

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

**ON NORMALIZED LAPLACIAN SPECTRUM OF SOME
SPECIAL GRAPHS**

NURSEMİN FERATLAR
A THESIS SUBMITTED FOR THE DEGREE OF
MASTER OF SCIENCE
DEPARTMENT OF MATHEMATICS

GEBZE
2018

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

**ON NORMALIZED LAPLACIAN
SPECTRUM OF SOME SPECIAL GRAPHS**

NURSEMİN FERATLAR
A THESIS SUBMITTED FOR THE DEGREE OF
MASTER OF SCIENCE
DEPARTMENT OF MATHEMATICS

THESIS SUPERVISOR
ASSIST. PROF. DR. ROGHAYEH HAFEZIEH

GEBZE
2018

T.C
GEBZE TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

**BAZI ÖZEL ÇİZGELERİN NORMALİZE
LAPLACIAN SPEKTRUMU**

**NURSEMİN FERATLAR
YÜKSEK LİSANS TEZİ
MATEMATİK ANABİLİM DALI**

DANIŞMANI
DR.ÖĞR. ÜYESİ ROGHAYEH HAFEZIEH

**GEBZE
2018**

GTÜ *Fen Bilimleri* Enstitüsü Yönetim Kurulu'nun *27/06/2018* tarih ve *2018/33* sayılı kararıyla oluşturulan jüri tarafından *20/09/2018* tarihinde tez savunma sınavı yapılan *Nursemin Feratlar*'ın tez çalışması *Matematik* Anabilim Dalında YÜKSEK LİSANS tezi olarak kabul edilmiştir.

JÜRİ

ÜYE

(TEZ DANIŞMANI)

: Dr. Öğr. Üyesi *Raghad HAFEZIEH*

20.09.2018

ÜYE

: Prof. Dr. Sibel *ÖZKAN*

ÜYE

: Dr. Öğr. Üyesi *İpek Tuvaç i. Kır*

ONAY

Gebze Teknik Üniversitesi Fen Bilimleri Enstitüsü Yönetim Kurulu'nun

...../...../..... tarih ve/..... sayılı kararı.

SUMMARY

Spectral graph theory looks at the interplay between the structure of a graph and the eigenvalues of a matrix associated with the graph. Many interesting graphs have rich structure which can help in determining eigenvalues associated with some particular matrix of a graph. There are many different matrices that are considered, including the adjacency matrix A whose entries indicate when two vertices are adjacent, Laplacian matrix L and normalized Laplacian matrix $\tilde{L}$.

Let Γ be an undirected graph without an isolated vertex, then the normalized Laplacian matrix $\tilde{L}(\Gamma)$ is defined as $\tilde{L}(\Gamma) = D^{-\frac{1}{2}} L(\Gamma) D^{-\frac{1}{2}}$ where D is diagonal matrix whose entries are degrees of vertices of Γ . The eigenvalues of $\tilde{L}(\Gamma)$ are called as the normalized Laplacian eigenvalues of Γ . Cavers in [Cavers, 2010] focused on graphs that are co-spectral with respect to the normalized Laplacian eigenvalues. Furthermore, he discussed properties of graphs with few normalized Laplacian eigenvalues. Almost at the same time, he and Fallat [Cavers et al., 2010] considered the energy of a simple graph with respect to its normalized Laplacian eigenvalues, which is called the $\tilde{L}$ -energy. Van Dam and Omid in [Dam and Omid, 2011] gave a combinatorial characterization of those graphs whose normalized Laplacian have three distinct eigenvalues.

The goal of this thesis is to investigate the spectrum of the normalized Laplacian of some special graphs, in particular, we will be looking at understanding how eigenvalues can be computed for some types of join graphs, as a note on [Hakimi-Nezhaad and Ghorbani, 2017].

Keywords: Adjacency Matrix, Normalized Laplacian Matrix, Join of Graphs, Integral Eigenvalue, Algebraic Graph Theory.

ÖZET

Spektral çizge teorisi, bir çizgenin yapısı ve çizgeye ait herhangi bir matrisin özdeğerleri arasındaki ilişkiyi inceler. Birçok ilginç çizge, çizgenin bazı özel matrisleri ile ilişkili özdeğerlerin belirlenmesine yardımcı olabilen zengin bir yapıya sahiptir.

İki düğümün ardışık olması durumuna göre elemanları belirlenen bitişiklik matrisi A dahil, Laplacian matris L ve normalize Laplacian matris $\tilde{L}$ gibi pek çok farklı matris vardır.

Γ , ayrık bir düğümü bulunmayan yönsüz bir çizge olsun. Bu durumda D , bu çizgedeki düğümlerin derecelerini içeren köşegen matris olmak üzere, normalize Laplacian matrisi $\tilde{L}(\Gamma) = D^{-\frac{1}{2}} L(\Gamma) D^{-\frac{1}{2}}$ olarak tanımlanır.

$\tilde{L}(\Gamma)$ matrisinin özdeğerleri, çizgenin normalize Laplacian özdeğerleri olarak adlandırılır. Cavers [Cavers, 2010]'de, normalize Laplacian özdeğerleri ile ilgili olarak ko-spektral olan çizgelere odaklanmıştır. Ayrıca, birkaç normalize Laplacian özdeğeri ile çizgelerin özelliklerini tartışmıştır. Aynı zamanda Cavers ve Fallat [Cavers et al., 2010]'de, $\tilde{L}(\Gamma)$ enerji adı verilen normalize Laplacian özdeğerleri ile ilgili basit bir çizgenin enerjisi üzerinde durmuştur. Van Dam ve Omid [Dam and Omid, 2011]'te, normalize Laplacian'ın üç farklı özdeğerine sahip olan bu çizgelerin bir kombinatoriyal karakterizasyonunu vermiştir.

Bu tezin amacı, bazı özel çizgelerin, normalize Laplacian spektrumunu araştırmaktır. Özellikle [Hakimi-Nezhaad and Ghorbani, 2017] içeriğinden faydalanan, bazı birleşik çizge türleri için, özdeğerlerin nasıl hesaplanabileceğini anlamaya çalışacağız.

Anahtar Kelimeler: Bitişiklik Matrisi, Normalize Laplacian Matris, Birleşik Çizgeler, Tamsayı Özdeğerler, Cebirsel Çizge Teorisi.

ACKNOWLEDGEMENTS

As a woman, my first great thank is for Mustafa Kemal Atatürk, who gave us rights to be educated and be able to exist in society.

I would like to thank my supervisor Assist.Prof. Roghayeh HAFEZIEH, who always supported me and tried to improve me despite my deficiencies through this research and my master degree education.

I would like to thank my friends, who always supported me. Especially i would like to thank Burak who has a great place in my life like a brother.

I would like to thank my mother and my father, who taught me how to love and to be loved and to respect others, for supporting me spiritually throughout writing this thesis and my life in general.

TABLE of CONTENTS

	<u>Page</u>
SUMMARY	v
ÖZET	vi
ACKNOWLEDMENTS	vii
TABLE of CONTENTS	viii
LIST of ABBREVIATIONS and ACRONYMS	ix
LIST of FIGURES	x
1. INTRODUCTION	1
2. PRELIMINARIES AND BASIC RESULTS	4
2.1. Preliminary Definitions and Results in Graph Theory	4
2.2. Preliminary Definitions and Results in Linear Algebra	9
3. SPECTRAL GRAPH THEORY	23
3.1. Matrix Representations of Graphs	23
4. NORMALIZED LAPLACIAN SPECTRUM	33
4.1. Properties of Laplacian Matrix	33
4.2. Properties of Normalized Laplacian Matrix	36
5. THE NORMALIZED LAPLACIAN SPECTRA OF TWO JOIN GRAPHS	39
6. FURTHER RESULTS ON TWO NEW JOIN GRAPHS	52
6.1. Integrality of Two New Join Graphs	52
6.2. Normalized Laplacian Energy	56
7. CONCLUSION	60
REFERENCES	61
BIOGRAPHY	64

LIST of ABBREVIATIONS and ACRONYMS

<u>Abbreviations</u>	<u>Explanations</u>
<u>and Acronyms</u>	
$A(\Gamma)$	: Adjacency matrix of a graph Γ .
$D(\Gamma)$	: Degree matrix of a graph Γ .
$\deg(v)$	: Degree of a vertex v in a graph Γ .
$\text{diam}(\Gamma)$	: Diameter of a graph Γ .
$\varepsilon(\Gamma)$	: The energy of a graph Γ .
$L(\Gamma)$	: Line graph of a graph Γ .
$\mathbb{L}(\Gamma)$	: Laplacian matrix of a graph Γ .
$\tilde{\mathbb{L}}(\Gamma)$	: Normalized Laplacian matrix of a graph Γ .
$\tilde{\mathbb{L}}_e(\Gamma)$	: The normalized Laplacian energy of a graph Γ .
$N(\Gamma)$	: The tree number of a graph Γ .
$\mathfrak{N}_\Gamma(v)$	: The neighbourhood of a vertex v in a graph Γ .
$R(\Gamma)$	: Incidence matrix of a graph Γ .
$\mathfrak{R}_e(\Gamma)$	: The incidence energy of a graph Γ .
$S(\Gamma)$	: The graph subdivision.
$\text{Spec}(\mathbb{L}(\Gamma))$	: The spectrum of Laplacian matrix for a graph Γ .
$\text{tr}(A)$	: Trace of a square matrix A .
$Q(\Gamma)$	: Signless Laplacian matrix of a graph Γ .

LIST of FIGURES

<u>Figure No:</u>	<u>Page</u>
2.1: A graph Γ and its edge.	5
2.2: The graph $\Gamma \setminus e$.	6
2.3: The graph Γ / e .	6
2.4: The graph Γ .	7
2.5: The subdivision graph of Γ .	8

1. INTRODUCTION

The infrastructure of spectral graph theory were laid in the middle of 19th century, though its origins can be dated back even earlier. Spectral graph theory starts by associating matrices to graphs, mostly, the adjacency matrix and the Laplacian matrix and following the general theme which is computing or estimating the eigenvalues of such matrices, and furthermore, discussing the structure of those graphs via their spectral information. The relationship between chemical eigenvalues and molecules is shown by using the spectral graph theory. Furthermore, it also appears that the graph spectra appeared in various problems of the theoretical physics and quantum mechanics.

In quantum chemistry, the concepts about spectral graph theory can be found in a paper which is published by Hückel [Hückel, 1931]. In this paper, the concept of the energy of a graph was first seen. The carbon atoms and hydrogen bonds between carbon atoms have represented as vertices and edges, respectively. To find the total energy of some electrons has used the eigenvalues.

The Matrix-Tree Theorem, which gives the number of spanning trees of a graph by using the matrix, is seen as a conclusion of spectral graph theory. This theorem was proved by Brooks, Smith, Stone and Tutte [Brooks et al., 1940] and independently by Trent [Trent, 1954] in 1954. According to many authors, the Matrix-Tree Theorem is indirectly contained in Kirchhoff's classic 1847 paper [Kirchhoff, 1847].

Cvetkovic', Rowlinson and Simic' [Cvetkovic', 1997] claim that the investigation between spectral and structural properties of graphs can be found in the unpublished thesis of Wei [Wei, 1952] from 1952 and an unpublished summary of a 1956 paper by Lihtenbaum [Lihtenbaum, 1956] communicated at the Third Congress of Mathematicians of the U.S.S.R.

Since then, spectral graph theory has been well documented in several surveys and books such as [Biggs, 1993], [Biggs et al., 1986], [Brouwer and Haemers, 2010], [Cvetkovic' et al., 1998], [Cvetkovic' et al., 1980], [Cvetkovic' et al., 1997], [Godsil and Royle, 2001], [Seidel, 1989].

In the last decade, the geometric approach has enhanced the area of spectral graph theory. For instance, Lubotzky-Phillips-Sarnak [Lubotzky et al., 1988] has

studied the eigenvalues and isoperimetric properties of graphs. Novel and powerful spectral information and techniques emerged. In brief, a new era has begun in the spectral graph theory. Verifying the fundamental properties and structure of a graph from its graph spectrum is one of the main goals in graph theory. Eigenvalues are closely related to a general graph. Especially eigenvalues play an important role in perceiving the structure of graphs. With examining the graph eigenvalues, an important interaction has been observed, especially between spectral graph theory and differential geometry.

There is an interesting analogy between spectral Riemannian geometry and spectral graph theory, see [Xiao and Hancock, 2004].

The concepts and methods of spectral geometry bring useful tools and crucial insights to the study of graph eigenvalues. Thus, new directions and results have emerged in spectral geometry. Algebraic spectral methods are also very useful, especially for extremal examples and constructions.

The recent progress on expander graphs and eigenvalues was initiated by problems in communication networks. Applications of graph eigenvalues occur in numerous areas and in different forms.

In this study, we investigate the relationship between the eigenvalues of the main graph and the associated eigenvalues for the s -vertex and s -edges graphs. The goal of this thesis is to investigate the spectrum of the normalized Laplacian of some special graphs, in particular, we will be looking at understanding how eigenvalues can be computed for some types of join graphs. We start by giving some preliminaries in graph theory and linear algebra. Then we mention some basic definitions and results that will be used throughout the thesis.

In third section we provide how the matrices of graphs is written, such as adjacency, Laplacian etc. We examine the Laplacian matrix of the some graphs depending on an edge e . We prove how the number of spanning tree of any undirected graph with at least one vertex is calculated by using the Laplacian matrix.

In fourth section we consider the spectrum of Laplacian matrix and normalized Laplacian matrix and the properties of them. After, we consider the relationship between the eigenvalues of signless Laplacian matrix and the eigenvalues of subdivision graph.

In fifth section we identify the normalized Laplacian spectrum for the s -vertex and s -edges graphs by using the eigenvalues of main graph. In addition, we

investigate how the normalized Laplacian spectrum changes if one of graphs the s-vertex and s-edges is isomorph to K_{n_2} .

In sixth section we investigate the some results on two new join graphs. We calculate both the normalized Laplacian energy and the degree Kirchhoff index for the s-vertex and s-edges graphs by defining the concepts of the $\tilde{L}$ -integral, normalized Laplacian energy and the degree Kirchhoff index.

2. PRELIMINARIES AND BASIC RESULTS

2.1. Preliminary Definitions and Results in Graph Theory

In this part we give those definitions and results in graph theory that will be used in other parts of this thesis. We follow the notations and results in [West, 2001]. We begin by the definition of a graph.

Definition 2.1: A graph Γ is an ordered pair $(V(\Gamma), E(\Gamma))$ with $V(\Gamma)$ as its vertex set and $E(\Gamma)$ as its edge set where each edge is associated with two vertices called the endpoints (not necessarily distinct). When u and v are the endpoints of an edge, they are called adjacent or neighbors. Adjacency of u and v is denoted by $\{u, v\}$ or $u - v$. For a vertex $v \in V(\Gamma)$, the neighbourhood of v is Γ , denoted by $\mathfrak{N}_\Gamma(v)$ is the following set:

$$\mathfrak{N}_\Gamma(v) = \{u \in V(\Gamma) : \{v, u\} \in E(\Gamma)\}. \quad (2.1)$$

In this case, the positive integer $|\mathfrak{N}_\Gamma(v)|$ is called the degree of v in Γ .

Definition 2.2: A loop is an edge whose endpoints are equal. Multiple edges are edges having the same pair of endpoints. A simple graph is a graph having no loops or multiple edges.

Definition 2.3: A walk in a graph is an arbitrary sequence of vertices and edges, say $v_0, e_1, v_1, e_2, \dots, e_k, v_k$ that begins and ends of a vertex. A path in a simple graph Γ is a walk where all the vertices $v_0, v_1, \dots, v_k$ are distinct. If u and v are two vertices of Γ , a path between these two vertices, is called a uv -path. A close path where only initial and final vertices repeat is called a cycle.

Definition 2.4: A graph Γ is connected if for every pair of vertices u and v of Γ there is a uv -path in Γ .

Definition 2.5: In a graph Γ , the distance between two vertices u and v is the length of a shortest path in Γ joining u and v . The diameter of Γ is the maximum distance among all pairs of vertices in Γ . It should be mentioned that if the graph is disconnected, we consider this value as ∞ .

Definition 2.6: A bipartite graph is a graph whose vertex set can be partitioned into two independent sets and no two graph vertices within the same set are adjacent. Bipartite graphs are mostly used in modeling relationships, especially between two entire separate classes of object.

Definition 2.7: A graph Γ is called r -regular if every vertex has precisely r edges, in other words, every vertex has degree r . Cycles are examples of connected 2-regular graph.

Definition 2.8: The complement $\bar{\Gamma}$ of Γ is the graph with the same vertex as Γ , where two distinct vertices are adjacent whenever they are nonadjacent in Γ .

Definition 2.9: A connected graph with no cycles is called a tree. A spanning tree of a graph Γ with n vertices, is a subset of $n - 1$ edges of Γ that form a tree. The number of distinct spanning trees in a graph Γ , denoted by $N(\Gamma)$, is called the tree number of Γ . For example; the spanning trees of the cycle graph C_4 are all pathes of length three.

Definition 2.10: With respect to a graph Γ and $e \in E(\Gamma)$ the graph obtained by deleting the edge e from Γ is denoted by $\Gamma \setminus \{e\}$, while the graph obtained by deleting the edge e of Γ and merging the endpoint of e is denoted by $\Gamma / \{e\}$.

Example 2.1: Consider the following graph:

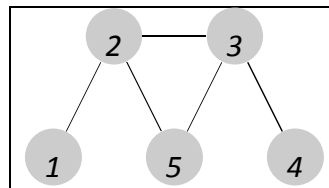


Figure 2.1: A graph Γ and its edge $\{2,3\}$.

Consider the edge $e := \{2,3\}$. With respect to the above definition $\Gamma \setminus e$ has the structure mentioned in Figure 2.2, while Γ / e is the graph in Figure 2.3, where 0 is the new vertex obtained by merging the vertices 2 and 3.

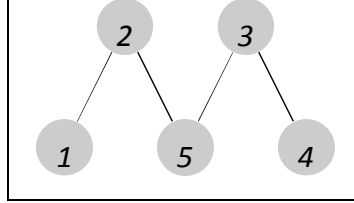


Figure 2.2: The graph $\Gamma \setminus e$.

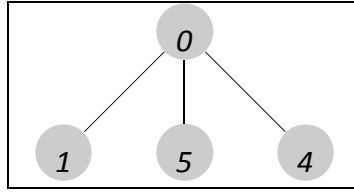


Figure 2.3: The graph Γ / e .

Lemma 2.1: Let Γ be a graph. For each $e \in E(\Gamma)$, we have:

$$N(\Gamma) = N(\Gamma \setminus e) + N(\Gamma / e). \quad (2.2)$$

Proof 2.1: There is a one to one correspondence between the set of spanning trees of Γ which contains e and the set of spanning trees of $\Gamma \setminus e$. Let

- $\alpha := \{ T : T \text{ is a spanning tree of } \Gamma \text{ passing } e \}$

and let

- $\beta := \{ T : T \text{ is a spanning tree of } \Gamma \text{ not passing } e \}$

Since $N(\Gamma) = |\alpha| + |\beta|$, $|\alpha| = N(\Gamma / e)$, and $N(\Gamma \setminus e) = |\beta|$, we have $N(\Gamma) = N(\Gamma \setminus e) + N(\Gamma / e)$.

Definition 2.11: [West, 2001] An isomorphism from a simple graph Γ to a simple graph Δ is a bijection $f : V(\Gamma) \rightarrow V(\Delta)$ such that $\{u, v\} \in E(\Gamma)$ if and only if $\{f(u), f(v)\} \in E(\Delta)$. If there is an isomorphism from Γ to Δ , these two graphs are called isomorphic and this property is denoted by $\Gamma \cong \Delta$.

Definition 2.12: Suppose Γ is a graph. Edge subdivision operation for an edge $\{u, v\} \in E(\Gamma)$ is the deletion of $\{u, v\}$ from Γ and the addition of two edges $\{u, w\}$ and $\{w, v\}$ along with the new vertex w . This operation generates a new graph Δ such that

- $\Delta = (V \cup \{w\}, E \cup \{u, v\} \cup \{\{u, w\}, \{w, v\}\})$.

Definition 2.13: For a graph Γ , the Graph subdivision, denoted by $S(\Gamma)$, is a graph obtained by applying a sequence of edge subdivision operations to all edges of the graph Γ .

Example 2.2: Consider the graph Γ as in Figure 2.4.

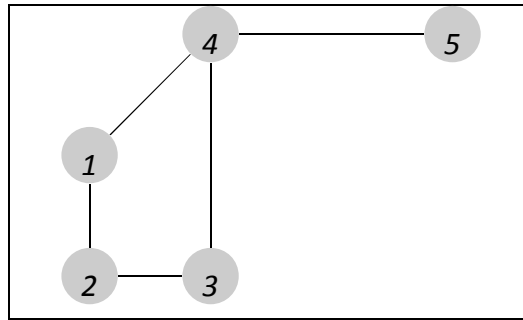


Figure 2.4: The graph Γ .

With respect to the Definition 2.13, the graph subdivision of the graph Γ will be the one in Figure 2.5.

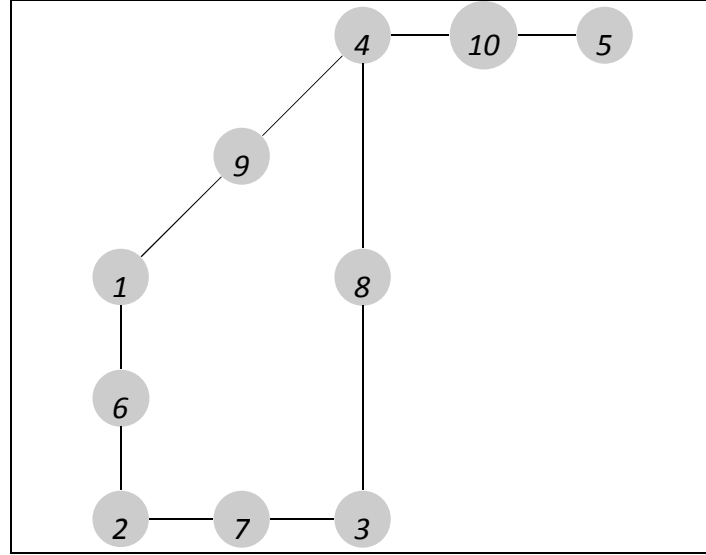


Figure 2.5: The subdivision graph.

Lemma 2.2: (Introduction in [Cvetkovic' et al., 1980]) *Subdivision graph of a graph is bipartite.*

Proof 2.2: Suppose Γ is a graph with vertex set $X := V(\Gamma)$ and edge set $Y := E(\Gamma)$. It is clear that $S(\Gamma)$ is a graph whose vertex set is $X \cup Y$ which is a bipartition for $S(\Gamma)$ as any edge of $S(\Gamma)$ has one vertex in X and one vertex in Y . Thus $S(\Gamma)$ is bipartite.

Definition 2.14: [Hakimi-Nezhaad and Ghorbani, 2017] The S_{vertex} join of two graphs Γ_1 and Γ_2 , denoted by $\Gamma_1 \cup \Gamma_2$, is the graph obtained from $S(\Gamma_1)$ and Γ_2 by joining all vertices of Γ_1 with all vertices of Γ_2 .

Definition 2.15: [Hakimi-Nezhaad and Ghorbani, 2017] The S_{edge} join of two graphs Γ_1 and Γ_2 , denoted by $\Gamma_1 \underline{\vee} \Gamma_2$ is the graph obtained from $S(\Gamma_1)$ and Γ_2 by joining all vertices of $S(\Gamma_1)$ corresponding to the edges of Γ_1 , (i.e. the new added vertices), with all vertices of Γ_2 .

Definition 2.16: Let Γ be a simple graph. The line graph, $L(\Gamma)$ of the graph Γ is the graph whose vertices are edges of the main graph Γ with the following properties:

- i) There is exactly one vertex $v(e)$ in $L(\Gamma)$ for each edge e in Γ .

- ii) For any two edges e and e' in Γ , $L(\Gamma)$ has an edge between $v(e)$ and $v(e')$, if and only if e and e' are incident with the same vertex in the graph Γ .

2.2. Preliminary Definitions and Results in Linear Algebra

In this part we give those definitions and results in linear algebra that will be used in other parts of this thesis. We follow the notations and results in [Hoffman and Kunze, 1971]. We begin by the notion of a binary operation.

Definition 2.17: On a nonempty set A , a binary operation is a map $f : A \times A \rightarrow A$ such that

- i) *f is defined for every element of $A \times A$, and*
- ii) *f uniquely associates each element of $A \times A$ to some element of A .*

It is easy to see that over the set $A := \mathbb{C}$ of complex numbers, addition $+$, subtraction $-$, and multiplication $\times$ are all binary operations.

Definition 2.18: Suppose F is a nonempty set together with two binary operations on it, called addition and multiplication. Furthermore, suppose addition and multiplication are both commutative and associative, multiplication is distributive over addition, there exist unique elements $0_F, 1_F \in F$ such that for each $c \in F$, we have $c = 0_F + c$ and $c = 1_F c$, and for each $x \in F$, there exists a unique element $x^{-1} \in F$ such that $xx^{-1} = 1_F$. Then F is called a field. The elements of F are called scalars. By easy verification, one can see that $\mathbb{R}$ and $\mathbb{C}$ are both fields with respect to their natural addition and multiplication.

Definition 2.19: Let V be a nonempty set and F be a field. V is called a vector space over the field F , if there exist two operations; a vector addition,

$$\begin{aligned} +: V \times V &\rightarrow V \\ (\alpha, \beta) &\mapsto \alpha + \beta \end{aligned} \tag{2.3}$$

and one scalar multiplication

$$\begin{aligned} \cdot : F \times V &\rightarrow V \\ (c, \beta) &\mapsto c\beta \end{aligned} \tag{2.4}$$

which satisfy the following properties:

- i) $\alpha + \beta = \beta + \alpha$ for each $\alpha, \beta \in V$;
- ii) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ for each $\alpha, \beta, \gamma \in V$;
- iii) There exists a unique zero vector, say $0_V = 0$ such that for each $\alpha \in V$ we have $0_V + \alpha = \alpha + 0_V = \alpha$;
- iv) For each vector α in V there is a unique vector $-\alpha$ in V such that $\alpha + (-\alpha) = 0_V$;
- v) For each $\alpha \in V, 1 \cdot \alpha = \alpha$;
- vi) $(c_1 c_2)\alpha = c_1(c_2\alpha)$ such that $c_1, c_2 \in F$ and $\alpha \in V$;
- vii) $c(\alpha + \beta) = c\alpha + c\beta$ where $c \in F$ and $\alpha, \beta \in V$;
- viii) $(c_1 + c_2)\alpha = c_1\alpha + c_2\alpha$ where $c_1, c_2 \in F$ and $\alpha \in V$.

Example 2.3: Let F be a field, and let

$$V := \{\alpha = (x_1, \dots, x_n) : x_i \in F, \text{ for each } 1 \leq i \leq n\}. \tag{2.5}$$

If $\beta = (y_1, \dots, y_n) \in V$, the sum of α and β is defined by

$$\alpha + \beta = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n). \tag{2.6}$$

The product of one scalar c and the vector α is defined by

$$c\alpha = (cx_1, \dots, cx_n). \tag{2.7}$$

It is easy to see that V , together with this vector addition and scalar multiplication is a vector space over the field F .

Definition 2.20: Suppose V is a vector space over a field F . A vector β in V is said to be a linear combination of the vectors $\alpha_1, \alpha_2, \dots, \alpha_n$ in V if there exist scalars $c_1, c_2, \dots, c_n$ in F such that

$$\beta = c_1 \alpha_1 + \dots + c_n \alpha_n = \sum_{k=1}^n c_k \alpha_k \quad (2.8)$$

Definition 2.21: Let V be a vector space over a field F . A subspace of V is a subset W of V which is itself a vector space over F with the operations of vector addition and scalar multiplication on V .

Theorem 2.1: [Hoffman and Kunze, 1971] A non-empty subset W of V is a subspace of V if and only if for each pair of vectors α, β in W and each scalar c in F the vector $c\alpha + \beta$ is again in W .

Example 2.4: Let A be an $m \times n$ matrix over a field F . The set of all $n \times 1$ (column) matrices X over F such that $AX = 0$ is a subspace of the space of all $n \times 1$ matrices over F .

Definition 2.22: Let S be a subset of V , where V is a vector space over a field F . The subspace spanned by S , denoted by $\langle S \rangle$, is defined to be the intersection of all subspaces of V containing S . One can see that $\langle S \rangle$ is the set of all linear combinations of elements of S . While S is a finite set of vectors, $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$, we have $\langle S \rangle = \langle \alpha_1, \alpha_2, \dots, \alpha_k \rangle$.

Definition 2.23: Let V be a vector space over a field F . A subset S of V is said to be linearly dependent, if there exist distinct vectors $\alpha_1, \alpha_2, \dots, \alpha_n$ in S and scalars $c_1, c_2, \dots, c_n$ in F , which are not all zero, such that

$$c_1 \alpha_1 + \dots + c_n \alpha_n = 0. \quad (2.9)$$

A set which is not linearly dependent is called linearly independent. If the set $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ we may call $\alpha_1, \alpha_2, \dots, \alpha_n$ as dependent (or independent) vectors.

Definition 2.24: Let V be a vector space over a field F . A linearly independent set of vectors in V which spans the whole vector space V is called a basis of V . If V has a finite basis, it is called finite dimensional.

One can see that for a vector space V over a field F , different bases have the same cardinality. So if β is a basis for V , we call $|\beta|$ as dimension of V , and denote it by $\dim_F V$, or simply by $\dim V$, if the field is clear for us.

Definition 2.25: The inner product of two column vectors x and y , of same order n , is scalar function denoted by (x, y) , as well as two other forms shown below, and is defined by

$$(x, y) = x^T y = y^T x = \sum_{i=1}^n x_i y_i. \quad (2.10)$$

A minor is defined as a value computed from the determinant of square matrix which is obtained after crossing out a row and a column corresponding to the element that is under consideration.

Suppose we want to compute the value of minor for the ij -th element in a matrix A , then we should form the determinant of the sub-matrix obtained by hiding i th row and j th column of A . This minor is known as (i, j) minor and usually denoted by $A(i|j)$. The scalar $(-1)^{i+j} \det(A(i|j))$, denoted by c_{ij} , is called the (i, j) -cofactor of A which is obviously defined as the signed minor. Suppose A is an $n \times n$ matrix over a field F . With respect to any column index j , the expansion of $\det(A)$ by cofactors of the j th column is as follows

$$\det(A) = \sum_{i=1}^n A_{ij} (-1)^{i+j} \det(A(i|j)) \quad (2.11)$$

Theorem 2.2: (Theorem 13 of [Hoffman and Kunze, 1971]) For an $n \times n$ matrix A over a field F , the following properties are equivalent:

- i) A is non-singular (or invertible);*
- ii) The homogeneous system $AX = 0$ has only trivial solution;*
- iii) $\det(A) \neq 0$.*

Definition 2.26: Let A be an $n \times n$ matrix. Suppose M, N, P , and Q are some submatrices of A of appropriate sizes. Then the following form is called the block form of A with respect to M, N, P , and Q :

$$\begin{bmatrix} M & N \\ P & Q \end{bmatrix}. \quad (2.12)$$

Lemma 2.3: [Hoffman and Kunze, 1971] Suppose a matrix A has the following $n \times n$ block form:

$$\begin{bmatrix} M_{r \times r} & N_{r \times s} \\ 0_{s \times r} & Q_{s \times s} \end{bmatrix}. \quad (2.13)$$

Then $\det(A) = \det(M)\det(Q)$.

Proposition 2.1: (Lemma 2.2 in [Cvetkovic' et al., 1980]) If M is a non-singular square matrix, and

$$A := \begin{bmatrix} M & N \\ P & Q \end{bmatrix}. \quad (2.14)$$

Then $\det(A) = \det(M)\det(Q - PM^{-1}N)$.

Proof 2.1: Since M is invertible, M^{-1} exists. Now by applying appropriate elementary row operations on the matrix A , we can obtain the following matrix:

$$B := \begin{bmatrix} M & N \\ 0 & Q - PM^{-1}N \end{bmatrix} \quad (2.15)$$

such that $\det(B)$ is the same as determinant of the main matrix A . Now Lemma 2.3 implies that $\det(A) = \det(B) = \det(M)\det(Q - PM^{-1}N)$.

Definition 2.27: If A is an $n \times n$ matrix over a field F , then a column vector $X \neq 0$ in $M_{n \times 1}(F) := F^{n \times 1}$ is an eigenvector of A if

$$AX = \lambda X \quad (2.16)$$

for some $\lambda \in F$ and λ is called an eigenvalue of A .

If A is an $n \times n$ square matrix and λ is an eigenvalue of A , then the union of the zero column vector $0_{n \times 1}$ and the set of all eigenvector corresponding to the eigenvalue λ is a subspace of $F^{n \times 1}$ known as the eigenspace of λ which is denoted by the following notation:

$$W_\lambda = \{V : AV = \lambda V\}. \quad (2.17)$$

Remark 2.1: Suppose $T : V \rightarrow V$ is a linear transformation and β is an ordered basis of V as a vector space over a field F . $\lambda \in F$ is called an eigenvalue of T if it is an eigenvalue of the associated matrix $A := [T]_\beta$. Suppose β' is another basis of V . As we know there exists an invertible matrix P such that $B := [T]_{\beta'} = P^{-1}[T]_\beta P = P^{-1}AP$.

Now we have:

$$\begin{aligned} p(B; \lambda) &= \det(\lambda I - B) = \det(\lambda I - P^{-1}AP) \\ &= \det(\lambda P^{-1}P - P^{-1}AP) \\ &= \det(P^{-1})\det(\lambda I - A)\det(P) = \det(\lambda I - A) = p(A; \lambda) \end{aligned} \quad (2.18)$$

Therefore, without loss of generality the characteristic polynomial of T can be defined uniquely as the characteristic polynomial of A (which is equal to the characteristic polynomial of B). So the characteristic polynomial of a linear transformation does not depend on the basis you work on.

Definition 2.28: For a matrix $A \in M_{n \times n}(F)$, where F is a field, the characteristic polynomial is defined as follows:

$$p(A; \lambda) = \det(\lambda I_n - A). \quad (2.19)$$

Eigenvalues of A are the root of this polynomial. Note that this is a monic polynomial of degree n .

Definition 2.29: For a matrix $A \in M_{n \times n}(F)$, where F is a field, the minimal polynomial is a monic polynomial denoted by $m(A; \lambda)$ with the following properties:

- i) $m(A; \lambda)$ annihilates A , i.e. $m(A; A) = 0_{n \times n}$.*
- ii) If $f(\lambda)$ is another polynomial with coefficients in the field F which annihilates A , then $m(A; \lambda)$ divides $f(\lambda)$.*

The following theorem is called the Cayley-Hamilton theorem.

Theorem 2.3: (Theorem 4 in [Hoffman and Kunze, 1971]) For a matrix $A \in M_{n \times n}(F)$, where F is a field, $m(A; \lambda)$ divides $p(A; \lambda)$. Furthermore, $m(A; \lambda)$ and $p(A; \lambda)$ have the same roots, but possibly with different multiplicities.

Proposition 2.2: Suppose $A \in M_{n \times n}(F)$, where F is a field and $\hat{\lambda} \in F$ is an eigenvalue of A . Then $\dim W_{\hat{\lambda}}$ is at most multiplicity of $\hat{\lambda}$ as a root of the characteristic polynomial $p(A; \lambda)$.

Proof 2.2: Suppose T is the linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ whose representing matrix with respect to the standard basis of $\mathbb{R}^n$ is A . Suppose $\hat{\lambda}$ is an eigenvalue of A and so an eigenvalue of T . Assume that $\dim_F W_{\hat{\lambda}} = m$ and multiplicity of $\hat{\lambda}$ in $p(A; \lambda) = p(T; \lambda) = d$. Since $\dim_F W_{\hat{\lambda}} = m$, there exists an ordered basis $\beta_{\hat{\lambda}} = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ for $W_{\hat{\lambda}}$. Since for each $(1 \leq i \leq m)$, α_i is an eigenvector of A with respect to the eigenvalue $\hat{\lambda}$, we have $T(\alpha_i) = \hat{\lambda}\alpha_i$. Extend this basis to obtain a basis for V , say $\beta = \{\alpha_1, \alpha_2, \dots, \alpha_m, \alpha_{m+1}, \dots, \alpha_n\}$. Now

$$A_{\hat{\lambda}} := [T]_{\beta} := \begin{bmatrix} [T]_{\beta_{\hat{\lambda}}} & M \\ 0 & N \end{bmatrix}_{n \times n} \quad (2.20)$$

such that

$$B_{\hat{\lambda}} := [T]_{\beta'} := \begin{bmatrix} \hat{\lambda} & \cdots & \hat{\lambda} \\ \vdots & \ddots & \vdots \\ \hat{\lambda} & \cdots & \hat{\lambda} \end{bmatrix}_{m \times m}. \quad (2.21)$$

Then by Lemma 2.3, and by Remark 2.1 we have:

$$\begin{aligned}
p(A; \lambda) &= p(T; \lambda) = p(A_{\hat{\lambda}}; \lambda) = \det(\lambda I_n - A_{\hat{\lambda}}) \\
&= \det(\lambda I_m - B) \det(\lambda I_{n \times m} - N) = p(B; \lambda) \det(\lambda I_{n \times m} - N) \\
&= (\lambda - \hat{\lambda})^m \det(\lambda I_{n \times m} - N).
\end{aligned} \tag{2.22}$$

Therefore, multiplicity of $\hat{\lambda} \geq m = \dim W_{\hat{\lambda}}$.

Lemma 2.4: If λ is an eigenvalue of A , then λ^2 is an eigenvalue of A^2 .

Proof 2.4: Since λ is an eigenvalue of A , by the definition there exists a nonzero column vector X in $F^{n \times 1}$ such that $AX = \lambda X$. Then we have:

$$A^2X = A(AX) = A(\lambda X) = \lambda(AX) = \lambda^2X \tag{2.23}$$

which implies that λ^2 is an eigenvalue of A^2 .

Proposition 2.3: Suppose $g(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_k x^k \in F[x]$ is polynomial over the field F , $A \in M_{n \times n}(F)$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of A , then $g(\lambda_1), g(\lambda_2), \dots, g(\lambda_n)$ are eigenvalues of $g(A)$.

Proof 2.3: Let $g(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_k x^k \in F[x]$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ be eigenvalues of $A \in M_{n \times n}(F)$. Suppose X_i is a nonzero eigenvector of A with respect to the eigenvalue λ_i . So by Definition 2.27, $AX_i = \lambda_i X_i$. We claim that $g(A)X_i = g(\lambda_i)X_i$ for $(1 \leq i \leq n)$. Then

$$\begin{aligned}
\bullet \quad g(A)X_i &= (\alpha_0 + \alpha_1 A + \dots + \alpha_k A^k)X_i = (\alpha_0 + \alpha_1 AX_i + \dots + \alpha_k A^k X_i) \\
&= (\alpha_0 + \alpha_1 (\lambda_i X_i) + \dots + \alpha_k A^{k-1}(\lambda_i X_i)).
\end{aligned}$$

As $X_i = \lambda_i X_i$, we can see that $A^t(\lambda_i X_i) = \lambda_i^{t+1} X_i$; for $(1 \leq t \leq k-1)$. Then

- $g(A)X_i = (\alpha_0 + \alpha_1\lambda_i X_i + \dots + \alpha_k\lambda_i^k X_i) = (\alpha_0 + \alpha_1\lambda_i + \dots + \alpha_k\lambda_i^k)X_i = g(\lambda_i)X_i.$

Definition 2.30. If A is a square $n \times n$ matrix, then the trace of A , denoted $\text{tr}(A)$ is the sum of all of the entries in the main diagonal, that is:

$$\text{tr}(A) = \sum_{i=1}^n a_{ii}. \quad (2.24)$$

Definition 2.31: A matrix $A \in M_{n \times n}(F)$ is called diagonalizable, if there exists a unique invertible matrix P and a diagonal matrix D whose diagonal entries are eigenvalues of A such that $A = P^{-1}DP$. The columns of P are linearly independent eigenvectors of A .

Proposition 2.4: Suppose $F = \mathbb{C}$, A is an $n \times n$ matrix over the field F , and A has n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. We have the following properties:

- i) $\sum_{i=1}^n \lambda_i = \text{tr}(A).$
- ii) $\prod_{i=1}^n \lambda_i = \det(A).$

Proof 2.4: i) Since A is an $n \times n$ matrix with n distinct eigenvalues, we can obtain a basis for $F^{n \times 1}$ whose elements are all eigenvectors of A with respect to distinct eigenvalues. Thus A is a diagonalizable matrix which implies that there exists an invertible $n \times n$ matrix P whose columns are the associated eigenvectors of A and a diagonal matrix D whose diagonal entries are eigenvalues of A , both over the field $\mathbb{C}$ such that A can be written as:

$$A = P^{-1}DP. \quad (2.25)$$

Taking the trace on both sides we have to obtain:

$$\text{tr}(A) = \text{tr}(P^{-1}DP) = \text{tr}(DPP^{-1}) = \text{tr}(D) = \sum_{i=1}^n \lambda_i. \quad (2.26)$$

ii) With respect to the points we mentioned in part i), applying determinant on the equation (2.25), we have:

$$\begin{aligned} \det(A) &= \det(P^{-1}DP) = \det(P^{-1})\det(D)\det(P) \\ &= \det(D)\det(P^{-1})\det(P) = \det(D) \end{aligned} \quad (2.27)$$

As D is a diagonal matrix whose diagonal entries are λ_i s, we conclude that

$$\det(D) = \prod_{i=1}^n \lambda_i = \det(A). \quad (2.28)$$

Proposition 2.5: If $\lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_0 = p(A; \lambda) = \det(\lambda I_n - A)$. Then

- i) $c_{n-1} = -\text{tr}(A)$.
- ii) $c_0 = (-1)^n \det(A)$.

Definition 2.32: A matrix whose elements are entirely of real numbers is called the real matrix.

Definition 2.33: A matrix $A \in M_{n \times n}(\mathbb{R})$ is symmetric, if $A = A^T$.

Theorem 2.4: (Theorem 2 in [Hoffman and Kunze, 1971]) Consider a matrix $A \in M_{n \times n}(F)$ where F is a field. The following properties are equivalent:

- i) A is diagonalizable;
- ii) $p(A; \lambda) = \prod_{i=1}^t (\lambda - \lambda_i)^{d_i}$, where $\lambda_1, \lambda_2, \dots, \lambda_t$ are distinct eigenvalues of A .
Furthermore for each i , $\dim W_{\lambda_i}$, the multiplicity of λ_i in $p(A; \lambda)$.
- iii) $m(A; \lambda) = \prod_{i=1}^t (\lambda - \lambda_i)$ where $\lambda_1, \lambda_2, \dots, \lambda_t$ are distinct eigenvalues of A .

Theorem 2.5: Let A be a symmetric matrix. Then the matrix A has the following properties;

- i) A has only real eigenvalues.
- ii) A is diagonalizable.

iii) A has orthogonal eigenvalues.

Proof 2.5: i) Suppose $\lambda \in \mathbb{C}$ is an eigenvalue of A . Then there exists a nonzero vector X such that $AX = \lambda X$. Let $X^* = (\bar{X})^T$. By multiplying both sides of the equality $AX = \lambda X$ by X^* , we obtain $X^*AX = X^*\lambda X = \lambda X^*X$. On the other hand, $\bar{\lambda}X^*X = (\lambda X^*X)^* = (X^*AX)^* = X^*A^*X = X^*AX = \lambda X^*X$. So $\bar{\lambda} = \lambda$ which implies that $\lambda \in \mathbb{R}$.

ii) Suppose $\lambda_1, \lambda_2, \dots, \lambda_t$ are the distinct eigenvalues of A . Since every polynomial over $\mathbb{C}$ can be written as the product of powers of linear terms, we may assume that:

$$m(a; \lambda) = (\lambda - \lambda_1)^{m_1} \dots (\lambda - \lambda_t)^{m_t}. \quad (2.29)$$

If A is not diagonalizable, Theorem 2.4 verifies that there exists $(1 \leq j \leq t)$ such that $m_j > 1$. As the minimal polynomial annihilates A , we have:

$$0 = m(A; A) = (A - \lambda_1 I)^{m_1} \dots (A - \lambda_j I)^{m_j} \dots (A - \lambda_t I)^{m_t}. \quad (2.30)$$

According to the choice of j , it is clear that

$$(A - \lambda_j I) \neq 0, \dots, (A - \lambda_j I)^{m_{j-1}} \neq 0, (A - \lambda_j I)^{m_j} = 0. \quad (2.31)$$

Hence there exists a nonzero vector Y such that $(A - \lambda_j I)^{m_{j-1}} Y \neq 0$. Now we have:

$$0 = Y^*(A - \lambda_j I)^{m_j} Y = Y^*(A - \lambda_j I)(A - \lambda_j I)^{m_{j-1}} Y. \quad (2.32)$$

But $(A - \lambda_j I)Y^* \neq 0$, which is a contradiction. So, for all $(1 \leq j \leq t)$, $m_j = 1$ which implies that A is diagonalizable.

iii) Let A be a symmetric matrix. By part ii) we know that A is diagonalizable. Thus there exists an invertible matrix P such that $A = P^{-1}DP$, where D is the diagonal matrix with diagonal entries D_{ii} as eigenvalues of A . Then

$$A^T = (P^{-1}DP)^T = P^T D^T P^{-T} = P^T DP^{-T}. \quad (2.33)$$

Since $A = A^T$, we have $P^{-1}DP = A = P^T DP^{-T}$ and $P^T = P^{-1}$. As the columns of P are eigenvectors of A and $P^T P = P^{-1}P = I$, we conclude that for $(1 \leq i, j \leq n)$, $P_j^T P_i = \delta_{ij}$, where P_i and P_j are the i th and j th columns of P , respectively. This implies that columns of P which are eigenvectors of A are orthogonal with respect to the standard inner product.

Definition 2.34: Assume A is an $n \times n$ matrix. Then

- i) A is positive definite if and only if $A = A^T$ and A has only positive eigenvalues.
- ii) A is positive semi definite if and only if $A = A^T$ and A has only nonnegative eigenvalues.

We have the following properties for such a matrix:

Proposition 2.6: A symmetric $n \times n$ real matrix M is positive definite if and only if for every nonzero column vector Z of n real numbers we have $Z^T M Z > 0$.

Proof 2.6: First suppose that M is positive definite and assume that there exists a nonzero column vectors Z such that $Z^T M Z \leq 0$. As M is symmetric, Theorem 2.5 implies that M is diagonalizable. Let $Y = PZ$ where P is the unique invertible matrix such that $M = P^{-1}DP$ and D is the diagonal matrix of eigenvalues of M . Then $P^{-1}Y = Z$. Since M and D are both symmetric matrices, we have:

$$P^{-1}DP = M = M^T = P^T D^T (P^T)^{-1}. \quad (2.34)$$

Now uniqueness property of P implies that $P^{-1} = P^T$ and we conclude that $P^T Y = Z \neq 0$.

Therefore:

$$0 \geq Z^T M Z = Z^T P^T D P Z = (P^T Y)^T P^T D P P^T Y = Y^T D Y. \quad (2.35)$$

On the other hand, $Y^T D Y = \sum_{i=1}^n \lambda_i Y_i^2 > 0$, which is a contradiction. So, $Z^T M Z > 0$.

Now suppose that $Z^T M Z > 0$, for each nonzero column vector Z . For $(1 \leq j \leq n)$, let Y_j be the following nonzero column vector:

$$Y_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 = j\text{th entry} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (2.36)$$

It is clear that $Z_j = P^{-1} Y_j \neq 0$. From the hypothesis we have:

$$\begin{aligned} 0 < Z_j^T M Z_j &= (P^T Y_j)^T P^T D P (P^T Y_j) = Y_j^T P P^T D P P^T Y_j = Y_j^T D Y_j \quad (2.37) \\ &= \sum_{i=1}^n \lambda_i Y_i^2 = \lambda_j \end{aligned}$$

Since j was arbitrary, we conclude that M is positive definite.

Definition 2.35: Diagonally dominant matrix is a complex matrix B with the property that

$$|b_{ii}| \geq \sum_{j \neq i, j=1}^n |b_{ij}| \text{ for all } i. \quad (2.38)$$

When all these inequalities are strict, the matrix is called strictly diagonally dominant matrix. So for such a matrix we have

$$|b_{ii}| > \sum_{j \neq i, j=1}^n |b_{ij}| \quad \text{for all } i. \quad (2.39)$$

Lemma 2.5: Strictly diagonally dominant matrices are non-singular.

Proof 2.5: Let B be a matrix which is strictly diagonally dominant and singular. Then by Theorem 2.2 we deduce that there exists a vector $u \neq 0$ with $Bu = 0$. As u is nonzero, it has some nonzero entry, say $u_i > 0$. Now for each j we have:

$$0 = (Bu)_i = \sum_j b_{ij}u_j. \quad (2.40)$$

So, $b_{ii}u_i = -\sum_{j \neq i} b_{ij}u_j$ and as u_i is nonzero, $b_{ii} = -\sum_{j \neq i} \frac{u_j}{u_i} b_{ij}$. By taking the absolute value of both sides of this equality, we obtain:

$$|b_{ii}| \leq \sum_{j \neq i} \left| b_{ij} \frac{u_j}{u_i} \right| = \sum_{j \neq i} |b_{ij}| \left| \frac{u_j}{u_i} \right|. \quad (2.41)$$

Consequently, $|b_{ii}| \leq \sum_{j \neq i} |b_{ij}|$, since $\left| \frac{u_j}{u_i} \right| \leq 1$. This is a contradiction. So, B is non-singular.

3. SPECTRAL GRAPH THEORY

3.1. Matrix Representations of Graphs

In this chapter, we give matrix representations of graphs such as adjacency matrix, incidence matrix, Laplacian matrix. Then we focus on the spectrum of Laplacian and normalized Laplacian matrix.

Let Γ be an undirected graph without loops with $V(\Gamma) = \{v_1, v_2, \dots, v_n\}$ and $E(\Gamma) = \{e_1, e_2, \dots, e_m\}$. We may associate the following matrices to the graph Γ .

Definition 3.1: The Adjacency Matrix of Γ , denoted by $A := A(\Gamma)$, is the $n \times n$ matrix defined as follows:

$$A_{ij} = \begin{cases} 1; & \{v_i, v_j\} \in E(\Gamma) \\ 0; & \text{otherwise} \end{cases} \quad (3.1)$$

Definition 3.2: The Incidence Matrix of Γ , denoted by $R := R(\Gamma)$ is the $n \times m$ following matrix:

$$R_{ij} = \begin{cases} 1; & v_i \text{ is incident to } e_j, \text{ i.e. } v_i \text{ is one of the endpoints of } e_j \\ 0; & \text{otherwise} \end{cases} \quad (3.2)$$

Definition 3.3: The Degree Matrix of Γ denoted by $D := D(\Gamma)$ is the diagonal matrix whose entries are degree of vertices of Γ .

Definition 3.4: The Oriented Incidence Matrix of Γ , denoted by $O := O(\Gamma)$, is the $n \times m$ matrix whose rows and columns are associated to the vertices and edges of Γ such that the ij -th entry which is the entry respected to the i th vertex of Γ , i.e. v_i and j th edge of Γ , i.e. e_j is defined as follows:

$$O_{ij} = \begin{cases} 1; & v_i \text{ is the head of } e \\ -1; & v_i \text{ is the tail of } e \\ 0; & \text{otherwise} \end{cases} \quad (3.3)$$

Definition 3.5: The Laplacian Matrix of Γ , denoted by $\mathbb{L} := \mathbb{L}(\Gamma)$, is the $n \times n$ matrix whose ij -th entry is defined as follows:

$$\mathbb{L}_{ij} = \begin{cases} \deg(v_i); & i = j \\ -1; & i \neq j \text{ and } \{v_i, v_j\} \in E(\Gamma) \\ 0; & \text{otherwise} \end{cases} \quad (3.4)$$

Definition 3.6: Signless Laplacian Matrix of Γ , denoted by $Q := Q(\Gamma)$, is the matrix $Q(\Gamma) = D(\Gamma) + A(\Gamma)$. The signless Laplacian eigenvalues of Γ which are the eigenvalues of $Q(\Gamma)$ are denoted by $q_1(\Gamma), q_2(\Gamma), \dots, q_n(\Gamma)$.

Proposition 3.1: Let Γ be regular graph of degree r with n vertices. Let A be adjacency matrix of Γ . Then we have the following properties;

- i) r is an eigenvalue of Γ .*
- ii) For any eigenvalue λ of Γ , we have $|\lambda| \leq r$.*
- iii) If Γ is connected, then multiplicity of r as a root of the characteristic polynomial of the adjacency matrix of Γ is 1.*
- iv) $\mathbf{1}_n$ is eigenvector corresponding to eigenvalue r of Γ .*

Proof 3.1: i) Suppose $V(\Gamma) = \{v_1, v_2, \dots, v_n\}$. Let $A := A(\Gamma)$ and $X^T = [1, \dots, 1]$.

Then for each i we have $(AX)_i = \sum_{j=1}^n A_{ij} X_j = \sum_{j=1}^n A_{ij}$. Since each entry of X is 1, it is clear that $(AX)_i$ is equal to the sum of the i th row of A which is the degree of the vertex v_i . As Γ is r -regular graph, $\deg(v_i) = r$. Hence for every $i \in \{1, \dots, n\}$, $(AX)_i = r = r[1, \dots, 1]^T = rX_i$. This implies that $AX = rX$. Consequently, r is an eigenvalue of A , the adjacency matrix of Γ , so we have part i).

ii) Let $Y \neq 0$ be an eigenvector with respect to an eigenvalue of Γ , say λ . Suppose $Y^T = [y_1, \dots, y_j, \dots, y_n]$ and j is the index for which y_j is largest in absolute value.

Let

$$I = \{i : \{v_i, v_j\} \text{ in } \Gamma\}. \quad (3.5)$$

Then r -regularity of Γ implies that $|I| = r$. As $AY = \lambda Y$, we have: $(AY)_j = \sum_{i=1}^n A_{ji} Y_i = \sum_{i \in I} \lambda y_i$. Therefore $|\sum_{i \in I} y_i| \leq \sum_{i \in I} |y_i| \leq r |y_j|$ which implies that $|\lambda| \leq r$.

iii) Let $X^T = [x_1, \dots, x_n]$ be an eigenvector with respect to r . Then $AX = rX$, so for every j ,

- $(AX)_i = \sum_{j=1}^n A_{ji} x_j = rx_j$.

Suppose $1 \leq j \leq n$ is such that $|x_j|$ is maximum among $\{|x_i| : 1 \leq i \leq n\}$. Then we have $|x_i| \leq |x_j|$, for every $1 \leq i \leq n$. Let

$$I = \{i : \{v_i, v_j\} \text{ in } \Gamma\}. \quad (3.6)$$

As for $i \in I$, $A_{ij} = 1$, we have

- $(AX)_i = \sum_{j=1}^n A_{ji} x_j = \sum_{i \in I} A_{ji} x_i = \sum_{i \in I} x_i = rx_j$.

Since $|I| = \deg(v_j) = r$ and $|x_i| \leq |x_j|$ for each i , the sum of r copies of x_i equals to rx_j if and only if $x_i = x_j$, for every $i \in I$.

Now let v_t be a neighbour of v_j in Γ , so $\deg(v_t) = r$, $t \in I$ and $x_i = x_j$. Let

$$J = \{i : \{v_i, v_t\} \text{ in } \Gamma\}. \quad (3.7)$$

It is obvious that $|J| = r$. Let $l \in J$ such that $|x_l| \geq |x_i|$, for every $i \in J$. Continuing as in the previous parts, we can conclude that $x_i = x_l$ for every $i \in J$. In particular, $j \in J$, so $x_l = x_j$. Since Γ is connected, continuing like this, we can conclude that $x_1 = x_2 = \dots = x_n := x$, hence $X = x[1, \dots, 1]$. As X was arbitrary, we deduce that $W_r = \langle X = [1, \dots, 1]^T \rangle$, thus $\dim W_r = 1$. Since A is a real symmetric matrix, it is diagonalizable, so by Theorem 2.4, $\dim W_r$ equals multiplicity of λ in the characteristic polynomial of Γ .

iv) Let $1_n^T = [1, \dots, 1]$ be eigenvector corresponding to eigenvalue r . We have $A1_n^T = r1_n^T$. This implies that $(A - rI)1_n^T = 0$. Then

$$\begin{bmatrix} -r & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & -r \end{bmatrix}_{n \times n} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1} = \begin{bmatrix} -r + \sum_{i=2}^n a_{1i} \\ \vdots \\ -r + \sum_{i=1}^{n-1} a_{ni} \end{bmatrix}_{n \times 1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1}. \quad (3.8)$$

The sum of entries of each row of A give the degree of vertex. Thus 1_n is eigenvector corresponding to eigenvalue r of Γ .

Lemma 3.1: [Web 1, 2018] The number of closed walks of length l in Γ equals:

$$\sum_{i=1}^n \lambda^l = 0 \quad (3.9)$$

Proof 3.1: The matrix A^l has in its uv -entry the number of walks from u to v . So a diagonal uv -entry gives the number of closed walks starting and ending at u , and the sum of the diagonal entries, $\text{tr}(A^l)$, gives the number of closed walks of length l . On the other hand, if λ is eigenvalue of A , then λ^l is eigenvalue of A^l with the same multiplicity.

Diagonalizing A^l as $S^{-1}A^l S = D$, where D is diagonal matrix with the eigenvalues of A^l on the diagonal, gives:

$$\text{tr}(A^l) = \text{tr}(S^{-1}A^l S) = \text{tr}(D) = \sum \lambda_i^l. \quad (3.10)$$

Lemma 3.2: [Web 1, 2018] A graph Γ is bipartite if and only if its spectrum is symmetric. (i.e if λ is eigenvalue, then $-\lambda$ is also eigenvalue and with the same multiplicity).

Proof 3.2: Let Γ be bipartite with parts K and L of sizes k and l . So, for some $k \times l$ matrix B we have:

$$A = \begin{bmatrix} 0_{kk} & B \\ B^T & 0_{ll} \end{bmatrix}. \quad (3.11)$$

As λ is eigenvalue,

$$\begin{bmatrix} \lambda v \\ \lambda w \end{bmatrix} = \lambda \begin{bmatrix} v \\ w \end{bmatrix} = A \begin{bmatrix} v \\ w \end{bmatrix} \begin{bmatrix} Bw \\ B^T v \end{bmatrix} \quad (3.12)$$

So, $Bw = \lambda v$ and $B^T v = \lambda w$. We claim that $-\lambda$ is also eigenvalue:

$$A \begin{bmatrix} v \\ -w \end{bmatrix} = \begin{bmatrix} -Bw \\ B^T v \end{bmatrix} = \begin{bmatrix} -\lambda v \\ \lambda w \end{bmatrix} = -\lambda \begin{bmatrix} v \\ \lambda w \end{bmatrix}. \quad (3.13)$$

Let λ be with multiplicity m . As the subspace spanned by the $[v \ w]^T$ will have the same dimension as that spanned by the corresponding $[v \ -w]^T$, $-\lambda$ has also multiplicity m . So spectrum is symmetric.

Now let the spectrum be symmetric. Then for any odd p we have :

$$\sum \lambda_i^p = 0, \quad (3.14)$$

as any term λ_i^p will be cancelled by a term $(-\lambda_i)^p = -\lambda_i^p$. By Lemma 3.1, there are no closed walks of odd length in Γ , so in particular there are no odd cycles. Therefore Γ is bipartite.

Lemma 3.3: (Lemma 3.6 in [Biggs, 1993]) Let Γ be a regular graph of degree r with n vertices and m edges. Let A and R be the adjacency matrix and the incidence matrix of Γ , respectively. Let $L(\Gamma)$ be its line graph. Then:

- i) $R^T R = A(L(\Gamma)) + 2I$,
- ii) $RR^T = A(\Gamma) + rI$.

Furthermore, if Γ has an eigenvalue equals to $-r$ with an eigenvector X , then Γ is bipartite and $R^T X = 0$.

Proof 3.3 : i) Suppose Γ is a regular graph of degree r . We have $(R^T R)_{ij} = \sum_{k=1}^n R_{ik}^t R_{kj} = \sum_{k=1}^n R_{ki} R_{kj}$. It is clear that $\sum_{k=1}^n R_{ki} R_{kj} \neq 0$ if and only if for some k , both terms R_{ki} and R_{kj} are nonzero. By the definition of the incidence matrix, one can see that

$$(R^T R)_{ij} = \begin{cases} A(L(\Gamma))_{ij}; & i \neq j \\ 2; & i = j \end{cases} = (A(L(\Gamma) + 2I_m))_{ij}. \quad (3.15)$$

ii) We will prove this part by a similar argument. We have:

$$(RR^T)_{ij} = \sum_{k=1}^n R_{ik} R_{kj}^t = \sum_{k=1}^n R_{ik} R_{jk}. \quad (3.16)$$

Similarly $\sum_{k=1}^n R_{ik} R_{jk} \neq 0$ if and only if for some k , we have $R_{ik} \neq 0$ and $R_{jk} \neq 0$ which holds for those k such that the edge e_k is incident with both vertices e_i and e_j . If $i = j$, then it is clear that $A_{ii} = 0$, $rI_{ii} = r$ and $(RR^T)_{ii}$ equals the number of neighbors of v_i which is r as Γ is regular of degree r . For the case $i \neq j$, $A_{ij} \neq 0$ while $\{v_i, v_j\} \in E(\Gamma)$. Let $e_i := \{v_i, v_j\}$. This implies that $R_{ik} R_{kj}^t = R_{il} R_{jl} = 1$. Hence

$$(RR^T)_{ij} = \begin{cases} A(L(\Gamma))_{ij}; & i \neq j \\ r; & i = j \end{cases} = (A(\Gamma) + rI_n)_{ij}. \quad (3.17)$$

If Γ has an eigenvalue equals to $-r$ with an eigenvector X , by Lemma 3.2, Γ is bipartite and so $R^T X = 0$.

Proposition 3.2: (Theorem 3.8 in [Biggs, 1993]) If Γ is regular graph of degree r with n vertices and $m = \frac{1}{2}nr$ edges, then

$$p(L(\Gamma); \lambda) = (\lambda + 2)^{m-n} p(\Gamma; \lambda + 2 - k). \quad (3.18)$$

Proof 3.2: We shall use the notation and results of Lemma 3.2. Let us define two matrices with $n + m$ rows and columns with the following block structures:

$$U = \begin{bmatrix} \lambda I_n & -R \\ 0 & I_m \end{bmatrix}, \quad V = \begin{bmatrix} I_n & R \\ R^T & \lambda I_m \end{bmatrix}. \quad (3.19)$$

Then we have:

$$UV = \begin{bmatrix} \lambda I_n - RR^T & 0 \\ R^T & \lambda I_m \end{bmatrix}, \quad VU = \begin{bmatrix} \lambda I_n & 0 \\ \lambda R^T & \lambda I_m - R^T R \end{bmatrix}. \quad (3.20)$$

Since $\det(UV) = \det(VU)$ we deduce that

$$\lambda^m \det(\lambda I_n - RR^T) = \lambda^m \det(\lambda I_m - R^T R). \quad (3.21)$$

Thus we may argue as follows:

- $p(L(\Gamma); \lambda) = \det(\lambda I_m - A_L) = \det((\lambda + 2)I_m - R^T R) = (\lambda + 2)^{m-n} \det((\lambda + 2)I_n - RR^T).$

As Γ is regular of degree r , Lemma 3.3 implies that $RR^T = A + rI$, thus

- $p(L(\Gamma); \lambda) = (\lambda + 2)^{m-n} \det((\lambda + 2 - r)I_n - A) = (\lambda + 2)^{m-n} p(\Gamma; \lambda + 2 - r).$

In other words, if the spectrum of Γ is $\{r^1, \lambda_1^{m_1}, \dots, \lambda_s^{m_s}\}$, then the spectrum of $L(\Gamma)$ is $\{[2r - 1]^1, [r - 2 + \lambda_1]^{m_1}, \dots, [r - 2 + \lambda_s]^{m_s}, [-2]^{m-n}\}.$

Lemma 3.4: Let Γ be a regular graph of degree r and let λ_i and q_i , for $(1 \leq i \leq n)$, be the eigenvalues of the adjacency matrix and signless Laplacian matrix of the graph Γ , respectively. Then the eigenvalues of the signless Laplacian matrix $Q(\Gamma)$ are $2r, r + \lambda_1, \dots$, and $r + \lambda_n$.

Proof 3.4: Let $R := R(\Gamma)$. By the definition and by Lemma 3.3, we know that $p(A(\Gamma), \lambda) = \det(\lambda I - A(\Gamma))$ and $RR^T = A(\Gamma) + rI$, therefore

- $p(A(\Gamma), \lambda) = \det(\lambda I - A(\Gamma)) = \det(\lambda I - (RR^T - rI))$
 $= \det((\lambda + r)I - Q(\Gamma)) = p(Q(\Gamma), \lambda + r).$

Consequently, $q_i = r + \lambda_i$ for $1 \leq i \leq n$.

Remark 3.1: [Cvetkovic' et al., 1980]) Consider the adjacency matrix of the subdivision graph of a graph Γ , i.e. $A(S(\Gamma))$. By Lemma 2.2 we know that $S(\Gamma)$ is a bipartite graph with bipartition $X | Y$, where $X = V(\Gamma)$ and $Y = E(\Gamma)$. It is an easy verification to see that

$$A(S(\Gamma)) = \begin{bmatrix} 0 & R^T \\ R & 0 \end{bmatrix} \quad (3.22)$$

where $R = R(\Gamma)$ is the incidence matrix of Γ .

Notation 3.1: Let E_y be the $n \times n$ matrix as follows;

$$(E_y)_{xz} = \begin{cases} 1; & x = y = z \\ 0; & \text{otherwise} \end{cases} \quad (3.23)$$

For a graph Γ and $S \subseteq V(\Gamma)$, $\mathbb{L}^S$ is obtained from $\mathbb{L}$ by deleting the rows and columns of $\mathbb{L}$ indexed by S .

Lemma 3.5: Let $\mathbb{L}_G$ be the Laplacian matrix with respect to a graph G and let $e = \{x, y\} \in E(\Gamma)$. Then we have the following properties:

- i) $\mathbb{L}_\Gamma^{\{x\}} = \mathbb{L}_{\Gamma \setminus e}^{\{x\}} + E_y$ and
- ii) $\mathbb{L}_\Gamma^{\{x, y\}} = \mathbb{L}_{\Gamma/e}^{\{y\}}$

Proof 3.5: The Laplacian matrix of $\mathbb{L}_{\Gamma \setminus e}$ differs from that of Γ only in the entries in positions (x, x) , (y, y) , (x, y) , and (y, x) . Three of these entries, since they belong to the row and column labelled by x , are absent from $\mathbb{L}_\Gamma^{\{x\}}$ and from $\mathbb{L}_{\Gamma \setminus e}^{\{x\}}$. The fourth represent the degree of v , which is greater by one in Γ than in $\Gamma \setminus e$. Hence the $(n - 1) \times (n - 1)$ matrices $\mathbb{L}_\Gamma^{\{x\}}$ and $\mathbb{L}_{\Gamma \setminus e}^{\{x\}}$ are related by the equation

$$\mathbb{L}_\Gamma^{\{x\}} = \mathbb{L}_{\Gamma \setminus e}^{\{x\}} + E_y \quad (3.24)$$

Let w and z be vertices of Γ (other than x and y). If $w \neq z$, the entry in the (w, z) position of $\mathbb{L}_{\Gamma}^{\{x,y\}}$ is -1 or 0 according as w is adjacent to z or not. Since w and z are adjacent in Γ if and only if they are adjacent in Γ/e , the (w, z) -entry of $\mathbb{L}_{\Gamma/e}^{\{y\}}$ is the same as that of $\mathbb{L}_{\Gamma}^{\{x,y\}}$. The $\{y, y\}$ -entry of $\mathbb{L}_{\Gamma}^{\{x,y\}}$ is the degree in Γ of w . This is the same as the degree of w in Γ/e (bearing in mind that a double edge from w to y in Γ/e contributes 2 to the degree of w in this graph). So the diagonal entries of $\mathbb{L}_{\Gamma}^{\{x,y\}}$ also coincide with those of $\mathbb{L}_{\Gamma/e}^{\{y\}}$. Then the following equality holds:

$$\mathbb{L}_{\Gamma}^{\{x,y\}} = \mathbb{L}_{\Gamma/e}^{\{y\}} \quad (3.25)$$

Proposition 3.3: (Proposition 1.3.4 in [Brouwer and Haemers, 2012]) *Let Γ be an undirected graph with at least one vertex and Laplace matrix $\mathbb{L}$ with eigenvalues $0 = \mu_1 \leq \mu_2 \leq \dots \leq \mu_n$. Let c_{xy} be the (x, y) -cofactor of the matrix $\mathbb{L}$. Then the number N of the spanning trees of Γ equals:*

$$N = c_{xy}, \text{ for any } x, y \in V(\Gamma). \quad (3.26)$$

Proof 3.3: By Notation 3.1, for $S \subseteq V(\Gamma)$, $\mathbb{L}^S$ is obtained from $\mathbb{L}$ by deleting the rows and columns indexed by S . In particular, if $S = \{x\}$, then $c_{xx} = \det \mathbb{L}_{\Gamma}^{\{x\}}$. We prove by induction on n .

For $n = 1$, we have $p(\mathbb{L}, t) = t^n = t^1$. Then c_{xx} equals to the determinant of the empty matrix which is one. On the other hand it is clear that in this case $N(\Gamma) = 1$. So we may assume that $n > 1$ and $c_{xx} = \det \mathbb{L}_{\Gamma}^{\{x\}} = N(\Gamma)$ for any graph Γ with at least one vertex which has less than n eigenvalues.

We claim that for each $x \in V(\Gamma)$, the following equality:

$$c_{xx} = \det \mathbb{L}_{\Gamma}^{\{x\}} = N(\Gamma), (n > 1). \quad (3.27)$$

To prove this, we use induction on the degree of x .

If $\deg_{\Gamma}(x) = 0$, which means that x is an isolated vertex in Γ , then Γ has no spanning tree. This implies that $N(\Gamma) = 0$ and so $c_{xx} = 0 = N(\Gamma)$. So suppose that

$\deg_\Gamma(x) \geq 1$ and for any graph $G \subseteq \Gamma$ such that $x \in V(G)$ and $\deg_G(x) < \deg_\Gamma(x)$, we have $\det \mathbb{L}_G^{\{x\}} = N(G)$. We show that $c_{xx} = \det \mathbb{L}_\Gamma^{\{x\}} = N(\Gamma)$. Since $\deg_{\Gamma \setminus e}(x) < \deg_\Gamma(x)$, by induction hypothesis, we deduce that

$$\det \mathbb{L}_{\Gamma \setminus e}^{\{x\}}(x) = N(\Gamma \setminus e). \quad (3.28)$$

As x has degree greater than one, there exists $y \in V(\Gamma)$ such that $\{x, y\} \in E(\Gamma)$. For such a vertex y we have:

$$c_{xx} = \det \mathbb{L}_\Gamma^{\{x\}} = \sum_{u \in V, w \neq x} (\mathbb{L}_\Gamma^{\{x\}})_{yw} \overline{c_{yw}} \quad (3.29)$$

where $\bar{c}$ is with respect to the matrix $\mathbb{L}_\Gamma^{\{x\}}$. We may rewrite the last sum as:

$$\begin{aligned} & \bullet \sum_{u \in V, w \neq x, y} (\mathbb{L}_\Gamma^{\{x\}})_{yw} \overline{c_{yw}} + (\mathbb{L}_\Gamma^{\{x\}})_{yy} \overline{c_{yy}} \\ &= \sum_{u \in V, w \neq x, y} (\mathbb{L}_\Gamma^{\{x\}})_{yw} \overline{c_{yw}} + \left[(\mathbb{L}_{\Gamma \setminus e}^{\{x\}})_{yy} + (E_y)_{yy} \right] \overline{c_{yy}}, \end{aligned}$$

since $\mathbb{L}_\Gamma^{\{x\}} = \mathbb{L}_{\Gamma \setminus e}^{\{x\}} + E_y$.

As

$$\bullet \sum_{u \in V, w \neq x, y} (\mathbb{L}_\Gamma^{\{x\}})_{yw} \overline{c_{yw}} + (\mathbb{L}_{\Gamma \setminus e}^{\{x\}})_{yy} \overline{c_{yy}} + \overline{c_{yy}} = \sum_{u \in V, w \neq x} (\mathbb{L}_{\Gamma \setminus e}^{\{x\}})_{yw} \overline{c_{yw}} + \overline{c_{yy}}$$

By the definition of determinant, we have:

$$c_{xx} = \det \mathbb{L}_{\Gamma \setminus e}^{\{x\}} + \det \left(\mathbb{L}_\Gamma^{\{x\}} \right)^{\{y\}} = N(\Gamma \setminus e) + N(\Gamma / e) = N(\Gamma). \quad (3.30)$$

4. NORMALIZED LAPLACIAN SPECTRUM

4.1. Properties of Laplacian Spectrum

Definition 4.1: The Laplacian Spectra

Let $\Gamma = (V, E)$ be a simple graph with n vertices and m edges. Let

$$\mu_1(\Gamma) \geq \mu_2(\Gamma) \geq \dots \geq \mu_n(\Gamma) \quad (4.1)$$

be the eigenvalues of the Laplacian matrix $\mathbb{L} = \mathbb{L}(\Gamma)$ of Γ . These eigenvalues are called the Laplacian eigenvalues of Γ . Let $\mu_1(\Gamma), \mu_2(\Gamma), \dots, \mu_s(\Gamma)$ be the distinct Laplacian eigenvalues of Γ such that for each $1 \leq i \leq s$, the eigenvalue $\mu_i(\Gamma)$ has multiplicity t_i . The multi-set

$$\text{Spec}(\mathbb{L}) := \{[\mu_1(\Gamma)]^{t_1}, [\mu_2(\Gamma)]^{t_2}, \dots, [\mu_s(\Gamma)]^{t_s}\} \quad (4.2)$$

of eigenvalues of $\mathbb{L}$ together with their multiplicities is called the spectrum of $\mathbb{L}(\Gamma)$, or the Laplacian spectra.

Theorem 4.1: Let Γ be a graph whose adjacency matrix and degree matrix are A and D , respectively. Then we have the following properties:

- i) $\mathbb{L} = D - A$.*
- ii) $Q = RR^T$ and $\mathbb{L} = OO^T$ where R is the incidence matrix and O is the oriented incidence matrix of Γ .*
- iii) $\mathbb{L}$ is positive semi-definite.*
- iv) Every row sum and column sum of $\mathbb{L}$ is zero.*
- v) The first Laplacian eigenvalue μ_0 of $\mathbb{L}$ corresponding to the eigenvector $[1, \dots, 1]^T$ is always 0. In addition, the multiplicity of μ_0 is 1.*
- vi) If Γ is a connected graph which has at least two vertices, then $\mu_2 > 0$.*

Proof 4.1: i) Let D and A be the degree matrix and adjacency matrix of an undirected graph Γ , respectively. Let $\deg(v_i)$ be the degree of the vertex v_i . Let i and j be two integers in $\{1, \dots, n\}$. We know that for $i = j$, we have $D_{ii} = \deg(v_i)$ and $A_{ii} = 0$, while for $i \neq j$, the ij -th entry of A denotes the relationship between the vertices v_i and v_j in Γ , hence A_{ij} is either 1 or 0 and $D_{ij} = 0$. Therefore:

$$\mathbb{L}_{ij} = D_{ij} - A_{ij} = \begin{cases} \deg(v_i) & ; & \text{if } i = j \\ -1 & ; & \text{if } i \neq j \text{ and } \{v_i, v_j\} \in E(\Gamma) \\ 0 & ; & \text{otherwise} \end{cases} \quad (4.3)$$

ii) Let i and j be two positive integers. Then

$$\mathbb{L}_{ij} = (OO^T)_{ij} = \sum_{k=1}^n O_{ik}(O^T)_{kj} = \sum_{k=1}^n O_{ik}O_{jk} \quad (4.4)$$

which is the inner product of ρ_i , the i th row, and ρ_j , the j th row of O . For $i \neq j$, $\sum_{k=1}^n O_{ik}O_{jk} \neq 0$ if and only if for some k , both $O_{ik} \neq 0$ and $O_{jk} \neq 0$ which implies that the vertices v_i and v_j are both on the edge e_k . Now we consider the following two cases:

Case 1: If either of O_{ik} or O_{jk} is positive, then the entry is 1.

Case 2: If the other is negative, then the entry is -1. Thus $O_{ik}O_{jk} = -1$.

Similarly, for $i = j$, $(OO^T)_{ii}$ is equal to inner product of the i th row vector ρ_i with itself which is also equal to the degree of v_i . By easy verification, one can deduce that $(OO^T)_{ij} = (D - A)_{ij} = (\mathbb{L})_{ij}$ for arbitrary i, j so, $\mathbb{L} = OO^T$.

Similarly for the signless Laplacian matrix we have:

$$Q_{ij} = (RR^T)_{ij} = \sum_{k=1}^n R_{ik}(R^T)_{kj} = \sum_{k=1}^n R_{ik}R_{jk} \quad (4.5)$$

which is the inner product of the i th and j th rows, i.e. ρ_i and ρ_j , of the matrix R . For $i \neq j$, $\sum_{k=1}^n R_{ik}R_{jk} \neq 0$ if and only if for some k , both $R_{ik} \neq 0$ and $R_{jk} \neq 0$ yields that v_i and v_j are both on the edge e_k , hence the entry Q_{ij} is 1. On the other hand, for $i = j$, the entry $(RR^T)_{ii}$ is equal to the inner product of

ρ_i with itself. This is also equal to degree of v_i . Consequently, $(RR^T)_{ij} = (D + A)_{ij} = (Q)_{ij}$ for every i, j . Therefore, $Q = RR^T$.

iii) Let X be an arbitrary vector and let Y be the vector $O^T X$. From part ii), we have $\mathbb{L} = OO^T$. Multiplying this equality by X^T and X from left and right, respectively, we obtain:

$$X^T \mathbb{L} X = X^T O O^T X = Y^T Y \geq 0, \quad (4.6)$$

as $Y^T Y$ is the inner product of Y by itself. As X is arbitrary, we conclude that $\mathbb{L}$ is symmetric positive matrix.

iv) Let Γ be a simple graph. For $(1 \leq i \leq n)$, the number of -1 s in the i th row of the Laplacian matrix is equal to the number of vertices which are adjacent to the vertex v_i . Indeed, this i th row have the degree of v_i . Then the sum of the elements of i th row is equal to zero. Since Laplacian matrix is symmetric, the sum of the elements of any of its columns is equal to zero.

v) Let $X = [1, \dots, 1]$ and let $M = \mathbb{L}X$. Let M_i and X_j be the i th and j th entries of the matrices M and X , respectively. As $M_i = \sum_j (\mathbb{L}X)_{ij} = \sum_j \mathbb{L}_{ij} X_j$ and $X_j = 1$, for each j , we obtain that $M_i = \sum_j \mathbb{L}_{ij}$ is equals to the sum of the i th row of $\mathbb{L}$. In particular, we have $\mathbb{L}X = 0X$ where $X \neq 0$, therefore $\mu_0 = 0$ is an eigenvalue of $\mathbb{L}$ corresponding to the eigenvector $X = [1, \dots, 1]$. Assume Z is an arbitrary nonzero eigenvector of $\mathbb{L}$ corresponding to the eigenvalue $\mu_0 = 0$. Hence $Z^T \mathbb{L} Z = 0$. On the other hand

$$\bullet \quad 0 = (Z^T \mathbb{L})Z = \sum_{\{u,v\} \in E(\Gamma)} (Z_u - Z_v)^2$$

which implies that for each $\{u, v\} \in E(\Gamma)$, $Z_u = Z_v$. In addition, if Γ is connected, then the eigenspace W_{μ_0} is spanned by the vector X and hence $\dim W_{\mu_0} = 1$. Since $\mathbb{L} = OO^T$ is symmetric and real, by Theorem 2.4 it is also diagonalizable. Now Theorem 4.1 verifies that the multiplicity of μ_0 is 1.

vi) By part v), if Γ is a connected graph, then the Laplacian eigenvalue $\mu_0 = 0$ has multiplicity one. Since $n \geq 2$, it is obvious that $\mu_1 > 0$.

4.2. Properties of Normalized Laplacian Spectrum

In this part after explaining the notion of Normalized Laplacian matrix associated to a graph, we discuss some properties, in particular the eigenvalues and spectrum, of this matrix.

Definition 4.2: Let Γ be an undirected graph without an isolated vertex. The normalized Laplacian matrix $\tilde{\mathbb{L}}$ is defined as $\tilde{\mathbb{L}} = D^{-\frac{1}{2}}\mathbb{L}D^{-\frac{1}{2}}$ where D and $\mathbb{L}$ are the diagonal and Laplacian matrices of Γ .

Remark 4.1: Let Γ be an undirected graph without an isolated vertex with diagonal and Laplacian matrices D and $\mathbb{L}$. Let i and j be two positive integers. Let $d_k := D_{kk}$, for each k . Since Γ has no isolated vertices, the integer d_k which is the degree of v_k is nonzero; so

$$\begin{aligned} \bullet \quad \tilde{\mathbb{L}}_{ij} &= \left(D^{-\frac{1}{2}}\mathbb{L}D^{-\frac{1}{2}} \right)_{ij} = \sum_{k=1}^n (D^{-\frac{1}{2}})_{ik} \left(\mathbb{L}D^{-\frac{1}{2}} \right)_{kj} \\ &= \sum_{k=1}^n (D^{-\frac{1}{2}})_{ik} \left(\sum_{l=1}^n \mathbb{L}_{kl} (D^{-\frac{1}{2}})_{lj} \right). \end{aligned}$$

We know that $D_{ik} \neq 0$ and $D_{lj} \neq 0$ if and only if $i = k$ and $l = j$, so

$$\tilde{\mathbb{L}}_{ij} = (D^{-\frac{1}{2}})_{ii} \left(\sum_{l=1}^n \mathbb{L}_{il} (D^{-\frac{1}{2}})_{lj} \right) = (D^{-\frac{1}{2}})_{ii} \mathbb{L}_{ij} (D^{-\frac{1}{2}})_{jj}. \quad (4.7)$$

If $i = j$, then from (4.7) we conclude that

$$\bullet \quad \tilde{\mathbb{L}}_{ii} = \frac{1}{\sqrt{d_i}} d_i \frac{1}{\sqrt{d_i}} = 1.$$

So consider the case where $i \neq j$: If $\{v_i, v_j\} \in E(\Gamma)$, then by (4.7) we conclude that

$$\bullet \quad \tilde{\mathbb{L}}_{ij} = (D^{-\frac{1}{2}})_{ii} \mathbb{L}_{ij} (D^{-\frac{1}{2}})_{jj} = -\frac{1}{\sqrt{d_i d_j}}.$$

In other situations, we can easily see that $\tilde{L}_{ij} = 0$. Thus the ij -th entry of the Normalized Laplacian matrix of Γ can be described as follows:

$$\tilde{L}_{ij} = \begin{cases} 1 & ; \quad \text{if } i = j \\ \frac{-1}{\sqrt{d_i d_j}} & ; \quad \text{if } i \neq j \text{ and } \{V_i, V_j\} \in E(\Gamma) \\ 0 & ; \quad \text{otherwise} \end{cases} \quad (4.8)$$

Definition 4.3: The Normalized Laplacian Spectra

For a graph Γ , the eigenvalues of $\tilde{L}(\Gamma)$ which are the roots of the polynomial $p(\tilde{L}(\Gamma); \delta) = \det(\delta I - \tilde{L}(\Gamma))$ are called the normalized Laplacian eigenvalues of Γ and are denoted by $\delta_1(\Gamma), \dots, \delta_n(\Gamma)$ where $\delta_n(\Gamma) = 0$ for any choice of Γ , [Butler, 2008], [Cavers, 2010], [Chung, 1997].

Theorem 4.2: (Lemma 1.7-8 in [Chung, 1997]) Suppose Γ is a graph on n vertices with $n \geq 2$. Let $\delta_i := \delta_n(\Gamma)$ be the normalized Laplacian eigenvalue of Γ , for $(1 \leq i \leq n)$. Then we have the following properties:

- i) $\delta_n = 0$.
- ii) Γ is bipartite if and only if $2 - \delta_i$ is an eigenvalue of Γ for each i .
- iii) $\delta_i \leq 2$ for all i . In particular, $\delta_1 = 2$ if and only if some connected component of Γ is a non-trivial bipartite graph.
- iv) $\delta_{n-1} > 0$ if Γ is connected.
- v) $\sum_i \delta_i \leq n$ with equality holding if and only if Γ has no isolated vertices.
- vi) $\delta_{n-1} \leq \frac{n}{n-1}$ with equality holding if and only if Γ is a complete graph on n vertices.
- vii) $\delta_1 \geq \frac{n}{n-1}$, Γ has no isolated vertices.

Proposition 4.1: Suppose Γ is a simple graph. Let n and m be the numbers of vertices and edges of Γ , respectively. Suppose $S(\Gamma)$ is the subdivision graph of the graph Γ . We have the following properties:

- i) $p(S(\Gamma); \lambda) = \lambda^{m-n} p(Q(\Gamma); \lambda^2)$, where $p(S(\Gamma); \lambda)$ and $p(Q(\Gamma); \lambda)$ denote the characteristic polynomial of $S(\Gamma)$ and the characteristic polynomial of $Q(\Gamma)$, respectively.
- ii) Furthermore; $\pm\sqrt{q_1(\Gamma)}, \pm\sqrt{q_2(\Gamma)}, \dots, \pm\sqrt{q_n(\Gamma)}$ and $[0]^{m-n}$ are all eigenvalues of $S(\Gamma)$.

Proof 4.1: i) By Remark 3.1, $S(\Gamma)$ is a bipartite graph whose adjacency matrix is

$$\begin{bmatrix} 0 & R \\ R^T & 0 \end{bmatrix}. \quad (4.9)$$

By the definition we have:

$$p(S(\Gamma); \lambda) = \det(\lambda I - A_S(\Gamma)) = \det \begin{bmatrix} \lambda I_n & -R \\ -R^T & \lambda I_m \end{bmatrix}. \quad (4.10)$$

Then by Proposition 2.1,

$$\begin{aligned} \bullet \quad p(S(\Gamma); \lambda) &= \det(\lambda I_m) \det(\lambda I_n - R(\lambda I_m)^{-1} R^T) = \lambda^m \det(\lambda I_n - \lambda^{-1} R R^T) \\ &= \lambda^m \lambda^{-n} \det(\lambda^2 I_n - R R^T) = \lambda^{m-n} p(Q(\Gamma); \lambda^2). \end{aligned}$$

- ii) By using part i), in the case of $\lambda^{m-n} = 0$, we obtain the eigenvalue 0 with multiplicity $m - n$ while in the case of $p(Q(\Gamma); \lambda)(\lambda^2) = \det(\lambda^2 I - Q) = 0$, we have $q_i = \lambda_i^2$, for each $(1 \leq i \leq n)$. Consequently, $\lambda_i = \pm q_i$, for each $(1 \leq i \leq n)$.

5. THE NORMALIZED LAPLACIAN SPECTRA OF TWO JOIN GRAPHS

In this section, we identify the normalized Laplacian spectrum of $\Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \vee \Gamma_2$, where Γ_i is r_i -regular ($i = 1, 2$). Consider two graphs Γ_1, Γ_2 with $V(\Gamma_1) = \{u_1, u_2, \dots, u_{n_1}\}$, $E(\Gamma_1) = \{e_1, e_2, \dots, e_{m_1}\}$ and $V(\Gamma_2) = \{v_1, v_2, \dots, v_{n_2}\}$. Then 1_k and 0_k are two vectors of order k with all elements equal to 1 and 0, respectively. $0_{i \times j}$ denotes an $i \times j$ matrix whose entries are zero, $J_{i \times j}$ is one whose entries are 1 and I_n is the identity matrix of order n .

Theorem 5.1: (Theorem 2.1 in [Hakimi- Nezhaad and Ghorbani, 2017]) *For $i = 1, 2$. Let Γ_i is r_i -regular graph with n_i vertices with incidence matrix R and $r_i = \lambda_1(\Gamma_i) \geq \lambda_2(\Gamma_i) \geq \dots \geq \lambda_s(\Gamma_i)$. Then the normalized Laplacian spectrum of $\Gamma_1 \cup \Gamma_2$ is*

$$\{[0]^1 [1 \pm \sqrt{\frac{\lambda_j(\Gamma_1) + r_1}{2(n_2 + r_1)}}]^1, [1 - \frac{\lambda_j(\Gamma_2)}{n_1 + r_2}]^1, [1]^{m_1 - n_1}\}, \quad (5.1)$$

($2 \leq j \leq n_1$), ($2 \leq k \leq n_2$), together with the root of

$$x^2 - \frac{3n_1 + 2r_2}{n_1 + r_2}x + \frac{2n_1n_2 + 2r_1n_1 + n_2r_2}{(n_1 + r_2)(n_2 + r_1)} = 0. \quad (5.2)$$

Proof 5.1: For ($1 \leq i \leq 2$), let Γ_i be a regular graph of degree r_i with n_i vertices. Let $A(\Gamma_i)$ be the associated adjacency matrix for each i . According to the definition of $\Gamma_1 \cup \Gamma_2$, we can calculate the associated adjacency matrix as follows:

$$\bullet A(\Gamma_1 \cup \Gamma_2) = \begin{bmatrix} 0_{n_1 \times n_1} & R_{n_1 \times m_1} & J_{n_1 \times n_2} \\ R_{m_1 \times n_1}^T & 0_{m_1 \times m_1} & 0_{m_1 \times n_2} \\ J_{n_2 \times n_1} & 0_{n_2 \times m_1} & A(\Gamma_2)_{n_2 \times n_2} \end{bmatrix}.$$

Furthermore, by easy verification one can see that

$$\bullet D(\Gamma_1 \cup \Gamma_2) = \begin{bmatrix} (n_2 + r_1)I_{n_1 \times n_1} & 0_{n_1 \times m_1} & 0_{n_1 \times n_2} \\ 0_{m_1 \times n_1} & 2I_{m_1 \times m_1} & 0_{m_1 \times n_2} \\ 0_{n_2 \times n_1} & 0_{n_2 \times m_1} & (n_1 + r_2)I_{n_2 \times n_2} \end{bmatrix}.$$

Let R be the incidence matrix of Γ_1 . With respect to the definition of Laplacian matrix of a graph, we verify that:

$$\bullet \quad \mathbb{L}(\Gamma_1 \cup \Gamma_2) = \begin{bmatrix} (n_2 + r_1)I_{n_1 \times n_1} & -R_{n_1 \times m_1} & -J_{n_1 \times n_2} \\ -R^T_{m_1 \times n_1} & 2I_{m_1 \times m_1} & 0_{m_1 \times n_2} \\ -J_{n_2 \times n_1} & 0_{n_2 \times m_1} & (n_1 + r_2)I_{n_2 \times n_2} - A(\Gamma_2)_{n_2 \times n_2} \end{bmatrix}.$$

As

$$\tilde{\mathbb{L}}(\Gamma_1 \cup \Gamma_2) = D^{-\frac{1}{2}} \mathbb{L}(\Gamma_1 \cup \Gamma_2) D^{-\frac{1}{2}} \quad (5.3)$$

we can see that, by a proper labeling of vertices, the normalized Laplacian matrix of $\Gamma_1 \cup \Gamma_2$ can be written as:

$$\bullet \quad \tilde{\mathbb{L}}(\Gamma_1 \cup \Gamma_2) = \begin{bmatrix} I_{n_1 \times n_1} & \frac{-R_{n_1 \times m_1}}{\sqrt{2(n_2 + r_1)}} & \frac{J_{n_1 \times n_2}}{\sqrt{(n_1 + r_2)(n_2 + r_1)}} \\ \frac{-R^T_{m_1 \times n_1}}{\sqrt{2(n_2 + r_1)}} & I_{m_1 \times m_1} & 0_{m_1 \times n_2} \\ \frac{-J_{n_2 \times n_1}}{\sqrt{(n_1 + r_2)(n_2 + r_1)}} & 0_{n_2 \times m_1} & I_{n_2 \times n_2} - \frac{A(\Gamma_2)_{n_2 \times n_2}}{(n_1 + r_2)} \end{bmatrix}.$$

Let $1_{n_1}^T = [1, \dots, 1]$ be an eigenvector corresponding to the eigenvalue r_1 of Γ_1 , Proposition 3.1. Since adjacency matrices are symmetric, we can see that other eigenvectors of Γ_1 are orthogonal to the vector 1_{n_1} . Consider an eigenvalue of Γ_1 , say λ_j where $(2 \leq j \leq n_1)$, where $\lambda_j \neq r_1$. Let X be an eigenvector of Γ_1 corresponding to this eigenvalue. We claim that the following vector Φ is an eigenvector of $\tilde{\mathbb{L}}(\Gamma_1 \cup \Gamma_2)$ corresponding to the normalized eigenvalue θ of graph $\Gamma_1 \cup \Gamma_2$:

$$\bullet \quad \Phi = \begin{bmatrix} \alpha X \\ R^T X \\ 0 \end{bmatrix}.$$

In order to prove our claim, we may prove that $\tilde{\mathbb{L}}\Phi = \theta\Phi$. With respect to the above mentioned formula for the normalized Laplacian of the join of these two graphs, we have:

$$\bullet \begin{bmatrix} I_{n_1 \times n_1} & \frac{-R_{n_1 \times m_1}}{\sqrt{2(n_2+r_1)}} & \frac{J_{n_1 \times n_2}}{\sqrt{(n_1+r_2)(n_2+r_1)}} \\ \frac{-R^T_{m_1 \times n_1}}{\sqrt{2(n_2+r_1)}} & I_{m_1 \times m_1} & 0_{m_1 \times n_2} \\ \frac{-J_{n_2 \times n_1}}{\sqrt{(n_1+r_2)(n_2+r_1)}} & 0_{n_2 \times m_1} & I_{n_2 \times n_2} - \frac{A(\Gamma_2)_{n_2 \times n_2}}{(n_1+r_2)} \end{bmatrix} \begin{bmatrix} \alpha X \\ R^T X \\ 0 \end{bmatrix} = \theta \begin{bmatrix} \alpha X \\ R^T X \\ 0 \end{bmatrix}.$$

It should be mentioned that since $\sum_{i=1}^{n_1} \lambda_i = 0$, we have the following result:

$$\frac{-J_{n_2 \times n_1}}{\sqrt{(n_1+r_2)(n_2+r_1)}} \alpha X = \frac{-\alpha}{\sqrt{(n_1+r_2)(n_2+r_1)}} [\sum_{i=1}^{n_1} \lambda_i, \dots, \sum_{i=1}^{n_1} \lambda_i] = 0. \quad (5.4)$$

Hence by solving the above matrix system, we conclude that:

$$\bullet \begin{bmatrix} (\alpha - \frac{RR^T}{\sqrt{2(n_2+r_1)}})X \\ (1 - \frac{\alpha}{\sqrt{2(n_2+r_1)}})R^T X \\ 0 \end{bmatrix} = \begin{bmatrix} \theta \alpha X \\ \theta R^T X \\ 0 \end{bmatrix}.$$

Now we consider the two cases for solution:

Case 1:

$$\bullet \left(1 - \frac{\alpha}{\sqrt{2(n_2+r_1)}}\right) R^T X = \theta R^T X. \text{ Then we have } \left(1 - \frac{\alpha}{\sqrt{2(n_2+r_1)}}\right) = \theta.$$

Case 2:

$$\bullet \left(\alpha - \frac{RR^T}{\sqrt{2(n_2+r_1)}}\right) X = \theta \alpha X. \text{ With distributivity of } X, \alpha X - \frac{RR^T}{\sqrt{2(n_2+r_1)}} X = \theta \alpha X.$$

Now we plug θ in equation, $\alpha X - \frac{RR^T}{\sqrt{2(n_2+r_1)}} X = \alpha X - \frac{\alpha^2}{\sqrt{2(n_2+r_1)}} X$. So, $\frac{RR^T}{\sqrt{2(n_2+r_1)}} X = \frac{\alpha^2}{\sqrt{2(n_2+r_1)}} X$. Then, $RR^T X = \alpha^2 X$. Consequently, $QX = \alpha^2 X$, since $Q = RR^T$. Then α^2 is the eigenvalue of signless Laplacian matrix Q . That is $\alpha = \pm \sqrt{q_i}$. We know that $q_i = \lambda(\Gamma_i) + r_i$. Then $\alpha = \pm \sqrt{\lambda_j(\Gamma_1) + r_1}$, ($2 \leq j \leq n_1$).

Therefore,

$$\bullet \theta_k = \begin{bmatrix} \pm \sqrt{\lambda_j(\Gamma_1) + r_1} X \\ R^T \\ 0 \end{bmatrix}$$

is an eigenvector of $\tilde{L}(\Gamma_1 \uplus \Gamma_2)$ corresponding to the normalized eigenvalues θ_k of graph $\Gamma_1 \uplus \Gamma_2$, where $\theta_k = 1 \pm \sqrt{\frac{\lambda_j(\Gamma_1) + r_1}{\sqrt{2(n_2 + r_1)}}}$, $k = 1, 2$.

Now consider the $m_1 - n_1$ linearly independent eigenvectors Z_1 ($1 \leq l \leq m_1 - n_1$) of $L(\Gamma_1)$ corresponding to the eigenvalue -2 . Then $RZ_1 = 0$. Consequently

$$\bullet \Omega_1 = \begin{bmatrix} 0 \\ Z_1 \\ 0 \end{bmatrix}$$

is an eigenvector of $\tilde{L}(\Gamma_1 \uplus \Gamma_2)$ with an eigenvalue 1, since

$$\bullet \tilde{L}\Omega_1 = \begin{bmatrix} I_{n_1 \times n_1} & \frac{-R_{n_1 \times m_1}}{\sqrt{2(n_2 + r_1)}} & \frac{J_{n_1 \times n_2}}{\sqrt{(n_1 + r_2)(n_2 + r_1)}} \\ \frac{-R^T_{m_1 \times n_1}}{\sqrt{2(n_2 + r_1)}} & I_{m_1 \times m_1} & 0_{m_1 \times n_2} \\ \frac{-J_{n_2 \times n_1}}{\sqrt{(n_1 + r_2)(n_2 + r_1)}} & 0_{n_2 \times m_1} & I_{n_2 \times n_2} - \frac{A(\Gamma_2)_{n_2 \times n_2}}{(n_1 + r_2)} \end{bmatrix} \begin{bmatrix} 0 \\ Z_1 \\ 0 \end{bmatrix} = I_{n_1 \times m_1} \begin{bmatrix} 0 \\ Z_1 \\ 0 \end{bmatrix}.$$

On the other hand, $1_{n_2}^T = [1, \dots, 1]$ is an eigenvector corresponding to eigenvalue r_2 of Γ and the other eigenvectors are orthogonal to 1_{n_2} , Proposition 3.1.

Let Y be an eigenvector of Γ_2 corresponding to an eigenvalue $\lambda_j(\Gamma_1) \neq r_2$, ($2 \leq j \leq n_2$).

We claim that:

$$\bullet \psi = \begin{bmatrix} 0 \\ 0 \\ Y \end{bmatrix}$$

is an eigenvector of $\tilde{L}(\Gamma_1 \uplus \Gamma_2)$ corresponding to the normalized eigenvalue $\kappa = 1 - \frac{\lambda_k(\Gamma_2)}{n_1 + r_2}$, ($2 \leq j \leq n_2$).

$$\bullet \tilde{\mathbf{L}}\psi = \begin{bmatrix} I_{n_1 \times n_1} & \frac{-R_{n_1 \times m_1}}{\sqrt{2(n_2+r_1)}} & \frac{J_{n_1 \times n_2}}{\sqrt{(n_1+r_2)(n_2+r_1)}} \\ \frac{-R^T_{m_1 \times n_1}}{\sqrt{2(n_2+r_1)}} & I_{m_1 \times m_1} & 0_{m_1 \times n_2} \\ \frac{-J_{n_2 \times n_1}}{\sqrt{(n_1+r_2)(n_2+r_1)}} & 0_{n_2 \times m_1} & I_{n_2 \times n_2} - \frac{A(\Gamma_2)_{n_2 \times n_2}}{(n_1+r_2)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ Y \end{bmatrix} = \kappa \begin{bmatrix} 0 \\ 0 \\ Y \end{bmatrix}$$

$$\bullet \begin{bmatrix} \frac{J_{n_1 \times n_2}}{\sqrt{(n_1+r_2)(n_2+r_1)}} Y \\ 0 \\ I_{n_2 \times n_2} - \frac{A(\Gamma_2)_{n_2 \times n_2}}{(n_1+r_2)} Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \kappa Y \end{bmatrix}$$

By solving this equations, $\kappa = 1 - \frac{\lambda_k(\Gamma_2)}{n_1+r_2}$, ($2 \leq j \leq n_2$).

Let

$$\bullet \Lambda = \begin{bmatrix} \alpha J_{n_1 \times 1} \\ \beta J_{m_1 \times 1} \\ \gamma J_{n_2 \times 1} \end{bmatrix}$$

for some vector $(\alpha, \beta, \gamma) \neq (0,0,0)$.

$$\bullet \begin{bmatrix} I_{n_1 \times n_1} & \frac{-R_{n_1 \times m_1}}{\sqrt{2(n_2+r_1)}} & \frac{J_{n_1 \times n_2}}{\sqrt{(n_1+r_2)(n_2+r_1)}} \\ \frac{-R^T_{m_1 \times n_1}}{\sqrt{2(n_2+r_1)}} & I_{m_1 \times m_1} & 0_{m_1 \times n_2} \\ \frac{-J_{n_2 \times n_1}}{\sqrt{(n_1+r_2)(n_2+r_1)}} & 0_{n_2 \times m_1} & I_{n_2 \times n_2} - \frac{A(\Gamma_2)_{n_2 \times n_2}}{(n_1+r_2)} \end{bmatrix} \begin{bmatrix} \alpha J_{n_1 \times 1} \\ \beta J_{m_1 \times 1} \\ \gamma J_{n_2 \times 1} \end{bmatrix} = x \begin{bmatrix} \alpha J_{n_1 \times 1} \\ \beta J_{m_1 \times 1} \\ \gamma J_{n_2 \times 1} \end{bmatrix}.$$

Consider the following term:

$$\alpha I_{n_1 \times n_1} J_{n_1 \times 1} - \beta \frac{R_{n_1 \times m_1}}{\sqrt{2(n_2+r_1)}} J_{m_1 \times 1} - \gamma \frac{J_{n_1 \times n_2}}{\sqrt{(n_1+r_2)(n_2+r_1)}} J_{n_2 \times 1}. \quad (5.5)$$

As

$$\bullet R_{n_1 \times m_1} J_{m_1 \times 1} = \begin{bmatrix} r_1 \\ \vdots \\ r_1 \end{bmatrix} = r_1 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = r_1 J_{n_1 \times 1}$$

and

- $J_{n_1 \times n_2} J_{n_2 \times 1} = \begin{bmatrix} n_2 \\ \vdots \\ n_2 \end{bmatrix} = n_2 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = n_2 J_{n_2 \times 1}.$

We can rewrite (5.5) as follows:

- $\alpha J_{n_1 \times 1} + \frac{-r_1}{\sqrt{2(n_2+r_1)}} \beta J_{n_1 \times 1} - \frac{n_2}{\sqrt{(n_1+r_2)(n_2+r_1)}} \gamma J_{n_1 \times 1}$

$$= \begin{bmatrix} 1 & \frac{-r_1}{\sqrt{2(n_2+r_1)}} & \frac{n_2}{\sqrt{(n_1+r_2)(n_2+r_1)}} \end{bmatrix} \begin{bmatrix} \alpha J_{n_1 \times 1} \\ \beta J_{m_1 \times 1} \\ \gamma J_{n_2 \times 1} \end{bmatrix}.$$

Similarly, we can see that

- $\alpha \frac{-R^T m_1 \times n_1}{\sqrt{2(n_2+r_1)}} J_{n_1 \times 1} + I_{m_1 \times m_1} \beta J_{m_1 \times 1} + 0_{m_1 \times n_2} \gamma J_{n_2 \times 1}$

$$= \begin{bmatrix} \frac{-2}{\sqrt{2(n_2+r_1)}} & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha J_{n_1 \times 1} \\ \beta J_{m_1 \times 1} \\ \gamma J_{n_2 \times 1} \end{bmatrix},$$

and

- $\alpha \frac{-J_{n_2 \times n_1}}{\sqrt{(n_1+r_2)(n_2+r_1)}} J_{n_1 \times 1} + 0_{n_2 \times m_1} \beta J_{m_1 \times 1} + (I_{n_2 \times n_2} - \frac{A(r_2)n_2 \times n_2}{(n_1+r_2)}) \gamma J_{n_2 \times 1}$

$$= \begin{bmatrix} \frac{-n_1}{\sqrt{(n_1+r_2)(n_2+r_1)}} & 0 & 1 - \frac{r_2}{n_1+r_2} \end{bmatrix} \begin{bmatrix} \alpha J_{n_1 \times 1} \\ \beta J_{m_1 \times 1} \\ \gamma J_{n_2 \times 1} \end{bmatrix}.$$

By the above notes, we deduce that the equation $\tilde{X} \Lambda = x \Lambda$ gives that Λ is an eigenvector of A corresponding to eigenvalue x if and only if

- $\zeta = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$

is an eigenvector of the matrix M , where

$$\bullet \quad M = \begin{bmatrix} 1 & \frac{-r_1}{\sqrt{2(n_2+r_1)}} & \frac{-n_2}{\sqrt{(n_1+r_2)(n_2+r_1)}} \\ \frac{-2}{\sqrt{2(n_2+r_1)}} & 1 & 0 \\ \frac{-n_1}{\sqrt{(n_1+r_2)(n_2+r_1)}} & 0 & 1 - \frac{r_2}{(n_1+r_2)} \end{bmatrix}.$$

The characteristic polynomial of M is

$$x^3 - \frac{3n_1+2r_2}{n_1+r_2}x^2 + \frac{2n_1n_2+2r_1n_1+n_2r_2}{(n_1+r_2)(n_2+r_1)}x = 0. \quad (5.6)$$

Corollary 5.1: (Corollary 2.2 in [Hakimi- Nezhaad and Ghorbani, 2017]) If $\Gamma_2 \cong \overline{K_{n_2}}$, then the normalized Laplacian spectrum of $\Gamma_1 \cup \Gamma_2$ is

$$\{ [0]^1, [1]^{m_1-n_1+n_2}, [1 \pm \sqrt{\frac{\lambda_j(\Gamma_1)+r_1}{2(n_2+r_1)}}]^1, [2]^1 \}, (2 \leq j \leq n_1). \quad (5.7)$$

Proof 5.1: If $\Gamma_2 \cong \overline{K_{n_2}}$, then $r_2 = 0$. We replace r_2 by 0.

We already have $\{ [0]^1, [1 \pm \sqrt{\frac{\lambda_j(\Gamma_1)+r_1}{2(n_2+r_1)}}]^1 \}$ from Theorem 5.1.

Now, since Γ_2 has no edge, then for each $(2 \leq k \leq n_2)$, $\lambda_{\Gamma_2} = 0$. Then

$$\left[1 - \frac{\lambda_j(\Gamma_2)}{n_1+r_2} \right] = [1 - 0] = 1 \quad (5.8)$$

repeating for each $(2 \leq k \leq n_2)$. Then we have $[1]^{n_2-1}$.

On the other hand, we had $[1]^{m_1-n_1}$, together with $[1]^{n_2-1}$. This give us $[1]^{m_1-n_1+n_2+1}$.

Now consider the roots of the polynomial

$$x^2 - \frac{3n_1+2r_2}{n_1+r_2}x + \frac{2n_1n_2+2r_1n_1+n_2r_2}{(n_1+r_2)(n_2+r_1)} = 0 \text{ for } r_2 = 0. \quad (5.9)$$

We obtain that $x^2 - 3x + 2 = 0$. The roots of this polynomial are 1 and 2.

We conclude that the eigenvalue 1 repeating $m_1 - n_1 + n_2$ times and we have the eigenvalue 2 which is repeating once.

Theorem 5.2: (Theorem 2.3 in [Hakimi-Nezhaad and Ghorbani, 2017]) For $i = 1, 2$. Let Γ_i is r_i -regular graph with n_i vertices with incidence matrix R and $r_i = \lambda_1(\Gamma_i) \geq \lambda_2(\Gamma_i) \geq \dots \geq \lambda_s(\Gamma_i)$. Then the normalized Laplacian spectrum of $\Gamma_1 \underline{\vee} \Gamma_2$ is

$$\{ [0]^1, [1 \pm \sqrt{\frac{\lambda_j(\Gamma_1) + r_1}{r_1(n_2 + 2)}}]^1, [1 - \frac{\lambda_j(\Gamma_2)}{m_1 + r_2}]^1, [1]^{m_1 - n_1} \}, \quad (5.10)$$

($2 \leq j \leq n_1$), ($2 \leq k \leq n_2$), together with the root of

$$x^2 - \frac{3m_1 + 2r_2}{m_1 + r_2}x + \frac{2m_1n_2 + 4m_1 + n_2r_2}{(m_1 + r_2)(2 + n_2)} = 0. \quad (5.11)$$

Proof 5.2: For ($1 \leq i \leq 2$), let Γ_i be a regular graph of degree r_i with n_i vertices, Proposition 3.1. Let $A(\Gamma_i)$ be the associated adjacency matrix for each i . According to the definition of $\Gamma_1 \underline{\vee} \Gamma_2$, we can calculate the associated adjacency matrix as follows:

$$\bullet A(\Gamma_1 \underline{\vee} \Gamma_2) = \begin{bmatrix} 0_{n_1 \times n_1} & R_{n_1 \times m_1} & 0_{n_1 \times n_2} \\ R_{m_1 \times n_1}^T & 0_{m_1 \times m_1} & J_{m_1 \times n_2} \\ 0_{n_2 \times n_1} & J_{n_2 \times m_1} & A(\Gamma_2)_{n_2 \times n_2} \end{bmatrix}.$$

Furthermore, by easy verification one can see that

$$\bullet D(\Gamma_1 \underline{\vee} \Gamma_2) = \begin{bmatrix} r_1 I_{n_1 \times n_1} & 0_{n_1 \times m_1} & 0_{n_1 \times n_2} \\ 0_{m_1 \times n_1} & (2 + n_2) I_{m_1 \times m_1} & 0_{m_1 \times n_2} \\ 0_{n_2 \times n_1} & 0_{n_2 \times m_1} & (m_1 + r_2) I_{n_2 \times n_2} \end{bmatrix}.$$

Let R be the incidence matrix of Γ_1 . With respect to the definition of Laplacian matrix of a graph, we verify that:

$$\bullet \mathbb{L}(\Gamma_1 \underline{\vee} \Gamma_2) = \begin{bmatrix} r_1 I_{n_1 \times n_1} & -R_{n_1 \times m_1} & 0_{n_1 \times n_2} \\ -R_{m_1 \times n_1}^T & (2 + n_2) I_{m_1 \times m_1} & J_{m_1 \times n_2} \\ 0_{n_2 \times n_1} & -J_{n_2 \times m_1} & (r_2 + m_1) I_{n_2 \times n_2} - A(\Gamma_2)_{n_2 \times n_2} \end{bmatrix}.$$

As

$$\tilde{\mathbb{L}}(\Gamma_1 \underline{\vee} \Gamma_2) = D^{-\frac{1}{2}} \mathbb{L}(\Gamma_1 \underline{\vee} \Gamma_2) D^{-\frac{1}{2}} \quad (5.12)$$

we can see that, by a proper labeling of vertices, the normalized Laplacian matrix of $\Gamma_1 \underline{\vee} \Gamma_2$ can be written as:

$$\bullet \quad \tilde{\mathbb{L}}(\Gamma_1 \underline{\vee} \Gamma_2) = \begin{bmatrix} I_{n_1 \times n_1} & \frac{-R_{n_1 \times m_1}}{\sqrt{r_1(n_2+2)}} & 0_{n_1 \times n_2} \\ \frac{-R^T_{m_1 \times n_1}}{\sqrt{r_1(n_2+2)}} & I_{m_1 \times m_1} & \frac{-J_{m_1 \times n_2}}{\sqrt{(m_1+r_2)(n_2+2)}} \\ 0_{n_2 \times n_1} & \frac{-J_{n_2 \times m_1}}{\sqrt{(m_1+r_2)(n_2+2)}} & I_{n_2 \times n_2} - \frac{A(\Gamma_2)_{n_2 \times n_2}}{(m_1+r_2)} \end{bmatrix}.$$

Let $1_{n_1}^T = [1, \dots, 1]$ be an eigenvector corresponding to the eigenvalue r_1 of Γ_1 , Proposition 3.1. Since adjacency matrices are symmetric, we can see that other eigenvectors of Γ_1 are orthogonal to the vector 1_{n_1} . Consider an eigenvalue of Γ_1 , say λ_j where $(2 \leq j \leq n_1)$, where $\lambda_j \neq r_1$. Let X' be an eigenvector of Γ_1 corresponding to this eigenvalue. We claim that the following vector Φ' is an eigenvector of $\tilde{\mathbb{L}}(\Gamma_1 \underline{\vee} \Gamma_2)$ corresponding to the normalized eigenvalue θ' of graph $\Gamma_1 \underline{\vee} \Gamma_2$:

$$\bullet \quad \Phi' = \begin{bmatrix} \alpha' X' \\ R^T X' \\ 0 \end{bmatrix}.$$

In order to prove our claim, we may prove that $\tilde{\mathbb{L}}\Phi' = \theta'\Phi'$. With respect to the above mentioned formula for the normalized Laplacian of the join of these two graphs, we have:

$$\bullet \quad \begin{bmatrix} I_{n_1 \times n_1} & \frac{-R_{n_1 \times m_1}}{\sqrt{r_1(n_2+2)}} & 0_{n_1 \times n_2} \\ \frac{-R^T_{m_1 \times n_1}}{\sqrt{r_1(n_2+2)}} & I_{m_1 \times m_1} & \frac{-J_{m_1 \times n_2}}{\sqrt{(m_1+r_2)(n_2+2)}} \\ 0_{n_2 \times n_1} & \frac{-J_{n_2 \times m_1}}{\sqrt{(m_1+r_2)(n_2+2)}} & I_{n_2 \times n_2} - \frac{A(\Gamma_2)_{n_2 \times n_2}}{(m_1+r_2)} \end{bmatrix} \begin{bmatrix} \alpha' X' \\ R^T X' \\ 0 \end{bmatrix} = \theta' \begin{bmatrix} \alpha' X' \\ R^T X' \\ 0 \end{bmatrix}$$

It should be mentioned that since $\sum_{i=1}^{n_1} \lambda_i = 0$, we have the following result:

$$\frac{-J_{n_2 \times m_1}}{\sqrt{(m_1+r_2)(n_2+2)}} \alpha' X' = \frac{-\alpha'}{\sqrt{(m_1+r_2)(n_2+2)}} [\sum_{i=1}^{n_1} \lambda_i, \dots, \sum_{i=1}^{n_1} \lambda_i] = 0. \quad (5.13)$$

Hence by solving the above matrix system, we conclude that:

$$\bullet \begin{bmatrix} (\alpha' - \frac{RR^T}{\sqrt{r_1(n_2+2)}})X' \\ (1 - \frac{\alpha'}{\sqrt{r_1(n_2+2)}})R^T X' \\ 0 \end{bmatrix} = \begin{bmatrix} \theta' \alpha' X' \\ \theta' R^T X' \\ 0 \end{bmatrix}.$$

Now we consider the two cases for solution:

Case 1:

$$\bullet \left(1 - \frac{\alpha'}{\sqrt{r_1(n_2+2)}}\right) R^T X' = \theta' R^T X'.$$

$$\text{Then we have } \left(1 - \frac{\alpha'}{\sqrt{r_1(n_2+2)}}\right) = \theta'.$$

Case 2:

$$\bullet \left(\alpha' - \frac{RR^T}{\sqrt{2(n_2+r_1)}}\right) X' = \theta' \alpha' X'.$$

With distributivity of X' , $\alpha' X' - \frac{RR^T}{\sqrt{r_1(n_2+2)}} X' = \theta' \alpha' X'$. Now we plug θ' in equation, $\alpha' X' - \frac{RR^T}{\sqrt{r_1(n_2+2)}} X' = \alpha' X' - \frac{\alpha'^2}{\sqrt{r_1(n_2+2)}} X'$.

So, $\frac{RR^T}{\sqrt{r_1(n_2+2)}} X' = \frac{\alpha'^2}{\sqrt{r_1(n_2+2)}} X'$. Then, $RR^T X' = \alpha'^2 X'$. Consequently,

$QX = \alpha'^2 X$, since $Q = RR^T$. Then α'^2 is the eigenvalue of signless Laplacian matrix Q . That is $\alpha' = \pm \sqrt{q_i}$. We know that $q_i = \lambda(\Gamma_i) + r_i$. Then $\alpha' = \pm \sqrt{\lambda_j(\Gamma_1) + r_1}$, ($2 \leq j \leq n_1$).

Therefore,

$$\bullet \theta'_k = \begin{bmatrix} \pm \sqrt{\lambda_j(\Gamma_1) + r_1} X' \\ R^T X' \\ 0 \end{bmatrix}$$

is an eigenvector of $\tilde{L}(\Gamma_1 \vee \Gamma_2)$ corresponding to the normalized eigenvalues θ'_k of graph $\Gamma_1 \vee \Gamma_2$, where $\theta'_k = 1 \pm \sqrt{\frac{\lambda_j(\Gamma_1) + r_1}{r_1(n_2 + 2)}}$, $k = 1, 2$.

Now consider the $m_1 - n_1$ linearly independent eigenvectors Z_1' ($1 \leq l \leq m_1 - n_1$) of $L(\Gamma_1)$ corresponding to the eigenvalue -2 . Then $RZ_1' = 0$. Consequently

$$\bullet \quad \Omega_1' = \begin{bmatrix} 0 \\ Z_1' \\ 0 \end{bmatrix}$$

is an eigenvector of $\tilde{L}(\Gamma_1 \cup \Gamma_2)$ with an eigenvalue 1, since

$$\bullet \quad \tilde{L}\Omega_1' = \begin{bmatrix} I_{n_1 \times n_1} & \frac{-R_{n_1 \times m_1}}{\sqrt{r_1(n_2 + 2)}} & 0_{n_1 \times n_2} \\ \frac{-R^T_{m_1 \times n_1}}{\sqrt{r_1(n_2 + 2)}} & I_{m_1 \times m_1} & \frac{-J_{m_1 \times n_2}}{\sqrt{(m_1 + r_2)(n_2 + 2)}} \\ 0_{n_2 \times n_1} & \frac{-J_{n_2 \times m_1}}{\sqrt{(m_1 + r_2)(n_2 + 2)}} & I_{n_2 \times n_2} - \frac{A(\Gamma_2)_{n_2 \times n_2}}{(m_1 + r_2)} \end{bmatrix} \begin{bmatrix} 0 \\ Z_2 \\ 0 \end{bmatrix} = I_{n_1 \times m_1} \begin{bmatrix} 0 \\ Z_1' \\ 0 \end{bmatrix}.$$

On the other hand, $1_{n_2} = [1, \dots, 1]$ is an eigenvector corresponding to eigenvalue r_2 of Γ and the other eigenvectors are orthogonal to 1_{n_2} , Proposition 3.1.

Let Y' be an eigenvector of Γ_2 corresponding to an eigenvalue $\lambda_j(\Gamma_1) \neq r_2$, ($2 \leq j \leq n_2$).

We claim that:

$$\bullet \quad \psi' = \begin{bmatrix} 0 \\ 0 \\ Y' \end{bmatrix}$$

is an eigenvector of $\tilde{L}(\Gamma_1 \cup \Gamma_2)$ corresponding to the normalized eigenvalue $\kappa' = 1 - \frac{\lambda_k(\Gamma_2)}{n_1 + r_2}$, ($2 \leq j \leq n_2$).

$$\bullet \tilde{\mathbb{L}}\psi' = \begin{bmatrix} I_{n_1 \times n_1} & \frac{-R_{n_1 \times m_1}}{\sqrt{r_1(n_2+2)}} & 0_{n_1 \times n_2} \\ \frac{-R^T_{m_1 \times n_1}}{\sqrt{r_1(n_2+2)}} & I_{m_1 \times m_1} & \frac{-J_{m_1 \times n_2}}{\sqrt{(m_1+r_2)(n_2+2)}} \\ 0_{n_2 \times n_1} & \frac{-J_{n_2 \times m_1}}{\sqrt{(m_1+r_2)(n_2+2)}} & I_{n_2 \times n_2} - \frac{A(\Gamma_2)_{n_2 \times n_2}}{(m_1+r_2)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ Y' \end{bmatrix} = \kappa' \begin{bmatrix} 0 \\ 0 \\ Y' \end{bmatrix}$$

$$\bullet \begin{bmatrix} \frac{-J_{m_1 \times n_2}}{\sqrt{(m_1+r_2)(n_2+2)}} Y' \\ 0 \\ I_{n_2 \times n_2} - \frac{A(\Gamma_2)_{n_2 \times n_2}}{(m_1+r_2)} Y' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \kappa' Y' \end{bmatrix}$$

By solving this equations, $\kappa' = 1 - \frac{\lambda_k(\Gamma_2)}{m_1+r_2}$, ($2 \leq j \leq n_2$).

Let

$$\bullet \Lambda' = \begin{bmatrix} \alpha' J_{n_1 \times 1} \\ \beta' J_{m_1 \times 1} \\ \gamma' J_{n_2 \times 1} \end{bmatrix}$$

for some vector $(\alpha', \beta', \gamma') \neq (0,0,0)$.

Similar to the previous part, we can see that the equation $\tilde{\mathbb{L}} \Lambda' = x \Lambda'$ gives that Λ' is an eigenvector of A corresponding to eigenvalue x' if and only if

$$\bullet \zeta' = \begin{bmatrix} \alpha' \\ \beta' \\ \gamma' \end{bmatrix}$$

is an eigenvector of the matrix S , where

$$\bullet S = \begin{bmatrix} 1 & \frac{-r_1}{\sqrt{r_1(n_2+2)}} & 0 \\ \frac{-2}{\sqrt{r_1(n_2+2)}} & 1 & \frac{-n_2}{\sqrt{(m_1+r_2)(2+n_2)}} \\ 0 & \frac{-m_1}{\sqrt{(m_1+r_2)(2+n_2)}} & 1 - \frac{r_2}{(m_1+r_2)} \end{bmatrix}.$$

The characteristic polynomial of S is

$$x^3 - \frac{3n_1+2r_2}{n_1+r_2} x^2 + \frac{2n_1n_2+2r_1n_1+n_2r_2}{(n_1+r_2)(n_2+r_1)} x = 0. \quad (5.14)$$

Corollary 5.2: (Corollary 2.4 in [Hakimi- Nezhaad and Ghorbani, 2017]) *If $\Gamma_2 \cong \overline{K_{n_2}}$, then the normalized Laplacian spectrum of $\Gamma_1 \vee \Gamma_2$ is*

$$\{ [0]^1, [1]^{m_1-n_1+n_2}, [1 \pm \sqrt{\frac{\lambda_j(\Gamma_1)+r_1}{r_1(n_2+2)}}]^1, [2]^1 \}, (2 \leq j \leq n_1). \quad (5.15)$$

Proof 5.2: *If $\Gamma_2 \cong \overline{K_{n_2}}$, then $r_2 = 0$. We replace r_2 by 0.*

We have $\{ [0]^1, [1 \pm \sqrt{\frac{\lambda_j(\Gamma_1)+r_1}{r_1(n_2+2)}}]^1 \}$ from Theorem 5.2.

Now, since Γ_2 has no edge, then for each $(2 \leq k \leq n_2)$, $\lambda_{\Gamma_2} = 0$. Then

$$\left[1 - \frac{\lambda_j(\Gamma_2)}{m_1+r_2} \right] = [1 - 0] = 1 \quad (5.16)$$

repeating for each $(2 \leq k \leq n_2)$. Then we have $[1]^{n_2-1}$. On the other hand, we had $[1]^{m_1-n_1}$, together with $[1]^{n_2-1}$. This give us $[1]^{m_1-n_1+n_2+1}$.

Now consider the roots of the polynomial

$$x^2 - \frac{3m_1+2r_2}{m_1+r_2}x + \frac{2m_1n_2+4m_1+n_2r_2}{(m_1+r_2)(2+n_2)} = 0 \text{ for } r_2 = 0. \quad (5.17)$$

We obtain that $x^2 - 3x + 2 = 0$. The roots of this polynomial are 1 and 2. We conclude that the eigenvalue 1 repeating $m_1 - n_1 + n_2$ times and we have the eigenvalue 2 which is repeating once.

6. FURTHER RESULTS ON TWO NEW JOIN GRAPHS

6.1. Integrality of Two New Join Graphs

In sixth section, we will define the concepts of $\tilde{\mathcal{L}}$ -integral. Accordingly, we will give some proposition and proofs related to $\tilde{\mathcal{L}}$ -integral. Then, we will define the normalized Laplacian energy and prove the normalized Laplacian energy formula for two new join graphs in some special cases. Finally, we will define Kirchhoff index and we will calculate the Kirchhoff index of the new join graphs by using normalized Laplacian eigenvalues.

Definition 6.1: The graph Γ is $\tilde{\mathcal{L}}$ -integral if all its $\tilde{\mathcal{L}}$ -eigenvalues are integer.

Theorem 6.1: (Example 1.2 in [Chung, 1991]) A graph Γ is $\tilde{\mathcal{L}}$ -integral if and only if its eigenvalues are 0, 1, 2.

Proof 6.1: If its eigenvalues are 0,1,2, then it is clear that Γ is $\tilde{\mathcal{L}}$ -integral.

Now suppose Γ is $\tilde{\mathcal{L}}$ -integral, so its $\tilde{\mathcal{L}}$ -eigenvalues are integer. By Theorem 4.2, we know that $\delta_n = 0$ and each $0 < \delta_i \leq 2$ must be integer and $0 < \delta_{n-1} \leq 1$. Since Γ is $\tilde{\mathcal{L}}$ -integral and δ_{n-1} is an integer, then we have $\delta_{n-1} = 1$.

Proposition 6.1: [Dam and Omid, 2011] A connected graph Γ has only two $\tilde{\mathcal{L}}$ -eigenvalues if and only if it is complete.

Proof 6.1: Let a connected graph Γ has only two $\tilde{\mathcal{L}}$ -eigenvalues. Then by Theorem 4.2, we have $\delta_n = 0$ and $\delta_{n-1} > 0$. Since this graph has no isolated vertices Theorem 4.2, $\sum_i \delta_i = 0$. Since we have two eigenvalues and $\delta_n = 0$, other eigenvalues are the same. Then we have $\sum_{i=1}^{n-1} \delta = n$. So , $\delta (n - 1 - 1 + 1) = n$ and thus $\delta = \frac{n}{n-1}$. Consequently, from Theorem 4.2, a connected graph which has only two $\tilde{\mathcal{L}}$ -eigenvalues is complete. Suppose Γ is complete graph on n vertices. By

Theorem 4.2, since K_n has no isolated vertices, the normalized Laplacian spectrum of K_n is $[0], [\frac{n}{n-1}]$. So Γ has only two $\tilde{\mathcal{L}}$ -eigenvalues.

Notation 6.1: Suppose u and v are two vertices of a connected graph Γ . Denote $\mathfrak{N}(u) \cap \mathfrak{N}(v)$ by λ_{uv} if u and v are adjacent, and by μ_{uv} otherwise.

Proposition 6.2: (Theorem 1 in [Dam and Omid, 2011]) Let Γ be a connected graph with e edges. Then Γ has three $\tilde{\mathcal{L}}$ -eigenvalues $0, \delta_1, \delta_2$ if and only if $\mu_{uv} = t \deg(u) \deg(v)$, for non adjacent vertices u and v where $t = \frac{\delta_1 \delta_2}{2e}$.

Proposition 6.2, verifies the following corollary:

Corollary 6.1: (Corollary 2 in [Dam and Omid, 2011]) A connected graph Γ with three $\tilde{\mathcal{L}}$ -eigenvalues $0, \delta_1, \delta_2$ has diameter two.

Proof 6.1: Let Γ be a connected graph with three $\tilde{\mathcal{L}}$ -eigenvalues. As δ_1, δ_2 and the degrees of vertices are positive numbers, by Proposition 6.2, we deduce that μ_{uv} is nonzero, for any pair u, v of nonadjacent vertices of Γ . On the other hand, Proposition 6.1, implies that Γ is not complete, so its diameter is greater than one. If $\text{diam}(\Gamma) > 2$, then there exist two vertices u' and v' in Γ such that $d(u', v') > 2$. Therefore $\mathfrak{N}(u') \cap \mathfrak{N}(v') = \emptyset$ and so $\mu_{u'v'} = 0$, a contradiction. Consequently, Γ has diameter two.

Proposition 6.3.: (Proposition 2 in [Dam and Omid, 2011]) Let be a connected graph. Then the following statements are equivalent.

- i) Γ is bipartite with three $\tilde{\mathcal{L}}$ -eigenvalues.
- ii) Γ is $\tilde{\mathcal{L}}$ -integral.
- iii) Γ is complete bipartite.

Proof 6.3: ($i \Rightarrow ii$) Suppose Γ is bipartite with three $\tilde{\mathcal{L}}$ -eigenvalues, we know that $\delta_1 = 2, \delta_n = 0, 0 \leq \delta_{n-1} \leq 2$ and Γ is bipartite, then $2 - \delta_i$ is eigenvalue of Γ for

every i . Then $2 - \delta_n - 1 = \delta_{n-1}$, from here $\delta_{n-1} = 1$ and we can say that $\tilde{\mathcal{L}}$ is integral.

(ii $\Rightarrow$ iii) Let Γ is $\tilde{\mathcal{L}}$ -integral. $\tilde{\mathcal{L}}$ -spectrum Γ is $\{[0]^1, [1]^{n-2}, [2]^1\}$ and hence Γ is bipartite. Γ is complete bipartite, since diameter equals two and Γ is connected.

(iii $\Rightarrow$ i) Clear.

Proposition 6.4: (Proposition 3.2 in [Hakimi-Nezhaad and Ghorbani, 2017]) For $i = 1, 2$, let Γ_i be r_i -regular connected graph, then $\Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \underline{\vee} \Gamma_2$ are $\tilde{\mathcal{L}}$ -integral if and only if $\Gamma_1 \cong K_2$ and $\Gamma_2 \cong \overline{K_{n_2}}$.

Proof 6.4: ($\Leftarrow$) Let $\Gamma_1 \cong K_2$ and $\Gamma_2 \cong \overline{K_{n_2}}$. Since $K_2 \cup \overline{K_{n_2}} = K_{2, n_2}$ and $K_2 \underline{\vee} \overline{K_{n_2}} = K_{1, n_2+2}$ are two graphs of order n with the normalized Laplacian eigenvalues $\{[0]^1, [1]^{n-2}, [2]^1\}$ these graphs are $\tilde{\mathcal{L}}$ -integral.

($\Rightarrow$) Conversely, suppose that for $i = 1, 2$, let Γ_i be r_i -regular connected graph and $\Gamma_1 \cup \Gamma_2$ is $\tilde{\mathcal{L}}$ -integral, $\Gamma_1 \not\cong K_2$ and $\Gamma_2 \not\cong \overline{K_{n_2}}$.

The normalized Laplacian spectrum of $\Gamma_1 \cup \Gamma_2$ is

$$\{[0]^1 [1 \pm \sqrt{\frac{\lambda_j(\Gamma_1) + r_1}{2(n_2 + r_1)}}]^1, [1 - \frac{\lambda_j(\Gamma_2)}{n_1 + r_2}]^1, [1]^{m_1 - n_1}\}, \quad (6.1)$$

($2 \leq j \leq n_1$), ($2 \leq k \leq n_2$) together with the root of

$$x^2 - \frac{3n_1 + 2r_2}{n_1 + r_2}x + \frac{2n_1n_2 + 2r_1n_1 + n_2r_2}{(n_1 + r_2)(n_2 + r_1)} = 0. \quad (6.2)$$

Since Γ_1 is connected, $\Gamma_1 \cup \Gamma_2$ is connected. Then, 0 is eigenvalue with multiplicity 1. If $\lambda_j(\Gamma_2) = 0$, ($2 \leq j \leq n_2$) and $\lambda_j(\Gamma_2) = r_2$, we have:

- $p_{\Gamma_2}(x) = (x - r_2)(x - 0)^{n_2-1} = x^{n_2} - r_2x^{n_2-1}.$

Since $\text{trace}(A(\Gamma_2)) = 0$ and $\text{trace}(A(\Gamma_2)) = -r$. We have, $r_2 = 0$. So, $\lambda_j(\Gamma_2) = 0$, for every i . Consequently, $\Gamma_2 \cong \overline{K_{n_2}}$. If $\lambda_j(\Gamma_1) = -r_1$, ($2 \leq j \leq n_1$), then Γ_1 is bipartite and spectrum is symmetric. Then

- $p_{\Gamma_1}(x) = (x - r_1)(x + r_1)^{n_1-1} = x^{n_1} - r_1 x^{n_1-1}$.

Since $n_1 = 2$, Γ_1 is connected. Thus $\Gamma_1 \cong K_2$. If $\lambda_j(\Gamma_1) = -r_1 + 2n_1 > 0$, ($2 \leq j \leq n_1$), then $n_1 > r_1$.

Proposition 6.5: (Proposition 3.3 in [Hakimi- Nezhaad and Ghorbani, 2017]) Let Γ_1 be an empty graph of order n_1 and Γ_2 be r_2 -regular graph of order n_2 , then $\Gamma_1 \uplus \Gamma_2$ and $\Gamma_1 \underline{\vee} \Gamma_2$ are $\tilde{\mathcal{L}}$ -integral if and only if $\Gamma_2 \cong \overline{K_{n_2}}$ or Γ_2 is complete bipartite graph.

Proof 6.5: Let Γ_1 be an empty graph of order n_1 and Γ_2 be r_2 -regular graph of order n_2 .

($\Rightarrow$) Suppose $\Gamma_1 \uplus \Gamma_2$ and $\Gamma_1 \underline{\vee} \Gamma_2$ are $\tilde{\mathcal{L}}$ -integral and $\Gamma_2 \not\cong \overline{K_{n_2}}$. From Proposition 6.3, these join graphs are complete bipartite. We know that $\Gamma_1 \uplus \Gamma_2$ is obtained by joining all of the vertices of Γ_1 with all vertices of Γ_2 . Since Γ_1 has only n_1 vertices and $\Gamma_1 \uplus \Gamma_2$ is complete bipartite, Γ_2 must be complete bipartite graph. Similarly we know that $\Gamma_1 \underline{\vee} \Gamma_2$ is obtained by joining the all of the vertices of $S(\Gamma_1)$ corresponding to the edges of Γ_1 with all of the vertices of Γ_2 . From hypothesis, since Γ_1 has no edges and $\Gamma_1 \underline{\vee} \Gamma_2$ is complete bipartite graph, then Γ_2 must be complete bipartite.

Now suppose $\Gamma_1 \uplus \Gamma_2$ and $\Gamma_1 \underline{\vee} \Gamma_2$ are $\tilde{\mathcal{L}}$ -integral and Γ_2 is not complete bipartite. Then we can say with the same arguments that $\Gamma_2 \cong \overline{K_{n_2}}$.

($\Leftarrow$) Suppose $\Gamma_2 \cong \overline{K_{n_2}}$ and Γ_2 is not complete bipartite graph. Then Γ_2 has only n_2 vertices. By using the definition of edge subdivision and graph subdivision, we can say that $\Gamma_1 \uplus \Gamma_2$ and $\Gamma_1 \underline{\vee} \Gamma_2$ are complete bipartite. Hence, from Proposition 6.3, these join graphs are $\tilde{\mathcal{L}}$ -integral.

Now suppose Γ_2 is complete bipartite graph and $\Gamma_2 \not\cong \overline{K_{n_2}}$. Then $\Gamma_1 \uplus \Gamma_2$ and $\Gamma_1 \underline{\vee} \Gamma_2$ are complete bipartite. Consequently these join graphs are $\tilde{\mathcal{L}}$ -integral from Proposition 6.3.

6.2. Normalized Laplacian Energy

In this chapter, we consider the energy of a simple graph with respect to its normalized Laplacian eigenvalues, which we call the $\tilde{\mathcal{L}}$ -energy.

Definition 6.2: The energy of a graph Γ is a graph invariant first introduced by Ivan Gutman [Gutman, 1978], [Gutman, 2001], [Lihtenbaum, 1956] as

$$\varepsilon(\Gamma) = \sum_{i=1}^n |\lambda_i(\Gamma)| \quad (6.3)$$

Definition 6.3: [Kirchhoff, 1847] For a graph Γ , the incidence energy of graph, denoted by $\Re\varepsilon(\Gamma)$, is introduced as

$$\Re\varepsilon(\Gamma) = \sum_{i=1}^n \sqrt{q_i(\Gamma)} \quad (6.4)$$

Definition 6.4: [Cavers et al., 2010], [Hakimi-Nezhaad and Ghorbani, 2017] For a graph Γ the normalized Laplacian energy, denoted by $\tilde{\mathcal{L}}_\varepsilon(\Gamma)$ is defined as

$$\tilde{\mathcal{L}}_\varepsilon(\Gamma) = \sum_{i=1}^n |\delta_i(\Gamma) - 1| \quad (6.5)$$

Let Γ_i be a graph for $i \in \{1, 2\}$. In the rest of this part, we will calculate the normalized Laplacian energy for the graphs $\Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \vee \Gamma_2$.

Proposition 6.6: (Proposition 3.4 in [Hakimi-Nezhaad and Ghorbani, 2017]) Let Γ_1 be a regular of graph of degree r_1 and order n_1 , and let $\Gamma_2 \cong \overline{K_{n_2}}$. Then

$$\begin{aligned} i) \quad \tilde{\mathcal{L}}_\varepsilon(\Gamma_1 \cup \Gamma_2) &= 2\left(1 + \frac{\Re\varepsilon(\Gamma_1) - \sqrt{2r_1}}{\sqrt{2(n_2 + r_1)}}\right), \\ ii) \quad \tilde{\mathcal{L}}_\varepsilon(\Gamma_1 \vee \Gamma_2) &= 2\left(1 + \frac{\Re\varepsilon(\Gamma_1) - \sqrt{2r_1}}{\sqrt{r_1(n_2 + 2)}}\right). \end{aligned}$$

Proof 6.6: Considering the structure of these two graphs and by the definition of the join graph, we have:

$$i) \tilde{\mathcal{L}}_\varepsilon(\Gamma_1 \cup \Gamma_2) = \sum_{i=1}^n |\delta_i(\Gamma_1 \cup \Gamma_2) - 1|$$

$$= |\delta_1(\Gamma_1 \cup \Gamma_2) - 1| + \sum_{j=2}^{n_1} |\delta_j(\Gamma_1 \cup \Gamma_2) - 1|.$$

Since $\delta_i(\Gamma_1 \cup \Gamma_2)$ is the eigenvalues of $\tilde{\mathcal{L}}_\varepsilon(\Gamma_1 \cup \Gamma_2)$, by Theorem 5.1,

$$\begin{aligned} \bullet \quad & |\delta_1(\Gamma_1 \cup \Gamma_2) - 1| + \sum_{j=2}^{n_1} \left| \pm \sqrt{\frac{\lambda_j(\Gamma_1) + r_1}{2(n_2 + r_1)}} \right| + |\delta_n(\Gamma_1 \cup \Gamma_2) - 1| \\ &= 1 + \sum_{j=2}^{n_1} \left| \pm \sqrt{\frac{\lambda_j(\Gamma_1) + r_1}{2(n_2 + r_1)}} \right| + 1 = 2 + 2 \sum_{j=2}^{n_1} \pm \sqrt{\frac{\lambda_j(\Gamma_1) + r_1}{2(n_2 + r_1)}}. \end{aligned}$$

we obtain the following equality:

$$\tilde{\mathcal{L}}_\varepsilon(\Gamma_1 \cup \Gamma_2) = 2 + 2 \sum_{j=2}^{n_1} \pm \sqrt{\frac{\lambda_j(\Gamma_1) + r_1}{2(n_2 + r_1)}} = 2 \left(1 + \frac{\Re \varepsilon(\Gamma_1) - \sqrt{2r_1}}{\sqrt{2(n_2 + r_1)}} \right) \quad (6.6)$$

where $\Re \varepsilon(\Gamma_1) - \sqrt{2r_1} = \sum_{j=2}^{n_1} |\pm \sqrt{\lambda_j(\Gamma_1) + r_1}|$.

$$ii) \tilde{\mathcal{L}}_\varepsilon(\Gamma_1 \underline{\vee} \Gamma_2) = \sum_{i=1}^n |\delta_i(\Gamma_1 \underline{\vee} \Gamma_2) - 1|$$

$$= |\delta_1(\Gamma_1 \underline{\vee} \Gamma_2) - 1| + \sum_{j=2}^{n_1} |\delta_j(\Gamma_1 \underline{\vee} \Gamma_2) - 1|.$$

Since $\delta_i(\Gamma_1 \underline{\vee} \Gamma_2)$ is the eigenvalues of $\tilde{\mathcal{L}}_\varepsilon(\Gamma_1 \underline{\vee} \Gamma_2)$, by Theorem 5.2,

$$\begin{aligned} \bullet \quad & |\delta_1(\Gamma_1 \underline{\vee} \Gamma_2) - 1| + \sum_{j=2}^{n_1} \left| \pm \sqrt{\frac{\lambda_j(\Gamma_1) + r_1}{r_1(n_2 + 2)}} \right| + |\delta_n(\Gamma_1 \underline{\vee} \Gamma_2) - 1| \\ &= 1 + \sum_{j=2}^{n_1} \left| \pm \sqrt{\frac{\lambda_j(\Gamma_1) + r_1}{r_1(n_2 + 2)}} \right| + 1 = 2 + 2 \sum_{j=2}^{n_1} \pm \sqrt{\frac{\lambda_j(\Gamma_1) + r_1}{r_1(n_2 + 2)}}. \end{aligned}$$

We obtain the following equality:

$$\tilde{\mathcal{L}}_\varepsilon(\Gamma_1 \underline{\vee} \Gamma_2) = 2 + 2 \sum_{j=2}^{n_1} \pm \sqrt{\frac{\lambda_j(\Gamma_1) + r_1}{r_1(n_2 + 2)}} = 2 \left(1 + \frac{\Re \varepsilon(\Gamma_1) - \sqrt{2r_1}}{\sqrt{r_1(n_2 + 2)}} \right) \quad (6.7)$$

where $\Re \varepsilon(\Gamma_1) - \sqrt{2r_1} = \sum_{j=2}^{n_1} |\pm \sqrt{\lambda_j(\Gamma_1) + r_1}|$.

Definition 6.5: Let Γ be a graph, $v_i, v_j \in V(\Gamma)$ and $|V(\Gamma)| = n$. Let $T := \mathbb{L} + \frac{1}{n}J$. Then the resistance distance between v_i, v_j in $V(\Gamma)$, denoted by Ω_{ij} is defined as follows:

$$\Omega_{ij} = (T^{-1})_{ii} + (T^{-1})_{jj} - 2(T^{-1})_{ij} \quad (6.8)$$

Then $\Omega = [\Omega_{ij}]$ is the resistance distance matrix which can be written explicitly as:

$$\bullet \quad \Omega = \begin{bmatrix} 2(T^{-1})_{11} & \cdots & (T^{-1})_{11} + (T^{-1})_{nn} \\ \vdots & \ddots & \vdots \\ (T^{-1})_{nn} + (T^{-1})_{11} & \cdots & 2(T^{-1})_{nn} \end{bmatrix}$$

Definition 6.6: The degree Kirchhoff index of a graph Γ of order n , denoted by $Kf^*(\Gamma)$ is first introduced by Chen in [Chen and Zhang, 2007] as:

$$Kf^*(\Gamma) = \sum_{i < j} d_i r_{ij} d_j, \quad (6.9)$$

where r_{ij} denotes the resistance-distance between the vertices v_i and v_j in a graph Γ .

$$Kf^*(\Gamma) = 2m \sum_{i=2}^n \frac{1}{\delta_i(\Gamma)}, \quad (6.10)$$

where for $(2 \leq i \leq n)$, $\delta_i(\Gamma)$ is the i th normalized Laplacian eigenvalues of Γ .

Proposition 6.7: (Proposition 3.5 in [Hakimi- Nezhaad and Ghorbani, 2017]) Suppose Γ_1 is a regular of graph of degree r_1 of order n_1 with edges and let $\Gamma_2 \cong \overline{K_{n_2}}$. Then considering the graphs $\Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \underline{\vee} \Gamma_2$, we have:

i) $Kf^*(\Gamma_1 \cup \Gamma_2)$ is equal to

$$2(2m_1 + n_1 n_2) \left(\frac{1}{2} + m_1 - n_1 + n_2 + \frac{(n_1-1) \sqrt{2(n_2+r_1)}}{(n_1-1)(\sqrt{2(n_2+r_1)} \pm (\Re \varepsilon(\Gamma) - \sqrt{2r_1}))} \right).$$

ii) $Kf^*(\Gamma_1 \underline{\vee} \Gamma_2)$ is equal to

$$2m_1(2 + n_2)\left(\frac{1}{2} + m_1 - n_1 + n_2 + \frac{(n_1-1)\sqrt{r_1(n_2+2)}}{(n_1-1)(\sqrt{r_1(n_2+2)} \pm (\Re \varepsilon(\Gamma) - \sqrt{2r_1}))}\right).$$

Proof 6.7: The number of edges of $\Gamma_1 \uplus \Gamma_2$ and $\Gamma_1 \underline{\vee} \Gamma_2$ are $|E(\Gamma_1 \uplus \Gamma_2)| = 2m_1 + n_1n_2$ and $|E(\Gamma_1 \underline{\vee} \Gamma_2)| = m_1(2 + n_2)$, respectively. By Definition 6.6, we plug the eigenvalues by using Corollary 5.1 and Corollary 5.2. Furthermore since the Γ_1 is a r_1 -regular, this implies that:

- $\Re \varepsilon(\Gamma_1) = \sum_{j=1}^{n_1} \sqrt{\lambda_j(\Gamma_1) + r_1}$, so $\Re \varepsilon(\Gamma_1) - \sqrt{2r_1} = \sum_{j=2}^{n_1} \sqrt{\lambda_j(\Gamma_1) + r_1}$.

7. CONCLUSION

As we expressed our aim in the introduction section, in this thesis, we wanted to calculate the normalized laplacian eigenvalues of two join graphs called s-vertex and s-edge. We mainly used the ideas in Normalized Laplacian spectrum of two new types of join graphs [Hakimi- Nezhaad and Ghorbani, 2017]). To do this, after discussing the general properties of normalized Laplacian eigenvalues in section four, we calculated the normalized Laplacian matrix of the the join graphs by using the Laplacian, adjacency and degree matrices of the main graphs. Specially, in Theorem 5.1 and 5.2, we considered the special case, where both graphs are regular. Considering r_i -regular graphs Γ_i , for $i= 1,2$, and the property that for each there exists an eigenvector $[1,1,\dots,1]^T$ corresponding to the eigenvalue r_i such that all the other eigenvectors are orthogonal to this eigenvector, we investigated the spectrum of the normalized Laplacian of these special graphs. In particular, we explained how eigenvalues can be computed for these types of join graphs.

To sum up, we investigated the relationship between the eigenvalues of the main graph and the associated eigenvalues for the s-vertex and s-edges graphs.

REFERENCES

- Biggs N., (1993), "Algebraic Graph Theory", 2nd Edition, Cambridge University Press, Cambridge.
- Biggs N., Lloyd E., and Wilson R., (1986), "Graph Theory", Oxford University Press, 1736-1936.
- Brooks R., Smith C., Stone A. and Tutte W., (1940), "Dissection of a rectangle into squares", Duke Math. J., 7 312-340.
- Brouwer A. E., Haemers W., (2010), "Spectra of Graphs", unpublished course notes.
- Brouwer A. E., Haemers W. H., (2012), "Spectra of Graphs", Universitext, Springer, New York.
- Butler S., (2010), "A note about cospectral graphs for the adjacency and normalized Laplacian matrices", Linear Multilinear Algebra, 58(3), 387-390.
- Butler S., (2008), "Eigenvalues and Structures of Graphs", Ph.D. dissertation, University of California, San Diego,.
- Cavers M., (2010), "The normalized Laplacian matrix and general Randić' index of graphs", Ph.D. University of Regina,.
- Cavers M., Fallat S., Kirkland S., (2010), "On the normalized Laplacian energy and general Randić index R_1 of graphs", Linear Algebra Appl. 433, 172-190.
- Chen H., Zhang F., (2007), "Resistance distance and the normalized Laplacian spectrum", Discr. Appl. Math., 155, 654-661.
- Chung F. R. K., (1997), "Spectral Graph Theory", American Math. Soc. Providence,.
- Chung F., Langlands R., (1996), "A combinatorial Laplacian with vertex weights", J. Combin. Theory Ser. A, 75, 316-327.
- Cvetkovic' D., Doob M., Gutman I., Torgasev A., (1998), "Recent Results in the Theory of Graph Spectra", Annals of Discrete Mathematics 36,.
- Cvetkovic' D., Doob M., Sachs H., (1980), "Spectra of Graphs: Theory and Applications", Academic Press, New York,.
- Cvetkovic' D., Rowlinson P., Simic S., (1997), "Eigenspaces of Graphs", Cambridge University Press, Cambridge,.
- Dam E. R. V., Omid G. R., (2011), "Graphs whose normalized Laplacian has three eigenvalues", Linear Algebra Appl. 435 (10), 2560-2569.

- Godsil C., Royle G., (2001), "Algebraic Graph Theory", Springer, New York,.
- Gutman I., (1978), "The energy of a graph", Steiermrkisches Mathematisches Symposium (Stift Rein, Graz, 1978), Ber. Math. Statist. Sect. Forsch. Graz 103, 1-22.
- Gutman I., Mohar B., (1996), "The Quasi-Wiener and the Kirchhoff indices coincide", J. Chem. Inf. Comput. Sci. 36, 982-985.
- Gutman I., (2001), "The energy of a graph: old and new results", in: A. Betten, A. Kohner, R. Laue, A. Wassermann (Eds.), Algebraic Combinatorics and Applications, Springer, Berlin, 196-211.
- Hakimi-Nezhaad M., Ashrafi A. R., Gutman I., (2013), "Note on degree Kirchhoff index of graphs", Trans. Comb. 2 (3), 43-52.
- Hakimi-Nezhaad M., Ashrafi A. R., (2014), "A note on normalized Laplacian energy of graphs", J. Contemp. Math. Anal. 49 (5), 207-211.
- Hakimi-Nezhaad M., Ghorbani M., (2017), "Normalized Laplacian spectrum of two new types of join graphs", Shahid Rajaee Teacher Training University, 16785-136, Iran.
- Hückel E., (1931), "Quantentheoretische Beiträge zum Benzolproblem", Z. Phys., 70, 204- 286.
- Hoffman K., Kunze R., (1971), "Linear Algebra", 2nd Edition, Prentice-Hall, Inc., Englewood Cliffs, New Jersey. E.
- Indulal G., (2012), "Spectrum of two new joins of graphs and infinite families of integral graphs", Kragujevac Journal of Mathematics, 36 (1), 133-139.
- Jooyandeh M. R., Kiani D., Mirzakhah M., (2009), "Incidence energy of a graph", MATCH Commun. Math. Comput. Chem. 62, 561-572.
- Kirchhoff G., (1847), "Über die Auflösung der Gleichungen, auf welche man bei der untersuchung der linearen verteilung galvanischer Ströme geführt wird", Annalen Der Physik Chemie, 72, 497-508.
- Klein D. J., Randic M., (1993), "Resistance distance", Journal of Mathematical Chemistry, 12, 81-95.
- Li X., Shi Y., Gutman I., (2012), "Graph energy", Springer, New York,.
- Lihtenbaum L.M., (1956), "Characteristic values of a simple graph" (Russian), Trud 3- go Vses. matem. sezda, tom 1, 135-136.
- Lubotzky A., Phillips R., Sarnak P., (1988), "Ramanujan Graphs", Combinatorica, 3, 261-277.

Xiao B., Hancock E.R., (2004), “Heat kernel, Riemannian manifolds and Graph Embedding”, LNCS, 313(8), 198–206.

Seidel J.J., (1989), “Graphs and their spectra”, Combinatorics and Graph Theory, PWN- Polish Scientific Publishers, Warsaw, 147-162.

Trent H., (1954), “A Note on the Enumeration and Listing of All Possible Trees in a Connected Linear Graph”, Proceedings of the National Academy of Sciences of the United States of America., 40, 1004-1007.

Web 1, (2018), <https://dgc.epfl.ch/files/content/sites/dgc/files/courses/2013%20-%20Advanced%20Discrete%20Mathematics/ADM-Eigenvalues-v3.pdf>, (Eriřim Tarihi: 12/04/2018)

Wei T.H., (1952), “The algebraic foundations of ranking theory”, Thesis, Cambridge University.

West D. B., (2001), “Introduction to Graph Theory”, 2nd Edition, University of Illinois-Urbana.

Zhou B., Trinajstić N., (2009), “On resistance-distance and Kirchhoff index”, Journal of Mathematical Chemistry 46, 283-289.

Zhou B., Wang Z., Bu C., (2016), “On resistance matrix of a graph”, Mathematics Subject Classifications.

BIOGRAPHY

Nursemin Feratlar was born on the 8th of August 1991 in İzmit, TURKEY. She graduated from Gazi High School in June 2008. She received her B.Sc from the Department of Mathematics at Kocaeli University in 2014. She started her M.Sc. in mathematics at the Graduate School of Natural and Applied Science in Gebze Technical University, Turkey.