ensure the local existence of solutions of (2.21) through any initial values $(t_0, U_0) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n)$.

Then, $D[U_0, \theta] < w_0$ implies that the largest interval of existence for any solution $U(t, t_0, U_0)$ of (2.21) is $[t_0, \infty)$.

Proof:

Assume that $U(t) = U(t, t_0, U_0)$ is any solution of (2.21) existing on $[t_0, \beta)$, $t_0 < \beta < \infty$, with $D[U_0, \theta] \leq w_0$, and the value of β cannot be increased.

Setting

$$m(t) = D[U(t), \theta] \quad (2.84)$$

We have

$$m(t_0) = D[U(t_0), \theta] = D[U_0, \theta] \leq w_0 \quad (2.85)$$

Then, given the assumption (ii), we have

$$\begin{aligned} D^+m(t) &= D^+D[U(t), \theta] \leq D[D_H U(t), \theta] = D[(QU)(t), \theta] \\ &\leq g(t, D_0[U, \theta](t)) \end{aligned} \quad (2.86)$$

Using the Comparison Theorem 1 in 2.3.1.1., we get

$$m(t) \leq r(t), \quad t_0 \leq t < \beta \quad (2.87)$$

For any t_1, t_2 such that $t_0 < t_1 < t_2 < \beta$, using (2.82) and the properties of Hausdorff metric D , we have

$$\begin{aligned}
D[U(t_1), U(t_2)] &= D \left[U_0 + \int_{t_0}^{t_1} (QU)(s) ds, U_0 + \int_{t_0}^{t_2} (QU)(s) ds \right] \\
&= D \left[\int_{t_0}^{t_1} (QU)(s) ds, \int_{t_0}^{t_2} (QU)(s) ds \right] \leq \int_{t_1}^{t_2} D[(QU)(s), \theta] ds \quad (2.88) \\
&\leq \int_{t_1}^{t_2} g(s, D_0[U, \theta](s)) ds
\end{aligned}$$

Employing the estimation (2.86) and the monotonicity of $g(t, w)$, we get

$$\begin{aligned}
D[U(t_1), U(t_2)] &\leq \int_{t_1}^{t_2} g(s, D_0[U, \theta](s)) ds \leq \int_{t_1}^{t_2} g(s, r(s)) ds \quad (2.89) \\
&= r(t_2) - r(t_1)
\end{aligned}$$

As $\lim_{t \rightarrow \beta^-} r(t, t_0, w_0)$ exists, by taking the limit as $t_1, t_2 \rightarrow \beta^-$, we obtain that $\{U(t_n)\}$ is a Cauchy sequence. Hence,

$$\lim_{t \rightarrow \beta^-} U(t, t_0, U_0) = U_\beta \quad (2.90)$$

exists.

Considering the IVP of the SDE with a causal operator

$$D_H U(t) = (QU)(t), \quad U(\beta) = U_\beta \in K_c(\mathbb{R}^n) \quad (2.91)$$

We supposed the local existence; However, $U(t, t_0, U_0)$ is possible to continue beyond β , which contradicts the premise regarding the value of β .

Therefore, every solution $U(t, t_0, U_0)$ of (2.21) such that $D[U_0, \theta] \leq w_0$ exists globally on $[t_0, \infty)$.

Or in other words; $D[U_0, \theta] < w_0$ implies that the largest interval of existence for any solution $U(t, t_0, U_0)$ of (2.21) is $[t_0, \infty)$; and this completes the proof. ■

Following the presentation of the two essential comparison theorems, as well as the local and global existence results of the solution $U(t)$ of the SDE with a causal

operator (2.21), we will present another comparison theorem by means of Lyapunov-like functions.

To utilize the approach of Lyapunov functions, first it is important to identify the minimum subsets of $E = C[[t_0, T], K_c(\mathbb{R}^n)]$ across which the Lyapunov function's derivative may be simply evaluated; let $L \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$ where

$$B = B(\theta, \rho) = \{U \in K_c(\mathbb{R}^n) : D[U, \theta] \leq \rho\} \quad (2.92)$$

and let us define the following set

$$E_\alpha = \{U \in K_c(\mathbb{R}^n) : L(s, U(s)) \alpha(s) \leq L(t, U(t)) \alpha(t), \quad t_0 \leq s \leq t\} \quad (2.93)$$

$$E_1 = \{U \in K_c(\mathbb{R}^n) : L(s, U(s)) \leq L(t, U(t)), \quad t_0 \leq s \leq t\} \quad (2.94)$$

$$E_0 = \{U \in K_c(\mathbb{R}^n) : L(s, U(s)) \leq f(L(t, U(t))), t_1 \leq s \leq t, t_1 \geq t_0\} \quad (2.95)$$

where

- i) $\alpha(t) > 0$ is a continuous function on $\mathbb{R}_+$,
- ii) $f(r)$ is non-decreasing in r , continuous on $\mathbb{R}_+$, in addition that $f(r) > r$ for any $r > 0$.

Before presenting the following comparison theorem by means of Lyapunov-like functions, it is worth mentioning the next well-known lemma from Lakshmikantham, V. and Rama Mohana Rao, M. [23].

2.3.1.6. Lemma

Let $g_0, g \in C[\mathbb{R}_+^2, \mathbb{R}]$ and for any pair $(t, w) \in \mathbb{R}_+^2$, we have

$$g_0(t, w) \leq g(t, w) \quad (2.96)$$

Then, the right maximal solution $r(t, t_0, w_0)$ of

$$w' = g(t, w), \quad w(t_0) = w_0 \geq 0 \quad (2.97)$$

and the left maximal solution $\eta(t, T_0, v_0)$ of

$$v' = g_0(t, v), \quad v(T_0) = v_0 \geq 0 \quad (2.98)$$

satisfy that $r(T_0, t_0, w_0) \leq v_0$ implies

$$r(t, t_0, w_0) \leq \eta(t, T_0, v_0), \quad t_0 \leq t \leq T_0 \quad (2.99)$$

2.3.1.7. Theorem (Comparison Theorem 3)

Assume that

- i) $L(t, W) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ be locally Lipschitzian in W ; i.e., given any W_1, W_2 in $K_c(\mathbb{R}^n)$, we guarantee the existence of a $k \in \mathbb{R}$, which is called a Lipschitzian constant, such that the following inequality holds.

$$|L(t, W_1) - L(t, W_2)| \leq k D[W_1, W_2], \quad t \in \mathbb{R}_+ \quad (2.100)$$

- ii) $g_0, g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$ such that

$$g_0(t, w) \leq g(t, w), \quad (t, w) \in \mathbb{R}_+^2 \quad (2.101)$$

- iii) For $t \in [t_0, \infty)$, there exists a maximal solution $r(t, t_0, w_0)$ of

$$w' = g(t, w), \quad w(t_0) = w_0 \geq 0 \quad (2.102)$$

- iv) For $t \in [t_0, T_0]$, there exists a left maximal solution $\eta(t, T_0, v_0)$ of

$$v' = g_0(t, v), \quad v(T_0) = v_0 \geq 0 \quad (2.103)$$

- v)

$$D_-L(t, U(t)) \leq g(t, L(t, U(t))) \text{ on } \Omega \quad (2.104)$$

where

$$\Omega = \left\{ U \in E : L(s, U(s)) \leq \eta(s, t, L(t, U(t))), \quad t_0 \leq s \leq t \right\} \quad (2.105)$$

and

$$D_-L(t, U(t)) = \liminf_{h \rightarrow 0^-} \frac{1}{h} [L(t+h, U(t) + h(QU)(t)) - L(t, U(t))] \quad (2.106)$$

Then,

$$L(t_0, U_0) \leq w_0 \quad (2.107)$$

implies

$$L(t, U(t, t_0, U_0)) \leq r(t, t_0, w_0), \quad t \geq t_0 \quad (2.108)$$

Proof:

Let us set

$$m(t) = L(t, U(t, t_0, U_0)), \quad t \geq t_0 \quad (2.109)$$

Then,

$$m(t_0) = L(t_0, U(t_0, t_0, U_0)) = L(t_0, U_0) \leq w_0 \quad (2.110)$$

For a sufficiently small enough $\varepsilon > 0$, suppose we have the scalar differential equation as follows:

$$w' = g(t, w) + \varepsilon, \quad w(t_0) = w_0 + \varepsilon \geq 0 \quad (2.111)$$

whose any solution $w(t, t_0, w_0, \varepsilon)$ of it satisfies that

$$\lim_{\varepsilon \rightarrow 0^+} w(t, t_0, w_0, \varepsilon) = r(t, t_0, w_0) \quad (2.112)$$

Then, proving that

$$m(t) \leq w(t, t_0, w_0, \varepsilon), \quad t \geq t_0 \quad (2.113)$$

is sufficient to imply (2.108).

If we suppose that (2.113) is not true, then considering that $m(t_0) < w_0 + \varepsilon$ (as $m(t_0) \leq w_0$), there exists a $t_1 > t_0$ such that

$$\begin{aligned} m(t) &< w(t, t_0, w_0, \varepsilon), \quad t_0 \leq t < t_1, \\ m(t_1) &= w(t_1, t_0, w_0, \varepsilon) \end{aligned} \quad (2.114)$$

Thus,

$$\begin{aligned} D_-m(t_1) &= D_-w(t_1, t_0, w_0, \varepsilon) \geq w'(t_1, t_0, w_0, \varepsilon) \\ &= g(t_1, w(t_1, t_0, w_0, \varepsilon)) + \varepsilon > g(t_1, w(t_1, t_0, w_0, \varepsilon)) \end{aligned} \quad (2.115)$$

On the other hand, considering the left maximal solution $\eta(s, t_1, m(t_1))$ of (2.103) with $v(t_1) = m(t_1)$ for $t_0 \leq s \leq t_1$. By the previous lemma in 2.3.1.6., the following inequality holds.

$$r(s, t_0, w_0) \leq \eta(s, t_1, m(t_1)), \quad t_0 \leq s \leq t_1 \quad (2.116)$$

Using (2.112) and (2.114) gives us that

$$r(t_1, t_0, w_0) = \lim_{\varepsilon \rightarrow 0^+} w(t_1, t_0, w_0, \varepsilon) = m(t_1) = v(t_1) = \eta(t_1, t_1, m(t_1)) \quad (2.117)$$

and for $t_0 \leq s \leq t_1$, we have

$$m(s) \leq w(s, t_0, w_0, \varepsilon) \quad (2.118)$$

Hence,

$$m(s) \leq r(s, t_0, w_0) \leq \eta(s, t_1, m(t_1)), \quad t_0 \leq s \leq t_1 \quad (2.119)$$

This implies that assumption (v) holds for $U(s, t_0, U_0)$ on $t_0 \leq s \leq t_1$. Thus, using (2.109) and the assumption (v), we have

$$D_-m(t_1) = D_-L(t_1, U(t_1, t_0, U_0)) \leq g(t_1, L(t_1, U(t_1))) = g(t_1, m(t_1)) \quad (2.120)$$

The last inequality gives us a contradiction to that inequality in (2.115).

Hence, we have $m(t) \leq w(t, t_0, w_0, \varepsilon)$, $t \geq t_0$, for any solution $w(t, t_0, w_0, \varepsilon)$ of the scalar differential equation (2.111). Taking $\varepsilon \rightarrow 0^+$, we have (2.112) holds. Therefore,

$$m(t) \leq r(t, t_0, w_0), \quad t \geq t_0 \quad (2.121)$$

This completes the proof. ■

2.3.1.8. Theorem (Comparison Theorem 4)

Assume that

- i) $L(t, W) \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$ is locally Lipschitzian in W ; i. e. for any W_1, W_2 in B , we guarantee the existence of a $k \in \mathbb{R}$, which is called a Lipschitzian constant, such that the following inequality holds.

$$|L(t, W_1) - L(t, W_2)| \leq k D[W_1, W_2], \quad t \in \mathbb{R}_+ \quad (2.122)$$

where $B = B(\theta, \rho) = \{U \in K_c(\mathbb{R}^n) : D[U, \theta] \leq \rho\}$.

- ii) $g(t, w) \in C[\mathbb{R}_+^2, \mathbb{R}_+]$ and for $U \in E_1$ the following inequality holds.

$$D_-L(t, U(t)) \leq g(t, L(t, U(t))), \quad t \geq t_0 \quad (2.123)$$

where

$$E_1 = \{U \in K_c(\mathbb{R}^n) : L(s, U(s)) \leq L(t, U(t)), \quad t_0 \leq s \leq t\} \quad (2.124)$$

and

$$D_-L(t, U(t)) = \liminf_{\varepsilon \rightarrow 0^-} \frac{1}{\varepsilon} [L(t + \varepsilon, U(t) + \varepsilon(QU)(t)) - L(t, U(t))], \quad (2.125)$$

- iii) A maximal solution $r(t) = r(t, t_0, w_0)$ of the scalar differential equation (2.22) exists for $t \geq t_0$.

Then, for any solution $U(t, t_0, U_0) \in B$ of the SDE with a causal operator (2.21) for $t_0 \leq t \leq t_1$, we have

$$L(t_0, U_0) \leq w_0 \text{ implies } L(t, U(t, t_0, U_0)) \leq r(t), \quad t_0 \leq t \leq t_1 \quad (2.126)$$

Proof:

Let $U(t, t_0, U_0) \in B$ be any solution of the SDE with a causal operator (2.21) for $t_0 \leq t \leq t_1$.

Let us set

$$m(t) = L(t, U(t, t_0, U_0)), \quad t_0 \leq t \leq t_1 \quad (2.127)$$

Then,

$$m(t_0) = L(t_0, U(t_0, t_0, U_0)) = L(t_0, U_0) \leq w_0 \quad (2.128)$$

For a sufficiently small enough $\varepsilon > 0$, suppose we have the scalar differential equation as follows:

$$w' = g(t, w) + \varepsilon, \quad w(t_0) = w_0 + \varepsilon \geq 0 \quad (2.129)$$

whose any solution $w(t, t_0, w_0, \varepsilon)$ of it satisfies that

$$\lim_{\varepsilon \rightarrow 0^+} w(t, t_0, w_0, \varepsilon) = r(t, t_0, w_0) \quad (2.130)$$

Then, proving that

$$m(t) \leq w(t, t_0, w_0, \varepsilon), \quad t_0 \leq t \leq t_1 \quad (2.131)$$

is sufficient to imply (2.126).

If we suppose that (2.131) is not true, then considering that $m(t_0) < w_0 + \varepsilon$ (as $m(t_0) \leq w_0$), there exists a t_2 such that $t_0 < t_2 < t_1$ and

$$\begin{aligned} m(t) &< w(t, t_0, w_0, \varepsilon), \quad t_0 \leq t < t_2, \\ m(t_2) &= w(t_2, t_0, w_0, \varepsilon) \end{aligned} \quad (2.132)$$

Thus,

$$\begin{aligned} D_- m(t_2) &= D_- w(t_2, t_0, w_0, \varepsilon) \geq w'(t_2, t_0, w_0, \varepsilon) \\ &= g(t_2, w(t_2, t_0, w_0, \varepsilon)) + \varepsilon > g(t_2, w(t_2, t_0, w_0, \varepsilon)) \end{aligned} \quad (2.133)$$

Considering the assumption on g , the solutions $w(t, t_0, w_0, \varepsilon)$ of (2.129) are increasing functions of t . Thus, as $m(t) = L(t, U(t, t_0, U_0))$ and given (2.132), we have

$$L(s, U(s, t_0, U_0)) \leq L(t_2, U(t_2, t_0, U_0)), \quad t_0 \leq s \leq t_2 \quad (2.134)$$

Therefore, $U(t, t_0, U_0) \in E_1$, $t_0 \leq t \leq t_2$.

For $h > 0$, the locally Lipschitzity of $L(t, W)$ in W gives us that

$$\begin{aligned}
m(t+h) - m(t) &= L(t+h, U(t+h, t_0, U_0)) - L(t, U(t, t_0, U_0)) \\
&= L(t+h, U(t+h, t_0, U_0)) \\
&\quad - L(t+h, U(t, t_0, U_0) + h(QU)(t)) \\
&\quad + L(t+h, U(t, t_0, U_0) + h(QU)(t)) - L(t, U(t, t_0, U_0)) \quad (2.135) \\
&\geq -kD[U(t+h, t_0, U_0), U(t, t_0, U_0) + h(QU)(t)] \\
&\quad + L(t+h, U(t, t_0, U_0) + h(QU)(t)) - L(t, U(t, t_0, U_0))
\end{aligned}$$

Then, taking the limit inf of both sides as $h \rightarrow 0^-$ gives us, because of the inequality (2.123) in (ii) and the fact that $D_H U(t)$ exists and is equal to $(QU)(t)$,

$$D_- m(t) = D_- L(t, U(t, t_0, U_0)) \leq g(t, L(t, U(t, t_0, U_0))) = g(t, m(t)) \quad (2.136)$$

Hence, for $t = t_2$, (2.136) with (2.132) gives us that

$$D_- m(t_2) \leq g(t_2, m(t_2)) = g(t_2, w(t_2, t_0, w_0, \varepsilon)) \quad (2.137)$$

which is a contradiction to (2.133).

Then, this proves (2.131), implying (2.126) and completing the proof. ■

2.3.1.9. Corollary

Assume that $L(t, W) \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$ is locally Lipschitzian in W where $B = B(\theta, \rho) = \{U \in K_c(\mathbb{R}^n) : D[U, \theta] \leq \rho\}$, and for $U \in E_0$ we have

$$D_- L(t, U(t)) \leq 0, \quad t \geq t_0 \quad (2.138)$$

where

$$E_0 = \left\{ U \in K_c(\mathbb{R}^n) : L(s, U(s)) \leq f(L(t, U(t))), t_1 \leq s \leq t, t_1 \geq t_0 \right\} \quad (2.139)$$

and $f(r)$ is non-decreasing in r , continuous on $\mathbb{R}_+$, in addition that $f(r) > r$ for any $r > 0$

Then, for any solution $U(t) = U(t, t_0, U_0)$ of (2.21) we have

$$L(t, U(t)) \leq L(t_0, U_0), \quad t \geq t_0 \quad (2.140)$$

Proof:

Continuing in an analogous manner to the proof of the Comparison Theorem 4 in 2.3.1.8., by considering $g(t, w) = 0$, we have for t_2 such that $t_0 < t_2 < t_1$, the following inequality

$$L(s, U(s)) \leq L(t_2, U(t_2)), \quad t_0 \leq s \leq t_2 \quad (2.141)$$

Then, since

$$\begin{aligned} L(t_2, U(t_2)) &= L(t_2, U(t_2, t_0, U_0)) = m(t_2) = w(t_2, t_0, w_0, \varepsilon) \\ &= L(t_0, U_0) + \varepsilon(t_2 - t_0) + \varepsilon > 0 \end{aligned} \quad (2.142)$$

We have

$$L(s, U(s)) \leq f(L(t_2, U(t_2))), \quad t_0 \leq s \leq t_2 \quad (2.143)$$

For $h > 0$, the locally Lipschitzity of $L(t, W)$ in W gives us that

$$\begin{aligned} m(t+h) - m(t) &= L(t+h, U(t+h, t_0, U_0)) - L(t, U(t, t_0, U_0)) = \\ &= L(t+h, U(t+h, t_0, U_0)) \\ &\quad - L(t+h, U(t, t_0, U_0) + h(QU)(t)) \\ &\quad + L(t+h, U(t, t_0, U_0) + h(QU)(t)) - L(t, U(t, t_0, U_0)) \\ &\geq -kD[U(t+h, t_0, U_0), U(t, t_0, U_0) + h(QU)(t)] \\ &\quad + L(t+h, U(t, t_0, U_0) + h(QU)(t)) - L(t, U(t, t_0, U_0)) \end{aligned} \quad (2.144)$$

Then, taking the limit inf of both sides as $h \rightarrow 0^-$ gives us, in view of the inequality (2.138) and the fact that $D_H U(t)$ exists and is equal to $(QU)(t)$,

$$D_- m(t) = D_- L(t, U(t, t_0, U_0)) \leq 0 \quad (2.145)$$

Hence, for $t = t_2$, (2.145) gives us that

$$D_- m(t_2) = D_- L(t_2, U(t_2, t_0, U_0)) \leq 0 \quad (2.146)$$

Which gives us a contradiction in view of (2.142).

This completes the proof. ■

2.3.1.10. Theorem (Comparison Theorem 5)

Given that all the assumptions of the Comparison Theorem 4 in 2.3.1.8. hold, except for (2.123). Instead of it, we assume that for $U \in E_\alpha$ we have

$$\alpha(t)D_-L(t, U(t)) + L(t, U(t))D_- \alpha(t) \leq w(t, L(t, U(t)) \alpha(t)), \quad t \geq t_0 \quad (2.147)$$

where the function $\alpha(t) > 0$ is continuous on $\mathbb{R}_+$, and

$$E_\alpha = \{U \in K_c(\mathbb{R}^n) : L(s, U(s)) \alpha(s) \leq L(t, U(t)) \alpha(t), \quad t_0 \leq s \leq t\} \quad (2.148)$$

and

$$D_-L(t, U(t)) = \liminf_{\varepsilon \rightarrow 0^-} \frac{1}{\varepsilon} [L(t + \varepsilon, U(t) + \varepsilon(QU)(t)) - L(t, U(t))], \quad (2.149)$$

and

$$D_- \alpha(t) = \liminf_{\varepsilon \rightarrow 0^-} \frac{\alpha(t + \varepsilon) - \alpha(t)}{\varepsilon} \quad (2.150)$$

Then,

$$\alpha(t_0)L(t_0, U_0) \leq w_0 \text{ implies } \alpha(t)L(t, U(t)) \leq r(t), \quad t \geq t_0 \quad (2.151)$$

Proof:

Let

$$M(t, U(t)) = L(t, U(t))\alpha(t) \quad (2.152)$$

Then,

$$M(t_0, U(t_0)) = L(t_0, U(t_0))\alpha(t_0) \leq w_0 \quad (2.153)$$

Let $U \in E_\alpha$. For a sufficiently small enough $\varepsilon > 0$, we have

$$\begin{aligned} M(t + \varepsilon, U(t) + \varepsilon(QU)(t)) - M(t, U(t)) \\ &= L(t + \varepsilon, U(t) + \varepsilon(QU)(t))\alpha(t + \varepsilon) - L(t, U(t))\alpha(t) \\ &= L(t + \varepsilon, U(t) + \varepsilon(QU)(t))(\alpha(t + \varepsilon) - \alpha(t)) \\ &\quad + [L(t + \varepsilon, U(t) + \varepsilon(QU)(t)) - L(t, U(t))] \alpha(t) \end{aligned} \quad (2.154)$$

Using the locally Lipschitzity of $L(t, W)$ in W , then taking the limit inf of both sides as $h \rightarrow 0^-$ with (2.147) gives us that

$$\begin{aligned} D_-M(t, U(t)) &= L(t, U(t))D_- \alpha(t) + \alpha(t) D_-L(t, U(t)) \\ &\leq w(t, L(t, U(t)) \alpha(t)) = w(t, M(t, U(t))) \end{aligned} \quad (2.155)$$

for $t_0 \leq t \leq t_1$ and $U \in E_1$, where E_1 here is defined as

$$E_1 = \{U \in K_c(\mathbb{R}^n) : M(s, U(s)) \leq M(t, U(t)), \quad t_0 \leq s \leq t\} \quad (2.156)$$

As $M(t, U)$ is locally Lipschitzian in U , we ascertain all the hypotheses of the Comparison Theorem 4 in 2.3.1.8. including the inequality in (2.123) with $L(t, U(t))$ being replaced by $M(t, U(t))$. Thus, by the Comparison Theorem 4 in 2.3.1.8. we conclude the requested implication (2.151), which completes the proof. ■

2.3.2. Stability Analysis

In this section, we restate the most important comparison results presented by Drici Z., Mcrae F. A., and Devi J. V. in [16] aimed to examine the qualitative behaviors of the solutions of SDEs with causal operators by means of Lyapunov-like functions that act as vehicles to translate the SDEs into corresponding scalar comparison systems, making it sufficient to derive the stability features of their solutions by evaluating the qualitative aspects of the simpler corresponding comparison systems, utilizing the comparison results presented in 2.3.1.

In what follows, in view of the local and global existence results (in 2.3.1.4. and 2.3.1.5.), let us consider $E = C[[t_0, \infty), K_c(\mathbb{R}^n)]$ with norm

$$\|U\| = \sup_{t \in [t_0, \infty)} \frac{D[U(t), \theta]}{h(t)} < \infty, \quad (2.157)$$

where $U \in E$; θ denotes the zero element in $\mathbb{R}^n$, and it is treated as a point set; and $h : [t_0, \infty) \rightarrow \mathbb{R}_+$ is continuous. E with such a norm is Banach space.

Let us start out with some definitions of the most fundamental terms.

2.3.2.1. Definition of Class $\mathcal{K}$ and Class $\mathcal{K}_\infty$

The function $\sigma \in \mathcal{C}[[0, \rho), \mathbb{R}_+]$ is said to be a function from class $\mathcal{K}$, and expressed as $\sigma \in \mathcal{K}$, if and only if for any λ such that $0 < \lambda < \rho$, $\sigma(\lambda)$ is strictly increasing in λ , and σ associates zero to zero.

If additionally, $\rho \rightarrow \infty$ and $\sigma(\lambda) \rightarrow \infty$ as $\lambda \rightarrow \infty$, then σ is said to be a function from the class $\mathcal{K}_\infty$, and expressed as $\sigma \in \mathcal{K}_\infty$.

2.3.2.2. Definition of Positive Definite and Decrescent Property

Let $L(t, U) \in \mathcal{C}[\mathbb{R}_+ \times B, \mathbb{R}_+]$, where $B = B(\theta, \rho) = \{U \in K_c(\mathbb{R}^n) : D[U, \theta] \leq \rho\}$.

$L(t, U)$ is said to be

- i) positive definite if and only if $L(t, 0) \equiv 0$ and we ensure the existence of a $b \in \mathcal{K}$ such that the following inequality holds.

$$b(D[U, \theta]) \leq L(t, U), \quad t \geq 0 \quad (2.158)$$

- ii) decrescent if and only if we ensure the existence of an $a \in \mathcal{K}$ such that the following inequality holds.

$$L(t, U) \leq a(D[U, \theta]), \quad t \geq 0 \quad (2.159)$$

2.3.2.3. Definition of Stability of the Trivial Solution

The trivial solution $U = \theta$ of (2.21) is said to be stable if and only if for each $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$D[U_0, \theta] \leq \delta \text{ implies } D[U(t), \theta] < \varepsilon, \quad t \geq t_0 \quad (2.160)$$

where $U(t) = U(t, t_0, U_0)$ is any solution of (2.21).

Before presenting the following theorems, discussing various stability features for the equilibrium points of the SDEs with causal operators, giving analogous results to Lyapunov original results; it is assumed that the solutions of the SDE (2.21) exist and that they are unique for all $t \geq t_0$.

Additionally, it is assumed that $U_0 = V_0 + W_0$ so we can ensure the existence of the Hukuhara difference $W_0 = U_0 - V_0$; and match the behavioral properties of solutions of the SDEs (2.21) with those properties of solutions of corresponding ordinary differential equations involving causal operators. Hence, $U(t) = U(t, t_0, U_0 - V_0) = U(t, t_0, W_0)$. With that in mind, let us look at the SDE with the causal operator shown below as a starting point for our analysis.

$$D_H U(t) = (QU)(t), \quad U(t_0) = W_0 \in K_c(\mathbb{R}^n) \quad (2.161)$$

Next, we will offer sufficient conditions for drawing various conclusions about the stability and other qualitative properties of the equilibrium point of the SDE (2.161), in view of the features of the solution of the IVP (2.22).

2.3.2.4. Theorem (Stability)

Assume that there exist $g(t, w) \in C[\mathbb{R}_+^2, \mathbb{R}_+]$ and $L(t, U) \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$ where

$$B = B(\theta, \rho) = \{U \in K_c(\mathbb{R}^n) : D[U, \theta] \leq \rho\} \quad (2.162)$$

Additionally, assume that

- i) $g(t, 0) \equiv 0$
- ii) $L(t, \theta) \equiv 0$ and $L(t, U)$ is positive definite, and locally Lipschitzian in U
- iii) Given any $U \in E_1 = \{U \in K_c(\mathbb{R}^n) : L(s, U(s)) \leq L(t, U(t)), \quad t_0 \leq s \leq t\}$,
we have

$$D_L(t, U(t)) \leq g(t, L(t, U(t))), \quad t \geq t_0 \quad (2.163)$$

where

$$D_-L(t, U(t)) = \liminf_{\varepsilon \rightarrow 0^-} \frac{1}{\varepsilon} [L(t + \varepsilon, U(t) + \varepsilon(QU)(t)) - L(t, U(t))], \quad (2.164)$$

Then, if the zero solution of (2.22) is stable, it implies that the zero solution of the SDE (2.161) is stable as well.

Proof:

Since $L(t, U)$ is positive definite, we ensure the existence of a $b \in \mathcal{K}$ such that the following inequality holds.

$$b(D[U, \theta]) \leq L(t, U), \quad (t, U) \in \mathbb{R}_+ \times B \quad (2.165)$$

Given that $0 < \varepsilon < \rho$ and $t_0 \in \mathbb{R}_+$. If it is assumed the stability of the zero solution of (2.22); then, by definition, given $b(\varepsilon) > 0, t_0 \in \mathbb{R}_+$, there exists a function $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$w_0 < \delta \text{ implies } w(t) < b(\varepsilon), \quad t \geq t_0 \quad (2.166)$$

where $w(t, t_0, w_0)$ is any solution of (2.22).

Choose $w_0 = L(t_0, W_0)$. Since $L(t, \theta) \equiv 0$ and $L(t, U)$ is continuous; then, there exists a function $\delta_1 = \delta_1(t_0, \varepsilon) > 0$ such that both of the following inequalities hold.

$$D[W_0, \theta] \leq \delta_1 \text{ and } L(t_0, W_0) \leq \delta \quad (2.167)$$

We aim to show that if $D[W_0, \theta] \leq \delta_1$, then $D[U(t), \theta] < \varepsilon$ for all $t \geq t_0$. If we assume that it is not, then this gives us the existence of a solution $U(t) = U(t, t_0, W_0)$ of (2.161) such that

$$D[U(t_2), \theta] = \varepsilon \text{ and } D[U(t), \theta] < \varepsilon \text{ for } t_0 < t < t_2, \quad t_2 \in (t_0, t_1) \quad (2.168)$$

Hence, in view of (2.165), this gives us that

$$b(\varepsilon) = b(D[U(t_2), \theta]) \leq L(t_2, U(t_2)) \quad (2.169)$$

Additionally, for $t_0 \leq t \leq t_2$ we have $U(t) \in B$.

Choosing $w_0 = L(t_0, W_0)$ and the inequality (2.163) in (iii) gives us, as a conclusion of the Comparison Theorem 4 in 2.3.1.8., that

$$L(t, U(t)) \leq r(t), \quad t_0 \leq t \leq t_2 \quad (2.170)$$

where $r(t) = r(t, t_0, w_0)$ is the maximal solution of (2.22).

Since the inequality (2.166) holds for any solution $w(t, t_0, w_0)$ of (2.22); then, it holds for the maximal solution $r(t) = r(t, t_0, w_0)$ of (2.22). That is,

$$w_0 < \delta \text{ implies } r(t) < b(\varepsilon), \quad t \geq t_0 \quad (2.171)$$

Now, from (2.169), (2.170) and (2.171), we have the following inequalities, respectively,

$$b(\varepsilon) \leq L(t_2, U(t_2)) \leq r(t_2) < b(\varepsilon) \quad (2.172)$$

We have a contradiction, which completes the proof. ■

The following theorem provides adequate criteria to conclude asymptotically stability of the zero solution of the SDE (2.161), given the stability of the zero solution of (2.22).

2.3.2.5. Theorem (Asymptotically Stability)

Assume that $g(t, w) \in C[\mathbb{R}_+^2, \mathbb{R}_+]$ and $L(t, U) \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$ where

$$B = B(\theta, \rho) = \{U \in K_c(\mathbb{R}^n) : D[U, \theta] \leq \rho\} \quad (2.173)$$

and that $g(t, w), L(t, U)$ satisfy the assumptions of the previous theorem in 2.3.2.4.

Assume further the existence of $\alpha(t) \in C[\mathbb{R}_+, \mathbb{R}_+]$ with $\alpha(t) > 0$ and $\lim_{t \rightarrow \infty} \alpha(t) = \infty$.

Additionally, assume that for $U \in E_\alpha$, we have

$$\alpha(t)D_-L(t, U(t)) + L(t, U(t))D_- \alpha(t) \leq w(t, L(t, U(t)) \alpha(t)), \quad t \geq t_0 \quad (2.174)$$

Then, if the zero solution of (2.22) is stable, it implies that the zero solution of the SDE (2.161) is asymptotically stable.

Proof:

Since $L(t, U)$ is positive definite, then we can ensure the existence of such a $b \in \mathcal{K}$ that satisfies the following inequality.

$$b(D[U, \theta]) \leq L(t, U), \quad (t, U) \in \mathbb{R}_+ \times B \quad (2.175)$$

Given that $0 < \varepsilon < \rho$ and $t_0 \in \mathbb{R}_+$. Given the stability of the zero solution of (2.22), we have, by definition, for any given $\varepsilon > 0, t_0 \in \mathbb{R}_+$, there exists a function $\delta = \delta(t_0, \varepsilon) > 0$ that gives us that

$$w_0 < \delta \text{ implies } w(t, t_0, w_0) < \varepsilon, \quad t \geq t_0 \quad (2.176)$$

where $w(t, t_0, w_0)$ is any solution of (2.22).

Let $\alpha_0 = \min_{t \in \mathbb{R}_+} \alpha(t)$; then, the assumptions on α give us that $\alpha_0 > 0$.

Given any $\varepsilon > 0, t_0 \in \mathbb{R}_+$, let us set

$$\varepsilon^* = \alpha_0 b(\varepsilon) \quad (2.177)$$

Then, the stability of the zero solution of (2.22) gives us that given $b(\varepsilon^*) > 0$ there exists a function $\delta^* = \delta^*(t_0, \varepsilon^*) > 0$ such that

$$w_0 < \delta^* \text{ implies } w(t, t_0, w_0) < b(\varepsilon^*), \quad t \geq t_0 \quad (2.178)$$

where $w(t, t_0, w_0)$ is any solution of (2.22).

We aim first to demonstrate the stability of the zero solution of the SDE (2.161) by continuing similarly to what we did in proving the previous theorem in 2.3.2.4.

Choose $w_0 = L(t_0, W_0)$. Then, since $L(t, \theta) \equiv 0$ and $L(t, U)$ is continuous; then, there exists a function $\delta_1 = \delta_1(t_0, \varepsilon^*) > 0$ such that both of the following inequalities hold.

$$D[W_0, \theta] \leq \delta_1 \text{ and } L(t_0, W_0) \leq \delta^* \quad (2.179)$$

We aim to show that if $D[W_0, \theta] \leq \delta_1$, then $D[U(t), \theta] < \varepsilon^*$ for any $t \geq t_0$; where $U(t) = U(t, t_0, W_0)$ is any solution of the SDE (2.161).

If we assume that it is not, then this gives us the existence of a solution $U(t) = U(t, t_0, W_0)$ of (2.161) such that

$$D[U(t_2), \theta] = \varepsilon^* \text{ and } D[U(t), \theta] < \varepsilon^* \text{ for } t_0 \leq t < t_2 \quad (2.180)$$

Hence, in view of (2.175), this gives us that

$$b(\varepsilon^*) = b(D[U(t_2), \theta]) \leq L(t_2, U(t_2)) \quad (2.181)$$

Additionally, for $t_0 \leq t \leq t_2$ we have $U(t) \in B$.

Choosing $w_0 = L(t_0, W_0)$ and the inequality (2.163) give us, as a conclusion of the Comparison Theorem 4 in 2.3.1.8., that

$$L(t, U(t)) \leq r(t), \quad t_0 \leq t \leq t_2 \quad (2.182)$$

where $r(t) = r(t, t_0, w_0)$ is the maximal solution of (2.22).

Since the inequality (2.178) holds for any solution $w(t, t_0, w_0)$ of (2.22); then, it holds for the maximal solution $r(t) = r(t, t_0, w_0)$ of (2.22). That is,

$$w_0 < \delta^* \text{ implies } r(t) < b(\varepsilon^*), \quad t \geq t_0 \quad (2.183)$$

Now, from (2.181), (2.182) and (2.183), we have the following inequalities, respectively,

$$b(\varepsilon^*) \leq L(t_2, U(t_2)) \leq r(t_2) < b(\varepsilon^*) \quad (2.184)$$

which gives us a contradiction proving that the zero solution of the SDE (2.161) is stable. Now, we aim to show its asymptotical stability.

The stability itself of the zero solution $U = \theta$ of the SDE (2.161) gives us that for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a function $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$D[W_0, \theta] \leq \delta \text{ implies } D[U(t), \theta] < \varepsilon, \quad t \geq t_0 \quad (2.185)$$

where $U(t) = U(t, t_0, W_0)$ is any solution of the SDE (2.161).

Then, for $\varepsilon = \frac{\rho}{2} > 0$ and $t_0 \in \mathbb{R}_+$, there exists a function $\delta_0 = \delta_0\left(t_0, \frac{\rho}{2}\right) > 0$ such that

$$D[W_0, \theta] \leq \delta_0 \text{ implies } D[U(t), \theta] < \frac{\rho}{2}, \quad t \geq t_0 \quad (2.186)$$

Since $\lim_{t \rightarrow \infty} \alpha(t) = \infty$, we can ensure the existence of such a $T = T(t_0, \varepsilon) > 0$ which gives the following inequality.

$$b(\varepsilon)\alpha(t) > \varepsilon^*, \quad t \geq t_0 + T \quad (2.187)$$

In addition to that, for any solution $U(t) = U(t, t_0, W_0)$ of the SDE (2.161) satisfying $D[W_0, \theta] \leq \delta_0$, the previous theorem in 2.3.2.4. and (2.175) give us that

$$\alpha(t) b(D[U(t), \theta]) \leq \alpha(t) L(t, U(t)) \leq r(t), \quad t \geq t_0 \quad (2.188)$$

If we suppose the zero solution $U = \theta$ of the SDE (2.161) is not asymptotically stable, then it would exist $\{t_k\}_{k \geq 1}$, such that $t_k \geq t_0 + T$ and $\lim_{k \rightarrow \infty} t_k = \infty$. Moreover, for some solutions $U(t) = U(t, t_0, W_0)$ of (2.161) satisfying $D[W_0, \theta] \leq \delta_0$ we have

$$D[U(t_k), \theta] \geq \varepsilon \quad (2.189)$$

Then,

$$\varepsilon^* = \alpha_0 b(\varepsilon) = \min_{t \in \mathbb{R}_+} \alpha(t) b(\varepsilon) \leq \alpha(t_k) b(\varepsilon) \quad (2.190)$$

which leads to a contradiction to (2.187). Therefore, it concludes the asymptotical stability of the zero solution $U = \theta$ of the SDE (2.147). ■

The following theorem gives adequate criteria for uniform asymptotically stability of the zero solution $U = \theta$ of the SDE (2.161).

2.3.2.6. Theorem (Uniformly Asymptotically Stability)

Suppose that there exists $L(t, U) \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$ where

$$B = B(\theta, \rho) = \{U \in K_c(\mathbb{R}^n) : D[U, \theta] \leq \rho\} \quad (2.191)$$

such that

- i) $L(t, U)$ is decrescent, positive definite, and locally Lipschitzian in U .
- ii) For $U \in E_0$, and $c \in \mathcal{K}$, where $f(r)$ is non-decreasing in r , continuous on $\mathbb{R}_+$, in addition that $f(r) > r$ for any $r > 0$; we have

$$D_- L(t, U(t)) \leq -c(D[U(t), \theta]), \quad t \geq t_0 \quad (2.192)$$

where

$$D_-L(t, U(t)) = \liminf_{\varepsilon \rightarrow 0^-} \frac{1}{\varepsilon} [L(t + \varepsilon, U(t) + \varepsilon(QU)(t)) - L(t, U(t))], \quad (2.193)$$

Then, the zero solution $U = \theta$ of the SDE (2.161) is uniformly asymptotically stable.

Proof:

Since $L(t, U)$ is decrescent and positive definite, then we can ensure the existence of such $a, b \in \mathcal{K}$ which they satisfy that for any $(t, U) \in \mathbb{R}_+ \times B$, the following inequalities hold.

$$b(D[U, \theta]) \leq L(t, U) \leq a(D[U, \theta]), \quad (2.194)$$

Given that $0 < \varepsilon < \rho$ and $t_0 \in \mathbb{R}_+$. We aim first to demonstrate that the zero solution of the SDE (2.161) is uniformly stable.

Let us choose $\delta = \delta(\varepsilon) > 0$ such that

$$a(\delta) < b(\varepsilon) \quad (2.195)$$

We aim to show that if $D[W_0, \theta] \leq \delta$, then $D[U(t), \theta] < \varepsilon$ for any $t \geq t_0$; where $U(t) = U(t, t_0, W_0)$ is any solution of the SDE (2.161).

If we assume that it is not, then this gives us the existence of a solution $U(t) = U(t, t_0, W_0)$ of the SDE (2.161) with $D[W_0, \theta] \leq \delta$ and a $t_2 > t_0$ such that

$$D[U(t_2), \theta] = \varepsilon \quad \text{and} \quad D[U(t), \theta] < \varepsilon \quad \text{for} \quad t_0 \leq t < t_2 \quad (2.196)$$

Hence, in view of (2.194), it gives us that

$$b(\varepsilon) = b(D[U(t_2), \theta]) \leq L(t_2, U(t_2)) \quad (2.197)$$

Additionally, as $\varepsilon < \rho$, we have $U(t) \in B$. Choosing $w_0 = L(t_0, W_0)$, then the inequality $D_-L(t, U(t)) \leq 0$ for $t \geq t_0$, $U \in E_0$, give us, as a conclusion of the corollary in 2.3.1.9., that

$$L(t, U(t)) \leq L(t_0, W_0), \quad t_0 \leq t < t_2 \quad (2.198)$$

Hence, the inequalities in the statements (2.197), (2.198), (2.194), and (2.195) together with $D[W_0, \theta] \leq \delta$ and $a \in \mathcal{K}$, give us that

$$\begin{aligned} b(\varepsilon) &= b(D[U(t_2), \theta]) \leq L(t_2, U(t_2)) \leq L(t_0, W_0) \\ &\leq a(D[W_0, \theta]) \leq a(\delta) < b(\varepsilon) \end{aligned} \quad (2.199)$$

which gives us a contradiction proving that the zero solution of the SDE (2.161) is uniformly stable. Now, we aim to show the uniform asymptotic stability.

The uniform stability itself of the zero solution $U = \theta$ of the SDE (2.161) gives us that for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a function $\delta = \delta(\varepsilon) > 0$ such that

$$D[W_0, \theta] \leq \delta \text{ implies } D[U(t), \theta] < \varepsilon, \quad t \geq t_0 \quad (2.200)$$

where $U(t) = U(t, t_0, W_0)$ is any solution of the SDE (2.161).

Then, for $\varepsilon = \frac{\rho}{2} > 0$ and $t_0 \in \mathbb{R}_+$, there exists a function $\delta_0 = \delta_0\left(\frac{\rho}{2}\right) > 0$ such that

$$D[W_0, \theta] \leq \delta_0 \text{ implies } D[U(t), \theta] < \frac{\rho}{2}, \quad t \geq t_0 \quad (2.201)$$

As $\varepsilon = \frac{\rho}{2} < \rho$, we have $U(t) \in B$ for all $t \geq t_0$.

Given $0 < \eta < \delta_0$; then we have

$$b(\eta) \leq a(\eta) \leq a(\delta_0) \quad (2.202)$$

Considering that the function $f(r)$, in the definition of E_0 , is non-decreasing in r , continuous on $\mathbb{R}_+$, in addition that $f(r) > r$ for any $r > 0$; then, in view of (2.202), there exists such a $\beta = \beta(\eta) > 0$ that

$$f(r) > r + \beta \quad \text{if } b(\eta) \leq r \leq a(\delta_0) \quad (2.203)$$

Additionally, it would exist an integer $N = N(\eta) > 0$ where

$$b(\eta) + N\beta > a(\delta_0) \quad (2.204)$$

Assuming that for some $t \geq t_0$ we have

$$b(\eta) \leq L(t, U(t)) \quad (2.205)$$

Then, from (2.194), we conclude that there exists a $\delta_2 = \delta_2(\eta) > 0$, such that

$$D[U(t), \theta] \geq \delta_2 \quad (2.206)$$

Hence, since $c \in \mathcal{K}$, we have

$$c(D[U(t), \theta]) \geq c(\delta_2) = \delta_3 \quad (2.207)$$

where $\delta_3 = \delta_3(\eta)$.

Let us designate $N + 1$ numbers

$$t_k = t_k(t_0, \eta), \quad k = 0, 1, 2, \dots, N \quad (2.208)$$

where

$$t_k(t_0, \eta) = \begin{cases} t_0(t_0, \eta) = t_0, & k = 0 \\ t_{k-1}(t_0, \eta) + \frac{\beta}{\delta_3}, & k = 1, 2, \dots, N \end{cases} \quad (2.209)$$

Setting

$$T_k(\eta) = \frac{k\beta}{\delta_3}, \quad k = 0, 1, 2, \dots, N \quad (2.210)$$

Then,

$$t_k(t_0, \eta) = t_0 + T_k(\eta) \quad (2.211)$$

To prove the uniform asymptotic stability, we would prove that

$$D[U(t), \theta] < \eta \quad \text{for all } t \geq t_0 + T_k(\eta) \quad (2.212)$$

Consequently, it is sufficient to show that

$$L(t, U(t)) < b(\eta) + (N - k)\beta, \quad t \geq t_k, \quad k = 0, 1, 2, \dots, N \quad (2.213)$$

Let us prove (2.213) by mathematical induction.

First, we can show that (2.213) is confirmed for $k = 0$. Considering (2.198), (2.194), and (2.204) together with $D[W_0, \theta] \leq \delta_0$ and $a \in \mathcal{K}$, gives us, for $t \geq t_0$, that

$$\begin{aligned} L(t, U(t)) &\leq L(t_0, W_0) \leq a(D[W_0, \theta]) \leq a(\delta_0) \\ &< b(\eta) + N\beta = b(\eta) + (N - 0)\beta \end{aligned} \quad (2.214)$$

Now, we assume for some $k > 0$ that

$$L(s, U(s)) < b(\eta) + (N - k)\beta, \quad s \geq t_k \quad (2.215)$$

Additionally, if possible, we assume that for $t_k \leq t \leq t_{k+1}$ we have

$$L(t, U(t)) \geq b(\eta) + (N - k - 1)\beta \quad (2.216)$$

Then, for $t_k < s < t$ and $t_k \leq t \leq t_{k+1}$, we have

$$a(\delta_0) \geq a(D[U(s), \theta]) \geq L(s, U(s)) \geq b(\eta) + N\beta - (k + 1)\beta \geq b(\eta) \quad (2.217)$$

Hence, from (2.217), we have

$$f(L(s, U(s))) \geq L(s, U(s)) + \beta > b(\eta) + (N - k)\beta > L(s, U(s)) \quad (2.218)$$

which gives us that $U(t) \in E_0$ for $t_k \leq t \leq t_{k+1}$.

Therefore, in view of (2.192) and (2.214), we have

$$\begin{aligned} L(t_{k+1}, U(t_{k+1})) &\leq L(t_k, U(t_k)) - \int_{t_k}^{t_{k+1}} c(D[U(s), \theta]) ds \\ &< b(\eta) + (N - k)\beta - \delta_3(t_{k+1} - t_k) < b(\eta) + (N - k)\beta \end{aligned} \quad (2.219)$$

In light of this contradiction, it concludes that there exists a t^* such that $t_k \leq t^* \leq t_{k+1}$ and

$$L(t^*, U(t^*)) < b(\eta) + (N - k - 1)\beta \quad (2.220)$$

Then, in view of (2.220), we aim to prove that

$$L(t, U(t)) < b(\eta) + (N - k - 1)\beta, \quad t \geq t^* \quad (2.221)$$

If we suppose on the contrary that (2.221) is not true, then we can ensure the existence of such a $t_1 > t^*$ that gives us the following

$$L(t_1, U(t_1)) = b(\eta) + (N - k - 1)\beta \quad (2.222)$$

or equivalently, we have the following inequality for a sufficiently small value $h < 0$

$$L(t_1 + h, U(t_1 + h)) < b(\eta) + (N - k - 1)\beta \quad (2.223)$$

which in its turn implies that

$$D_-L(t_1, U(t_1)) \geq 0 \quad (2.224)$$

Similar to earlier, we have $U(s) \in B$, for $t^* \leq s \leq t_1$ and

$$D_-L(t_1, U(t_1)) \leq -\delta_3 < 0 \quad (2.225)$$

which gives us a contradiction to (2.224). Consequently,

$$L(t, U(t)) < b(\eta) + (N - k - 1)\beta, \quad t \geq t_{k+1} \quad (2.226)$$

The proof of the theorem is therefore completed. ■

3. ITD STABILITY OF PERTURBED AND UNPERTURBED SDES INVOLVING CAUSAL OPERATORS

3.1. Introduction

In this chapter, we investigate ITD stability for SDEs with causal operators through Lyapunov functions and functionals. We present adequate criteria for ITD stability as well as ITD asymptotic stability.

In Section 3.2, we pave the way for our findings in this chapter by presenting the essential concepts and conclusions about ITD stability and the stability of the null solution of SDEs involving causal operators.

In Section 3.3, we discuss the challenges we would encounter during attempting to derive ITD stability features from the characteristics of a null solution. In order to do so, we examine the ideas of stability associated with both classical and ITD contexts for the SDEs with causal operators and contrast them in terms of these two notions of stability.

In Section 3.4, we develop comparison theorems that resolve the previously mentioned challenges when attempting to derive ITD stability features from the characteristics of a null solution using the conventional notion of stability when dealing with different initial states.

In Section 3.5, we utilize the comparison theorems established in Section 3.4 to solve the challenges and issues associated with inferring stability and other qualitative features from the characteristics of a null solution in the context of the ITD conventional notion of stability for SDEs with causal operators, and we do so by employing an appropriate comparison system.

3.2. Basic Definitions

Consider the following differential equations:

$$D_H U = (QU)(t), \quad U(t_0) = U_0 \quad \text{for } U_0 \in K_c(\mathbb{R}^n) \text{ and } t \geq t_0 \geq 0 \quad (3.1)$$

$$D_H U = (QU)(t), \quad U(\tau_0) = V_0 \quad \text{for } V_0 \in K_c(\mathbb{R}^n) \text{ and } t \geq \tau_0 \geq 0 \quad (3.2)$$

$$D_H V = (PV)(t), \quad V(\tau_0) = V_0 \quad \text{for } V_0 \in K_c(\mathbb{R}^n) \text{ and } t \geq \tau_0 \quad (3.3)$$

$$D_H W = (SW)(t), \quad W(\tau_0) = V_0 - U_0 \quad \text{for } W(\tau_0) = W_0 \in K_c(\mathbb{R}^n) \text{ and } t \geq \tau_0 \quad (3.4)$$

where $Q, P, S : E \rightarrow E$ are causal operators, and each of them satisfies the local Lipschitzity condition on $\mathbb{R}_+ \times S_\rho$ where $E = C[[t_0, \infty), K_c(\mathbb{R}^n)]$ and $S_\rho = \{U \in K_c(\mathbb{R}^n) : D[U, \tilde{0}] < \rho < \infty\}$.

It is clear that (3.1) and (3.2) are different in the starting parameters (time and position). Moreover, if $(PV)(t)$ in (3.3) is written as $(PV)(t) = (QV)(t) + (RV)(t)$; Then, we consider (3.3) as the perturbed form that corresponds to the un-perturbed form (3.2) with the perturbation term $(RV)(t)$.

Assuming that $(Q\tilde{0})(t) \equiv \tilde{0}$ for $t \geq 0$, and assuming the necessary smoothness of each of P, Q and R to ensure the existence and uniqueness of the solution $U(t) = U(t, t_0, U_0)$ of (3.1) through (t_0, U_0) for all $t \geq t_0$, and those of the solution $V(t) = V(t, \tau_0, V_0)$ of (3.3) through (τ_0, V_0) for all $t \geq \tau_0$, along with their continuous dependence on the starting conditions in the initial states.

If $U \in C^1[J_1, K_c(\mathbb{R}^n)]$ where $J_1 = [t_0, t_0 + T_1]$, then it is said to be a solution of (3.1) on J_1 if it satisfies (3.1) on J_1 .

If U, V and $W \in C^1[J_2, K_c(\mathbb{R}^n)]$ where $J_2 = [t_0, t_0 + T_2]$, then they are said to be solutions of (3.2), (3.3), and (3.4) on J_2 provided that they satisfy (3.2), (3.3), and (3.4) on J_2 , respectively.

The continuous differentiability of $U(t), V(t)$ and $W(t)$ enables us to write

$$U(t) = U_0 + \int_{t_0}^t D_H U(s) ds, \quad U_0 \in K_c(\mathbb{R}^n) \text{ and } t \in J_1 \quad (3.5)$$

$$U(t) = U_0 + \int_{\tau_0}^t D_H U(s) ds, \quad U_0 \in K_c(\mathbb{R}^n) \text{ and } t \in J_2 \quad (3.6)$$

$$V(t) = V_0 + \int_{\tau_0}^t D_H V(s) ds, \quad V_0 \in K_c(\mathbb{R}^n) \text{ and } t \in J_2 \quad (3.7)$$

$$W(t) = W_0 + \int_{\tau_0}^t D_H W(s) ds, \quad W_0 \in K_c(\mathbb{R}^n) \text{ and } t \in J_2 \quad (3.8)$$

Following this, we may derive the following equivalent equations to initial value problems (IVPs) of (3.1), (3.2), (3.3), and (3.4), respectively,

$$U(t) = U_0 + \int_{t_0}^t (Q U(s))(s) ds, \quad U_0 \in K_c(\mathbb{R}^n) \text{ and } t \in J_1 \quad (3.9)$$

$$U(t) = U_0 + \int_{\tau_0}^t (Q U(s))(s) ds, \quad U_0 \in K_c(\mathbb{R}^n) \text{ and } t \in J_2 \quad (3.10)$$

$$V(t) = V_0 + \int_{\tau_0}^t (P V(s))(s) ds, \quad V_0 \in K_c(\mathbb{R}^n) \text{ and } t \in J_2 \quad (3.11)$$

$$W(t) = W_0 + \int_{\tau_0}^t (S W(s))(s) ds, \quad W_0 \in K_c(\mathbb{R}^n) \text{ and } t \in J_2 \quad (3.12)$$

where the Hukuhara integrals are utilized in these analogous equations.

It is important to note that $U(t)$ is the solution of (3.1) if and only if it satisfies (3.9) on J_1 . Similarly, $U(t)$, $V(t)$, and $W(t)$ are the solutions of (3.2), (3.3), and (3.4) if and only if they satisfy (3.10), (3.11), and (3.12) on J_2 , respectively.

Moreover, to get stability results and other qualitative behavioral features of the solution of (3.1) analogous to those of an ordinary counterpart one, in addition to guaranteeing the existence of the Hukuhara difference $U_0^* = U_0 - Z_0$, we assume that $U_0 = Z_0 + U_0^*$. Therefore, we have $U(t, t_0, U_0 - Z_0) = U(t, t_0, U_0^*)$ and the corresponding equation

$$D_H U = (QU)(t), \quad U(t_0) = U_0^* \text{ for } U_0^* \in K_c(\mathbb{R}^n) \text{ and } t \geq t_0 \geq 0 \quad (3.13)$$

Further, to guarantee the existence of the Hukuhara difference $V_0^* = V_0 - S_0$, we assume that $V_0 = S_0 + V_0^*$. Therefore, we have $V(t, t_0, V_0 - S_0) = U(t, t_0, V_0^*)$ and the corresponding equation

$$D_H U = (QU)(t), \quad U(\tau_0) = V_0^* \text{ for } V_0^* \in K_c(\mathbb{R}^n) \text{ and } t \geq \tau_0 \geq 0 \quad (3.14)$$

and the corresponding perturbed system of (3.14)

$$D_H V = (PV)(t), \quad V(\tau_0) = V_0^* \text{ for } V_0^* \in K_c(\mathbb{R}^n) \text{ and } t \geq \tau_0 \geq 0 \quad (3.15)$$

There is no need to use the asterisk symbols in (3.13), (3.14), and (3.15) as long as we are certain that the abovementioned Hukuhara differences exist and there is no room for misunderstanding in the contextual clarity. If so, we proceed with the typical notion as in (3.1), (3.2), and (3.3).

Before developing comparison results in this context, it is necessary to begin by defining the following fundamental concepts regarding ITD stability.

3.2.1. Definitions (in the Context of ITD)

Let $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, where $U(t, t_0, U_0)$ solves (3.1) for $t \geq \tau_0$, $\tau_0 = \eta + t_0$ and $t_0, \tau_0 \in \mathbb{R}_+$. The solution $V(t, \tau_0, V_0)$ of (3.3) for $t \geq \tau_0$, $\tau_0 \in \mathbb{R}_+$, considering the initial parameters (τ_0, V_0) , is said to be

(S1) ITD stable w. r. t. the solution $\tilde{U}(t, \tau_0, U_0)$ if and only if, for any given $\varepsilon > 0$, we would be able to designate $\delta_1 = \delta_1(\varepsilon, \tau_0) > 0$ and $\delta_2 = (\varepsilon, \tau_0) > 0$, which gives us the following inequality

$$D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] < \varepsilon \quad (3.16)$$

whenever $D[V_0 - U_0, \tilde{0}] < \delta_1$ and $d_s[\tau_0 - t_0, 0] < \delta_2$ for $t \geq \tau_0$.

(S2) ITD uniformly stable w. r. t. the solution $\tilde{U}(t, \tau_0, U_0)$ if and only if both δ_1, δ_2 in (S1) do not depend on the initial time $\tau_0 \in \mathbb{R}$.

(S3) ITD attractive w. r. t. the solution $\tilde{U}(t, \tau_0, U_0)$ if and only if for any $\varepsilon, \tau_0 > 0$, we would be able to designate $\delta_1 = \delta_1(\tau_0) > 0$, $\delta_2 = \delta_2(\tau_0) > 0$ and a $T = T(\varepsilon, \tau_0) > 0$, which gives us the following inequality

$$D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] < \varepsilon \quad (3.17)$$

provided that $D[V_0 - U_0, \tilde{0}] < \delta_1$ and $d_s[\tau_0 - t_0, 0] < \delta_2$ for $t \geq \tau_0 + T$.

(S4) ITD uniformly attractive w. r. t. the solution $\tilde{U}(t, \tau_0, U_0)$ if and only if δ_1, δ_2 and T in (S3) does not depend on the initial time $\tau_0 \in \mathbb{R}_+$.

(S5) ITD asymptotically stable w. r. t. the solution $\tilde{U}(t, \tau_0, U_0)$ if and only if both (S1) and (S3) are satisfied at the same time, or equivalently if (S1) holds and we can ensure the existence of such $\gamma_1(\tau_0) > 0$ and $\gamma_2(\tau_0) > 0$ so that

$$\lim_{t \rightarrow \infty} D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] = 0 \quad (3.18)$$

for any $V(t, \tau_0, V_0)$ and $U(t - \eta, t_0, U_0)$ with

$$D[V_0 - U_0, \tilde{0}] < \gamma_1 \text{ and } d_s[\tau_0 - t_0, 0] < \gamma_2, \text{ for } t \geq \tau_0 \quad (3.19)$$

(S6) ITD uniformly asymptotically stable w. r. t. the solution $\tilde{U}(t, \tau_0, U_0)$ if and only if both (S2) and (S4) are satisfied at the same time, or equivalently if along with (S2), we would be able to designate $\gamma_1 = \gamma_1(\varepsilon) > 0$ and $\gamma_2 = \gamma_2(\varepsilon) > 0$ and a $T(\varepsilon) > 0$, given any $\varepsilon > 0$, so that

$$D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] < \varepsilon \quad (3.20)$$

provided that $D[V_0 - U_0, \tilde{0}] < \gamma_1$ and $d_s[\tau_0 - t_0, 0] < \gamma_2$ for each $t \geq \tau_0 + T(\varepsilon)$; or if both γ_1 and γ_2 in (S5) do not depend on the initial time $\tau_0 \in \mathbb{R}_+$.

(S7) ITD exponentially asymptotically stable w. r. t. the solution $\tilde{U}(t, \tau_0, U_0)$ if and only if we can ensure the existence of such $\alpha > 0$ so that

$$D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] \leq D[V_0 - U_0, \tilde{0}] \exp[-\alpha(t - \tau_0)] \quad (3.21)$$

for $t \geq \tau_0 \in \mathbb{R}_+$.

3.2.2. Definitions (Dini and Dini-like Derivatives)

Let $L \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$,

(a) The Dini derivatives of L are defined as

$$D^+L(t, V) = \lim_{\varepsilon \rightarrow 0^+} \sup \frac{1}{\varepsilon} [L(t + \varepsilon, V + \varepsilon(QV)(t)) - L(t, V)] \quad (3.22)$$

$$D_-L(t, V) = \lim_{\varepsilon \rightarrow 0^-} \inf \frac{1}{\varepsilon} [L(t + \varepsilon, V + \varepsilon(QV)(t)) - L(t, V)] \quad (3.23)$$

for $(t, V) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n)$.

(b) The generalized (Dini-like) derivatives of L are defined as

$$\begin{aligned} D_*^+ L(t, V - \tilde{U}) \\ = \lim_{\varepsilon \rightarrow 0^+} \sup \frac{1}{\varepsilon} \left[L\left(t + \varepsilon, V - \tilde{U} + \varepsilon \left((PV)(t) - (Q\tilde{U})(t) \right) \right) \right. \\ \left. - L(t, V - \tilde{U}) \right] \end{aligned} \quad (3.24)$$

$$\begin{aligned} D_{*-} L(t, V - \tilde{U}) \\ = \lim_{\varepsilon \rightarrow 0^-} \inf \frac{1}{\varepsilon} \left[L\left(t + \varepsilon, V - \tilde{U} + \varepsilon \left((PV)(t) - (Q\tilde{U})(t) \right) \right) \right. \\ \left. - L(t, V - \tilde{U}) \right] \end{aligned} \quad (3.25)$$

where $V(t, \tau_0, V_0)$ solves (3.3); and $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$ where $\tau_0 = \eta + t_0$ and $U(t, t_0, U_0)$ solve (3.1); $t_0, \tau_0 \in \mathbb{R}_+$.

(c) The generalized derivatives of a $L(t, W) \in C[\mathbb{R}_+ \times E, \mathbb{R}_+]$ are defined as

$$\begin{aligned} D^+ L(t, V(t) - \tilde{U}(t)) \\ = \lim_{\varepsilon \rightarrow 0^+} \sup \frac{1}{\varepsilon} \left[L\left(t + \varepsilon, V(t + \varepsilon, t, V) - \tilde{U}(t + \varepsilon, t, \tilde{U}) \right) \right. \\ \left. - L(t, V(t) - \tilde{U}(t)) \right] \end{aligned} \quad (3.26)$$

$$\begin{aligned} D_- L(t, V(t) - \tilde{U}(t)) \\ = \lim_{\varepsilon \rightarrow 0^-} \inf \frac{1}{\varepsilon} \left[L\left(t + \varepsilon, V(t + \varepsilon, t, V) - \tilde{U}(t + \varepsilon, t, \tilde{U}) \right) \right. \\ \left. - L(t, V(t) - \tilde{U}(t)) \right] \end{aligned} \quad (3.27)$$

where $V(t + \varepsilon, t, V)$ solves (3.3); and $\tilde{U}(t + \varepsilon, t, \tilde{U})$ solves (3.1). Furthermore, $\tilde{U}(t) = U(t - \eta)$, where $\tau_0 = \eta + t_0$ and $U(t) = U(t, t_0, U_0)$ solves (3.1); $t_0, \tau_0 \in \mathbb{R}_+$.

Following that, we will try to offer a comparative examination on how to deduce stability and other qualitative attributes of solutions of SDEs with causal operators from those of the null solution in the conventional classical setting of stability. Employing the same approach in the framework of ITD stability, on the other hand, would be incompatible.

3.3. Stability of SDEs involving Causal Operators (Conventional notion vs. ITD)

3.3.1. Conventional Notion of Stability of SDEs involving Causal Operators

Let us take the following form of SDE involving a causal operator into consideration:

$$D_H V = (QV)(t), \quad V(\tau_0) = V_0 \text{ for } V_0 \in K_c(\mathbb{R}^n) \text{ and } t \geq \tau_0, \tau_0 \in \mathbb{R}_+ \quad (3.28)$$

where $Q : E \rightarrow E$ and $S_\rho = \{V \in K_c(\mathbb{R}^n) : D[V, \tilde{0}] < \rho < \infty\}$.

Let $V(t, t_0, V_0)$ be any solution of (3.28) and suppose that $(Q\tilde{0})(t) = \tilde{0}$, $t \in \mathbb{R}_+$ and that the zero solution $V \equiv 0$ satisfies (3.28) through the starting parameters $(t_0, 0)$.

3.3.1.1. Definitions (Stability of the Trivial Solution)

The zero solution $V = 0$ of (3.28) is said to be

(S1) stable if and only if for any given $\varepsilon > 0, \tau_0 \in \mathbb{R}_+$, we would be able to designate $\delta_1 = \delta_1(\varepsilon, \tau_0) > 0$ and $\delta_2 = \delta_2(\varepsilon, \tau_0) > 0$, which are continuous in τ_0 for each ε , and which give us

$$D[V(t, \tau_0, V_0), \tilde{0}] < \varepsilon \quad (3.29)$$

for $t \geq \tau_0$, given that $D[V_0, \tilde{0}] < \delta_1$ and $d_s[\tau_0, 0] < \delta_2$.

(S2) uniformly stable if, in addition to (S1), we have that both δ_1 and δ_2 in (S1) do not depend on the starting time τ_0 .

(S3) quasi-equi asymptotically stable if and only if for any given ε, t_0 we would be able to designate $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \varepsilon) > 0$ so

$$D[V(t, \tau_0, V_0), \tilde{0}] < \varepsilon \quad (3.30)$$

holds for $t \geq t_0 + T$ provided that $D[V_0, \tilde{0}] < \delta_0$.

(S4) quasi-uniformly asymptotically stable if in addition to (S3) we have that both of δ_0 and T in (S3) do not depend on the starting time t_0 .

(S5) equi-asymptotically stable if and only if the statements in (S1) and (S3) are both satisfied.

(S6) uniformly asymptotically stable if and only if the statements in (S2) and (S4) are both satisfied.

(S7) quasi-equi asymptotically stable if and only if for any given positive values t_0, ε, α we would be able to designate $T = T(t_0, \varepsilon, \alpha) > 0$ which gives us

$$D[V(t, \tau_0, V_0), \tilde{0}] < \varepsilon \quad (3.31)$$

for $t \geq t_0 + T$, provided that $D[V_0, \tilde{0}] \leq \alpha$.

(S8) quasi-uniformly asymptotically stable if, in addition to (S7), we have that T does not depend on the starting time t_0 .

(S9) completely stable if and only if (S1) is satisfied and (S7) is also satisfied for any given $\alpha, 0 \leq \alpha < \infty$.

(S10) uniformly completely stable if and only if (S2) is satisfied and (S8) is also satisfied given any $\alpha, 0 \leq \alpha < \infty$.

(S11) unstable if and only if the statement in (S1) fails to be satisfied.

3.3.1.2. Definitions (Stability w. r. t. Other Solution)

The solution $V(t, \tau_0, V_0)$ of (3.28) through the initial conditions (τ_0, V_0) is described to be stable w. r. t. the solution $V(t, \tau_0, U_0)$ of (3.28) for $t \geq \tau_0 \geq 0$ if and only if for each $\varepsilon > 0$, we would be able to designate $\delta_1 = \delta_1(\varepsilon, \tau_0) > 0$ and $\delta_2 = \delta_2(\varepsilon, \tau_0) > 0$ which are continuous in τ_0 for each ε so

$$D[V(t, \tau_0, V_0) - V(t, \tau_0, U_0), \tilde{0}] < \varepsilon \quad (3.32)$$

holds for $t \geq \tau_0$, provided that $D[V_0 - U_0, \tilde{0}] < \delta_1$ and $d_s[\tau_0, 0] < \delta_2$.

If in addition to that each of δ_1 and δ_2 does not depend on the starting time $\tau_0 \in \mathbb{R}_+$, then the solution $V(t, \tau_0, V_0)$ of (3.28) is described to be uniformly stable w. r. t. the solution $V(t, \tau_0, U_0)$ of (3.28).

3.3.1.3. Note

Let $V(t, \tau_0, V_0)$ be any given solution of (3.28). We aim to investigate the stability of this solution using the change of variable method as follows.

For $t \geq t_0$, let

$$\Omega(t, t_0, V_0 - U_0) = V(t, t_0, V_0) - V(t, t_0, U_0) \quad (3.33)$$

Then,

$$D_H \Omega(t, t_0, V_0 - U_0) = D_H V(t, t_0, V_0) - D_H V(t, t_0, U_0) \quad (3.34)$$

$$\begin{aligned} D_H \Omega(t, t_0, V_0 - U_0) &= (Q\Omega(t, t_0, V_0 - U_0))(t) + (QV(t, t_0, U_0))(t) - (QV(t, t_0, U_0))(t) \\ &= (Q(\Omega(t, t_0, V_0 - U_0) + V(t, t_0, U_0)))(t) - (QV(t, t_0, U_0))(t) \\ &= (\tilde{Q}\Omega(t, t_0, V_0 - U_0))(t) \end{aligned} \quad (3.35)$$

If $V_0 - U_0 = \tilde{0}$, we can ascertain that $\Omega(t, t_0, V_0 - U_0) \equiv \tilde{0}$ is a solution of the transformed SDE. Therefore, it implies that $(Q\tilde{0})(t) = \tilde{0}$ for $t \geq t_0 \geq 0$.

As $D_H \Omega = \tilde{0}$ and $\Omega(t, t_0, \tilde{0}) = \tilde{0}$, the solution $V(t, t_0, U_0)$ of (3.28) corresponds to the identically zero solution of $D_H \Omega = (Q\Omega)(t)$ where

$$\begin{aligned} &(Q\Omega(t, t_0, V_0 - U_0))(t) \\ &= (Q(\Omega(t, t_0, V_0 - U_0) + V(t, t_0, U_0)))(t) - (QV(t, t_0, U_0))(t) \end{aligned} \quad (3.36)$$

As a result, with no loss of generality, it is sufficient to restrict the scope of the stability analysis study to the zero solution $V(t, t_0, U_0) \equiv \tilde{0}$ of (3.28). Unfortunately, due to the nature of ITD stability analysis, employing the same approach in the framework of ITD stability, on the other hand, would be incompatible.

3.3.2. ITD Stability of SDEs with Causal Operators

Suppose that $U(t, \tau_0, V_0)$ solves (3.2) and let $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$ where $U(t, t_0, U_0)$ solves (3.1) for $t \geq \tau_0 \geq 0$.

Analogous to the technique described in (3.35), we will follow the same steps, starting with setting

$$\Omega(t, \tau_0, V_0 - U_0) = U(t, \tau_0, V_0) - U(t - \eta, t_0, U_0) \text{ for } t \geq \tau_0 \quad (3.37)$$

Then,

$$D_H \Omega(t, \tau_0, V_0 - U_0) = D_H U(t, \tau_0, V_0) - D_H U(t - \eta, t_0, U_0) \quad (3.38)$$

$$D_H \Omega(t, \tau_0, V_0 - U_0) = (Q\Omega(t, \tau_0, V_0 - U_0))(\eta; t) \quad (3.39)$$

Considering that even when $V_0 = U_0$, $\Omega(t, \tau_0, \tilde{0})$ is neither zero nor it solves the translated equation; additionally, $U(t - \eta, t_0, U_0)$ is not the null solution of $D_H \Omega = (Q\Omega)(\eta; t)$. As a result, inferring ITD stability features and other qualitative attributes of solutions of SDEs with causal operators from those of the null solution by employing the same manner used in the conventional classical setting of stability is not a viable choice.

Consequently, we are driven to seek out new appropriate techniques for examining stability and other behavioral properties.

3.4. Comparison Theorems for ITD Stability of SDEs with Causal Operators

According to the earlier discussion, besides additional analyses and findings presented by others [8], [6], [13], because of the incompatibility between the ITD concept of stability and the conventional notion, it comes to our attention that we can not undertake the same behavioral analysis study of the null solution of SDEs involving causal operators in those two quite different contexts.

In the following, we will provide an attempt to untangle the challenges we encountered when trying to derive ITD stability features from the characteristics of a null solution, allowing us to deduce stability and other qualitative attributes of the solution $V(t, \tau_0, V_0)$ of (3.3) w. r. t. $\tilde{U}(t) = U(t - \eta, t_0, U_0)$ where $U(t, t_0, U_0)$ solves (3.1) and $\tau_0 = \eta + t_0$.

3.4.1. Theorem

Let

i) $L(t, W) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ be locally Lipschitzian in W ; i.e., given any W_1, W_2 in $K_c(\mathbb{R}^n)$, we guarantee the existence of a $k \in \mathbb{R}$, which is called a Lipschitzian constant, such that the following inequality holds.

$$|L(t, W_1) - L(t, W_2)| \leq kD[W_1 - W_2, \tilde{0}] \quad (3.40)$$

ii) For $(t, V), (t, \tilde{U}) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n)$ and $G \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$,

$$D_*^+ L(t, V - \tilde{U}) \leq G\left(t, L\left(t, V(t) - \tilde{U}(t)\right)\right), \quad t \geq \tau_0 \quad (3.41)$$

where

$$U \in E_1 = \{\Omega \in K_c(\mathbb{R}^n) : L(s, \Omega(s)) \leq L(t, \Omega(t)), \quad t_0 \leq s \leq t\} \quad (3.42)$$

and

$$\begin{aligned} D_*^+ L(t, V(t) - \tilde{U}(t)) \\ = \lim_{\varepsilon \rightarrow 0^+} \sup \frac{1}{\varepsilon} \left[L\left(t + \varepsilon, V(t) - \tilde{U}(t)\right) \right. \\ \left. + \varepsilon \left((PV)(t) - (Q\tilde{U})(t) \right) \right] - L\left(t, V(t) - \tilde{U}(t)\right) \end{aligned} \quad (3.43)$$

iii) $r(t) = r(t, \tau_0, z_0)$ is the maximal solution for

$$z' = G(t, z), \quad z(\tau_0) = z_0 \geq 0 \quad \text{for } t \geq \tau_0 \quad (3.44)$$

existing on $[\tau_0, \infty)$.

And let $\tilde{U}(t) = U(t - \eta, t_0, U_0)$, $\tau_0 = \eta + t_0$, where $U(t) = U(t, t_0, U_0)$ solves (3.1) and $V(t) = V(t, \tau_0, V_0)$ solves (3.3) for $t \geq \tau_0$, and assume $L(\tau_0, V_0 - U_0) \leq z_0$; then,

$$L\left(t, V(t) - \tilde{U}(t)\right) \leq r(t), \quad \text{for } t \geq \tau_0 \quad (3.45)$$

Proof:

Let

$$m(t) = L\left(t, V(t) - \tilde{U}(t)\right), \text{ for } t \geq \tau_0 \quad (3.46)$$

For $t \geq \tau_0$, let us show that the inequality $m(t) \leq r(t)$ holds.

As a matter of fact,

$$\tilde{U}(\tau_0) = U(\tau_0 - \eta, t_0, U_0), \quad \tau_0 = \eta + t_0 \quad (3.47)$$

Substituting the last statement together with $\eta = \tau_0 - t_0$ in $m(\tau_0)$, and considering the theorem's assumptions give us that

$$\begin{aligned} m(\tau_0) &= L\left(\tau_0, V(\tau_0) - \tilde{U}(\tau_0)\right) = L\left(\tau_0, V_0 - U(\tau_0 - \eta, t_0, U_0)\right) \\ &= L\left(\tau_0, V_0 - U(\tau_0 - \tau_0 + t_0, t_0, U_0)\right) \\ &= L\left(\tau_0, V_0 - U(t_0, t_0, U_0)\right) = L(\tau_0, V_0 - U_0) \leq z_0. \end{aligned} \quad (3.48)$$

Let $\varepsilon > 0$ be sufficiently small, and considering

$$z' = G(t, z) + \varepsilon, \quad z(\tau_0) = z_0 + \varepsilon, \text{ for } t \geq \tau_0 \quad (3.49)$$

whose we ensure the existence of its solutions $z(t, \varepsilon) = z(t, \tau_0, z_0, \varepsilon)$ up to $r(t)$.

Let us conclude this proof by showing that

$$m(t) < z(t, \varepsilon), \text{ for } t \geq \tau_0, \quad (3.50)$$

To do so, let us suppose on the contrary that there is a $t^* \geq \tau_0$ where

$$m(t) < z(t, \varepsilon), \text{ for } t^* > t \geq \tau_0, \quad m(t^*) = z(t^*, \varepsilon) \quad (3.51)$$

Thus, it follows the inequality

$$D^+m(t^*) \geq m'(t^*) = z'(t^*, \varepsilon) = G(t^*, z(t^*, \varepsilon)) + \varepsilon \quad (3.52)$$

Considering the assumptions about G besides that $G(t, z) \geq 0$, it implies that $z(t, \varepsilon)$ are nondecreasing in t .

As $m(s) = L\left(s, V(s) - \tilde{U}(s)\right)$ for $s \geq \tau_0$, this gives us that

$$L\left(s, V(s) - \tilde{U}(s)\right) \leq z(t^*, \varepsilon) \text{ for } t^* \geq s \geq \tau_0 \quad (3.53)$$

Consequently, $V - \tilde{U} \in E_1$.

For sufficiently small enough $\delta < 0$, taking into consideration the statement in (i) regarding the local Lipschitz property of $L(t, W)$ in W , gives us that

$$\begin{aligned}
m(t + \delta) - m(t) &= L(t + \delta, V(t + \delta) - \tilde{U}(t + \delta)) - L(t, V(t) - \tilde{U}(t)) \\
&\leq L(t + \delta, V(t + \delta) - \tilde{U}(t + \delta)) \\
&\quad - L(t + \delta, V(t) - \tilde{U}(t) + \delta((PV)(t) - (Q\tilde{U})(t))) \\
&\quad + L(t + \delta, V(t) - \tilde{U}(t) + \delta((PV)(t) - (Q\tilde{U})(t))) \\
&\quad - L(t, V(t) - \tilde{U}(t)) \\
&\leq KD \left[[V(t + \delta) - V(t)] - [\tilde{U}(t + \delta) - \tilde{U}(t)] \right. \\
&\quad \left. - \delta((PV)(t) - (Q\tilde{U})(t)), \tilde{0} \right] \\
&\quad + L(t + \delta, V(t) - \tilde{U}(t) + \delta((PV)(t) - (Q\tilde{U})(t))) \\
&\quad - L(t, V(t) - \tilde{U}(t))
\end{aligned} \tag{3.54}$$

Hence, we obtain

$$\begin{aligned}
m(t + \delta) - m(t) &\leq K[\varepsilon_1(\delta) - \varepsilon_2(\delta)] \\
&\quad + L(t + \delta, V(t) - \tilde{U}(t) + \delta((PV)(t) - (Q\tilde{U})(t))) \\
&\quad - L(t, V(t) - \tilde{U}(t))
\end{aligned} \tag{3.55}$$

where ε_1 and ε_2 refer to the error terms.

As a result, we may estimate the Dini derivative of $m(t)$ as follows:

$$\begin{aligned}
D^+m(t) &\leq \lim_{\delta \rightarrow 0^+} \sup \frac{1}{\delta} K[\varepsilon_1(\delta) - \varepsilon_2(\delta)] \\
&\quad + \lim_{\delta \rightarrow 0^+} \sup \frac{1}{\delta} \left[L(t + \delta, V(t) - \tilde{U}(t) \right. \\
&\quad \left. + \delta((PV)(t) - (Q\tilde{U})(t))) - L(t, V(t) - \tilde{U}(t)) \right] \\
&\leq G(t, L(t, V(t) - \tilde{U}(t))) = G(t, m(t))
\end{aligned} \tag{3.56}$$

for $\tau_0 \leq t \leq t^* < \infty$.

Since $\lim_{\delta \rightarrow 0^+} \frac{\varepsilon_1(\delta) - \varepsilon_2(\delta)}{\delta} = 0$, when $t = t^*$ we have

$$\begin{aligned} D^+m(t^*) &\leq D_*^+L\left(t^*, V(t^*) - \tilde{U}(t^*)\right) \leq G\left(t^*, L\left(t, V(t^*) - \tilde{U}(t^*)\right)\right) \\ &= G(t^*, m(t^*)) < G(t^*, z(t^*, \varepsilon)) + \varepsilon \end{aligned} \quad (3.57)$$

which contradicts (3.52).

Consequently, we have $m(t) < z(t, \varepsilon)$. Therefore, by taking the limit as $\varepsilon \rightarrow 0$, we obtain the required statement

$$m(t) \leq r(t, \tau_0, z_0), \text{ for } t \geq \tau_0 \quad (3.58)$$

3.4.2. Corollary

Assume that L in the theorem in 3.4.1. satisfies the conditions mentioned above in addition to $G(t, z) \equiv 0$ and

$$U \in E_1 = \{\Omega \in K_c(\mathbb{R}^n) : L(s, \Omega(s)) \leq L(t, \Omega(t)), \quad t_0 \leq s \leq t\} \quad (3.59)$$

Then,

$$L\left(t, V(t) - \tilde{U}(t)\right) \leq L(\tau_0, V_0 - U_0) \text{ for } t \geq \tau_0 \quad (3.60)$$

where $U(t)$ and $V(t)$ solves (3.1) and (3.2), respectively.

Equivalently, for $\tau_0 \leq t^* \leq \tau^* < \infty$, we have

$$L\left(\tau^*, V(\tau^*) - \tilde{U}(\tau^*)\right) < L\left(t^*, V(t^*) - \tilde{U}(t^*)\right) \quad (3.61)$$

3.4.3. Theorem

Let

- i) $L(t, W) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ be locally Lipschitzian in W ; i.e., given any W_1, W_2 in $K_c(\mathbb{R}^n)$, we guarantee the existence of a $k \in \mathbb{R}$, which is called a Lipschitzian constant, such that the following inequality holds.

$$|L(t, W_1) - L(t, W_2)| \leq kD[W_1 - W_2, \tilde{0}] \quad (3.62)$$

ii) For any $(t, V), (t, \tilde{U}) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n)$ besides $G \in \mathcal{C}[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$, we have

$$\begin{aligned} A(t) D_*^+ L(t, V(t) - \tilde{U}(t)) + L(t, V(t) - \tilde{U}(t)) A'(t) \\ \leq G(t, L(t, V(t) - \tilde{U}(t)) A(t)) \end{aligned} \quad (3.63)$$

for $t \geq \tau_0$; and $U \in E_A$ where

$$E_A = \{\Omega \in K_c(\mathbb{R}^n) : L(s, \Omega(s)) A(s) \leq L(t, \Omega(t)) A(t), \quad t_0 \leq s \leq t\} \quad (3.64)$$

and

$$\begin{aligned} D_*^+ L(t, V(t) - \tilde{U}(t)) \\ = \lim_{\varepsilon \rightarrow 0^+} \sup \frac{1}{\varepsilon} \left[L(t + \varepsilon, V(t) - \tilde{U}(t)) \right. \\ \left. + \varepsilon \left((PV)(t) - (Q\tilde{U})(t) \right) \right] - L(t, V(t) - \tilde{U}(t)) \end{aligned} \quad (3.65)$$

iii) $r(t) = r(t, \tau_0, z_0)$ is the maximal solution of

$$z' = G(t, z), \quad z(\tau_0) = z_0 \geq 0, \quad \text{for } t \geq \tau_0 \quad (3.66)$$

existing on $[\tau_0, \infty)$.

And let $\tilde{U}(t) = U(t - \eta, t_0, U_0)$, $\tau_0 = \eta + t_0$, where $U(t) = U(t, t_0, U_0)$ solves (3.1) and $V(t) = V(t, \tau_0, V_0)$ solves (3.3) for $t \geq \tau_0$, and assume

$$A(\tau_0)L(\tau_0, V_0 - U_0) \leq z_0 \quad (3.67)$$

Then,

$$L(t, V(t) - \tilde{U}(t)) A(t) \leq r(t) \quad \text{for } t \geq \tau_0 \quad (3.68)$$

Proof:

Let

$$M(t, V(t) - \tilde{U}(t)) = L(t, V(t) - \tilde{U}(t)) A(t) \quad \text{for } t \geq \tau_0 \quad (3.69)$$

and assume $U \in E_A$.

Hence, taking into consideration the assumptions of the theorem, we get

$$\begin{aligned}
D^+M(t, V(t) - \tilde{U}(t)) &\leq A(t) D_*^+L(t, V(t) - \tilde{U}(t)) + L(t, V(t) - \tilde{U}(t)) A'(t) \\
&\leq G(t, L(t, V(t) - \tilde{U}(t)) A(t)) \\
&= G(t, M(t, V(t) - \tilde{U}(t)))
\end{aligned} \tag{3.70}$$

The utilization of the theorem in 3.4.1., in turn, provides us with the estimate we seek.

$$L(t, V(t) - \tilde{U}(t)) A(t) = M(t, V(t) - \tilde{U}(t)) \leq r(t) \text{ for } t \geq \tau_0 \tag{3.71}$$

Now that we have presented the preceding comparative findings regarding the SDEs involving causal operators, we are in the position to utilize them to establish a number of stability criteria in the context of ITD for the solution $U(t) = U(t, t_0, U_0)$ of the SDE (3.1).

3.5. ITD Stability of SDE with Causal Operator

3.5.1. ITD Stability of SDEs with Causal Operators by Means of Lyapunov Functions

First, utilizing Lyapunov functions and considering the assumption that the solution $U(t) = U(t, t_0, U_0)$ of (3.1) exists and is unique for all $t \geq \tau_0$, we will start by presenting adequate criteria for ITD stability of this solution $U(t) = U(t, t_0, U_0)$ of (3.1).

3.5.1.1. Theorem

Let

- i) $L(t, W) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ be locally Lipschitzian in W ; i.e., given any W_1, W_2 in $K_c(\mathbb{R}^n)$, we guarantee the existence of a $k \in \mathbb{R}$, which is called a Lipschitzian constant, such that the following inequality holds.

$$|L(t, W_1) - L(t, W_2)| \leq kD[W_1 - W_2, \tilde{0}] \quad (3.72)$$

ii) For any $(t, V), (t, \tilde{U}) \in \mathbb{R}_+ \times S(\rho)$ and for $t \geq \tau_0$ and $U \in E_1$, we have

$$D_*^+ L(t, V - \tilde{U}) \leq 0 \quad (3.73)$$

where

$$\begin{aligned} & D_*^+ L(t, V(t) - \tilde{U}(t)) \\ &= \lim_{\varepsilon \rightarrow 0^+} \sup \frac{1}{\varepsilon} \left[L\left(t + \varepsilon, V(t) - \tilde{U}(t) + \varepsilon \left((PV)(t) - (Q\tilde{U})(t) \right)\right) \right. \\ & \quad \left. - L(t, V(t) - \tilde{U}(t)) \right] \end{aligned} \quad (3.74)$$

iii) $L(t, W)$ be decrescent and positive definite on $\mathbb{R}_+ \times S(\rho)$; i. e., we can ensure the existence of such $a, b \in \mathcal{K}$ which they satisfy that for any $(t, W) \in \mathbb{R}_+ \times S(\rho)$, the following inequalities hold.

$$a(D[W, \tilde{0}]) \leq L(t, W) \leq b(D[W, \tilde{0}]) \quad (3.75)$$

Then, the solution $V(t, \tau_0, V_0)$ of (3.3) is ITD stable w. r. t. the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, where $U(t, t_0, U_0)$ solves (3.1) for $t \geq t_0$.

Proof:

For a sufficiently small given value $\varepsilon > 0$ and $t \geq \tau_0$, let us designate $\delta_1(\varepsilon, \tau_0) > 0$ and $\delta_2(\varepsilon, \tau_0) > 0$, under the assumption that $b(\delta) < a(\varepsilon)$ holds, where

$$\delta = \delta(\varepsilon, \tau_0) = \max\{\delta_1(\varepsilon, \tau_0), \delta_2(\varepsilon, \tau_0)\} > 0 \quad (3.76)$$

Next, using this δ , we aim to demonstrate the stability of the solution $U(t, t_0, U_0)$ of (3.1) for $t \geq t_0$.

If the opposite was to be assumed, then there would exist solutions $V(t, \tau_0, V_0)$ of (3.3) and $\tilde{U}(t, \tau_0, U_0)$ of (3.1) for $t \geq \tau_0$, and $\tau^* > t^* > \tau_0$ so

$$\begin{aligned} D[V(t^*) - \tilde{U}(t^*), \tilde{0}] &= \delta, & D[V(\tau^*) - \tilde{U}(\tau^*), \tilde{0}] &= \varepsilon, \\ \delta &\leq D[V(t) - \tilde{U}(t), \tilde{0}] \leq \varepsilon & \text{for } t^* \leq t \leq \tau^* \end{aligned} \quad (3.77)$$

By the assumption (ii) and the corollary in 3.4.2., we obtain the following estimation

$$L\left(\tau^*, V(\tau^*) - \tilde{U}(\tau^*)\right) \leq L\left(t^*, V(t^*) - \tilde{U}(t^*)\right) \text{ for } t^* \leq t \leq \tau^* < \infty \quad (3.78)$$

Therefore, by using the prior assumptions, particularly (3.77), (3.78), and (iii), as well as with the choice of δ , we conclude that

$$\begin{aligned} a(\varepsilon) &= a(D[V(\tau^*) - \tilde{U}(\tau^*), \tilde{0}]) \leq L\left(\tau^*, V(\tau^*) - \tilde{U}(\tau^*)\right) \\ &\leq L\left(t^*, V(t^*) - \tilde{U}(t^*)\right) \leq b(D[V(t^*) - \tilde{U}(t^*), \tilde{0}]) = b(\delta) \\ &< a(\varepsilon) \end{aligned} \quad (3.79)$$

which gives us a contradiction.

As a result, we conclude that the solution $U(t, t_0, U_0)$ of (3.1) for $t \geq t_0$ is stable. Hence, the solution $V(t, \tau_0, V_0)$ of (3.3) is ITD stable w. r. t. the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$ of (3.1), assuming $U(t, t_0, U_0)$ solves (3.1).

If additionally both $\delta_1 = \delta_1(\varepsilon, \tau_0)$ and $\delta_2 = \delta_2(\varepsilon, \tau_0)$ do not depend on the starting time τ_0 , then the solution $U(t, t_0, U_0)$ of (3.1) is uniformly stable for $t \geq t_0$.

3.5.1.2. Theorem

Let

- i) $L(t, W) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ be locally Lipschitzian in W ; i.e., given any W_1, W_2 in $K_c(\mathbb{R}^n)$, we guarantee the existence of a $k \in \mathbb{R}$, which is called a Lipschitzian constant, such that the following inequality holds.

$$|L(t, W_1) - L(t, W_2)| \leq kD[W_1 - W_2, \tilde{0}] \quad (3.80)$$

- ii) For any $(t, V), (t, \tilde{U}) \in \mathbb{R}_+ \times S(\rho)$ and for $t \geq \tau_0$ and $U \in E_A$, the following inequality holds.

$$D_*^+ L\left(t, V(t) - \tilde{U}(t)\right) A(t) + L\left(t, V(t) - \tilde{U}(t)\right) A'(t) \leq 0 \quad (3.81)$$

where

$$E_A = \{\Omega \in K_c(\mathbb{R}^n) : L(s, \Omega(s)) A(s) \leq L(t, \Omega(t)) A(t), \quad t_0 \leq s \leq t\} \quad (3.82)$$

and

$$\begin{aligned} D_*^+ L(t, V(t) - \tilde{U}(t)) \\ = \lim_{\varepsilon \rightarrow 0^+} \sup \frac{1}{\varepsilon} \left[L(t + \varepsilon, V(t) - \tilde{U}(t)) \right. \\ \left. + \varepsilon \left((PV)(t) - (Q\tilde{U})(t) \right) \right] - L(t, V(t) - \tilde{U}(t)) \end{aligned} \quad (3.83)$$

Additionally, $A(\tau_0) = 1$, $A'(t)$ is continuous for $t \geq \tau_0$, $A(t) \geq 1$, and $\lim_{t \rightarrow \infty} A(t) = \infty$.

iii) $L(t, W)$ is decrescent and positive definite on $\mathbb{R}_+ \times S(\rho)$; i. e., we can ensure the existence of such $a, b \in \mathcal{K}$ which they satisfy that for any $(t, W) \in \mathbb{R}_+ \times S(\rho)$, the following inequalities hold.

$$a(D[W, \tilde{0}]) \leq L(t, W) \leq b(D[W, \tilde{0}]) \quad (3.84)$$

Then, the solution $V(t, \tau_0, V_0)$ of (3.3) is ITD asymptotically stable w. r. t. the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, where $U(t, t_0, U_0)$ solves (3.1) for $t \geq t_0$.

Proof:

The theorem in 3.4.3. yields

$$L(t, V(t) - \tilde{U}(t)) A(t) \leq L(\tau_0, V_0 - U_0) \quad \text{for } t \geq \tau_0 \quad (3.85)$$

Furthermore, by the theorem in 3.5.1.1., we have the stability of the solution $U(t, t_0, U_0)$ of (3.1) for $t \geq t_0$. Consequently, proving its quasi-asymptotic stability is sufficient.

To make this happen, let us set $\varepsilon = \rho$ such that

$$\delta_0 = \delta(\rho, \tau_0) = \max\{\delta_1(\rho, \tau_0), \delta_2(\rho, \tau_0)\} > 0 \quad (3.86)$$

Choose $D[V_0 - U_0, \tilde{0}] < \delta_0$ and $d_s[\tau_0 - t_0, 0] < \delta_2$. Next, employing the prior assumptions, particularly (ii), (iii), and (3.85), and considering that $\lim_{t \rightarrow \infty} A(t) = \infty$,

i. e., $\lim_{t \rightarrow \infty} A^{-1}(t) = 0$, we conclude that given any $\varepsilon > 0$ and $\tau_0 \geq 0$, a $T = T(\varepsilon, \tau_0) > 0$ can be designated in order to satisfy

$$\begin{aligned} a(D[V(t) - \tilde{U}(t), \tilde{0}]) &\leq L(t, V(t) - \tilde{U}(t)) \leq L(\tau_0, V_0 - U_0) A^{-1}(t) \\ &\leq b(D[V_0 - U_0, \tilde{0}]) A^{-1}(t) < b(\delta_0) A^{-1}(t) < \varepsilon \end{aligned} \quad (3.87)$$

given that $D[V_0 - U_0, \tilde{0}] < \delta_0$ for $t \geq \tau_0 + T$.

Therefore, the solution $U(t, t_0, U_0)$ of (3.1) is a quasi-asymptotically stable solution for $t \geq t_0$.

Hence, the solution $V(t, \tau_0, V_0)$ of (3.3) is ITD asymptotically stable w. r. t. the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$ of (3.1), or the solution $U(t, t_0, U_0)$ of (3.1) is an asymptotically stable solution for $t \geq t_0$.

If in addition to that, $\delta_1 = \delta_1(\varepsilon, \tau_0)$, $\delta_2 = \delta_2(\varepsilon, \tau_0)$ and $T = T(\varepsilon, \tau_0)$ are all independent of τ_0 , then the solution $U(t, t_0, U_0)$ of (3.1) is a uniformly asymptotically stable solution for $t \geq t_0$.

3.5.2. ITD Stability of SDEs with Causal Operators by Means of Lyapunov Functionals

Afterward, we establish sufficient criteria to uniformly asymptotically stability for solutions of SDEs involving causal operators in the ITD context, employing Lyapunov functional within an analogous approach to that of Lyapunov second method in the stability analysis of nonlinear systems.

3.5.2.1. Theorem

Let

i) $L(t, W) \in C[\mathbb{R}_+ \times E, \mathbb{R}_+]$ in addition to satisfying the following estimation

$$D^+L(t, V(t) - \tilde{U}(t)) \leq -c(D[V(t) - \tilde{U}(t), \tilde{0}]), \quad c \in \mathcal{K} \quad (3.88)$$

ii) $L(t, W)$ is decrescent and positive definite on $\mathbb{R}_+ \times E$; i. e., we can ensure the existence of such $a, b \in \mathcal{K}$ which they satisfy that for any $(t, W) \in \mathbb{R}_+ \times E$, the following inequalities hold.

$$a(D[W, \tilde{0}]) \leq L(t, W) \leq b(D[W, \tilde{0}]) \quad (3.89)$$

Then, the solution $V(t, \tau_0, V_0)$ of (3.3) is ITD uniformly asymptotically stable w. r. t. the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$ of (3.1) where $U(t, t_0, U_0)$ solves (3.1) for $t \geq t_0$.

Proof:

For a sufficiently small given value $\varepsilon > 0$ and $t \geq \tau_0$, let us designate $\delta_1(\varepsilon, \tau_0) = \delta_1(\varepsilon) > 0$ and $\delta_2(\varepsilon, \tau_0) = \delta_2(\varepsilon) > 0$ under the assumption that

$$b(\delta) < a(\varepsilon) \quad (3.90)$$

holds, where $\delta = \delta(\varepsilon) = \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\} > 0$.

Next, using these δ and ε , we aim to demonstrate the stability of the solution $U(t, t_0, U_0)$ of (3.1) for $t \geq t_0$.

If the opposite was to be assumed, then there would exist solutions $V(t, \tau_0, V_0)$ of (3.3), and $\tilde{U}(t, \tau_0, U_0)$ of (3.1) for $t \geq \tau_0$, and $t_1 > \tau_0$ so

$$D[V(t_1, \tau_0, V_0) - \tilde{U}(t_1, \tau_0, U_0), \tilde{0}] = \varepsilon, \quad (3.91)$$

$$D[V(t, \tau_0, V_0) - \tilde{U}(t, \tau_0, U_0), \tilde{0}] \leq \varepsilon, \text{ for } \tau_0 \leq t \leq t_1$$

holds.

Then, the assumption (ii) and the corollary in 3.4.2. give us the following estimation

$$L(t, V(t, \tau_0, V_0) - \tilde{U}(t, \tau_0, U_0)) \leq L(\tau_0, V_0 - U_0) \quad (3.92)$$

for $\tau_0 \leq t \leq t_1 < \infty$.

Hence, in view of (3.90), (3.91), (3.92), and (ii), as well as the assumptions about δ , we arrive to the following conclusion

$$\begin{aligned}
a(\varepsilon) &= a(D[V(t_1, \tau_0, V_0) - \tilde{U}(t_1, \tau_0, U_0), \tilde{0}]) \\
&\leq L(t_1, V(t_1, \tau_0, V_0) - \tilde{U}(t_1, \tau_0, U_0)) \leq L(\tau_0, V_0 - U_0) \\
&\leq b(D[V_0 - U_0, \tilde{0}]) = b(\delta) < a(\varepsilon)
\end{aligned} \tag{3.93}$$

which gives us a contradiction. Consequently, it gives us the uniform stability of the solution $U(t, t_0, U_0)$ of (3.1) for $t \geq t_0$.

Let us now demonstrate the uniformly asymptotic stability of the solution $U(t, t_0, U_0)$ of (3.1) for $t \geq t_0$, to finish the proof. Setting that $\varepsilon = \rho$ and $\delta_0 = \delta(\rho)$ so $D[V_0 - U_0, \tilde{0}] < \delta_0$, $d_s[\tau_0 - t_0, 0] < \delta_0$ gives us

$$D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] < \rho \quad \text{for } t \geq \tau_0 \tag{3.94}$$

Considering the uniform stability of the solution $U(t, t_0, U_0)$ of (3.1) for $t \geq t_0$, this gives us that there exists a $t^* \in (\tau_0, \tau_0 + T)$ such that

$$D[V_0 - U_0, \tilde{0}] < \delta_0 \quad \text{and} \quad D[V(t^*, \tau_0, V_0) - \tilde{U}(t^*, \tau_0, U_0), \tilde{0}] < \delta(\varepsilon) \tag{3.95}$$

with $\delta = \delta(\varepsilon) > 0$ corresponds to $\varepsilon > 0$ for uniform stability.

If the opposite was to be assumed, then considering

$$\delta \leq D[V(t, \tau_0, V_0) - \tilde{U}(t, \tau_0, U_0), \tilde{0}], \quad t \in [\tau_0, \tau_0 + T] \tag{3.96}$$

gives us, in light of (i), that

$$D^+L(t, V(t, \tau_0, V_0) - \tilde{U}(t, \tau_0, U_0)) \leq c(D[V(t, \tau_0, V_0) - \tilde{U}(t, \tau_0, U_0), \tilde{0}]) \tag{3.97}$$

Hence, for $t \in [\tau_0, \tau_0 + T]$, it is seen that

$$\begin{aligned}
&L(t, V(t, \tau_0, V_0) - \tilde{U}(t, \tau_0, U_0)) \\
&\leq L(\tau_0, V_0 - U_0) - \int_{\tau_0}^t c(V(s, \tau_0, V_0) - \tilde{U}(s, \tau_0, U_0)) \, ds
\end{aligned} \tag{3.98}$$

Since $b \in \mathcal{K}$, i. e. $b(\lambda)$ is strictly increasing in λ , and it associates zero to zero and taking into account that $D[V_0 - U_0, \tilde{0}] < \delta_0$ we obtain

$$b(D[V_0 - U_0, \tilde{0}]) \leq b(\delta_0) \quad (3.99)$$

We conclude from (3.98), with each t being substituted by $\tau_0 + T$, and with the choice of T as

$$T = 1 + \frac{b(\delta_0)}{c(\delta)} \quad (3.100)$$

with the above assumptions, we have

$$\begin{aligned} 0 &\leq L\left(\tau_0 + T, V(\tau_0 + T, \tau_0, V_0) - \tilde{U}(\tau_0 + T, \tau_0, U_0)\right) \\ &\leq L(\tau_0, V_0 - U_0) - \int_{\tau_0}^{\tau_0+T} c\left(V(s, \tau_0, V_0) - \tilde{U}(s, \tau_0, U_0)\right) ds \\ &\leq b(D[V_0 - U_0, \tilde{0}]) - \int_{\tau_0}^{\tau_0+T} c\left(V(s, \tau_0, V_0) - \tilde{U}(s, \tau_0, U_0)\right) ds \quad (3.101) \\ &\leq b(\delta_0) - c(\delta) T = b(\delta_0) - c(\delta) \left(1 + \frac{b(\delta_0)}{c(\delta)}\right) \\ &= b(\delta_0) - c(\delta) - b(\delta_0) = -c(\delta) < 0 \end{aligned}$$

which gives us a contradiction. Consequently, it follows the existence of such a $t^* > \tau_0$ so that

$$D[V(t^*, \tau_0, V_0) - U(t^* - \eta, t_0, U_0), \tilde{0}] < \delta \quad (3.102)$$

which gives us, in turn, along with the stability of the solution $U(t, t_0, U_0)$ of (3.1) for $t \geq t_0$, that

$$D[V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0), \tilde{0}] < \varepsilon \quad (3.103)$$

for $t \geq \tau_0 + T$, given that

$$D[V_0 - U_0, \tilde{0}] < \delta, \quad d_s[\tau_0 - t_0, 0] < \delta \quad (3.104)$$

3.6. Conclusion

Regardless of all the challenges we encountered when attempting to deduce ITD stability features and other qualitative attributes of solutions of SDEs involving

causal operators from the those of the null solution, via the change of variables method, we were able to overcome these challenges by establishing comparison theorems employed in proving various types of ITD stability criteria for SDEs involving causal operators, with taking into account the differences in the starting parameters in the initial states of the solutions of these equations.

In this chapter, through utilizing Lyapunov functions and functionals, we attempt to establish adequate criteria to infer stability features in the ITD context of solutions of perturbed forms of SDEs with causal operators in regards to the unperturbed form of these equations; in an analogous approach to that of Lyapunov second method in the analysis of the stability of nonlinear systems, and in viewing a manner of inferring stability and other qualitative features from those of the zero solution of appropriate comparison systems in the conventional context of stability.

4. STABILITY ANALYSIS IN TERMS OF TWO MEASURES WITH ITD FOR SDES INVOLVING CAUSAL OPERATORS

4.1. Introduction

In this chapter, we study the stability properties in terms of two measures for SDEs with causal operators taking into account the differences in the starting parameters in the initial states.

Section 4.2 present the basic definitions regarding the stability properties for SDEs involving causal operators in terms of two measures, and we laid the foundation to examine the ITD stability features in terms of two measures for SDEs with causal operators.

Section 4.3 restate the basic definitions and the most relevant results concluded by Naidu Ch. A., Dhaigude D. B., and Devi J. V. in [47] trying to study the stability in terms of two measures for SDEs with causal operators.

Section 4.4 presents the basic definitions in the ITD context, aiming to obtain generalized stability properties for SDEs involving causal operators in terms of two measures with ITD.

Section 4.5 establishes generalized ITD variational comparative results for SDEs involving causal operators in terms of two measures.

Section 4.6 employs those comparative findings in Section 4.5 to investigate the practical stability, equi-boundedness, equi-attractiveness in the large, and Lagrange stability properties in terms of two measures with ITD for solutions of perturbed forms of SDEs with causal operators in regard to their un-perturbed forms.

4.2. Basic Definitions

Consider the following differential equations:

$$D_H U = (QU)(t), \quad U(t_0) = U_0 \quad \text{for } U_0 \in K_c(\mathbb{R}^n) \text{ and } t \geq t_0 \geq 0 \quad (4.1)$$

$$D_H U = (QU)(t), \quad U(\tau_0) = V_0 \quad \text{for } V_0 \in K_c(\mathbb{R}^n) \text{ and } t \geq \tau_0 \geq 0 \quad (4.2)$$

$$D_H V = (PV)(t), \quad V(\tau_0) = V_0 \text{ for } V_0 \in K_c(\mathbb{R}^n) \text{ and } t \geq \tau_0 \quad (4.3)$$

$$D_H W = (SW)(t), \quad W(\tau_0) = V_0 - U_0 \text{ for } W(\tau_0) = W_0 \in K_c(\mathbb{R}^n) \text{ and } t \geq \tau_0 \quad (4.4)$$

where $Q, P, S : E \rightarrow E$ are causal operators, and each of them satisfies the local Lipschitzity condition on $\mathbb{R}_+ \times S_\rho$ where $E = C[[t_0, \infty), K_c(\mathbb{R}^n)]$ and $S_\rho = \{U \in K_c(\mathbb{R}^n) : D[U, \tilde{0}] < \rho < \infty\}$.

It is clear that (4.1) and (4.2) are different in the starting parameters (time and position). Moreover, if $(PV)(t)$ in (4.3) is written as $(PV)(t) = (QV)(t) + (RV)(t)$; Then, we consider (4.3) as the perturbed form corresponding to the un-perturbed form (4.2) with the perturbation term $(RV)(t)$.

Assuming that $(Q\tilde{0})(t) \equiv \tilde{0}$ for $t \geq 0$, and assuming the necessary smoothness of each of P, Q and R to ensure the existence and uniqueness of the solution $U(t) = U(t, t_0, U_0)$ of (4.1) through (t_0, U_0) for all $t \geq t_0$, and those of the solution $V(t) = V(t, \tau_0, V_0)$ of (4.3) through (τ_0, V_0) for all $t \geq \tau_0$, along with their continuous dependence on the starting conditions in the initial states.

If $U \in C^1[J_1, K_c(\mathbb{R}^n)]$ where $J_1 = [t_0, t_0 + T_1]$, then it is said to be a solution of (4.1) on J_1 if it satisfies (4.1) on J_1 .

If U, V and $W \in C^1[J_2, K_c(\mathbb{R}^n)]$ where $J_2 = [t_0, t_0 + T_2]$, then they are said to be solutions of (4.2), (4.3), and (4.4) on J_2 provided that they satisfy (4.2), (4.3), and (4.4) on J_2 , respectively.

Now, let us define an order relation in the metric space $K_c(\mathbb{R}^n)$ embedded with the Hausdorff metric D .

First, we start by defining a cone in $K_c(\mathbb{R}^n)$.

4.2.1. Definition of the Cone in $K_c(\mathbb{R}^n)$

The subfamily $K \subset K_c(\mathbb{R}^n)$ is said to be a cone in $K_c(\mathbb{R}^n)$ if it consists of sets $U \in K_c(\mathbb{R}^n)$ such that all the elements $u \in U$ are written as a non-negative n-component vector $u = (u_1, u_2, \dots, u_n)$ satisfying $u_i \geq 0$ for $i = 1 \dots n$.

The subfamily $K^0 \subset K_c(\mathbb{R}^n)$, that consists of sets $U \in K_c(\mathbb{R}^n)$ such that any $u \in U$ is a positive n -component vector $u = (u_1, u_2, \dots, u_n)$ satisfying $u_i > 0$ for $i = 1 \dots n$, is the nonempty interior of the cone K .

4.2.2. Definition of Order Relation in $K_c(\mathbb{R}^n)$

For any $U, V \in K_c(\mathbb{R}^n)$, if there exists $Z \in K_c(\mathbb{R}^n)$ so that $Z \in K$ and $U = V + Z$ then it is said that $U \geq V$ or $V \leq U$. Similarly, if there exists $Z \in K_c(\mathbb{R}^n)$ so that $Z \in K^0$ and $U = V + Z$ then it is said that $U > V$ or $V < U$.

Below are some essential needed classes in establishing stability criteria in terms of two measures.

$$\mathcal{K} = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] : a(u) \text{ is strictly increasing in } u \text{ and } a(0) = 0\} \quad (4.5)$$

$$\mathcal{L} = \left\{ \sigma \in C[\mathbb{R}_+, \mathbb{R}_+] : \sigma(u) \text{ is strictly decreasing in } u \text{ and } \lim_{u \rightarrow \infty} \sigma(u) = 0 \right\} \quad (4.6)$$

$$\mathcal{CK} = \left\{ a \in C[\mathbb{R}_+^2, \mathbb{R}_+] : a(t, s) \in \mathcal{K} \text{ for each } t \right. \\ \left. \text{and } a(t, s) \text{ is continuous for each } s \right\} \quad (4.7)$$

$$\Gamma = \left\{ h \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+] : \inf_{(t, U)} h(t, U) = 0 \right\} \quad (4.8)$$

$$\Gamma_0 = \left\{ h \in \Gamma : \inf_U h(t, U) = 0, \text{ for each } t \in \mathbb{R}_+ \right\} \quad (4.9)$$

Next, to introduce a Lyapunov-like function, we present some definitions needed in the qualitative analysis in terms of two measures.

4.2.3. Definition of “ h -positive Definite,” “ h -decreascent,” “ h -weakly Decreascent”

Let $L \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$, then L is called

- i) h -positive definite if we were able to ensure the existence of such a $\rho > 0$ and a $b \in \mathcal{K}$ so that

$$h(t, U) < \rho \text{ implies } b(h(t, U)) \leq L(t, U) \quad (4.10)$$

- ii) h -decrecent if we were able to ensure the existence of such a $\rho > 0$ and an $a \in \mathcal{K}$ so that

$$h(t, U) < \rho \text{ implies } L(t, U) \leq a(h(t, U)) \quad (4.11)$$

- iii) h -weakly decrecent if we were able to ensure the existence of such a $\rho > 0$ and an $a \in \mathcal{CK}$ so that

$$h(t, U) < \rho \text{ implies } L(t, U) \leq a(t, h(t, U)) \quad (4.12)$$

4.2.4. Definition of “Finer,” “Uniformly Finer”

Suppose that $h_0, h \in \Gamma$, then h_0 is described to be finer than h if we were able to ensure the existence of such a $\rho > 0$ and a $\phi \in \mathcal{CK}$ so that

$$h_0(t, U) \leq \rho \text{ implies } h(t, U) \leq \phi(t, h_0(t, U)) \quad (4.13)$$

h_0 is described as uniformly finer than h if the function ϕ in the above definition does not depend on the parameter t .

4.3. Stability Results in Terms of Two Measures

In this section, we restate the essential definitions, and the most relevant findings concluded by Naidu Ch. A., Dhaigude D. B. and Devi J. V. in [47] trying to examine the stability in terms of two measures for SDEs with causal operators.

4.3.1. Definitions (Stability in Terms of Two Measures)

Suppose that the trivial solution $U(t) = \theta$ solves (4.1) with the initial conditions (t_0, θ) , and let $U(t) = U(t, t_0, U_0)$ solves (4.1).

Then, the differential system (4.1) is called

- i) (h_0, h) -equi stable if and only if for any given positive real values ε, t_0 , there exists a function $\delta = \delta(t_0, \varepsilon) > 0$ continuous in t_0 and

$$h_0(t_0, U_0) < \delta \text{ implies } h(t, U(t)) \leq \varepsilon, \quad t \geq t_0 \quad (4.14)$$

- ii) (h_0, h) -uniformly stable if the implication in (i) is confirmed with δ being independent of the initial time t_0 .

- iii) (h_0, h) -equi attractive if and only if for any given positive real values ε, t_0 , there exists $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \varepsilon) > 0$ such that

$$h_0(t_0, U_0) < \delta_0 \text{ implies } h(t, U(t)) < \varepsilon, \quad t \geq t_0 + T \quad (4.15)$$

- iv) (h_0, h) -uniformly attractive if the implication in (iii) is confirmed with δ_0 and T being independent of the initial time t_0 .

- v) (h_0, h) -equi asymptotically stable if and only if both (i) and (iii) are confirmed.

- vi) (h_0, h) -uniformly asymptotically stable if and only if both (ii) and (iv) are confirmed.

4.3.2. Comparison Results

First, let us state the comparison results used in developing the stability criteria for SDEs with causal operators in terms of two measures

4.3.2.1. Theorem

Assume that

- i) $L(t, U) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ be locally Lipschitzian in U ; i.e., given any U_1, U_2 in $K_c(\mathbb{R}^n)$, we guarantee the existence of a $k \in \mathbb{R}$, which is called a Lipschitzian constant, such that the following inequality holds.

$$|L(t, U_1) - L(t, U_2)| \leq kD[U_1 - U_2, \tilde{0}] \quad (4.16)$$

ii) For any $U \in E_1$, where

$$E_1 = \{\Omega \in K_c(\mathbb{R}^n) : L(s, \Omega(s)) \leq L(t, \Omega(t)), \quad t_0 \leq s \leq t\} \quad (4.17)$$

we have

$$D^+L(t, U(t)) \leq g(t, L(t, U(t))), \quad t \geq t_0 \quad (4.18)$$

where

$$D^+L(t, U(t)) = \lim_{\varepsilon \rightarrow 0^+} \sup \frac{1}{\varepsilon} (L(t + \varepsilon, U(t) + \varepsilon(QU)(t)) - L(t, U(t))) \quad (4.19)$$

iii) $r(t) = r(t, t_0, u_0)$ is the maximal solution of

$$u' = g(t, u), \quad u(t_0) = u_0 \geq 0 \quad (4.20)$$

where $g \in C[\mathbb{R}_+^2, \mathbb{R}_+]$ existing on $[t_0, \infty)$.

iv) $U(t) = U(t, t_0, U_0)$ is any solution of (4.1) for $t \geq t_0$,

Then,

$$L(t_0, U_0) \leq u_0 \text{ implies } L(t, U(t)) \leq r(t), \quad t \geq t_0 \quad (4.21)$$

4.3.2.2. Corollary

Under the assumptions of the theorem in 4.3.2.1., if moreover $g(t, u) = 0$, and $U(t) = U(t, t_0, U_0)$ solves (4.1) for $t \geq t_0$ and $U(t) \in E_1$; then,

$$L(t, U(t)) \leq L(t_0, U_0), \quad t \geq t_0 \quad (4.22)$$

4.3.3. Stability Results

In what follows, we will give the major stability findings in terms of two measures for SDEs with causal operators by stating the adequate criteria for equi-stability, uniformly stability, and uniformly asymptotically stability.

4.3.3.1. Theorem

Assume that

- i) $h, h_0 \in \Gamma$,
- ii) $L(t, U) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ be locally Lipschitzian in U and h -positive definite,
- iii) For $(t, U) \in S(h, \rho)$

$$D^+L(t, U(t)) \leq 0 \quad (4.23)$$

where $U \in E_1$ and

$$S(h, \rho) = \{(t, U) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n), h(t, U) < \rho, \rho > 0\} \quad (4.24)$$

Then,

- If additionally h_0 is finer than h , in addition to that $L(t, U)$ is h_0 -weakly decrescent, then (4.1) is (h_0, h) -equi stable.
- If additionally h_0 is uniformly finer than h , in addition to that $L(t, U)$ is h_0 -decrescent, then (4.1) is (h_0, h) -uniformly stable.

4.3.3.2. Theorem

Assume that

- i) $h, h_0 \in \Gamma$, besides that h_0 is uniformly finer than h ,
- ii) $L(t, U) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ satisfies the locally Lipschitzity condition in U , and it is h_0 -decrescent and h -positive definite,
- iii) For $(t, U) \in S(h, \rho)$

$$D^+L(t, U(t)) \leq -c(h_0(t, U)) \quad (4.25)$$

where $U \in E_1$, $c \in \mathcal{K}$ and $S(h, \rho)$ is defined as in (4.24).

Then, (4.1) is (h_0, h) - uniformly asymptotically stable.

4.4. Basic Definitions in the ITD context

In this section, we introduce the definition of generalized Dini-like derivatives of L . Next, we introduce the definitions of the initial time difference (ITD) practically stability, equi-boundedness, equi-attractiveness in the large, and Lagrange stability in terms of two measures before proceeding through our main results.

4.4.1. Definition of Dini-like Derivatives

We define the generalized derivative (Dini-like derivatives) for a function $L \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ as follows:

$$D_*^+ L(t, s, U) = \lim_{h \rightarrow 0^+} \sup \frac{1}{h} \left[L\left(s + h, V\left(t, s + h, U + h(Q\tilde{U})(s)\right)\right) - L(s, V(t, s, U)) \right] \quad (4.26)$$

$$D_{*-} L(t, s, U) = \lim_{h \rightarrow 0^-} \inf \frac{1}{h} \left[L\left(s + h, V\left(t, s + h, U + h(Q\tilde{U})(s)\right)\right) - L(s, V(t, s, U)) \right] \quad (4.27)$$

for $t, s \in \mathbb{R}_+$ and $U \in K_c(\mathbb{R}^n)$.

4.4.2. Definition of ITD Stability in Terms of Two Measures

Let $U(t, t_0, U_0)$ solves (4.1) for $t \geq t_0 \geq 0$, and in addition to that let $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, for $\eta = \tau_0 - t_0$.

Then, the solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ is called

- i) ITD (h_0, h) -practically stable w. r. t. the solution $\tilde{U}$ if and only if given any (λ, A) such that $0 < \lambda < A$ and given any $\tau_0 \in \mathbb{R}_+$ so that $h_0(\tau_0, V_0 - U_0) \leq \lambda$, it implies

$$h(t, V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)) \leq A, \quad t \geq \tau_0 \quad (4.28)$$

ii) ITD (h_0, h) -uniformly practically stable w. r. t. the solution $\tilde{U}$ if the previous implication in (i) holds for every $\tau_0 \in \mathbb{R}_+$.

iii) ITD (h_0, h) -practically quasi-stable w. r. t. the solution $\tilde{U}$ if and only if given any given $(\lambda, B, T) > 0$ such that $0 < \lambda < B$ and given any $\tau_0 \in \mathbb{R}_+$ so that $h_0(\tau_0, V_0 - U_0) \leq \lambda$, it implies

$$h(t, V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)) \leq B, \quad t \geq \tau_0 + T \quad (4.29)$$

iv) ITD (h_0, h) -uniformly practically quasi-stable w. r. t. the solution $\tilde{U}$ if the previous implication in (iii) holds for every $\tau_0 \in \mathbb{R}_+$.

v) ITD (h_0, h) -equi-bounded w. r. t. the solution $\tilde{U}$ if and only if given any $\alpha > 0$ and $\tau_0 \in \mathbb{R}_+$, there exists $\beta = \beta(\alpha, \tau_0) > 0$ such that $h_0(\tau_0, V_0 - U_0) < \beta$ implies

$$h(t, V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)) < \alpha, \quad t \geq \tau_0 \quad (4.30)$$

vi) ITD (h_0, h) -uniformly equi-bounded w. r. t. the solution $\tilde{U}$ if the previous implication in (i) holds for every $\tau_0 \in \mathbb{R}_+$, or in other words, $\beta = \beta(\alpha, \tau_0) > 0$ is independent of τ_0 .

It is worth pointing out that if β in (v) satisfies that $\beta(\cdot, \tau_0) \in \mathcal{K}$, then the solution $V(t, \tau_0, V_0)$ of (4.3) is ITD (h_0, h) -stable w. r. t. the solution $\tilde{U}$. In fact, for any $\varepsilon > 0$ there exists a function $\delta = \delta(\varepsilon, \tau_0) > 0$ continuous in τ_0 , such that whenever $\alpha < \delta$, we have $\beta = \beta(\alpha, \tau_0) < \varepsilon$.

vii) ITD (h_0, h) -equi-attractive in the large w. r. t. the solution $\tilde{U}$ if and only if given any $\varepsilon, \alpha > 0$ and $\tau_0 \in \mathbb{R}_+$, there exists a $T = T(\tau_0, \varepsilon, \alpha) > 0$ such that $h_0(\tau_0, V_0 - U_0) < \alpha$ implies

$$h(t, V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)) < \varepsilon, \quad t \geq \tau_0 + T(\tau_0, \varepsilon, \alpha) \quad (4.31)$$

viii) ITD (h_0, h) -uniform equi-attractive in the large w. r. t. the solution $\tilde{U}$ if the previous implication in (iii) holds for every $\tau_0 \in \mathbb{R}_+$, or in other words, $T = T(\tau_0, \varepsilon, \alpha) > 0$ is independent of τ_0 .

- ix) ITD (h_0, h) -Lagrange stable w. r. t. the solution $\tilde{U}$ if and only if it is ITD (h_0, h) -equi-bounded as well as ITD (h_0, h) -equi-attractive in the large w. r. t. the solution $\tilde{U}$.
- x) ITD (h_0, h) -uniform Lagrange stable w. r. t. the solution $\tilde{U}$ if and only if it is ITD (h_0, h) -Lagrange stable along with that both $\beta = \beta(\alpha, \tau_0) > 0$ in (i) and $T = T(\tau_0, \varepsilon, \alpha) > 0$ in (iii) are independent of τ_0 .

4.5. ITD Variational Comparison Results

In what follows, let us present generalized variational comparison results to examine the stability properties in terms of two measures for solutions of SDEs with causal operators, taking into account the differences in the starting parameters in the initial states.

4.5.1. Theorem

Assume that

- i) Both $L(t, \Omega) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+^N]$ and $\|W(t, s, \Omega)\|$ satisfy the local Lipschitzity condition in Ω given any t, s ; where $W(t) = W(t, \tau_0, V_0 - U_0)$ is the solution of (4.4) for $t \geq \tau_0$, $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, for $\eta = \tau_0 - t_0$, $U(t, t_0, U_0)$ is any solution of (4.1) for $t \geq t_0$, and $V(t) = V(t, \tau_0, V_0)$ is the solution of (4.3) for $t \geq \tau_0$; and let $\Omega(t) = V(t) - \tilde{U}(t)$.

ii)

$$D_{*-}L(t, s, \Omega) \leq g\left(t, s, L(s, W(t, s, \Omega))\right) \quad (4.32)$$

where

$$\begin{aligned} D_{*-}L(t, s, \Omega) = \lim_{\delta \rightarrow 0^+} \inf \frac{1}{\delta} & \left(L\left(s \right. \right. \\ & + \delta, W\left(t, s + \delta, \Omega + \delta \left((PV)(s) - (Q\tilde{U})(s) \right) \right) \\ & \left. \left. - L(s, W(t, s, \Omega)) \right) \right) \end{aligned} \quad (4.33)$$

iii) $g \in C[\mathbb{R}_+ \times \mathbb{R}_+^N, \mathbb{R}^N]$, $g(t, s, u)$ is quasi-monotone non-decreasing in u for any t, s ; i.e., if $u \leq v$, $u_i = v_i$ for some i so that $1 \leq i \leq N$, then $g_i(t, s, u) \leq g_i(t, s, v)$, given any $t, s \in \mathbb{R}_+$ (In this context, the inequality symbol used in the vectorial inequalities is understood to denote component-wise inequality [51]); and $r(t, s, \tau_0, V_0)$ is the maximal solution of

$$\frac{du(s)}{ds} = g(t, s, u(s)), \quad u(\tau_0) = u_0 \geq 0 \quad (4.34)$$

existing for $\tau_0 \leq s \leq t < \infty$.

Then, $L(\tau_0, W(t, \tau_0, V_0 - U_0)) = u_0$ implies

$$L(t, \Omega(t, \tau_0, V_0 - U_0)) \leq r_0(t, \tau_0, L(\tau_0, W(t, \tau_0, V_0 - U_0))) \quad (4.35)$$

where $r_0(t, \tau_0, u_0) = r(t, t, \tau_0, u_0)$.

Proof:

Let us set

$$m(t, s) = L(s, W(t, s, \Omega(s))) \text{ for } \tau_0 \leq s \leq t \quad (4.36)$$

Then, we have

$$\begin{aligned} m(t, \tau_0) &= L(\tau_0, W(t, \tau_0, \Omega(\tau_0))) = L(\tau_0, W(t, \tau_0, V(\tau_0) - \tilde{U}(\tau_0))) \\ &= L(\tau_0, W(t, \tau_0, V_0 - U_0)) = u_0 \end{aligned} \quad (4.37)$$

For a sufficiently small positive value δ , we have

$$\begin{aligned} m(t, s + \delta) - m(t, s) &= L(s + \delta, W(t, s + \delta, \Omega(s + \delta))) - L(s, W(t, s, \Omega(s))) \\ &= L(s + \delta, W(t, s, \Omega(s)) + \delta(SW(t, s, \Omega(s)))(s) + \varepsilon(\delta)) \\ &\quad - L(s, W(t, s, \Omega(s))) \end{aligned} \quad (4.38)$$

where ε stands for error and $\lim_{\delta \rightarrow 0^+} \frac{\varepsilon(\delta)}{\delta} = 0$.

Taking into consideration the assumptions in (i) regarding the locally Lipschitz property of $L(t, \Omega)$ and $\|W(t, s, \Omega)\|$ in Ω , it is seen that

$$\begin{aligned}
m(t, s + \delta) - m(t, s) &\leq k(\varepsilon_1(\delta) - \varepsilon_2(\delta)) \\
&+ L \left(s + \delta, W \left(t, s, V(s) - \tilde{U}(s) \right) \right. \\
&\left. + \delta \left((PV)(s) - (Q\tilde{U})(s) \right) \right) - L \left(s, W \left(t, s, V(s) - \tilde{U}(s) \right) \right)
\end{aligned} \tag{4.39}$$

where $\varepsilon_1, \varepsilon_2$ stand for errors, k stands for a Lipschitz constant.

The inequality in the assumption (ii) enables us to estimate the Dini derivative of $m(t, s)$ as follows:

$$\begin{aligned}
D_{*-}m(t, s) &\leq \liminf_{\delta \rightarrow 0^-} \frac{1}{\delta} k(\varepsilon_1(\delta) - \varepsilon_2(\delta)) \\
&+ \liminf_{\delta \rightarrow 0^-} \frac{1}{\delta} L \left(s + \delta, W \left(t, s, V(s) - \tilde{U}(s) \right) \right. \\
&\left. + \delta \left((PV)(s) - (Q\tilde{U})(s) \right) \right) \\
&- \liminf_{\delta \rightarrow 0^-} \frac{1}{\delta} L \left(s, W \left(t, s, V(s) - \tilde{U}(s) \right) \right) \\
&\leq g \left(t, s, L \left(s, W \left(t, s, V(s) - \tilde{U}(s) \right) \right) \right) \\
&= g \left(t, s, L \left(s, W(t, s, \Omega(s)) \right) \right) = g(t, s, m(t, s))
\end{aligned} \tag{4.40}$$

for $\tau_0 \leq s \leq t < \infty$.

A comparison result [Theorem 1.7.1] from [6] gives us the following inequality

$$m(t, s) \leq r \left(t, s, \tau_0, L(\tau_0, W(t, \tau_0, V_0 - U_0)) \right) \text{ for } \tau_0 \leq s \leq t \tag{4.41}$$

Choosing $s = t$ in the right-hand side in the previous inequality (4.41), gives us that

$$\begin{aligned}
m(t, s) &\leq r \left(t, t, \tau_0, L(\tau_0, W(t, \tau_0, V_0 - U_0)) \right) \\
&= r_0 \left(t, \tau_0, L(\tau_0, W(t, \tau_0, V_0 - U_0)) \right)
\end{aligned} \tag{4.42}$$

which yields the desired estimation in (4.35) completing the proof. ■

4.5.2. Theorem

Under the assumptions of the theorem in 4.5.1. with $N = 1$ along with $g(t, s, u) \equiv 0$, we have

$$L(t, \Omega(t, \tau_0, V_0 - U_0)) \leq L(\tau_0, W(t, \tau_0, V_0 - U_0)), \quad t \geq \tau_0 \quad (4.43)$$

Furthermore, we assume

$$D_{*-}L(t, s, \Omega) \leq -c \left(h(s, W(t, s, \Omega)) \right), \quad \tau_0 \leq s \leq t < \infty \quad (4.44)$$

where $c \in \mathcal{K} = \{\phi \in C[\mathbb{R}_+, \mathbb{R}_+] \text{ such that } \phi(0) = 0 \text{ and } \phi(s) \text{ is increasing in } s\}$ and $h \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$.

Then, for $t \geq \tau_0$

$$\begin{aligned} L(t, \Omega(t, \tau_0, V_0 - U_0)) \\ \leq L(\tau_0, W(t, \tau_0, V_0 - U_0)) - \int_{\tau_0}^t c \left(h(s, W(t, s, \Omega(s))) \right) ds \end{aligned} \quad (4.45)$$

Proof:

Starting from the statement (4.40) in the proof of the theorem in 3.5.1.,

$$D_{*-}m(t, s) \leq g(t, s, m(t, s)) \text{ for } \tau_0 \leq s \leq t < \infty \quad (4.46)$$

Then, since $g(t, s, u) \equiv 0$, we get by integrating the two sides of the previous inequality (4.46), for $s \in [\tau_0, t]$,

$$\int_{\tau_0}^t D_{*-}m(t, s) ds = L(t, W(t, t, \Omega(t))) - L(\tau_0, W(t, \tau_0, \Omega(\tau_0))) \leq 0 \quad (4.47)$$

Hence, we have

$$L(t, \Omega(t, \tau_0, V_0 - U_0)) \leq L(\tau_0, W(t, \tau_0, V_0 - U_0)) \text{ for } t \geq \tau_0 \quad (4.48)$$

Now, let us set

$$\begin{aligned}
& M(s, W(t, s, \Omega(s))) \\
& \equiv L(s, W(t, s, \Omega(s))) + \int_{\tau_0}^s c(h(\xi, W(t, \xi, \Omega(\xi)))) d\xi
\end{aligned} \tag{4.49}$$

Next, if we take both sides' Dini derivatives in (4.49), we have, in view of (4.44),

$$\begin{aligned}
D_{*-}M(t, s, \Omega(s)) &= D_{*-}L(t, s, \Omega(s)) + c(h(s, W(t, s, \Omega(s)))) \\
&\quad - c(h(\tau_0, W(t, \tau_0, \Omega(\tau_0)))) \\
&\leq D_{*-}L(t, s, \Omega(s)) + c(h(s, W(t, s, \Omega(s)))) \\
&\leq -c(h(s, W(t, s, \Omega(s)))) + c(h(s, W(t, s, \Omega(s)))) = 0
\end{aligned} \tag{4.50}$$

Thus, $D_{*-}M(t, s, \Omega(s)) \leq 0$, in view of (4.48), gives us $t \geq \tau_0$,

$$M(t, \Omega(t, \tau_0, V_0 - U_0)) \leq M(\tau_0, W(t, \tau_0, V_0 - U_0)) \tag{4.51}$$

By the definition of M , this implies, for $t \geq \tau_0$,

$$\begin{aligned}
& L(t, \Omega(t, \tau_0, V_0 - U_0)) + \int_{\tau_0}^t c(h(\xi, W(t, \xi, \Omega(\xi)))) d\xi \\
& \leq L(\tau_0, W(t, \tau_0, V_0 - U_0)) + \int_{\tau_0}^{\tau_0} c(h(\xi, W(t, \xi, \Omega(\xi)))) d\xi
\end{aligned} \tag{4.52}$$

$$\begin{aligned}
& L(t, \Omega(t, \tau_0, V_0 - U_0)) + \int_{\tau_0}^t c(h(\xi, W(t, \xi, \Omega(\xi)))) d\xi \\
& \leq L(\tau_0, W(t, \tau_0, V_0 - U_0))
\end{aligned} \tag{4.53}$$

Moving the integral term to the right-hand side gives us the desired estimation (4.45), which completes the proof. ■

4.6. Main ITD Stability Results in Terms of Two Measures

First, let us employ the comparison results in section 4.5. to prove the following theorems giving appropriate requirements for ITD practical stability in terms of two measures for the solutions of perturbed forms of SDEs with causal operators regarding their un-perturbed forms.

The following theorem gives adequate criteria for ITD (h_0, h) -practically stability of the solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ through (τ_0, V_0) w. r. t. the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, for $\eta = \tau_0 - t_0$, where $U(t) = U(t, t_0, U_0)$ is the solution of (4.1) through (t_0, U_0) for $t \geq t_0$.

4.6.1. Theorem (Practical Stability Theorem 1)

Assume that

- i) $0 < \lambda < A$;
- ii) $h_0, h \in \Gamma$ and h_0 is uniformly finer than h , i. e., we can ensure the existence of such a $\phi \in \mathcal{K}$ so that

$$h_0(t, \Omega) \leq \lambda \text{ implies } h(t, \Omega) \leq \phi(h_0(t, \Omega)) \quad (4.54)$$

- iii) $L(t, \Omega) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ satisfies the locally Lipschitzity property in Ω along with

$$h(t, \Omega) \leq A \text{ implies } b(h(t, \Omega)) \leq L(t, \Omega), \quad b \in \mathcal{K} \quad (4.55)$$

$$h_0(t, \Omega) \leq \lambda \text{ implies } L(t, \Omega) \leq a(h_0(t, \Omega)), \quad a \in \mathcal{K} \quad (4.56)$$

and the inequality

$$D^+L(t, \Omega) \leq g\left(t, L(t, W(t, s, \Omega))\right), \quad (t, \Omega) \in S(h, A) \quad (4.57)$$

where $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$, in addition to that $W(t) = W(t, \tau_0, V_0 - U_0)$ is the solution of (4.4) and

$$\Omega(t, \tau_0, V_0 - U_0) = V(t) - \tilde{U}(t) \text{ for } t \geq \tau_0 \quad (4.58)$$

iv) Each of the inequalities $\varphi(\lambda) < A$ and $a(\lambda) < b(A)$ holds.

Then, the practical stability of the solution of

$$\frac{du(s)}{ds} = g(t, s, u(s)), \quad u(\tau_0) = u_0 \geq 0 \quad (4.59)$$

for $\tau_0 \leq s \leq t < \infty$, with $g(t, 0) = 0$ implies that the corresponding ITD (h_0, h) -practical stability property of the solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ w. r. t. the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, for $\eta = \tau_0 - t_0$, where $U(t) = U(t, t_0, U_0)$ is the solution of (4.1) through (t_0, U_0) for $t \geq t_0$.

Proof:

Consider the practical stability of the solution $u(t, \tau_0, u_0)$ of (4.59) for $t \geq \tau_0$, with the assumption in (iv), that $a(\lambda) < b(A)$, we may write, for $t \geq \tau_0$,

$$u_0 < a(\lambda) \text{ implies } u(t, \tau_0, u_0) < b(A) \quad (4.60)$$

We shall prove that the solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ is ITD (h_0, h) -practically stable w. r. t. the solution $\tilde{U}$ corresponding to (λ, A) .

If we assume this statement is not confirmed, then there would exist such a $t_1 > \tau_0$ and a solution $W(t) = W(t, \tau_0, V_0 - U_0)$ of (4.4) for $t \geq \tau_0$ so that

$$\begin{aligned} h_0(\tau_0, V_0 - U_0) &< \lambda, \\ h(t_1, V(t_1, \tau_0, V_0) - U(t_1 - \eta, t_0, U_0)) &= A \text{ and} \\ h(t, V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)) &< A \text{ for } \tau_0 \leq t < t_1 \end{aligned} \quad (4.61)$$

Since $\phi \in \mathcal{K}$ is strictly monotone increasing, the assumptions (ii) and (iv) give us

$$h(\tau_0, V_0 - U_0) \leq \phi(h_0(\tau_0, V_0 - U_0)) < \phi(\lambda) < A \quad (4.62)$$

Thus, in view of (4.57), and by using [Theorem 1.3.1] in [46], it gives us the following inequality

$$L(t, V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)) \leq \gamma(t, \tau_0, u_0), \quad \tau_0 \leq t \leq t_1 \quad (4.63)$$

where $\gamma(t, \tau_0, u_0)$ is the maximal solution of (4.59) for $t \geq \tau_0$ and $u_0 = L(\tau_0, V_0 - U_0)$.

Since $a \in \mathcal{K}$ is strictly monotone increasing, the assumptions (iii) and (iv) give us

$$u_0 = L(\tau_0, V_0 - U_0) \leq a(h_0(\tau_0, V_0 - U_0)) < a(\lambda) \quad (4.64)$$

Considering that (4.60) holds for any solution $u(t, \tau_0, u_0)$ of (4.59) for $t \geq \tau_0$, so it holds for the maximal solution $\gamma(t, \tau_0, u_0)$ of (4.59) for $t \geq \tau_0$, this yield

$$\gamma(t, \tau_0, u_0) < b(A), \quad t \geq \tau_0 \quad (4.65)$$

The last inequality holds for any $t \geq \tau_0$ so it holds for $t = t_1 \geq \tau_0$.

Thus, employing (iii), (4.61), (4.63) and (4.65), we have

$$\begin{aligned} b(A) &= b\left(h(t_1, V(t_1, \tau_0, V_0) - U(t_1 - \eta, t_0, U_0))\right) \\ &\leq L(t_1, V(t_1, \tau_0, V_0) - U(t_1 - \eta, t_0, U_0)) \leq \gamma(t_1, \tau_0, u_0) \\ &< b(A) \end{aligned} \quad (4.66)$$

This contradiction proves the assertion. Hence, the solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ is ITD (h_0, h) -practically stable w. r. t. the solution $\tilde{U}$.

The next theorem gives adequate requirements to the ITD (h_0, h) -practically stability of the solution $V(t, \tau_0, V_0)$ of (4.3) through (τ_0, V_0) for $t \geq \tau_0$ w. r. t. the solution $\tilde{U}(t) = U(t - \eta, t_0, U_0)$ for $t \geq t_0$, where $U(t) = U(t, t_0, U_0)$ is the solution of (4.1) through (t_0, U_0) for $t \geq t_0$; provided that the solution $V(t, \tau_0, V_0)$ of (4.3) is ITD (h_0, h_0) -practically stable w. r. t. $\tilde{U}$.

4.6.2. Theorem (Practical Stability Theorem 2)

Assume that

- i) Both $L(t, \Omega) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ and $\|W(t, s, \Omega)\|$ satisfy a local Lipschitz condition in Ω given any t, s ; where $W(t) = W(t, \tau_0, V_0 - U_0)$ is the solution of (4.4) for $t \geq \tau_0$ and

$$\Omega(t, \tau_0, V_0 - U_0) = V(t) - \tilde{U}(t) \text{ for } t \geq \tau_0 \quad (4.67)$$

ii)

$$D_{*-}L(t, s, \Omega) \leq 0 \text{ in } S(h, M) \quad (4.68)$$

where

$$S(h, M) = \{(t, \Omega) : h(t, \Omega) < M \text{ for some } h \in \Gamma \text{ and } M > 0\} \quad (4.69)$$

and

$$\begin{aligned} D_{*-}L(t, s, \Omega) = \lim_{\delta \rightarrow 0^-} \inf \frac{1}{\delta} & \left(L \left(s \right. \right. \\ & \left. \left. + \delta, W \left(t, s + \delta, \Omega + \delta \left((PV)(s) - (Q\tilde{U})(s) \right) \right) \right) \right. \\ & \left. \left. - L(s, W(t, s, \Omega)) \right) \right) \end{aligned} \quad (4.70)$$

iii) For any $b \in \mathcal{K}$ and $a_1, a_0 \in \mathcal{CK}$,

$$b(h(t, \Omega)) \leq L(t, \Omega) \text{ in } S(h, M) \text{ and} \quad (4.71)$$

$$L(t, \Omega) \leq a_1(t, h(t, \Omega)) + a_0(t, h_0(t, \Omega)) \text{ in } S(h, M) \cap S(h_0, M)$$

iv) h_0 is finer than h , i. e., we can ensure the existence of such a $\phi \in \mathcal{K}$ so that

$$h_0(t, \Omega) \leq M_0 \text{ implies } h(t, \Omega) \leq \phi(h_0(t, \Omega)) \quad (4.72)$$

for some M_0 with $\phi(M_0) \leq M$;

v) The solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ is ITD (h_0, h_0) -practically stable w. r. t. the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, for $\eta = \tau_0 - t_0$.

Then this implies that the ITD (h_0, h) -practical stability of the solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ w. r. t. the solution $\tilde{U}$.

Proof:

We shall show that the solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ is ITD (h_0, h) -practically stable w. r. t. the solution $\tilde{U}$, that is, for any given (λ, A) such that $0 < \lambda < A$ and for any $\tau_0 \in \mathbb{R}_+$, we have

$$h_0(\tau_0, V_0 - U_0) < \lambda \text{ implies } h(t, \Omega(t)) < A \text{ for } t \geq \tau_0 \quad (4.73)$$

If we assume that is not, then this gives us the existence of solutions $\tilde{U}(t) = U(t - \eta, t_0, U_0)$, where $U(t, t_0, U_0)$ is the solution of (4.1) for $t \geq t_0$; and $V(t) = V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$, and $t_1 > \tau_0$ such that

$$\begin{aligned} h_0(\tau_0, V_0 - U_0) &< \lambda, \\ h(t_1, \Omega(t_1)) &= A \end{aligned} \quad (4.74)$$

$$\text{and } h(t, \Omega(t)) \leq A, \text{ for } \tau_0 \leq t \leq t_1$$

where $\Omega(t) = V(t) - \tilde{U}(t)$ for $t \geq \tau_0$.

Employing the theorem in 4.5.2. gives us that

$$L(t, \Omega(t)) \leq L(\tau_0, W(t, \tau_0, V_0 - U_0)), \text{ for } \tau_0 \leq t \leq t_1 \quad (4.75)$$

Starting from the statement of $b(A)$ and using the assumptions (iii), (4.74) and (4.75), we obtain

$$\begin{aligned} b(A) &= b(h(t_1, \Omega(t_1))) \leq L(t_1, \Omega(t_1)) \leq L(\tau_0, W(t_1, \tau_0, V_0 - U_0)) \\ &\leq a_1(\tau_0, h(\tau_0, W(t_1, \tau_0, V_0 - U_0))) \\ &\quad + a_0(\tau_0, h_0(\tau_0, W(t_1, \tau_0, V_0 - U_0))) \end{aligned} \quad (4.76)$$

We aim to reach a contradiction to finish the proof. We will use assumption (v) for this purpose.

Given $0 < A < M$ and that there exists a M_0 with $\phi(M_0) \leq M$.

Choosing $N_1 = N_1(\tau_0, A)$ such that $0 < N_1(\tau_0, A) < M_0$, and

$$h_0(t, \Omega(t)) < N_1 \text{ implies } a_0(t, h_0(t, \Omega(t))) < \frac{b(A)}{2} \text{ for } t \geq \tau_0 \quad (4.77)$$

By assumption (v), corresponding to this N_1 , there exists a $\lambda_1 = \lambda_1(\tau_0, N_1) > 0$ such that

$$h_0(\tau_0, V_0 - U_0) < \lambda_1 \text{ implies } h_0(t, \Omega(t)) < N_1 \text{ for } t \geq \tau_0 \quad (4.78)$$

Thus (4.77) and (4.78) give us

$$h_0(\tau_0, V_0 - U_0) < \lambda_1 \text{ implies } a_0(t, h_0(t, \Omega(t))) < \frac{b(A)}{2} \text{ for } t \geq \tau_0 \quad (4.79)$$

Similarly, we choose $N_2 = N_2(\tau_0, A)$ such that $0 < N_2(\tau_0, A) < M_0$ as well as

$$h(t, \Omega(t)) < N_2 \text{ implies } a_1\left(t, h(t, \Omega(t))\right) < \frac{b(A)}{2} \text{ for } t \geq \tau_0 \quad (4.80)$$

By the assumptions (iv) and (v) correspond to $\phi^{-1}(N_2)$, there exists a $\lambda_2 = \lambda_2(\tau_0, N_2) > 0$ such that

$$h_0(\tau_0, V_0 - U_0) < \lambda_2 \text{ implies } h_0(t, \Omega(t)) < \phi^{-1}(N_2) \text{ for } t \geq \tau_0 \quad (4.81)$$

Since $\phi \in \mathcal{K}$ is strictly monotone increasing; then, we have by taking the composition of ϕ of both sides of the inequality $h_0(t, \Omega(t)) < \phi^{-1}(N_2)$ in (4.81), with considering (4.72),

$$h_0(\tau_0, V_0 - U_0) < \lambda_2 \text{ implies} \quad (4.82)$$

$$h(t, \Omega(t)) \leq \phi\left(h_0(t, \Omega(t))\right) < \phi(\phi^{-1}(N_2)) = N_2 \text{ for } t \geq \tau_0$$

So, (4.80) and (4.82) give us, for $t \geq \tau_0$,

$$h_0(\tau_0, V_0 - U_0) < \lambda_2 \text{ implies } a_1\left(t, h(t, \Omega(t))\right) < \frac{b(A)}{2} \quad (4.83)$$

Let $\lambda = \min\{\lambda_1, \lambda_2\}$, then with this λ the following statement holds.

$$h_0(\tau_0, V_0 - U_0) < \lambda \text{ implies}$$

$$a_0\left(t, h_0(t, \Omega(t))\right) < \frac{b(A)}{2} \text{ and } a_1\left(t, h(t, \Omega(t))\right) < \frac{b(A)}{2} \text{ for } t \geq \tau_0 \quad (4.84)$$

Hence, when $t = t_1$, using (4.84), the statement (4.76), it is possible to express it as

$$\begin{aligned} b(A) &= b\left(h(t_1, \Omega(t_1))\right) \leq L(t_1, \Omega(t_1)) \leq L(\tau_0, W(t_1, \tau_0, V_0 - U_0)) \\ &\leq a_1\left(\tau_0, h(\tau_0, W(t_1, \tau_0, V_0 - U_0))\right) \\ &+ a_0\left(\tau_0, h_0(\tau_0, W(t_1, \tau_0, V_0 - U_0))\right) < \frac{b(A)}{2} + \frac{b(A)}{2} \\ &= b(A) \end{aligned} \quad (4.85)$$

This contradiction demonstrates that the solution $V(t, \tau_0, V_0)$ of (4.3) through (τ_0, V_0) for $t \geq \tau_0$ is ITD (h_0, h) -practically stable w. r. t. the solution $\tilde{U}$. ■

Now, let us employ the comparison results in section 4.5. to prove the following theorems giving sufficient conditions for equi-boundedness, equi-attractiveness in the large, and Lagrange stability in terms of two measures for the solutions of perturbed forms of SDEs involving causal operators in regards to their un-perturbed forms.

The following theorem gives sufficient conditions to the ITD (h_0, h) -equi-boundedness of the solution $V(t, \tau_0, V_0)$ of (4.3) through (τ_0, V_0) for $t \geq \tau_0$ w. r. t. the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, for $\eta = \tau_0 - t_0$, where $U(t) = U(t, t_0, U_0)$ is the solution of (4.1) through (t_0, U_0) for $t \geq t_0$; provided that the solution $V(t, \tau_0, V_0)$ of (4.3) is ITD (h_0, h_0) -equi-bounded w. r. t. $\tilde{U}$.

4.6.3. Theorem (Equi-Boundedness Theorem)

Assume that

- i) Both $L(t, \Omega) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ and $\|W(t, s, \Omega)\|$ satisfy a local Lipschitz condition in Ω given any t, s ; where $W(t) = W(t, \tau_0, V_0 - U_0)$ is the solution of (4.4) for $t \geq \tau_0$ and

$$\Omega(t, \tau_0, V_0 - U_0) = V(t) - \tilde{U}(t) \text{ for } t \geq \tau_0 \quad (4.86)$$

- ii)

$$D_{*-}L(t, s, \Omega) \leq -c \left(h \left(s, W(t, s, \Omega(s)) \right) \right) \text{ in } S(h, M) \quad (4.87)$$

where $c \in \mathcal{K} = \{\phi \in C[\mathbb{R}_+, \mathbb{R}_+] \text{ such that } \phi(0) = 0 \text{ and } \phi(s) \text{ is increasing in } s\}$ and

$$S(h, M) = \{(t, \Omega) : h(t, \Omega) < M \text{ for some } h \in \Gamma \text{ and } M > 0\} \quad (4.88)$$

and

$$\begin{aligned}
D_{*-}L(t, s, \Omega) = \lim_{\delta \rightarrow 0^-} \inf \frac{1}{\delta} & \left(L \left(s \right. \right. \\
& + \delta, W \left(t, s + \delta, \Omega + \delta \left((PV)(s) - (Q\tilde{U})(s) \right) \right) \\
& \left. \left. - L(s, W(t, s, \Omega)) \right) \right)
\end{aligned} \tag{4.89}$$

iii) For $b \in \mathcal{K}$ and $a_1, a_0 \in \mathcal{CK}$,

$$\begin{aligned}
b(h(t, \Omega)) + \int_{\tau_0}^t c \left(h \left(s, W(t, s, \Omega(s)) \right) \right) ds & \leq L(t, \Omega) \text{ in } S(h, M) \text{ and} \\
L(t, \Omega) & \leq a_1(t, h(t, \Omega)) + a_0(t, h_0(t, \Omega)) \text{ in } S(h, M) \cap S(h_0, M)
\end{aligned} \tag{4.90}$$

iv) h_0 is finer than h , i. e., there exists a $\phi \in \mathcal{K}$ such that

$$h_0(t, \Omega) \leq M_0 \text{ implies } h(t, \Omega) \leq \phi(h_0(t, \Omega)) \tag{4.91}$$

for some M_0 with $\phi(M_0) \leq M$;

v) The solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ is ITD (h_0, h_0) -equi-bounded w. r. t. the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, for $\eta = \tau_0 - t_0$.

Then this implies the ITD (h_0, h) -equi-boundedness of the solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ w. r. t. the solution $\tilde{U}$.

Proof:

We shall show that the solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ is ITD (h_0, h) -equi-bounded w. r. t. the solution $\tilde{U}$, that is, given any $\alpha > 0$ and for some $\tau_0 \in \mathbb{R}_+$, there exists $\beta = \beta(\alpha, \tau_0) > 0$ such that $h_0(\tau_0, V_0 - U_0) < \beta$ implies

$$h(t, V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)) < \alpha \text{ for } t \geq \tau_0 \tag{4.92}$$

Assume that (4.92) is not true, then there exist solutions $\tilde{U}(t) = U(t - \eta, t_0, U_0)$, where $U(t, t_0, U_0)$ is the solution of (4.1) for $t \geq t_0$; and $V(t) = V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$, and $t_1 > \tau_0$ such that

$$\begin{aligned}
h_0(\tau_0, V_0 - U_0) & < \beta, \\
h(t_1, \Omega(t_1)) & = \alpha \\
\text{and } h(t, \Omega(t)) & \leq \alpha, \text{ for } \tau_0 \leq t \leq t_1
\end{aligned} \tag{4.93}$$

where $\Omega(t) = V(t) - \tilde{U}(t)$ for $t \geq \tau_0$.

By the theorem in 4.5.2., we have, for $\tau_0 \leq t \leq t_1$,

$$L(t, \Omega(t)) \leq L(\tau_0, W(t, \tau_0, V_0 - U_0)) - \int_{\tau_0}^t c \left(h(s, W(t, s, \Omega(s))) \right) ds \quad (4.94)$$

Then, using the assumptions (iii), (4.93) and (4.94), we obtain when $t = t_1$,

$$\begin{aligned} b(\alpha) + \int_{\tau_0}^{t_1} c \left(h(s, W(t_1, s, \Omega(s))) \right) ds \\ &= b(h(t_1, \Omega(t_1))) + \int_{\tau_0}^{t_1} c \left(h(s, W(t_1, s, \Omega(s))) \right) ds \\ &\leq L(t_1, \Omega(t_1)) \\ &\leq L(\tau_0, W(t_1, \tau_0, V_0 - U_0)) \\ &\quad - \int_{\tau_0}^{t_1} c \left(h(s, W(t_1, s, \Omega(s))) \right) ds \\ &\leq L(\tau_0, W(t_1, \tau_0, V_0 - U_0)) \\ &\quad + \int_{\tau_0}^{t_1} c \left(h(s, W(t_1, s, \Omega(s))) \right) ds \\ &\leq a_1 \left(\tau_0, h(\tau_0, W(t_1, \tau_0, V_0 - U_0)) \right) \\ &\quad + a_0 \left(\tau_0, h_0(\tau_0, W(t_1, \tau_0, V_0 - U_0)) \right) \\ &\quad + \int_{\tau_0}^{t_1} c \left(h(s, W(t_1, s, \Omega(s))) \right) ds \end{aligned} \quad (4.95)$$

We aim to reach a contradiction to finish the proof. We will use assumption (v) for this purpose.

Given $0 < \alpha < M$ and that there exists a M_0 with $\phi(M_0) \leq M$.

Choosing $N_1 = N_1(\tau_0, \alpha)$ such that $0 < N_1(\tau_0, \alpha) < M_0$, and

$$h_0(t, \Omega(t)) < N_1 \quad \text{implies} \quad a_0 \left(t, h_0(t, \Omega(t)) \right) < \frac{b(\alpha)}{2} \quad \text{for } t \geq \tau_0 \quad (4.96)$$

By assumption (v), corresponding to this N_1 , there exists a $\beta_1 = \beta_1(\tau_0, N_1) > 0$ such that

$$h_0(\tau_0, V_0 - U_0) < \beta_1 \text{ implies } h_0(t, \Omega(t)) < N_1 \text{ for } t \geq \tau_0 \quad (4.97)$$

Thus (4.96) and (4.97) give us

$$h_0(\tau_0, V_0 - U_0) < \beta_1 \text{ implies } a_0\left(t, h_0(t, \Omega(t))\right) < \frac{b(\alpha)}{2} \text{ for } t \geq \tau_0 \quad (4.98)$$

Similarly, we choose $N_2 = N_2(\tau_0, \alpha)$ such that $0 < N_2(\tau_0, \alpha) < M_0$ and

$$h(t, \Omega(t)) < N_2 \text{ implies } a_1\left(t, h(t, \Omega(t))\right) < \frac{b(\alpha)}{2} \text{ for } t \geq \tau_0 \quad (4.99)$$

By assumptions (iv) and (v) correspond to $\phi^{-1}(N_2)$, there exists a $\beta_2 = \beta_2(\tau_0, N_2) > 0$ such that

$$h_0(\tau_0, V_0 - U_0) < \beta_2 \text{ implies } h_0(t, \Omega(t)) < \phi^{-1}(N_2) \text{ for } t \geq \tau_0 \quad (4.100)$$

Since $\phi \in \mathcal{K}$ is strictly monotone increasing; then, we have by taking the composition of ϕ of both sides of the inequality $h_0(t, \Omega(t)) < \phi^{-1}(N_2)$ in (4.100), with considering (4.91),

$$h_0(\tau_0, V_0 - U_0) < \beta_2 \text{ implies } h(t, \Omega(t)) \leq \phi\left(h_0(t, \Omega(t))\right) < \phi(\phi^{-1}(N_2)) = N_2 \text{ for } t \geq \tau_0 \quad (4.101)$$

So, (4.99) and (4.101) give us, for $t \geq \tau_0$,

$$h_0(\tau_0, V_0 - U_0) < \beta_2 \text{ implies } a_1\left(t, h(t, \Omega(t))\right) < \frac{b(\alpha)}{2} \quad (4.102)$$

Let $\beta = \min\{\beta_1, \beta_2\}$, then with this β the following statement holds.

$$h_0(\tau_0, V_0 - U_0) < \beta \text{ implies } a_0\left(t, h_0(t, \Omega(t))\right) < \frac{b(\alpha)}{2} \text{ and } a_1\left(t, h(t, \Omega(t))\right) < \frac{b(\alpha)}{2} \text{ for } t \geq \tau_0 \quad (4.103)$$

Hence, when $t = t_1$, using (4.103), it is possible to write the statement (4.95)

as

$$\begin{aligned}
b(\alpha) + \int_{\tau_0}^{t_1} c \left(h \left(s, W(t_1, s, \Omega(s)) \right) \right) ds \\
&= b(h(t_1, \Omega(t_1))) + \int_{\tau_0}^{t_1} c \left(h \left(s, W(t_1, s, \Omega(s)) \right) \right) ds \\
&\leq L(t_1, \Omega(t_1)) \\
&\leq L(\tau_0, W(t_1, \tau_0, V_0 - U_0)) - \int_{\tau_0}^{t_1} c \left(h \left(s, W(t_1, s, \Omega(s)) \right) \right) ds \\
&\leq L(\tau_0, W(t_1, \tau_0, V_0 - U_0)) + \int_{\tau_0}^{t_1} c \left(h \left(s, W(t_1, s, \Omega(s)) \right) \right) ds \quad (4.104) \\
&\leq a_1 \left(\tau_0, h(\tau_0, W(t_1, \tau_0, V_0 - U_0)) \right) \\
&\quad + a_0 \left(\tau_0, h_0(\tau_0, W(t_1, \tau_0, V_0 - U_0)) \right) \\
&\quad + \int_{\tau_0}^{t_1} c \left(h \left(s, W(t_1, s, \Omega(s)) \right) \right) ds \\
&< \frac{b(\alpha)}{2} + \frac{b(\alpha)}{2} + \int_{\tau_0}^{t_1} c \left(h \left(s, W(t_1, s, \Omega(s)) \right) \right) ds \\
&= b(\alpha) + \int_{\tau_0}^{t_1} c \left(h \left(s, W(t_1, s, \Omega(s)) \right) \right) ds
\end{aligned}$$

This contradiction demonstrates that the solution $V(t, \tau_0, V_0)$ of (4.3) through (τ_0, V_0) for $t \geq \tau_0$ is ITD (h_0, h) -equi-bounded w. r. t. the solution $\tilde{U}$. ■

The next theorem gives sufficient conditions to the ITD equi-attractiveness in the large of the solution $V(t, \tau_0, V_0)$ of (4.3) through (τ_0, V_0) for $t \geq \tau_0$ w. r. t. the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, for $\eta = \tau_0 - t_0$, where $U(t) = U(t, t_0, U_0)$ is the solution of (4.1) through (t_0, U_0) for $t \geq t_0$; provided that the solution $V(t, \tau_0, V_0)$ of (4.3) is ITD (h_0, h_0) -equi-attractive in the large w. r. t. $\tilde{U}$.

4.6.4. Theorem (Equi-Attractiveness Theorem)

Assume that

- i) Both $L(t, \Omega) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ and $\|W(t, s, \Omega)\|$ satisfy a local Lipschitz condition in Ω given any t, s ; where $W(t) = W(t, \tau_0, V_0 - U_0)$ is the solution of (4.4) for $t \geq \tau_0$ and

$$\Omega(t, \tau_0, V_0 - U_0) = V(t) - \tilde{U}(t) \text{ for } t \geq \tau_0 \quad (4.105)$$

ii)

$$D_{*-}L(t, s, \Omega) \leq -c \left(h \left(s, W(t, s, \Omega(s)) \right) \right) \text{ in } S(h, M) \quad (4.106)$$

where $c \in \mathcal{K} = \{\phi \in C[\mathbb{R}_+, \mathbb{R}_+] \text{ such that } \phi(0) = 0 \text{ and } \phi(s) \text{ is increasing in } s\}$ as well as

$$S(h, M) = \{(t, \Omega) : h(t, \Omega) < M \text{ for some } h \in \Gamma \text{ and } M > 0\} \quad (4.107)$$

and

$$\begin{aligned} D_{*-}L(t, s, \Omega) = \lim_{\delta \rightarrow 0^-} \inf \frac{1}{\delta} & \left(L \left(s \right. \right. \\ & \left. \left. + \delta, W \left(t, s + \delta, \Omega + \delta \left((PV)(s) - (Q\tilde{U})(s) \right) \right) \right) \right. \\ & \left. \left. - L(s, W(t, s, \Omega)) \right) \right) \end{aligned} \quad (4.108)$$

iii) For any $b \in \mathcal{K}$ and $a_1, a_0 \in \mathcal{CK}$,

$$\begin{aligned} b(h(t, \Omega)) + \int_{\tau_0}^t c \left(h \left(s, W(t, s, \Omega(s)) \right) \right) ds & \leq L(t, \Omega) \text{ in } S(h, M) \text{ and} \\ L(t, \Omega) & \leq a_1(t, h(t, \Omega)) + a_0(t, h_0(t, \Omega)) \text{ in } S(h, M) \cap S(h_0, M) \end{aligned} \quad (4.109)$$

iv) h_0 is finer than h , i. e., we can ensure the existence of such a $\phi \in \mathcal{K}$ so that

$$h_0(t, \Omega) \leq M_0 \text{ implies } h(t, \Omega) \leq \phi(h_0(t, \Omega)) \quad (4.110)$$

for some M_0 with $\phi(M_0) \leq M$;

v) The solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ is ITD (h_0, h_0) -equi-attractive in the large w. r. t. the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, for $\eta = \tau_0 - t_0$.

Then this implies the ITD (h_0, h) -equi-attractiveness in the large of the solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ w. r. t. the solution $\tilde{U}$.

Proof:

We shall show that the solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ is ITD (h_0, h) -equi-attractive in the large w. r. t. the solution $\tilde{U}$, that is, given any $\varepsilon, \alpha > 0$ and $\tau_0 \in \mathbb{R}_+$, there exists a $T = T(\tau_0, \varepsilon, \alpha) > 0$ such that $h_0(\tau_0, V_0 - U_0) < \alpha$ implies

$$h(t, V(t, \tau_0, V_0) - U(t - \eta, t_0, U_0)) < \varepsilon, \quad t \geq \tau_0 + T(\tau_0, \varepsilon, \alpha) \quad (4.111)$$

Assume that (4.111) is not true, then there exist solutions $\tilde{U}(t) = U(t - \eta, t_0, U_0)$, where $U(t, t_0, U_0)$ is the solution of (4.1) for $t \geq t_0$; and $V(t) = V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$, and a sequence $\{t_k\}_{k \geq 1}$, $t_k \geq \tau_0 + T$ and $\lim_{k \rightarrow \infty} t_k = \infty$ such that

$$\begin{aligned} h_0(\tau_0, V_0 - U_0) &< \alpha, \\ h(t_k, \Omega(t_k)) &\geq \varepsilon, \text{ for } t_k \geq \tau_0 + T \end{aligned} \quad (4.112)$$

where $\Omega(t) = V(t) - \tilde{U}(t)$ for $t \geq \tau_0$.

By the theorem in 4.5.2., we have, for $t \geq \tau_0$,

$$L(t, \Omega(t)) \leq L(\tau_0, W(t, \tau_0, V_0 - U_0)) - \int_{\tau_0}^t c \left(h \left(s, W(t, s, \Omega(s)) \right) \right) ds \quad (4.113)$$

Then, using the assumptions (iii), (4.112) and (4.113), we obtain

$$\begin{aligned}
b(\varepsilon) + \int_{\tau_0}^t c \left(h \left(s, W(t_k, s, \Omega(s)) \right) \right) ds \\
\leq b \left(h(t_k, \Omega(t_k)) \right) + \int_{\tau_0}^t c \left(h \left(s, W(t_k, s, \Omega(s)) \right) \right) ds \\
\leq L(t_k, \Omega(t_k)) \\
\leq L(\tau_0, W(t_k, \tau_0, V_0 - U_0)) \\
- \int_{\tau_0}^t c \left(h \left(s, W(t_k, s, \Omega(s)) \right) \right) ds \\
\leq L(\tau_0, W(t_k, \tau_0, V_0 - U_0)) \\
+ \int_{\tau_0}^t c \left(h \left(s, W(t_k, s, \Omega(s)) \right) \right) ds \\
\leq a_1 \left(\tau_0, h(\tau_0, W(t_k, \tau_0, V_0 - U_0)) \right) \\
+ a_0 \left(\tau_0, h_0(\tau_0, W(t_k, \tau_0, V_0 - U_0)) \right) \\
+ \int_{\tau_0}^t c \left(h \left(s, W(t_k, s, \Omega(s)) \right) \right) ds
\end{aligned} \tag{4.114}$$

We aim to reach a contradiction to finish the proof. We will use assumption (v) for this purpose.

Given $0 < \varepsilon < M$ and that there exists a M_0 with $\phi(M_0) \leq M$. Choosing $N_1 = N_1(\tau_0, \varepsilon)$ such that $0 < N_1(\tau_0, \varepsilon) < M_0$, and

$$h_0(t, \Omega(t)) < N_1 \text{ implies } a_0 \left(t, h_0(t, \Omega(t)) \right) < \frac{b(\varepsilon)}{2} \text{ for } t \geq \tau_0 \tag{4.115}$$

By assumption (v), corresponding to this N_1 , there exists a $\alpha_1 > 0$ and a $T_1 = T_1(\tau_0, N_1, \alpha_1) > 0$ such that

$$h_0(\tau_0, V_0 - U_0) < \alpha_1 \text{ implies } h_0(t, \Omega(t)) < N_1 \text{ for } t \geq \tau_0 + T_1 \tag{4.116}$$

Thus (4.115) and (4.116) give us

$$h_0(\tau_0, V_0 - U_0) < \alpha_1 \text{ implies } a_0 \left(t, h_0(t, \Omega(t)) \right) < \frac{b(\varepsilon)}{2} \text{ for } t \geq \tau_0 + T_1 \tag{4.117}$$

Similarly, we choose $N_2 = N_2(\tau_0, \varepsilon)$ such that $0 < N_2(\tau_0, \varepsilon) < M_0$ and

$$h(t, \Omega(t)) < N_2 \text{ implies } a_1\left(t, h(t, \Omega(t))\right) < \frac{b(\varepsilon)}{2} \text{ for } t \geq \tau_0 \quad (4.118)$$

By the assumptions (iv) and (v) correspond to $\phi^{-1}(N_2)$, there exists a $\alpha_2 > 0$ and a $T_2 = T_2(\tau_0, N_2, \alpha_2) > 0$ such that

$$h_0(\tau_0, V_0 - U_0) < \alpha_2 \text{ implies } h_0(t, \Omega(t)) < \phi^{-1}(N_2) \text{ for } t \geq \tau_0 + T_2 \quad (4.119)$$

Since $\phi \in \mathcal{K}$ is strictly monotone increasing; then, we have by taking the composition of ϕ of both sides of the inequality $h_0(t, \Omega(t)) < \phi^{-1}(N_2)$ in (4.119), with considering (4.110),

$$h_0(\tau_0, V_0 - U_0) < \alpha_2 \text{ implies} \quad (4.120)$$

$$h(t, \Omega(t)) \leq \phi\left(h_0(t, \Omega(t))\right) < \phi(\phi^{-1}(N_2)) = N_2 \text{ for } t \geq \tau_0 + T_2$$

So, (4.118) and (4.120) give us, for $t \geq \tau_0 + T_2$,

$$h_0(\tau_0, V_0 - U_0) < \alpha_2 \text{ implies } a_1\left(t, h(t, \Omega(t))\right) < \frac{b(\varepsilon)}{2} \quad (4.121)$$

Let $\alpha = \min\{\alpha_1, \alpha_2\}$, and $T = \max\{T_1, T_2\}$, then,

$$T = T(T_1, T_2) = T(\tau_0, N_1, \alpha_1, N_2, \alpha_2) = T(\tau_0, \varepsilon, \alpha) \quad (4.122)$$

Therefore, with these α, T the following statement holds.

$$h_0(\tau_0, V_0 - U_0) < \alpha \text{ implies} \quad (4.123)$$

$$a_0\left(t, h_0(t, \Omega(t))\right) < \frac{b(\varepsilon)}{2} \text{ and } a_1\left(t, h(t, \Omega(t))\right) < \frac{b(\varepsilon)}{2} \text{ for } t \geq \tau_0 + T$$

Hence, when $t = t_1$, using (4.123), the statement (4.114) can be written as

$$\begin{aligned}
b(\varepsilon) + \int_{\tau_0}^{t_1} c \left(h \left(s, W(t_k, s, \Omega(s)) \right) \right) ds \\
\leq b \left(h(t_k, \Omega(t_k)) \right) + \int_{\tau_0}^{t_1} c \left(h \left(s, W(t_k, s, \Omega(s)) \right) \right) ds \\
\leq L(t_k, \Omega(t_k)) \\
\leq L(\tau_0, W(t_k, \tau_0, V_0 - U_0)) \\
- \int_{\tau_0}^{t_1} c \left(h \left(s, W(t_k, s, \Omega(s)) \right) \right) ds \\
\leq L(\tau_0, W(t_k, \tau_0, V_0 - U_0)) \\
+ \int_{\tau_0}^{t_1} c \left(h \left(s, W(t_k, s, \Omega(s)) \right) \right) ds \tag{4.124} \\
\leq a_1 \left(\tau_0, h(\tau_0, W(t_k, \tau_0, V_0 - U_0)) \right) \\
+ a_0 \left(\tau_0, h_0(\tau_0, W(t_k, \tau_0, V_0 - U_0)) \right) \\
+ \int_{\tau_0}^{t_1} c \left(h \left(s, W(t_k, s, \Omega(s)) \right) \right) ds \\
< \frac{b(\varepsilon)}{2} + \frac{b(\varepsilon)}{2} + \int_{\tau_0}^{t_1} c \left(h \left(s, W(t_k, s, \Omega(s)) \right) \right) ds \\
= b(\varepsilon) + \int_{\tau_0}^{t_1} c \left(h \left(s, W(t_k, s, \Omega(s)) \right) \right) ds
\end{aligned}$$

This contradiction proves the ITD (h_0, h) -equi-attractiveness in the large of the solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0 + T(\tau_0, \varepsilon, \alpha)$ w. r. t. the solution $\tilde{U}$. ■

The next theorem gives adequate requirements to the ITD (h_0, h) -Lagrange stability of the solution $V(t, \tau_0, V_0)$ of (4.3) through (τ_0, V_0) for $t \geq \tau_0$ w. r. t. the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, for $\eta = \tau_0 - t_0$, where $U(t) = U(t, t_0, U_0)$ is the solution of (4.1) through (t_0, U_0) for $t \geq t_0$; provided that the solution $V(t, \tau_0, V_0)$ of (4.3) is ITD (h_0, h_0) -Lagrange stable w. r. t. $\tilde{U}$.

4.6.5. Theorem (Lagrange Stability Theorem)

Assume that

- i) Both $L(t, \Omega) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ and $\|W(t, s, \Omega)\|$ satisfy a local Lipschitz condition in Ω given any t, s ; where $W(t) = W(t, \tau_0, V_0 - U_0)$ is the solution of (4.4) for $t \geq \tau_0$ and

$$\Omega(t, \tau_0, V_0 - U_0) = V(t) - \tilde{U}(t) \text{ for } t \geq \tau_0 \quad (4.125)$$

ii)

$$D_{*-}L(t, s, \Omega) \leq -c \left(h \left(s, W(t, s, \Omega(s)) \right) \right) \text{ in } S(h, M) \quad (4.126)$$

where $c \in \mathcal{K} = \{\phi \in C[\mathbb{R}_+, \mathbb{R}_+] \text{ such that } \phi(0) = 0 \text{ and } \phi(s) \text{ is increasing in } s\}$ and

$$S(h, M) = \{(t, \Omega) : h(t, \Omega) < M \text{ for some } h \in \Gamma \text{ and } M > 0\} \quad (4.127)$$

and

$$\begin{aligned} D_{*-}L(t, s, \Omega) = \lim_{\delta \rightarrow 0^-} \inf \frac{1}{\delta} & \left(L \left(s \right. \right. \\ & \left. \left. + \delta, W \left(t, s + \delta, \Omega + \delta \left((PV)(s) - (Q\tilde{U})(s) \right) \right) \right) \right) \\ & - L(s, W(t, s, \Omega)) \end{aligned} \quad (4.128)$$

- iii) For $b \in \mathcal{K}$ and $a_1, a_0 \in \mathcal{CK}$,

$$\begin{aligned} b(h(t, \Omega)) + \int_{\tau_0}^t c \left(h \left(s, W(t, s, \Omega(s)) \right) \right) ds & \leq L(t, \Omega) \text{ in } S(h, M) \\ \text{and } L(t, \Omega) & \leq a_1(t, h(t, \Omega)) + a_0(t, h_0(t, \Omega)) \text{ in } S(h, M) \cap S(h_0, M) \end{aligned} \quad (4.129)$$

- iv) h_0 is finer than h , i.e., we can ensure the existence of such a $\phi \in \mathcal{K}$ so that

$$h_0(t, \Omega) \leq M_0 \text{ implies } h(t, \Omega) \leq \phi(h_0(t, \Omega)) \quad (4.130)$$

for some M_0 with $\phi(M_0) \leq M$;

v) The solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ is ITD (h_0, h_0) -Lagrange stable w. r. t. the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, for $\eta = \tau_0 - t_0$.

Then this implies the ITD (h_0, h) -Lagrange stability of the solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ w. r. t. the solution $\tilde{U}$.

Proof:

The ITD (h_0, h_0) -Lagrange stability of the solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ w. r. t. the solution $\tilde{U}$ gives us by definition of the ITD (h_0, h_0) -equi-boundedness and the ITD (h_0, h_0) -equi-attractiveness in the large of the solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ w. r. t. the solution $\tilde{U}$. Hence, by applying

In Theorem 3 and Theorem 4, respectively, we obtain the ITD (h_0, h) -equi-boundedness and ITD (h_0, h) -equi-attractiveness in the large of the solution $V(t, \tau_0, V_0)$ of (4.3) for $t \geq \tau_0$ w. r. t. the solution $\tilde{U}$. That is, it is ITD (h_0, h) -Lagrange stable w. r. t. the solution $\tilde{U}$, by definition. ■

4.7. Conclusion

In this chapter, we have provided adequate criteria for ITD practical stability, ITD equi-boundedness, equi-attractiveness in the large, and Lagrange stability for the solutions of perturbed forms of SDEs with causal operators regarding their unperturbed forms; in the context of “two measures” with considering the differences in the starting parameters in the initial states. We proved the sufficiency of these criteria using ITD variational comparative results.

5. FUTURE PROSPECTS

This dissertation examines a preparatory study addressing the stability properties of SDEs involving causal operators. The difference in initial conditions of perturbed forms of SDEs involving causal operators regarding un-perturbed forms has been considered. Analogous results to Lyapunov's second method in stability have been presented in studying the qualitative behavior of solutions of SDEs with causal operators. Different types of stability properties have been investigated using generalized variational comparison results to examine the stability aspects in terms of two measures for solutions of SDEs with causal operators.

The generalized variational comparison results we concluded may be used to investigate a variety of other sorts of stability properties and additional qualitative features in terms of two measures later on.

Furthermore, many exceptional cases of the SDEs involving causal operators can be discussed to achieve reducible findings from the ones we found.

Moreover, integrating those mentioned above and theoretical presented ideas with concrete real-world problems can give more practical numerical approaches for studying the qualitative and quantitative properties of solutions of SDEs involving causal operators.

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BIOGRAPHY

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APPENDIX

Publications Based on the Thesis

Yakar C., Talab H., (2021), "Stability of perturbed set differential equations involving causal operators in regard to their unperturbed ones considering difference in initial conditions", *Advances in Mathematical Physics*, 2021, Article ID 9794959, 1–12.

Yakar C., Talab H., (2022), "Practical stability in terms of two measures with initial time difference for set differential equations involving causal operators", *Konuralp Journal of Mathematics*, 10, 149–158.

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