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GEBZE INSTITUTE OF TECHNOLOGY

INSTITUTE OF SCIENCE AND ENGINEERING

COPURE (m,n)-INJECTIVE MODULES AND (α,m,n)-COTORSION MODULES

SERAP ŞAHİNKAYA

MASTER THESIS

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YÜKSEK LİSANS TEZİ JÜRİ ONAY SAYFASI

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ÖZET

TEZ BAŞLIĞI: EŞ-SAF (m,n)-INJEKTİF MODÜLLER VE (α,m,n)- EŞ-BURULMALI MODÜLLER

TEZ YAZARI: Serap ŞAHİNKAYA

Bu tez beş bölümden oluşmaktadır.

1. bölüm tezin giriş kısmıdır.

2. bölümde , bazı temel tanım ve kavramları verdik.

3. bölümde, Mao and Ding [11] tarafından yazılan 'Relative Copure Injective and Copure Flat Modules' makalesinin bazı bölümlerini çalıştık.

4. bölümde, Mao and Ding [13] tarafından yazılan 'Relative cotorsion Modules and Relative Flat Modules' makalesinin bazı bölümleri üzerinde durduk.

Sonuncu bölümde, eş-saf (m,n)-injektif modülleri tanımladık ve çalıştık.. Bu tanım (m,n)-injektif modül gibi, modüllerin injektifliği ile ilgili çeşitli tanımları birleştirir. Ayrıca (α,m,n)-eş-burulmalı modülleri çalıştık ve (α,m,n)-eş-burulmalı modüllerin bazı özelliklerini elde ettik.

SUMMARY

THESIS TITLE: COPURE (m,n)-INJECTIVE MODULES AND (α,m,n)-COTORSION MODULES

THESIS AUTHOR: Serap ŞAHİNKAYA

This thesis is consists of five chapter.

Chapter 1 is the introduction part of the thesis.

In chapter 2, we present some basic definitions and useful notions.

In chapter 3, we study some part of paper 'Relative Copure Injective and Copure Flat Modules' which was written by Mao and Ding [11].

In chapter 4, we focus on some part of paper 'Relative cotorsion Modules and Relative Flat Modules' again written by Mao and Ding [13].

In the last chapter, we introduce and study copure (m,n)-injective modules. This definiton unifies several definitions of injectivity of modules such as (m,n)-injective modules. Also we define (α,m,n)-cotorsion modules and we obtain some properties of (α,m,n)-cotorsion modules.

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0.1 List of symbols

$Mod-R$	The class of all right R-modules
$R-Mod$	The class of all left R-modules
$\mathcal{P}$	The class of all modules of finite projective dimension
$\mathcal{P}_n$	The class of all modules of projective dimension $\leq n$
$\mathcal{I}$	The class of all modules of finite injective dimension
$\mathcal{I}_n$	The class of all modules of injective dimension $\leq n$
$\mathcal{F}$	The class of all modules of finite flat dimension
$\mathcal{F}_n$	The class of all modules of flat dimension $\leq n$
$\Omega^i(M)$	The class of all the i-th syzygies occuring in all projective resolutions of module M
$\Omega^{-i}(M)$	The class of all the i-th cosyzygies occuring in all injective resolutions of a module M
$\oplus_{i\in I}M_i$	Direct sum of modules M_i
$\mathcal{C}^{\perp}$	$= KerExt^1_R(\mathcal{C},-)(=\{N\in Mod-R \mid Ext^1_R(C,N)=0 \ \ for \ \ all \ \ C\in\mathcal{C}\})$
$\mathcal{C}^{\perp i}$	$= KerExt^i_R(\mathcal{C},-)$
$\mathcal{C}^{\perp\infty}$	$= \cap_{1\leq i<w}\mathcal{C}^{\perp i}$
${}^{\perp}\mathcal{C}$	$= KerExt^1_R(-,\mathcal{C})(=\{N\in Mod-R \mid Ext^1_R(N,C)=0 \ \ for \ \ all \ \ C\in\mathcal{C}\})$
${}^{\perp i}\mathcal{C}$	$= KerExt^i_R(-,\mathcal{C})$
${}^{\perp\infty}\mathcal{C}$	$= \cap^{\perp i}_{1\leq i<w}\mathcal{C}$
$\mathbb{Z}$	The rings of integers
Ab	The category of all abelian groups
$E(M)$	Injective envelope of a module M
$J(R)$	The radical of an R-module R
$rad(M)$	The radical of an R-module M
$cokerf$	$= N/imf$, where $f: M\to N$ is an R-map
$kerf$	The kernel of the homomorphism f

1 INTRODUCTION

A submodule A of B is said to be *copure* if B/A is injective. A left R-module M is said to be *copure injective* if it is injective with respect to all copure submodules. It is easy to see that copure injective modules [6] are precisely those left R-modules M such that $Ext^1_R(E, M) = 0$ for all injective left R-modules E. Every injective R-module is copure injective but copure injective R-module need not to be injective. Before giving example about this, we need some definitions.

Definition 1.1 ([9]) (i) An R-module M is said to be *cofinitely generated* if $E(M) = E(S_1) \oplus E(S_2) \oplus \cdots \oplus E(S_n)$, where $S_1, \ldots, S_n$ are simple R-modules, where $E(X)$ denotes the injective hull of an R-module X.

(ii) An R-module M is said to be *cofree* if M is isomorphic to $\prod\{E(S_\alpha) : S_\alpha \text{ is a simple R}-\text{module}, \alpha \in \wedge\}$, for some index set $\wedge$.

(iii) An R-module A is said to be *cofinitely related* if there is an exact sequence $0 \to A \to B \to C \to 0$ of R-modules with B cofinitely generated, cofree and C cofinitely generated.

Example 1.2 ([9]) Let $G = \mathbb{Z}(n)$ be a cyclic group of order n. Since G is finite it is cofinitely generated and hence, cofinitely related as a $\mathbb{Z}$-module. Since every cofinitely related R-module is copure injective, G is copure injective as a $\mathbb{Z}$-module. But we know that G is not injective as a $\mathbb{Z}$-module.

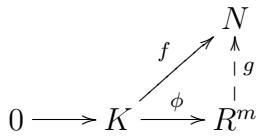
A module M is *copure flat* if it is flat with respect to all copure submodules. Again, it is easy to see that a right R-module F is copure flat [5] if $Tor^R_1(F, N) = 0$ for all injective left R-modules N. Copure injective modules and copure flat

modules were discovered when studying injective precovers and flat preenvelopes.

The notion of n-copure injective and n-copure flat were introduced by Mao and Ding in [11]. Let n be a nonnegative integer and $\mathcal{I}_n$ ($\mathcal{F}_n$) the class of all left (right) R-modules of injective (flat) dimension at most n. A left R-module M is called *n-copure injective* if $Ext^1_R(N, M) = 0$ for any $N \in \mathcal{I}_n$. A right R-module F is called *n-copure flat* if $Tor^R_1(F, N) = 0$ for any $N \in \mathcal{I}_n$. Mao and Ding showed that a left R-module M over any ring R is n-copure injective if and only if M is a kernel of an $\mathcal{I}_n$-precover $f\colon A \to B$ of a left R-module B with A injective. Also they showed that M is n-copure flat if and only if M^+ is n-copure injective,where $M^+ = Hom(M, \mathbb{Q}/\mathbb{Z})$, if and only if $M \in C^+$, where $C = \{B^+ : B \in \mathcal{I}_n\}$.

The notions of right (m, n)-injective modules and right (m, n)-injective rings were introduced and studied by Chen, Ding, Li and Zhou in [2]. For the fixed positive integers m and n, a right R-module M is called *(m, n)-injective* if every right R-homomorphism from an n-generated submodule of R^m to M extends to one from R^m to M. So a ring R is said to be *right (m, n)-injective* if R_R is an (m, n)-injective R-module.

In section 5.1, we introduce and study copure (m, n)-injective modules and obtain some properties of copure (m, n)-injective modules. Now, we recall that an R-module M is called *(m, n)-presented* if there exists an exact sequence of right R-modules $0 \to K \to R^m \to M \to 0$, where K is an n-generated. Let $\mathcal{I}_{(m,n)}$ denote the class of all (m, n)-injective modules. So a module N belongs to $\mathcal{I}_{(m,n)}$ if and only if the following diagram is commutative for any map $f : K \to N$ where K is n-generated.



So call M *a copure* (m,n)*-injective module* if $\mathrm{Ext}_R(N,M)=0$ for all (m,n)-injective module N. This definition unifies several definitions of injectivity of modules such as (m,n)-injective modules.

We call the module M_R *copure* (m,n)*-flat* if $Tor_1(M,N)=0$ for every (m,n)-injective left R-module N.

The module M_R is called *cotorsion* if $Ext^1_R(F,M)=0$ for any flat right R-module F. M_R is called α*-cotorsion* if $Ext^{\alpha+1}_R(F,M)=0$ for any flat right R-module F. M is said to be an α*-flat module* if $Ext^1_R(M,N)=0$ for any α-cotorsion right R-module N. Note that M is 0-cotorsion (respectively, 0-flat) if and only if M is cotorsion (respectively, flat). In [13], Mao and Ding had some applications of n-flat and n-cotorsion modules. They showed that every right R-module is n-cotorsion if and only if every $(n-)$flat right R-modules are n-cotorsion. Dually they gave characterizations of those rings such that every right R-module is n-flat. It was shown that every right R-module is n-flat if and only if every n-cotorsion right R-module is n-flat.

In section 5.2 we define (α,m,n)-cotorsion modules and we obtain some properties of (α,m,n)-cotorsion modules. The module M_R is called (α,m,n)*-cotorsion* if $\mathrm{Ext}^{\alpha+1}_R(N,M)=0$ for any copure (m,n)-flat right R-module N.

2 DEFINITIONS AND PRELIMINARIES

2.1 Categories and Functors

Definition 2.1 A *category* $\mathcal{C}$ consists of:

(i) a class $\mathrm{Ob}\,\mathcal{C}$, whose elements will be called the *objects* of $\mathcal{C}$;

(ii) for each pair (A, B) of objects of $\mathcal{C}$, a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$, whose elements will be called *morphisms* of A into B;

(iii) for each triple (A, B, C) of objects of $\mathcal{C}$, a mapping

$$\circ\colon \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C),$$

called *composition*.

Before stating the axioms for categories, we introduce some notation. Instead of writing $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, we will often write $f\colon A \to B$. For the composition $\circ\colon \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$ and morphisms $f\colon A \to B$, $g\colon B \to C$, we will denote the composite morphism by gf or $g \circ f$.

The axioms to which a category must satisfy are:

(1) If A, B, C, D are objects of $\mathcal{C}$ and $f\colon A \to B$, $g\colon B \to C$, $h\colon C \to D$ are morphisms, then $(hg)f = h(gf)$ (associativity of composition).

(2) For every $A \in \mathrm{Ob}\,\mathcal{C}$, there exists an element of $\mathrm{Hom}_{\mathcal{C}}(A, A)$, which we will denote 1_A, such that $f \circ 1_A = f$ and $1_A \circ g = g$ for every $B \in \mathrm{Ob}\,\mathcal{C}$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ [Enochs and Jenda, 2000].

Examples 2.2 (1) *The category* **Set**. The objects of **Set** are all sets. If A, B are sets, the morphisms $f\colon A \to B$ are exactly all mappings $f\colon A \to B$, that is $\mathrm{Hom}_{\mathbf{Set}}(A, B) := B^A$. The composition is the composition of mappings. Then

Set is a category, in which $1_A\colon A \to A$ is the identity mapping defined by $1_A(a) = a$ for every $a \in A$.

(2) *The category* **Grp**. The objects of **Grp** are all groups. If G, H are groups, the morphisms $f\colon G \to H$ are the "usual" group morphisms of G into H, that is, the mappings $f\colon G \to H$ such that $f(xy) = f(x)f(y)$ for every $x, y \in G$. The composition is the composition of mappings. Then **Grp** turns out to be a category.

(3) *The category* **Rng**. The objects of **Rng** are all rings with identity. The morphisms $f\colon R \to S$ are the ring morphisms of R into S. The composition is the composition of mappings.

(4) The category **Ab** of abelian groups has as objects all abelian additive groups, as morphisms $G \to H$ the group morphisms $G \to H$, and as composition the composition of mappings.

Definition 2.3 Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* (or a *covariant functor*) $F\colon \mathcal{C} \to \mathcal{D}$ assigns to every object $C \in \operatorname{Ob}\mathcal{C}$ an object $F(C) \in \operatorname{Ob}\mathcal{D}$, and to every morphism $f\colon C \to C'$ in $\mathcal{C}$ a morphism $F(f)\colon F(C) \to F(C')$ in $\mathcal{D}$, and the following axioms are satisfied:

(i) if $f\colon C \to C'$ and $g\colon C' \to C''$ are morphisms in $\mathcal{C}$, then

$$F(g \circ f) = F(g) \circ F(f);$$

(ii) $F(1_C) = 1_{F(C)}$ for every $C \in \operatorname{Ob}\mathcal{C}$

[Enochs and Jenda, 2000].

Definition 2.4 Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *contravariant functor* $F\colon \mathcal{C} \to \mathcal{D}$ assigns to every $C \in \mathrm{Ob}\,\mathcal{C}$ an object $F(C) \in \mathrm{Ob}\,\mathcal{D}$, and to every morphism $f\colon C \to C'$ a morphism $F(f)\colon F(C') \to F(C)$, and the following axioms are satisfied:

(i) $F(g \circ f) = F(f) \circ F(g)$ for every $f\colon C \to C'$ and $g\colon C' \to C''$;

(ii) $F(1_C) = 1_{F(C)}$ for every $C \in \mathrm{Ob}\,\mathcal{C}$ [Enochs and Jenda, 2000].

Examples 2.5 (1) Clearly, a contravariant functor $F\colon \mathcal{C} \to \mathcal{D}$ is essentially the same thing of a covariant functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, where $\mathcal{C}^{\mathrm{op}}$ denotes the dual category.

(2) *The identity functor.* Let $\mathcal{C}$ be an arbitrary category. The functor $I\colon \mathcal{C} \to \mathcal{C}$, defined by $I(C) = C$ for every object $C \in \mathrm{Ob}\,\mathcal{C}$ and $I(f) = f$ for every morphism $f\colon A \to B$ in $\mathcal{C}$, is trivially a covariant functor. It is often denoted as $I_{\mathcal{C}}$ or $1_{\mathcal{C}}$.

(3) *The opposite functor.* The functor $F\colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$, defined by $F(C) = C$ for every object $C \in \mathrm{Ob}\,\mathcal{C}$ and $F(f) = f$ for every morphism $f\colon A \to B$ in $\mathcal{C}$, is a contravariant functor.

(4) Let **Grp** be the category of groups (Example 2.2(2)). Fix a group $\overline{G} \in \mathrm{Ob}\,\mathbf{Grp}$, and define $F\colon \mathbf{Grp} \to \mathbf{Set}$ as follows: on objects, $F(G) = \mathrm{Hom}_{\mathbf{Grp}}(\overline{G}, G)$ for every $G \in \mathrm{Ob}\,\mathbf{Grp}$; on morphisms, $F(f)\colon \mathrm{Hom}_{\mathbf{Grp}}(\overline{G}, G) \to \mathrm{Hom}_{\mathbf{Grp}}(\overline{G}, G')$ is, for every morphism $f\colon G \to G'$, the mapping $F(f)\colon \alpha \mapsto f \circ \alpha$. Then F is a functor of **Grp** into **Set**, because if $f\colon G \to G', g\colon G' \to G''$ are morphisms in **Grp**, then for every $\alpha \in \mathrm{Hom}_{\mathbf{Grp}}(\overline{G}, G)$, $F(g \circ f)(\alpha) = (g \circ f) \circ \alpha = g \circ (f \circ \alpha) = (F(g) \circ F(f))(\alpha)$, so that $F(g \circ f) = F(g) \circ F(f)$. Moreover, $F(1_G)(\alpha) = 1_G \circ \alpha = \alpha$

for all $\alpha \in \mathrm{Hom}_{\mathbf{Grp}}(\overline{G}, G)$, so that $F(1_G) = 1_{\mathrm{Hom}_{\mathbf{Grp}}(\overline{G},G)}$.

(5) *Composition of functors.* Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories and $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{E}$ be functors. The composite functor $GF\colon \mathcal{C} \to \mathcal{E}$ is defined in the obvious way: $GF(C) = G(F(C))$ for every object C in $\mathcal{C}$; $GF(f) = G(F(f))$ for every morphism $f\colon C \to C'$ in $\mathcal{C}$. Notice that if F and G are both covariant or both contravariant, the composite functor GF is covariant. If one is covariant and the other is contravariant, the composite functor GF is contravariant.

(6) *Isomorphism of categories.* An *isomorphism* of a category $\mathcal{C}$ into a category $\mathcal{D}$ is a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ with GF the identity functor $1_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ and FG the identity functor $1_{\mathcal{D}}\colon \mathcal{D} \to \mathcal{D}$. If there is an isomorphism $F\colon \mathcal{C} \to \mathcal{D}$ we will say that the two categories $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic.*

2.2 Tensor product of modules

Definition 2.6 Let R be a ring, M_R a right R-module, ${}_RN$ a left R-module and G an additive abelian group. A mapping $\beta\colon M_R \times {}_RN \to G$ is said to be a *balanced mapping* (or an *R-balanced mapping*) if, for every $m, m' \in M_R, n, n' \in {}_RN$ and $r \in R$,

(i) $\beta(m + m', n) = \beta(m, n) + \beta(m', n)$.

(ii) $\beta(m, n + n') = \beta(m, n) + \beta(m, n')$.

(iii) $\beta(mr, n) = \beta(m, rn)$.

[Enochs and Jenda, 2000].

If $\beta\colon M_R \times {}_RN \to G$ is a balanced mapping, then, for any fixed $m \in M_R$, the mapping $\beta(m,-)\colon N \to G$, defined by $n \mapsto \beta(m,n)$ for every $n \in N$, is a group homomorphism. It follows that we have that $\beta(m,0) = 0$, $\beta(m,-n) = -\beta(m,n)$ and $\beta(m,zn) = z\beta(m,n)$ for every $m \in M$, $n \in N$ and $z \in \mathbb{Z}$. Similarly, $\beta(0,n) = 0$, $\beta(-m,n) = -\beta(m,n)$ and $\beta(zm,n) = z\beta(m,n)$ for every $m \in M$, $n \in N$ and $z \in \mathbb{Z}$.

For instance, for every ring R, the multiplication $R_R \times {}_RR \to R$, $(r,s) \mapsto rs$ is a balanced mapping.

Definition 2.7 A *tensor product* of a right R-module M_R and a left R-module ${}_RN$ is a pair (T,τ), where T is an abelian group, $\tau\colon M_R \times {}_RN \to T$ is a balanced mapping, and the following universal property holds: for any abelian group G and any balanced mapping $\beta\colon M_R \times {}_RN \to G$ there exists a unique group homomorphism $\tilde{\beta}\colon T \to G$ such that $\beta = \tilde{\beta} \circ \tau$, that is, such that the diagram

$$\begin{array}{ccc} M_R \times {}_RN & \stackrel{\tau}{\to} & T \\ {\scriptstyle\beta}\downarrow & \swarrow {\scriptstyle\tilde{\beta}} & \\ G & & \end{array}$$

commutes [Enochs and Jenda, 2000].

Thus (T,τ) is a tensor product of M_R and ${}_RN$ if for every abelian group G the mapping

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Ab}}(T,G) & \to & \mathrm{Bal}(M_R \times_R N, G) \\ f & \mapsto & f \circ \tau \end{array}$$

is a bijection. Here $\mathrm{Bal}(M_R \times {}_RN, G)$ denotes the set of all balanced mappings $M_R \times {}_RN \to G$.

Proposition 2.8 Given any ring R and any two modules M_R and ${}_RN$, a tensor product (T,τ) of M_R and ${}_RN$ always exists [Enochs and Jenda, 2000].

Proposition 2.9 (Uniqueness of the tensor product up to isomorphism) Let R be a ring, M_R a right R-module, ${}_RN$ a left R-module, and $(T,\tau),(T',\tau')$ two tensor products of M_R and ${}_RN$. Then there is a unique group isomorphism $\varphi\colon T \to T'$ for which the diagram

$$\begin{array}{ccc} M\times N & & \\ {\scriptstyle\tau}\downarrow & \searrow{\scriptstyle\tau'} & \\ T & \overset{\varphi}{\to} & T' \end{array}$$

commutes. [Enochs and Jenda, 2000]

Given a ring R and two modules M_R and ${}_RN$, if (T,τ) is a tensor product of M and N then T will be denoted by $M\otimes_R N$ and $\tau(m,n)$ by $m\otimes n$ for any $(m,n)\in M\times N$.

Since τ is balanced, in this notation we have, for every $m\in M$, $n\in N$ and $r\in R$, that $(m+m')\otimes n = m\otimes n + m'\otimes n$, $m\otimes(n+n') = m\otimes n + m\otimes n'$, $(mr)\otimes n = m\otimes(rn)$, $m\otimes 0 = 0$, $m\otimes(-n) = -(m\otimes n)$, etc.

Remark 2.10 Any element of $M\otimes_R N$ can be written in the form

$$\sum_{i=1}^{t} m_i\otimes n_i$$

for suitable $m_i\in M,\ n_i\in N,\ t\geq 1$ [Rotman, 2008].

Note that $M_R\otimes - : R-Mod \to \mathbf{Ab}$ is a functor for any right R-module M.

2.3 Exact sequences

We will denote the zero module with one element by 0.

We will now consider *sequences of modules*, where by a sequence of modules

$$\ldots \to M_{i-1} \overset{f_{i-1}}{\to} M_i \overset{f_i}{\to} M_{i+1} \overset{f_{i+1}}{\to} \ldots,$$

we mean a family of modules M_i indexed by integers and a set of module morphisms $f_i\colon M_i \to M_{i+1}$. Sequences can be either finite or infinite on one side or both sides.

Definition 2.11 A sequence of modules is called a 0-*sequence* (or *a complex of modules*, or *a chain complex of modules*) if $f_i(M_i) \subseteq \ker f_{i+1}$ for every index i. Equivalently, if $f_{i+1}f_i = 0$ for every i [Enochs and Jenda, 2000].

Definition 2.12 A sequence is *exact in* M_i if $f_{i-1}(M_{i-1}) = \ker f_i$. A sequence is *exact* if it is exact in M_i for every i [Rotman, 2008].

Lemma 2.13 Let $f\colon M \to N$ be any module homomorphism.

(1) The sequence $0 \to M \xrightarrow{f} N$ is exact if and only if f is injective.

(2) The sequence $M \xrightarrow{f} N \to 0$ is exact if and only if f is surjective.

(3) A sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact if and only if f is injective, g is surjective and $f(A) = \ker g$ [Anderson and Fuller, 1992].

Exact sequences of the form

$$0 \to A \to B \to C \to 0$$

are called *short exact sequences* (s.e.s., for short).

Examples 2.14 (a) For every submodule N of M, there is a short exact sequence

$$0 \to N \xrightarrow{\iota} M \xrightarrow{\pi} M/N \to 0,$$

where ι denotes the embedding of N into M and π denotes the canonical projection of M onto N.

(b) For every pair of modules M and N, there is a short exact sequence

$$0 \to M \xrightarrow{\varepsilon} M \oplus N \xrightarrow{\pi} N \to 0,$$

where $\varepsilon(m) = (m, 0)$ and $\pi(m, n) = n$ for every $m \in M$ and $n \in N$.

Lemma 2.15 Let $f\colon M \to N$ and $g\colon N \to M$ be homomorphisms with $gf = \iota_M$. Then $N = f(M) \oplus \ker g$ [Anderson and Fuller, 1992].

Proposition 2.16 The following conditions are equivalent for a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$:

(i) There exists a homomorphism $f'\colon B \to A$ such that $f'f = \iota_A$ (i.e., f is left invertible).

(ii) There exists a homomorphism $g'\colon C \to B$ such that $gg' = \iota_C$ (i.e., g is right invertible).

(iii) $f(A) = \ker g$ is a direct summand of B.

Moreover, if these three equivalent conditions hold, then $B \cong A \oplus C$ [Anderson and Fuller, 1992].

A short exact sequence satisfying the three equivalent conditions in the statement of Proposition 2.16 is called a *split* exact sequence.

Example 2.17 *(A short exact sequence $0 \to A \to B \to C \to 0$ which does not split, although $B \cong A \oplus C$.)* Let R be the ring $\mathbb{Z}$, so that R-modules are exactly abelian groups. Set $A := \mathbb{Z}$, let C be a countably dimensional vector space over the field $\mathbb{Z}/2\mathbb{Z}$, so that we can write C in the form $C = \bigoplus_{i\geq 0}(\mathbb{Z}/2\mathbb{Z})v_i$, where $\{\, v_i \mid i \geq 0 \,\}$ is a basis of the $\mathbb{Z}/2\mathbb{Z}$-vector space C. Set $B := \mathbb{Z} \oplus \big(\bigoplus_{i=0}^{+\infty}(\mathbb{Z}/2\mathbb{Z})v_i\big)$.

Consider the exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0, \tag{1}$$

where $f\colon \mathbb{Z} \to \mathbb{Z} \oplus \left(\bigoplus_{i=0}^{+\infty}(\mathbb{Z}/2\mathbb{Z})v_i\right)$ is defined by $f(z) = (2z, \bar{0}, \bar{0}, \ldots)$, where $\bar{t} = t + 2\mathbb{Z}$ for every $t \in \mathbb{Z}$. The image of f is $2\mathbb{Z} \oplus 0$. Let the map $g\colon \mathbb{Z} \oplus \left(\bigoplus_{i=0}^{+\infty}(\mathbb{Z}/2\mathbb{Z})v_i\right) \to \bigoplus_{i=0}^{+\infty}(\mathbb{Z}/2\mathbb{Z})v_i$ be defined by $g\colon (z, \bar{a_0}, \bar{a_1}, \bar{a_2}, \ldots) \mapsto (\bar{z}, \bar{a_0}, \bar{a_1}, \ldots)$. Then $\ker g = 2\mathbb{Z} \oplus 0$, hence (1) is a short exact sequence.

Assume that (1) splits. Then there exists a homomorphism

$$f'\colon \mathbb{Z} \oplus \left(\bigoplus_{i=0}^{+\infty}(\mathbb{Z}/2\mathbb{Z})v_i\right) \to \mathbb{Z}$$

with $f'f = \iota_{\mathbb{Z}}$. Hence there exist homomorphisms

$$f_1\colon \mathbb{Z} \to \mathbb{Z} \qquad \text{and} \qquad f_2\colon \bigoplus_{i=0}^{+\infty}(\mathbb{Z}/2\mathbb{Z})v_i \to \mathbb{Z}$$

such that $f'(z, v) = f_1(z) + f_2(v)$. Then we have for all $t \in \mathbb{Z}$ that $t = \iota_{\mathbb{Z}}(t) = f'f(t) = f'(2t, 0) = f_1(2t) + f_2(0) = f_1(2t)$. Setting $t = 1$, we obtain $1 = f_1(2) = 2f_1(1) \in 2\mathbb{Z}$, which implies that 1 is even, a contradiction.

2.4 Special Modules

Projective modules

Definition 2.18 Let R be a ring. A right R-module P_R is *projective* if for every epimorphism $f\colon M_R \to N_R$ and every homomorphism $g\colon P_R \to N_R$, there exists a morphism $h\colon P \to M$ with $f \circ h = g$ [Rotman, 2008].

The situation in the previous definition is described by the following commutative diagram in which the row is exact:

$$\begin{array}{ccccccc} & & P_R & & & & \\ & & {\scriptstyle h}\downarrow & \overset{g}{\searrow} & & & \\ M_R & \overset{f}{\rightarrow} & N_R & \rightarrow & 0 & & \end{array}$$

Lemma 2.19 (i) Every free module is projective.

(ii) Every direct summands and direct sums of a projective modules are projective

[Rotman, 2008].

Proposition 2.20 The following conditions are equivalent for a right R-module P_R:

(i) The module P_R is projective.

(ii) Every short exact sequence of the form $0 \rightarrow M_R \rightarrow N_R \rightarrow P_R \rightarrow 0$ splits.

(iii) The module P_R is isomorphic to a direct summand of a free module.

[Rotman, 2008].

Thus projective modules are exactly the modules isomorphic to direct summands of free modules. Since every free module is projective and every module is a homomorphic image of a free module, we get that every module is a homomorphic image of a projective module. Similarly, every finitely generated module is a homomorphic image of a finitely generated projective module.

Corollary 2.21 A module P_R is a finitely generated projective module if and only if it is isomorphic to a direct summand of R_R^n for some $n \geq 0$ [Rotman, 2008].

Theorem 2.22 Every projective module is a direct sum of countably generated projective modules [Rotman, 2008].

Injective modules

Fix two modules M_R and N_R. We already know that there are a covariant functor

$$\mathrm{Hom}(M_R, -)\colon \text{Mod-}R \to \mathbf{Ab}$$

and a contravariant functor

$$\mathrm{Hom}(-, N_R)\colon \text{Mod-}R \to \mathbf{Ab}.$$

Also, we have already known that these functors Hom are "left exact", in the sense that, for every fixed module M_R, if $0 \to N'_R \to N_R \to N''_R$ is exact, then so is $0 \to \mathrm{Hom}(M_R, N'_R) \to \mathrm{Hom}(M_R, N_R) \to \mathrm{Hom}(M_R, N''_R)$, and, for every fixed module N_R, if $M'_R \to M_R \to M''_R \to 0$ is exact, then so is $0 \to \mathrm{Hom}(M''_R, N_R) \to \mathrm{Hom}(M_R, N_R) \to \mathrm{Hom}(M'_R, N_R)$.

In general, these functors $\mathrm{Hom}(M_R, -)$ and $\mathrm{Hom}(-, N_R)$ are not "exact", that is, it is not always true that, for every fixed module M_R, if $0 \to N'_R \to N_R \to N''_R \to 0$ is a short exact sequence, then $0 \to \mathrm{Hom}(M_R, N'_R) \to \mathrm{Hom}(M_R, N_R) \to \mathrm{Hom}(M_R, N''_R) \to 0$ is necessarily exact, and, for every fixed module N_R, if $0 \to M'_R \to M_R \to M''_R \to 0$ is exact, then $0 \to \mathrm{Hom}(M''_R, N_R) \to \mathrm{Hom}(M_R, N_R) \to$

$\mathrm{Hom}(M'_R, N_R) \to 0$ is necessarily exact. It is easily seen that a module M_R is projective if and only if the functor $\mathrm{Hom}(M_R, -)$ is exact.

Proposition 2.23 The following conditions are equivalent for an R-module E_R:

(i) The functor $\mathrm{Hom}(-, E_R)\colon \text{Mod-}R \to \mathbf{Ab}$ is exact.

(ii) For every monomorphism $M'_R \to M_R$ of right R-modules, $\mathrm{Hom}(M_R, E_R) \to \mathrm{Hom}(M'_R, E_R)$ is an epimorphism of abelian groups.

(iii) For every submodule M'_R of a right R-module M_R, every morphism $M'_R \to E_R$ can be extended to a morphism $M_R \to E_R$.

(iv) For every monomorphism $f\colon M'_R \to M_R$ and every homomorphism $g\colon M'_R \to E_R$, there exists a morphism $h\colon M_R \to E_R$ with $h \circ f = g$ [Rotman, 2008].

A module E_R is *injective* if it satisfies the equivalent conditions of Proposition 2.23.

Condition (iv) is described by the following commutative diagram, in which the row is exact:

$$\begin{array}{ccccc} 0 & \to & M'_R & \xrightarrow{f} & M_R \\ & & & \overset{g}{\searrow} & \downarrow h \\ & & & & E_R \end{array}$$

Essentially, the unique characterization of projective modules that cannot be immediately dualized to injective modules is the characterization of projective modules as direct summands of free modules. In the next Proposition we give a further criterion to recognize injective modules, that is, a further characterization of injective modules, which does not have an analog for projective modules.

Proposition 2.24 (Baer's criterion) A right module M over a ring R is injective if and only if for every right ideal I of R, every morphism $\sigma\colon I \to M$ can be

extended to a morphism $\sigma^*\colon R \to M$ [Rotman, 2008].

Let us determine injective $\mathbb{Z}$-modules, that is, injective abelian groups.

Definition 2.25 An abelian group G is *divisible* if $nG = G$ for every non-zero integer n. Thus G is divisible if and only if for every $g \in G$ and $n > 0$ there exists $h \in G$ such that $nh = g$ [Rotman, 2008].

For instance, the abelian group $\mathbb{Z}$ is not divisible, and the abelian group $\mathbb{Q}$ is divisible. Homomorphic images of a divisible abelian groups are divisible.

Proposition 2.26 A $\mathbb{Z}$-module G is injective if and only if it is a divisible abelian group [Rotman, 2008].

In particular, a direct sum of finitely many injective modules is an injective module.

Proposition 2.27 If R is a ring and G is a divisible abelian group, then

$$\mathrm{Hom}(R_{\mathbb{Z}}, G_{\mathbb{Z}})$$

is an injective right R-module [Rotman, 2008].

Here the right R-module structure on $\mathrm{Hom}(R_{\mathbb{Z}}, G_{\mathbb{Z}})$ is that induced by the bimodule structure on ${}_R R_{\mathbb{Z}}$. Hence, for every $f \in \mathrm{Hom}(R_{\mathbb{Z}}, G_{\mathbb{Z}})$ and every $r \in R$, it is defined by $fr := f \circ \lambda_r$, where $\lambda_r\colon R \to R$ denotes left multiplication by r.

Theorem 2.28 Every right R-module can be embedded in an injective right R-module [Rotman, 2008].

Proposition 2.29 Direct summands and direct products of injective modules are injective [Rotman, 2008].

Corollary 2.30 The following conditions are equivalent for a right R-module N_R:

(i) The module N_R is injective.

(ii) Every short exact sequence that begins with N_R splits, that is, every short exact sequence of right R-modules of the form $0 \to N_R \to B_R \to C_R \to 0$ splits.

(iii) The module N_R is a direct summand of every module of which it is a submodule [Rotman, 2008].

Flat modules

Definition 2.31 Let R and S be rings. A covariant (contravariant) functor F: Mod-$R \to$ Mod-S is said to be an *exact functor* if for every short exact sequence $0 \to A_R \xrightarrow{\alpha} B_R \xrightarrow{\beta} C_R \to 0$ in Mod-R, the short sequence $0 \to F(A_R) \xrightarrow{F(\alpha)} F(B_R) \xrightarrow{F(\beta)} F(C_R) \to 0$ ($0 \to F(C_R) \xrightarrow{F(\beta)} F(B_R) \xrightarrow{F(\alpha)} F(A_R) \to 0$, respectively) is exact in Mod-S [Rotman, 2008].

Lemma 2.32 Let R and S be rings. The following conditions are equivalent for a covariant functor F: Mod-$R \to$ Mod-S:

(i) For every short exact sequence $0 \to A_R \xrightarrow{\alpha} B_R \xrightarrow{\beta} C_R \to 0$ in Mod-R, the sequence $F(A_R) \xrightarrow{F(\alpha)} F(B_R) \xrightarrow{F(\beta)} F(C_R) \to 0$ is exact in Mod-S.

(ii) For every exact sequence $A_R \xrightarrow{\alpha} B_R \xrightarrow{\beta} C_R \to 0$ in Mod-R, the sequence $F(A_R) \xrightarrow{F(\alpha)} F(B_R) \xrightarrow{F(\beta)} F(C_R) \to 0$ is exact in Mod-S [Rotman, 2008].

We say that the covariant functor $F\colon \text{Mod-}R \to \text{Mod-}S$ is *right exact* if it satisfies the equivalent conditions of the Lemma 2.24. There are obvious variants for R-Mod, S-Mod, contravariant functors and left exactness instead of right exactness. The equivalences of these conditions will be generalized, and become trivial, when we will introduce abelian categories and exact or right exact functors between them.

For instance, we know that, for every fixed right R-module M_R, the functor $M_R \otimes -\colon R\text{-Mod} \to Ab$ is right exact and $\text{Hom}(M_R, -)\colon \text{Mod-}R \to Ab$ is left exact. Hence $M \otimes_R -\colon R\text{-Mod} \to Ab$ right exact means that, given any short exact sequence $0 \to A \to B \to C \to 0$ of left R-modules, the induced sequence $M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ of abelian groups is exact.

Definition 2.33 A right R-module M_R is *flat* if the functor $M_R \otimes_R -\colon R\text{-Mod} \to \mathbf{Ab}$ is exact. Equivalently, if, for every left R-module monomorphism $\varphi\colon {}_RA \to {}_RB$, the group homomorphism $1_M \otimes \varphi\colon M \otimes_R A \to M \otimes_R B$ is also a monomorphism. Similarly, a left R-module ${}_RM$ is *flat* if the functor $- \otimes_R M\colon \text{Mod-}R \to \mathbf{Ab}$ is exact, or, equivalently, if for every right R-module monomorphism $\varphi\colon A \to B$, the abelian group morphism $\varphi \otimes 1_M\colon A_R \otimes_R M \to B \otimes_R M$ is a monomorphism [Rotman, 2008].

In order to give examples of flat modules and examples of modules that are not flat, we now introduce the notions of torsion, torsion-free, and divisible modules over a commutative integral domain.

Definition 2.34 Let R be a commutative integral domain.

(i) For any R-module M_R, the *torsion submodule* $t(M_R)$ of M_R is consists of all elements $x \in M_R$ for which there exists a non-zero element $r \in R$ such that $xr = 0$. A module M_R is *torsion* if $t(M_R) = M_R$.

(ii) A module M_R is *torsion-free* if $t(M_R) = 0$, that is, if for every $r \in R$ and $x \in M_R$, $xr = 0$ implies $r = 0$ or $x = 0$. Equivalently, a module M_R is torsion-free if for every non-zero element $r \in R$, multiplication by r is an injective homomorphism $\lambda_r \colon M_R \to M_R$ [Rotman, 2008].

Every free module over a commutative integral domain is torsion-free. For every module M_R over an arbitrary integral domain R, the submodule $t(M_R)$ is a torsion module, and the quotient $M_R/t(M_R)$ is torsion-free.

Definition 2.35 An R-module M_R is *divisible* if for every $r \in R$, $r \neq 0$, and every $x \in M_R$ there exists $y \in M_R$ with $x = yr$. Equivalently, a module M_R is divisible if for every non-zero element $r \in R$, multiplication by r is a surjective homomorphism $\lambda_r \colon M_R \to M_R$ [Rotman, 2008].

Notice that we are extending the notion of divisible abelian group to modules over arbitrary integral domains.(See Definition 2.25)

Example 2.36 We will now show that if R is a commutative integral domain, T is a torsion R-module and D is a divisible R-module, then $T \otimes_R D = 0$.

The elements of $T \otimes_R D$ are finite sums of elements of the form $t \otimes d$ with $t \in T$ and $d \in D$. Now if $t \in T$, there exists a non-zero $r \in R$ such that $tr = 0$. But D is divisible, so that there exists $d' \in D$ with $d = rd'$. Then $t \otimes d = t \otimes rd' = tr \otimes d' = 0 \otimes d' = 0$. Hence $T \otimes_R D = 0$.

Example 2.37 Here is an example of module that is not flat. Consider the embedding $\varphi\colon \mathbb{Z} \hookrightarrow \mathbb{Q}$, so that $1_{\mathbb{Z}/2\mathbb{Z}} \otimes \varphi$ is a group homomorphism of $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$ into $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$. Now $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ because $\mathbb{Z}/2\mathbb{Z}$ is a torsion $\mathbb{Z}$-module and ${}_{\mathbb{Z}}\mathbb{Q}$ is divisible. Thus the mapping $1_{\mathbb{Z}/2\mathbb{Z}} \otimes \varphi\colon \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a morphism from a non-zero group into a zero group. Thus $1_{\mathbb{Z}/2\mathbb{Z}} \otimes \varphi$ is not an injective mapping. Hence $Z/2\mathbb{Z}$ is not a flat $\mathbb{Z}$-module. We will characterize flat $\mathbb{Z}$-modules, more generally flat modules over principal ideal domains, in Proposition 2.40(2).

Example 2.38 For any ring R, the right R-module $R_R$is flat. To see this, recall that $R \otimes_R M \cong M$ for every left R-module ${}_RM$.

Let us pass to describe the basic properties of flat modules.

Lemma 2.39 (1) Every direct sum of flat modules is flat.

(2) Every direct summmand of a flat module is flat.

(3) Every projective module is flat [Rotman, 2008]

Recall that every finitely generated module over a commutative principal ideal domain R is a finite direct sums of cyclic modules (Elementary divisors theorem). A cyclic module R/rR is torsion for $r \neq 0$ and is torsion-free for $r = 0$. It follows that a finitely generated module over a commutative principal ideal domain is torsion-free if and only if it is free.

Proposition 2.40 (1) Every flat module over a commutative integral domain R is torsion-free.

(2) If R is a commutative principal ideal domain, an R-module is flat if and only if it is torsion-free [Rotman, 2008]

2.5 Covers and Envelopes

Projective covers

Definition 2.41 A homomorphism $\phi : M \to C$ is a *$\mathcal{C}$-precover* if $C \in \mathcal{C}$ and the abelian group homomorphism $Hom_R(C', \phi) : Hom(C', C) \to Hom(C', M)$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-precover $\phi : M \to C$ is said to be *$\mathcal{C}$-cover* if every endomorphism $g : C \to C$ such that $\phi g = \phi$ is an isomorphism. [Xu, 1996]

Name of the covers (precovers) depend on the name of the class of $\mathcal{C}$, that is if $\mathcal{C}$ consists of projective modules then $\mathcal{C}$-cover (precover) will be called projective cover (precover). Similarly if $\mathcal{C}$ consists of injective modules then $\mathcal{C}$-cover (precover) will be called injective cover (precover). Projective covers don't always exists. We need the following definition for the equivalent condition of projective cover.

Definition 2.42 A submodule K of M is *small (superfluous)in M*,abbreviated $K \ll M$, in case for every submodule $L \leq M$, $K + L = M$ implies $L = M$. An epimorphism $g : M \to N$ is superfluous in case $Kerg \ll M$ [Anderson and Fuller, 1992].

Definition 2.43 The radical of an R-module M, denoted $rad(M)$, is the sum of all superfluous submodules of M. Moreover, we will denote radical of a module R, by $J(R)$ [Anderson and Fuller, 1992].

Proposition 2.44 For any morphism $f : M \to N$, $(rad(M))f \subset radN$; moreover, if $Kerf \subset radM$, then $(radM)f = rad(Mf)$ [Anderson and Fuller, 1992].

Proposition 2.45 $\phi : P \to M$ is a projective cover if and only if P is projective, ϕ is surjective and $Ker\phi \ll P$ [Xu, 1996].

Remark 2.46 We will denote projective cover of a module M_R by a pair (P_R, ϕ) where P_R is a projective right R-module and $\phi\colon P \to M$ is a superfluous epimorphism (that is $Ker\phi \ll P$).

Examples 2.47 (1) Let us show that a $\mathbb{Z}$-module has a projective cover if and only if it is free.

($\Leftarrow$) If G is a free $\mathbb{Z}$-module then $G \cong \mathbb{Z}^{(X)}$ for some set X. Take $\varphi\colon \mathbb{Z}^{(X)} \to G$ isomorphism. Then $(\mathbb{Z}^{(X)}, \varphi)$ is a projective cover of G, because 0 is always superfluous (in general, every projective module has a projective cover, the trivial one).

($\Rightarrow$) Assume that the $\mathbb{Z}$-module G has a projective cover $(P_{\mathbb{Z}}, p)$. $P_{\mathbb{Z}}$ is free because $\mathbb{Z}$ is hereditary (every projective $\mathbb{Z}$-module is free). Let $\varphi\colon P_{\mathbb{Z}} \to \mathbb{Z}^{(X)}$ be an isomorphism. Let, for $x \in X$, $\pi_x\colon \mathbb{Z}^{(X)} \to \mathbb{Z}$ be the canonical projection. Then $\pi_x\varphi\colon P_{\mathbb{Z}} \to \mathbb{Z}$. Since $\pi_x\varphi(rad(P_{\mathbb{Z}})) \subseteq rad(\mathbb{Z}_{\mathbb{Z}}) = 0$ holds for every $x \in X$, $rad(P_{\mathbb{Z}}) = 0$, so the only superfluous submodule of $P_{\mathbb{Z}}$ is 0. So $\ker(p) = 0$, i.e. p is an isomorphism. Hence G is free.

(2) Let R be a ring. Let $\pi\colon R \to R/J(R)$ be the canonical projection. Then (R_R, π) is a projective cover of $R/J(R)$. In fact, the module R_R is projective, π is onto, and $\ker(\pi) = J(R)$ is superfluous in R by the Nakayama Lemma applied to the module R_R.

The same proof shows that:

(3) Let P_R be a finitely generated projective module and $\pi\colon P \to P/PJ(R)$ be the canonical projection. Then (P, π) is a projective cover of the right R-module $P/PJ(R)$.

Theorem 2.48 (Uniqueness of projective covers up to isomorphism) Projective covers, when they exist, are unique up to isomorphism in the following sense. If $(P, p), (Q, q)$ are two projective covers of a right R-module M, there is an isomorphism $h\colon Q \to P$ such that $p \circ h = q$ [Xu,1996].

Injective envelopes

Definition 2.49 A homomorphism $\phi : M \to C$ is a *$\mathcal{C}$-preenvelope* if $C \in \mathcal{C}$ and the abelian group homomorphism $Hom_R(\phi, C') : Hom(C, C') \to Hom(M, C')$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-preenvelope $\phi : M \to C$ is said to be *$\mathcal{C}$-envelope* if every endomorphism $g : C \to C$ such that $g\phi = \phi$ is an isomorphism [Xu,1996].

Name of the envelopes (preenvelopes) depend on the name of the class of $\mathcal{C}$, for example if $\mathcal{C}$ consists of injective modules then $\mathcal{C}$-envelope (preenvelope) will be called injective envelope (preenvelope). We need the following definition for the equivalent condition of injective envelope.

Definition 2.50 A submodule N of a module M_R is *essential* (or *large*) in M_R if, for every submodule L of M_R, $N \cap L = 0$ implies $L = 0$. In this case, we will write $N \leq_e M_R$. A monomorphism $f\colon N_R \to M_R$ is said to be *essential* if its image $f(N_R)$ is an essential submodule of M_R. [Anderson and Fuller, 1992]

Let M be a right R-module. An *extension* of M is a pair (N, f), where N is a right R-module and $f\colon M \to N$ is a monomorphism. An *essential extension* of M is an extension (N, f) where $f\colon M \to N$ is an essential monomorphism. An extension (N, f) is *proper* if f is not an isomorphism.

Proposition 2.51 $\phi : M \to E$ is an injective envelope if and only if E is injective, ϕ is monomorphism and $Im\phi \leq_e E$ [Xu, 1996].

Remark 2.52 We will denote injective envelope of a module M_R by a pair (E_R, ϕ) were E_R is an injective right R-module and $\phi\colon M_R \to E_R$ is an essential monomorphism.

For example, if i is the inclusion of $\mathbb{Z}_\mathbb{Z}$ into $\mathbb{Q}_\mathbb{Z}$, then $(\mathbb{Q}_\mathbb{Z}, i)$ is an injective envelope of $\mathbb{Z}_\mathbb{Z}$.

Theorem 2.53 Every right R-module has an injective envelope, which is unique up to isomorphism in the following sense: if (E, i) and (E', i) are both injective envelopes of M, then there exists an isomorphism $h\colon E \to E'$ such that $hi = i'$ [Xu,1996].

Like for the tensor product of two modules, the injective envelope of a module M_R, which is unique up to isomorphism, will not be usually denoted as a pair. We will usually omit to indicate the embedding of M_R into the injective module. The injective envelope of a module M_R will be usually denoted by $E(M_R)$.

2.6 Projective Resolutions, Injective Resolutions

Definition 2.54 A *projective resolution* of a module M_R is a complex $\mathbb{P}$ such that $P_i = 0$ for every $i < 0$, P_i is projective for every $i \in \mathbb{Z}$ and there exists an epimorphism $\varepsilon\colon P_0 \to M$ such that the complex

$$\cdots P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M_R \longrightarrow 0$$

is an exact sequence [Rotman, 2008].

Definition 2.55 If $\mathbb{P}$ is a projective resolution of M, then its *deleted projective resolution* is the complex

$$\mathbb{P}_M \cdots P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} 0$$

[Rotman, 2008].

Lemma 2.56 For every module M_R there is an exact sequence

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M_R \longrightarrow 0$$

with the projective modules P_i [Rotman, 2008].

Definition 2.57 An *injective resolution* of a module M_R is a complex $\mathbb{E}$ such that $E_i = 0$ for every $i < 0$, E_i is injective for every $i \in \mathbb{Z}$ and there exists a monomorphism $\varepsilon\colon M \to E_0$ such that the complex

$$0 \longrightarrow M_R \xrightarrow{\varepsilon} E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots$$

is an exact sequence [Rotman, 2008].

Definition 2.58 If $\mathbb{E}$ is an injective resolution of M, then its *deleted injective resolution* is the complex

$$\mathbb{E}_M : 0 \longrightarrow E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots$$

[Rotman, 2008].

Lemma 2.59 For every module M_R there is an exact sequence

$$0 \longrightarrow M_R \longrightarrow E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots$$

with the E_i's injective modules [Rotman, 2008].

Definition 2.60

$$\cdots C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} M_R \longrightarrow 0$$

be a left $\mathcal{C}$-resolution of M_R. Let $K_0 = ker\varepsilon$ and $K_n = kerd_n$. We call K_n the *nth $\mathcal{C}$-syzygy* of M [Mao and Ding, 2007].

Definition 2.61

$$\cdots P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M_R \longrightarrow 0$$

be a projective resolution of M_R. Let $K_0 = ker\varepsilon$ and $K_n = kerd_n$. We call K_n the *nth syzygy of M* [Rotman, 2008].

Definition 2.62

$$0 \longrightarrow M_R \xrightarrow{\varepsilon} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} E_2 \longrightarrow \cdots$$

be an right $\mathcal{C}$-resolution of M_R. Let $L^0 = coker\varepsilon$ and $L^n = cokerd^{n-1}$. We call L^n the *nth $\mathcal{C}$-cosyzygy* of M [Mao and Ding, 2007].

Definition 2.63

$$0 \longrightarrow M_R \xrightarrow{\varepsilon} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} E_2 \longrightarrow \cdots$$

be an injective resolution of M_R. Let $L^0 = coker\varepsilon$ and $L^n = cokerd^{n-1}$. We call V^n the *nth cosyzygy of M* [Rotman, 2008].

2.7 The torsion functor $\mathrm{Tor}_i^R(M, N)$

Fix a left module ${}_RN$. $- \otimes_R N$: Mod-$R \to$ **Ab** is a covariant, additive and right exact functor. Hence we can write,

$$\mathrm{Tor}_i^R(-, N)\colon \text{Mod-}R \to \mathbf{Ab}.$$

For every M_R fix a deleted projective resolution $\mathbb{P}_M$, and apply $-\otimes_R N$ getting a complex

$$\cdots \overset{d_2\otimes 1_N}{\to} P_1 \otimes_R N \overset{d_1\otimes 1_N}{\to} P_0 \otimes_R N \to 0.$$

Then

$$\mathrm{Tor}_i^R(M, N) = H_i(\mathbb{P}_M \otimes_R N) = ker(d_i \otimes 1_N)/(d_{i+1} \otimes 1_N)(P_{i+1} \otimes N)$$

Since $-\otimes_R N$ is right exact, $M \otimes_R N \cong \mathrm{Tor}_0^R(M, N)$. Obviously, $\mathrm{Tor}_i^R(M, N) = 0$ for every $i < 0$.

If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of right R-modules, there is a long exact sequence

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathrm{Tor}_1^R(M', N) & \longrightarrow & \mathrm{Tor}_1^R(M, N) & \longrightarrow & \mathrm{Tor}_1^R(M'', N) & \\
 & & & & & \downarrow & \\
0 \longleftarrow & M'' \otimes_R N & \longleftarrow & M \otimes_R N & \longleftarrow & M' \otimes_R N &
\end{array}$$

Theorem 2.64 The following are equivalent for a right module M_R:

1. M_R is flat, i.e. $M \otimes_R -$ is an exact functor.

2. $\mathrm{Tor}_i^R(M, N) = 0$ for every $i > 0$, ${}_RN$.

3. $\mathrm{Tor}_1^R(M, N) = 0$ for every ${}_RN$.

[Rotman, 2008]

2.8 The $\mathrm{Hom}_R(M, -)$ and $\mathrm{Hom}_R(-, N)$ functors

Let N_R be fixed. Then there is a contravariant functor $Hom_R(-, N)$: Mod-$R \to$ **Ab**. Consider the deleted projective resolution $\mathbb{P}_M$ and apply the $Hom_R(-, N)$ functor

$$0 \longrightarrow Hom_R(P_0, N) \xrightarrow{d_1^*} Hom_R(P_1, N) \xrightarrow{d_2^*} Hom_R(P_2, N) \longrightarrow \cdots$$

Then

$$\mathrm{Ext}_R^i(M, N) = H_i(Hom_R(\mathbb{P}_M, N)) = ker(d_{i+1}^*)/Im(d_i^*)$$

Any short exact sequence of right R-modules

$$0 \to M' \to M \to M'' \to 0$$

induces a long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & Hom_R(M'', N) & \longrightarrow & Hom_R(M, N) & \longrightarrow & Hom_R(M', N) \\ & & & & & & \downarrow \\ \cdots & \longleftarrow & Ext_R^1(M', N) & \longleftarrow & Ext_R^1(M, N) & \longleftarrow & Ext_R^1(M'', N) \end{array}.$$

Fix M_R. There is a covariant functor $Hom_R(M,-)$: Mod-$R \to \mathbf{Ab}$. Consider a deleted injective resolution of N_R, and apply to it the functor $Hom_R(M,-)$,

$$0 \longrightarrow Hom_R(M,E_0) \xrightarrow{d_1^*} Hom_R(M,E_1) \xrightarrow{d_2^*} Hom_R(M,E_2) \longrightarrow \cdots$$

Then,

$$\mathrm{Ext}_R^i(M,N) = H_i(Hom_R(M,\mathbb{E}_N)) = ker(d_{i+1}^*)/Im(d_i^*)$$

Any short exact sequence

$$0 \to N' \to N \to N'' \to 0$$

induces a long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & Ext_R^0(M,N') & \longrightarrow & Ext_R^0(M,N) & \longrightarrow & Ext_R^0(M,N'') \\ & & & & & & \downarrow \\ \cdots & \longleftarrow & Ext_R^1(M,N'') & \longleftarrow & Ext_R^1(M,N) & \longleftarrow & Ext_R^1(M,N') \end{array}$$

Since $H_M = Hom(M,-)$ is left exact, $Ext_R^0(M,N) \cong Hom_R(M,N)$.

Theorem 2.65 The following are equivalent for M_R:

1. M_R is projective.
2. $Ext_R^i(M,N_R) = 0$ for every $i > 0$, N_R.
3. $Ext_R^1(M,N_R) = 0$ for every ${}_RN$.

[Rotman, 2008].

3 RELATIVE COPURE INJECTIVE AND COPURE FLAT MODULES

In this section, all things we used without reference, belongs to [11].

Definition 3.1 Let R be a ring, n a fixed nonnegative integer and $\mathcal{I}_n$ the class of all left $R-$modules of injective dimension at most n. A left $R-$module M is called $n-$*copure injective* if $Ext^1(N, M) = 0$ for any $N \in \mathcal{I}_n$. A right $R-$module F is said to be $n-$*copure flat* if $Tor_1(F, N) = 0$ for any $N \in \mathcal{I}_n$. For an R-module M *the character module* $Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+.

Proposition 3.2 Let R be any ring.

(1) If $Ext^i_R(N, M) = 0$ for any i with $1 \leq i \leq n+1$ and any injective left $R-$module N, then every kth cosyzygy of M is $(n-k)-$copure injective for any k with $0 \leq k \leq n$, in particular, M is $n-$copure injective.

(2) If $Tor^R_i(M, N) = 0$ for any i with $1 \leq i \leq n+1$ and any injective left $R-$module N, then every kth syzygy of M is $(n-k)-$copure flat for any k with $0 \leq k \leq n$, in particular, M is $n-$copure flat.

Proof . (1) Let k be an integer with $0 \leq k \leq n$, L^k a kth cosyzygy of M and N a left $R-$module with $id(N) \leq n-k$. Then $Ext^1_R(N, L^k) \cong Ext^{k+1}_R(N, M)$. On the other hand, there is an exact sequence $0 \to N \to E^0 \to E^1 \to \ldots \to E^{n-k} \to 0$ with each E^i injective (for $id(N) \leq n-k$), and so $Ext^{k+1}_R(N, M) \cong Ext^{n+1}_R(E^{n-k}, M) = 0$ by assumption. Thus $Ext^1_R(N, L^k) = 0$, and hence L^k is $(n-k)-$copure injective.

(2) The proof is similar to that of (1). ■

Proposition 3.3 The following are equivalent for a left $R-$module M:

(1) M is $n-$copure injective.

(2) For every exact sequence $0 \to M \to E \to L \to 0$ with $E \in I_n$, $E \to L$ is an $\mathcal{I}_n$-precover of L.

(3) M is a kernel of an $\mathcal{I}_n$-precover $f : A \to B$ with A injective.

(4) M is injective with respect to every exact sequence $0 \to A \to B \to C \to 0$ with $C \in \mathcal{I}_n$.

PROOF . (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4) are clear by definition.

(2) $\Rightarrow$ (3) Since there is a short exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$ and $E(M) \in \mathcal{I}_n$, then (3) follows from (2).

(3) $\Rightarrow$ (1) Let M be a kernel of an $\mathcal{I}_n$-precover $f : A \to B$ with A injective. Then we have an exact sequence $0 \to M \to A \to A/M \to 0$. So, for any $N \in \mathcal{I}_n$, the sequence $Hom(N, A) \to Hom(N, A/M) \to Ext^1(N, M) \to 0$ is exact. It is easy to verify that $Hom(N, A) \to Hom(N, A/M) \to 0$ is exact by (3). Thus $Ext^1(N, M) = 0$, and so (1) follows.

(4) $\Rightarrow$ (1) For each $N \in \mathcal{I}_n$, there exists a short exact sequence $0 \to K \to P \to N \to 0$ with P projective, which induces an exact sequence $Hom(P, M) \to Hom(K, M) \to Ext^1(N, M) \to 0$. Note that $Hom(P, M) \to Hom(K, M) \to 0$ is exact by (4). Hence $Ext^1(N, M) = 0$, as desired. ■

Corollary 3.4 *Let R be a left noetherian ring. Then every $(n+1)$th $\mathcal{I}_0$-syzygy of any left R-module is n-copure injective.*

PROOF . Let $\ldots \to E_1 \to E_0 \to M \to 0$ be a left $\mathcal{I}_0$-resolution of a left R-module M. By [7, Lemma 8.4.34], $E_n \to K_n$ is an $\mathcal{I}_n$-precover, where K_n is the nth $\mathcal{I}_0$-

syzygy of M, and so the $(n+1)$th $\mathcal{I}_0$-syzygy K_{n+1} of M is n-copure injective by Proposition 2.1.3 ■

Proposition 3.5 The following are equivalent for a right R-module M :

(1) M is n-copure flat.

(2) M^+ is n-copure injective.

(3) $M \in^{\perp} C$, where $C = \{B^+ : B \in I_n\}$.

(4) For every exact sequence $0 \to A \to B \to C \to 0$ of left R-modules with $C \in I_n$, the functor $M \otimes -$ preserves the exactness.

Proof . By [1, VI.5.1], there are the following standard isomorphisms:

$$Ext^1(N, M^+) \cong Tor_1(M, N)^+ \cong Ext^1(M, N^+)$$

for any left R-module N. Thus (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow.

(1) $\Leftrightarrow$ (4) is easy. ■

4 RELATIVE COTORSION MODULES AND FLAT MODULES

In this section, all things we used without reference, belongs to [13].

Definition 4.1 Given a class $\mathcal{L}$ of right R-modules, *the right orthogonal class* of $\mathcal{L}$ is

$$\mathcal{L}^{\perp} = \{C \in Mod - R \mid \mathrm{Ext}^1(L, C) = 0 \ \text{ for all } \ L \in \mathcal{L}\}$$

and *the left orthogonal class* of $\mathcal{L}$ is

$${}^{\perp}\mathcal{L} = \{C \in Mod - R \mid \mathrm{Ext}^1(C, L) = 0 \ \text{ for all } \ L \in \mathcal{L}\}$$

[Xu, 1996].

Definition 4.2 M_R is called *n-cotorsion* if $Ext_R^{n+1}(N, M) = 0$ for any flat right R-module N. M_R is said to be *n-flat* if $Ext^1(M, N) = 0$ for any n-cotorsion right R-module N. Let $\sigma_R(M) = sup\{n : \ M \ is \ n - flat\}$. Then $\sigma_R(M) = -1$ if $Ext_R^1(M, N) \neq 0$ for some cotorsion right R-module N. A monomorphism $\alpha : M \to C$ with $C \in \mathcal{C}$ is said to be *special $\mathcal{C}$-preenvelope* of M if $coker(\alpha) \in^{\perp} C$. An epimorphism $\alpha : C \to M$ with $C \in \mathcal{C}$ is said to be *special $\mathcal{C}$-precover* of M if $ker(\alpha) \in C^{\perp}$. A pair $(\mathcal{F}, \mathcal{C})$ of classes of right R-modules is called *cotorsion theory* if $\mathcal{F}^{\perp} = \mathcal{C}$ and ${}^{\perp}\mathcal{C} = \mathcal{F}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called *complete* if every right R-module has a special $\mathcal{C}$-preenvelope, and every right R-module has a special $\mathcal{F}$-precover. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is said to be *hereditary* if whenever $0 \to L' \to L \to L'' \to 0$ is exact with $L, L'' \in \mathcal{F}$, then L' is also in $\mathcal{F}$.

Proposition 4.3 The following are equivalent for a right R-module M and an integer $n \geq 0$:

(1) M is n-cotorsion;

(2) If the sequence $0 \to M \to C^0 \to C^1 \to \ldots \to C^{n-1} \to C^n \to 0$ is exact with $C^0, C^1, \ldots, C^{n-1}$ cotorsion, then C^n is also cotorsion;

(3) There exists an exact sequence $0 \to M \to C^0 \to C^1 \to \ldots \to C^{n-1} \to C^n \to 0$ with $C^0, C^1, \ldots, C^{n-1}, C^n$ cotorsion;

(4) The flat cover of M is n-cotorsion.

PROOF . (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) is standard homological algebra fare.

(1) $\Leftrightarrow$ (4) Let K be the kernel of the flat cover $N \to M$, then we have the exact sequence $0 \to K \to N \to M \to 0$ with K cotorsion by Wakamatsu's Lemma (Xu, 1996, Lemma 2.1.1). Note that $Ext^j(F, K) = 0$ for all $j \geq 1$ and flat modules F by the proof of Xu (1996, Proposition 3.1.2), so (1) $\Leftrightarrow$ (4) follows. ■

Remark 4.4 Let m and n be integers with $m \geq n > 0$. Proposition 4.2 shows that any n-cotorsion right R-module is m-cotorsion. It follows that any m-flat right R-module is n-flat.

Proposition 4.5 Let R be a ring and $0 \to A \to B \to C \to 0$ an exact sequence of right R-modules. If $\sigma_R(C) \geq 0$, then $\sigma_R(A) \geq inf\{\sigma_R(B), \sigma_R(C) + 1\}$.

Lemma 4.6 Let M and N be right $R-$modules, n a positive integer and M the union of a continuous chain of submodules $(M_\alpha)_{\alpha<\lambda}$. If $Ext^n_R(M_0, N) = 0$ and $Ext^n_R(M_{\alpha+1}/M_\alpha, N) = 0$ whenever $\alpha < \lambda$, then $Ext^n_R(M, N) = 0$.

Theorem 4.7 Let R be a ring and $n \geq 0$. Then $(\mathcal{F}_n, \mathcal{C}_n)$ is a complete hereditary cotorsion theory.

PROOF . Let F be a flat right R-module. By [7, Lemma 5.3.12], if $CardR \leq \mathfrak{N}_\beta$, then for each $x \in F$, there is a pure submodule S of F with $x \in S$ such that $CardS \leq \mathfrak{N}_\beta$. So we can write F as a union of a continuous chain $(F_\alpha)_{\alpha \leq \lambda}$ of pure submodules of F such that $CardF_0 \leq \mathfrak{N}_\beta$ and $CardF_{\alpha+1/F_\alpha} \leq \mathfrak{N}_\beta$ whenever $\alpha + 1 \leq \lambda$.

By Lemma 4.5, if N is a right R-module such that $Ext_R^{n+1}(F_0, N) = 0$ and $Ext_R^{n+1}(F_{\alpha+1}/F_\alpha, N) = 0$ whenever $\alpha + 1 \leq \lambda$, then $Ext_R^{n+1}(F, N) = 0$. Denote by F_1 the nth syzygy module of F. Then $Ext_R^{n+1}(F, N) = 0$ if and only if $Ext_R^1(F_1, N) = 0$. Let X be the set of representatives of nth syzygy modules of all flat right R-modules G with $CardG \leq \mathfrak{N}_\beta$. Then $C_n = X^\perp$, and so $(\mathcal{F}_n, \mathcal{C}_n)$ is complete by [7, Theorem 7.4.1].

On the other hand, let $0 \to A \to B \to C \to 0$ be exact with $A, B \in \mathcal{C}_n$. Then $C \in \mathcal{C}_n$ by Remark 4.3. So $(\mathcal{F}_n, \mathcal{C}_n)$ is hereditary by Enochs et al. (2004, Proposition 1.2). ■

Theorem 4.8 Let R be a ring and n a fixed non-negative integer. Then the following are equivalent:

(1) Every right R-module is n-cotorsion;

(2) Every $(n-)$flat right R-module is n-cotorsion;

(3) Every n-flat right R-module is projective;

(4) $Ext_R^{n+1}(M, N) = 0$ for all flat right $R-$modules M and N.

PROOF . (1) $\Rightarrow$ (4) $\Rightarrow$ (2) is obvious. (1) $\Leftrightarrow$ (3) is clear since $(\mathcal{F}_n, \mathcal{C}_n)$ is a cotorsion theory.

(2) $\Rightarrow$ (1) Let M be a right R-module. By Theorem 4.6, M has a special $\mathcal{F}_n$-precover, and hence there is a short exact sequence $0 \to K \to N \to M \to 0$, where K is n-cotorsion and N is n-flat. Since N is n-cotorsion by (2), M is n-cotorsion by Remark 4.3. So (1) follows. ■

Theorem 4.9 Let R be a ring and n a fixed non-negative integer. Then the following are equivalent:

(1) Every right R-module is n-flat;

(2) Every n-cotorsion right R-module is n-flat;

(3) Every n-cotorsion right R-module is injective.

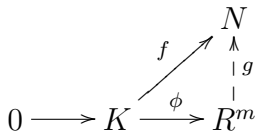
PROOF . (1) $\Rightarrow$ (2) is obvious. (1) $\Leftrightarrow$ (3) follows from Theorem 4.6.

(2) $\Rightarrow$ (1) For any right R-module M, by Theorem 4.6, there is a short exact sequence $0 \to M \to N \to L \to 0$, where N is n-cotorsion and L is n-flat. Since N is n-flat by (2), M is n-flat by Proposition 4.4. Hence (1) follows. ■

5 COPURE (m,n)-INJECTIVE MODULES AND (α,m,n)-COTORSION MODULES

All the things of this section is the original part of the thesis.

Now, we recall that an R-module M is called (m,n)-*presented* if there exists an exact sequence of right R-modules $0 \to K \to R^m \to M \to 0$, where K is an n-generated. So call M *a copure* (m,n)-*injective module* if $\mathrm{Ext}_R(N,M) = 0$ for all (m,n)-injective module N. Let $\mathcal{I}_{(m,n)}$ denote the class of all (m,n)-injective modules. So a module N belongs to $\mathcal{I}_{(m,n)}$ if and only if the following diagram is commutative for any map $f : K \to N$ where K is n-generated



5.1 Copure (m,n)-injective modules

The following theorem generalizes [11, Proposition 2.4] to (m,n)-injective modules.

Theorem 5.1 The following are equivalent for an R-module M.

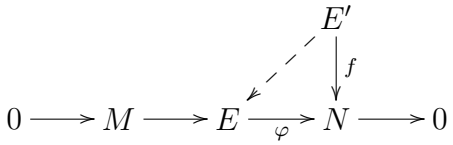
(1) M is a copure (m,n)-injective module.

(2) For any exact sequence $0 \to M \to E \xrightarrow{\varphi} N \to 0$ with $E \in \mathcal{I}_{(m,n)}$, $E \xrightarrow{\varphi} N \to 0$ is a precover of N in $\mathcal{I}_{(m,n)}$.

(3) M is a kernel of an $\mathcal{I}_{(m,n)}$-precover $\pi : A \to A/M$ with A injective.

(4) M is an injective module with respect to $0 \to A \to B \to C \to 0$ with $C \in \mathcal{I}_{(m,n)}$.

PROOF (1) $\Rightarrow$ (2) Let $0 \to M \to E \xrightarrow{\varphi} N \to 0$ be any exact sequence and let $E' \in \mathcal{I}_{(m,n)}$ where $E \in \mathcal{I}_{(m,n)}$. Consider the following diagram.



If we apply $\mathrm{Hom}_R(E',)$, we can obtain that, by (1),

$$\mathrm{Hom}_R(E', M) \to \mathrm{Hom}_R(E', E) \xrightarrow{\varphi^*} \mathrm{Hom}_R(E', N) \to \mathrm{Ext}^1_R(E', M) = 0$$

and E' is (m,n)-injective by assumption. Hence there exists a $g \in \mathrm{Hom}_R(E', E)$ such that $\varphi^*(g) = f$ so $g\varphi = f$.

(2) $\Rightarrow$ (3) Clearly, the sequence

$$0 \to M \to E(M) \xrightarrow{\varphi} E(M)/M \to 0$$

is exact. Since $E(M)$ is injective, it is (m,n)-injective. By (2),

$$E(M) \xrightarrow{\varphi} E(M)/M$$

is an $\mathcal{I}_{(m,n)}$-precover.

(3) $\Rightarrow$ (1) By (3), M is a kernel of an $\mathcal{I}_{(m,n)}$-precover $\beta : E \to E/M$. So there is an exact sequence $0 \to M \xrightarrow{\alpha} E \xrightarrow{\beta} E/M \to 0$ where E is injective. Let N be any (m,n)-injective module. Then the sequence

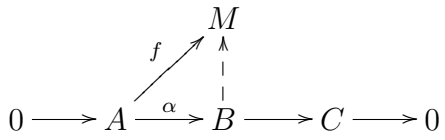
$$\mathrm{Hom}_R(N, E) \xrightarrow{\beta^*} \mathrm{Hom}_R(N, E/M) \xrightarrow{\alpha^*} \mathrm{Ext}^1_R(N, M) \to 0$$

is exact. Since $\beta : E \to E/M$ is an $\mathcal{I}_{(m,n)}$-precover with E injective, every map from N to E/M lifts to E. Therefore

$$\mathrm{Hom}_R(N, E) \xrightarrow{\beta^*} \mathrm{Hom}_R(N, E/M) \xrightarrow{\alpha^*} 0.$$

Hence $\mathrm{Ext}^1_R(N, M) = 0$, and so M is an (m, n)-injective module.

(1) $\Rightarrow$ (4) Consider the following diagram with $C \in \mathcal{I}_{(m,n)}$.



If we apply $\mathrm{Hom}_R(, M)$, we can obtain that

$$\mathrm{Hom}_R(B, M) \xrightarrow{\alpha^*} \mathrm{Hom}_R(A, M) \to \mathrm{Ext}^1_R(C, M) = 0$$

and C is (m, n)-injective by (1). Hence there exists a $g \in \mathrm{Hom}_R(B, M)$ such that $\alpha^*(g) = f$, i.e., $g\alpha = f$.

(4) $\Rightarrow$ (1). Let $0 \to M \xrightarrow{f} E \to N \to 0$ be an exact sequence such that N is an (m, n)-injective module. By (4), the identity map 1_M of M extends to an $f : E \to M$ such that $gf = 1_M$ and so $\mathrm{Ext}^1_R(N, M) = 0$.

■

Corollary 5.2 Let R be a right noetherian ring. Then every nth $\mathcal{I}_0$-syzygy of any right R-module is copure (m, n)-injective.

Proof Let $\cdots \to E_1 \to E_0 \to M \to 0$ be a right $\mathcal{I}_0$-resolution of a right R-module M. By [7, Lemma 8.3.34], $E_i \to K_i$ is an $\mathcal{I}_n$-precover, where K_i is nth

$\mathcal{I}_0$-syzygy of M, that is $K_i = \ker(E_{i-1} \to E_{i-2})$, and so, by Theorem 5.1, the module K_i is copure (m,n)-injective for $i \geq 1$. ■

According to Enochs and Jenda [5], a right R-module F is said to be *copure flat* if $\mathrm{Tor}_1(F,N) = 0$ for every injective left R-module N. So we call the module M_R *copure* (m,n)*-flat* if $\mathrm{Tor}_1(M,N) = 0$ for every (m,n)-injective left R-module N.

The following theorem generalizes Theorem 2.6 in [11].

Theorem 5.3 The following are equivalent for any right R-module M.

(1) M is a copure (m,n)-flat module.

(2) M^+ is a copure (m,n)-injective module.

(3) M belongs to the class ${}^{\perp}(\mathcal{I}^+_{(m,n)})$ where $\mathcal{I}^+_{(m,n)} = \{N^+ \mid N \in \mathcal{I}_{(m,n)}\}$.

(4) For any exact sequence $0 \to A \to B \to C \to 0$ of left R-modules with $C \in \mathcal{I}_{(m,n)}$, the functor $M \otimes_R -$ is exact.

Proof By [1, VI, 5.1] or [16, p. 360], there are the following standard isomorphisms:

$$\mathrm{Ext}^1_R(N, M^+) \cong \mathrm{Tor}_1(M,N)^+ \cong \mathrm{Ext}^1_R(M, N^+)$$

for any (m,n)-injective left R-module N. Thus (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow.

(1) $\Rightarrow$ (4) Let $0 \to A \to B \to C \to 0$ be any sequence of left R-modules with $C \in \mathcal{I}_{(m,n)}$. Apply the functor $M \otimes_R -$ to this sequence to obtain

$$\cdots \to \mathrm{Tor}^R_1(M,C) \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0.$$

Since C is (m,n)-injective, by (1) $\mathrm{Tor}_1^R(M,C) = 0$ for all $C \in \mathcal{I}_{(m,n)}$. Hence $M \otimes -$ is exact. This is (4).

(4) $\Rightarrow$ (1) To prove (1), for any $C \in \mathcal{I}_{(m,n)}$ we show $\mathrm{Tor}_1^R(M,C) = 0$. So let C be a (m,n)-injective module with any exact sequence $0 \to A \to B \to C \to 0$. By (4) and applying the functor $M \otimes_R -$ to this sequence we obtain

$$0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0.$$

Hence $\mathrm{Tor}_1^R(M,C) = 0$. This completes the proof. ∎

5.2 (α, m, n)-cotorsion modules

Let M be a right R-module and α a fixed nonnegative integer. Recall that M is called a *cotorsion module* if $\mathrm{Ext}_R^1(F,M) = 0$ for any flat right R-module F [4], and the module M is called *α-cotorsion* if $\mathrm{Ext}_R^{\alpha+1}(N,M) = 0$ for any flat right R-module N. So we call the module M_R *(α,m,n)-cotorsion* if $\mathrm{Ext}_R^{\alpha+1}(N,M) = 0$ for any copure (m,n)-flat right R-module N.

Theorem 5.4 If for any right R-homomorphism $f{:}M_1 \to M_2$ such that M_1 and M_2 are (α,m,n)-cotorsion modules, $ker(f)$ is (α,m,n)-cotorsion then every right R-module is (α,m,n)-cotorsion.

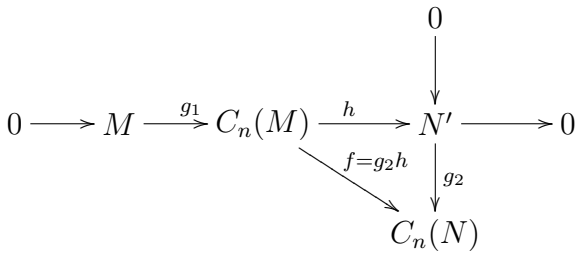
Proof We will show that every copure (m,n)-flat right R-module is (α,m,n)-cotorsion. Let M be a copure (m,n)-flat module R-module. Then we have the sequences

$$0 \longrightarrow M \xrightarrow{g_1} C_n(M) \longrightarrow 0$$

and

$$0 \longrightarrow N \xrightarrow{g_2} C_n(N).$$

Hence we can obtain the following exact commutative diagram



and so

$$0 \longrightarrow M \longrightarrow C_n(M) \longrightarrow C_n(N)$$

is exact such that $C_n(M)$ and $C_n(M')$ are copure (m,n)-flat modules. This implies that $M = ker(g_2) = ker(f)$ is a copure (m,n)-flat module by (3). ■

Theorem 5.5 The following are equivalent for a ring R:

(1) Every right R-module is copure (m,n)-flat.

(2) For any right R-homomorphism $f{:}M_1 \to M_2$ such that M_1 and M_2 are copure (m,n)-flat modules, $coker(f)$ is copure (m,n)-flat.

(3) Every non-zero right R-module contains a non-zero copure (m,n)-flat submodule.

PROOF (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are clear.

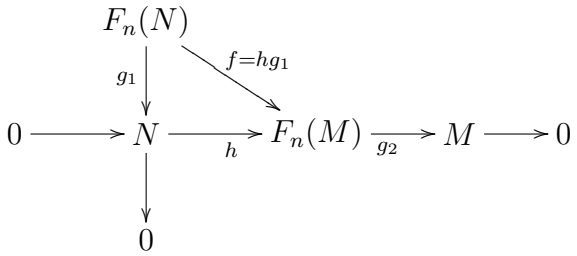
(2) $\Rightarrow$ (1) We will show that Lemma 5.4 (2-b) is satisfied. Let M be an (α, m, n)-cotorsion right R-module. Then we have the sequences

$$F_n(N) \xrightarrow{g_1} N \longrightarrow 0$$

and

$$F_n(M) \xrightarrow{g_2} M \longrightarrow 0\ .$$

Hence we can obtain the following exact commutative diagram



and so

$$F_n(K) \longrightarrow F_n(M) \longrightarrow M \longrightarrow 0$$

is exact with $F_n(N)$ and $F_n(M)$ n-flat, then $M \cong \mathrm{coker}(h) \cong \mathrm{coker}(f)$ is copure (m, n)-flat by (2).

(3) $\Rightarrow$ (1). We will show that Lemma 5.4 (2-c) is satisfied. Assume that $0 \to A \to B \to C \to 0$ is any exact sequence and A is a submodule of B. Let M be an (α, m, n)-cotorsion right R-module and $f{:}A \to M$ be any homomorphism. By Zorn's Lemma, we have $g{:}D \to M$ where $A \subseteq D \subseteq B$, $g|_A = f$, such that g can not be extended to any submodule of B properly containing D. We claim that $D = B$. By (3), there exists a non-zero submodule N/D of B/D such that N/D is copure (m, n)-flat. Since M is (α, m, n)-cotorsion, there is an $h{:}N \to M$ such

that $h|_D = g$. It is obvious that h extends g, we get the desired contradiction, and so M is injective. ■

Theorem 5.6 Let $\mu : N \to M$ be a monomorphism. If $\mathrm{coker}(\mu)$ is copure (m,n)-flat, then $i\mu : N \to H$ is an (α, m, n)-cotorsion preenvelope of N whenever $i : M \to H$ is an (α, m, n)-cotorsion preenvelope of M.

PROOF Let L be an (α, m, n)-cotorsion right R-module and $g : N \to L$ be any R-homomorphism. The exactness of the sequence

$$0 \longrightarrow N \xrightarrow{\mu} M \longrightarrow coker(\mu) \longrightarrow 0$$

induces an exact sequence

$$\mathrm{Hom}_R(M, L) \longrightarrow \mathrm{Hom}_R(N, L) \longrightarrow \mathrm{Ext}^1_R(coker(\mu), L)\ .$$

Since $\mathrm{coker}(\mu)$ is copure (m,n)-flat, $\mathrm{Hom}_R(M, L) \longrightarrow \mathrm{Hom}_R(N, L)$ is epic. Therefore there exists a $\beta : M \to L$ with $g = \beta\mu$. Then there exists a $\gamma : H \to L$ such that $\beta = \gamma i$. Hence $\gamma(i\mu) = (\gamma i)\mu = \beta\mu = g$. ■

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