

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL

DERIVATIONS OF STRUCTURAL MATRIX ALGEBRAS

SERDAR DEMIR

A THESIS OF MASTER OF SCIENCE
DEPARTMENT OF MATHEMATICS

ADVISOR: PROF. DR. MUSTAFA AKKURT

FEBRUARY 2025

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ABSTRACT

Matrix algebra derivations have been a focal point of study for numerous researchers over the years. In 1993, Coelho and Milies [1] explored this topic in depth, presenting a significant finding in their work. They demonstrated that every derivation of upper triangular matrix rings, defined over a ring R with unity, could be expressed as a combination of two components: an inner derivation and a derivation derived from the ring R itself. This result provided a foundational understanding of how derivations behave within this specific algebraic structure.

Two years later, in 1995, S. Jøndrup contributed to this area of research by offering a result that paralleled the findings of Coelho and Milies. His work focused on triangular matrices as well, and he distinguished his approach by providing a concise and elementary proof for both his result and the earlier one established by Coelho and Milies. Jøndrup's techniques streamlined the reasoning, making the proofs more accessible while reinforcing the same core insight about the structure of derivations.

Now, consider a quasi-order ρ defined on the set $\{1, 2, \dots, n\}$. Within the full matrix algebra $M_n(R)$, the subset $M_n(R, \rho)$ consists of all matrices where the indices (i, j) satisfy the relation ρ . This subset forms an algebra known as a structural matrix algebra, which introduces a more specialized framework for studying derivations. The goal of this investigation is to determine the conditions under which derivations of matrix algebras, particularly those of structural matrix algebras like $M_n(R, \rho)$, can be decomposed into a sum of an inner derivation and a derivation stemming from the ring R .

To achieve this, the strategy is to adapt the argument originally put forth by Coelho and Milies in their theorem from 1993 [1]. Their proof, tailored to upper triangular matrix rings, serves as a model for this exploration. By carefully modifying their reasoning to suit the context of $M_n(R, \rho)$, the aim is to establish that any derivation of this structural matrix algebra can similarly be expressed as the sum of an inner derivation and a derivation obtained from R . This adaptation seeks to preserve the essence of their approach while extending its applicability to a broader class of matrix algebras, shedding light on the nature of derivations in these structured settings. Through this process, the interplay between the algebraic properties of R and the constraints imposed by ρ will be crucial in shaping the final result.

Keywords: Derivation, Structural Matrix Algebras, Upper Triangular Matrix

ÖZET

Matris cebiri türevleri, yıllar boyunca birçok araştırmacı için bir odak noktası olmuştur. 1993 yılında Coelho ve Milies [1] bu konuyu derinlemesine incelemiş ve çalışmalarında önemli bir bulgu sunmuşlardır. Birim elemana sahip bir R halkası üzerinde tanımlı üst üçgen matris halkalarının her türevinin, iki bileşenin bir kombinasyonu olarak ifade edilebileceğini göstermişlerdir: bir iç türev ve R halkasından türetilen bir türev. Bu sonuç, bu özel cebirsel yapıda türevlerin nasıl davrandığına dair temel bir anlayış sağlamıştır.

İki yıl sonra, 1995'te, S. Jøndrup bu araştırma alanına katkıda bulunarak Coelho ve Milies'in bulgularıyla paralellik gösteren bir sonuç sunmuştur. Onun çalışması da üçgen matrislere odaklanmış ve yaklaşımını, hem kendi sonucu hem de Coelho ve Milies tarafından daha önce ortaya konan sonuç için kısa ve temel bir kanıt sunarak ayırt etmiştir. Jøndrup'un teknikleri, akıl yürütmeyi sadeleştirerek kanıtları daha erişilebilir hale getirmiş ve türevlerin yapısına dair aynı temel içgörüyü pekiştirmiştir.

Şimdi, $\{1, 2, \dots, n\}$ kümesi üzerinde tanımlı bir yarı-sıra ρ düşünelim. Tam matris cebiri $M_n(R)$ içinde, indeksleri (i, j) ρ ilişkisini sağlayan tüm matrislerden oluşan $M_n(R, \rho)$ alt kümesi, yapısal matris cebiri olarak bilinen bir cebir oluşturur ve bu, türevlerin incelenmesi için daha özel bir çerçeve sunar. Bu araştırmanın amacı, matris cebirlerinin, özellikle $M_n(R, \rho)$ gibi yapısal matris cebirlerinin türevlerinin, bir iç türev ile R halkasından kaynaklanan bir türevin toplamı olarak ayrıştırılabileceği koşulları belirlemektir.

Bunu başarmak için strateji, Coelho ve Milies tarafından 1993'te teoremlerinde ileri sürülen argümanı uyarlamaktır [1]. Üst üçgen matris halkalarına özel olarak hazırlanmış kanıtları, bu keşif için bir model teşkil eder. Onların akıl yürütmeleri, $M_n(R, \rho)$ bağlamına dikkatlice uyarlanarak, bu yapısal matris cebirinin herhangi bir türevinin benzer şekilde bir iç türev ile R halkasından elde edilen bir türevin toplamı olarak ifade edilebileceği gösterilmeye çalışılacaktır. Bu uyarlama, onların yaklaşımının özünü korurken, uygulanabilirliğini daha geniş bir matris cebirleri sınıfına genişletmeyi amaçlar ve bu yapılandırılmış ortamlardaki türevlerin doğasına ışık tutar. Bu süreçte, R halkasının cebirsel özellikleri ile ρ tarafından dayatılan kısıtlamalar arasındaki etkileşim, nihai sonucu şekillendirmede kritik olacaktır.

Anahtar Kelimeler: Derivasyon, Yapısal Matris Cebirleri, Üst Üçgensel Matris

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LIST of ABBREVIATIONS and ACRONYMS

R	:	Ring
G	:	Group
F	:	Field
$T_n(R)$	:	Upper Triangular Matrix Ring over a Ring R
$M_n(R)$	:	A Ring of $n \times n$ Matrices over a Ring R
e_{ij}	:	Matrix Units
$M_n(R, \rho)$	:	Structural Matrix Algebra

1. INTRODUCTION

Derivatives of Matrix Algebras have been studied by many authors. In 1981, Benkart and Osborn studied that derivations and automorphism of non-associative matrix algebras, and showed that $M_n(A)$ has anti-automorphisms if and only if A does. In here, $M_n(A)$, $n \times n$ matrices over an arbitrary non-associative algebra A with multiplicative identity 1 [2].

In 1993, Coelho and Milies showed in 1993 that each derivations of the upper triangular matrix rings defined on the ring R with unity can be written as the sum of an inner derivation and the derivation obtained from the R ring [1].

In 2019, Umut Sayın and Feride Kuzucuoğlu studied that Jordan Derivations of Special Sub-rings of Matrix Rings and they described that K be a 2-torsion free ring with identity and $R_n(K, J)$ be the ring of all $n \times n$ matrices over K such that the entries on and above the main diagonal are elements of an ideal J of K . Also they described all Jordan derivations of the matrix ring $R_n(K, J)$ in their paper. The main result states that every Jordan derivation Δ of $R_n(K, J)$ is of the form $\Delta = D + \Omega$ where D is a derivation of $R_n(K, J)$ and Ω is an extremal Jordan derivation of $R_n(K, J)$ [3].

In 2020, Oksana Bezushchak studied that Derivations of rings of infinite matrices and described derivations of several important associative and Lie rings of infinite matrices over general rings of coefficients [4].

In 2020, Oksana Bezushchak studied that Automorphisms Derivations of algebras of infinite matrices and described derivations automorphisms and derivations of several important associative and Lie algebras of infinite matrices over a field [5].

In 2023, Vladeva studied that Derivations of triangular matrix rings and gave a description of the derivations of $UTM_n(R)$, the ring of upper triangular matrices over an associative ring R with identity. The main result states that if D is an arbitrary derivation of the ring $UTM_n(R)$ and $A \in UTM_n(R)$, then there are matrices, such that the derivative $D(A)$ is a linear combination of the values of well-known derivations of these matrices [6].

In this thesis, we will try to show under what conditions the derivations of matrix al-

gebras, especially structural matrix algebras, can be written as the sum of the inner derivations and the derivations obtained from the ring R . Our work is to examine the articles in the literature, to investigate the relationships of the results there with our subject, and to prove the following claim in the light of these.

Claim: Let R be a ring, part of ρ be a reflexive and transitive relation on $N = \{1, 2, \dots, n\}$ and let $d : M_n(R, \rho) \rightarrow M_n(R, \rho)$ be the derivation. In this case, there is a derivation $\delta : R \rightarrow R$ such that $d = \bar{\delta} + d_A$ and an inner derivation d_A such that $A \in M_n(R, \rho)$.

In this study, we will first consider the ρ is partial order then we will try to demonstrate our claim above by examining ρ is symmetric and its general case in the light of these.

2. PRELIMINARIES

Definition 2.1: [7] A *group* is an ordered pair $(G, \triangle)$ where G is a set and $\triangle$ is a binary operation on G satisfying the following axioms:

1. $(a \triangle b) \triangle c = a \triangle (b \triangle c)$, for all $a, b, c \in G$, i.e., $\triangle$ is associative.
2. There exists an element $e \in G$, called an *identity* of G , such that for all $a \in G$
 $a \triangle e = e \triangle a = a$.
3. For each $a \in G$ there is an element a^{-1} of G , called an *inverse* of a , such that
 $a \triangle a^{-1} = a^{-1} \triangle a = e$.

The group $(G, \triangle)$ is called *abelian* (or commutative) if for every $a, b \in G$, $a \triangle b = b \triangle a$.

Definition 2.2: [7] Let $(G, *)$ and $(H, \circ)$ be groups. A map ϕ from G to H such that

$$\phi(x * y) = \phi(x) \circ \phi(y), \quad \text{for all } x, y \in G$$

is called a *homomorphism*.

Example 1: [8] H be the group of all real 2×2 matrices $\begin{pmatrix} k & l \\ m & n \end{pmatrix}$ such that $kn - lm \neq 0$, under matrix multiplication and $\bar{H}$ be the group of all nonzero real numbers under multiplication.

$$\lambda : H \rightarrow \bar{H} \quad \text{and} \quad \lambda \left(\begin{pmatrix} k & l \\ m & n \end{pmatrix} \right) = kn - lm.$$

For every 2×2 matrices of H , λ is a homomorphism of H onto $\bar{H}$. To verify that this mapping is a homomorphism we must check to see whether $\lambda(AB) = \lambda(A)\lambda(B)$, $A, B \in H$

$$\begin{aligned} A &= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \\ \lambda(AB) &= \lambda \left(\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \right) = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix} \\ &= (a_1b_1 + a_2b_3) \cdot (a_3b_2 + a_4b_4) - (a_1b_2 + a_2b_4) \cdot (a_3b_1 + a_4b_3) \\ &= (a_1b_1a_3b_2 + a_1b_1a_4b_4 + a_2b_3a_3b_2 + a_2b_3a_4b_4) - (a_1b_2a_3b_1 + a_1b_2a_4b_3 + a_2b_4a_3b_1 + \end{aligned}$$

$$\begin{aligned}
& a_2b_4a_4b_3) \\
& = a_1b_1a_3b_2 + a_1b_1a_4b_4 + a_2b_3a_3b_2 + a_2b_3a_4b_4 - a_1b_2a_3b_1 - a_1b_2a_4b_3 - a_2b_4a_3b_1 - a_2b_4a_4b_3 \\
& = b_1b_4(a_1a_4 - a_2a_3) + b_2b_3(a_2a_3 - a_1a_4) \\
& = (a_1a_4 - a_2a_3)(b_1b_4 - b_2b_3) \\
& = \lambda(A)\lambda(B).
\end{aligned}$$

Definition 2.3: [7] A *ring* R is a set together with two binary operations “+” and “ $\cdot$ ” (called addition and multiplication) satisfying the following axioms:

- $(R, +)$ is an *abelian* group,
- $\cdot$ is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$,
- the distributive laws hold in R : for all $a, b, c \in R$

$$(a + b) \cdot c = (a \cdot c) + (b \cdot c) \quad \text{and} \quad a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

We shall usually write simply ab rather than $a \cdot b$ for $a, b \in R$. The additive identity of R will always be denoted by 0_R and the additive inverse of the ring element a will be denoted by $-a$. If the multiplication in R satisfies $a \cdot b = b \cdot a$ for every a, b in R , then we call R a *commutative ring*. Moreover, if there is an element 1_R in R such that $a \cdot 1_R = 1_R \cdot a = a$ for every a in R , then we describe R as a *ring with unit element*.

Definition 2.4: [7] Let R be a ring. A nonzero a of R is called a *zero divisor* if there is a nonzero element b in R such that $ab = 0_R$ or $ba = 0_R$.

Definition 2.5: [7] A commutative ring with identity $1_R \neq 0_R$ is called an *integral domain* if it has no zero divisors.

Assume a, b and c are elements of any ring with a not a zero divisor. If $ab = ac$, then either $a = 0_R$ or $b = c$ (i.e., if $a \neq 0_R$ we can cancel the a 's). In particular, if a, b, c are any elements in an integral domain and $ab = ac$, then either $a = 0_R$ or $b = c$.

Proof: If $ab = ac$ then $a(b - c) = 0_R$ so either $a = 0_R$ or $b - c = 0_R$. The second statement follows from the first and the definition of an integral domain.

Definition 2.6: [7] Let R be a ring, let I be a subset of R and let $r \in R$.

1. $rI = \{ra : a \in I\}$ and $Ir = \{ar : a \in I\}$.
2. A subset I of R is a left ideal of R if
 - (i) I is a subring of R , and

- (ii) I is closed under left multiplication by elements from R , i.e., $rI \subseteq I$ for all $r \in R$.

Similarly I is a right ideal if (i) holds and in place of (ii) one has

- (ii)' I is closed under right multiplication by elements from R , i.e., $Ir \subseteq I$ for all $r \in R$.

3. A subset I that is both a left ideal and a right ideal is called an *ideal* (or, for added emphasis, a two-sided ideal) of R .

Example 2: Let $R = M_2(\mathbb{Z}_2)$. This is a finite non-commutative ring with identity

$1_R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and zero element $0_R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. We give an example to show it is non-commutative:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ but } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Definition 2.7: [7] A ring R with identity 1_R where, $1_R \neq 0_R$ is called a *division ring* (or skew field) if every nonzero element $a \in R$ has a multiplicative inverse, i.e., there exists $b \in R$ such that $ab = ba = 1_R$.

Definition 2.8: [7] Let R be a ring. Define $Z(R) = \{a \in R \mid ar = ra \text{ for all } r \in R\}$, the set of elements commuting with all the elements of R . This subset of R is called the *center* of R .

Definition 2.9: [7] Let R be a ring. An element x in R is called *nilpotent* if $x^m = 0$ for some m in positive integer.

Definition 2.10: [7] Let I be an ideal of the commutative ring R and define

$$\text{rad } I = \{r \in R \mid r^n \in I \text{ for some } n \text{ in positive integer}\}$$

called the *radical* of I .

Definition 2.11: [7] Let R and S be rings.

1. A ring *homomorphism* is a map $\varphi : R \rightarrow S$ satisfying

- $\varphi(a+b) = \varphi(a) + \varphi(b)$ for all $a, b \in R$ (so φ is a group homomorphism on the additive groups) and
- $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in R$.

2. The kernel of the ring homomorphism φ , denoted $\ker \varphi$, is the set of elements of R that map 0_S (i.e., the kernel of φ viewed as a homomorphism of additive groups.)
3. A bijective ring homomorphism is called an *isomorphism*.

Definition 2.12: [7]

1. A field is a set F together with two binary operations "+" and "·" on F such that $(F, +)$ is an abelian group (call its identity 0) and $(F - \{0\}, \cdot)$ is also an abelian group, and the following distributive law holds:

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c), \quad \text{for all } a, b, c \in F.$$

2. For any field F let $F^\times = F - \{0\}$.

Definition 2.13: [9] A *vector space* (or linear space) consists of the following:

1. a field F of scalars;
2. a set V of objects, called vectors;
3. a rule (or operation), called vector addition, which associates with each pair of vectors α, β in V a vector $\alpha + \beta$ in V , called the sum of α and β , in such a way that
 - (a) addition is commutative, $\alpha + \beta = \beta + \alpha$;
 - (b) addition is associative, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$;
 - (c) there is a unique vector 0 in V , called the zero vector, such that $\alpha + 0 = \alpha$ for all α in V ;
 - (d) for each vector α in V there is a unique vector $-\alpha$ in V such that $\alpha + (-\alpha) = 0$;
4. a rule (or operation), called scalar multiplication, which associates with each scalar c in F and vector α in V a vector $c\alpha$ in V , called the product of c and α , in such a way that
 - (a) $1\alpha = \alpha$ for every α in V ;
 - (b) $(c_1c_2)\alpha = c_1(c_2\alpha)$;
 - (c) $c(\alpha + \beta) = c\alpha + c\beta$;

$$(d) (c_1 + c_2)\alpha = c_1\alpha + c_2\alpha.$$

Definition 2.14: [8] If V is a vector space over F and if $v_1, \dots, v_n \in V$ then any elements of the form $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$, where the $\alpha_i \in F$, is a *linear combination* over F of $v_1, \dots, v_n \in V$.

Definition 2.15: [8] If S is a nonempty subset of the vector space V , then $L(S)$, the *linear span* of S , is the set of all linear combinations of finite sets of elements of S .

Definition 2.16: [9] Let V and W be vector space over field F . A linear transformation from V into W is a function T from V into W such that

$$T(c\alpha + \beta) = cT(\alpha) + T(\beta)$$

for all α and β in V and all scalars c in F .

Definition 2.17: [9] If V and W are vector spaces over field F , any one to one linear transformation T of V onto W is called an *isomorphism* of V onto W .

Definition 2.18: [7] Let R be a commutative ring with identity. An R -algebra is a ring A with identity together with a ring homomorphism $f : R \rightarrow A$ mapping 1_R to 1_A such that the subring $f(R)$ of A is contained in the center of A .

Definition 2.19: [10] A block diagonal matrix is a square matrix of the form

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{pmatrix},$$

where $A_1, \dots, A_m$ are square matrices lying along the diagonal and all other entries of the matrix equal 0.

Definition 2.20: [11] An additive map $\delta : R \rightarrow R$ on a ring R is a *derivation* on R if

$$\delta(ab) = \delta(a)b + a\delta(b), \quad \text{for all } a, b \text{ in } R.$$

The set of all derivations on R is denoted by $Der(R)$. The set $Der(R)$ has the structure of a Lie ring with multiplication given by

$$[\delta, \lambda] = \delta\lambda - \lambda\delta, \quad \text{for all } \delta, \lambda \text{ in } Der(R).$$

If $a \in R$ and $\iota_a : R \rightarrow R$ is defined by $\iota_a(r) = ar - ra$, for all r in R , then ι_a is an *inner derivation*. The set of all inner derivations by $In(Der((R)))$. It is easy to verify that $InDer((R))$ is an ideal of $Der(R)$.

If we take the $M_n(R)$ matrix ring and redefine it as above definition following:

Definition 2.21: An additive map $d : M_n(R) \rightarrow M_n(R)$ on a ring $n \times n$ matrix is a *derivation* on $n \times n$ matrix if

$$d(AB) = d(A)B + Ad(B), \quad \forall A, B \in M_n(R)$$

The set of all derivations on $M_n(R)$ is denoted by $Der(M_n(R))$. The set $Der(M_n(R))$ has the structure of a Lie ring with multiplication given by

$$[d, \delta] = d\delta - \delta d, \quad \forall d, \delta \in Der(M_n(R)).$$

If $K \in M_n(R)$ and

$$\begin{aligned} \iota_K : M_n(R) &\rightarrow M_n(R) \\ L &\mapsto \iota_K(L) = KL - LK, \quad \forall L \in M_n(R) \end{aligned}$$

then ι_K is an *inner derivation*. The set of all inner derivations is denoted by $In(Der(M_n(R)))$. $In(Der(M_n(R)))$ is an ideal of $Der(M_n(R))$.

Now, let $d \in Der(M_n(R))$ and $\iota_A \in In(Der(M_n(R)))$ be two derivations, and let $B \in M_n(R)$. We need to show $[d, \iota_A] \in In(Der(M_n(R)))$

$$\begin{aligned} [d, \iota_A](B) &= (d\iota_A - \iota_A d)(B) = (d\iota_A)(B) - (\iota_A d)(B) \\ &= d(\iota_A(B)) - \iota_A(d(B)) \\ &= d(AB - BA) - Ad(B) + d(B)A \\ &= d(A)B + Ad(B) - d(B)A - Bd(A) - Ad(B) + d(B)A \\ &= d(A)B - Bd(A) \\ &= \iota_{d(A)}(B) \end{aligned}$$

which shows that $[d, \iota_A] = \iota_{d(A)}$.

The essential definitions and theorems can be found in *Algebra* by Thomas William Hungerford[12] and *Abstract Algebra* by David Steven Dummit and Richard Michael Foote[7], which serve as fundamental references on the subject.

3. STRUCTURAL MATRIX ALGEBRAS

Let ρ be reflexive, transitive relation i.e. quasi-order relation on the set $\{1, \dots, n\}$ for some $n \geq 2$, then $(\{1, \dots, n\}, \rho)$ is called a *quasi-ordered set* (short quoset). This also in accordance with [13]. Note that each quoset $(\{1, \dots, n\}, \rho)$ gives rise to the partition $\{[1], \dots, [n]\}$ of $\{1, \dots, n\}$ whereby $[i] := \{j \mid (i, j) \in \rho \text{ and } (j, i) \in \rho\}$, and that $(\{[1], \dots, [n]\}, \leq)$, $[i] \leq [j]$ defined by $(i, j) \in \rho$, is a *partially ordered set* (short poset). Thus the relation " $\leq$ " is reflexive and anti-symmetric (i.e. $[i] \leq [j]$, $[j] \leq [i]$ implies $[i] = [j]$).

Let R be a ring with unity and $(\{1, \dots, n\}, \rho)$ a quoset (i.e. reflexive, transitive). The set

$$M_n(R, \rho) = \{A = [a_{ij}] \in M_n(R) : a_{ij} = 0 \text{ whenever } (i, j) \notin \rho\}.$$

is the subset of $M_n(R)$ the algebra of all matrices over R . Clearly this set is closed under the matrix addition. Let e_{ij} be matrix unit that is the matrices with a 1 in a position (i, j) and 0 elsewhere for $(i, j) \in \rho$. Then the set $\{e_{ij}\}$ spans (generates) $M_n(R, \rho)$. Now let $T \in M_n(R, \rho)$ and $S \in M_n(R, \rho)$. $M_n(R, \rho)$ has as a basis all e_{ij} such that $a_{ij} = 1$, that is $(i, j) \in \rho$. Then for

$$T = \sum_{(i,j) \in \rho} t_{ij} e_{ij}, \quad \text{where } t_{ij} \in R,$$

and

$$S = \sum_{(i,j) \in \rho} s_{ij} e_{ij}, \quad \text{where } s_{ij} \in R,$$

we have

$$TS = \sum_{(i,j) \in \rho} \sum_{(p,q) \in \rho} t_{ip} s_{pq} e_{ij} e_{pq}.$$

Since $e_{ij} e_{pq} = \delta_{jp} e_{iq}$ we have two cases. If $j \neq p$ then $e_{ij} e_{pq} = 0$ and if $j = p$ then $e_{ij} e_{pq} = e_{iq} \in M_n(R, \rho)$ since ρ is transitive. Thus $TS \in M_n(R, \rho)$ and $M_n(R, \rho)$ is closed under matrix multiplication. Since ρ is transitive and reflexive this is an algebra (with identity I).

Definition 3.1: Let R be a ring with unity, $M_n(R)$ be the algebra of all matrices over R and ρ a quoset (reflexive, transitive) on $\{1, \dots, n\}$. The subset $M_n(R, \rho)$ of $M_n(R)$ of all matrices with $(i, j) \in \rho$, forms an algebra called a *structural matrix algebra*, that is,

$$M_n(R, \rho) = \{A = [a_{ij}] \in M_n(R) : a_{ij} = 0 \text{ whenever } (i, j) \notin \rho\}.$$

Example 3: For $n = 3$ and $\rho = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$, then the structural matrix algebra is

$$M_3(R, \rho) = \left\{ \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix} : a_{ij} \in R \right\}.$$

3.1. Construction of block upper triangular form of structural matrix algebras

For a structural matrix algebra $M_n(R, \rho)$ we define an equivalence relation $\bar{\rho}$ by

$$(i, j) \in \bar{\rho} \text{ if and only if } (i, j), (j, i) \in \rho.$$

Let $[r_1], [r_2], \dots, [r_p]$ denote the distinct equivalence classes of $\bar{\rho}$ with representatives $r_1, r_2, \dots, r_p$. And we construct a permutation π in [14] as follows. For

$$[r_1] = \{r_1 = r_{11}, r_{12}, \dots, r_{1m_1}\}$$

let

$$\pi(1) = \pi(r_{11}) = 1, \pi(r_{12}) = 2, \dots, \pi(r_{1m_1}) = m_1$$

and in general if

$$[r_k] = \{r_{k1}, r_{k2}, \dots, r_{km_k}\}$$

then

$$\pi(r_{kj}) = m_1 + m_2 + \dots + m_{k-1} + j.$$

If we apply this permutation similarity to $M_n(R, \rho)$ we have the following relation ρ' where

$$(i, j) \in \rho' \iff (\pi^{-1}(i), \pi^{-1}(j)) \in \rho$$

and $M_n(R, \rho')$ consists of all matrices of the form

$$\begin{bmatrix} M_{m_1}(F) & M_{m_1 \times m_2}(R) & \cdots & M_{m_1 \times m_p}(R) \\ 0 & M_{m_2}(F) & \cdots & M_{m_2 \times m_p}(R) \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & M_{m_p}(F) \end{bmatrix},$$

where R is either field F or 0.

And we define

$$[i]_{\rho'} \leq [j]_{\rho'} \quad (1)$$

to mean $(i, j) \in \rho'$. Then $\forall p_1 \in [i], p_2 \in [j]$,

$$(p_1, p_2) \in \rho' \text{ if and only if } (i, j) \in \rho'.$$

Next, let $[t_1], [t_2], \dots, [t_q]$ be the classes which are incomparable with any other class, i.e., for any $t_k, k = 1, \dots, q$, for any $r_j \neq t_k$ we have

$$[r_j] \not\leq [t_k] \quad (\text{that is } (r_j, t_k) \notin \rho')$$

and

$$[t_k] \not\leq [r_j] \quad (\text{that is } (t_k, r_j) \notin \rho').$$

Then we performed a permutation similarity of the block triangular matrix algebras which makes the diagonal blocks indexed by the indices of $[t_1], [t_2], \dots, [t_q]$ the last q diagonal blocks. Denote the corresponding permutation by π_1 and the quasi-order relation by σ . Further the last q σ -classes, which we denote by $[t_1]_\sigma, [t_2]_\sigma, \dots, [t_q]_\sigma$, are such that no one of these classes is comparable to any other σ -class (with $\leq$ defined by (1) and ρ' replaced by σ). This means that $R = 0$ for all the $M_{m_j \times m_k}(R)$ with $k = p - q + 1, \dots, p$.

To summarize this, the classes comparable to some other class are $[r_j]$ ($j = 1, \dots, l$) while the classes comparable to no other class are $[t_k]$ ($k = 1, \dots, q$) each of cardinality

m_l and m_q respectively. Also $l + q = p$. Hence we have our diagonal form

$$\begin{bmatrix} M_{m_1}(F) & M_{m_1 \times m_2}(R) & \cdots & M_{m_1 \times m_l}(R) & \cdots 0 \\ 0 & M_{m_2}(F) & \cdots & M_{m_2 \times m_l}(R) & \cdots 0 \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & M_{m_l}(F) & \cdots 0 \\ 0 & 0 & 0 & 0 & M_{m_p}(F) \end{bmatrix}.$$

Note that the way we have chosen the r_j and t_k in the construction of π_1 we have that

- (i) each r_j and each t_k is minimal in its class,
- (ii) $r_1 < \dots < r_l$ and without loss of generality we can have $t_1 < \dots < t_q$.

Now, if we consider a block $M_{m_a \times m_b}(R)$ with $a \leq l$ and $b \leq l$, then the row and the column indices are π_1 classes, say $[r_a]$ and $[r_b]$ respectively with $a < b$. If $[r_a] \leq [r_b]$ we have $R = F$; otherwise $R = 0$.

For notational simplicity, we may write M_i or M_{ij} for M_{m_i} or $M_{m_i \times m_j}$ respectively. The subset of $M_n(R, \rho)$ which has elements of M_{ij} (or M_i) in the i^{th} block row and j^{th} block column (or i^{th} block diagonal) and zero elsewhere will be denoted by $\overline{M}_{ij}$. Analogously we let A_{ij} denote an $m_i \times m_j$ matrix in M_{ij} while $\overline{A}_{ij}$ is the corresponding element of $\overline{M}_{ij}$. If $i = j$ we write M_i , A_i , $\overline{M}_i$, and $\overline{A}_i$, for the corresponding sets and matrices.

We have two special subsets of $M_n(R, \rho)$. Let $\mathcal{D}$ denote the subset of all block diagonal elements. That is

$$\mathcal{D} = \{\text{diag}(A_1, \dots, A_p) : A_j \in M_j\}.$$

Before we proceed, we need a definition of a properly nilpotent element of an algebra. Let $\mathcal{A}$ be an algebra, for every $z \in \mathcal{A}$ if az is nilpotent then an element a of $\mathcal{A}$ is called a *properly nilpotent element* of $\mathcal{A}$. [15]

Next, let J be the set of block strictly upper triangular matrices in $M_n(R, \rho)$. Any $A \in J$ is properly nilpotent [15], and conversely we can show that any properly nilpotent element in $M_n(R, \rho)$ is actually an element of J . But the radical consists of all properly nilpotent elements, whence J is a radical of $M_n(R, \rho)$.

Next, note that $M_n(R, \rho) = \mathcal{D}$ when and only when the radical J is zero, that is when and only when $M_n(R, \rho)$ is semi-simple. However, $M_n(R, \rho) = \mathcal{D}$ if and only if ρ is symmetric. Thus ρ is symmetric if and only if $M_n(R, \rho)$ is semi-simple.

If necessarily we may choose the ordering, so that the elements of $M_n(R, \rho)$ are in block upper triangular form, see above or [14].

Example 4: [16] Let $N = \{1, 2, 3, 4, 5, 6\}$ and

$$\rho = \{(1, 1), (1, 5), (1, 6), (2, 2), (2, 3), (3, 2), (3, 3), \\ (4, 4), (5, 1), (5, 5), (5, 6), (6, 1), (6, 5), (6, 6)\}$$

$\bar{\rho} = \rho$ then the structural matrix algebra is

$$M_6(F, \rho) = \left\{ \begin{bmatrix} F & 0 & 0 & 0 & F & F \\ 0 & F & F & 0 & 0 & 0 \\ 0 & F & F & 0 & 0 & 0 \\ 0 & 0 & 0 & F & 0 & 0 \\ F & 0 & 0 & 0 & F & F \\ F & 0 & 0 & 0 & F & F \end{bmatrix} : F \text{ is a field} \right\}.$$

We can construct equivalence classes and define permutation π as follows,

$$[r_1] = \{r_{11}, r_{12}, r_{13}\}, [r_2] = \{r_{21}, r_{22}\}, [r_3] = \{r_{31}\},$$

where $r_1 = 1, r_{12} = 5, r_{13} = 6, r_{21} = 2, r_{22} = 3, r_{31} = 4$, i.e.

$$[1] = \{1, 5, 6\}, [2] = \{2, 3\}, [4] = \{4\}.$$

$\pi(1) = 1, \pi(5) = 2, \pi(6) = 3, \pi(2) = 4, \pi(3) = 5, \pi(4) = 6$, i.e.

$\pi = (24635) \Rightarrow \pi^{-1} = (25364)$. Using $(i, j) \in \rho' \iff (\pi^{-1}(i), \pi^{-1}(j)) \in \rho$,

So we have

$$\rho' = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), \\ (3, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5), (6, 6)\}$$

and the structural matrix algebra related to ρ' is

$$M_6(F, \rho') = \left\{ \begin{bmatrix} F & F & F & 0 & 0 & 0 \\ F & F & F & 0 & 0 & 0 \\ F & F & F & 0 & 0 & 0 \\ 0 & 0 & 0 & F & F & 0 \\ 0 & 0 & 0 & F & F & 0 \\ 0 & 0 & 0 & 0 & 0 & F \end{bmatrix} : F \text{ is a field} \right\}.$$

Then the structural matrix algebra $M_6(F, \rho)$ is permutation similarity to the structural matrix algebra $M_6(F, \rho')$. Since $M_6(F, \rho')$ in block diagonal form and hence semi-simple, then we have $\overline{M}_1, \overline{M}_2$, and $\overline{M}_3$ are ideals which are simple and we can follow the proof of Theorem 1 in [14] and write

$$M_6(F, \rho') = \overline{M}_1 \oplus \overline{M}_2 \oplus \overline{M}_3$$

Example 5: Let $N = \{1, 2, 3, 4\}$ and

$$\rho = \{(1, 1), (1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 3), (3, 4), (4, 4)\}$$

$$\bar{\rho} = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$

and the structural matrix algebra is

$$M_4(F, \rho) = \left\{ \begin{bmatrix} F & F & 0 & F \\ 0 & F & 0 & F \\ 0 & F & F & F \\ 0 & 0 & 0 & F \end{bmatrix} : F \text{ is a field} \right\}.$$

We can construct equivalence classes and define permutation π as follows,

$$[r_1] = \{r_{11}\}, [r_2] = \{r_{21}\}, [r_3] = \{r_{31}\}, [r_4] = \{r_{41}\}$$

where $r_{11} = 1, r_{21} = 2, r_{31} = 3, r_{41} = 4$

$$[1] = \{1\}, [2] = \{2\}, [3] = \{3\}, [4] = \{4\}.$$

$\pi(1) = 1, \pi(2) = 3, \pi(3) = 2, \pi(4) = 4$, i.e. $\pi = (23) \Rightarrow \pi^{-1} = (32)$. Using $(i, j) \in \rho' \iff (\pi^{-1}(i), \pi^{-1}(j)) \in \rho$,

So we have

$$\rho' = \{(1, 1), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 1)\}$$

and the structural matrix algebra related to ρ' is

$$M_4(F, \rho') = \left\{ \begin{bmatrix} F & 0 & F & F \\ 0 & F & F & F \\ 0 & 0 & F & F \\ 0 & 0 & 0 & F \end{bmatrix} : F \text{ is a field} \right\}.$$

Then the structural matrix algebra $M_4(F, \rho)$ is permutation similarity to the structural matrix algebra $M_4(F, \rho') = T_4(F, \rho')$.

Conjecture 3.2: Let R be a ring with identity and $M_n(R)$ be the algebra of all $n \times n$ matrices over R and ρ be quasi order on the set $\{1, \dots, n\}$ let $d : M_n(R, \rho) \rightarrow M_n(R, \rho)$ be a derivation. Then there exists a derivation $\delta : R \rightarrow R$ and a matrix $A \in M_n(R, \rho)$ such that $d = \bar{\delta} + d_A$. Note: $\delta : R \rightarrow R$ is a derivation $\bar{\delta} : M_n(R, \rho) \rightarrow M_n(R, \rho)$ defined by $\bar{\delta}([x_{ij}]) = (\delta(x_{ij}))$, $\forall [x_{ij}] \in M_n(R, \rho)$ is a derivation.

Suppose now ρ is a partial order. Then $M_n(R, \rho)$ is in upper triangular form and we write $T_n(R, \rho)$. Thus, if $(i, j) \in \rho$, then $i \leq j$. If $(i \leq j)$ implies $(i, j) \in \rho$ then $T_n(R, \rho)$ becomes $T_n(R)$ the R -algebra of all upper triangular matrices. For this case, we have the result (Theorem page 263) [1]) that every derivations d of $T_n(R)$ can be written as $d = \bar{\delta} + d_A$, where $\bar{\delta}$ and d_A defined as in conjecture. In the present case, we wish to verify that [1]'s proof carries over to this case.

Theorem 3.3: Let R be a ring with identity and ρ be partially order on the set $\{1, \dots, n\}$ let $d : T_n(R, \rho) \rightarrow T_n(R, \rho)$ be a derivation. Then there exists a derivation $\delta : R \rightarrow R$ and a matrix $A \in T_n(R, \rho)$ such that $d = \bar{\delta} + d_A$. Note: $\delta : R \rightarrow R$ is a derivation $\bar{\delta} : T_n(R, \rho) \rightarrow T_n(R, \rho)$ defined by $\bar{\delta}([x_{ij}]) = [\delta(x_{ij})]$, $\forall [x_{ij}] \in T_n(R, \rho)$ is a derivation.

Proof. Proof of this theorem we are follow same as the Lemma 1 and Lemma 2 in [1]. Notice that a matrix unit e_{ij} belongs to $T_n(R, \rho)$ provided that $(i, j) \in \rho$. Since e_{kk} is an idempotent, we have that $d(e_{kk}) = d(e_{kk}e_{kk}) = e_{kk}d(e_{kk}) + d(e_{kk})e_{kk}$. Thus, if we

write $d(e_{kk}) = \sum_{(i,j) \in \rho} x_{ij}(k)e_{ij}$ we have that,

$$d(e_{kk}) = \sum_{(i,j) \in \rho} x_{ij}(k)e_{ij} = \sum_{(k,j) \in \rho} x_{kj}(k)e_{kj} + \sum_{(i,k) \in \rho} x_{ik}(k)e_{ik}$$

Which shows readily that $x_{kk}(k) = 0$ and that $x_{ij}(k) = 0$ if $(i, j) \in \rho$, and i, j are both different from k . Given a matrix $X \in T_n(R, \rho)$, we shall denote by $[X]_{ij}$ the i, j entry of this matrix. If $i < j$, we define $a_{ij} = -[e_{ii}d(e_{jj})e_{ij}]_{ij} = -x_{ij}(i)$. Also we set $a_{ij} = 0$ if $i > j$. Let A be a matrix with entries chosen as above and arbitrary diagonal entries. Set $h = d - d_A$: then:

i $h(e_{kk}) = 0, \quad 1 \leq k \leq n$

ii Furthermore, if we set $a_{jj} = -[d(e_{1j})]_{1j}, \quad 1 < j \leq n$ then $h(e_{ij}) = 0$, for $i < j$.

To proof. (i) we compute

$$d_A(e_{kk}) = Ae_{kk} - e_{kk}A = \sum_{(i,k) \in \rho} a_{ik}(k)e_{ik} - \sum_{(k,j) \in \rho} a_{kj}(k)e_{kj}$$

By our definition $-a_{kj} = x_{kj}(k)$. Also, notice that if $e, f \in T_n(R)$ are orthogonal idempotents, meaning that $e^2 = e, f^2 = f$, and $ef = fe = 0$, then we have that $ed(f) + d(e)f = 0$, which implies $ed(f)f = -ed(e)f$. Hence, in particular, $e_{ii}d(e_{kk})e_{kk} = -e_{ii}d(e_{ii})e_{kk}$, so if $i < k$ then $a_{ik} = -x_{ik}(i)$. Hence $d_A(e_{kk}) = d(e_{kk}), 1 \leq k \leq n$, and (i) follows.

To prove (ii) we shall first that $h(e_{1j}) = 0, \quad 1 < j \leq n$. Since $e_{1j} = e_{11}e_{1j}e_{jj}$, using (i) we obtain

$$h(e_{1j}) = e_{11}h(e_{1j})e_{jj} = e_{11}(d(e_{1j}) - Ae_{1j} + e_{1j}A)e_{jj} = -a_{jj}e_{1j} - a_{11}e_{1j} + a_{jj}e_{1j} = 0.$$

Now we prove that $h(e_{ij}) = 0, \quad 1 < i < j$. In fact, since $e_{1j} = e_{1i}e_{ij}$. we see that

$$0 = h(e_{1j}) = e_{1i}h(e_{1j}) = e_{1i}d(e_{1j}) - e_{1i}Ad(e_{ij}) + e_{1i}e_{ij}A.$$

Hence

$$[d(e_{ij})]_{ij} = a_{ii} - a_{jj}$$

Finally, we have that $e_{ij} = e_{ii}e_{ij}e_{jj}$, so we can write

$$h(e_{ij}) = e_{ii}h(e_{ij})e_{jj} = e_{ii}d(e_{ij}) - e_{ii}Ae_{ij} + e_{ij}Ae_{jj} = [d(e_{ij})]_{ij}e_{ij} - a_{ii}e_{ij} + a_{jj}e_{ij} = 0.$$

From now on, we shall assume that the elements on the diagonal of A are chosen as (ii) above, and hence that $h(e_{ij}) = 0, (i, j) \in \rho$.

Now we need to show that there exists a derivation $\delta : R \rightarrow R$ such that $h(rI) = \delta(r)I, \forall r \in R$. Set $r \in R$. Notice that $h(re_{ii}) = h(rI)e_{ii} = e_{ii}h(rI), 1 \leq i \leq n$, and for every pair $(i, j) \in \rho$, we have that $h(re_{ij}) = h(re_{ii}e_{ij}) = h(re_{ii})e_{ij} = h(e_{ij}re_{jj}) = e_{ij}h(re_{jj})$. The first set of equations shows that $h(rI)$ is a diagonal matrix and hence, by the second set of equations, $h(rI) = r'I$. It is now easy to see that the map $\delta : R \rightarrow R$ given by $r \rightarrow r'$ is a derivation of R .

Now set $\alpha = (r_{ij})_{(i,j) \in \rho} \in T_n(R, \rho)$. We have that

$$\begin{aligned} (d - d_A)(\alpha) &= h(\alpha) = h\left(\sum_{(i,j) \in \rho} r_{ij} I e_{ij}\right) \\ &= \sum_{(i,j) \in \rho} \delta(r_{ij}) I e_{ij} = \sum_{(i,j) \in \rho} \delta(r_{ij}) e_{ij} = \bar{\delta}(\alpha) \end{aligned}$$

Hence, $d(\alpha) = d_A(\alpha) + \bar{\delta}(\alpha), \forall \alpha \in T_n(R, \rho)$. We proved the theorem. $\square$

Example 6: Let $N = \{1, 2, 3\}$ and

$$\rho = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$$

$$d : M_3(R, \rho) \rightarrow M_3(R, \rho)$$

$$[a_{ij}]_{3 \times 3} \mapsto [|i - j| \cdot a_{ij}]_{3 \times 3}$$

on a ring 3×3 matrix is a derivation.

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \rightarrow d(A) = \begin{bmatrix} 0 & a_{12} & 2a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix} \rightarrow d(B) = \begin{bmatrix} 0 & b_{12} & 2b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ 0 & a_{22}b_{22} & a_{22}b_{23} + a_{23}b_{33} \\ 0 & 0 & a_{33}b_{33} \end{bmatrix}$$

$$\begin{aligned}
d(AB) &= \begin{bmatrix} 0 & a_{11}b_{12} + a_{12}b_{22} & 2(a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}) \\ 0 & 0 & a_{22}b_{23} + a_{23}b_{33} \\ 0 & 0 & 0 \end{bmatrix} \\
d(A)B + Ad(B) &= \begin{bmatrix} 0 & a_{12} & 2a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} 0 & b_{12} & 2b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & a_{11}b_{12} + a_{12}b_{22} & 2a_{11}b_{13} + 2a_{12}b_{23} + 2a_{13}b_{33} \\ 0 & 0 & a_{22}b_{23} + a_{23}b_{33} \\ 0 & 0 & 0 \end{bmatrix} = d(AB) \\
\Rightarrow d(AB) &= d(A)B + Ad(B)
\end{aligned}$$

Hence, there exist a matrix $A \in M_3(R, \rho)$ and a derivation $\delta : R \rightarrow R$ such that $d = \bar{\delta} + d_A$. In this situation,

$$a_{jj} = -[d(e_{1j})]_{1j}$$

Our matrix is 3×3 , therefore we should calculate a_{22} and a_{33} .

$$\begin{aligned}
d(e_{12}) &= d \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow -[d(e_{12})]_{12} = -1 = a_{22} \\
d(e_{13}) &= d \left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow -[d(e_{13})]_{13} = -2 = a_{33}
\end{aligned}$$

$a_{ij} = -x_{ij}(i) = -b_{ij}(i)$, so we can choose matrix A as follows

$$A = \begin{bmatrix} 0 & -b_{12} & -b_{13} \\ 0 & -1 & -b_{23} \\ 0 & 0 & -2 \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}$$

$$\begin{aligned}
d_A(B) &= AB - BA \\
&= \begin{bmatrix} 0 & -b_{12} & -b_{13} \\ 0 & -1 & -b_{23} \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix} \begin{bmatrix} 0 & -b_{12} & -b_{13} \\ 0 & -1 & -b_{23} \\ 0 & 0 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & b_{12} + b_{11}b_{12} - b_{12}b_{22} & 2b_{13} + b_{11}b_{13} - b_{13}b_{33} \\ 0 & 0 & b_{23} + b_{22}b_{23} - b_{23}b_{33} \\ 0 & 0 & 0 \end{bmatrix} \\
d(B) &= \begin{bmatrix} 0 & b_{12} & 2b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{bmatrix} \\
d(B) - d_A(B) &= \begin{bmatrix} 0 & b_{12} & 2b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & b_{12} + b_{11}b_{12} - b_{12}b_{22} & 2b_{13} + b_{11}b_{13} - b_{13}b_{33} \\ 0 & 0 & b_{23} + b_{22}b_{23} - b_{23}b_{33} \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & b_{12}b_{22} - b_{11}b_{12} & b_{13}b_{33} - b_{11}b_{13} \\ 0 & 0 & b_{23}b_{33} - b_{22}b_{23} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b_{12}(b_{22} - b_{11}) & b_{13}(b_{33} - b_{11}) \\ 0 & 0 & b_{23}(b_{33} - b_{22}) \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \delta(b_{12}) & \delta(b_{13}) \\ 0 & 0 & \delta(b_{23}) \\ 0 & 0 & 0 \end{bmatrix} = \bar{\delta}(B),
\end{aligned}$$

$\delta : R \rightarrow R$ is a derivation as follows:

$$\delta(b_{ij}) = b_{ij}(b_{jj} - b_{ii}), 1 \leq i < j \leq 3$$

If we generalize the derivation d as follows:

Let ρ be quasi order on the set $\{1, \dots, n\}$

$$\begin{aligned}
d : M_n(R, \rho) &\rightarrow M_n(R, \rho) \\
[a_{ij}]_{n \times n} &\mapsto [|i - j| \cdot a_{ij}]_{n \times n}
\end{aligned}$$

on a ring $n \times n$ matrix is a derivation

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \Rightarrow d(A) = \begin{bmatrix} 0 & a_{12} & \cdots & (n-1)a_{1n} \\ 0 & 0 & \cdots & (n-2)a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} \Rightarrow d(B) = \begin{bmatrix} 0 & b_{12} & \cdots & (n-1)b_{1n} \\ 0 & 0 & \cdots & (n-2)b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} + a_{12}b_{22} & \cdots & \sum_{k=1}^n a_{1k}b_{kn} \\ 0 & a_{22}b_{22} & \cdots & \sum_{k=1}^n a_{2k}b_{kn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}b_{nn} \end{bmatrix}$$

$$d(AB) = \begin{bmatrix} 0 & a_{11}b_{12} + a_{12}b_{22} & \cdots & (n-1)\sum_{k=1}^n a_{1k}b_{kn} \\ 0 & 0 & \cdots & (n-2)\sum_{k=1}^n a_{2k}b_{kn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\begin{aligned} d(A)B + Ad(B) &= \begin{bmatrix} 0 & a_{12} & \cdots & (n-1)a_{1n} \\ 0 & 0 & \cdots & (n-2)a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 0 & b_{12} & \cdots & (n-1)b_{1n} \\ 0 & 0 & \cdots & (n-2)b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_{12}b_{22} & \cdots & \sum_{k=1}^n (k-1)a_{1k}b_{kn} \\ 0 & 0 & \cdots & \sum_{k=1}^n (k-2)a_{2k}b_{kn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{11}b_{12} & \cdots & \sum_{k=1}^n (n-k)a_{1k}b_{kn} \\ 0 & 0 & \cdots & \sum_{k=1}^n (n-k)a_{2k}b_{kn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_{11}b_{12} + a_{12}b_{22} & \cdots & (n-1)\sum_{k=1}^n a_{1k}b_{kn} \\ 0 & 0 & \cdots & (n-2)\sum_{k=1}^n a_{2k}b_{kn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \end{aligned}$$

$$\Rightarrow d(AB) = d(A)B + Ad(B)$$

Hence, there exist a matrix $A \in M_n(R, \rho)$ and a derivation $\delta : R \rightarrow R$ such that

$d = \bar{\delta} + d_A$. In this situation,

$$a_{jj} = -[d(e_{1j})]_{1j}$$

$$d(e_{12}) = d \left(\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \rightarrow a_{22} = -1$$

$$d(e_{13}) = d \left(\begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \rightarrow a_{33} = -2$$

We continue as follows

$$d(e_{1n}) = d \left(\begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 & \cdots & (n-1) \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \rightarrow a_{nn} = -(n-1)$$

$a_{ij} = -x_{ij}(i) = -b_{ij}(i)$, so we can generalize and choose matrix A as follows

$$A = \begin{bmatrix} 0 & -b_{12} & \cdots & -b_{1n} \\ 0 & -1 & \cdots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -(n-1) \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix}$$

$$d_A(B) = AB - BA$$

$$= \begin{bmatrix} 0 & -b_{12} & \cdots & -b_{1n} \\ 0 & -1 & \cdots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -(n-1) \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} 0 & -b_{12} & \cdots & -b_{1n} \\ 0 & -1 & \cdots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -(n-1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -b_{12}b_{22} + b_{11}b_{12} + b_{12} & \cdots & -b_{1n}b_{nn} + b_{11}b_{1n} + (n-1)b_{1n} \\ 0 & 0 & \cdots & -b_{2n}b_{nn} + b_{22}b_{2n} + (n-2)b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\begin{aligned}
d(B) &= \begin{bmatrix} 0 & b_{12} & \cdots & (n-1)b_{1n} \\ 0 & 0 & \cdots & (n-2)b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\
d(B) - d_A(B) &= \begin{bmatrix} 0 & b_{12} & \cdots & (n-1)b_{1n} \\ 0 & 0 & \cdots & (n-2)b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} 0 & -b_{12}b_{22} + b_{11}b_{12} + b_{12} & \cdots & -b_{1n}b_{nn} + b_{11}b_{1n} + (n-1)b_{1n} \\ 0 & 0 & \cdots & -b_{2n}b_{nn} + b_{22}b_{2n} + (n-2)b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & b_{12}b_{22} - b_{11}b_{12} & \cdots & b_{1n}b_{nn} - b_{11}b_{1n} \\ 0 & 0 & \cdots & b_{2n}b_{nn} - b_{22}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & b_{12}(b_{22} - b_{11}) & \cdots & b_{1n}(b_{nn} - b_{11}) \\ 0 & 0 & \cdots & b_{2n}(b_{nn} - b_{22}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \delta(b_{12}) & \delta(b_{13}) & \cdots & \delta(b_{1n}) \\ 0 & 0 & \delta(b_{23}) & \cdots & \delta(b_{2n}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \delta(b_{n-1n}) \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \bar{\delta}(B).
\end{aligned}$$

$$\delta : R \rightarrow R$$

$$\delta(b_{ij}) = b_{ij}(b_{jj} - b_{ii}), 1 \leq i < j \leq n$$

4. CONCLUSION

The main objective of this thesis is to realize the derivation of structural matrix algebras as the sum of the inner derivation and the derivations obtained from the ring R . We give the some conditions on ρ and we reproof of the theorem in [1] for our cases with the following theorem.

Theorem 4.1: Let R be a ring with identity and ρ be partially order on the set $\{1, 2, 3, \dots, n\}$ let $d : T_n(R, \rho) \rightarrow T_n(R, \rho)$ be a derivation. Then there exists a derivation $\delta : R \rightarrow R$ and a matrix $A \in T_n(R, \rho)$ such that $d = \bar{\delta} + d_A$. Note: $\delta : R \rightarrow R$ is a derivation $\bar{\delta} : T_n(R, \rho) \rightarrow T_n(R, \rho)$ defined by $\bar{\delta}([x_{ij}]) = [\delta(x_{ij})], \forall [x_{ij}] \in T_n(R, \rho)$ is a derivation.

For the future, we will first consider the case ρ is a symmetric; in this case $M_n(R, \rho)$ is block diagonal form, that is $M_n(R, \rho) = \mathcal{D} = \{diag(A_1, \dots, A_p) : A_j \in M_j\}$. Next, we will consider the general case where ρ is a quasi-order.

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BIOGRAPHY

Serdar DEMİR, ilk ve orta öğrenimini Kütahya’da tamamladıktan sonra Çanakkale Onsekiz Mart Üniversitesi Fen - Edebiyat Fakültesi Matematik Bölümü’ne yerleşti. 2014 yılında Matematik bölümünden birincilikle mezun olduktan sonra Milli Eğitim Bakanlığı bünyesine Matematik öğretmeni olarak atandı. Halen görevine devam etmektedir.

APPENDICES

A. Python Codes

A.1. Upper Triangular Matrix and Derivation Functions

The following Python code demonstrates an algorithm for constructing upper triangular matrices and performing derivative operations:

```
1 from sympy import symbols, Matrix, Abs, simplify
2 # Derivasyon fonksiyonu
3 def der(matrix):
4     n = matrix.shape[0]
5     der_matrix = Matrix.zeros(n, n) # n x n sıfır matrisi
6
7     for i in range(n):
8         for j in range(n):
9             der_matrix[i, j] = Abs(i - j) * matrix[i, j]
10    return der_matrix
11 # Upper triangular matrix üretme fonksiyonu
12 def upperA(n):
13     # n x n boyutunda bir üst üçgen matrix oluşturuyoruz
14     matrix_elements = []
15     for i in range(n):
16         row = []
17         for j in range(n):
18             # Sadece üst üçgen matris içindeki elemanlar sembolik
19             # olacak
20             if j >= i:
21                 row.append(symbols(f'a{i+1}{j+1}'))
22             else:
23                 row.append(0) # Alt üçgen elemanlar sıfır
24         matrix_elements.append(row)
25     return Matrix(matrix_elements)
26 def upperB(n):
27     # n x n boyutunda bir üst üçgen matrix oluşturuyoruz
28     matrix_elements = []
29     for i in range(n):
30         row = []
31         for j in range(n):
32             # Sadece üst üçgen matris içindeki elemanlar sembolik
33             # olacak
34             if j >= i:
35                 row.append(symbols(f'b{i+1}{j+1}'))
36             else:
37                 row.append(0) # Alt üçgen elemanlar sıfır
38         matrix_elements.append(row)
39     return Matrix(matrix_elements)
40 # İlk matrixin boyutunu sorma
41 n = int(input("Matrixin boyutunu girin: "))
42 # Boyut n'e göre üst üçgen matrix üretimi
```

```

41 matrix1 = upperA(n)
42 # İkinci matrisin boyutunu sorma
43 # m = int(input("Matrisin boyutunu girin (m): "))
44 # Boyut m'e göre üst üçgen matris üretimi
45 matrix2 = upperB(n)
46
47 # Matris çarpma şilemi
48 carpm = matrix1 * matrix2
49 dcarpm = der(carpm)
50 toplam = der(matrix1) * matrix2 + matrix1 * der(matrix2)
51 # Matris şğıeitlii kontrolü
52 derivation = simplify(dcarpm.equals(toplam))
53 # 1. Matrisin düzgün şekilde ıııyazdrılmas
54 print(f"\şnOluturululanAMatrisi(UpperTriangular):")
55 for i in range(n):
56     print(' '.join(str(matrix1[i, j]) for j in range(n)))
57 # 2. Matrisin düzgün şekilde ıııyazdrılmas
58 print(f"\şnOluturululanBMatrisi(UpperTriangular):")
59 for i in range(n):
60     print(' '.join(str(matrix2[i, j]) for j in range(n)))
61 # ıÇarpm Matrisin düzgün şekilde ıııyazdrılmas
62 print(f"\şnOluturululanABMatrisi:")
63 for i in range(min(n, n)): # min(n, m) çünkü ıçarpanların
    boyutlarına ğıbal olarak ıııyazdryoruz
64     print(' '.join(str(carpm[i, j]) for j in range(n)))
65 # ııÇarpmn Derivation Matrisin düzgün şekilde ıııyazdrılmas
66 print(f"\şnOluturululand(AB)Matrisi:")
67 for i in range(min(n, n)): # min(n, m) yine boyutlar dikkate
    alınarak ıııyazdrılır
68     print(' '.join(str(dcarpm[i, j]) for j in range(n)))
69 # Derivation Matrisin düzgün şekilde ıııyazdrılmas
70 print(f"\şnOluturululand(A)B+Ad(B)Matrisi:")
71 for i in range(min(n, n)): # min(n, m) yine boyutlar dikkate
    alınarak ıııyazdrılır
72     print(' '.join(str(toplam[i, j]) for j in range(n)))
73 print(f"\şnOluturululanA:")
74 for i in range(min(n, n)): # min(n, m) yine boyutlar dikkate
    alınarak ıııyazdrılır
75     print(' '.join(str(der(matrix1)[i, j]) for j in range(n)))
76 if derivation:
77     print("\nSonuç: İki matris şğıettir.")
78 else:
79     print("\nSonuç: İki matris şğıeit ğdeildir.")

```

Listing A.1: Python Code - Calculating Upper Triangular Matrices and Derivation