

T.C.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL

**WHIRL FLUTTER STABILITY ANALYSIS FOR DISTRIBUTED
PROPULSION SYSTEMS ON ELASTIC WINGS**



YUNUS EMRE DOĞRU

A THESIS OF MASTER OF SCIENCE
DEPARTMENT OF MECHANICAL ENGINEERING

ADVISOR: ASSIST. PROF. DR. ALİ EKEN

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SİSTEMİ İÇİN DÖNME ÇİRPİNMA KARARSIZLIĞI
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Dedicated to those who made this work possible.

ABSTRACT

Increased passenger traffic and the need to reduce environmental emissions are pushing the aviation industry toward new technologies. To reduce carbon emissions, distributed electric propulsion (DEP) systems are being developed as an alternative to traditional engines. This system relies on multiple electric motors driven by propellers being placed along the wing to generate thrust.

When the propeller-nacelle system is attached to a rigid structure, the gyroscopic forces generated by the rotation of the propeller cause pitching and yawing motions, resulting in two distinct modes known as backward whirl and forward whirl. Under the influence of aerodynamic forces, this system can become aeroelastically unstable above a certain critical speed. When the propeller-nacelle system is integrated onto an elastic wing, the interaction between the wing and propeller modes can cause the divergence, flutter, and whirl flutter speeds to change. The position of the propeller-nacelle system on the wing directly affects these critical speeds.

To investigate these effects, this thesis study has performed a comprehensive analytical modeling for the aeroelastic analysis of distributed propulsive systems. Euler-Bernoulli beam theory was used for structural modeling, the unsteady Theodorsen theory for wing aerodynamics, and the Houbolt-Reed theory modified to include unsteady effects for propeller aerodynamics; these theories were integrated using the Galerkin Method. The system matrices were derived using the Mathematica symbolic programming language, and eigenvalue analyses were performed in the MATLAB environment. The developed code was successfully validated by comparing it with experimental and analytical data in the literature.

The study followed a systematic and step-by-step validation process. First, the clean wing (without propeller) model was validated, and the natural frequencies of the wing, as well as the static and dynamic aeroelastic instability limits, were examined. Subsequently, the aerodynamic derivatives of the propeller were validated using the FORTRAN code underlying the NASTRAN software and the literature data. In the whirl flutter analysis performed for the isolated propeller-nacelle system, the speed-dependent stability conditions of the backward and forward whirl modes were analyzed. When the propeller-nacelle structure is added to the wing as a rigid mass, it is observed that the natural frequencies of the wing bending and torsion change. When the system is connected flexibly and aerodynamic forces are neglected, the effect of gyroscopic forces generated by the rotation of the propeller on the natural frequencies is evaluated. In the subsequent aeroelastic analyses, the critical speeds at which the modes were driven into instability were determined. Finally, structural and aeroelastic analyses were performed on a wing model incorporating multiple propeller-nacelle systems to investigate mode interactions and instability limits.

Keywords: Whirl Flutter, Aeroelasticity, Distributed Propulsion, Elastic Wings, Unsteady Aerodynamics

ÖZET

Artan yolcu hareketliliği ve çevresel emisyonların azaltılması gerekliliği, havacılık endüstrisini yeni teknolojilere yöneltmektedir. Karbon emisyonlarını düşürmek amacıyla, konvansiyonel motorlara alternatif olarak dağıtılmış elektrikli itki (DEP) sistemleri geliştirilmektedir. Bu sistem, itki üretimi için pervane ile tahrik edilen birden fazla elektrik motorunun kanat boyunca yerleştirilmesine dayanmaktadır.

Pervane-nacelle sisteminin rijit bir yapıya bağlı olduğu durumda, pervanenin dönmesiyle oluşan jiroskopik kuvvetler nedeniyle yunuslama ve sapma hareketleri etkileşime girerek, "geriye dönme" (backward whirl) ve "ileriye dönme" (forward whirl) olarak adlandırılan iki farklı mod ortaya çıkmaktadır. Aerodinamik kuvvetlerin etkisi altında bu sistem, belirli bir kritik hızın üzerinde aeroelastik olarak kararsız hale geçebilmektedir. Pervane-nacelle sistemi elastik bir kanat üzerine entegre edildiğinde ise kanat ve pervane modlarının etkileşimi sonucunda; kanat ıraksaması (divergence), kanat çırpınması (flutter) ve dönme çırpınması (whirl flutter) hızları değişebilmektedir. Pervane-nacelle sisteminin kanat üzerindeki konumu, bu kritik hızları doğrudan etkilemektedir.

Bu etkileri incelemek amacıyla, bu tez çalışmasında dağıtık itki sistemlerinin aeroelastik analizi için kapsamlı bir analitik modelleme gerçekleştirilmiştir. Yapısal modelleme için Euler-Bernoulli kiriş teorisi, kanat aerodinamiği için kararsız (unsteady) Theodorsen teorisi ve pervane aerodinamiği için kararsız etkileri içerecek şekilde modifiye edilen Houbolt-Reed teorisi kullanılmış; bu teoriler Galerkin Yöntemi ile entegre edilmiştir. Sistem matrisleri Mathematica sembolik programlama dili ile türetilmiş ve özdeğer (eigenvalue) analizleri MATLAB ortamında gerçekleştirilmiştir. Geliştirilen bu kod, literatürdeki deneysel ve analitik verilerle karşılaştırılarak başarılı bir şekilde doğrulanmıştır.

Çalışmada sistematik ve aşamalı bir doğrulama süreci izlenmiştir. Öncelikle temiz kanat (pervanesiz) modeli doğrulanarak kanadın doğal frekansları, statik ve dinamik aeroelastik kararsızlık sınırları incelenmiştir. Ardından, pervane aerodinamik türevleri NAS-TRAN yazılımının temel aldığı FORTRAN kodu ve literatür verileri ile doğrulanmıştır. İzole pervane-nacelle sistemi için yapılan whirl flutter analizinde, geriye ve ileriye dönme modlarının hıza bağlı kararlılık durumları analiz edilmiştir. Pervane-nacelle yapısı kanat üzerine rijit kütle olarak eklendiğinde, kanadın eğilme (bending) ve burulma (torsion) doğal frekanslarındaki değişimler gözlemlenmiştir. Sistemin esnek olarak bağlandığı ve aerodinamik kuvvetlerin ihmal edildiği durumda, pervanenin dönmesiyle oluşan jiroskopik kuvvetlerin doğal frekanslar üzerindeki etkisi değerlendirilmiştir. Takip eden aeroelastik analizlerde ise modların hangi kritik hızlarda kararsızlığa sürüklendiği tespit edilmiştir. Son olarak, birden fazla pervane-nacelle sisteminin entegre edildiği kanat modeli üzerinde yapısal ve aeroelastik analizler gerçekleştirilerek, mod etkileşimleri ve kararsızlık sınırları incelenmiştir.

Anahtar Kelimeler: Dönme Çırpınması, Aeroelastisite, Dağıtılmış İtki, Elastik Kanatlar, Kararsız Aerodinamik

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LIST OF SYMBOLS AND ABBREVIATIONS

DEP	: Distributed Electric Propulsion
GM	: Galerkin Method
MAC	: Modal Assurance Criterion
MATLAB	: Matrix Laboratory
MTOW	: Maximum Take-off Weight
NASTRAN	: NASA Structural Analysis
$[A]$	: State matrix
$[C]$	: Damping matrix
$[C_{ae}]$	: Aerodynamic damping matrix
$[C_s]$	: Structural damping matrix
$[K]$	: Stiffness matrix
$[K_{ae}]$	: Aerodynamic stiffness matrix
$[K_s]$	: Structural stiffness matrix
$[M]$	: Mass matrix
$[M_{ae}]$	: Aerodynamic mass matrix
$[M_s]$	: Structural mass matrix
$\{N\}$	: Shape function vector
$\{q\}$	: Generalized coordinate vector
$\{Q\}$	: Generalized force vector
$[Q(ik)]$	: Generalized aerodynamic force matrix
$\{x\}$	: State vector
$\{z\}$	: State space eigenvector / Mode shape vector
a	: Elastic axis location (dimensionless); Pivot offset ratio
b	: Semichord length
b_r	: Reference semichord length
c	: Chord length (wing or blade)
$\bar{c}$	: Reference chord length
C_{l_α}	: Section lift-curve slope
$C(k)$	: Theodorsen function
C_{y_θ}, C_{y_ψ}	: Propeller stability derivatives
D	: Dissipation function
e	: Mass offset distance
EI	: Bending stiffness
F_P	: Propeller reference area
g_θ, g_ψ	: Structural damping coefficients (pitch, yaw)
GJ	: Torsional stiffness
h	: Vertical displacement (plunge)
$H_n^{(2)}$	: Hankel function of the second kind
I_X	: Polar moment of inertia of propeller
I_Y, I_Z	: Moments of inertia of propeller-pylon system
I_ϕ	: Mass moment of inertia per unit length
k	: Reduced frequency
l	: Lift force per unit span (blade)
L	: Wing span

m	: Mass per unit length
M_ϕ	: Aerodynamic moment per unit span
p	: Complex eigenvalue
P	: Lift force per unit span (wing); Propeller force
q_∞	: Dynamic pressure
R	: Propeller radius
S_θ, S_ψ	: Rotational stiffness constants (pitch, yaw)
T	: Kinetic Energy
U	: Potential Energy; Resultant velocity
V, V_∞	: Airspeed / Freestream velocity
x_{cg}	: Distance between elastic axis and center of gravity
y_h, z_h	: Lateral and vertical hub displacements
α	: Angle of attack; Real part of eigenvalue
β	: Blade pitch angle; Imaginary part of eigenvalue
γ	: Induced inflow angle; Damping parameter
ζ	: Modal damping ratio
η	: Normalized spanwise coordinate
θ	: Pitch angle (propeller)
λ	: Eigenvalue
ρ	: Air density
ϕ	: Torsional angle (wing twist)
ψ	: Yaw angle (propeller)
ω	: Natural frequency / Circular frequency
Ω	: Propeller rotational speed

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1. INTRODUCTION

1.1. Motivation

Today's aviation industry faces the major challenge of meeting growing passenger demand while rapidly reducing environmental emissions. The European Commission's Flightpath 2050 vision targets a 75% reduction in CO₂ emissions, a 90% reduction in NO_x emissions, and a 65% reduction in noise emissions by 2050 [6]. In addition, the Destination 2050 strategy aims to achieve net-zero CO₂ emissions in European aviation by 2050 [7]. To eliminate carbon emissions from flights, transitioning to alternative propulsion systems such as electric or hybrid engines instead of traditional engines has become a primary goal.

Distributed Electric Propulsion (DEP) systems are considered one of the leading innovative solution approaches for achieving these goals. In DEP systems, unlike conventional gas-turbine configurations, multiple small propulsion units are positioned intermittently along the wing span. This distributed arrangement offers notable advantages in aerodynamic and overall aircraft design. The simplified integration of electric motors—eliminating the need for fuel lines, tanks, and other complex subsystems—provides designers with expanded geometric and aerodynamic flexibility. Beyond enabling zero-emission propulsion, DEP also has the potential to enhance the lift-to-drag ratio and improve the overall performance of the aircraft. The most significant obstacle to the widespread adoption of electric aircraft is the limited energy density of current battery technologies. To address this limitation, it is essential to develop mass-optimized aircraft structures designed to minimize the maximum takeoff weight (MTOW). This reduction in structural stiffness leads to increased dynamic aeroelastic interactions due to aerodynamic loads and the gyroscopic forces of the rotating propellers. This can lead to an earlier occurrence of dynamic instabilities, such as wing flutter and whirl flutter. Therefore, it is important to check aeroelastic stability early in the design of these next-generation configurations.

In aviation, the ongoing need to improve fuel efficiency and reduce drag has led to the use of wings with high aspect ratios. This is mainly because a higher aspect ratio reduces induced drag, which improves the lift-to-drag ratio and allows the aircraft to

fly farther [8]. As wings become longer and thinner to improve efficiency, it becomes necessary to keep the structural weight as low as possible. This requirement makes the wings less stiff and more flexible compared to conventional designs. As a result, the wings can undergo large, low-stress deformations even under normal operating conditions.

1.2. Research Objectives and Questions

The move to electric propulsion and the use of flexible, high-aspect-ratio wings increase the risk of early dynamic aeroelastic instabilities, such as Wing Flutter and Whirl Flutter. Therefore, designing next-generation DEP systems safely requires checking aeroelastic stability during the early design stages. The main goal of this study is to create a fast and efficient analytical model that can predict *wing flutter* and *whirl flutter*, and to use this model to study how DEP wing design parameters affect stability.

The combined effects of propeller placement, weight variables, and wing elasticity on whirl instability will be examined. Since the whirl instability is very sensitive to the propeller arrangement, the results obtained from the multi-propeller analysis will be a novel contribution to the literature.

1.3. Thesis Outline

The structure of this thesis is organized as follows:

- **Chapter 2** provides a comprehensive review of aeroelastic phenomena, covering the classification of instabilities, fundamental concepts of wing flutter and whirl flutter, and the specific challenges associated with next-generation distributed electric propulsion aircraft.
- **Chapter 3** establishes the theoretical framework and mathematical modeling for the aeroelastic stability analysis. It details the derivation of equations of motion using the Lagrangian approach, discretization via the Galerkin Method, unsteady aerodynamic modeling using Theodorsen theory, and the implementation of the p-k method for stability solution.
- **Chapter 4** presents the validation and analysis phases. It begins with the verification of the clean wing and isolated propeller models, followed by a detailed

investigation of the coupled wing-propeller system. The chapter discusses the effects of propeller positioning, gyroscopic coupling, and aerodynamic forces on the stability boundaries.

- **Chapter 5** summarizes the key findings of the study, highlights the contributions to the field, and provides recommendations for future research directions.



2. AEROELASTIC PHENOMENA IN AIRCRAFT SYSTEMS

2.1. Classification of Aeroelastic Instabilities

Aeroelasticity is a multidisciplinary field in aerospace engineering that studies the physical phenomena resulting from the interactions among inertial, elastic, and aerodynamic forces. This discipline describes the interactions among elastic forces arising from structural deformations, inertial forces resulting from flight dynamics, and aerodynamic forces generated by the kinematics and geometry of the aircraft. Unlike classical approaches, aeroelasticity does not treat aerodynamic and structural analyses separately. Instead, it accounts for how aerodynamic forces deform the structure, and how these deformations change the aerodynamic pressures. This process continues until an equilibrium is reached. The field of aeroelasticity is most clearly illustrated by Collar's famous aeroelastic triangle (Figure 2.1), which depicts the interactions among the three main forces: inertial, elastic, and aerodynamic.

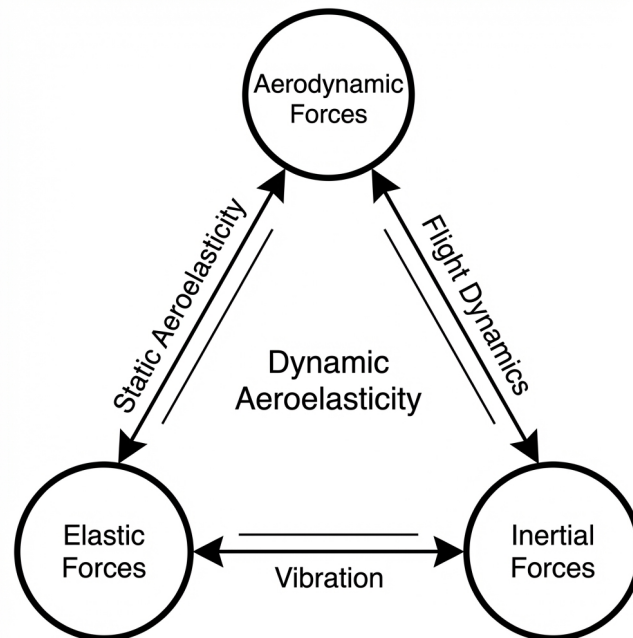


Figure 2.1: Collar's Aeroelastic Triangle.

Aeroelastic phenomena are generally classified into two main categories:

Static Aeroelasticity: Studies the non-oscillating effects of aerodynamic forces on flexible structures. It includes phenomena such as *divergence*, where wing twist increases and may cause structural failure, and *control surface reversal*, where control surfaces lose effectiveness or act in the opposite direction.

Dynamic Aeroelasticity: Focuses on time-dependent interactions and oscillatory behaviors in flexible structures, where inertial forces play a crucial role. This field covers dynamic instabilities such as *wing flutter* and *whirl flutter*.

2.2. Wing Flutter Mechanism

Wing flutter is an important dynamic aeroelastic phenomenon that happens when the inertial, elastic, and unsteady aerodynamic forces on a lifting surface interact and create a self-excited oscillation. As airspeed increases, the effects of aeroelastic forces become more pronounced, eventually reaching the critical flutter speed (V_F). At this speed, the energy added to the structure by aerodynamic forces matches the energy dissipated through structural damping, resulting in neutral stability with constant-amplitude oscillations. When the speed exceeds this critical level, the oscillation amplitudes grow exponentially. This phenomenon can cause the vibration amplitude of the wing to grow over time, creating the potential for serious structural damage or even failure.

The primary physical mechanism responsible for classical wing flutter is the coupling between bending and torsional vibration modes, commonly referred to as binary coupling or bending–torsion coupling [9]. In this mechanism, a vertical bending deformation of the wing induces a change in the local angle of attack, which modifies the aerodynamic lift and pitching moment. These altered aerodynamic loads then excite torsional motion of the wing. The resulting torsional deformation further changes the aerodynamic loading, feeding energy back into the bending motion. If the phase relationship between these coupled motions is such that aerodynamic energy input exceeds structural damping, a dynamic instability develops and flutter occurs.

The strength of bending–torsion coupling is influenced by several structural and aerodynamic parameters, including the relative location of the elastic axis, mass distribution, stiffness properties, aerodynamic center location, and flow speed. When the natural frequencies of the bending and torsional modes become sufficiently close, strong modal interaction can occur, significantly reducing the flutter speed. Accurate modeling of

this coupling is therefore essential for reliable flutter prediction and aeroelastic design.

2.3. Propeller Whirl Flutter Mechanism

Whirl flutter is a specific and critical type of dynamic aeroelastic instability, commonly observed in turboprop aircraft and in modern DEP systems. This instability can occur in an isolated system, but it also interacts dynamically with wing modes when connected to flexible wings. Although this phenomenon was analytically described by Taylor and Browne in 1938 [10], it became a major practical concern after the failures experienced by the Lockheed Electra II aircraft in the 1960s. This instability arises from the complex interaction between the gyroscopic effects of the rotating propeller, and the aerodynamic forces and moments acting on rotors or propellers supported by elastic structures.

When the propeller rotates, its independent pitch (θ) and yaw (ψ) modes are coupled due to gyroscopic effects, resulting in two main whirl modes, shown in Figure 2.2:

- (a) **Backward Whirl Mode:** Lower in frequency, with oscillation opposite to the propeller's rotation.
- (b) **Forward Whirl Mode:** Higher in frequency, with oscillation in the same direction as the propeller's rotation.

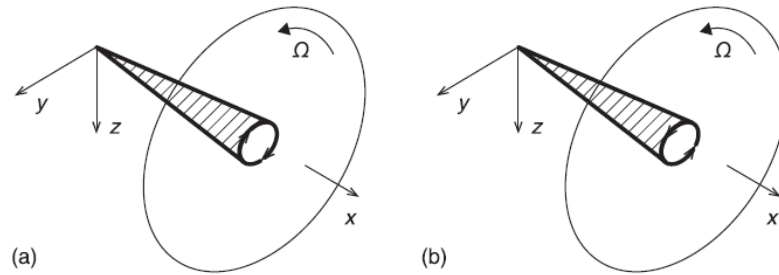


Figure 2.2: Propeller Whirl Modes: (a) Backward Whirl, (b) Forward Whirl [1].

In rigid-wing or isolated systems, when the propeller blade itself is rigid, the instability has been found to develop in the backward whirl mode [11]. In flexible-wing systems, even the forward whirl mode—which is stable in an isolated system—can become unstable due to interactions with wing modes and wing aerodynamics, assuming the propeller blades are rigid. In this case, the resulting instability represents a combination

of wing flutter and whirl flutter rather than a pure whirl flutter [5].

2.4. Aeroelastic Phenomena Related to Next-Generation Aircraft

The transition to distributed electric propulsion (DEP) and the use of high-aspect-ratio flexible wings introduce new aeroelastic challenges. The interaction between the flexible wing dynamics and the multiple propeller-pylon systems creates complex stability landscapes that differ from conventional aircraft. In these configurations, the coupling between the wing's bending and torsion modes and the propeller's whirl modes becomes more pronounced. This can lead to novel instability mechanisms where wing flutter and whirl flutter coalesce or influence each other's onset speeds. Understanding these coupled phenomena is crucial for the safe design of next-generation electric aircraft.

3. ANALYTICAL FRAMEWORK FOR AEROELASTIC STABILITY ANALYSIS

The methodology employed in this thesis combines numerical and analytical techniques to investigate the dynamic instabilities—specifically wing flutter and whirl flutter—of highly flexible wings integrated with DEP systems. First, the equations of motion for the flexible wing structure are derived and discretized using the Galerkin Method (GM) [12]. Unsteady aerodynamic forces acting on the wing are modeled based on Theodorsen’s Unsteady Aerodynamic Theory [13], while the complex propeller-wing interaction effects, which are critical for whirl flutter analysis, are captured using the Houbolt-Reed Aerodynamics approach [14]. The Mathematica software package is utilized to analytically derive and organize the resulting complex system matrices and the final State-Space equations. Subsequently, the dynamic instability speeds are reliably determined using the widely accepted $p - k$ method [15]. Finally, the accuracy and robustness of the developed aeroelastic model are validated against existing experimental and numerical data available in the literature, ensuring the scientific rigor of the findings.

3.1. Governing Equations of Motion

The methods used for the dynamic analysis of mechanical systems and the derivation of equations of motion generally fall into one of two categories based on the assumptions of their founder - vectorial approaches are the application of Newton’s laws to derive equations of motion whereas analytic techniques are based on principles such as energy conservation. For aeronautical structures with complex boundary conditions and multiple degrees of freedom (such as wing-propeller systems), scalar energy methods (that is, methods that utilize scalar energy quantities (kinetic and potential energy)) instead of vectorial methods, offer significant advantages. The basis of these forms of methodological developments was formulated by scientists, such as Huygens, Leibniz, Bernoulli, and Lagrange, inasmuch as these general principles are enough to provide us with the possibility of deducing Newton’s second law from them.

The Lagrangian Approach was employed to derive the aeroelastic equations of motion

for the wing-propeller system. This method offers significant convenience in defining inertial loads and coupling effects, particularly for complex dynamic systems, and allows the system to be expressed in terms of generalized coordinates (q_i). The differential equations governing the dynamic behavior of the system are formulated using the general form of Lagrange's equation as given in Eq. 3.1.

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i \quad (i = 1, 2, \dots, N) \quad (3.1)$$

where:

- T : Total Kinetic Energy of the system,
- U : Total Potential Energy (Strain Energy) of the system,
- D : The Dissipation Function of the system (representing damping effects),
- q_i : The i -th generalized coordinate,
- $\dot{q}_i$: Generalized velocity,
- Q_i : The Generalized Force corresponding to the i -th generalized coordinate.

3.2. Discretization via Galerkin Method

In this study, the Galerkin Method (GM) is employed to discretize the governing equations of motion and to solve the aeroelastic system. As a robust approximate solution technique it is particularly effective for the dynamic analysis of thin-walled beams and wing structures characterized by complex boundary conditions [16]. The GM is also useful for adding discrete masses—such as motors or external stores attached to the wing—and their inertial effects to the model. By using Dirac delta functions (δ), the location and influence of these point masses can be included easily inside the integral expressions of the formulation [17]. The main advantage of the Galerkin Method is that the chosen weighting functions only need to satisfy the geometric boundary conditions. The method does not require the natural boundary conditions related to forces or moments to be satisfied exactly. Because of this, the approach provides more flexibility and often gives better convergence when modeling coupled bending–torsion behavior in aeroelastic systems. For the structural model, the bending $v(y, t)$ and torsion $\phi(y, t)$ displacements are written using spatial shape functions and time-dependent generalized coordinates. Following the approach used in the literature, these displacement fields are expressed as shown in Eqs. 3.2 and 3.3.

$$v(y, t) = \{N_v\}^T(y)\{q_v\}(t) \quad (3.2)$$

$$\phi(y, t) = \{N_\phi\}^T(y)\{q_\phi\}(t) \quad (3.3)$$

Here, $\{N\}(y)$ is the vector of shape functions, which are polynomial functions chosen to satisfy the geometric boundary conditions. The vector $\{q\}(t)$ contains the generalized coordinates that describe how the system moves in time.

3.3. State-Space Formulation

Aeroelastic stability analysis is based on the equations of motion (EoM), which describe the dynamic behavior of the system. In this study, the model is formulated using a minimal set of variables, expressed through the generalized coordinate vector $\{q\}$. For flexible structures, the classical form of the equations of motion is given by Eq. 3.4

$$[M_s]\{\ddot{q}\} + [C_s]\{\dot{q}\} + [K_s]\{q\} = [M_a]\{\ddot{q}\} + [C_a]\{\dot{q}\} + [K_a]\{q\} \quad (3.4)$$

where:

- $[M_s]$, $[C_s]$, and $[K_s]$ represent the structural mass, damping, and stiffness matrices, respectively.
- $[M_a]$, $[C_a]$, and $[K_a]$ represent the aerodynamic mass, damping, and stiffness matrices, respectively.

To analyze the aeroelastic stability of the system, the equations of motion must be rewritten in a form suitable for numerical evaluation. State-space system is a common way to rewrite the second-order equations as a first-order. This lets us write the system as coupled first-order equations that can be analyzed using standard eigenvalue methods. The general form of the state-space representation is given in Eq. 3.5

$$\{\dot{x}\} = [A]\{x\} \quad (3.5)$$

where $[A]$ is the system state matrix.

To express the system in state-space form, the state vector $\{x\}$ is defined by introducing the generalized coordinates and their time derivatives as given in Eq. 3.6:

$$\{x\} = \begin{bmatrix} \{q\} \\ \{\dot{q}\} \end{bmatrix} \quad (3.6)$$

Using this definition, the governing equations can be rewritten in first-order form as shown in Eq. 3.7:

$$\{\dot{x}\} = \begin{bmatrix} [0] & [I] \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix} \{x\} \quad (3.7)$$

where:

- $[M] = [M_s] - [M_a]$
- $[C] = [C_s] - [C_a]$
- $[K] = [K_s] - [K_a]$
- $[0]$ and $[I]$ are zero and identity matrices, respectively.

3.4. Eigenvalue-Based Stability Analysis

The stability of the aeroelastic system is evaluated by performing an eigenvalue analysis of the state matrix $[A]$ given in Eq. 3.8. This analysis involves examining the eigenvalues and corresponding eigenmodes of the system.

$$[A] = \begin{bmatrix} [0] & [I] \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix} \quad (3.8)$$

In the state-space approach, the system response is written in the form $\{y(t)\} = \{z\}e^{\lambda t}$. When this expression is substituted into the state-space equation, it leads to the eigenvalue problem $[A]\{z\} = \lambda\{z\}$. If the system has n degrees of freedom, the state-space form doubles the number of variables, so the state matrix $[A]$ has a size of $2n \times 2n$. This results in $2n$ eigenvalues, which generally appear in complex conjugate pairs. Each eigenvalue can be expressed as

$$\lambda_i = \alpha_i + i\beta_i \quad (3.9)$$

where α_i is the real part and β_i is the imaginary part. The imaginary part gives the modal frequency, while the real part determines the modal damping. Specifically,

$$\omega_i = \sqrt{\alpha_i^2 + \beta_i^2} \quad (3.10)$$

$$\zeta_i = -\frac{\alpha_i}{\sqrt{\alpha_i^2 + \beta_i^2}} \quad (3.11)$$

The stability of the system can be categorized based on the real and imaginary parts of the eigenvalues, as summarized in Table 3.1.

Table 3.1: Stability classification based on eigenvalues.

Eigenvalue Type	Stability Condition	Resulting Phenomenon
Real ($\lambda_i \in \mathbb{R}$)	Unstable ($\alpha_i > 0$)	Divergence
Complex ($\lambda_i = \alpha_i + i\beta_i$)	Unstable ($\alpha_i > 0$)	Flutter
Any	Stable ($\alpha_i < 0$)	Stable oscillation (decaying)
Any	Neutrally stable ($\alpha_i = 0$)	Constant amplitude oscillation

3.5. Flutter Prediction Using P-K Method

The discretization process described in Section 3.2 leads to a set of equations of motion with the aerodynamic forces as dependent on the oscillation frequency. Since the frequency is part of the solution itself, these equations cannot be solved directly as a standard eigenvalue problem. Although the conventional k-method, also known as the "American Method", is computationally efficient, it is known for produce damping values that have no physical meaning at speeds other than the flutter speed.

This study employs the p-k method, also known as the "British Method", which is first formalized by Hassig [18]. A more realistic estimate of damping is obtained with this method over the classical k-method. Unlike the k-method, which assumes simple harmonic motion for the aerodynamic terms even when the structure is damped, the p-k method approximates the motion as an exponentially decaying or growing sine wave. The fundamental aeroelastic equation of motion in the p-k method is expressed as:

$$\left[[M]p^2 + [C]p + [K] - \frac{1}{2}\rho V^2 [Q(ik)] \right] \{q\} = 0 \quad (3.12)$$

where:

- $[M]$, $[C]$, and $[K]$ are the structural mass, damping, and stiffness matrices respectively.
- p is the complex eigenvalue, defined as $p = \omega(\gamma + i)$
- V is the true airspeed.
- ρ is the air density.
- $[Q(ik)]$ is the generalized aerodynamic force matrix, which is a function of the reduced frequency $k = \frac{\omega b}{V}$.
- $\{q\}$ is the vector of generalized coordinates.

While Hassig established the theoretical basis, the specific numerical implementation adopted in this thesis follows the formulation developed by Rodden et al. [15, 19], which is widely used in commercial aeroelastic solvers such as MSC Nastran.

In Rodden's approach, the frequency-dependent aerodynamic influence coefficient matrix $[Q(ik)]$, is decomposed into its real (Q^R) and imaginary (Q^I) components:

$$[Q(ik)] = [Q^R(k)] + i[Q^I(k)] \quad (3.13)$$

Unlike the general form where the aerodynamic term is treated as a separate forcing function, Rodden's formulation incorporates these aerodynamic components directly into the system's damping and stiffness matrices. This transformation allows the aeroelastic equation to be written in a canonical quadratic form:

$$[[M]p^2 + [\bar{B}]p + [\bar{K}]] \{q\} = 0$$

where the augmented damping ($[\bar{B}]$) and stiffness ($[\bar{K}]$) matrices are defined as:

$$[\bar{B}] = [C] - \frac{1}{4}\rho V \bar{c} \frac{[Q^I(k)]}{k} \quad (3.14)$$

$$[\bar{K}] = [K] - \frac{1}{2}\rho V^2 [Q^R(k)] \quad (3.15)$$

where:

- $\bar{c}$ is the reference chord.

This formulation effectively converts the imaginary part of the aerodynamic forces into an equivalent aerodynamic damping and the real part into an aerodynamic stiffness.

Since the aerodynamic matrices $[Q^R]$ and $[Q^I]$ depend on the reduced frequency k , which is itself a function of the unknown eigenvalue p , an iterative solution is required.

For a fixed velocity V :

1. **Estimation:** An initial reduced frequency k is assumed.
2. **Assembly:** The aerodynamic terms are calculated, and the matrices $[\bar{B}]$ and $[\bar{K}]$ are assembled.
3. **Eigenvalue Solution:** The quadratic eigenvalue problem is solved to find p .
4. **Consistency Check:** The oscillation frequency derived from the eigenvalue ($\text{Im}(p)$) is compared with the assumed frequency used to calculate k .
5. **Iteration:** The process is repeated until the reduced frequency converges, ensuring that the aerodynamics are consistent with the structural motion.

This implementation allows for a robust tracking of the system's modal damping (γ) and frequency (ω) versus airspeed, enabling the precise identification of whirl flutter instabilities. The iterative process is visualized in Figure 3.1, where s represents the number of the oscillatory mode.

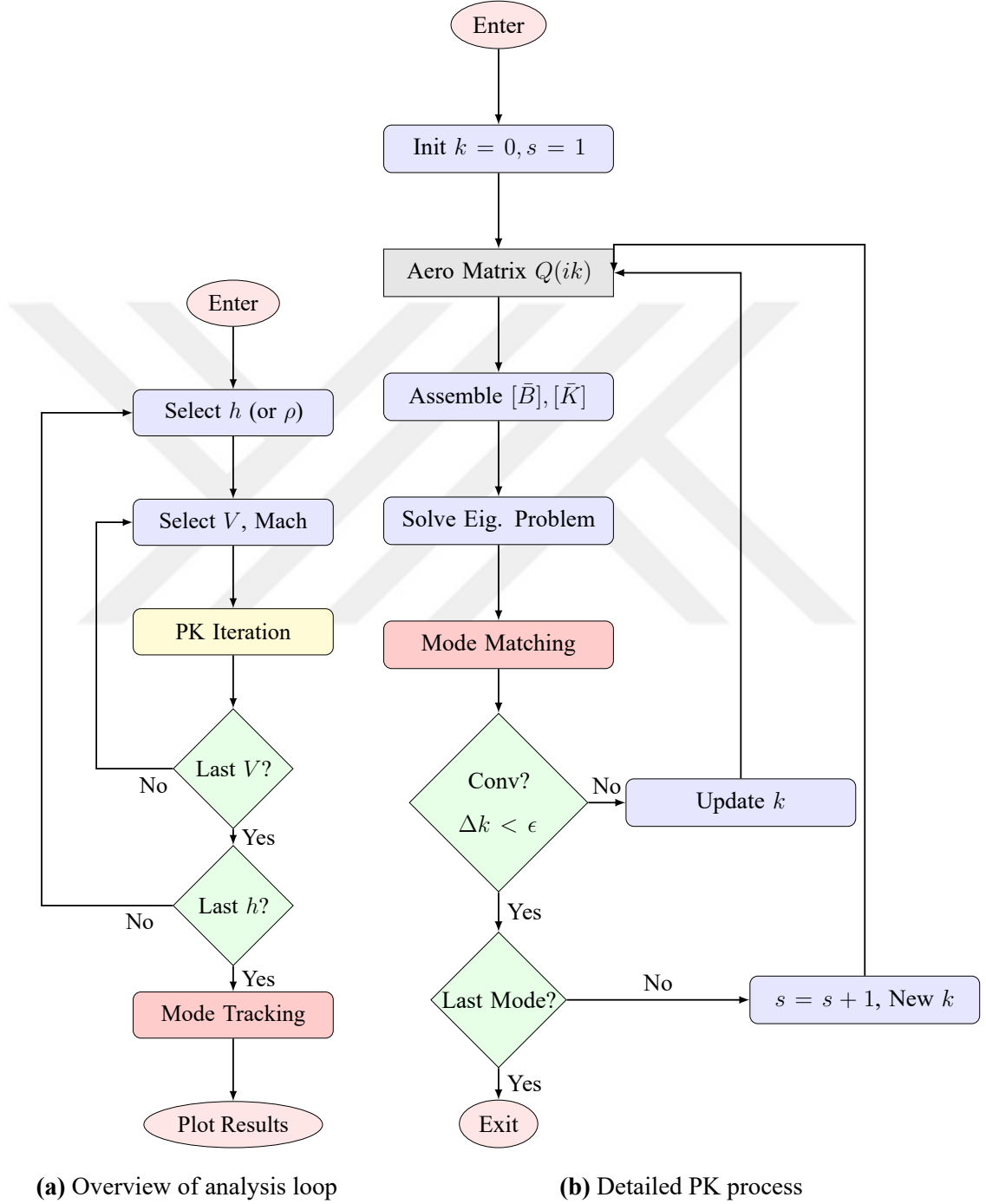


Figure 3.1: Flowcharts of (a) stability analysis and (b) p-k iteration process.

3.6. MAC Based Mode Tracking

In the aeroelastic stability analysis (Section 3.5) performed using the p-k method, the system's eigenvalues and eigenvectors are calculated at discrete intervals of airspeed. As the airspeed increases, the frequencies of different modes (for example, wing bending, torsion, or propeller whirl modes) may approach each other or cross. Simply organizing the resulting modes based on the magnitude of their frequencies can lead to "mode switching" phenomena, where it can lead to wrong mode identification and give misleading results when defining the aircraft's flight envelope [20]. To keep track of the aeroelastic modes and follow how the damping of the mode changes, the Modal Assurance Criterion (MAC) [21] is used as a mode-tracking tool.

The Modal Assurance Criterion, originally proposed by Allemang and Brown [22], provides a statistical indicator of the similarity between two mode shape vectors. In the context of this study, it is used to compare the i -th mode shape vector obtained at a current velocity step, $z_i^{(V)}$, with the j -th mode shape vector from the previous velocity step, $z_j^{(V-\Delta V)}$.

The MAC value is calculated as follows:

$$MAC(i, j) = \frac{|\{z\}_i^T \{z\}_j|^2}{(\{z\}_i^T \{z\}_i) (\{z\}_j^T \{z\}_j)} \quad (3.16)$$

where $\{z\}_i^T$ and $\{z\}_j^T$ are the transposes of the i -th and j -th mode shape vectors, respectively. The MAC value ranges from 0 to 1, with higher values indicating greater similarity between the two mode shape vectors. A MAC value of 1 indicates that the two mode shape vectors are identical, while a MAC value of 0 indicates that the two mode shape vectors are orthogonal.

4. NUMERICAL PREDICTION OF WHIRL FLUTTER

This chapter focuses on the practical construction, implementation, and verification of the aeroelastic models based on the theoretical framework established in Chapter 3. To give accurate final results, a step-by-step method is used. Modeling is done in three phases. First, a clean flexible wing is analyzed to validate structural dynamics and classical flutter prediction. Second, a standalone propeller model is studied to validate the gyroscopic and aerodynamic force formulations through an isolated whirl flutter analysis. Finally, the wing and propeller models are combined into a fully coupled system to evaluate the complete stability boundaries. This incremental approach makes sure that each instability mechanism is correctly understood before examining the more complex behavior of the coupled configuration.

4.1. Cantilever Wing Aeroelastic Model

As the first step of the validation process, a uniform flexible cantilever wing is modeled without any propulsion units. The basis for this model is to check the accuracy of the structural dynamics formulation and the aeroelastic solution procedure. The computed natural frequencies and flutter speeds are then compared with benchmark results from the literature. This comparison helps confirm that the core aeroelastic solver works reliably.

4.1.1. Wing Structural Model

The structure of the clean wing is modeled as the Euler-Bernoulli beam, treating the lifting surface as a continuous cantilever beam. Figure 4.1 shows a cross section of the wing.

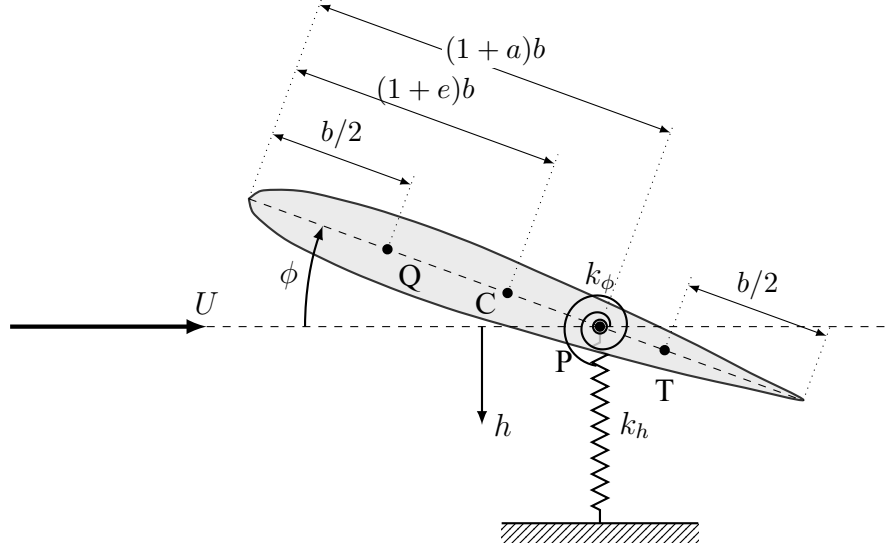


Figure 4.1: 2-DOF Aeroelastic Wing Model (adapted from [2]).

The kinematics of the wing are defined through the vertical translation of the elastic axis, h (positive downwards), coupled with the rotational twist of the airfoils, ϕ , around the same axis.

The total kinetic energy of the wing is obtained by integrating along the span L :

$$T = \int_0^L \left(\frac{1}{2} m \dot{h}^2 + m x_{cg} \dot{h} \dot{\phi} + \frac{1}{2} I_\phi \dot{\phi}^2 \right) dy \quad (4.1)$$

Similarly, the total potential energy is given by:

$$U = \int_0^L \left[\frac{1}{2} EI \left(\frac{\partial^2 h}{\partial y^2} \right)^2 + \frac{1}{2} GJ \left(\frac{\partial \phi}{\partial y} \right)^2 \right] dy \quad (4.2)$$

Using the Extended Galerkin Method, the displacements $h(y, t)$ and $\phi(y, t)$ are approximated by shape functions vectors $\{N_h\}$ and $\{N_\phi\}$ multiplied by the generalized coordinates vectors $\{q_h\}$ and $\{q_\phi\}$.

$$h(y, t) = \{N_h\}^T \{q_h\}, \quad \phi(y, t) = \{N_\phi\}^T \{q_\phi\} \quad (4.3)$$

where the shape function vectors are defined as:

$$\{N_h\} = [y^2, y^3, y^4, \dots, y^{n+1}]^T \quad (4.4)$$

$$\{N_\phi\} = [y, y^2, y^3, \dots, y^n]^T \quad (4.5)$$

By applying Lagrange's equations, the structural mass matrix $[M_s]$ and stiffness matrix $[K_s]$ are derived:

$$[M_s] = \begin{bmatrix} \int_0^L m \{N_h\} \{N_h\}^T dy & \int_0^L m x_{cg} \{N_h\} \{N_\phi\}^T dy \\ \int_0^L m x_{cg} \{N_\phi\} \{N_h\}^T dy & \int_0^L I_\phi \{N_\phi\} \{N_\phi\}^T dy \end{bmatrix} \quad (4.6)$$

$$[K_s] = \begin{bmatrix} \int_0^L EI \{N_h''\} \{N_h''\}^T dy & 0 \\ 0 & \int_0^L GJ \{N_\phi'\} \{N_\phi'\}^T dy \end{bmatrix} \quad (4.7)$$

where $\{N_h''\}$ represents the second derivative vector and $\{N_\phi'\}$ represents the first derivative vector with respect to y .

4.1.2. Unsteady Aerodynamic Loads using Theodorsen's Theory

In order to verify the clean wing model, the unsteady aerodynamic loads are calculated by employing Theodorsen's generalized theory of aerodynamic instability [13]. The methodological approach lays in presumption of strip theory whereby the airflow over each spanwise segment of the wing is treated as a two-dimensional, incompressible, and potential flow over an oscillating thin airfoil [23].

The aerodynamic forces are deduced in accordance with the kinematics of the airfoil, which are characterized by the vertical displacement h and the torsional angle ϕ . The comprehensive aerodynamic force, referred to as Lift (P) per unit span acting on the airfoil is the totality of non-circulatory components that originated from the apparent mass effects, and circulatory components associated with the vorticity that exist in the wake.

According to Theodorsen's formulation, the downward force (P) and the aerodynamic moment about the elastic axis (M_ϕ) are expressed as:

$$P = -\rho b^2 (V \pi \dot{\phi} + \pi \ddot{h} - \pi b a \ddot{\phi}) - 2\pi \rho V b C(k) \left(V \phi + \dot{h} + b \left(\frac{1}{2} - a \right) \dot{\phi} \right) \quad (4.8)$$

$$M_\phi = -\rho b^2 \left(\pi \left(\frac{1}{2} - a \right) V b \dot{\phi} + \pi b^2 \left(\frac{1}{8} + a^2 \right) \ddot{\phi} - a \pi b \ddot{h} \right) + 2\pi \rho V b^2 \left(\frac{1}{2} + a \right) C(k) \left(V \phi + \dot{h} + b \left(\frac{1}{2} - a \right) \dot{\phi} \right) \quad (4.9)$$

where:

- ρ is the air density,

- V is the airspeed,
- b is the semichord length,
- a is the dimensionless distance of the elastic axis from the midchord (positive towards the trailing edge),
- $C(k)$ is the Theodorsen function, a complex-valued term that accounts for the phase lag between the airfoil motion and the aerodynamic response.

The Theodorsen function depends on the reduced frequency $k = \omega b/V$ and is defined using Hankel functions of the second kind, $H_n^{(2)}$:

$$C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)} = F(k) + iG(k) \quad (4.10)$$

The two-dimensional aerodynamic loads are integrated across the wingspan using the mode shapes defined in the Extended Galerkin Method. This process results in the formulation of the generalized aerodynamic influence coefficient matrix, denoted as $[Q(ik)]$, which is indispensable for the p-k stability analysis described in Section 3.5.

$$[Q(ik)] = q_\infty (-k^2[M_{ae}] + ik[C_{ae}] + [K_{ae}]) \quad (4.11)$$

where the aerodynamic mass ($[M_{ae}]$), damping ($[C_{ae}]$), and stiffness ($[K_{ae}]$) matrices are assembled from their sub-blocks:

$$[M_{ae}] = \begin{bmatrix} M_{ae11} & M_{ae12} \\ M_{ae21} & M_{ae22} \end{bmatrix}, \quad [C_{ae}] = \begin{bmatrix} C_{ae11} & C_{ae12} \\ C_{ae21} & C_{ae22} \end{bmatrix}, \quad [K_{ae}] = \begin{bmatrix} K_{ae11} & K_{ae12} \\ K_{ae21} & K_{ae22} \end{bmatrix} \quad (4.12)$$

The components are calculated via integration over the normalized span $\eta \in [0, 1]$:

$$\begin{aligned} M_{ae11} &= \int_0^1 \frac{-2\pi b^2}{b_r^2} \{N_h\} \{N_h\}^T d\eta \\ M_{ae12} &= \int_0^1 \frac{2\pi ab^3}{b_r^2} \{N_h\} \{N_\phi\}^T d\eta \\ M_{ae21} &= \int_0^1 \frac{2\pi ab^3}{b_r^2} \{N_\phi\} \{N_h\}^T d\eta \\ M_{ae22} &= \int_0^1 -\frac{\pi b^4(1 + 8a^2)}{4b_r^2} \{N_\phi\} \{N_\phi\}^T d\eta \end{aligned} \quad (4.13)$$

$$\begin{aligned}
C_{ae11} &= \int_0^1 \frac{-2bC(k)C_{L\alpha}}{b_r} \{N_h\} \{N_h\}^T d\eta \\
C_{ae12} &= \int_0^1 \frac{b^2[(2a-1)C(k)C_{L\alpha} - 2\pi]}{b_r} \{N_h\} \{N_\phi\}^T d\eta \\
C_{ae21} &= \int_0^1 \frac{(1+2a)b^2C(k)C_{L\alpha}}{b_r} \{N_\phi\} \{N_h\}^T d\eta \\
C_{ae22} &= \int_0^1 \frac{b^3[C_L C(k)(1-4a^2) - 2\pi(1-2a)]}{2b_r} \{N_\phi\} \{N_\phi\}^T d\eta
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
K_{ae11} &= 0 \\
K_{ae12} &= \int_0^1 -2bC(k)C_{L\alpha} \{N_h\} \{N_\phi\}^T d\eta \\
K_{ae21} &= 0 \\
K_{ae22} &= \int_0^1 (1+2a)b^2C(k)C_{L\alpha} \{N_\phi\} \{N_\phi\}^T d\eta
\end{aligned} \tag{4.15}$$

Finally, the total system matrices used in the p-k flutter solution are assembled by combining the structural and aerodynamic contributions:

$$\begin{aligned}
[\overline{M}] &= [M_s] \\
[\overline{C}] &= [C_s] - \frac{\text{Im}([Q(ik)])}{kV/b_r} \\
[\overline{K}] &= [K_s] - \text{Re}([Q(ik)])
\end{aligned} \tag{4.16}$$

4.2. Propeller–Pylon Aeroelastic Model

An examination of the stability characteristics of the isolated propeller system is conducted prior to the integration of the propulsion unit with the elastic wing. In this section, the propeller-nacelle configuration is idealized as a rigid body affixed to a flexible pivot characterized by specified pitch and yaw stiffnesses. The aim is to conduct a whirl flutter analysis on this simplified system to validate the generation of the propeller aerodynamic ($[A_{prop}]$) and gyroscopic ($[G]$) matrices. By comparing the predicted instability boundaries with analytical results from extant literature, the capability of the developed solver is well demonstrated to accurately capture the fundamental physics of whirl flutter in the absence of wing elastic coupling.

4.2.1. Structural Dynamics and Kinematics

The coordinate system employed to describe the motion of the propeller system is depicted in Figure 4.2, 4.3. Two sets of orthogonal axes are defined: the space-fixed axes (X_s, Y_s, Z_s) with the origin at the gimbal center, and the body-fixed axes (X, Y, Z) which pitch and yaw with the powerplant but do not rotate with the propeller. The Euler angles relating the body system to the space-fixed coordinates are the pitch angle θ and the yaw angle ψ . The transformation sequence from the space-fixed frame to the body-fixed frame is as follows:

- (a) Pitch through an angle θ about the space-fixed Y_s -axis to define the intermediate coordinates (X_a, Y_s, Z).
- (b) Yaw through an angle ψ about the body-fixed Z -axis to obtain the final body coordinates (X, Y, Z).

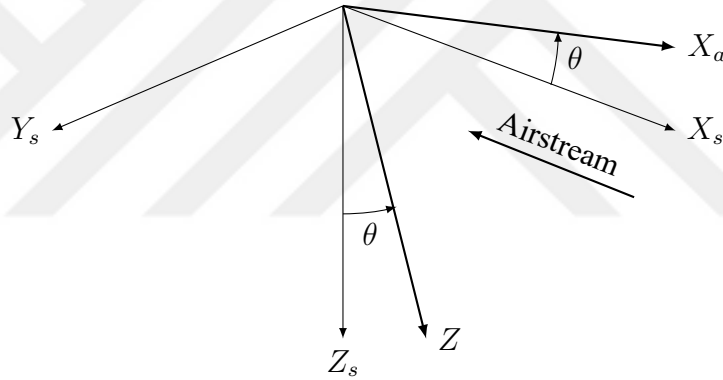


Figure 4.2: Coordinate system for propeller whirl: θ rotation.

Positive directions follow the right-hand rule. The total angular velocity vector of the system comprises the pitch rate $\dot{\theta}$, the yaw rate $\dot{\psi}$, and the propeller rotational speed Ω . Resolving these vectors along the body axes yields the components ω_X, ω_Y , and ω_Z . For small pitch and yaw angles, the relations between the angular velocities in the body-fixed system and those in the space-fixed system are approximated as:

$$\begin{aligned}\omega_X &= \Omega + \dot{\theta} \sin \psi \approx \Omega + \dot{\theta} \psi \\ \omega_Y &= \dot{\theta} \cos \psi \approx \dot{\theta} \\ \omega_Z &= \dot{\psi}\end{aligned}\tag{4.17}$$

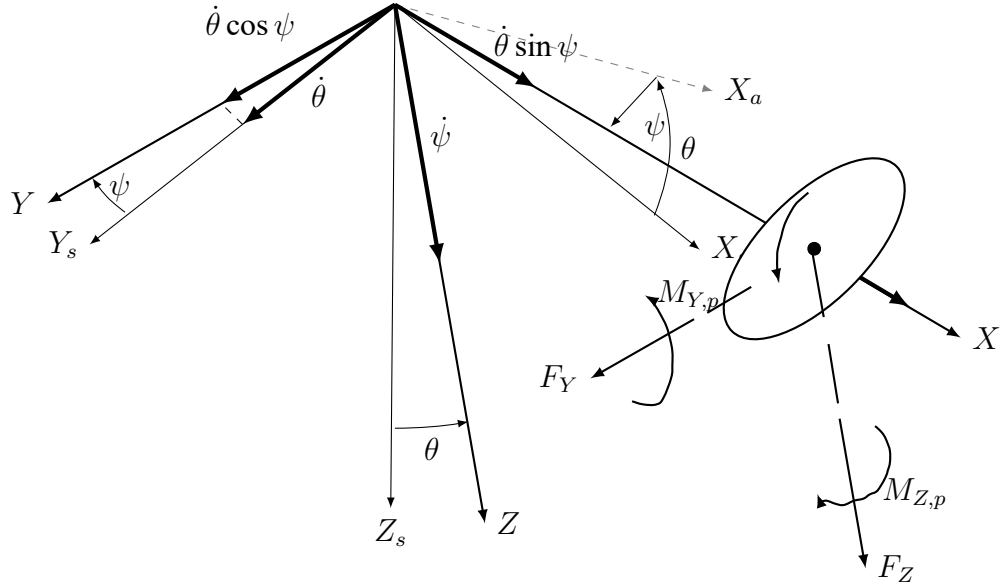


Figure 4.3: Coordinate system for propeller whirl: ψ rotation.

The total kinetic energy of the system can be expressed as:

$$T = \frac{1}{2}I_X\omega_X^2 + \frac{1}{2}I_Y\omega_Y^2 + \frac{1}{2}I_Z\omega_Z^2 \quad (4.18)$$

where I_X denotes the polar moment of inertia of the propeller, while I_Y and I_Z represent the moments of inertia of the complete system about the Y - and Z -axes, respectively. In this analysis, it is assumed that the system is symmetric such that $I_Z = I_Y$. Thus, substituting the angular velocity expressions from Equation 4.17 into Equation 4.18, the expression for the total kinetic energy of the system for small pitch and yaw deflections about the gimbal axes becomes:

$$T = \frac{1}{2}I_X(\Omega^2 + 2\Omega\dot{\theta}\psi) + \frac{1}{2}I_Y(\dot{\theta}^2 + \dot{\psi}^2) \quad (4.19)$$

The potential energy associated with pitching and yawing deflections is:

$$U = \frac{1}{2}S_\theta\theta^2 + \frac{1}{2}S_\psi\psi^2 \quad (4.20)$$

where, as indicated in the system schematic, S_θ and S_ψ are the rotational spring constants for pitch and yaw, respectively.

Assume for the present that the damping in the engine mount system is defined by the structural damping concept commonly employed in flutter analysis. This type of damping produces a force in phase with the velocity of oscillation but proportional to

the elastic restoring force [24]. A dissipation function applicable to structural damping with sinusoidal motion may be written as:

$$D = \frac{1}{2} \frac{S_\theta g_\theta}{\omega} \dot{\theta}^2 + \frac{1}{2} \frac{S_\psi g_\psi}{\omega} \dot{\psi}^2 \quad (4.21)$$

where ω is the frequency of oscillation, and g_θ and g_ψ are the damping coefficients for the uncoupled pitch and yaw modes, respectively.

By applying Lagrange's equations with generalized coordinates θ and ψ , and corresponding generalized aerodynamic forces M_Y and M_Z , the equations of motion are obtained as:

$$\begin{aligned} I_Y \ddot{\theta} + I_X \Omega \dot{\psi} + S_\theta \theta + \frac{S_\theta g_\theta}{\omega} \dot{\theta} &= M_Y \\ I_Y \ddot{\psi} - I_X \Omega \dot{\theta} + S_\psi \psi + \frac{S_\psi g_\psi}{\omega} \dot{\psi} &= M_Z \end{aligned} \quad (4.22)$$

These equations illustrate the coupling between the pitch and yaw motions. The terms can be categorized as follows:

- **Inertia terms:** $I_Y \ddot{\theta}$ and $I_Y \ddot{\psi}$ represent the inertial resistance to angular acceleration.
- **Gyroscopic terms:** $I_X \Omega \dot{\psi}$ and $-I_X \Omega \dot{\theta}$ arise from the rotation of the propeller and couple the pitch and yaw motions. These are velocity-dependent but conservative forces, typical of gyroscopic systems.
- **Stiffness terms:** $S_\theta \theta$ and $S_\psi \psi$ represent the elastic restoring moments from the flexible pivot.
- **Damping terms:** $\frac{S_\theta g_\theta}{\omega} \dot{\theta}$ and $\frac{S_\psi g_\psi}{\omega} \dot{\psi}$ represent the structural damping forces, which are proportional to displacement but in phase with velocity.

4.2.2. Propeller Aerodynamic Loads: Houbolt-Reed Formulation

The analytical method developed by Houbolt and Reed [14] is used to calculate the aerodynamic forces on the propeller. This approach follows quasi-steady strip theory, where the blade is separated into small segments of width dr . Aerodynamic forces are calculated for each 2D section and integrated along the entire span. The method assumes that each section acts independently and focuses on the steady response in incompressible flow.

Consider a propeller with a radius R turning at a constant speed Ω . As illustrated in Figure 4.4a, the hub is positioned at a distance aR from the pylon's pivot point. The

system undergoes small angular movements in pitch (θ) and yaw (ψ). We can represent the vertical and lateral displacements of the hub as:

$$\begin{aligned} y_h &= aR\psi \\ z_h &= aR\theta \end{aligned} \quad (4.23)$$

For any section of the blade at a radial distance r , its coordinates change during rotation:

$$\begin{aligned} y &= y_h + r \cos(\Omega t) \\ z &= z_h + r \sin(\Omega t) \end{aligned} \quad (4.24)$$

In these formulas, Ωt represents the azimuth angle of the blade. When the propeller axis is tilted relative to the air velocity V_∞ , the local angle of attack changes. This change, called the geometric incidence α_g , is calculated by projecting the free stream velocity vector:

$$\alpha_g = \psi \sin(\Omega t) - \theta \cos(\Omega t) \quad (4.25)$$

The motion of the hub and its orientation create extra velocities at the blade section. We define $\dot{s}$ as the velocity in the rotation plane and $\dot{w}$ as the velocity perpendicular to it, as shown in Figure 4.4b. These values are obtained by transforming the hub velocities into the blade coordinate system:

$$\begin{aligned} \dot{s} &= -aR\dot{\psi} \sin(\Omega t) + aR\dot{\theta} \cos(\Omega t) \\ \dot{w} &= aR\dot{\psi} \cos(\Omega t) + aR\dot{\theta} \sin(\Omega t) + V_\infty \alpha_g \end{aligned} \quad (4.26)$$

By inserting α_g from Equation (4.25) into the axial velocity formula, we see that it depends on both the angles and the rates of movement.

These extra velocities alter the total velocity U seen by the propeller section. To find the lift and drag, we focus on the velocity components that are perpendicular and parallel to the main velocity U . The component normal to U is v_n :

$$v_n = \dot{w} \cos \gamma - \dot{s} \sin \gamma \quad (4.27)$$

where γ is the inflow angle ($\tan \gamma = V_\infty / \Omega r$). This velocity component changes the section's angle of attack. For small perturbations, this change, α_1 , is estimated as:

$$\alpha_1 \approx \frac{v_n}{U} = \frac{\dot{w} \cos \gamma - \dot{s} \sin \gamma}{U} \quad (4.28)$$

The final effective angle of attack becomes $\alpha_{eff} = \alpha_0 + \alpha_1$, where $\alpha_0 = \beta - \gamma$ represents the steady state. These perturbation velocities also change the velocity magnitude U_{eff} :

$$U_{eff} \approx U + \dot{s} \cos \gamma + \dot{w} \sin \gamma \quad (4.29)$$

(a) Blade element geometry.

Figure 4.4: Propeller blade element geometry and velocity diagram (adapted from [3]).

$$l = \frac{1}{2} \rho U_e^2 c C_{l_\alpha} \alpha \quad (4.30)$$

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and angle leads to:

$$l \approx \frac{1}{2} \rho c C_{l_\alpha} \left(U + \dot{w} \frac{V}{U} + \dot{s} \frac{\Omega r}{U} \right)^2 \left(\alpha_0 + \alpha_1 - \dot{w} \frac{\Omega r}{U^2} + \dot{s} \frac{V}{U^2} \right) \quad (4.31)$$

Ignoring higher-order terms, the lift formula expands to:

$$\begin{aligned} l &\approx \frac{1}{2} \rho c C_{l_\alpha} \left[U^2 + 2U\dot{w} \frac{V}{U} + 2U\dot{s} \frac{\Omega r}{U} \right] \left(\alpha_0 + \alpha_1 - \dot{w} \frac{\Omega r}{U^2} + \dot{s} \frac{V}{U^2} \right) \\ &\approx \underbrace{\frac{1}{2} \rho c C_{l_\alpha} U^2 \alpha_0}_{\text{steady-state lift}} + \underbrace{\frac{1}{2} \rho c C_{l_\alpha} U^2 \left[\alpha_1 - \frac{\Omega r}{U^2} \dot{w} + \frac{V}{U^2} \dot{s} + \left(\frac{2V}{U} \dot{w} + \frac{2\Omega r}{U} \dot{s} \right) \frac{\alpha_0}{U} \right]}_{\text{non-steady lift}} \end{aligned} \quad (4.32)$$

While non-steady lift depends on the initial angle α_0 , these terms are generally small. For a propeller without thrust, we assume $\alpha_0 = 0$. This means that without pitching or yawing, the inflow angle matches the blade angle. By using the expressions for α_1 , $\dot{s}$, and $\dot{w}$, the lift for the first blade is:

$$\begin{aligned} l_1 &= \frac{1}{2} \rho c C_{l_\alpha} \{ U^2 \alpha_1 - \dot{w} \Omega r + \dot{s} V \} \\ &= -\frac{1}{2} \rho c C_{l_\alpha} (V^2 \theta - V \dot{z}_h - \Omega r^2 \psi) \cos \Omega t + \frac{1}{2} \rho c C_{l_\alpha} (V^2 \psi - V \dot{y}_h + \Omega r^2 \theta) \sin \Omega t \\ &= -f_1 \cos \Omega t + f_2 \sin \Omega t \end{aligned} \quad (4.33)$$

The lift on the other blades is found by adjusting phase angles in Equation (4.33). Given the system's symmetry:

$$l_2 = -l_1, \quad l_3 = f_1 \sin \Omega t + f_2 \cos \Omega t, \quad l_4 = -l_3 \quad (4.34)$$

The lift can also be separated into thrust and torque components:

$$\text{thrust: } l_1 \cos \psi \approx l_1 \frac{\Omega r}{U}, \quad \text{torque: } l_1 \sin \psi \approx l_1 \frac{V}{U} \quad (4.35)$$

By summing the contributions of all four blades, the side forces (l_Y, l_Z) and moments ($\Delta M_Y, \Delta M_Z$) are:

$$\begin{aligned} l_Y &= 2 \frac{V}{U} (l_1 \sin \Omega t + l_3 \cos \Omega t) \\ l_Z &= 2 \frac{V}{U} (-l_1 \cos \Omega t + l_3 \sin \Omega t) \\ \Delta M_Y &= -r \frac{\Omega r}{V} l_Y \\ \Delta M_Z &= r \frac{\Omega r}{V} l_Z \end{aligned} \quad (4.36)$$

By using the definition for the first blade, the explicit formulas for the sections loads are:

$$\begin{aligned}
l_Y &= 2 \frac{V}{U} f_2 = \rho c a \frac{V}{U} (V^2 \psi - V \dot{y}_h + \Omega r^2 \theta) \\
l_Z &= 2 \frac{V}{U} f_1 = \rho c a \frac{V}{U} (V^2 \theta - V \dot{z}_h - \Omega r^2 \psi) \\
\Delta M_Y &= -2 \frac{\Omega r^2}{U} f_2 = -\rho c a \frac{\Omega r^2}{U} (V^2 \psi - V \dot{y}_h + \Omega r^2 \theta) \\
\Delta M_Z &= 2 \frac{\Omega r^2}{U} f_1 = \rho c a \frac{\Omega r^2}{U} (V^2 \theta - V \dot{z}_h - \Omega r^2 \psi)
\end{aligned} \tag{4.37}$$

The total aerodynamic loads at the hub are determined by summing these components:

$$\begin{aligned}
l_Y &= 2 \frac{V}{U} f_2 = a \rho c \frac{V}{U} (V^2 \psi - V \dot{y}_h + \Omega r^2 \theta) \\
l_Z &= 2 \frac{V}{U} f_1 = a \rho c \frac{V}{U} (V^2 \theta - V \dot{z}_h - \Omega r^2 \psi) \\
\Delta M_Y &= -2 \frac{\Omega r^2}{U} f_2 = -a \rho c \frac{\Omega r^2}{U} (V^2 \psi - V \dot{y}_h + \Omega r^2 \theta) = -\frac{\Omega r^2}{V} l_Y \\
\Delta M_Z &= 2 \frac{\Omega r^2}{U} f_1 = a \rho c \frac{\Omega r^2}{U} (V^2 \theta - V \dot{z}_h - \Omega r^2 \psi) = \frac{\Omega r^2}{V} l_Z
\end{aligned} \tag{4.38}$$

We integrate these loads from the blade root (r_0) to the tip (R). Assuming the ideal lift-curve slope of 2π , the total hub forces and moments are:

$$\begin{aligned}
L_Y &= \frac{1}{2} \rho V^2 S' \left[\left(\frac{4\Omega c_0}{V} A_1 \right) \left(\psi - \frac{\dot{y}_h}{V} \right) + \left(\frac{4\Omega c_0}{V} A_2 \right) \left(\frac{\dot{\theta} R}{V} \right) \right] \\
L_Z &= \frac{1}{2} \rho V^2 S' \left[\left(\frac{4\Omega c_0}{V} A_1 \right) \left(\theta - \frac{\dot{z}_h}{V} \right) - \left(\frac{4\Omega c_0}{V} A_2 \right) \left(\frac{\dot{\psi} R}{V} \right) \right] \\
M_Y &= -\frac{1}{2} \rho V^2 S' R \left[\left(\frac{4\Omega c_0}{V} A_2 \right) \left(\psi - \frac{\dot{y}_h}{V} \right) + \left(\frac{4\Omega c_0}{V} A_3 \right) \left(\frac{\dot{\theta} R}{V} \right) \right] \\
M_Z &= \frac{1}{2} \rho V^2 S' R \left[\left(\frac{4\Omega c_0}{V} A_2 \right) \left(\theta - \frac{\dot{z}_h}{V} \right) - \left(\frac{4\Omega c_0}{V} A_3 \right) \left(\frac{\dot{\psi} R}{V} \right) \right]
\end{aligned} \tag{4.39}$$

where $S' = \pi R^2$ is the disk area. The factors A_i represent integrals along the blade span, defined as:

$$\begin{aligned}
A_1 &= \int_{\eta_0}^1 \frac{c}{c_0} \frac{\mu^2}{\sqrt{\mu^2 + \eta^2}} d\eta \\
A_2 &= \int_{\eta_0}^1 \frac{c}{c_0} \frac{\mu \eta}{\sqrt{\mu^2 + \eta^2}} d\eta \\
A_3 &= \int_{\eta_0}^1 \frac{c}{c_0} \frac{\eta^2}{\sqrt{\mu^2 + \eta^2}} d\eta
\end{aligned} \tag{4.40}$$

In these expressions, $\mu = V/(\Omega R)$ is the advance ratio, $\eta = r/R$ is the normalized radius, and $\eta_0 = r_0/R$ is the hub ratio.

To increase the accuracy for high tip speeds and finite blades, we modify the lift-curve slope for compressibility and induction effects. While the basic theory uses incompressible flow and a 2π slope, real cases require corrections for Mach number and aspect ratio. This is done by applying the Glauert-Prandtl factor:

$$\frac{1}{\sqrt{1 - M_e^2}} \quad (4.41)$$

and the aspect ratio (AR) correction factor for compressible flow:

$$\frac{AR}{AR\sqrt{1 - M_e^2} + 2} \quad (4.42)$$

where $M_e = \sqrt{V^2 + (\Omega r)^2} / a_{sound}$ is the local Mach number. These adjustments are integrated into A_i to ensure the model remains valid at high speeds.

The forces and moments on a propeller during whirling are typically described using aerodynamic derivatives related to the pitch/yaw angles and rates. These follow the coordinate system shown in Figure 4.4. The expressions for these loads are:

$$\begin{aligned} F_Z &= \frac{1}{2}\rho V^2 S' \left(C_{Z_\theta} \bar{\theta} + C_{Z_\psi} \bar{\psi} + C_{Z_q} \frac{\dot{\theta} R}{V} + C_{Z_r} \frac{\dot{\psi} R}{V} \right) \\ M_{Y,p} &= \rho V^2 S' R \left(C_{m_\theta} \bar{\theta} + C_{m_\psi} \bar{\psi} + C_{m_q} \frac{\dot{\theta} R}{V} + C_{m_r} \frac{\dot{\psi} R}{V} \right) \\ F_Y &= \frac{1}{2}\rho V^2 S' \left(C_{Y_\theta} \bar{\theta} + C_{Y_\psi} \bar{\psi} + C_{Y_q} \frac{\dot{\theta} R}{V} + C_{Y_r} \frac{\dot{\psi} R}{V} \right) \\ M_{Z,p} &= \rho V^2 S' R \left(C_{n_\theta} \bar{\theta} + C_{n_\psi} \bar{\psi} + C_{n_q} \frac{\dot{\theta} R}{V} + C_{n_r} \frac{\dot{\psi} R}{V} \right) \end{aligned} \quad (4.43)$$

where the effective angles $\bar{\theta}$ and $\bar{\psi}$ are:

$$\begin{aligned} \bar{\theta} &= \theta - \frac{\dot{z}_h}{V} \\ \bar{\psi} &= \psi - \frac{\dot{y}_h}{V} \end{aligned} \quad (4.44)$$

The model uses sixteen aerodynamic derivatives. Due to the symmetry between pitch and yaw and the assumption that the wing does not interfere with the propeller, several derivatives are mathematically related:

$$\begin{aligned} C_{Z_\theta} &= C_{Y_\psi} & C_{Z_{\dot{\theta}}} &= C_{Y_{\dot{\psi}}} \\ C_{m_\psi} &= -C_{n_\theta} & C_{m_{\dot{\psi}}} &= -C_{n_{\dot{\theta}}} \\ C_{Y_\theta} &= -C_{Z_\psi} & C_{Y_{\dot{\theta}}} &= -C_{Z_{\dot{\psi}}} \\ C_{n_\theta} &= C_{m_\psi} & C_{n_{\dot{\theta}}} &= C_{m_{\dot{\psi}}} \end{aligned} \quad (4.45)$$

If we ignore second-order time derivatives and compare Equation (4.39) with Equation (4.43), we get the following derivatives:

$$\begin{aligned}
C_{Y_{\dot{\psi}}} &= -C_{Z_{\dot{\theta}}} = \frac{4\Omega c_0}{V} A_1 \\
C_{Y_{\ddot{\psi}}} &= C_{Z_{\ddot{\theta}}} = -\frac{4\Omega c_0}{V} A_2 \\
C_{m_{\dot{\psi}}} &= -C_{n_{\dot{\theta}}} = -\frac{2\Omega c_0}{V} A_2 \\
C_{m_{\ddot{\psi}}} &= C_{n_{\ddot{\theta}}} = \frac{2\Omega c_0}{V} A_3
\end{aligned} \tag{4.46}$$

These eight derivatives represent the primary aerodynamic loads. The remaining derivatives describe the influence of wake effects, which are detailed below.

The quasi-steady model mentioned earlier does not account for the delay in lift caused by shed vortices. For a precessing propeller, this oscillatory lift creates helical wakes. This causes a "lift lag" that affects both the magnitude and the direction of the forces. To include these effects, we multiply the lift by the Theodorsen function $C(k) = F(k) + iG(k)$. For an oscillatory motion with frequency ω , the lift components f_1 and f_2 are:

$$\begin{aligned}
f_1 &= \frac{1}{2} \rho c a_l [V^2 \bar{\theta} - \Omega r^2 \bar{\psi}] \\
f_2 &= \frac{1}{2} \rho c a_l [V^2 \bar{\psi} + \Omega r^2 \bar{\theta}]
\end{aligned} \tag{4.47}$$

The lift lag introduces two frequency components ($\Omega + \omega$ and $\Omega - \omega$), leading to a modified lift l'_1 :

$$l'_1 = \frac{F_1 + iG_1}{2} (f_2 + i f_1) e^{i(\Omega + \omega)t} - \frac{F_2 - iG_2}{2} (f_2 - i f_1) e^{-i(\Omega - \omega)t} \tag{4.48}$$

where the reduced frequencies k_1 and k_2 define the factors F and G :

$$k_1 = \frac{(\Omega + \omega)c}{2\sqrt{\Omega^2 r^2 + V^2}}, \quad k_2 = \frac{(\Omega - \omega)c}{2\sqrt{\Omega^2 r^2 + V^2}} \tag{4.49}$$

The side forces and moments can be simplified assuming ω is small compared to Ω and using a single average reduced frequency $k = \Omega c / 2V$. The lag-modified loads are then:

$$\begin{aligned}
l'_Y &= F l_Y + G l_Z \\
l'_Z &= F l_Z - G l_Y \\
\Delta M'_Y &= -r(F l'_Y + G l'_Z) \approx F \Delta M_Y - G \Delta M_Z \\
\Delta M'_Z &= r(F l'_Z - G l'_Y) \approx F \Delta M_Z + G \Delta M_Y
\end{aligned} \tag{4.50}$$

Integrating these loads provides the final set of derivatives, including the wake influence:

$$\begin{aligned}
C_{Y\psi} &= -C_{Z\theta} = \frac{4\Omega c_0}{V} I_1 & C_{Y\theta} &= C_{Z\psi} = -\frac{4\Omega c_0}{V} J_1 \\
C_{Yq} &= C_{Zr} = -\frac{4\Omega c_0}{V} I_2 & C_{Yr} &= -C_{Zq} = -\frac{4\Omega c_0}{V} J_2 \\
C_{m\psi} &= -C_{n\theta} = \frac{2\Omega c_0}{V} I_2 & C_{m\theta} &= C_{n\psi} = -\frac{2\Omega c_0}{V} J_2 \\
C_{mq} &= C_{nr} = -\frac{2\Omega c_0}{V} I_3 & C_{mr} &= -C_{nq} = -\frac{2\Omega c_0}{V} J_3
\end{aligned} \tag{4.51}$$

where the blade integrals I and J are:

$$\begin{aligned}
I_1 &= \frac{\mu^2 A}{c_0} \int_{\eta_0}^1 \frac{c(\eta) F(k)}{\sqrt{\mu^2 + \eta^2} \left[2 + A\sqrt{1 - M^2(1 + \eta^2/\mu^2)} \right]} d\eta \\
I_2 &= \frac{\mu A}{c_0} \int_{\eta_0}^1 \frac{\eta^2 c(\eta) F(k)}{\sqrt{\mu^2 + \eta^2} \left[2 + A\sqrt{1 - M^2(1 + \eta^2/\mu^2)} \right]} d\eta \\
I_3 &= \frac{A}{c_0} \int_{\eta_0}^1 \frac{\eta^4 c(\eta) F(k)}{\sqrt{\mu^2 + \eta^2} \left[2 + A\sqrt{1 - M^2(1 + \eta^2/\mu^2)} \right]} d\eta \\
J_1 &= \frac{\mu^2 A}{c_0} \int_{\eta_0}^1 \frac{c(\eta) G(k)}{\sqrt{\mu^2 + \eta^2} \left[2 + A\sqrt{1 - M^2(1 + \eta^2/\mu^2)} \right]} d\eta \\
J_2 &= \frac{\mu A}{c_0} \int_{\eta_0}^1 \frac{\eta^2 c(\eta) G(k)}{\sqrt{\mu^2 + \eta^2} \left[2 + A\sqrt{1 - M^2(1 + \eta^2/\mu^2)} \right]} d\eta \\
J_3 &= \frac{A}{c_0} \int_{\eta_0}^1 \frac{\eta^4 c(\eta) G(k)}{\sqrt{\mu^2 + \eta^2} \left[2 + A\sqrt{1 - M^2(1 + \eta^2/\mu^2)} \right]} d\eta
\end{aligned} \tag{4.52}$$

These integrals account for compressibility and finite blade lengths. If the number of

blades N_b is different from four, the integrals are adjusted as follows:

$$\begin{aligned}
I_1 &= \frac{N_b \mu^2 A}{4 c_0} \int_{\eta_0}^1 \frac{c(\eta) F(k)}{\sqrt{\mu^2 + \eta^2} \left[2 + A \sqrt{1 - M^2(1 + \eta^2/\mu^2)} \right]} d\eta \\
I_2 &= \frac{N_b \mu A}{4 c_0} \int_{\eta_0}^1 \frac{\eta^2 c(\eta) F(k)}{\sqrt{\mu^2 + \eta^2} \left[2 + A \sqrt{1 - M^2(1 + \eta^2/\mu^2)} \right]} d\eta \\
I_3 &= \frac{N_b A}{4 c_0} \int_{\eta_0}^1 \frac{\eta^4 c(\eta) F(k)}{\sqrt{\mu^2 + \eta^2} \left[2 + A \sqrt{1 - M^2(1 + \eta^2/\mu^2)} \right]} d\eta \\
J_1 &= \frac{N_b \mu^2 A}{4 c_0} \int_{\eta_0}^1 \frac{c(\eta) G(k)}{\sqrt{\mu^2 + \eta^2} \left[2 + A \sqrt{1 - M^2(1 + \eta^2/\mu^2)} \right]} d\eta \\
J_2 &= \frac{N_b \mu A}{4 c_0} \int_{\eta_0}^1 \frac{\eta^2 c(\eta) G(k)}{\sqrt{\mu^2 + \eta^2} \left[2 + A \sqrt{1 - M^2(1 + \eta^2/\mu^2)} \right]} d\eta \\
J_3 &= \frac{N_b A}{4 c_0} \int_{\eta_0}^1 \frac{\eta^4 c(\eta) G(k)}{\sqrt{\mu^2 + \eta^2} \left[2 + A \sqrt{1 - M^2(1 + \eta^2/\mu^2)} \right]} d\eta
\end{aligned} \tag{4.53}$$

Further modification is possible to include the actual lift-curve slope distribution $a(\eta)$ along the blade span:

$$\begin{aligned}
I_1 &= \frac{N_b}{4} \frac{1}{2\pi} \frac{\mu^2}{c_0} \int_{\eta_0}^1 \frac{a(\eta) c(\eta) F(k)}{\sqrt{\mu^2 + \eta^2}} d\eta \\
I_2 &= \frac{N_b}{4} \frac{1}{2\pi} \frac{\mu}{c_0} \int_{\eta_0}^1 \frac{a(\eta) \eta^2 c(\eta) F(k)}{\sqrt{\mu^2 + \eta^2}} d\eta \\
I_3 &= \frac{N_b}{4} \frac{1}{2\pi} \frac{1}{c_0} \int_{\eta_0}^1 \frac{a(\eta) \eta^4 c(\eta) F(k)}{\sqrt{\mu^2 + \eta^2}} d\eta \\
J_1 &= \frac{N_b}{4} \frac{1}{2\pi} \frac{\mu^2}{c_0} \int_{\eta_0}^1 \frac{a(\eta) c(\eta) G(k)}{\sqrt{\mu^2 + \eta^2}} d\eta \\
J_2 &= \frac{N_b}{4} \frac{1}{2\pi} \frac{\mu}{c_0} \int_{\eta_0}^1 \frac{a(\eta) \eta^2 c(\eta) G(k)}{\sqrt{\mu^2 + \eta^2}} d\eta \\
J_3 &= \frac{N_b}{4} \frac{1}{2\pi} \frac{1}{c_0} \int_{\eta_0}^1 \frac{a(\eta) \eta^4 c(\eta) G(k)}{\sqrt{\mu^2 + \eta^2}} d\eta
\end{aligned} \tag{4.54}$$

Research by Cecrdle [1] indicates that using the actual lift-curve slope distribution leads to more conservative stability predictions. Although some derivatives like C_{mr} and C_{Yr} are often ignored in literature due to their small magnitude, they are kept in this study for mathematical completeness. Steady aerodynamics contribute to $C_{Y\psi}$, $C_{Z\theta}$, while unsteady effects are responsible for $C_{Y\theta}$, $C_{Z\psi}$ and others.

While Theodorsen's flat-wake theory works well for high advance ratios, low advance ratios (common in helicopters) may require Loewy's helical wake model [25] for better accuracy.

Despite the complex derivation, implementing the Houbolt-Reed method is straightforward. The total hub loads are found by combining Equations (4.43) through (4.54). The final set of forces and moments, including the flexibility of the pylon, can be written as:

$$\begin{aligned}
F_Z &= \frac{1}{2}\rho V^2 S' \left(C_{Z_\theta} \bar{\theta} + C_{Z_q} \frac{\dot{\theta} R}{V} + C_{Z_\psi} \bar{\psi} + C_{Z_r} \frac{\dot{\psi} R}{V} \right) \\
M_{Y,p} &= \rho V^2 S' R \left(C_{m_\theta} \bar{\theta} + C_{m_q} \frac{\dot{\theta} R}{V} + C_{m_\psi} \bar{\psi} + C_{m_r} \frac{\dot{\psi} R}{V} \right) \\
F_Y &= \frac{1}{2}\rho V^2 S' \left(C_{Y_\theta} \bar{\theta} + C_{Y_q} \frac{\dot{\theta} R}{V} + C_{Y_\psi} \bar{\psi} + C_{Y_r} \frac{\dot{\psi} R}{V} \right) \\
M_{Z,p} &= \rho V^2 S' R \left(C_{n_\theta} \bar{\theta} + C_{n_q} \frac{\dot{\theta} R}{V} + C_{n_\psi} \bar{\psi} + C_{n_r} \frac{\dot{\psi} R}{V} \right)
\end{aligned} \tag{4.55}$$

with effective angles:

$$\begin{aligned}
\bar{\theta} &= \theta - \frac{e_\theta \dot{\theta}}{V} \\
\bar{\psi} &= \psi - \frac{e_\psi \dot{\psi}}{V}
\end{aligned} \tag{4.56}$$

The full set of aerodynamic derivatives is:

$$\begin{aligned}
C_{Y_\psi} &= -C_{Z_\theta} = \frac{4\Omega c_0}{V} I_1 & C_{Y_\theta} &= C_{Z_\psi} = -\frac{4\Omega c_0}{V} J_1 \\
C_{Y_q} &= C_{Z_r} = -\frac{4\Omega c_0}{V} I_2 & C_{Y_r} &= -C_{Z_q} = -\frac{4\Omega c_0}{V} J_2 \\
C_{m_\psi} &= -C_{n_\theta} = \frac{2\Omega c_0}{V} I_2 & C_{m_\theta} &= C_{n_\psi} = -\frac{2\Omega c_0}{V} J_2 \\
C_{m_q} &= C_{n_r} = -\frac{2\Omega c_0}{V} I_3 & C_{m_r} &= -C_{n_q} = -\frac{2\Omega c_0}{V} J_3
\end{aligned} \tag{4.57}$$

In summary, this model uses strip theory and 2D steady/unsteady aerodynamics, neglecting drag and induced hub velocities. Wake effects are added via Theodorsen functions, and the propeller is assumed to be in a non-thrusting (windmilling) state with a linear twist distribution.

There are slight differences between these results and those in Rodden and Rose [26], specifically regarding the sign of C_{nq} . In this study, we found:

$$C_{nq} = -C_{mr} = \frac{2\Omega c_0}{V} J_3 \tag{4.58}$$

whereas Rodden and Rose [19] reported a negative sign. However, since they set this parameter to zero in their final model, the difference does not affect qualitative comparisons.

4.2.3. Equations of Motion for the Aeroelastic System

The equations of motion for the aeroelastic system can be derived using Lagrange's equations and expressed in a linearized matrix form as:

$$[M_s - M_a] \begin{Bmatrix} \ddot{\theta} \\ \ddot{\psi} \end{Bmatrix} + [C_s - C_a] \begin{Bmatrix} \dot{\theta} \\ \dot{\psi} \end{Bmatrix} + [K_s - K_a] \begin{Bmatrix} \theta \\ \psi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (4.59)$$

where the subscripts s and a denote structural and aerodynamic contributions, respectively. The structural matrices are defined as follows:

$$M_s = \begin{bmatrix} I_Y & 0 \\ 0 & I_Z \end{bmatrix}, \quad C_s = \begin{bmatrix} \frac{K_{\theta} g_{\theta}}{\omega} & I_X \Omega \\ -I_X \Omega & \frac{K_{\psi} g_{\psi}}{\omega} \end{bmatrix}, \quad K_s = \begin{bmatrix} K_{\theta} & 0 \\ 0 & K_{\psi} \end{bmatrix} \quad (4.60)$$

where I_X, I_Y, I_Z are the moments of inertia of the propeller, Ω is the rotational speed, K_{θ}, K_{ψ} are the stiffness values, and g_{θ}, g_{ψ} represent the structural damping coefficients. The aerodynamic matrices derived from the Houbolt-Reed method, including the effects of the hub offset (e_{θ}, e_{ψ}), are given by:

$$M_a = \begin{bmatrix} -k_2 C_{mq} \frac{R}{V} \frac{e_{\theta}}{V} + e_{\theta} k_1 C_{Zq} \frac{R}{V} \frac{e_{\theta}}{V} & -k_2 C_{mr} \frac{R}{V} \frac{e_{\psi}}{V} + e_{\theta} k_1 C_{Zr} \frac{R}{V} \frac{e_{\psi}}{V} \\ -k_2 C_{nq} \frac{R}{V} \frac{e_{\theta}}{V} - e_{\psi} k_1 C_{Yq} \frac{R}{V} \frac{e_{\theta}}{V} & -k_2 C_{nr} \frac{R}{V} \frac{e_{\psi}}{V} - e_{\psi} k_1 C_{Yr} \frac{R}{V} \frac{e_{\psi}}{V} \end{bmatrix} \quad (4.61)$$

$$C_a = \begin{bmatrix} -k_2 C_{m\theta} \frac{e_{\theta}}{V} + k_2 C_{mq} \frac{R}{V} + e_{\theta} k_1 C_{Z\theta} \frac{e_{\theta}}{V} - e_{\theta} k_1 C_{Zq} \frac{R}{V} & -k_2 C_{m\psi} \frac{e_{\psi}}{V} + k_2 C_{mr} \frac{R}{V} + e_{\theta} k_1 C_{Z\psi} \frac{e_{\psi}}{V} - e_{\theta} k_1 C_{Zr} \frac{R}{V} \\ -k_2 C_{n\theta} \frac{e_{\theta}}{V} + k_2 C_{nq} \frac{R}{V} - e_{\psi} k_1 C_{Y\theta} \frac{e_{\theta}}{V} + e_{\psi} k_1 C_{Yq} \frac{R}{V} & -k_2 C_{n\psi} \frac{e_{\psi}}{V} + k_2 C_{nr} \frac{R}{V} - e_{\psi} k_1 C_{Y\psi} \frac{e_{\psi}}{V} + e_{\psi} k_1 C_{Yr} \frac{R}{V} \end{bmatrix} \quad (4.62)$$

$$K_a = \begin{bmatrix} k_2 C_{m\theta} - e_{\theta} k_1 C_{Z\theta} & k_2 C_{m\psi} - e_{\theta} k_1 C_{Z\psi} \\ k_2 C_{n\theta} + e_{\psi} k_1 C_{Y\theta} & k_2 C_{n\psi} + e_{\psi} k_1 C_{Y\psi} \end{bmatrix} \quad (4.63)$$

where the aerodynamic constants are defined as $k_1 = \frac{1}{2} \rho V^2 S'$ and $k_2 = \rho V^2 S' R$.

4.3. Coupled Wing–Propeller System Model

In this section, the aeroelastic modeling of a wing-propeller system is presented.

4.3.1. Structural Model

The structural model is derived using the Galerkin Method, adopting the Generalized Coordinate approach. The system is discretized by expanding the deformations into

products of spatial shape functions vectors $\{N\}$ (denoted with transpose ^{T}) and time-dependent generalized coordinate vectors $\{q\}$.

The displacements for wing bending $h(y, t)$, wing torsion $\phi(y, t)$, nacelle pitch $\theta(t)$, and nacelle yaw $\psi(t)$ are defined as:

$$\begin{aligned} h(y, t) &= \{N_h(y)\}^T \{q_h(t)\} \\ \phi(y, t) &= \{N_\phi(y)\}^T \{q_\phi(t)\} \\ \theta(t) &= \{N_\theta\}^T \{q_\theta(t)\} \\ \psi(t) &= \{N_\psi\}^T \{q_\psi(t)\} \end{aligned} \quad (4.64)$$

For the specific quantities at the pylon location y_P , the definitions are:

$$h_P(t) = \{N_h(y_P)\}^T \{q_h(t)\}, \quad \phi_P(t) = \{N_\phi(y_P)\}^T \{q_\phi(t)\} \quad (4.65)$$

The total kinetic energy is the sum of the wing energy (T_W) and the propeller/pylon energy (T_P), which includes the gyroscopic effects of the rotating propeller.

Wing Kinetic Energy (T_W) Integrating over the wing span L :

$$T_W = \int_0^L \left(\frac{1}{2} m_w \dot{h}^2 + m_w x_{ab} \dot{h} \dot{\phi} + \frac{1}{2} m_w x_{ab}^2 \dot{\phi}^2 \right) dy \quad (4.66)$$

where m_w is the wing mass per unit length and x_{ab} represents the distance between the elastic axis and the center of mass.

Propeller-Pylon Kinetic Energy (T_P) The kinetic energy contribution from the propeller and pylon system, including inertia terms and gyroscopic coupling, is given by:

$$\begin{aligned} T_P &= \frac{1}{2} M_P \dot{h}_P^2 + \frac{1}{2} I_{\phi, P} \dot{\phi}_P^2 + S_{\phi, P} \dot{h}_P \dot{\phi}_P \\ &\quad + \frac{1}{2} I_{\theta, P} \dot{\theta}^2 + S_{\theta, P} \dot{\theta} \dot{h}_P + I_{\theta\phi, P} \dot{\theta} \dot{\phi}_P + \frac{1}{2} I_{\psi, P} \dot{\psi}^2 \\ &\quad + \frac{1}{2} I_\Omega \left(\Omega^2 + 2\Omega \dot{\theta} \psi + 2\Omega \dot{\phi}_P \psi \right) \end{aligned} \quad (4.67)$$

Here, M_P , $I_{\phi, P}$, $I_{\theta, P}$, $I_{\psi, P}$ are the mass and moments of inertia of the pylon/engine assembly. $S_{\phi, P}$, $S_{\theta, P}$ and $I_{\theta\phi, P}$ utilize the coupling terms arising from the offset center of gravity. I_Ω is the polar moment of inertia of the propeller, and the last term represents the gyroscopic energy contribution involving coupling between pitch/torsion rates and yaw displacement.

The total potential energy (U) consists of the strain energy of the wing (U_W) and the potential energy of the pylon springs (U_P):

$$U = U_W + U_P \quad (4.68)$$

Wing Potential Energy (U_W)

$$U_W = \int_0^L \left(\frac{1}{2} EI (h'')^2 + \frac{1}{2} GJ (\phi')^2 \right) dy \quad (4.69)$$

Pylon Potential Energy (U_P)

$$U_P = \frac{1}{2} K_\theta \theta^2 + \frac{1}{2} K_\psi \psi^2 \quad (4.70)$$

The structural damping in the pylon pitch and yaw modes is modeled using a dissipation function D :

$$D = \frac{1}{2} K_\theta \frac{g_\theta}{\omega} \dot{\theta}^2 + \frac{1}{2} K_\psi \frac{g_\psi}{\omega} \dot{\psi}^2 \quad (4.71)$$

Applying Lagrange's equations to the energy expressions and organizing the terms into matrix form yields the structural system matrices. The generalized coordinate vector is ordered as $\{q\} = [\{q_\theta\}^T, \{q_\psi\}^T, \{q_h\}^T, \{q_\phi\}^T]^T$.

Structural Mass Matrix ($[M_s]$)

$$[M_s] = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \quad (4.72)$$

where the mass matrix elements are defined as follows (note that $[0]$ represents a zero

matrix of appropriate dimensions):

$$\begin{aligned}
M_{11} &= I_{\theta,P} \{N_\theta\} \{N_\theta\}^T \\
M_{12} &= [0] \\
M_{13} &= S_{\theta,P} \{N_\theta\} \{N_h(y_P)\}^T \\
M_{14} &= I_{\theta\phi,P} \{N_\theta\} \{N_\phi(y_P)\}^T \\
M_{21} &= [0] \\
M_{22} &= I_{\psi,P} \{N_\psi\} \{N_\psi\}^T \\
M_{23} &= [0] \\
M_{24} &= [0] \\
M_{31} &= S_{\theta,P} \{N_h(y_P)\} \{N_\theta\}^T \\
M_{32} &= [0] \\
M_{33} &= M_P \{N_h(y_P)\} \{N_h(y_P)\}^T + \int_0^L m_w \{N_h\} \{N_h\}^T dy \\
M_{34} &= S_{\phi,P} \{N_h(y_P)\} \{N_\phi(y_P)\}^T \\
M_{41} &= I_{\theta\phi,P} \{N_\phi(y_P)\} \{N_\theta\}^T \\
M_{42} &= [0] \\
M_{43} &= S_{\phi,P} \{N_\phi(y_P)\} \{N_h(y_P)\}^T \\
M_{44} &= I_{\phi,P} \{N_\phi(y_P)\} \{N_\phi(y_P)\}^T + \int_0^L I_\alpha \{N_\phi\} \{N_\phi\}^T dy
\end{aligned} \tag{4.73}$$

Gyroscopic/Damping Matrix ($[C_s]$)

$$[C_s] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix} \tag{4.74}$$

where the elements are:

$$\begin{aligned}
C_{11} &= [0] \\
C_{12} &= I_{\Omega} \Omega \{N_{\theta}\} \{N_{\psi}\}^T \\
C_{13} &= [0] \\
C_{14} &= [0] \\
C_{21} &= -I_{\Omega} \Omega \{N_{\psi}\} \{N_{\theta}\}^T \\
C_{22} &= [0] \\
C_{23} &= [0] \\
C_{24} &= -I_{\Omega} \Omega \{N_{\psi}\} \{N_{\phi}(y_P)\}^T \\
C_{31} &= [0] \\
C_{32} &= [0] \\
C_{33} &= [0] \\
C_{34} &= [0] \\
C_{41} &= [0] \\
C_{42} &= I_{\Omega} \Omega \{N_{\phi}(y_P)\} \{N_{\psi}\}^T \\
C_{43} &= [0] \\
C_{44} &= [0]
\end{aligned} \tag{4.75}$$

Structural Stiffness Matrix ($[K_s]$)

$$[K_s] = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \tag{4.76}$$

where the elements are:

$$\begin{aligned}
K_{11} &= (1 + ig_\theta)K_\theta\{N_\theta\}\{N_\theta\}^T \\
K_{12} &= [0] \\
K_{13} &= [0] \\
K_{14} &= [0] \\
K_{21} &= [0] \\
K_{22} &= (1 + ig_\psi)K_\psi\{N_\psi\}\{N_\psi\}^T \\
K_{23} &= [0] \\
K_{24} &= [0] \\
K_{31} &= [0] \\
K_{32} &= [0] \\
K_{33} &= \int_0^L EI\{N_h''\}\{N_h''\}^T dy \\
K_{34} &= [0] \\
K_{41} &= [0] \\
K_{42} &= [0] \\
K_{43} &= [0] \\
K_{44} &= \int_0^L GJ\{N_\phi'\}\{N_\phi'\}^T dy
\end{aligned} \tag{4.77}$$

The generalized aerodynamic forces $\{Q_i\}$ driving the aeroelastic system are composed of contributions from the unsteady wing aerodynamics (Q_W) and the propeller aerodynamic loads (Q_P).

$$\{Q\} = \{Q_P\} + \{Q_W\} \tag{4.78}$$

4.3.2. Wing Unsteady Aerodynamics

The aerodynamic loads on the wing are modeled using Theodorsen's unsteady strip theory. The lift per unit span (L) and the pitching moment about the elastic axis (M_{EA}) are defined as functions of the wing bending (h) and torsion (ϕ). Note that in the context of wing aerodynamics, the torsional deformation variable is ϕ (denoted as 'thetaAE' in some formulations).

The Lift per unit span is given by:

$$L = C_{L\alpha} \rho V b C(k) \left[\dot{h} + V\dot{\phi} + b \left(\frac{1}{2} - a \right) \ddot{\phi} \right] + \pi \rho b^2 \left[\ddot{h} + V\dot{\phi} - ba\ddot{\phi} \right] \quad (4.79)$$

The Moment about the Elastic Axis is:

$$M_{EA} = M_{AC} + \left(\frac{1}{2} + a \right) bL \quad (4.80)$$

where the Moment about the Aerodynamic Center (M_{AC}) is:

$$M_{AC} = -\pi \rho b^3 \left[\frac{1}{2} \ddot{h} + V\dot{\phi} + b \left(\frac{1}{8} - \frac{a}{2} \right) \ddot{\phi} \right] \quad (4.81)$$

Here, $C(k)$ is the Theodorsen function, b is the semichord, and a is the elastic axis location. The contribution of these distributed loads to the generalized forces for wing bending (Q_{hW}) and torsion ($Q_{\phi W}$) is found by integrating over the span with the corresponding shape functions:

$$Q_{hW} = - \int_0^L L N_h(y) dy$$

$$Q_{\phi W} = \int_0^L M_{EA} N_\phi(y) dy \quad (4.82)$$

Note the sign convention: Lift is defined positive up, while h is positive down (or consistent with coordinate system), hence the negative sign for Q_{hW} .

4.3.3. Propeller Aerodynamics

The aerodynamic forces generated by the propeller depend on the effective angle of attack and sideslip seen by the propeller disk. These effective angles, denoted as Θ^* and Ψ^* , include the geometric angles of the nacelle as well as the rates of motion of the pylon attachment point.

The effective pitch angle Θ^* includes the nacelle pitch θ , wing torsion at the pylon ϕ_P , and rate-induced terms:

$$\Theta^* = \theta + \phi_P + \frac{e_\theta}{V} \dot{\theta} + \frac{e_\phi}{V} \dot{\phi}_P + \frac{1}{V} \dot{h}_P \quad (4.83)$$

The effective yaw angle Ψ^* includes the nacelle yaw ψ and its rate:

$$\Psi^* = \psi + \frac{e_\psi}{V} \dot{\psi} \quad (4.84)$$

The propeller aerodynamic forces (Side force P_Y , Normal force P_Z) and moments (M_{YP} , M_{ZP}) are expressed using the propeller stability derivatives defined in previous section:

$$\begin{aligned}
P_Y &= q_\infty F_P \left(C_{y_\theta} \Theta^* + C_{y_\psi} \Psi^* + C_{y_q} \frac{\dot{\Theta}^* R}{V} + C_{y_r} \frac{\dot{\Psi}^* R}{V} \right) \\
P_Z &= q_\infty F_P \left(C_{z_\theta} \Theta^* + C_{z_\psi} \Psi^* + C_{z_q} \frac{\dot{\Theta}^* R}{V} + C_{z_r} \frac{\dot{\Psi}^* R}{V} \right) \\
M_{YP} &= q_\infty F_P D_P \left(C_{m_\theta} \Theta^* + C_{m_\psi} \Psi^* + C_{m_q} \frac{\dot{\Theta}^* R}{V} + C_{m_r} \frac{\dot{\Psi}^* R}{V} \right) \\
M_{ZP} &= q_\infty F_P D_P \left(C_{n_\theta} \Theta^* + C_{n_\psi} \Psi^* + C_{n_q} \frac{\dot{\Theta}^* R}{V} + C_{n_r} \frac{\dot{\Psi}^* R}{V} \right)
\end{aligned} \tag{4.85}$$

where $F_P = \pi R^2$ is the disk area, $D_P = 2R$ is the diameter, and $q_\infty = \frac{1}{2} \rho V^2$ is the dynamic pressure.

The generalized forces due to the propeller loads ($Q_{\theta P}$, $Q_{\psi P}$, Q_{hP} , $Q_{\phi P}$) act on the nacelle and are transferred to the wing tip:

$$\begin{aligned}
Q_{\theta P} &= (M_{YP} + e_\theta P_Z) N_\theta \\
Q_{\psi P} &= (M_{ZP} - e_\psi P_Y) N_\psi \\
Q_{hP} &= P_Z N_h(y_P) \\
Q_{\phi P} &= (M_{YP} + e_\phi P_Z) N_\phi(y_P)
\end{aligned} \tag{4.86}$$

4.3.4. Total Generalized Forces

Combining the contributions from the wing and the propeller, the final generalized aerodynamic force vector $\{Q\}$ components are:

$$\begin{aligned}
Q_\theta &= Q_{\theta P} \\
Q_\psi &= Q_{\psi P} \\
Q_h &= Q_{hP} + Q_{hW} \\
Q_\phi &= Q_{\phi P} + Q_{\phi W}
\end{aligned} \tag{4.87}$$

Assuming harmonic motion $q(t) = \bar{q} e^{i\omega t}$ where $\omega = kV/b_r$, the generalized aerodynamic forces can be expanded into aerodynamic mass $[M_{ae}]$, damping $[C_{ae}]$, and stiffness $[K_{ae}]$ matrices.

Aerodynamic Mass Matrix ($[M_{ae}]$)

$$[M_{ae}] = \begin{bmatrix} M_{ae11} & M_{ae12} & M_{ae13} & M_{ae14} \\ M_{ae21} & M_{ae22} & M_{ae23} & M_{ae24} \\ M_{ae31} & M_{ae32} & M_{ae33} & M_{ae34} \\ M_{ae41} & M_{ae42} & M_{ae43} & M_{ae44} \end{bmatrix} \quad (4.88)$$

where the elements are:

$$\begin{aligned} M_{ae11} &= \frac{\pi q_\infty R^3 (C_{z_q} e_\theta + 2C_{m_q} R) V^2}{b_r^2 V^2} e_\theta \{N_\theta\} \{N_\theta\}^T \\ M_{ae12} &= \frac{\pi q_\infty R^3 (C_{z_r} e_\theta + 2C_{m_r} R) V^2}{b_r^2 V^2} e_\psi \{N_\theta\} \{N_\psi\}^T \\ M_{ae13} &= \frac{\pi q_\infty R^3 (C_{z_q} e_\theta + 2C_{m_q} R) V^2}{b_r^2 V^2} \{N_\theta\} \{N_h(y_P)\}^T \\ M_{ae14} &= \frac{\pi q_\infty R^3 (C_{z_q} e_\theta + 2C_{m_q} R) V^2}{b_r^2 V^2} e_\phi \{N_\theta\} \{N_\phi(y_P)\}^T \\ M_{ae21} &= \frac{\pi q_\infty R^3 (-C_{y_q} e_\psi + 2C_{n_q} R) V^2}{b_r^2 V^2} e_\theta \{N_\psi\} \{N_\theta\}^T \\ M_{ae22} &= \frac{\pi q_\infty R^3 (-C_{y_r} e_\psi + 2C_{n_r} R) V^2}{b_r^2 V^2} e_\psi \{N_\psi\} \{N_\psi\}^T \\ M_{ae23} &= \frac{\pi q_\infty R^3 (-C_{y_q} e_\psi + 2C_{n_q} R) V^2}{b_r^2 V^2} \{N_\psi\} \{N_h(y_P)\}^T \\ M_{ae24} &= \frac{\pi q_\infty R^3 (-C_{y_q} e_\psi + 2C_{n_q} R) V^2}{b_r^2 V^2} e_\phi \{N_\psi\} \{N_\phi(y_P)\}^T \\ M_{ae31} &= \frac{C_{z_q} e_\theta \pi q_\infty R^3 V^2}{b_r^2 V^2} \{N_h(y_P)\} \{N_\theta\}^T \\ M_{ae32} &= \frac{C_{z_r} e_\psi \pi q_\infty R^3 V^2}{b_r^2 V^2} \{N_h(y_P)\} \{N_\psi\}^T \\ M_{ae33} &= \frac{C_{z_q} \pi q_\infty R^3 V^2}{b_r^2 V^2} \{N_h(y_P)\} \{N_h(y_P)\}^T \\ &\quad - \int_0^1 \frac{b^2 \pi \rho V^2}{b_r^2} \{N_h\} \{N_h\}^T d\eta \\ M_{ae34} &= \frac{C_{z_q} e_\phi \pi q_\infty R^3 V^2}{b_r^2 V^2} \{N_h(y_P)\} \{N_\phi(y_P)\}^T \\ &\quad + \int_0^1 \frac{ab^3 \pi \rho V^2}{b_r^2} \{N_h\} \{N_\phi\}^T d\eta \\ M_{ae41} &= \frac{\pi q_\infty R^3 (C_{z_q} e_\phi + 2C_{m_q} R) V^2}{b_r^2 V^2} e_\theta \{N_\phi(y_P)\} \{N_\theta\}^T \\ M_{ae42} &= \frac{\pi q_\infty R^3 (C_{z_r} e_\phi + 2C_{m_r} R) V^2}{b_r^2 V^2} e_\psi \{N_\phi(y_P)\} \{N_\psi\}^T \\ M_{ae43} &= \frac{\pi q_\infty R^3 (C_{z_q} e_\phi + 2C_{m_q} R) V^2}{b_r^2 V^2} \{N_\phi(y_P)\} \{N_h(y_P)\}^T \end{aligned} \quad (4.89)$$

$$\begin{aligned}
& + \int_0^1 \frac{ab^3\pi\rho V^2}{b_r^2} \{N_\phi\} \{N_h\}^T d\eta \\
M_{ae44} = & \frac{\pi q_\infty R^3 (C_{z_q} e_\phi + 2C_{m_q} R) V^2}{b_r^2 V^2} e_\phi \{N_\phi(y_P)\} \{N_\phi(y_P)\}^T \\
& - \int_0^1 \frac{b^4(1+8a^2)\pi\rho V^2}{8b_r^2} \{N_\phi\} \{N_\phi\}^T d\eta
\end{aligned}$$

Aerodynamic Damping Matrix ($[C_{ae}]$)

$$[C_{ae}] = \begin{bmatrix} C_{ae11} & C_{ae12} & C_{ae13} & C_{ae14} \\ C_{ae21} & C_{ae22} & C_{ae23} & C_{ae24} \\ C_{ae31} & C_{ae32} & C_{ae33} & C_{ae34} \\ C_{ae41} & C_{ae42} & C_{ae43} & C_{ae44} \end{bmatrix} \quad (4.90)$$

where the elements are:

$$\begin{aligned}
C_{ae11} &= \frac{\pi q_\infty R^2 (C_{z_\theta} e_\theta^2 + R(2C_{m_\theta} e_\theta + C_{z_q} e_\theta + 2C_{m_q} R)) V}{b_r V} \times \\
& \quad \{N_\theta\} \{N_\theta\}^T \\
C_{ae12} &= \frac{\pi q_\infty R^2 (C_{z_\psi} e_\psi e_\theta + R(2C_{m_\psi} e_\psi + C_{z_r} e_\theta + 2C_{m_r} R)) V}{b_r V} \times \\
& \quad \{N_\theta\} \{N_\psi\}^T \\
C_{ae13} &= \frac{\pi q_\infty R^2 (C_{z_\theta} e_\theta + 2C_{m_\theta} R) V}{b_r V} \{N_\theta\} \{N_h(y_P)\}^T \\
C_{ae14} &= \frac{\pi q_\infty R^2 (C_{z_\theta} e_\phi e_\theta + R(2C_{m_\theta} e_\phi + C_{z_q} e_\theta + 2C_{m_q} R)) V}{b_r V} \times \\
& \quad \{N_\theta\} \{N_\phi(y_P)\}^T \\
C_{ae21} &= \frac{\pi q_\infty R^2 (-C_{y_\theta} e_\psi e_\theta + R(-C_{y_q} e_\psi + 2C_{n_\theta} e_\theta + 2C_{n_q} R)) V}{b_r V} \times \\
& \quad \{N_\psi\} \{N_\theta\}^T \\
C_{ae22} &= \frac{\pi q_\infty R^2 (-C_{y_\psi} e_\psi^2 + R(2C_{n_\psi} e_\psi - C_{y_r} e_\psi + 2C_{n_r} R)) V}{b_r V} \times \\
& \quad \{N_\psi\} \{N_\psi\}^T \\
C_{ae23} &= \frac{\pi q_\infty R^2 (-C_{y_\theta} e_\psi + 2C_{n_\theta} R) V}{b_r V} \{N_\psi\} \{N_h(y_P)\}^T \\
C_{ae24} &= \frac{\pi q_\infty R^2 (-C_{y_\theta} e_\phi e_\psi + R(2C_{n_\theta} e_\phi - C_{y_q} e_\psi + 2C_{n_q} R)) V}{b_r V} \times \quad (4.91)
\end{aligned}$$

$$\begin{aligned}
& \{N_\psi\} \{N_\phi(y_P)\}^T \quad (4.92) \\
C_{ae31} &= \frac{\pi q_\infty R^2 (C_{z_\theta} e_\theta + C_{z_q} R) V}{b_r V} \{N_h(y_P)\} \{N_\theta\}^T
\end{aligned}$$

$$\begin{aligned}
C_{ae32} &= \frac{\pi q_\infty R^2 (C_{z_\psi} e_\psi + C_{z_r} R) V}{b_r V} \{N_h(y_P)\} \{N_\psi\}^T \\
C_{ae33} &= \frac{\pi q_\infty R^2 C_{z_\theta} V}{b_r V} \{N_h(y_P)\} \{N_h(y_P)\}^T \\
&\quad - \int_0^1 \frac{b C(k) C_{L_\alpha} \rho V^2}{b_r} \{N_h\} \{N_h\}^T d\eta \\
C_{ae34} &= \frac{\pi q_\infty R^2 (C_{z_\theta} e_\phi + C_{z_q} R) V}{b_r V} \{N_h(y_P)\} \{N_\phi(y_P)\}^T \\
&\quad + \int_0^1 \frac{b^2 ((2a-1)C(k)C_{L_\alpha} - 2\pi) \rho V^2}{2b_r} \{N_h\} \{N_\phi\}^T d\eta \\
C_{ae41} &= \frac{\pi q_\infty R^2 (C_{z_\theta} e_\phi e_\theta + R(C_{z_q} e_\phi + 2C_{m_\theta} e_\theta + 2C_{m_q} R)) V}{b_r V} \times \\
&\quad \{N_\phi(y_P)\} \{N_\theta\}^T \\
C_{ae42} &= \frac{\pi q_\infty R^2 (C_{z_\psi} e_\phi e_\psi + R(C_{z_r} e_\phi + 2C_{m_\psi} e_\psi + 2C_{m_r} R)) V}{b_r V} \times \\
&\quad \{N_\phi(y_P)\} \{N_\psi\}^T \\
C_{ae43} &= \frac{\pi q_\infty R^2 (C_{z_\theta} e_\phi + 2C_{m_\theta} R) V}{b_r V} \{N_\phi(y_P)\} \{N_h(y_P)\}^T \\
&\quad + \int_0^1 \frac{(1+2a)b^2 C(k) C_{L_\alpha} \rho V^2}{2b_r} \{N_\phi\} \{N_h\}^T d\eta \\
C_{ae44} &= \frac{\pi q_\infty R^2 (C_{z_\theta} e_\phi^2 + R(2C_{m_\theta} e_\phi + C_{z_q} e_\phi + 2C_{m_q} R)) V}{b_r V} \times \\
&\quad \{N_\phi(y_P)\} \{N_\phi(y_P)\}^T \\
&\quad + \int_0^1 \frac{b^3 C(k) C_{L_\alpha} (1-4a^2) - 2b^3 \pi (1-2a) \rho V^2}{4b_r} \{N_\phi\} \{N_\phi\}^T d\eta
\end{aligned}$$

Aerodynamic Stiffness Matrix ($[K_{ae}]$)

$$[K_{ae}] = \begin{bmatrix} K_{ae11} & K_{ae12} & K_{ae13} & K_{ae14} \\ K_{ae21} & K_{ae22} & K_{ae23} & K_{ae24} \\ K_{ae31} & K_{ae32} & K_{ae33} & K_{ae34} \\ K_{ae41} & K_{ae42} & K_{ae43} & K_{ae44} \end{bmatrix} \quad (4.93)$$

where the elements are:

$$\begin{aligned}
K_{ae11} &= \pi q_\infty R^2 (C_{z_\theta} e_\theta + 2C_{m_\theta} R) \{N_\theta\} \{N_\theta\}^T \\
K_{ae12} &= \pi q_\infty R^2 (C_{z_\psi} e_\theta + 2C_{m_\psi} R) \{N_\theta\} \{N_\psi\}^T \\
K_{ae13} &= [0] \\
K_{ae14} &= \pi q_\infty R^2 (C_{z_\theta} e_\theta + 2C_{m_\theta} R) \{N_\theta\} \{N_\phi(y_P)\}^T \\
K_{ae21} &= \pi q_\infty R^2 (-C_{y_\theta} e_\psi + 2C_{n_\theta} R) \{N_\psi\} \{N_\theta\}^T
\end{aligned}$$

$$\begin{aligned}
K_{ae22} &= \pi q_\infty R^2 (-C_{y_\psi} e_\psi + 2C_{n_\psi} R) \{N_\psi\} \{N_\psi\}^T \\
K_{ae23} &= [0] \\
K_{ae24} &= \pi q_\infty R^2 (-C_{y_\theta} e_\psi + 2C_{n_\theta} R) \{N_\psi\} \{N_\phi(y_P)\}^T \\
K_{ae31} &= \pi q_\infty R^2 C_{z_\theta} \{N_h(y_P)\} \{N_\theta\}^T \\
K_{ae32} &= \pi q_\infty R^2 C_{z_\psi} \{N_h(y_P)\} \{N_\psi\}^T \\
K_{ae33} &= [0] \\
K_{ae34} &= \pi q_\infty R^2 C_{z_\theta} \{N_h(y_P)\} \{N_\phi(y_P)\}^T \\
&\quad - \int_0^1 bC(k) C_{L_\alpha} \rho V^2 \{N_h\} \{N_\phi\}^T d\eta \\
K_{ae41} &= \pi q_\infty R^2 (C_{z_\theta} e_\phi + 2C_{m_\theta} R) \{N_\phi(y_P)\} \{N_\theta\}^T \\
K_{ae42} &= \pi q_\infty R^2 (C_{z_\psi} e_\phi + 2C_{m_\psi} R) \{N_\phi(y_P)\} \{N_\psi\}^T \\
K_{ae43} &= [0] \\
K_{ae44} &= \pi q_\infty R^2 (C_{z_\theta} e_\phi + 2C_{m_\theta} R) \{N_\phi(y_P)\} \{N_\phi(y_P)\}^T \\
&\quad + \int_0^1 \frac{(1+2a)b^2 C(k) C_{L_\alpha} \rho V^2}{2} \{N_\phi\} \{N_\phi\}^T d\eta
\end{aligned} \tag{4.94}$$

Finally, the total system matrices for the p-k flutter analysis are assembled by combining the structural and aerodynamic components:

$$[\bar{M}] = [M_s] \tag{4.95}$$

$$[\bar{C}] = [C_s] - \frac{\text{Im}([A_{total}])}{kV/b_r} \tag{4.96}$$

$$[\bar{K}] = [K_s] - \text{Re}([A_{total}]) \tag{4.97}$$

where the total aerodynamic matrix $[A_{total}]$ is defined as:

$$[A_{total}] = -k^2 [M_{ae}] + ik [C_{ae}] + [K_{ae}] \tag{4.98}$$

The explicit forms of the system matrices for the two-propeller model configuration are provided in Appendix A.

4.4. Verification of Presented Framework

4.4.1. Verification of the Clean Wing Model (No Propellers)

Before studying the complex coupling effects of the propeller–pylon system, the basic aeroelastic solver must first be validated using a simple reference case. For this reason,

the “Basic Wing Model” described in the reference study is chosen as the benchmark configuration. The goal of this verification step has two parts:

- **Structural validation:** to check that the Galerkin Method correctly captures the wing’s modal properties, such as natural frequencies and mode shapes.
- **Aeroelastic validation:** to confirm that Theodorsen’s unsteady aerodynamic model and the p–k method are implemented correctly by predicting the flutter speed and flutter frequency.

By matching the results of this well-known benchmark case, the accuracy of the core computational model is confirmed, providing a solid and reliable basis for the later coupled whirl flutter analysis.

To ensure the accuracy of the aeroelastic solver, the structural and geometric characteristics of the wing have been chosen to correspond to the benchmark model depicted in the reference study. In contrast to a simplified uniform beam, the validation wing exhibits a linearly tapered planform, thereby requiring the incorporation of aerodynamic and structural properties that vary along the span.

The geometry of the wing is defined by its semi-span (L), root chord (c_{root}), and tip chord (c_{tip}). The local chord length $c(y)$ at any spanwise station y is determined by the taper ratio ($\lambda = c_{tip}/c_{root}$) using the linear relationship:

$$c(y) = c_{root} \left[1 - (1 - \lambda) \frac{y}{L} \right] \quad (4.99)$$

This variation in chord length directly influences the aerodynamic influence coefficients calculated via Theodorsen’s theory, as well as the structural mass and stiffness distributions if they are scaled with the geometry. The specific geometric and material properties used for this validation case are summarized in Table 4.1.

Table 4.1: Geometric and structural properties of the baseline wing.

Parameter	Symbol	Value	Units
<i>Flight Condition</i>			
Altitude	H	8000	ft
Air Density	ρ_∞	0.9629	kg/m ³
Cruise Speed	V_∞	77	m/s
<i>Wing Geometry</i>			
Semi-span	L	5.7	m
Root Chord	c_{root}	1.25	m
Tip Chord	c_{tip}	0.80	m
Taper Ratio	λ	0.64	-
<i>Structural & Aerodynamic Properties</i>			
Mass per unit length	m_w	25	kg/m
Bending Rigidity	EI	7.0×10^5	N·m ²
Torsional Rigidity	GJ	2.0×10^5	N·m ²
Elastic Axis (from LE)	x_{ea}	0.50	c
Center of Gravity (from LE)	x_{cg}	0.50	c
Radius of Gyration	r_α	0.25	c
Lift Curve Slope	C_{L_α}	2π	rad ⁻¹

To evaluate the fidelity of the structural model, the natural frequencies of the tapered wing are computed under conditions of zero speed ($V=0$). The eigenvalues obtained from the structural equations of motion correspond to the system's natural modes. Table 4.2 presents the comparison of these natural frequencies. The percent error is calculated to quantify the agreement between the present code and the reference values.

Table 4.2: Comparison of natural frequencies (Hz) for the baseline wing.

Mode	Ref. [5]	Ref. [27]	Ref. [28]	Present
1st Bending	2.88	2.882	2.88	2.882
1st Torsion	16.68	16.679	16.68	16.679
2nd Bending	18.06	18.062	18.06	18.060
2nd Torsion	46.59	46.499	40.49	46.483
3rd Bending	50.57	-	50.58	50.587

To verify the correct implementation of the boundary conditions and the spanwise property variations, the computed mode shapes are visualized in Figure 4.5. Due to the coincidence of the center of gravity and the elastic axis ($x_\alpha = 0$), the structural equations of motion are uncoupled. Therefore, the resulting eigenvectors represent pure bending and pure torsional modes, with no inertial cross-coupling [29].

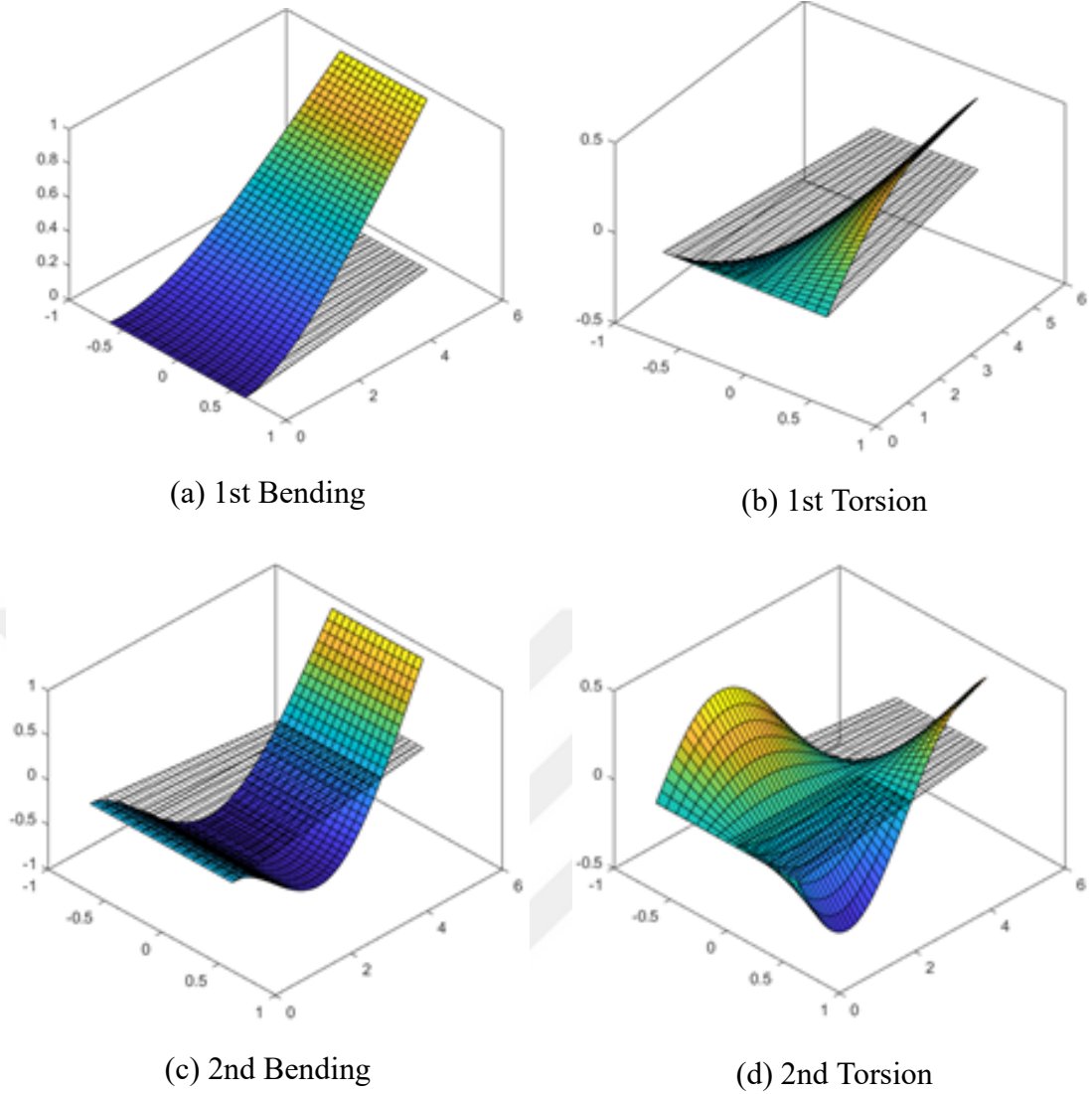


Figure 4.5: Visualized mode shapes of the baseline wing.

- **1st Bending Mode (Figure 4.5a):** This mode exhibits pure flexural deformation. Since there is no inertial coupling, the wing undergoes vertical displacement without any induced twisting. The shape corresponds to the fundamental mode of a tapered cantilever beam.
- **1st Torsion Mode (Figure 4.5b):** This mode displays a pure rotational oscillation about the elastic axis. The twist angle (ϕ) increases towards the tip, while the vertical displacement (h) remains zero throughout the span. This confirms the correct decoupling of the structural matrices.
- **2nd Bending Mode (Figure 4.5c):** This mode represents the second pure bending harmonic, characterized by a nodal line along the span where the displace-

ment is zero. Similar to the first mode, the torsional component is identically zero.

- **2nd Torsion Mode (Figure 4.5d):** The second torsional mode exhibits a node in the twist distribution, representing a higher-order pure torsional vibration.

The aeroelastic stability limits of the baseline tapered wing are found using the p–k method described in Chapter 3. In the analysis, the airspeed is gradually increased from 0 m/s to 350 m/s. Two types of instability are observed: dynamic instability (flutter) and static instability (divergence).

To verify the accuracy of the current aeroelastic model, the critical velocities and frequencies are compared with three distinct reference studies from the literature [5, 27, 28]. Table 4.3 summarizes the flutter speed (V_f), flutter frequency (f_f), and the divergence speed (V_{div}) obtained from the present analysis alongside the benchmark data. Values in parentheses indicate the absolute percentage difference relative to the present results.

Table 4.3: Comparison of aeroelastic stability limits for the baseline wing.

Parameter	Ref. [5]	Ref. [27]	Ref. [28]	Present
V_f (m/s)	152 (2.6%)	162.5 (8.9%)	151 (2.0%)	148
f_f (Hz)	8.08 (2.4%)	-	8.41 (6.2%)	7.89
V_{div} (m/s)	154 (2.6%)	175 (14.3%)	-	150

The results indicate excellent agreement with the established literature. The maximum percentage error remains below 14.3%, which is within the acceptable range for numerical aeroelastic simulations.

The evolution of the system’s eigenvalues with respect to airspeed is visualized in the V-g (Velocity vs. Damping) and V-f (Velocity vs. Frequency) diagrams presented in Figure 4.6.

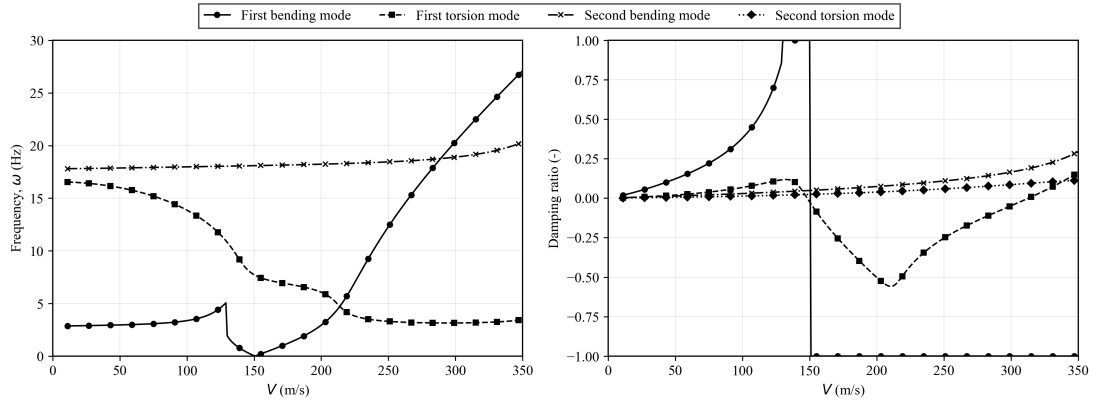


Figure 4.6: Aeroelastic stability plots for the baseline wing.

- **V-f Plot (Left):** This graph displays the variation of modal frequencies. It can be observed that the frequencies of the first bending and first torsion modes approach each other as the dynamic pressure increases. This frequency coalescence (merging) facilitates the energy transfer between the modes, triggering the classical bending-torsion flutter.
- **V-g Plot (Right):** This graph tracks the modal damping (g) of the first four aeroelastic modes. As the velocity increases, the aerodynamic damping of the first torsion mode (or the coupled mode) decreases rapidly. The flutter point is identified at $V = 148$ m/s, where the damping curve crosses the zero axis ($g = 0$) into the unstable positive region.

4.4.2. Verification of the Propeller Aerodynamic Method

The reference propeller selected for this verification is the 1/8-scale dynamic model described in Bennett & Bland's investigation. This configuration represents a typical four-bladed turboprop propeller. The key geometric and operational parameters used in the analysis are summarized in Table 4.4.

Table 4.4: Geometric and aerodynamic parameters of the reference propeller.

Parameter	Value	Unit
Radius (R)	10.1256	in
Diameter (D)	20.2512	in
Reference Chord (c_r)	4.3752	in
Aspect Ratio (A)	3.47	-
Lift Curve Slope (a_0)	2π	rad^{-1}
Max. Transonic Slope (a_M)	4π	rad^{-1}
Speed of Sound (a_∞)	1116	ft/s

The spanwise distribution of the blade geometric properties, including the non dimensional chord and local radius, is presented in Table 4.5.

Table 4.5: Blade geometry distribution of the reference propeller [4].

η	c/c_r	c (in)	r (in)
0.17 (η_0)	0.25	0.8750	1.7214
0.20	0.33	1.4438	2.0251
0.21	0.40	1.7501	2.1264
0.22	0.50	2.1876	2.2276
0.23	0.70	3.0626	2.3289
0.24	0.80	3.5002	2.4301
0.25	0.90	3.9377	2.5314
0.27	1.00	4.3752	2.7339
1.00	1.00	4.3752	10.1256

The reference blade angle for the windmilling propeller is a function of the advance ratio (J), which relates the forward speed (V) to the rotational velocity (Ω). The advance ratio is defined as $J = \pi V / (\Omega R)$, and alternatively as $\mu = J / \pi = V / (\Omega R)$. The variation of the blade angle at 75% radius ($\beta_{0.75R}$) with respect to J is illustrated in Figure 4.7. In the absence of reported RPM/velocity combinations in the reference study, a

typical rotational speed of 1800 RPM is assumed. This assumption is considered valid as the measurements were conducted at low Mach numbers. The experimental data points describing this relationship are approximated using a third-degree polynomial curve fit as follows:

$$J = 3.31 \times 10^{-5} \beta_{0.75R}^3 - 2.37 \times 10^{-3} \beta_{0.75R}^2 + 0.106 \beta_{0.75R} - 0.463 \quad (4.100)$$

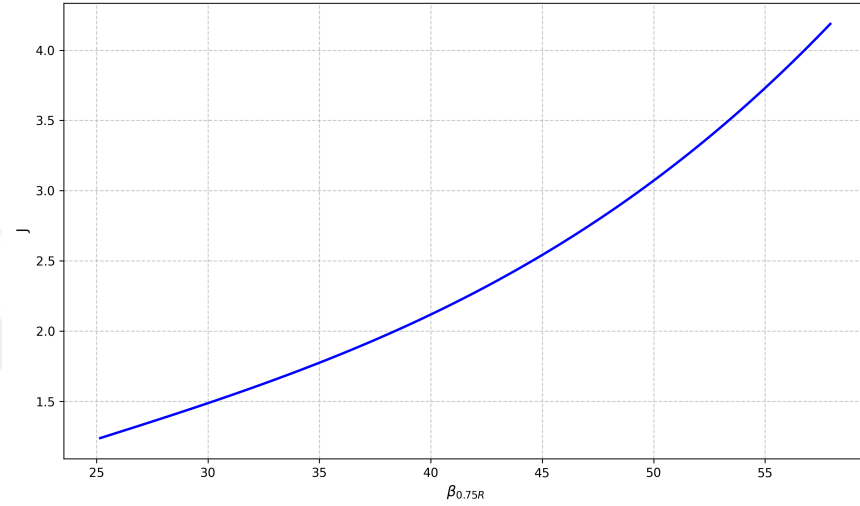


Figure 4.7: Curve fitting for the experimental data.

The aerodynamic derivatives are calculated for these blade angles. The specific flight conditions corresponding to select blade angles used for the calculation of aerodynamic coefficients are listed in Table 4.6.

Table 4.6: Flight condition data for the aerodynamic coefficients [4].

$\beta_{0.75R}$ (deg)	J	μ	V (ft/s)
25	1.25	0.3979	63.28
35	1.80	0.5730	91.13
46	2.65	0.8435	134.16
52	3.32	1.0568	168.08
58	4.20	1.3369	212.64

To validate the propeller aerodynamic model, the calculated unsteady aerodynamic

derivatives are compared with reference data. In the following comparative analysis, the first two columns represent values taken from the literature, the third column displays results from the present analytical approach, and the final column shows values computed using the prop.f FORTRAN code from MSC Nastran.

The comparison for the vertical force derivative with respect to pitch, $C_{Z\theta}$, is presented in Table 4.7.

Table 4.7: Comparison of $C_{Z\theta}$ derivative.

$\beta_{0.75R}$ (deg)	Ref. [26]	Ref. [5]	Analytic	NASTRAN
25	-0.331	-0.328	-0.33033	-0.33097
35	-0.418	-0.420	-0.41661	-0.41752
46	-0.512	-0.516	-0.51096	-0.51197
52	-0.566	-0.575	-0.56444	-0.56552
58	-0.619	-0.622	-0.61766	-0.61884

Similarly, Table 4.8 summarizes the results for the pitching moment derivative with respect to pitch, $C_{m\theta}$.

Table 4.8: Comparison of $C_{m\theta}$ derivative.

$\beta_{0.75R}$ (deg)	Ref. [26]	Ref. [5]	Analytic	NASTRAN
25	0.0400	0.040	0.039988	0.040002
35	0.0362	0.036	0.036182	0.036192
46	0.0304	0.030	0.030419	0.030426
52	0.0264	0.026	0.026430	0.026436
58	0.0221	0.022	0.022075	0.022081

The verification of the damping-related derivative C_{mq} is shown in Table 4.9.

Table 4.9: Comparison of C_{mq} derivative.

$\beta_{0.75R}$ (deg)	Ref. [26]	Ref. [5]	Analytic	NASTRAN
25	-0.2160	-0.219	-0.21573	-0.21609
35	-0.1391	-0.138	-0.13897	-0.13913
46	-0.0833	-0.082	-0.08321	-0.08332
52	-0.0603	-0.057	-0.06019	-0.06027
58	-0.0422	-0.042	-0.04212	-0.04218

Next, the lift due to yawing, $C_{Z\psi}$, is compared in Table 4.10.

Table 4.10: Comparison of $C_{Z\psi}$ derivative.

$\beta_{0.75R}$ (deg)	Ref. [26]	Ref. [5]	Analytic	NASTRAN
25	0.0877	0.087	0.087427	0.087646
35	0.1106	0.111	0.110294	0.110589
46	0.1320	0.133	0.131655	0.131957
52	0.1408	0.143	0.140512	0.140810
58	0.1461	0.147	0.145807	0.146096

Finally, the side moment due to yawing, $C_{m\psi}$, is validated in Table 4.11.

Table 4.11: Comparison of $C_{m\psi}$ derivative.

$\beta_{0.75R}$ (deg)	Ref. [26]	Ref. [5]	Analytic	NASTRAN
25	0.1506	0.152	0.150596	0.150648
35	0.1373	0.138	0.137240	0.137278
46	0.1188	0.119	0.118769	0.118800
52	0.1067	0.105	0.106638	0.106667
58	0.0937	0.094	0.093654	0.093682

The calculated aerodynamic derivatives (Analytic) exhibit excellent agreement with

both the benchmark literature [5, 26] and NASTRAN results across the entire range of blade pitch angles. This high degree of correlation validates the fidelity of the implemented propeller aerodynamic model.

4.4.3. Verification of the Isolated Propeller Model

4.4.3.1. Validation Case 1: Bennett and Bland Model

The validation of the present whirl flutter analysis is performed by comparing the computed flutter speeds and frequencies with the experimental measurements reported by Bennett and Bland [4]. In their study, Bennett and Bland conducted an extensive experimental campaign to determine the whirl flutter characteristics of an isolated, four-bladed windmilling propeller.

Two distinct configurations were investigated in these tests. The primary difference between these configurations lies in their structural characteristics, specifically the location of the system's elastic center (pivot point) relative to the propeller plane.

- **Configuration A:** The elastic center (pivot) is located at a distance of $0.346R$ from the rotor plane.
- **Configuration B:** The elastic center is positioned $0.691R$ from the rotor plane.

The structural and geometric characteristics of the model are summarized in Table 4.12.

Table 4.12: Summary of model characteristics [4].

Parameter	Symbol	Value
Polar moment of inertia	I_X	0.00858 slug-ft ²
Diametral moment of inertia ($l_0/R = 0.346$)	I_Y	0.0634 slug-ft ²
Diametral moment of inertia ($l_0/R = 0.691$)	I_Y	0.0937 slug-ft ²
Propeller radius	R	0.8438 ft
Blade chord at 75% radius	$b_{0.75R}$	0.1823 ft

The specific structural settings and flight conditions for the analyzed test cases are detailed in Table 4.13. The set of 13 cases covers a range of blade angles and structural frequencies for both configurations.

Table 4.13: Definition of the test cases for whirl flutter verification.

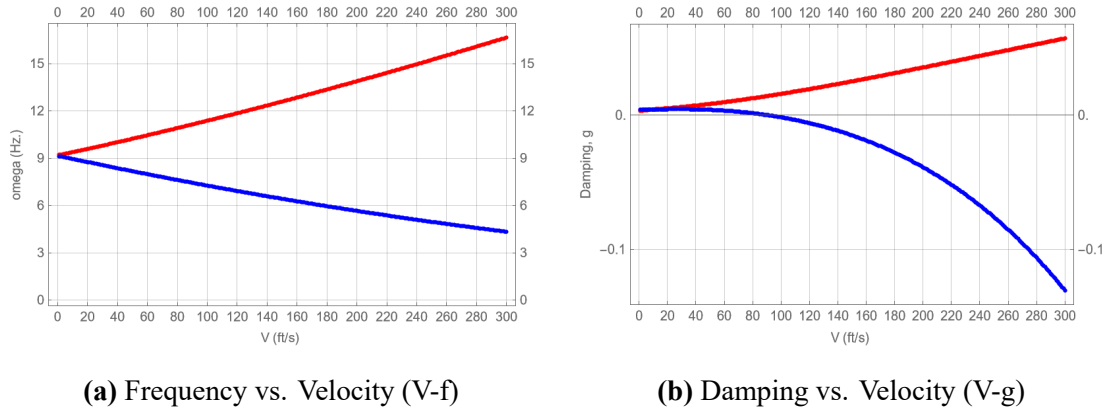
Conf.	Case	$\beta_{0.75R}$ (deg)	f_θ (Hz)	f_ψ (Hz)	g_θ	g_ψ	ρ (slug/ft ³)
A	1	35	9.20	9.12	0.0060	0.0090	0.00211
	2	35	10.96	10.96	0.0150	0.0238	0.00217
	3	46	10.96	10.96	0.0150	0.0238	0.00218
	4	52	10.96	10.96	0.0150	0.0238	0.00209
	5	58	10.96	10.96	0.0150	0.0238	0.00209
	6	58	10.71	10.28	0.0300	0.0393	0.00220
B	7	35	7.60	7.60	0.0056	0.0053	0.00227
	8	46	7.60	7.60	0.0056	0.0053	0.00226
	9	52	7.60	7.60	0.0056	0.0053	0.00226
	10	27.5	7.64	7.68	0.0060	0.0074	0.00226
	11	35	7.64	7.68	0.0060	0.0074	0.00220
	12	46	7.64	7.68	0.0060	0.0074	0.00220
	13	58	7.64	7.68	0.0060	0.0062	0.00220

A comparison of the predicted whirl flutter speeds and frequencies against the experimental data for both configurations is presented in Table 4.14.

Table 4.14: Comparison of whirl flutter speeds and frequencies with literature data.

Case	Experimental [4]		Present Study (Analytic)				Reference Study [5]			
	V_f (ft/s)	f_f (Hz)	V_f (ft/s)	Err (%)	f_f (Hz)	Err (%)	V_f (ft/s)	Err (%)	f_f (Hz)	Err (%)
1	120	6.88	90	25.0	7.44	-8.1	94	21.7	7.25	-5.4
2	159	7.58	142	10.7	8.30	-9.5	152	4.4	8.01	-5.7
3	167	8.30	156	6.6	8.86	-6.7	165	1.2	8.64	-4.1
4	186	8.63	170	8.6	9.03	-4.6	181	2.7	8.89	-3.0
5	210	8.75	186	11.4	9.17	-4.8	196	6.7	9.07	-3.7
6	231	8.10	217	6.1	8.34	-3.0	232	-0.4	8.23	-1.6
7	145	5.27	137	5.5	5.73	-8.7	100	31.0	6.18	-17.3
8	171	5.57	128	25.1	6.30	-13.1	115	32.7	6.41	-15.1
9	172	5.66	151	12.2	6.25	-10.4	131	23.8	6.47	-14.3
10	144	4.73	98	31.9	5.93	-25.4	100	30.6	5.88	-24.3
11	134	5.50	110	17.9	6.16	-12.0	106	20.9	6.15	-11.8
12	147	5.81	135	8.2	6.29	-8.3	123	16.3	6.39	-10.0
13	183	6.09	194	-6.0	6.05	0.7	157	14.2	6.49	-6.6

Representative whirl flutter stability results for Case 1 (Configuration A) are presented in Figure 4.8, showing the evolution of frequency and damping with airspeed. Detailed V-g and V-f plots for all analyzed cases (Case 1–13) are provided in Appendix B.

**Figure 4.8:** Whirl flutter stability results for Case 1 (Configuration A).

4.4.3.2. Validation Case 2: NASA X-57 Baseline Propeller

In addition to the Bennett and Bland model, a second isolated propeller model is verified. The baseline propeller is based on the wingtip-mounted propeller (cruise pro-

propeller) from the NASA X-57 Maxwell aircraft [5, 30]. It consists of a tractor propeller mounted on the wing's leading edge. The propeller is composed of a spinner, three blades, a motor, and a nacelle. The general parameters of the propeller are listed in Table 4.15.

Table 4.15: General propeller parameters for NASA X-57 validation case [5].

Parameter	Value	Units
Number of blades	3	[-]
Rotor diameter	5 (1.524)	ft (m)
Hub diameter	0.824 (0.25)	ft (m)
Motor diameter	14 (0.3556)	in (m)
Blade chord	0.308 (0.094)	ft (m)
Cruise RPM	2250	rev/min
Airfoil	MH117	[-]
Geometric collective pitch angle, $\beta_{0.75R}$	35	[°]
Advance ratio, J	1.96	[-]

Table 4.16: Structural parameters for NASA X-57 validation case [5].

Parameter	Value	Units
Rotor mass	8	kg
Motor-nacelle mass	35	kg
Position of rotor CG	-1.16	m
Position of motor CG	-0.86	m
Pitch axis = yaw axis		
$l_\theta = l_\psi$	0	m
$e_\theta = e_\psi$	-1.16	m
Pitch stiffness, f_θ	7	[Hz]
Yaw stiffness, f_ψ	7	[Hz]
Pitch damping, g_θ	0.005	[-]
Yaw damping, g_ψ	0.005	[-]
Polar moment of inertia, I_Ω	1.55	kg·m ²
Mom. of inertia, $I_{\theta,P}/I_{\psi,P}$	36.65	kg·m ²
Static unbalance, $S_{\theta,P}/S_{\alpha,P}$	-39.38	kg·m

The computed whirl flutter stability limits are compared with the reference results in Table 4.17. The present analytical model demonstrates excellent agreement with the reference data.

Table 4.17: Comparison of whirl flutter results for NASA X-57 validation case.

Parameter	Ref. [5]	Present Study	Error (%)
Whirl flutter speed (m/s)	221	223	0.90
Whirl flutter frequency (Hz)	5	4.98	-0.40
Rotational speed at flutter (RPM)	4433	4472	0.88

The V-g and V-f plots corresponding to the NASA X-57 validation case are illustrated in Figure 4.9.

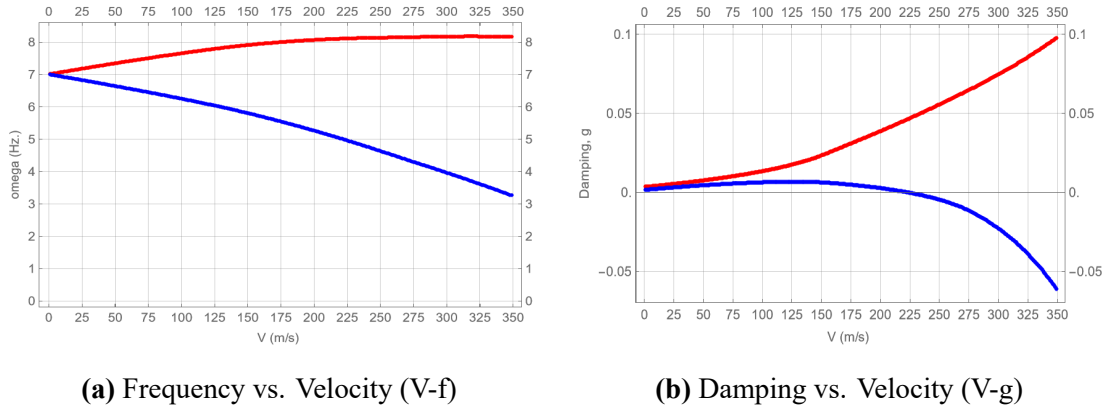


Figure 4.9: Whirl flutter stability results for NASA X-57 validation case.

4.4.4. Verification of Propeller-Wing Interaction (Coupled Model)

In this section, the coupling between the wing and the propeller is verified in a step-wise manner. The verification process is divided into three stages: first, the addition of a rigid propeller to the wing tip is analyzed to verify the effects of mass and inertia on structural frequencies and wing flutter limits (excluding propeller aerodynamics). Second, the influence of gyroscopic forces resulting from propeller rotation on the system modes is demonstrated. Finally, the fully coupled system with a flexibly mounted propeller is structurally verified, followed by a complete aeroelastic analysis.

For the coupled analysis, the propeller is positioned at a spanwise distance of 1.6 m from the wing root (Figure 4.10). The mass distribution is modeled using concentrated masses: the rotor assembly (blades and spinner) is located at the rotor plane center, 0.6 m upstream of the wing leading edge (corresponding to -1.16 m from the wing mid-chord axis), while the motor and nacelle assembly is positioned 0.3 m upstream of the leading edge (-0.86 m from the mid-chord axis). The pivot points for both pitch and yaw motions are aligned with the wing mid-chord axis.

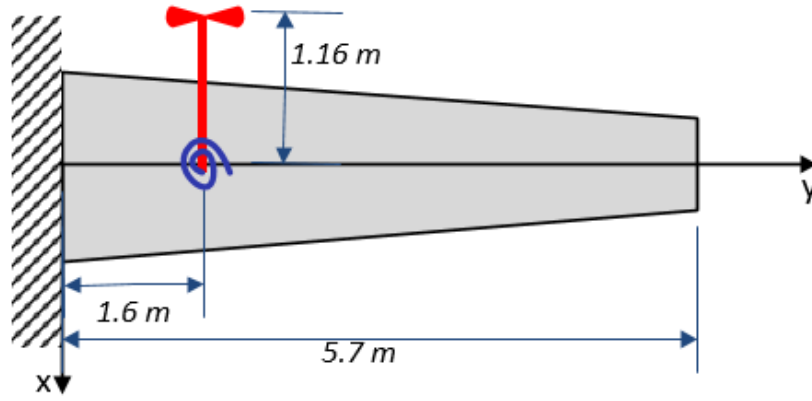


Figure 4.10: Configuration of the single propeller integrated on the flexible wing.

The propeller blades are characterized by a constant chord and a linearly varying twist distribution, representing a fixed-pitch propeller configuration. The whirl flutter analysis is conducted under windmilling conditions. For this regime, the relationship between the advance ratio (J) and the geometric collective pitch angle at the 75% radius station ($\beta_{0.75R}$) is approximated as:

$$J \approx \pi \tan(\beta_{0.75R} - 3^\circ) \quad (4.101)$$

where 3° represents the zero-lift angle of attack of the airfoil at this station.

In this first stage, the propeller is treated as a rigid mass positioned at a spanwise distance of 1.6 m from the wing root. The objective is to validate the structural modeling of the propeller's mass and inertia and their impact on the wing's dynamic characteristics. In the baseline wing configuration, the bending and torsion modes were dynamically uncoupled due to the coincidence of the elastic axis and the center of gravity. However, placing the propeller mass forward of the elastic axis introduces a significant static unbalance (offset). This configuration alters the mass distribution and forces the previously independent bending and torsional modes to couple intensely. The most striking effect of this inertial coupling is observed in the torsional modes, where the frequency drops by nearly half compared to the baseline wing, while the bending frequencies show only a slight decrease.

The resulting natural frequencies for this initial structural configuration are summarized in Table 4.18.

Table 4.18: Structural frequencies of the wing with rigid propeller.

Mode	Ref. Frequency (Hz) [5]	Present Frequency (Hz)	Error (%)
First wing mode	2.85	2.85	0.00
Second wing mode	8.19	8.39	2.44
Third wing mode	17.84	17.84	0.00
Fourth wing mode	26.23	27.27	3.97
Fifth wing mode	49.98	49.99	0.02

The effect of this addition on the wing's natural frequencies is presented in Table 4.18. To quantify the accuracy of this modeling approach, the obtained natural frequencies were compared with the reference data [5]. The results demonstrate excellent agreement, particularly for the first and third wing modes, which exhibited a 0.00% error, indicating an exact match with the literature. Although slight deviations were observed in the second (2.44%) and fourth (3.97%) modes, the overall correlation remains high. This confirms that the inertial effects of the propeller and the resulting modal coupling are modeled correctly.

The shift in modal characteristics observed in the structural analysis directly dictates the aeroelastic stability boundaries. The dramatic reduction in torsional frequency causes the bending and torsion modes to coalesce at a much lower airspeed compared to the baseline wing, resulting in a significant decrease in the critical flutter speed. Conversely, the divergence speed remains practically unaffected. This insensitivity arises because divergence is a static instability governed by the balance between elastic stiffness and aerodynamic moments, which is independent of the system's inertial mass distribution.

The calculated flutter and divergence limits for this configuration are summarized in Table 4.19.

Table 4.19: Aeroelastic stability results for the wing with rigid propeller.

Parameter	Ref. [5]	Present Study	Error (%)
Flutter Speed, V_f (m/s)	142	138	-2.82
Flutter Frequency, ω_f (rad/s)	6.30	6.22	-1.27
Divergence Speed, V_{div} (m/s)	153	150	-1.96

The critical flutter speed (V_f), flutter frequency (ω_f), and divergence speed (V_{div}) obtained from the analysis are presented in Table 4.19. The results indicate a high degree of accuracy in the aeroelastic modeling. The critical flutter speed was calculated as 138 m/s, corresponding to a deviation of -2.82% from the reference value. Similarly, the divergence speed was found to be 150 m/s with a relative error of -1.96%. These low error margins confirm that the present method correctly captures both the dynamic (flutter) and static (divergence) aeroelastic instabilities of the propeller-wing assembly. In the next verification stage, the propeller is flexibly mounted and allowed to rotate, considering the full coupling effects. The natural modes of the flexible mounts (pitch and yaw) couple with the wing's bending and torsional modes, modifying the system's natural vibration characteristics. The resulting natural frequencies are presented in Table 4.20.

The propeller pitch mode frequency is observed to decrease due to structural coupling with wing torsion. Conversely, since there are no in-plane motions of the wing modeled, the propeller yaw mode remains largely unchanged. Introducing a flexibly mounted propeller appears to slightly increase the clean wing's natural frequencies, with the exception of the first wing mode. However, when comparing these frequencies to the configuration with a rigidly mounted non-rotating propeller, a significant shift is observed; notably, the second wing mode frequency approximately doubles due to the introduction of flexible mounts, and other wing mode frequencies also show an increase.

Table 4.20: Structural frequencies of the flexibly mounted propeller-wing system.

Mode	Flexible Mounts (Hz)	Clean Wing & Isolated Prop. (Hz)	Rigid Mounts Non-rotating (Hz)
First wing mode	2.85	2.88	2.85
Propeller pitch mode	5.55	7	-
Propeller yaw mode	7	7	-
Second wing mode	17.75	16.68	8.39
Third wing mode	19.57	18.06	17.84
Fourth wing mode	49.26	46.48	27.26
Fifth wing mode	51.46	50.58	49.99

The influence of propeller rotation on the system's dynamic characteristics is further investigated by analyzing the modal frequencies across a rotational speed range of 0–3000 RPM. The evolution of these frequencies, illustrating the gyroscopic coupling effects, is presented in Figure 4.11. The results indicate that the wing's structural modes (1st through 5th) remain relatively stable, exhibiting minimal frequency variation. Specifically, the first wing mode fluctuates between approximately 2–3 Hz, the third mode between 20–27 Hz, and the fifth mode between 51–55 Hz.

In contrast, the propeller modes show distinct frequency trends driven by gyroscopic effects. The first propeller mode demonstrates a backward whirl behavior, where the natural frequency decreases from 6 Hz to 3 Hz as the rotational speed increases. Conversely, the second propeller mode exhibits a forward whirl characteristic, with its frequency increasing from 7 Hz to 15 Hz over the same speed range.

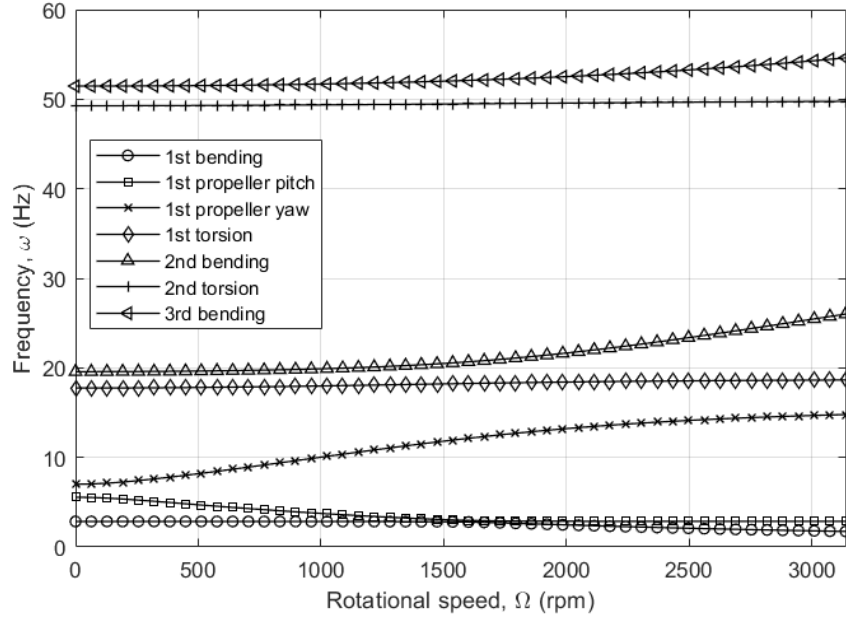


Figure 4.11: Campbell diagram of the system.

Following the establishment of the structural behavior and gyroscopic mode splitting of the flexible system in the preceding phase, the analysis advances to a comprehensive aeroelastic characterization. The current final phase of the coupled model has aerodynamic influence coefficients, associated with both the wing and the propeller, incorporated into the existing equations of gyroscopic motion. The stability boundaries are determined by resolving the complex eigenvalue problem for the system's complete matrices, denoted as $[M]$, $[C]$, $[K]$, across a range of airspeeds. An analysis was performed to provide information concerning the interactions of the whirl modes - already analyzed in a vacuum - and the aerodynamic environment. Specifically, the comprehensive nature of the coupled model allows for the simultaneous detection of distinct instability mechanisms, including classical wing bending-torsion flutter, propeller whirl flutter, and static divergence. The primary objective of this analysis is to monitor the evolution of all system eigenvalues to determine which of these competing phenomena becomes unstable first, thereby dictating the critical safety boundary of the aircraft.

The V-g and V-f diagrams, which characterize the aeroelastic stability boundaries and track the evolution of the system's eigenvalues, are presented in Figure 4.12.

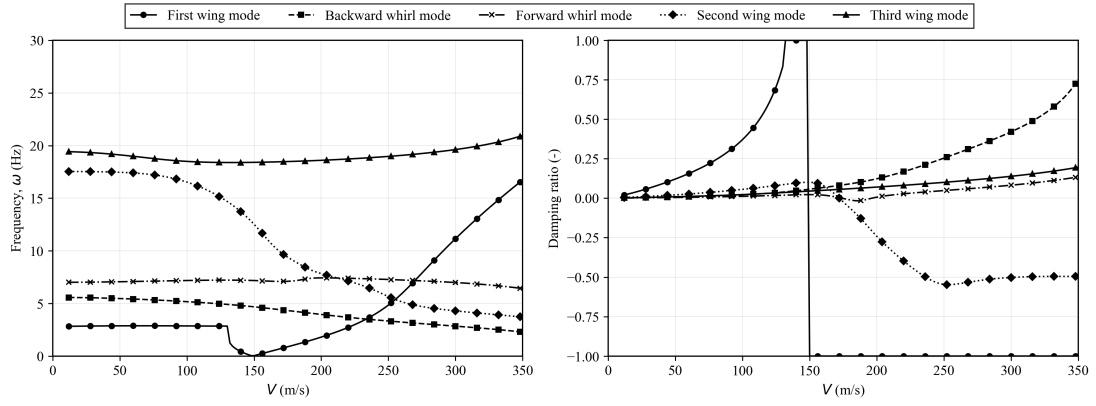


Figure 4.12: Aeroelastic stability results for the fully coupled system.

The critical speeds and frequencies for divergence, wing flutter, and whirl flutter are summarized and compared with the reference data in Table 4.21.

Table 4.21: Comparison of aeroelastic stability limits for the fully coupled system.

Parameter	Ref. [5]	Present Study	Error (%)
Divergence speed (m/s)	152	149	-1.97
Wing flutter speed (m/s)	174	172	-1.15
Wing flutter frequency (Hz)	9.48	9.65	1.79
Whirl flutter speed (m/s)	181	174	-3.87
Whirl flutter frequency (Hz)	7.17	7.09	-1.12

To investigate the aeroelastic implications of distributed propulsion systems, the modeling framework is extended to a dual-propeller configuration mounted on a single semi-span wing. This model integrates two distinct propeller-nacelle assemblies at different spanwise stations along the same flexible lifting surface. In this configuration, the cantilever wing supports two separate propulsion units. The integration follows the principle of superposition, where the inertial, aerodynamic, and gyroscopic contributions of each propeller are added to the global system matrices at their respective attachment nodes.

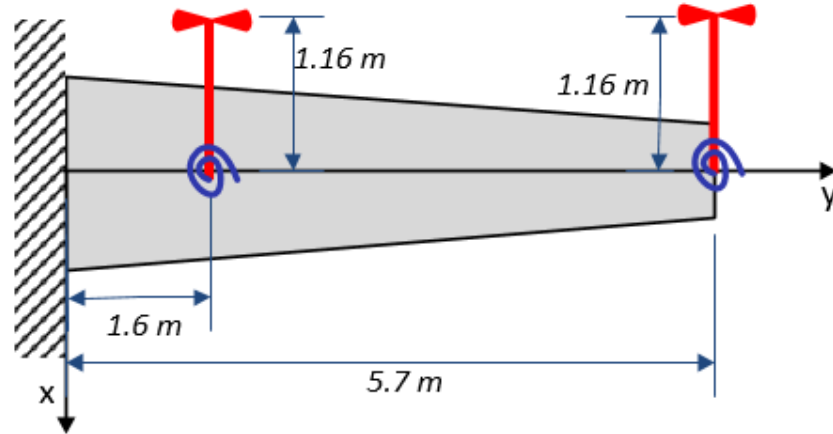


Figure 4.13: Configuration of the two propellers integrated on the flexible wing.

For this analysis, one propeller is positioned 1.60 m from the wing root, while the other is located at the wing tip (Figure 4.13). Both propellers possess the same characteristics as the baseline propeller, and the underlying structure is the standard baseline wing. Summary of structural frequencies for the wing-two propeller system are presented in Table 4.22.

Table 4.22: Structural frequencies of the wing-two propeller system.

Mode	Ref. Frequency (Hz) [5]	Present Frequency (Hz)	Error (%)
First wing mode	1.80	1.79	-0.56
First backward mode	5.39	5.43	0.74
Second backward mode	6.21	6.23	0.32
First forward mode	7	7	0.00
Second forward mode	7	7	0.00
Second wing mode	18.64	18.67	0.16
Third wing mode	26.71	26.78	0.26
Fourth wing mode	50.59	50.63	0.08
Fifth wing mode	54.12	54.15	0.06

The natural frequencies obtained from the free vibration analysis indicate a high level of agreement between the present model and the reference study [5]. The calculated

frequencies for the forward whirl modes showed no deviation (0.00% error), while the maximum relative error observed was limited to 0.74% for the first backward mode. The error rates for all other wing and whirl modes remained below 0.60%. These results confirm that the structural properties of the system have been modeled accurately. The evolution of eigenvalues for the wing-two propeller system is illustrated in Figure 4.14.

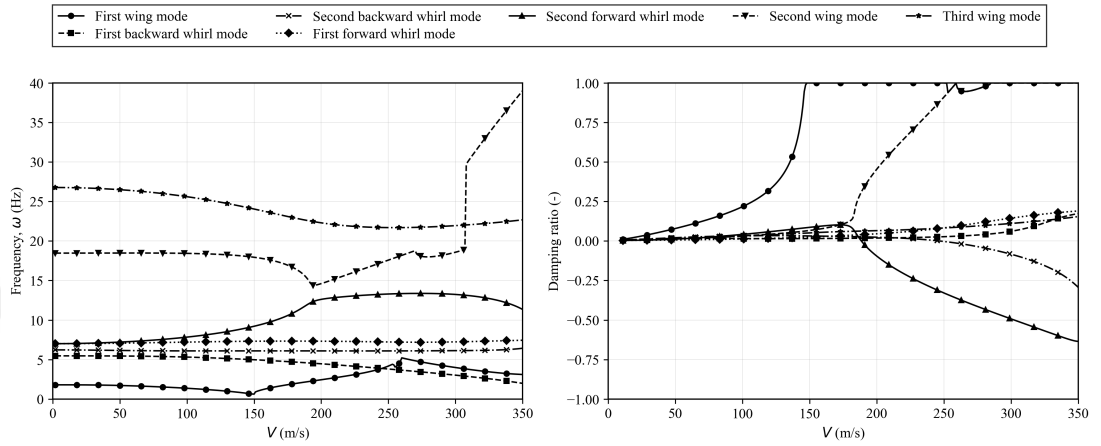


Figure 4.14: Aeroelastic stability results for the wing-two propeller system.

The aeroelastic stability boundaries are determined by tracking the evolution of the system eigenvalues. The V-g and V-f diagrams for the two-propeller configuration are presented in Figure 4.14. An examination of the wing modes reveals that neither divergence nor wing flutter occurred in the two-propeller model. However, two modes were observed to become unstable within the analyzed velocity range: the forward whirl mode and the backward whirl mode.

Specifically, forward whirl flutter arises from the interaction between the second forward whirl mode and the second wing mode, causing the second forward whirl mode to become unstable. Backward whirl flutter originates from the second backward whirl mode due to its interaction with both the first backward whirl mode and the first wing mode. The critical aeroelastic stability limits for this dual-propeller configuration are summarized in Table 4.23.

Table 4.23: Comparison of aeroelastic stability limits for the wing-two propeller system.

Parameter	Ref. [5]	Present Study	Error (%)
Whirl flutter speed (2nd forward whirl) (m/s)	184	196	6.52
Whirl flutter frequency (2nd forward whirl) (Hz)	12.62	12.45	-1.35
Whirl flutter speed (2nd backward whirl) (m/s)	251	244	-2.79
Whirl flutter frequency (2nd backward whirl) (Hz)	6.28	6.08	-3.18

Following the identification of these instability modes, a validation study was conducted to assess the accuracy of the predictions. As shown in Table 4.23, the calculated whirl flutter speeds and frequencies are in close agreement with those reported in Ref. [5]. Although a slight deviation of 6.52% exists in the forward whirl speed, the overall low error rates confirm that the present model effectively captures the aeroelastic behavior of the system.

5. CONCLUSION

5.1. Summary of the Study and Key Findings

This thesis presents a comprehensive analytical study for the aeroelastic analysis of distributed propulsion systems. To model the structure, Euler-Bernoulli beam theory, unsteady Theodorsen theory for wing aerodynamics, and Houbolt Reed theory modified to include unsteady effects for propeller aerodynamics were integrated using the Galerkin method. The system matrices were derived using the Mathematica Symbolic Programming Language, and eigenvalue analysis was performed in MATLAB. The developed in-house code was successfully validated with experimental and analytical data from the literature.

The work involved systematic step-by-step validation studies. First, the clean wing was validated, and the natural frequencies of the wing, as well as static and dynamic aeroelastic instability conditions, were examined. Subsequently, the aerodynamic derivatives of the propeller were validated using FORTRAN code, which is also used by the NASTRAN program. In the standalone whirl flutter analysis of the propeller-nacelle, the stability of the backward and forward coupled modes was examined as a function of speed. This analysis revealed that the forward whirl mode was unstable under no conditions. When the propeller-nacelle was rigidly attached to the wing, a decrease in the wing's bending and torsion natural frequencies was observed. When the propeller-nacelle system was attached flexibly and aerodynamic forces were neglected, it was observed that with the rotation of the propeller and gyroscopic effects, the natural frequencies of the bending modes remained almost constant, but the torsion modes showed a slight increase. Subsequently, in the aeroelastic analysis, it was observed that there was no significant change in the divergence speed. In this analysis, it was observed that the forward whirl mode passed the unstable point. It was observed that the forward whirl mode occurred before the wing flutter, indicating that whirl flutter is critical.

5.2. Contributions and Future Work

Based on the findings and limitations of this study, the following areas are identified for future research to enhance the analysis of propeller-wing systems:

- **Integration into Full-Aircraft Model:** The current research uses a cantilevered wing model. Future studies should expand this to a full-aircraft aeroelastic model. Including rigid body modes will enable the investigation of the coupling between flight mechanics and aeroelastic stability.
- **Elastic Blade Modeling:** In this study, propeller blades are modeled as rigid bodies. To improve simulation fidelity, future work should incorporate elastic modeling of the blades. This would capture the interaction between blade and wing structural modes, providing more accurate stability boundaries.
- **Thrust Effects (Powered Conditions):** The analyses presented here are conducted under wind-milling conditions. Future research should extend the model to include thrust generation. Modeling thrust as a follower force is essential to understand its stabilizing or destabilizing effects on wing dynamics during powered flight.
- **Design Optimization:** Since the spanwise location of the propeller significantly affects aeroelastic behavior, design optimization studies should be conducted. The goal is to determine the optimal placement of propulsion units to maximize critical flutter speed and expand the flight envelope.

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BIOGRAPHY

Yunus Emre Dogru graduated from Samsun University with a degree in Aerospace Engineering in 2023. Currently, he is a master's student in Mechanical Engineering at Gebze Technical University and works as a research assistant in the Department of Aeronautical Engineering. His research focuses on aeroelasticity and structural dynamics.



PUBLICATIONS AND PRESENTATIONS FROM THE THESIS

[1] Doğru Y. E., Küçükali M., Eken A., Eken S., (2025), “Aeroelastic Analysis of Flexible Wings with Distributed Electric Propulsion”, IMECE2025, Memphis, TN, USA.



APPENDICES

APPENDIX-A: SYSTEM MATRICES

In this appendix, the explicit forms of the system mass, damping, gyroscopic, and stiffness matrices derived for the multi-propeller aeroelastic model are presented. These matrices govern the coupled dynamics of the elastic wing and the rotating propeller systems.

Mass Matrix Elements (M_{ij})

$$\begin{aligned} M_{11} &= N_{\theta 1}(I_{\theta P} N_{\theta 1} N_{\theta T1}) \\ M_{12} &= 0 \\ M_{13} &= 0 \\ M_{14} &= 0 \\ M_{15} &= N_{\theta 1}(N_{\theta 1} N_{yPT1} S_{\theta P}) \\ M_{16} &= N_{\theta 1}(I_{\theta \phi P} N_{\phi PT1} N_{\theta 1}) \\ M_{21} &= 0 \\ M_{22} &= N_{\psi 1}(I_{\psi P} N_{\psi 1} N_{\psi T1}) \\ M_{23} &= 0 \\ M_{24} &= 0 \\ M_{25} &= 0 \\ M_{26} &= 0 \\ M_{31} &= 0 \\ M_{32} &= 0 \\ M_{33} &= N_{\theta 2}(I_{\theta P} N_{\theta 2} N_{\theta T2}) \\ M_{34} &= 0 \\ M_{35} &= N_{\theta 2}(N_{\theta 2} N_{yPT2} S_{\theta P}) \\ M_{36} &= N_{\theta 2}(I_{\theta \phi P} N_{\phi PT2} N_{\theta 2}) \\ M_{41} &= 0 \\ M_{42} &= 0 \\ M_{43} &= 0 \\ M_{44} &= N_{\psi 2}(I_{\psi P} N_{\psi 2} N_{\psi T2}) \\ M_{45} &= 0 \\ M_{46} &= 0 \\ M_{51} &= N_{yP1}(N_{\theta T1} S_{\theta P}) \\ M_{52} &= 0 \\ M_{53} &= N_{yP2}(N_{\theta T2} S_{\theta P}) \\ M_{54} &= 0 \\ M_{55} &= N_{yP1} N_{yPT1}(M_P) + N_{yP2} N_{yPT2}(M_P) \end{aligned}$$

$$\begin{aligned}
& + \int_0^L N_y (M_P (N_{yPT1} + N_{yPT2}) + m_w N_{yT}) dy \\
M_{56} &= N_{yP1} N_{\phi PT1} (S_{\phi P}) + N_{yP2} N_{\phi PT2} (S_{\phi P}) \\
& + \int_0^L N_y (N_{\phi PT1} S_{\phi P} + N_{\phi PT2} S_{\phi P} + m_w N_{\phi T} x_{ab}) dy \\
M_{61} &= N_{\phi P1} (I_{\theta \phi P} N_{\theta T1}) \\
M_{62} &= 0 \\
M_{63} &= N_{\phi P2} (I_{\theta \phi P} N_{\theta T2}) \\
M_{64} &= 0 \\
M_{65} &= N_{\phi P1} N_{yPT1} (S_{\phi P}) + N_{\phi P2} N_{yPT2} (S_{\phi P}) \\
& + \int_0^L N_{\phi} (N_{yPT1} S_{\phi P} + N_{yPT2} S_{\phi P} + m_w N_{yT} x_{ab}) dy \\
M_{66} &= N_{\phi P1} N_{\phi PT1} (I_{\phi P}) + N_{\phi P2} N_{\phi PT2} (I_{\phi P}) \\
& + \int_0^L N_{\phi} (I_{\phi P} (N_{\phi PT1} + N_{\phi PT2}) + m_w N_{\phi T} x_{ab}^2) dy
\end{aligned}$$

Damping/Gyroscopic Matrix Elements (C_{ij})

$$\begin{aligned}
C_{11} &= N_{\theta 1} \left(\frac{g_{\theta} K_{\theta} N_{\theta 1} N_{\theta T1}}{\omega} \right) \\
C_{12} &= N_{\theta 1} (I_{\Omega} N_{\psi T1} N_{\theta 1} \Omega) \\
C_{13} &= 0 \\
C_{14} &= 0 \\
C_{15} &= 0 \\
C_{16} &= 0 \\
C_{21} &= N_{\psi 1} (-I_{\Omega} N_{\psi 1} N_{\theta T1} \Omega) \\
C_{22} &= N_{\psi 1} \left(\frac{g_{\psi} K_{\psi} N_{\psi 1} N_{\psi T1}}{\omega} \right) \\
C_{23} &= 0 \\
C_{24} &= 0 \\
C_{25} &= 0 \\
C_{26} &= N_{\psi 1} (-I_{\Omega} N_{\phi PT1} N_{\psi 1} \Omega) \\
C_{31} &= 0 \\
C_{32} &= 0 \\
C_{33} &= N_{\theta 2} \left(\frac{g_{\theta} K_{\theta} N_{\theta 2} N_{\theta T2}}{\omega} \right) \\
C_{34} &= N_{\theta 2} (I_{\Omega} N_{\psi T2} N_{\theta 2} \Omega) \\
C_{35} &= 0 \\
C_{36} &= 0 \\
C_{41} &= 0 \\
C_{42} &= 0
\end{aligned}$$

$$\begin{aligned}
C_{43} &= N_{\psi 2}(-I_{\Omega} N_{\psi 2} N_{\theta T 2} \Omega) \\
C_{44} &= N_{\psi 2} \left(\frac{g_{\psi} K_{\psi} N_{\psi 2} N_{\psi T 2}}{\omega} \right) \\
C_{45} &= 0 \\
C_{46} &= N_{\psi 2}(-I_{\Omega} N_{\phi P T 2} N_{\psi 2} \Omega) \\
C_{51} &= 0 \\
C_{52} &= 0 \\
C_{53} &= 0 \\
C_{54} &= 0 \\
C_{55} &= 0 \\
C_{56} &= 0 \\
C_{61} &= 0 \\
C_{62} &= N_{\phi P 1}(I_{\Omega} N_{\psi T 1} \Omega) \\
C_{63} &= 0 \\
C_{64} &= N_{\phi P 2}(I_{\Omega} N_{\psi T 2} \Omega) \\
C_{65} &= 0 \\
C_{66} &= 0
\end{aligned}$$

Stiffness Matrix Elements (K_{ij})

$$\begin{aligned}
K_{11} &= N_{\theta 1}(K_{\theta} N_{\theta 1} N_{\theta T 1}) \\
K_{12} &= 0 \\
K_{13} &= 0 \\
K_{14} &= 0 \\
K_{15} &= 0 \\
K_{16} &= 0 \\
K_{21} &= 0 \\
K_{22} &= N_{\psi 1}(K_{\psi} N_{\psi 1} N_{\psi T 1}) \\
K_{23} &= 0 \\
K_{24} &= 0 \\
K_{25} &= 0 \\
K_{26} &= 0 \\
K_{31} &= 0 \\
K_{32} &= 0 \\
K_{33} &= N_{\theta 2}(K_{\theta} N_{\theta 2} N_{\theta T 2}) \\
K_{34} &= 0 \\
K_{35} &= 0 \\
K_{36} &= 0 \\
K_{41} &= 0 \\
K_{42} &= 0
\end{aligned}$$

$$\begin{aligned}
K_{43} &= 0 \\
K_{44} &= N_{\psi 2}(K_{\psi} N_{\psi 2} N_{\psi T 2}) \\
K_{45} &= 0 \\
K_{46} &= 0 \\
K_{51} &= 0 \\
K_{52} &= 0 \\
K_{53} &= 0 \\
K_{54} &= 0 \\
K_{55} &= \int_0^L N_y(EI \cdot N_{yT}'''') dy \\
K_{56} &= 0 \\
K_{61} &= 0 \\
K_{62} &= 0 \\
K_{63} &= 0 \\
K_{64} &= 0 \\
K_{65} &= 0 \\
K_{66} &= \int_0^L N_{\phi}(-GJ \cdot N_{\phi T}'') dy
\end{aligned}$$

Aerodynamic Mass Matrix Elements ($M_{ae,ij}$)

$$\begin{aligned}
M_{ae,11} &= N_{\theta 1} \left(\frac{e_{\theta} N_{\theta T 1} \pi q_{\infty} R^3 (c_{zq} e_{\theta} + 2c_{mq} R)}{V_{\infty}^2} \right) \\
M_{ae,12} &= N_{\theta 1} \left(\frac{e_{\psi} N_{\psi T 1} \pi q_{\infty} R^3 (c_{zr} e_{\theta} + 2c_{mr} R)}{V_{\infty}^2} \right) \\
M_{ae,13} &= 0 \\
M_{ae,14} &= 0 \\
M_{ae,15} &= N_{\theta 1} \left(\frac{N_{yPT1} \pi q_{\infty} R^3 (c_{zq} e_{\theta} + 2c_{mq} R)}{V_{\infty}^2} \right) \\
M_{ae,16} &= N_{\theta 1} \left(\frac{e_{\phi} N_{\phi PT1} \pi q_{\infty} R^3 (c_{zq} e_{\theta} + 2c_{mq} R)}{V_{\infty}^2} \right) \\
M_{ae,21} &= N_{\psi 1} \left(\frac{e_{\theta} N_{\theta T 1} \pi q_{\infty} R^3 (-c_{yq} e_{\psi} + 2c_{nq} R)}{V_{\infty}^2} \right) \\
M_{ae,22} &= N_{\psi 1} \left(\frac{e_{\psi} N_{\psi T 1} \pi q_{\infty} R^3 (-c_{yr} e_{\psi} + 2c_{nr} R)}{V_{\infty}^2} \right) \\
M_{ae,23} &= 0 \\
M_{ae,24} &= 0 \\
M_{ae,25} &= N_{\psi 1} \left(\frac{N_{yPT1} \pi q_{\infty} R^3 (-c_{yq} e_{\psi} + 2c_{nq} R)}{V_{\infty}^2} \right) \\
M_{ae,26} &= N_{\psi 1} \left(\frac{e_{\phi} N_{\phi PT1} \pi q_{\infty} R^3 (-c_{yq} e_{\psi} + 2c_{nq} R)}{V_{\infty}^2} \right)
\end{aligned}$$

$$\begin{aligned}
M_{ae,31} &= 0 \\
M_{ae,32} &= 0 \\
M_{ae,33} &= N_{\theta 2} \left(\frac{e_{\theta} N_{\theta T 2} \pi q_{\infty} R^3 (c_{zq} e_{\theta} + 2c_{mq} R)}{V_{\infty}^2} \right) \\
M_{ae,34} &= N_{\theta 2} \left(\frac{e_{\psi} N_{\psi T 2} \pi q_{\infty} R^3 (c_{zr} e_{\theta} + 2c_{mr} R)}{V_{\infty}^2} \right) \\
M_{ae,35} &= N_{\theta 2} \left(\frac{N_{yPT 2} \pi q_{\infty} R^3 (c_{zq} e_{\theta} + 2c_{mq} R)}{V_{\infty}^2} \right) \\
M_{ae,36} &= N_{\theta 2} \left(\frac{e_{\phi} N_{\phi PT 2} \pi q_{\infty} R^3 (c_{zq} e_{\theta} + 2c_{mq} R)}{V_{\infty}^2} \right) \\
M_{ae,41} &= 0 \\
M_{ae,42} &= 0 \\
M_{ae,43} &= N_{\psi 2} \left(\frac{e_{\theta} N_{\theta T 2} \pi q_{\infty} R^3 (-c_{yq} e_{\psi} + 2c_{nq} R)}{V_{\infty}^2} \right) \\
M_{ae,44} &= N_{\psi 2} \left(\frac{e_{\psi} N_{\psi T 2} \pi q_{\infty} R^3 (-c_{yr} e_{\psi} + 2c_{nr} R)}{V_{\infty}^2} \right) \\
M_{ae,45} &= N_{\psi 2} \left(\frac{N_{yPT 2} \pi q_{\infty} R^3 (-c_{yq} e_{\psi} + 2c_{nq} R)}{V_{\infty}^2} \right) \\
M_{ae,46} &= N_{\psi 2} \left(\frac{e_{\phi} N_{\phi PT 2} \pi q_{\infty} R^3 (-c_{yq} e_{\psi} + 2c_{nq} R)}{V_{\infty}^2} \right) \\
M_{ae,51} &= 0 \\
M_{ae,52} &= 0 \\
M_{ae,53} &= 0 \\
M_{ae,54} &= 0 \\
M_{ae,55} &= \int_0^L N_y N_{yT} (-b^2 \pi \rho) dy \\
M_{ae,56} &= \int_0^L N_y N_{\phi T} (ab^3 \pi \rho) dy \\
M_{ae,61} &= 0 \\
M_{ae,62} &= 0 \\
M_{ae,63} &= 0 \\
M_{ae,64} &= 0 \\
M_{ae,65} &= \int_0^L N_{\phi} N_{yT} (ab^3 \pi \rho) dy \\
M_{ae,66} &= \int_0^L N_{\phi} N_{\phi T} \left(-\frac{1}{8} (1 + 8a^2) b^4 \pi \rho \right) dy
\end{aligned}$$

Aerodynamic Damping Matrix Elements ($C_{ae,ij}$)

$$C_{ae,11} = \frac{N_{\theta 1}}{V_{\infty}} (N_{\theta T 1} \pi q_{\infty} R^2 (c_{z\theta} e_{\theta}^2 + R(2c_{m\theta} e_{\theta} + c_{zq} e_{\theta} + 2c_{mq} R)))$$

$$\begin{aligned}
C_{ae,12} &= \frac{N_{\theta 1}}{V_{\infty}} (N_{\psi T 1} \pi q_{\infty} R^2 (c_{z\psi} e_{\psi} e_{\theta} + R(2c_{m\psi} e_{\psi} + c_{zr} e_{\theta} + 2c_{mr} R))) \\
C_{ae,13} &= 0 \\
C_{ae,14} &= 0 \\
C_{ae,15} &= \frac{N_{\theta 1}}{V_{\infty}} (N_{yPT1} \pi q_{\infty} R^2 (c_{z\theta} e_{\theta} + 2c_{m\theta} R)) \\
C_{ae,16} &= \frac{N_{\theta 1}}{V_{\infty}} (N_{\phi PT1} \pi q_{\infty} R^2 (c_{z\theta} e_{\phi} e_{\theta} + R(2c_{m\theta} e_{\phi} + c_{zq} e_{\theta} + 2c_{mq} R))) \\
C_{ae,21} &= \frac{N_{\psi 1}}{V_{\infty}} (N_{\theta T 1} \pi q_{\infty} R^2 (-c_{y\theta} e_{\psi} e_{\theta} + R(-c_{yq} e_{\psi} + 2c_{n\theta} e_{\theta} + 2c_{nq} R))) \\
C_{ae,22} &= \frac{N_{\psi 1}}{V_{\infty}} (N_{\psi T 1} \pi q_{\infty} R^2 (-c_{y\psi} e_{\psi}^2 + R(2c_{n\psi} e_{\psi} - c_{yr} e_{\psi} + 2c_{nr} R))) \\
C_{ae,23} &= 0 \\
C_{ae,24} &= 0 \\
C_{ae,25} &= \frac{N_{\psi 1}}{V_{\infty}} (N_{yPT1} \pi q_{\infty} R^2 (-c_{y\theta} e_{\psi} + 2c_{n\theta} R)) \\
C_{ae,26} &= \frac{N_{\psi 1}}{V_{\infty}} (N_{\phi PT1} \pi q_{\infty} R^2 (-c_{y\theta} e_{\phi} e_{\psi} + R(2c_{n\theta} e_{\phi} - c_{yq} e_{\psi} + 2c_{nq} R))) \\
C_{ae,31} &= 0 \\
C_{ae,32} &= 0 \\
C_{ae,33} &= \frac{N_{\theta 2}}{V_{\infty}} (N_{\theta T 2} \pi q_{\infty} R^2 (c_{z\theta} e_{\theta}^2 + R(2c_{m\theta} e_{\theta} + c_{zq} e_{\theta} + 2c_{mq} R))) \\
C_{ae,34} &= \frac{N_{\theta 2}}{V_{\infty}} (N_{\psi T 2} \pi q_{\infty} R^2 (c_{z\psi} e_{\psi} e_{\theta} + R(2c_{m\psi} e_{\psi} + c_{zr} e_{\theta} + 2c_{mr} R))) \\
C_{ae,35} &= \frac{N_{\theta 2}}{V_{\infty}} (N_{yPT2} \pi q_{\infty} R^2 (c_{z\theta} e_{\theta} + 2c_{m\theta} R)) \\
C_{ae,36} &= \frac{N_{\theta 2}}{V_{\infty}} (N_{\phi PT2} \pi q_{\infty} R^2 (c_{z\theta} e_{\phi} e_{\theta} + R(2c_{m\theta} e_{\phi} + c_{zq} e_{\theta} + 2c_{mq} R))) \\
C_{ae,41} &= 0 \\
C_{ae,42} &= 0 \\
C_{ae,43} &= \frac{N_{\psi 2}}{V_{\infty}} (N_{\theta T 2} \pi q_{\infty} R^2 (-c_{y\theta} e_{\psi} e_{\theta} + R(-c_{yq} e_{\psi} + 2c_{n\theta} e_{\theta} + 2c_{nq} R))) \\
C_{ae,44} &= \frac{N_{\psi 2}}{V_{\infty}} (N_{\psi T 2} \pi q_{\infty} R^2 (-c_{y\psi} e_{\psi}^2 + R(2c_{n\psi} e_{\psi} - c_{yr} e_{\psi} + 2c_{nr} R))) \\
C_{ae,45} &= \frac{N_{\psi 2}}{V_{\infty}} (N_{yPT2} \pi q_{\infty} R^2 (-c_{y\theta} e_{\psi} + 2c_{n\theta} R)) \\
C_{ae,46} &= \frac{N_{\psi 2}}{V_{\infty}} (N_{\phi PT2} \pi q_{\infty} R^2 (-c_{y\theta} e_{\phi} e_{\psi} + R(2c_{n\theta} e_{\phi} - c_{yq} e_{\psi} + 2c_{nq} R))) \\
C_{ae,51} &= 0 \\
C_{ae,52} &= 0 \\
C_{ae,53} &= 0 \\
C_{ae,54} &= 0 \\
C_{ae,55} &= \int_0^L N_y N_{yT} (-bC(k) c_L \rho V) dy
\end{aligned}$$

$$\begin{aligned}
C_{ae,56} &= \int_0^L N_y N_{\phi T} \left(\frac{b^2((-1+2a)C(k)c_L - 2\pi)\rho V}{2} \right) dy \\
C_{ae,61} &= 0 \\
C_{ae,62} &= 0 \\
C_{ae,63} &= 0 \\
C_{ae,64} &= 0 \\
C_{ae,65} &= \int_0^L N_{\phi} N_{yT} \left(\frac{(1+2a)b^2 C(k)c_L \rho V}{2} \right) dy \\
C_{ae,66} &= \int_0^L N_{\phi} N_{\phi T} \cdot \left(\frac{b^3 C(k)c_L \rho V V_{\infty} - 4a^2 b^3 C(k)c_L \rho V V_{\infty} - 2b^3 \pi \rho V V_{\infty} + 4ab^3 \pi \rho V V_{\infty}}{4V_{\infty}} \right) dy
\end{aligned}$$

Aerodynamic Stiffness Matrix Elements ($K_{ae,ij}$)

$$\begin{aligned}
K_{ae,11} &= N_{\theta 1}(N_{\theta T 1} \pi q_{\infty} R^2 (c_{z\theta} e_{\theta} + 2c_{m\theta} R)) \\
K_{ae,12} &= N_{\theta 1}(N_{\psi T 1} \pi q_{\infty} R^2 (c_{z\psi} e_{\theta} + 2c_{m\psi} R)) \\
K_{ae,13} &= 0 \\
K_{ae,14} &= 0 \\
K_{ae,15} &= 0 \\
K_{ae,16} &= N_{\theta 1}(N_{\phi PT 1} \pi q_{\infty} R^2 (c_{z\theta} e_{\theta} + 2c_{m\theta} R)) \\
K_{ae,21} &= N_{\psi 1}(N_{\theta T 1} \pi q_{\infty} R^2 (-c_{y\theta} e_{\psi} + 2c_{n\theta} R)) \\
K_{ae,22} &= N_{\psi 1}(N_{\psi T 1} \pi q_{\infty} R^2 (-c_{y\psi} e_{\psi} + 2c_{n\psi} R)) \\
K_{ae,23} &= 0 \\
K_{ae,24} &= 0 \\
K_{ae,25} &= 0 \\
K_{ae,26} &= N_{\psi 1}(N_{\phi PT 1} \pi q_{\infty} R^2 (-c_{y\theta} e_{\psi} + 2c_{n\theta} R)) \\
K_{ae,31} &= 0 \\
K_{ae,32} &= 0 \\
K_{ae,33} &= N_{\theta 2}(N_{\theta T 2} \pi q_{\infty} R^2 (c_{z\theta} e_{\theta} + 2c_{m\theta} R)) \\
K_{ae,34} &= N_{\theta 2}(N_{\psi T 2} \pi q_{\infty} R^2 (c_{z\psi} e_{\theta} + 2c_{m\psi} R)) \\
K_{ae,35} &= 0 \\
K_{ae,36} &= N_{\theta 2}(N_{\phi PT 2} \pi q_{\infty} R^2 (c_{z\theta} e_{\theta} + 2c_{m\theta} R)) \\
K_{ae,41} &= 0 \\
K_{ae,42} &= 0 \\
K_{ae,43} &= N_{\psi 2}(N_{\theta T 2} \pi q_{\infty} R^2 (-c_{y\theta} e_{\psi} + 2c_{n\theta} R)) \\
K_{ae,44} &= N_{\psi 2}(N_{\psi T 2} \pi q_{\infty} R^2 (-c_{y\psi} e_{\psi} + 2c_{n\psi} R)) \\
K_{ae,45} &= 0 \\
K_{ae,46} &= N_{\psi 2}(N_{\phi PT 2} \pi q_{\infty} R^2 (-c_{y\theta} e_{\psi} + 2c_{n\theta} R)) \\
K_{ae,51} &= 0 \\
K_{ae,52} &= 0
\end{aligned}$$

$$\begin{aligned}
K_{ae,53} &= 0 \\
K_{ae,54} &= 0 \\
K_{ae,55} &= 0 \\
K_{ae,56} &= \int_0^L N_y N_{\phi T} (-bC(k)c_L \rho V^2) dy \\
K_{ae,61} &= 0 \\
K_{ae,62} &= 0 \\
K_{ae,63} &= 0 \\
K_{ae,64} &= 0 \\
K_{ae,65} &= 0 \\
K_{ae,66} &= \int_0^L N_{\phi} N_{\phi T} \left(\frac{(1+2a)b^2 C(k)c_L \rho V^2}{2} \right) dy
\end{aligned}$$

Combined System Matrices

The final system matrices used in the flutter analysis, combining the structural (M_s, C_s, K_s) and aerodynamic terms, are defined as:

$$[\overline{M}] = [M_s] \quad (\text{A.1})$$

$$[\overline{C}] = [C_s] - \frac{\text{Im}([A_{total}])}{kV/b_r} \quad (\text{A.2})$$

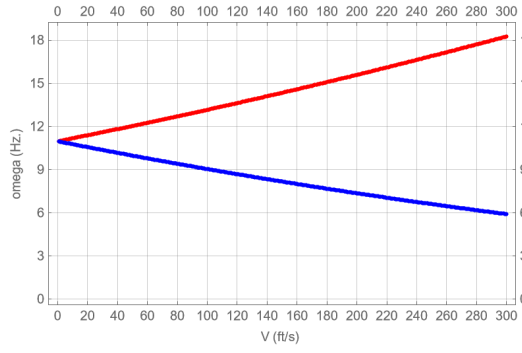
$$[\overline{K}] = [K_s] - \text{Re}([A_{total}]) \quad (\text{A.3})$$

where the total aerodynamic matrix $[A_{total}]$ is constructed from the aerodynamic mass, damping, and stiffness matrices evaluated at a specific reduced frequency k :

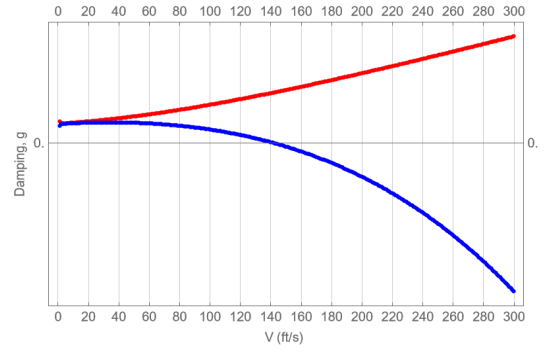
$$[A_{total}] = -k^2[M_{ae}] + ik[C_{ae}] + [K_{ae}] \quad (\text{A.4})$$

APPENDIX-B: WHIRL FLUTTER V-G AND V-F PLOTS

Detailed frequency and damping plots for the analyzed whirl flutter test cases (Cases 2–13) are presented in this appendix.

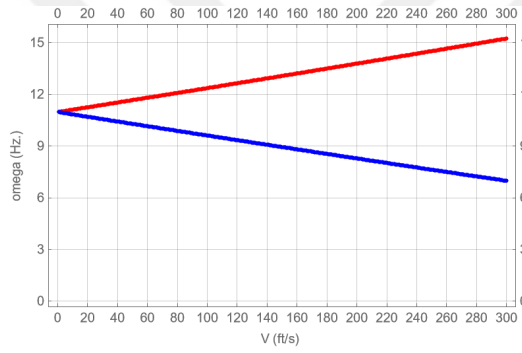


(a) Case 2 - Frequency

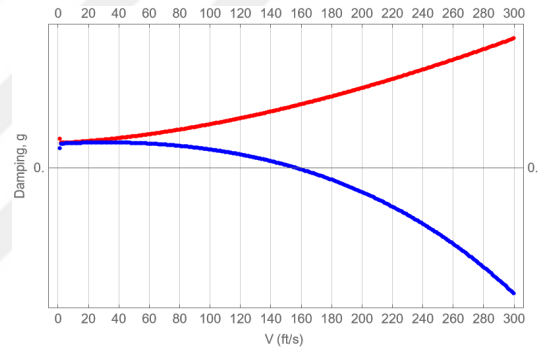


(b) Case 2 - Damping

Figure B.1: Whirl flutter results for Case 2.

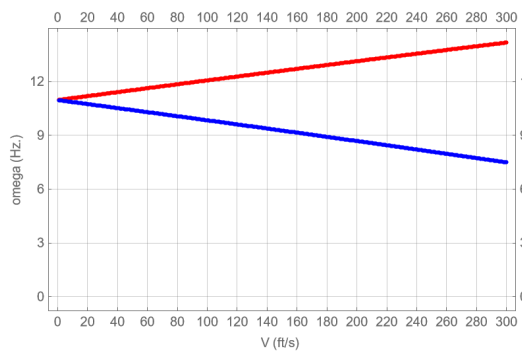


(a) Case 3 - Frequency

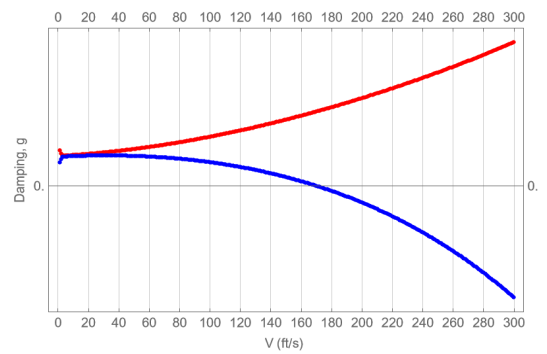


(b) Case 3 - Damping

Figure B.2: Whirl flutter results for Case 3.

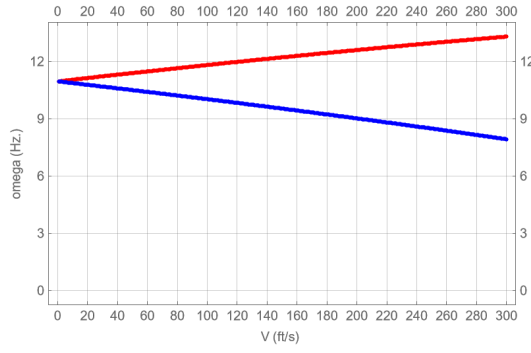


(a) Case 4 - Frequency

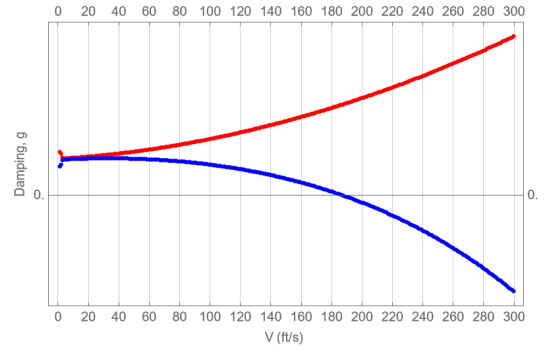


(b) Case 4 - Damping

Figure B.3: Whirl flutter results for Case 4.

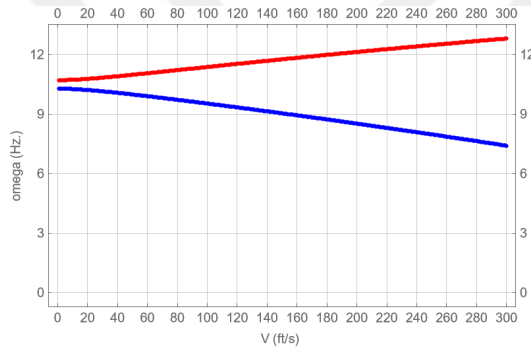


(a) Case 5 - Frequency

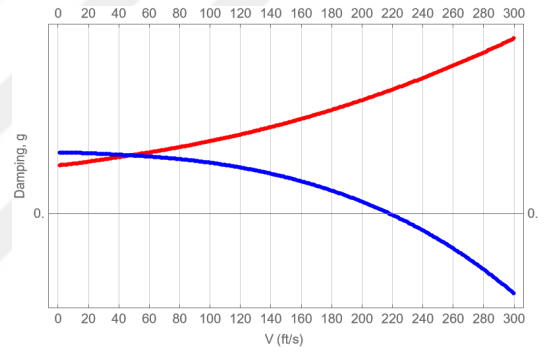


(b) Case 5 - Damping

Figure B.4: Whirl flutter results for Case 5.

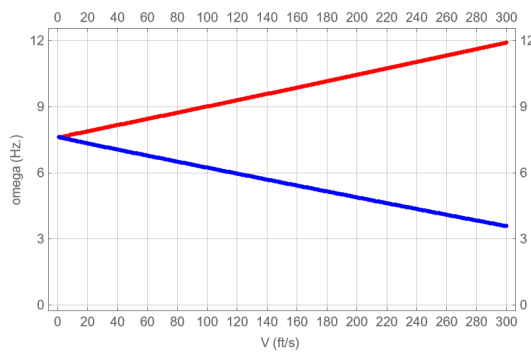


(a) Case 6 - Frequency

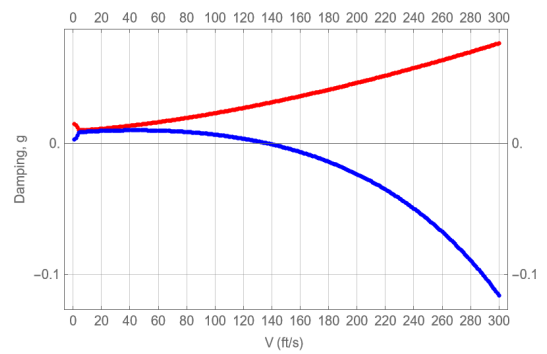


(b) Case 6 - Damping

Figure B.5: Whirl flutter results for Case 6.

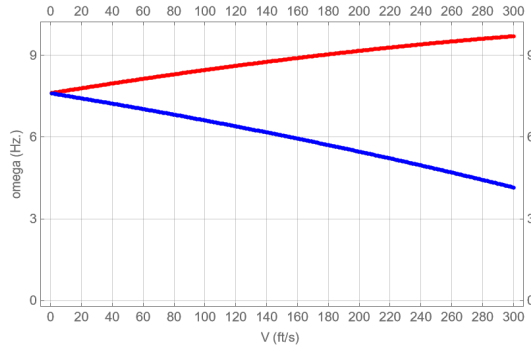


(a) Case 7 - Frequency

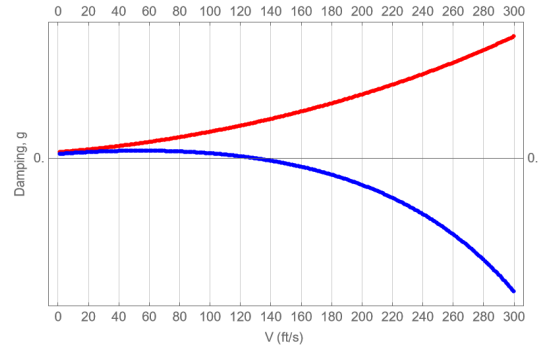


(b) Case 7 - Damping

Figure B.6: Whirl flutter results for Case 7.

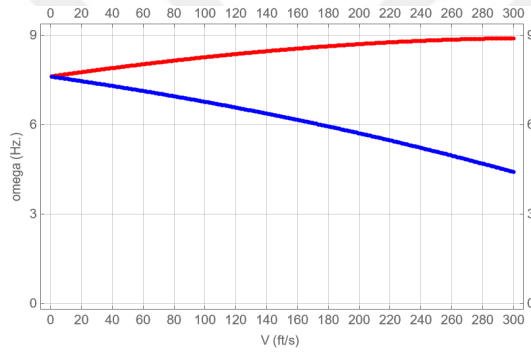


(a) Case 8 - Frequency

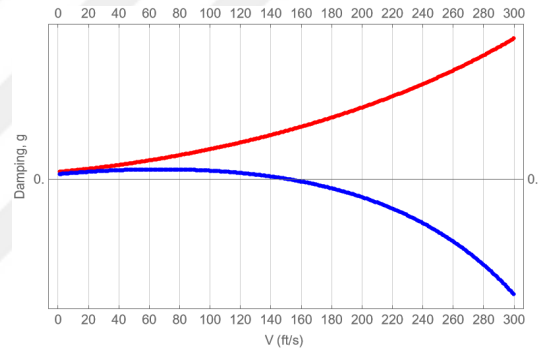


(b) Case 8 - Damping

Figure B.7: Whirl flutter results for Case 8.

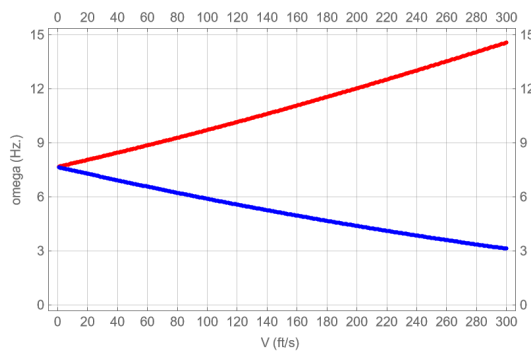


(a) Case 9 - Frequency

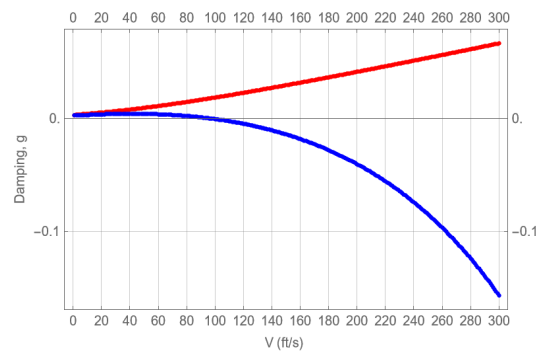


(b) Case 9 - Damping

Figure B.8: Whirl flutter results for Case 9.

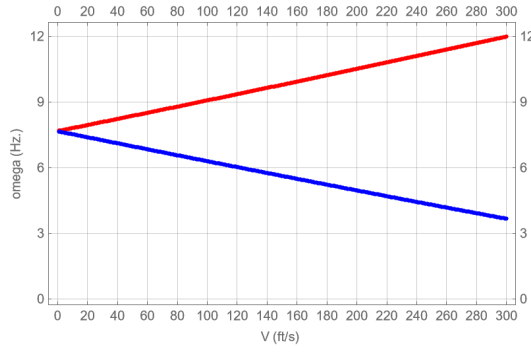


(a) Case 10 - Frequency

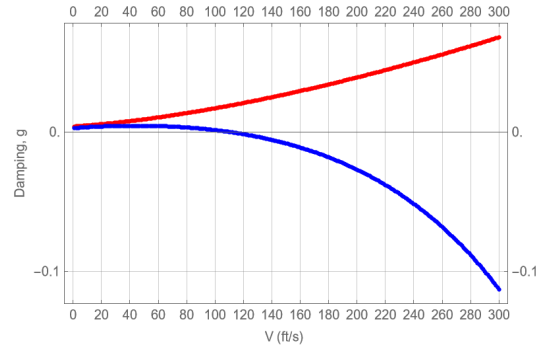


(b) Case 10 - Damping

Figure B.9: Whirl flutter results for Case 10.

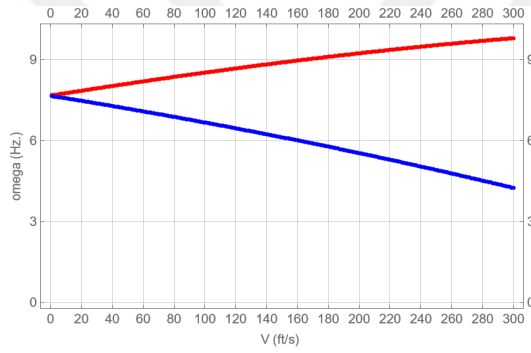


(a) Case 11 - Frequency

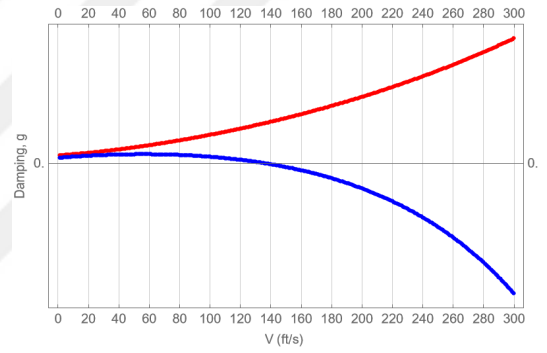


(b) Case 11 - Damping

Figure B.10: Whirl flutter results for Case 11.

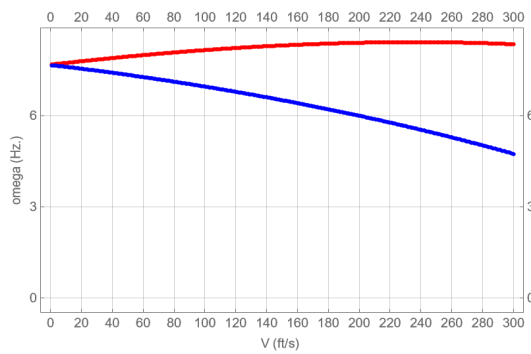


(a) Case 12 - Frequency

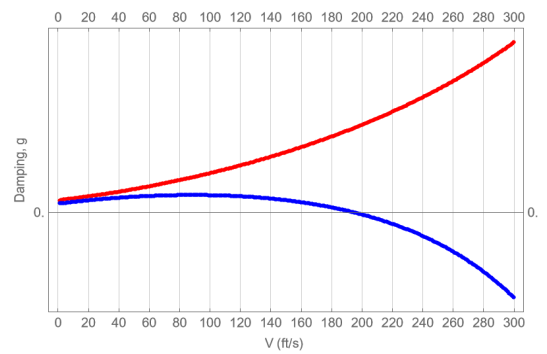


(b) Case 12 - Damping

Figure B.11: Whirl flutter results for Case 12.



(a) Case 13 - Frequency



(b) Case 13 - Damping

Figure B.12: Whirl flutter results for Case 13.