

**T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL**

**ANALYSIS OF ELECTROMAGNETIC WAVE PROPAGATION
IN COAXIAL WAVEGUIDES WITH IMPEDANCE COATED
STEP DISCONTINUITY**

ZEYNEP SAZAK

**A THESIS OF MASTER OF SCIENCE
DEPARTMENT OF MATHEMATICS**

ADVISOR: ASSOC. PROF. DR. HULYA OZTURK

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GEBZE TEKNİK ÜNİVERSİTESİ
LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ**

**EMPEDANS KAPLI BASAMAK SÜREKSİZLİĞİNE
SAHİP KOAKSİYEL DALGA KILAVUZLARINDA
ELEKTROMANYETİK DALGA YAYILIMININ ANALİZİ**

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ABSTRACT

In this thesis, using the Mode-Matching technique and the Wiener-Hopf technique, the propagation of the TEM wave along a coaxial waveguide with an impedance coated step discontinuity in the inner conductor is mathematically analyzed. Such configurations are of substantial interest in high-frequency applications, including radar and telecommunication technologies. Firstly, the Mode-Matching technique is applied to the problem and in this context, the structure is divided into two different regions and solution modes are defined for each region. Following the technique-specific formulation, an infinite system of equations is obtained for the problem and these equations are solved simultaneously. After the analytical study, programs are written in the MATLAB programming language and the results are presented with generated graphs. The problem is then analyzed using the Wiener-Hopf technique, applying the Fourier transform to the Helmholtz equation and considering boundary and continuity conditions in the same domain to obtain the modified Wiener-Hopf equation for the problem. The solution of this equation is obtained in the form of an infinite system of unknown coefficients, which results in an infinite system of linear algebraic equations that are truncated at an appropriate cut-off number. Finally, the reflected field and the transmitted field are obtained and the results are presented alongside the produced graphs. Some of these graphs show the variation of the reflection coefficient according to different parameters, while others describe the variation of the transmission coefficient. The results obtained show that both methods validate each other and provide reliable analytical approaches that can be used in high-frequency waveguide designs.

Keywords: Wiener-Hopf Technique, Electromagnetic Wave Scattering, Electromagnetic Wave Propagation, Mode-Matching Technique, Coaxial Waveguides

ÖZET

Bu tezde, Mod-Eşleştirme tekniği ve Wiener-Hopf tekniği kullanılarak, TEM dalgasının iç iletkende empedans kaplı bir basamak süreksizliği olan bir koaksiyel dalga kılavuzu boyunca yayılması matematiksel olarak analiz edilmiştir. Bu tür konfigürasyonlar, radar ve telekomünikasyon teknolojileri de dahil olmak üzere yüksek frekans uygulamalarında büyük ilgi görmektedir. İlk olarak probleme Mod-Eşleştirme tekniği uygulanmış ve bu kapsamda yapı iki farklı bölgeye ayrılarak her bir bölge için çözüm modları tanımlanmıştır. Tekniğe özgü formülasyonun ardından problem için sonsuz denklem sistemi elde edilmiştir ve bu denklemler eş zamanlı olarak çözülmüştür. Analitik çalışmanın ardından MATLAB programlama dilinde programlar yazılmış ve sonuçlar oluşturulan grafiklerle sunulmuştur. Daha sonra problem Wiener-Hopf tekniği kullanılarak analiz edilmiş, Helmholtz denklemine Fourier dönüşümü uygulanmış ve problem için modifiye Wiener-Hopf denklemini elde etmek için sınır ve süreklilik koşulları aynı alanda dikkate alınmıştır. Bu denklemin çözümü, bilinmeyen katsayılardan oluşan sonsuz bir sistem şeklinde elde edilmiştir ve bu da uygun bir kesme sayısında kesilen sonsuz bir doğrusal cebirsel denklem sistemi ile sonuçlanmıştır. Son olarak, yansıyan alan ve iletilen alan elde edilmiştir ve sonuçlar, üretilen grafiklerle birlikte sunulmuştur. Bu grafiklerden bazıları yansıma katsayısının farklı parametrelere göre değişimini ortaya koyarken, bazılarıda iletim katsayısının değişimlerini betimlemektedir. Elde edilen bulgular, iki yöntemin de birbirini doğruladığını göstermekte olup, yüksek frekanslı dalga kılavuzu tasarımlarında kullanılabilecek güvenilir analitik yaklaşımlar sunduğunu ortaya koymaktadır.

Anahtar Kelimeler: Wiener-Hopf Tekniği, Elektromanyetik Dalga Yayılımı, Elektromanyetik Dalga Saçınımı, Mod-Eşleştirme Tekniği, Koaksiyel Dalga Kılavuzları

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I dedicate this thesis to my mother Kıymet SAZAK...

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LIST of ABBREVIATIONS and ACRONYMS

MWHE	:	Modified Wiener-Hopf Equation
TEM	:	Transverse Electromagnetic
k	:	Wave number
a	:	Radius of inner cylinder
b	:	Radius of outer cylinder
c	:	Height of the groove on the inner cylinder
ω	:	Angular frequency
η	:	Surface impedance of the inner cylinder
N	:	Truncation number
Hz	:	Hertz
MHz	:	MegaHertz
GHz	:	GigaHertz

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1. INTRODUCTION

Structural discontinuities in coaxial waveguides represent a key element in microwave engineering and have been widely studied due to their relevance in numerous practical systems. These variations in geometry play a vital role in the accurate evaluation of microwave and transmission line behaviors. Furthermore, they are crucial components in the development of filters, measurement setups, and waveguide-based radiating structures. However, structural features like step changes or impedance coatings introduce challenges in the analytical modeling of wave behavior, particularly at high frequencies. As a result, many studies, employing both analytical and computational approaches, have focused on examining their impact [Bobrovnikov et al. (1966)]. Whinnery and his team, in a 1944 study, explored the effects of step discontinuities in coaxial waveguides and developed equivalent circuit models using a Mode Matching-based methodology [Whinnery et al. (1944)]. In 1965, Green numerically analyzed various configurations of step discontinuities in coaxial waveguides by means of the finite difference technique [Green (1967)]. Later, in 1971, Vijayaraghavan and Arora examined the issue of a coaxial waveguide with an inductively reactive inner surface, employing the Wiener-Hopf method for their analysis [Vijayaraghavan & Arora (1971)]. The formulation of the dyadic Green's function for a coaxial structure featuring a step change on the inner wall was presented by Fallahi and Rashed-Mohassel in 2006 [Arya & Jalil (2006)].

In this thesis, the transmission of TEM waves is examined in a coaxial waveguide structure that includes an impedance-coated step discontinuity on its inner surface. The problem is addressed through separate applications of the Mode-Matching and Wiener-Hopf techniques.

In chapter 2, Mode-Matching technique described in [Öztürk (2021)] is used to analyze this problem. The technique known as Mode-Matching is commonly utilized for formulating boundary value problems in waveguide theory. It has been extensively applied in past research that dealt with step discontinuities in waveguides [Wexler (1967)-Mittra & Lee (1971)]. Specifically, this method has been employed in studies involving

coaxial waveguides, particularly when the discontinuities are positioned on the inner wall [Aksimsek et al. (2013)]. The same problem is analyzed by the Wiener-Hopf technique, which is especially powerful when dealing with semi-infinite domains and complex boundary conditions, such as those found in coated coaxial geometries with step discontinuities by considering direct Fourier transform of the Helmholtz equation, boundary conditions, and continuity relations, and then a unique type of Wiener-Hopf equation, which is solved by the factorization and decomposition procedure as in [Rawlins (1995)], is determined following a similar procedure mentioned in [Hacıvelioğlu et al. (2009), Kuryliak et al. (2000), Tiryakioğlu (2020)]. The equation known as the Modified Wiener-Hopf Equation (MWHE) incorporates a specific kernel function that describes the characteristics of the step discontinuity present on the inner wall [Öztürk (2020)]. Solving the MWHE involves determining an infinite number of unknown coefficients that meet a set of linear algebraic equations. The scattering coefficients are then derived upon completion of the analysis.

After determining the analytical results, a computer code is developed in MATLAB for each problem and each technique. Chapter 3 discusses the numerical results obtained using the scattering parameters in the TEM mode. A computational comparison in terms of accuracy and convergence speed between Wiener-Hopf and Mode Matching is also presented in this section.

The entire analysis is based on the assumption of a time dependence $\exp(-i\omega t)$, where ω is the angular frequency. The conductor walls in the problem are perfect electrical conductors (PEC).

2. STEP DISCONTINUITY WITH IMPEDANCE TYPE COATING

2.1. Mode-Matching Analysis

Consider the propagation of an incident field along a waveguide. Figure 2.1 illustrates the division of the coaxial channel into two regions, where the first region is defined as $\varrho \in (a, b)$, $-\infty < z < 0$, the second region is defined as $\varrho \in (c, b)$, $0 < z < \infty$. The horizontal interface of the coaxial structure at $\varrho = c$ and $z > 0$, is characterized by a surface impedance $Z = \eta Z_0$, while the outer wall is considered to be perfectly conducting.

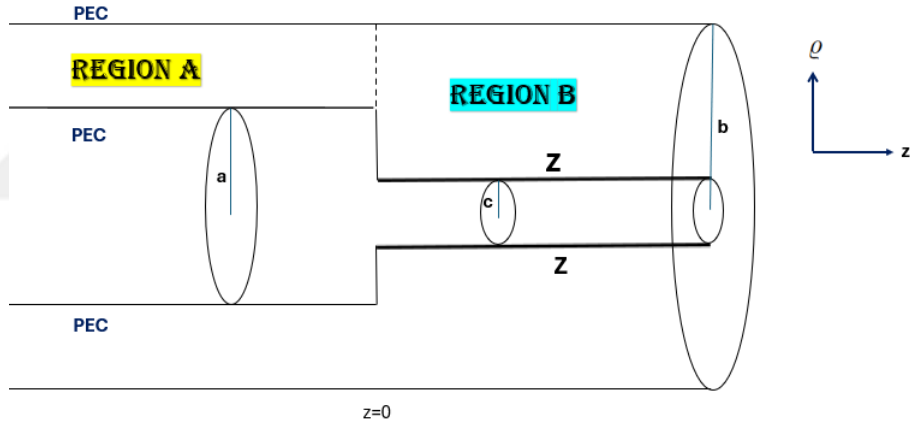


Figure 2.1: Regions for Mode Matching Technique.

In order to facilitate the solution, the total field $\mathcal{Y}_T(\varrho, z)$ can be expressed as a piecewise-defined function, as illustrated below

$$\mathcal{Y}_T(\varrho, z) = \begin{cases} \mathcal{Y}_i(\varrho, z) + \mathcal{Y}_1(\varrho, z), & a < \varrho < b, \quad -\infty < z < 0 \\ \mathcal{Y}_2(\varrho, z) & , \quad c < \varrho < b, \quad 0 < z < \infty \end{cases} \quad (2.1)$$

where $\mathcal{Y}_j(\varrho, z)$, $j=1,2$ are the scattered fields in Region A and Region B, respectively, and the given function $\mathcal{Y}_i(\varrho, z) = \frac{e^{ikz}}{\varrho}$ applies to the incident TEM mode propagating

in the positive direction along the z-axis with k representing the propagation constant and satisfy the following Helmholtz equation :

$$\left[\frac{\partial^2}{\partial \varrho^2} + \frac{1}{\varrho} \frac{\partial}{\partial \varrho} + \frac{\partial^2}{\partial z^2} + \left(K^2 - \frac{1}{\varrho^2} \right) \right] \mathcal{Y}_j(\varrho, z) = 0 \quad , \quad j = 1, 2 \quad (2.2)$$

The field terms satisfy the below boundary conditions;

$$\mathcal{Y}_1(b, z) + b \frac{\partial}{\partial \varrho} \mathcal{Y}_1(b, z) = 0, \quad z \in (-\infty, 0) \quad (2.3)$$

$$\mathcal{Y}_1(a, z) + a \frac{\partial}{\partial \varrho} \mathcal{Y}_1(a, z) = 0, \quad z \in (-\infty, 0) \quad (2.4)$$

$$\left(\eta + \frac{1}{ikc} \right) \mathcal{Y}_2(c, z) + \frac{1}{ik} \frac{\partial}{\partial \varrho} \mathcal{Y}_2(c, z) = 0, \quad z \in (0, \infty) \quad (2.5)$$

To apply the Mode-Matching method, the waveguide is partitioned into two distinct regions. In each region, the corresponding eigenfunction expansions are identified based on cylindrical waveguide theory, along with their associated orthogonality relations.

Region A : $-\infty < z < 0$, $\varrho \in (a, b)$

Using the expansion of eigenfunction, the solution in this region can be written in the form of

$$\mathcal{Y}_1(\varrho, z) = \sum_{n=0}^{\infty} a_n \vartheta_n(\varrho) e^{-i\alpha_n z} \quad (2.6)$$

where

$$\vartheta_n(\varrho) = \frac{\pi}{2} K_n \left[J_1(K_n \varrho) Y_0(K_n a) - J_0(K_n a) Y_1(K_n \varrho) \right] \quad (2.7)$$

are the eigenfunction, a_n is the magnitude of the reflected wave and α_n 's being the

zeros of $K^2 \left[J_0(Ka)Y_0(Kb) - J_0(Kb)Y_0(Ka) \right]$ satisfying

$$K_n^2 \left[J_0(K_n a)Y_0(K_n b) - J_0(K_n b)Y_0(K_n a) \right] = 0 \quad (2.8)$$

with

$$\alpha_n = \sqrt{k^2 - K_n^2}, \quad n = 1, 2, 3, \dots \quad (2.9)$$

The Bessel and Neumann functions are shown above J_n and Y_n ($n = 0, 1$). K_n are eigenvalues and α_n are the associated eigenvalues.

Region B : $0 < z < \infty$, $\varrho \in (c, b)$

The solution in this region, $\mathcal{Y}_2(\varrho, z)$ can be expressed as

$$\mathcal{Y}_2(\varrho, z) = \sum_{n=0}^{\infty} b_n \Upsilon_n(\varrho) e^{i\zeta_n z} \quad (2.10)$$

where

$$\Upsilon_n(\varrho) = \frac{\pi}{2} \mu_n \left[J_1(\mu_n \varrho)Y_0(\mu_n b) - J_0(\mu_n b)Y_1(\mu_n \varrho) \right] \quad (2.11)$$

are the eigenfunction, where b_n is the magnitude of the transmitted duct mode and ζ_n is the root of the equation

$$\begin{aligned} \mu_n \eta \left[J_1(\mu_n c)Y_0(\mu_n b) - J_0(\mu_n b)Y_1(\mu_n c) \right] \\ + \frac{\mu_n^2}{ik} \left[J_0(\mu_n c)Y_0(\mu_n b) - J_0(\mu_n b)Y_0(\mu_n c) \right] = 0 \end{aligned} \quad (2.12)$$

with

$$\zeta_n = \sqrt{k^2 - \mu_n^2}, \quad n = 1, 2, 3, \dots \quad (2.13)$$

Continuity conditions at $z = 0$ serve as the basis for employing the Mode-Matching approach by taking into account the continuity of the tangential components of the

electric and magnetic fields, to find the unknown coefficients a_n and b_n in the series expansions (2.6) and (2.10), respectively.

$$E_\varrho(\varrho, 0^+) = \begin{cases} E_\varrho(\varrho, 0^-) , & a < \varrho < b \\ 0 & , \quad c < \varrho < a \end{cases} \quad (2.14)$$

and $E_\varrho(\varrho, z) = \frac{1}{i\omega\epsilon} \frac{\partial}{\partial z} \mathcal{V}(\varrho, z)$ then we obtain

$$\sum_{n=0}^{\infty} b_n \zeta_n \Upsilon_n(\varrho) = - \sum_{n=0}^{\infty} a_n \alpha_n \vartheta_n(\varrho) + \frac{ik}{\varrho}, \quad \varrho \in (a, b) \quad (2.15)$$

Multiplying (2.15) by " $\varrho \Upsilon_m(\varrho)$ " and integrating along $\varrho \in (c, b)$, give

$$\begin{aligned} \sum_{n=0}^{\infty} b_n \zeta_n \int_c^b \Upsilon_n(\varrho) \Upsilon_m(\varrho) \varrho d\varrho \\ = ik \int_a^b \Upsilon_m(\varrho) d\varrho - \sum_{n=0}^{\infty} a_n \alpha_n \int_a^b \vartheta_n(\varrho) \Upsilon_m(\varrho) \varrho d\varrho \end{aligned} \quad (2.16)$$

The series expansions in (2.16) exist if and only if the associated eigenfunction satisfies the orthogonality integral given below:

$$\int_c^b \Upsilon_n(\varrho) \Upsilon_m(\varrho) d\varrho = 0, \quad m \neq n \quad (2.17)$$

with

$$\chi_m^{(1)} = \int_a^b \Upsilon_m(\varrho) d\varrho \quad (2.18)$$

$$\nabla_{mn}^{(1)} = \int_a^b \vartheta_n(\varrho) \Upsilon_m(\varrho) \varrho d\varrho \quad (2.19)$$

$$\theta_m^{(1)} = \int_c^b \Upsilon_n(\varrho) \Upsilon_m(\varrho) \varrho d\varrho \quad (2.20)$$

Similarly, by taking into account $H_\phi(\varrho, 0^+) = H_\phi(\varrho, 0^-)$, $\varrho \in (a, b)$, we get

$$\sum_{n=0}^{\infty} b_n \Upsilon_n(\varrho) = \sum_{n=0}^{\infty} a_n \vartheta_n(\varrho) + \frac{1}{\varrho}, \quad \varrho \in (a, b) \quad (2.21)$$

Multiplying (2.21) by " $\varrho \vartheta_m(\varrho)$ " and integrating along $\varrho \in (a, b)$, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} b_n \int_a^b \Upsilon_n(\varrho) \vartheta_m(\varrho) \varrho d\varrho \\ = \int_a^b \vartheta_m(\varrho) d\varrho + \sum_{n=0}^{\infty} a_n \int_a^b \vartheta_n(\varrho) \vartheta_m(\varrho) \varrho d\varrho \end{aligned} \quad (2.22)$$

with

$$\chi_m^{(2)} = \int_a^b \vartheta_m(\varrho) d\varrho \quad (2.23)$$

$$\nabla_{mn}^{(2)} = \int_a^b \Upsilon_n(\varrho) \vartheta_m(\varrho) \varrho d\varrho \quad (2.24)$$

$$\theta_m^{(2)} = \int_a^b \vartheta_n(\varrho) \vartheta_m(\varrho) \varrho d\varrho \quad (2.25)$$

When $z=0$, the following system is obtained from the equation (2.16);

$$b_m \zeta_m \theta_m^{(1)} = k \chi_m^{(1)} - \sum_{n=0}^{\infty} a_n \alpha_n \nabla_{mn}^{(1)}, \quad m = 0, 1, 2, \dots, \infty \quad (2.26)$$

and the following system is obtained from the equation (2.22);

$$\sum_{n=0}^{\infty} b_n \nabla_{mn}^{(2)} = a_m \theta_m^{(2)} + \chi_m^{(2)}, \quad m = 0, 1, 2, \dots, \infty \quad (2.27)$$

Consequently, it is necessary to view the sets of equations provided in (2.26) and (2.27) as a doubly infinite set of equations that need to be solved at the same time. Solving this doubly infinite set of equations requires truncating the first and second equations

using a specific truncation number (N) to obtain $\mathcal{S}$ and $\mathcal{T}$, respectively.

$$b_m \zeta_m \theta_m^{(1)} + \sum_{n=0}^{\mathcal{T}} a_n \alpha_n \nabla_{mn}^{(1)} = k \chi_m^{(1)}, \quad m = 0, 1, 2, \dots, \mathcal{S} \quad (2.28)$$

and

$$a_m \theta_m^{(2)} - \sum_{n=0}^{\mathcal{S}} b_n \nabla_{mn}^{(2)} = -\chi_m^{(2)}, \quad m = 0, 1, 2, \dots, \mathcal{T} \quad (2.29)$$

It should be noted that this truncation process results in a $(N + 2) \times (N + 2)$ algebraic system, where N is defined as the sum of $\mathcal{S}$ and $\mathcal{T}$. It is also important to observe that $\mathcal{S}$ must be greater than $\mathcal{T}$ since the first set of equations involves more information than the latter. The explicit expressions for $\chi_m^{(1)}$, $\chi_m^{(2)}$, $\theta_m^{(1)}$, $\theta_m^{(2)}$, $\nabla_{mn}^{(1)}$, $\nabla_{mn}^{(2)}$, are

$$\chi_m^{(1)} = \frac{\pi}{2} \left[J_0(\mu_m a) Y_0(\mu_m b) - J_0(\mu_m b) Y_0(\mu_m a) \right] \quad (2.30)$$

$$\chi_m^{(2)} = \begin{cases} \log(b/a) & , \quad m = 0 \\ 0 & , \quad m \neq 0 \end{cases} \quad (2.31)$$

$$\theta_m^{(1)} = \begin{cases} 0 & , \quad m \neq n \\ \mathfrak{V}_m & , \quad m = n \end{cases} \quad (2.32)$$

where $\mathfrak{V}_m = \frac{1}{2} - \frac{\pi^2}{4} \mu_m^2 \frac{c^2}{2} \left[J_1(\mu_m c) Y_0(\mu_m b) - J_0(\mu_m b) Y_1(\mu_m c) \right]^2 \left[1 - (ik\eta)^2 + \frac{2ik\eta}{c\mu_m^2} \right]$

$$\theta_m^{(2)} = \begin{cases} \log(b/a) & , \quad m = 0 \\ \frac{[Y_0^{(2)}(\mu_m a) - Y_0^{(2)}(\mu_m b)]}{2Y_0^{(2)}(\mu_m b)} & , \quad m \neq 0 \end{cases} \quad (2.33)$$

$$\nabla_{mn}^{(1)} = \begin{cases} \frac{-\pi\mu_0^2}{2(K_n^2 - \mu_0^2)} \left[J_0(\mu_0 a) Y_0(\mu_0 b) - J_0(\mu_0 b) Y_0(\mu_0 a) \right] \frac{J_0(K_n b)}{J_0(K_n a)} & , \quad m = 0, \quad n \neq 0 \\ \frac{\pi}{2} \left[J_0(\mu_0 a) Y_0(\mu_0 b) - J_0(\mu_0 b) Y_0(\mu_0 a) \right] & , \quad m = 0, \quad n = 0 \\ \frac{\pi}{2} \left[J_0(\mu_m a) Y_0(\mu_m b) - J_0(\mu_m b) Y_0(\mu_m a) \right] & , \quad m \neq 0, \quad n = 0 \\ \frac{-\pi\mu_m^2}{2(K_n^2 - \mu_m^2)} \left[J_0(\mu_m a) Y_0(\mu_m b) - J_0(\mu_m b) Y_0(\mu_m a) \right] \frac{J_0(K_n b)}{J_0(K_n a)}, & m \neq 0, \quad n \neq 0 \end{cases} \quad (2.34)$$

$$\nabla_{mn}^{(2)} = \begin{cases} \frac{-\pi\mu_0^2}{2(K_m^2 - \mu_0^2)} \left[J_0(\mu_0 a) Y_0(\mu_0 b) - J_0(\mu_0 b) Y_0(\mu_0 a) \right] \frac{J_0(K_m b)}{J_0(K_m a)} & , \quad m \neq 0, \quad n = 0 \\ \frac{\pi}{2} \left[J_0(\mu_0 a) Y_0(\mu_0 b) - J_0(\mu_0 b) Y_0(\mu_0 a) \right] & , \quad m = 0, \quad n = 0 \\ \frac{\pi}{2} \left[J_0(\mu_n a) Y_0(\mu_n b) - J_0(\mu_n b) Y_0(\mu_n a) \right] & , \quad m = 0, \quad n \neq 0 \\ \frac{-\pi\mu_n^2}{2(K_m^2 - \mu_n^2)} \left[J_0(\mu_n a) Y_0(\mu_n b) - J_0(\mu_n b) Y_0(\mu_n a) \right] \frac{J_0(K_m b)}{J_0(K_m a)} & , \quad m \neq 0, \quad n \neq 0 \end{cases} \quad (2.35)$$

2.2. Wiener-Hopf Analysis

2.2.1. Formulation of the Problem

Consider a coaxial waveguide whose inner and outer conductors have radii $\varrho = a$ and $\varrho = b$, respectively. A groove of height $\varrho = c$ is located on the inner conductor.

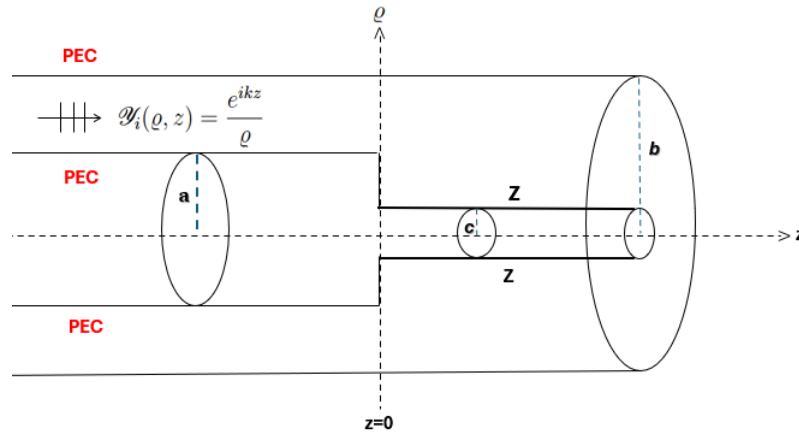


Figure 2.2: Regions for the Wiener-Hopf Technique.

Taking the geometry into account, the problem domain can be described in the following manner:

$$\mathcal{Y}_T(\varrho, z) = \begin{cases} \frac{e^{ikz}}{\varrho} + \mathcal{Y}_1(\varrho, z) , & a < \varrho < b, \quad -\infty < z < \infty \\ \mathcal{Y}_2(\varrho, z) , & c < \varrho < a, \quad 0 < z < \infty \end{cases} \quad (2.36)$$

where $\mathcal{Y}_1(\varrho, z)$ and $\mathcal{Y}_2(\varrho, z)$ are the scattered fields which satisfy the Helmholtz equation (2.2). Considering the geometry of the problem, it is possible to write the necessary boundary conditions for the fields $\mathcal{Y}_1(\varrho, z)$ and $\mathcal{Y}_2(\varrho, z)$;

$$\mathcal{Y}_1(b, z) + b \frac{\partial}{\partial \varrho} \mathcal{Y}_1(b, z) = 0, \quad z \in (-\infty, \infty) \quad (2.37)$$

$$\mathcal{Y}_1(a, z) + a \frac{\partial}{\partial \varrho} \mathcal{Y}_1(a, z) = 0, \quad z \in (-\infty, 0) \quad (2.38)$$

$$\frac{\partial}{\partial z} \mathcal{Y}_2(\varrho, 0) = 0, \quad \varrho \in (c, a) \quad (2.39)$$

$$\left(\eta + \frac{1}{ikc} \right) \mathcal{Y}_2(c, z) + \frac{1}{ik} \frac{\partial}{\partial \varrho} \mathcal{Y}_2(c, z) = 0, \quad z \in (0, \infty) \quad (2.40)$$

which originate from the condition that the tangential components of the electric field vanish along the waveguide walls, and the continuity relations at $\varrho = a$;

$$\mathcal{Y}_1(a, z) + a \frac{\partial}{\partial \varrho} \mathcal{Y}_1(a, z) = \mathcal{Y}_2(a, z) + a \frac{\partial}{\partial \varrho} \mathcal{Y}_2(a, z), \quad z \in (0, \infty) \quad (2.41)$$

$$\frac{e^{ikz}}{a} + \mathcal{Y}_1(a, z) = \mathcal{Y}_2(a, z), \quad z \in (0, \infty) \quad (2.42)$$

which indicates the continuity of the tangential components of both electric and magnetic fields within the specified region. Ensuring the uniqueness of the obtained solu-

tion requires that the radiation condition be properly imposed [Alkumru et al. (2002)]

$$\mathcal{Y}_T(\varrho, z) = \mathcal{O}(e^{ik|z|}), \quad |z| \rightarrow \infty \quad (2.43)$$

and the edge conditions;

$$\mathcal{Y}_T(a, z) = \mathcal{O}(z^{2/3}), \quad z \rightarrow 0 \quad (2.44)$$

$$\frac{\partial}{\partial \varrho} \mathcal{Y}_T(a, z) = \mathcal{O}(z^{-1/3}), \quad z \rightarrow 0 \quad (2.45)$$

When the Fourier transform is applied to the Helmholtz equation governing $\mathcal{Y}_1(\varrho, z)$ in the entire range from $(-\infty, \infty)$ with respect to z , it yields

$$\left[\frac{\partial^2}{\partial \varrho^2} + \frac{1}{\varrho} \frac{\partial}{\partial \varrho} + \left(K^2 - \frac{1}{\varrho^2} \right) \right] \mathcal{H}(\varrho, \alpha) = 0 \quad (2.46)$$

Here the function $K(\alpha) = \sqrt{k^2 - \alpha^2}$ is defined as a branch of the square root in the complex- α -plane, with branch cuts extending from $\alpha = k$ to $\alpha = k + i\infty$ and from $\alpha = -k$ to $\alpha = -k - i\infty$, such that $K(0) = k$ as illustrated in Figure 2.3.

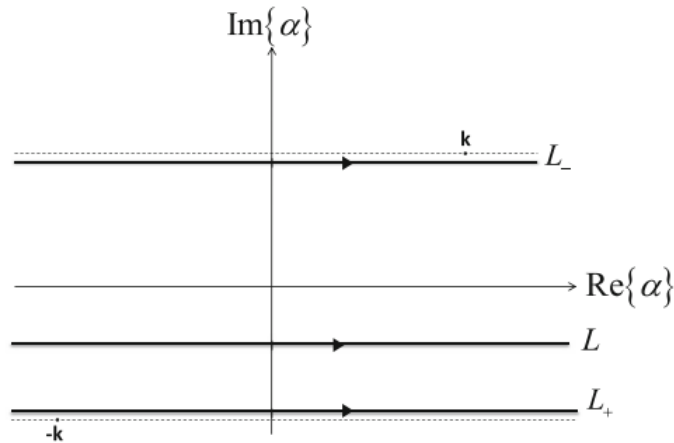


Figure 2.3: Complex α -plane.

The Fourier transform is given by

$$\mathcal{H}(\varrho, \alpha) = \mathcal{H}_+(\varrho, \alpha) + \mathcal{H}_-(\varrho, \alpha) \quad (2.47)$$

where

$$\mathcal{H}_\pm(\varrho, \alpha) = \pm \int_0^{\pm\infty} \mathcal{Y}_1(\varrho, z) e^{i\alpha z} dz \quad (2.48)$$

The functions $\mathcal{H}_+(\varrho, \alpha)$ and $\mathcal{H}_-(\varrho, \alpha)$ are unknown analytic functions, defined to be regular in the respective half-planes where $\text{Im}(\alpha) > \text{Im}(-k)$ and $\text{Im}(\alpha) < \text{Im}(k)$.

A structured method allows the determination of the general solution corresponding to equation (2.46) [Ross (2007)] :

$$\mathcal{H}(\varrho, \alpha) = \mathcal{A}(\alpha) J_1(K\varrho) + \mathcal{B}(\alpha) Y_1(K\varrho) \quad (2.49)$$

The symbols $J_1(K\varrho)$ and $Y_1(K\varrho)$ represent the standard Bessel functions of the first and second types, respectively. On the other hand, by using the boundary conditions (2.37) and (2.38), which are satisfied on the perfect electric conductor (PEC) of the coaxial waveguide, the unknown spectral coefficients $\mathcal{A}(\alpha)$ and $\mathcal{B}(\alpha)$ are determined.

First, applying the Fourier transform to the boundary condition (2.37) yields

$$\int_{-\infty}^{\infty} \mathcal{Y}_1(b, z) e^{i\alpha z} dz + b \frac{\partial}{\partial \varrho} \int_{-\infty}^{\infty} \mathcal{Y}_1(b, z) e^{i\alpha z} dz = 0 \quad (2.50)$$

$$\mathcal{H}(b, \alpha) + b \mathcal{H}'(b, \alpha) = 0 \quad (2.51)$$

Substituting (2.49) into (2.51):

$$\begin{aligned} [\mathcal{A}(\alpha) J_1(Kb) + \mathcal{B}(\alpha) Y_1(Kb)] + b \left[\mathcal{A}(\alpha) \left(K J_0(Kb) - \frac{1}{b} J_1(Kb) \right) \right. \\ \left. + \mathcal{B}(\alpha) \left(K Y_0(Kb) - \frac{1}{b} Y_1(Kb) \right) \right] = 0 \end{aligned} \quad (2.52)$$

$$\mathcal{A}(\alpha) (K J_0(Kb)) + \mathcal{B}(\alpha) (K Y_0(Kb)) = 0 \quad (2.53)$$

$$\mathcal{B}(\alpha) = -\mathcal{A}(\alpha) \frac{J_0(Kb)}{Y_0(Kb)} \quad (2.54)$$

Substituting (2.54) into (2.49), one gets the following equation;

$$\mathcal{H}(\varrho, \alpha) = \frac{\mathcal{A}(\alpha)}{Y_0(Kb)} \left[J_1(K\varrho)Y_0(Kb) - J_0(Kb)Y_1(K\varrho) \right] \quad (2.55)$$

Secondly, application of the Fourier transform to the boundary condition (2.38) yields

$$\int_{-\infty}^0 \mathcal{Y}_1(a, z) e^{i\alpha z} dz + a \frac{\partial}{\partial \varrho} \int_{-\infty}^0 \mathcal{Y}_1(a, z) e^{i\alpha z} dz = 0 \quad (2.56)$$

$$\mathcal{H}_-(a, \alpha) + a \mathcal{H}'_-(a, \alpha) = 0 \quad (2.57)$$

On the other hand, it can be written that

$$\mathcal{H}(a, \alpha) + a \mathcal{H}'(a, \alpha) = \mathcal{H}_-(a, \alpha) + a \mathcal{H}'_-(a, \alpha) + \mathcal{H}_+(a, \alpha) + a \mathcal{H}'_+(a, \alpha) \quad (2.58)$$

Substituting (2.57) into (2.58), one can obtain the following statement

$$\mathcal{H}(a, \alpha) + a \mathcal{H}'(a, \alpha) = \mathcal{H}_+(a, \alpha) + a \mathcal{H}'_+(a, \alpha) \quad (2.59)$$

In addition, substituting (2.55) into (2.59) for $\varrho = a$ yields

$$\mathcal{A}(\alpha) = \frac{[\mathcal{H}_+(a, \alpha) + a \mathcal{H}'_+(a, \alpha)] Y_0(Kb)}{Ka [J_0(Ka) Y_0(Kb) - J_0(Kb) Y_0(Ka)]} \quad (2.60)$$

Hence,

$$\mathcal{H}(\varrho, \alpha) = \mathcal{P}_+(\alpha) \frac{[J_1(K\varrho)Y_0(Kb) - J_0(Kb)Y_1(K\varrho)]}{Ka [J_0(Ka)Y_0(Kb) - J_0(Kb)Y_0(Ka)]} \quad (2.61)$$

where $\mathcal{P}_+(\alpha)$ stands for

$$\mathcal{P}_+(\alpha) = \mathcal{H}_+(a, \alpha) + a\mathcal{H}'_+(a, \alpha) \quad (2.62)$$

Similarly, the scattered field $\mathcal{Y}_2(\varrho, z)$ also obeys the Helmholtz equation in the region defined by $\varrho \in (c, a)$ and $z \in (0, \infty)$. Its Fourier transform with respect to z results in

$$\left[\frac{\partial^2}{\partial \varrho^2} + \frac{1}{\varrho} \frac{\partial}{\partial \varrho} + K^2(\alpha) - \frac{1}{\varrho^2} \right] \mathcal{G}_+(\varrho, \alpha) = -i\alpha \mathcal{F}(\varrho) \quad (2.63)$$

where the boundary condition (2.39) is considered. $\mathcal{G}_+(\varrho, \alpha)$ stands for

$$\mathcal{G}_+(\varrho, \alpha) = \int_0^\infty \mathcal{Y}_2(\varrho, z) e^{i\alpha z} dz \quad (2.64)$$

and

$$\mathcal{F}(\varrho) = \mathcal{Y}_2(\varrho, 0) \quad (2.65)$$

In order to derive the complete solution of a non-homogeneous differential equation, one needs to determine a particular solution corresponding to the non-homogeneous term and then add it to the general solution of the homogeneous equation derived from the same differential operator.

$$\mathcal{G}_+(\varrho, \alpha) = \mathcal{G}_+^H(\varrho, \alpha) + \mathcal{G}_+^P(\varrho, \alpha) \quad (2.66)$$

Here, $\mathcal{G}_+^H(\varrho, \alpha)$ and $\mathcal{G}_+^P(\varrho, \alpha)$ represent the homogeneous and particular solutions, respectively. A particular solution of (2.63) can be expressed in terms of

$$\mathcal{G}_+^P(\varrho, \alpha) = -i\alpha \int_c^a \mathcal{F}(t) \mathcal{Q}(t, \varrho, \alpha) t dt \quad (2.67)$$

following the procedure described in [Bender & Orszag (1999)]. In equation (2.67), $\mathcal{Q}(t, \varrho, \alpha)$ is the Green's function associated with this differential equation provides a

framework for formulating a particular solution to equation (2.67)

$$\left[\frac{\partial^2}{\partial \varrho^2} + \frac{1}{\varrho} \frac{\partial}{\partial \varrho} + K^2(\alpha) - \frac{1}{\varrho^2} \right] \mathcal{G}(\varrho, t, \alpha) = \frac{\delta(\varrho - t)}{t} \quad (2.68)$$

that satisfies the necessary conditions for $\varrho, t \in (c, a)$ with the associated continuity relations and the boundary conditions, which are given as follows

$$\mathcal{G}(t+0, t, \alpha) - \mathcal{G}(t-0, t, \alpha) = 0 \quad (2.69)$$

$$\frac{\partial}{\partial \varrho} \mathcal{G}(t+0, t, \alpha) - \frac{\partial}{\partial \varrho} \mathcal{G}(t-0, t, \alpha) = \frac{1}{t} \quad (2.70)$$

$$\left(\eta + \frac{1}{ikc} \right) \mathcal{G}(c, t, \alpha) + \frac{1}{ik} \frac{\partial}{\partial \varrho} \mathcal{G}(c, t, \alpha) = 0 \quad (2.71)$$

$$\mathcal{G}(a, t, \alpha) + a \frac{\partial}{\partial \varrho} \mathcal{G}(a, t, \alpha) = 0 \quad (2.72)$$

Note that the continuity relations (2.69) and (2.70) are written in the range described by $\varrho \in (c, a)$, $z \in (0, \infty)$. The general solution of this Green's function in (2.68) has the form

$$\mathcal{G}(\varrho, t, \alpha) = \begin{cases} \mathcal{C}(t)J_1(K\varrho) + \mathcal{D}(t)Y_1(K\varrho), & \varrho \in [c, t] \\ \tilde{\mathcal{C}}(t)J_1(K\varrho) + \tilde{\mathcal{D}}(t)Y_1(K\varrho), & \varrho \in [t, a] \end{cases} \quad (2.73)$$

where $\mathcal{C}(t)$, $\mathcal{D}(t)$, $\tilde{\mathcal{C}}(t)$, $\tilde{\mathcal{D}}(t)$ are the unknown spectral coefficients. Taking into account the conditions given in (2.69)-(2.72), Green's function is defined as

$$\mathcal{G}(\varrho, t, \alpha) = \frac{1}{ikK\mathfrak{E}(a, c, \alpha)} \mathcal{Q}(\varrho, t, \alpha) \quad (2.74)$$

with

$$\mathcal{Q}(\varrho, t, \alpha) = \frac{\pi}{2} \begin{cases} ikK \left[-J_1(Kt)Y_0(Ka) + J_0(Ka)Y_1(Kt) \right] \mathfrak{D}(\varrho, c, \alpha), & \varrho \in [c, t] \\ ikK \left[-J_1(K\varrho)Y_0(Ka) + J_0(Ka)Y_1(K\varrho) \right] \mathfrak{D}(t, c, \alpha), & \varrho \in [t, a] \end{cases} \quad (2.75)$$

where

$$\mathfrak{C}(\varrho, c, \alpha) = \eta \left[J_0(K\varrho)Y_1(Kc) - J_1(Kc)Y_0(K\varrho) \right] + \frac{K}{ik} \left[J_0(K\varrho)Y_0(Kc) - J_0(Kc)Y_0(K\varrho) \right] \quad (2.76)$$

and

$$\mathfrak{D}(\varrho, c, \alpha) = \eta \left[J_1(K\varrho)Y_1(Kc) - J_1(Kc)Y_1(K\varrho) \right] + \frac{K}{ik} \left[J_1(K\varrho)Y_0(Kc) - J_0(Kc)Y_1(K\varrho) \right] \quad (2.77)$$

Substituting (2.74) into (2.67) allows one to get a particular solution

$$\mathcal{G}_+^P(\varrho, \alpha) = -\frac{1}{\mathfrak{C}(a, c, \alpha)} i\alpha \int_c^a \mathcal{F}(t) \mathcal{Q}(t, \varrho, \alpha) t dt \quad (2.78)$$

On the other hand, the homogeneous solution $\mathcal{G}_+^H(\varrho, \alpha)$ is determined as

$$\mathcal{G}_+^H(\varrho, \alpha) = \frac{1}{ikK\mathfrak{C}(a, c, \alpha)} \left[\mathfrak{W}_1(\alpha)J_1(K\varrho) + \mathfrak{W}_2(\alpha)Y_1(K\varrho) \right] \quad (2.79)$$

where $\mathfrak{W}_1(\alpha)$ and $\mathfrak{W}_2(\alpha)$ are the unknown spectral coefficients. The Fourier transform of the boundary condition (2.40), with

$$\mathfrak{W}_1(\alpha) = \frac{\mathcal{P}_+(\alpha)}{Ka} \left[\eta Y_1(Kc) + \frac{K}{ik} Y_0(Kc) \right] ikK \quad (2.80)$$

and

$$\mathfrak{W}_2(\alpha) = -\mathfrak{W}_1(\alpha) \frac{[\eta J_1(Kc) + \frac{K}{ik} J_0(Kc)]}{[\eta Y_1(Kc) + \frac{K}{ik} Y_0(Kc)]} \quad (2.81)$$

Therefore,

$$\begin{aligned} \mathcal{G}_+(\varrho, \alpha) = \frac{\mathfrak{W}_1(\alpha)}{ikK\mathfrak{C}(a, c, \alpha)} & \left[J_1(K\varrho) - \frac{[\eta J_1(Kc) + \frac{K}{ik} J_0(Kc)]}{[\eta Y_1(Kc) + \frac{K}{ik} Y_0(Kc)]} Y_1(K\varrho) \right] \\ & - \frac{1}{ikK\mathfrak{C}(a, c, \alpha)} i\alpha \int_c^a \mathcal{F}(t) \mathcal{Q}(t, \varrho, \alpha) t dt \end{aligned} \quad (2.82)$$

The Fourier transform of the continuity relation (2.41) allows to write

$$\mathcal{H}_+(a, \alpha) + a\mathcal{H}'_+(a, \alpha) = \mathcal{G}_+(\varrho, \alpha) + a\mathcal{G}'_+(\varrho, \alpha) \quad (2.83)$$

and the general solution of equation (2.64) can be represented as follows

$$\begin{aligned} \mathcal{G}_+(\varrho, \alpha) = \frac{1}{ikK\mathfrak{C}(a, c, \alpha)} & \left[\frac{\mathcal{P}_+(\alpha)}{a} \mathfrak{D}(\varrho, c, \alpha) ik \right. \\ & \left. - i\alpha \int_c^a \mathcal{F}(t) \mathcal{Q}(t, \varrho, \alpha) t dt \right] \end{aligned} \quad (2.84)$$

Although the left-hand side of equation (84) is analytic throughout the upper half of the complex plane, the right-hand side fails to retain this property due to the presence of simple poles at $\alpha = \alpha_m$, corresponding to the zeros of $ikK_m\mathfrak{C}(a, c, \alpha_m)$. To maintain analyticity, these poles are eliminated by enforcing the vanishing of their residues. This gives

$$\left[\frac{\mathcal{P}_+(\alpha)}{a} \mathfrak{D}(\varrho, c, \alpha) ik - i\alpha \int_c^a \mathcal{F}(t) \mathcal{Q}(t, \varrho, \alpha) t dt \right]_{\varrho=a} = 0 \quad (2.85)$$

Since the function $\mathcal{F}(t)$ meets the conditions of absolute integrability and is consistent with Dini's criterion, it admits a representation of the Fourier-Bessel series as shown

below [Demir & Büyükaksoy (2005)]

$$\sum_{m=1}^{\infty} \mathcal{F}_m \frac{\pi}{2} \left[J_1(K_m t) Y(c, \alpha_m) - Y_1(K_m t) J(c, \alpha_m) \right] \quad (2.86)$$

This series expansion allows one to write

$$\mathcal{F}_m = \frac{1}{s_m} \int_c^a \frac{\pi}{2} \mathcal{F}(t) \left[J_1(K_m t) Y(c, \alpha_m) - Y_1(K_m t) J(c, \alpha_m) \right] t dt \quad (2.87)$$

where s_m is the norm of the relevant series expansion determined using the orthogonality integral given by:

$$s_m = \frac{\pi^2}{4} \int_c^a \mathcal{F}(t) \left[J_1(K_m t) Y(c, \alpha_m) - Y_1(K_m t) J(c, \alpha_m) \right]^2 t dt \quad (2.88)$$

Equation (2.88) is evaluated as:

$$s_m = \frac{\pi^2}{4} \left[\frac{a^2}{2} [ik\mathfrak{D}(a, c, \alpha_m)]^2 + \frac{2k^2\eta^2c - 2cK_m^2 - 4ik\eta}{\pi^2 K_m^2 c} \right] \quad (2.89)$$

Then

$$\mathcal{F}_m = -ik \frac{\pi}{2} \frac{\mathcal{P}_+(\alpha_m) \mathfrak{D}(a, c, \alpha_m)}{i\alpha_m s_m} \quad (2.90)$$

The Fourier transform of continuity relation (2.41) allows to write the following

$$\mathcal{H}_+(a, \alpha) - \frac{1}{ia(k + \alpha)} = \mathcal{G}_+(a, \alpha) \quad (2.91)$$

Making use of (2.49) and (2.63) can be written as follows:

$$\begin{aligned} -\frac{1}{ia(k + \alpha)} + \frac{\mathcal{P}_+(\alpha)}{K^2} \frac{2}{\pi a^2} \frac{\mathfrak{E}(b, c, \alpha)}{\mathfrak{E}(a, c, \alpha) [J_0(Ka)Y_0(Kb) - J_0(Kb)Y_0(Ka)]} \\ - \mathcal{H}_-(a, \alpha) - i\alpha \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{ik\mathcal{F}_m \mathfrak{D}(a, c, \alpha_m)}{(\alpha^2 - \alpha_m^2)} \frac{\pi^2}{4} = 0 \end{aligned} \quad (2.92)$$

and then,

$$\frac{\mathcal{P}_+(\alpha)}{K^2} \frac{2}{\pi a^2 \mathcal{M}(\alpha)} - \mathcal{H}_-(a, \alpha) = \frac{1}{ia(k + \alpha)} - i\alpha \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{ik \mathcal{F}_m \mathfrak{D}(a, c, \alpha_m)}{(\alpha^2 - \alpha_m^2)} \frac{\pi^2}{4} \quad (2.93)$$

with

$$\mathcal{M}(\alpha) = \frac{2}{\pi K a} \frac{K \mathfrak{C}(a, c, \alpha) [J_0(Ka)Y_0(Kb) - J_0(Kb)Y_0(Ka)]}{K^2 a \mathfrak{C}(b, c, \alpha)} \quad (2.94)$$

The equation (2.93) represents the modified Wiener-Hopf equation that needs to be solved. It is important to understand that $\mathcal{M}(\alpha)$ serves as the kernel function of the Wiener-Hopf equation, which describes the impedance coated step discontinuity present on the inner wall of the coaxial waveguide.

2.2.2. The Solution of Wiener-Hopf Equation

According to the standard Wiener-Hopf procedure, the kernel $\mathcal{M}(\alpha)$ is factorized into $\mathcal{M}_+(\alpha)$ and $\mathcal{M}_-(\alpha)$. This factorization is performed according to the procedures outlined in [Rawlins (1995)] as follows:

$$\mathcal{M}(\alpha) = \mathcal{M}_+(\alpha) \mathcal{M}_-(\alpha) \quad \text{and} \quad \mathcal{M}_-(-\alpha) = \mathcal{M}_+(\alpha) \quad (2.95)$$

and

$$\mathcal{M}_+(\alpha) = \sqrt{\frac{2}{\pi K a} \frac{\mathfrak{C}_+(a, c, \alpha) [J_0(Ka)Y_0(Kb) - J_0(Kb)Y_0(Ka)]}{K \mathfrak{C}_+(b, c, \alpha)}} \times e^{P(\alpha)} \prod_{m=1}^{\infty} \left(1 + \frac{\alpha}{\mu_m}\right) e^{\frac{\alpha}{\kappa_m}} \quad (2.96)$$

with

$$\mathcal{M}_+(\alpha) = \sqrt{M(0)} \prod_{m=1}^{\infty} \frac{(1 + \frac{\alpha}{\mu_m})(1 + \frac{\alpha}{\gamma_m^1})(1 + \frac{\alpha}{\kappa_m})}{(1 + \frac{\alpha}{\xi_m})(1 + \frac{\alpha}{\gamma_m^2})} \exp(\frac{i\alpha(b-a)}{m\pi}) \times \exp(\frac{-2\alpha}{\pi i}(b-a)[1 - C + \log(\frac{2\pi i}{k(b-a)})]) \quad (2.97)$$

where $\mathfrak{C}_+(b, c, \alpha)$ stands for the factorization of $\mathfrak{C}(b, c, \alpha)$ which is done by following the procedure where C in (2.97) is the Euler's constant given with $C = 0,57721566...$

Then, equation (2.93) becomes

$$\begin{aligned} \frac{\mathcal{P}_+(\alpha)}{(k+\alpha)} \frac{2}{\pi} \frac{1}{a\mathcal{M}_+(\alpha)} - \mathcal{H}_-(a, \alpha) ia\mathcal{M}_-(\alpha)(k-\alpha) \\ = \frac{\mathcal{M}_-(\alpha)(k-\alpha)}{(k+\alpha)} - \alpha\mathcal{M}_-(\alpha)(k-\alpha)a \sum_{m=1}^{\infty} \frac{ik\mathcal{F}_m\mathfrak{D}(a, c, \alpha_m)}{(\alpha^2 - \alpha_m^2)} \frac{\pi}{2} \end{aligned} \quad (2.98)$$

The determination of the unknown function $\mathcal{P}_+(\alpha)$ requires expressing the left-hand side of equation (2.98) as the sum of two functions regular in the regions $\mathcal{B}_+$ and $\mathcal{B}_-$. Note that $\mathcal{B}_+$ and $\mathcal{B}_-$ demonstrate the regions described by $Im(\alpha) > Im(k)$ and $Im(\alpha) < Im(k)$, respectively. To eliminate the singularities on the right-hand side of (2.98), a decomposition is performed by applying the Cauchy Residue Theorem [Williams & Meylan (2011)]. The second phase in the Wiener-Hopf solution process involves applying the decomposition method [Greenwood (1975)]. Multiply each term of (2.98) by $i\sqrt{\frac{\pi}{2}}a\mathcal{M}_-(\alpha)(k-\alpha)$:

$$\begin{aligned} -\sqrt{\frac{\pi}{2}} \frac{\mathcal{M}_-(\alpha)(k-\alpha)}{(k+\alpha)} + \sqrt{\frac{2}{\pi}} \frac{\mathcal{P}_+(\alpha)}{(k+\alpha)} \frac{1}{a\mathcal{M}_+(\alpha)} \\ - ia\sqrt{\frac{\pi}{2}} \mathcal{H}_-(a, \alpha)\mathcal{M}_-(\alpha)(k-\alpha) \\ + a\alpha\mathcal{M}_-(\alpha)(k-\alpha)\sqrt{\frac{\pi}{2}} \sum_{m=1}^{\infty} \frac{ik\mathcal{F}_m\mathfrak{D}(a, c, \alpha_m)}{(\alpha^2 - \alpha_m^2)} \frac{\pi}{2} = 0 \end{aligned} \quad (2.99)$$

The non-analytic components of equation (2.99) are decomposed into a sum of functions analytic in the upper and lower halves of the complex plane. Firstly, if we consider

the equation $d_1(\alpha)$ where $\alpha = -k$ is the pole point

$$d_1(\alpha) = -\sqrt{\frac{\pi}{2}} \frac{\mathcal{M}_-(\alpha)(k - \alpha)}{(k + \alpha)} \quad (2.100)$$

$d_1(\alpha) = d_1^+(\alpha) + d_1^-(\alpha)$ are obtained

$$d_1^+(\alpha) = \frac{1}{2\pi i} \int_{L_+} \sqrt{\frac{\pi}{2}} \frac{\mathcal{M}_-(v)(k - v)}{(k + v)(v - \alpha)} dv = \sqrt{\frac{\pi}{2}} \frac{\mathcal{M}_+(k)2k}{(-k - \alpha)} \quad (2.101)$$

$$d_1^-(\alpha) = -\sqrt{\frac{\pi}{2}} \frac{\mathcal{M}_-(\alpha)(k - \alpha)}{(k + \alpha)} + \sqrt{\frac{\pi}{2}} \frac{\mathcal{M}_+(k)2k}{(k + \alpha)} \quad (2.102)$$

If the residue of the pole point $\alpha = -\alpha_m$ is considered in the decomposition for the other term,

$$d_2(\alpha) = a\alpha\mathcal{M}_-(\alpha)(k - \alpha) \sqrt{\frac{\pi}{2}} \sum_{m=1}^{\infty} \frac{ik\mathcal{F}_m\mathfrak{D}(a, c, \alpha_m)}{(\alpha^2 - \alpha_m^2)} \frac{\pi}{2} \quad (2.103)$$

$d_2(\alpha) = d_2^+(\alpha) + d_2^-(\alpha)$ are found

$$\begin{aligned} d_2^+(\alpha) &= \frac{1}{2\pi i} \int_{L_+} \sum_{m=1}^{\infty} \frac{av\mathcal{M}_-(v)(k - v) \sqrt{\frac{\pi}{2}} \sum_{m=1}^{\infty} ik\mathcal{F}_m\mathfrak{D}(a, c, \alpha_m)}{(v^2 - \alpha_m^2)(v - \alpha)} \frac{\pi}{2} dv \\ &= \sqrt{\frac{\pi}{2}} a \sum_{m=1}^{\infty} \frac{\mathcal{M}_+(\alpha_m)(k + \alpha_m) ik\mathcal{F}_m\mathfrak{D}(a, c, \alpha_m)}{2(\alpha_m + \alpha)} \frac{\pi}{2} \end{aligned} \quad (2.104)$$

$$d_2^-(\alpha) = d_2(\alpha) - d_2^+(\alpha) \quad (2.105)$$

According to this, we get

$$\begin{aligned} &\frac{\mathcal{M}_+(k)2k}{(-k - \alpha)} \sqrt{\frac{\pi}{2}} + d_1^-(\alpha) + \frac{\mathcal{P}_+(\alpha)}{(k + \alpha)} \sqrt{\frac{2}{\pi}} \frac{1}{a\mathcal{M}_+(\alpha)} - ia\sqrt{\frac{\pi}{2}} \mathcal{H}_-(a, \alpha) \mathcal{M}_-(\alpha)(k - \alpha) \\ &+ \sqrt{\frac{\pi}{2}} a \sum_{m=1}^{\infty} \frac{\mathcal{M}_+(\alpha_m)(k + \alpha_m) ik\mathcal{F}_m\mathfrak{D}(a, c, \alpha_m)}{2(\alpha_m + \alpha)} \frac{\pi}{2} + d_2^-(\alpha) = 0 \end{aligned} \quad (2.106)$$

Hence

$$\begin{aligned}
& \sqrt{\frac{2}{\pi}} \frac{\mathcal{P}_+(\alpha)}{(k+\alpha)} \frac{1}{a\mathcal{M}_+(\alpha)} + \sqrt{\frac{\pi}{2}} a \sum_{m=1}^{\infty} \frac{\mathcal{M}_+(\alpha_m)(k+\alpha_m)ik\mathcal{F}_m\mathfrak{D}(a, c, \alpha_m)}{2(\alpha_m + \alpha)} \frac{\pi}{2} \\
& \quad + \sqrt{\frac{\pi}{2}} \frac{\mathcal{M}_+(k)2k}{(-k-\alpha)} \\
& = ia\sqrt{\frac{\pi}{2}} \mathcal{H}_-(a, \alpha)\mathcal{M}_-(\alpha)(k-\alpha) + \sqrt{\frac{\pi}{2}} \frac{\mathcal{M}_-(\alpha)(k-\alpha)}{(k+\alpha)} - \sqrt{\frac{\pi}{2}} \frac{\mathcal{M}_+(k)2k}{(k+\alpha)} \\
& \quad - a\alpha \sum_{m=1}^{\infty} \frac{ik\mathcal{F}_m\mathfrak{D}(a, c, \alpha_m)}{(\alpha^2 - \alpha_m^2)} \frac{\pi}{2} \left[\frac{\mathcal{M}_-(\alpha)(k-\alpha)}{(\alpha^2 - \alpha_m^2)} - \frac{\mathcal{M}_+(\alpha_m)(k+\alpha_m)}{2(\alpha_m + \alpha)} \right] \quad (2.107)
\end{aligned}$$

The left-hand side is the sum of terms which are regular on region $\mathcal{B}_+$, while the right-hand side is the sum of terms which are regular on $\mathcal{B}_-$. This equation can be separated to two different equations symbolized by $\xi_+(\alpha)$ and $\xi_-(\alpha)$.

$$\begin{aligned}
\xi_+(\alpha) &= \frac{\mathcal{P}_+(\alpha)}{(k+\alpha)} \frac{1}{a\mathcal{M}_+(\alpha)} + \frac{\mathcal{M}_+(k)2k}{(-k-\alpha)} \\
& \quad + \sum_{m=1}^{\infty} a \frac{\mathcal{M}_+(\alpha_m)(k+\alpha_m)ik\mathcal{F}_m\mathfrak{D}(a, c, \alpha_m)}{2(\alpha_m + \alpha)} \frac{\pi}{2}, \\
& \qquad \qquad \qquad \alpha \in \mathcal{B}_+ \quad (2.108)
\end{aligned}$$

and

$$\begin{aligned}
\xi_-(\alpha) &= ia\mathcal{H}_-(a, \alpha)\mathcal{M}_-(\alpha)(k-\alpha) + \frac{\mathcal{M}_-(\alpha)(k-\alpha)}{(k+\alpha)} - \frac{\mathcal{M}_+(k)2k}{(k+\alpha)} \\
& \quad - a\alpha \sum_{m=1}^{\infty} \frac{ik\mathcal{F}_m\mathfrak{D}(a, c, \alpha_m)}{(\alpha^2 - \alpha_m^2)} \frac{\pi}{2} \left[\frac{\mathcal{M}_-(\alpha)(k-\alpha)}{(\alpha^2 - \alpha_m^2)} - \frac{\mathcal{M}_+(\alpha_m)(k+\alpha_m)}{2(\alpha_m + \alpha)} \right], \\
& \qquad \qquad \qquad \alpha \in \mathcal{B}_- \quad (2.109)
\end{aligned}$$

which each of them is analytic on the related domains. At this point, the equality $\xi_+(\alpha) = \xi_-(\alpha)$ is valid on $\mathcal{B}_+ \cap \mathcal{B}_-$. Then, it can be said that, $\xi_+(\alpha)$ is the analytical continuation of $\xi_-(\alpha)$ to domain $\mathcal{B}_+$, and vice versa. In addition, it is required to evaluate the asymptotic behavior of the entire function $\xi(\alpha)$. In accordance with Liouville's theorem, $\xi(\alpha)$ is 0 for all α -values, since $\xi(\alpha)$ is asymptotically equivalent

to 0 as $\alpha \rightarrow \infty$, and, it allows one to write $\mathcal{P}_+(\alpha)$ as follows:

$$\mathcal{P}_+(\alpha) = \frac{\pi a(k + \alpha)\mathcal{M}_+(\alpha)}{2i} \left(\frac{\mathcal{M}_+(k)2k}{(k + \alpha)} - \sum_{m=1}^{\infty} \frac{a\pi}{2} \frac{\mathcal{M}_+(\alpha_m)(k + \alpha_m)ik\mathcal{F}_m\mathfrak{D}(a, c, \alpha_m)}{2(\alpha_m + \alpha)} \right) \quad (2.110)$$

2.2.3. Scattered Field Analysis

The scattered field $\mathcal{Y}_1(\varrho, z)$ can be reconstructed by performing the inverse Fourier transform of the corresponding expression of the spectral domain, which involves evaluating a contour integral in the complex plane.

$$\mathcal{Y}_1(\varrho, z) = \frac{1}{2\pi} \int_L \mathcal{H}(\varrho, \alpha) e^{-i\alpha z} d\alpha \quad (2.111)$$

Here the curve L lies in the series $Im(-k) < Im(\alpha) < Im(k)$. Due to the term $e^{-i\alpha z}$ it is necessary to close the curve from below. The poles in this region are $\alpha = -k$ from the function K^2 and $\alpha = -\alpha_m$ from $ikK\mathfrak{C}(b, c, \alpha)$. The residues of the existing pole points are calculated by considering (2.63) and (2.110), the above integral becomes

$$\begin{aligned} \mathcal{Y}_1(\varrho, z) = & \frac{1}{2\pi} \int_L \frac{\pi a(k + \alpha)\mathcal{M}_+(\alpha)}{2i} \left(\frac{\mathcal{M}_+(k)2k}{(k + \alpha)} \right. \\ & \left. - a \frac{\pi}{2} \sum_{m=1}^{\infty} \frac{\mathcal{M}_+(\alpha_m)(k + \alpha_m)ik\mathcal{F}_m\mathfrak{D}(a, c, \alpha_m)}{2(\alpha_m + \alpha)} \right) \\ & \times \frac{[J_1(K\varrho)Y_0(Kb) - J_0(Kb)Y_1(K\varrho)]K}{[J_0(Ka)Y_0(Kb) - J_0(Kb)Y_0(Ka)]K^2a} e^{-i\alpha z} d\alpha \quad (2.112) \end{aligned}$$

or

$$\begin{aligned} \mathcal{Y}_1(\varrho, z) = & \frac{1}{4i} \int_L \frac{(k + \alpha)\mathfrak{C}(a, c, \alpha)ikK}{\mathcal{M}_-(\alpha)} \left(\frac{\mathcal{M}_+(k)2k}{(k + \alpha)} \right. \\ & \left. - a \frac{\pi}{2} \sum_{m=1}^{\infty} \frac{\mathcal{M}_+(\alpha_m)(k + \alpha_m)ik\mathcal{F}_m\mathfrak{D}(a, c, \alpha_m)}{2(\alpha_m + \alpha)} \right) \\ & \times \frac{[J_1(K\varrho)Y_0(Kb) - J_0(Kb)Y_1(K\varrho)]K}{K^2\mathfrak{C}(b, c, \alpha)ikK} e^{-i\alpha z} d\alpha \quad (2.113) \end{aligned}$$

Hence, the transmission coefficient is

$$\mathcal{T}_0 = \frac{-i\pi}{2} [\mathcal{H}_+(a, -k) + a\mathcal{H}'_+(a, -k)] \quad (2.114)$$

Here, $\mathcal{T}_0$ represents the transmission coefficient corresponding to the dominant TEM mode.

To analyze the component of the field reflected into the original region $z < 0$, when the aforementioned L curve is closed from above in a similar way as above, the zero of K^2 in the upper half region is $\alpha = k$ and the zero of $[J_0(Ka)Y_0(Kb) - J_0(Kb)Y_0(Ka)]$ is $\alpha = \mu_n$.

$$\begin{aligned} \mathcal{Y}_1(\varrho, z) = & \frac{1}{4i} \int_L (k + \alpha) M_+(\alpha) \left(\frac{\mathcal{M}_+(k) 2k}{(k + \alpha)} \right. \\ & \left. - a \frac{\pi}{2} \sum_{m=1}^{\infty} \frac{\mathcal{M}_+(\alpha_m) (k + \alpha_m) i k \mathcal{F}_m \mathfrak{D}(a, c, \alpha_m)}{2(\alpha_m + \alpha)} \right) \\ & \times \frac{[J_1(K\varrho)Y_0(Kb) - J_0(Kb)Y_1(K\varrho)]K}{K^2 \mathfrak{C}(b, c, \alpha) i k K} e^{-i\alpha z} d\alpha \quad (2.115) \end{aligned}$$

By applying the residue theorem, the integral given in (2.115) may be represented as a contour integral over a closed contour formed by the infinite semicircular arc C_R in the upper half of the complex plane, together with the real-axis contour L, as illustrated below:

$$\int_L \frac{2}{\pi a} \mathfrak{g}(\varrho, \alpha) e^{-i\alpha z} d\varrho + \int_{C_R} \frac{2}{\pi a} \mathfrak{g}(\varrho, \alpha) e^{-i\alpha z} d\varrho = 2\pi i \sum_{n=1}^{\infty} \text{Re}z(\alpha_n) \quad (2.116)$$

where $\mathfrak{g}(\varrho, \alpha)$ is the integrand of (2.115). By making use of Jordan's lemma, it can be easily shown that C_R contour integral in (2.117) satisfies

$$\lim_{R \rightarrow \infty} \int_{C_R} \mathfrak{g}(\varrho, \alpha) e^{-i\alpha z} d\varrho \rightarrow 0 \quad (2.117)$$

for

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{2}{\pi a} \mathfrak{g}(\varrho, \alpha) e^{-i\alpha\alpha} d\varrho \rightarrow 0 \quad (2.118)$$

Therefore, the integral along the real axis can be evaluated as the sum of the complex residues enclosed by the chosen contour, and is given by:

$$\mathcal{Y}_1(\varrho, z) = \frac{1}{2\pi} \int_L \frac{2}{\pi a} \mathfrak{g}(\varrho, \alpha) e^{-i\alpha z} d\varrho = i \sum_{n=1}^{\infty} \text{Rez}(\alpha_n) \quad (2.119)$$

The right-hand side of (2.119) is evaluated by

$$\begin{aligned} i \sum_{n=1}^{\infty} \text{Rez}(\alpha_n) &= \lim_{\alpha \rightarrow \alpha_n} (\alpha - \alpha_n) \frac{1}{4i} (k + \alpha) M_+(\alpha) \left(\frac{\mathcal{M}_+(k) 2k}{(k + \alpha)} \right. \\ &\quad \left. - a \frac{\pi}{2} \sum_{m=1}^{\infty} \frac{\mathcal{M}_+(\alpha_m) (k + \alpha_m) i k \mathcal{F}_m \mathfrak{D}(a, c, \alpha_m)}{2(\alpha_m + \alpha)} \right) \\ &\quad \times \frac{[J_1(K\varrho)Y_0(Kb) - J_0(Kb)Y_1(K\varrho)]K}{K^2 \mathfrak{C}(b, c, \alpha) i k K} e^{-i\alpha z} \end{aligned} \quad (2.120)$$

or

$$\begin{aligned} i \sum_{n=1}^{\infty} \text{Rez}(\alpha_n) &= \lim_{\alpha \rightarrow k} (\alpha - k) \\ &\quad \times \frac{(k + \alpha) M_+(\alpha) [J_1(K\varrho)Y_0(Kb) - J_0(Kb)Y_1(K\varrho)] K e^{-i\alpha z}}{(k - \alpha) [J_0(Ka)Y_0(Kb) - J_0(Kb)Y_0(Ka)] (k + \alpha)} \\ &\quad \times \left(\frac{\mathcal{M}_+(k) 2k}{(k + \alpha)} - a \frac{\pi}{2} \sum_{m=1}^{\infty} \frac{\mathcal{M}_+(\alpha_m) (k + \alpha_m) i k \mathcal{F}_m \mathfrak{D}(a, c, \alpha_m)}{2(\alpha_m + \alpha)} \right) \end{aligned} \quad (2.121)$$

Defining the reflected field as

$$\mathcal{R}_0 = \frac{-i\pi M_+(k)}{2i \ln(b/a)} \left(\mathcal{M}_+(k) - a \frac{\pi}{2} \sum_{m=1}^{\infty} \mathcal{M}_+(\alpha_m) i k \mathcal{F}_m \mathfrak{D}(a, c, \alpha_m) \right) \quad (2.122)$$

Here, $\mathcal{R}_0$ represents the reflection coefficient corresponding to the dominant TEM mode, which characterizes the reflected portion of the incident wave in the coaxial waveguide.

3. NUMERICAL RESULTS

This chapter evaluates and compares the results obtained through both Mode-Matching and Wiener-Hopf techniques numerically. It also involves constructing the systems of linear algebraic equations, as discussed in Chapter 2.

The Figure 3.1 shows the variation of the reflection coefficient with respect to the truncation number (N) at a fixed frequency of 100 MHz. It can be observed that increasing the truncation number beyond N=10 has a negligible effect on the accuracy of the reflection coefficient.

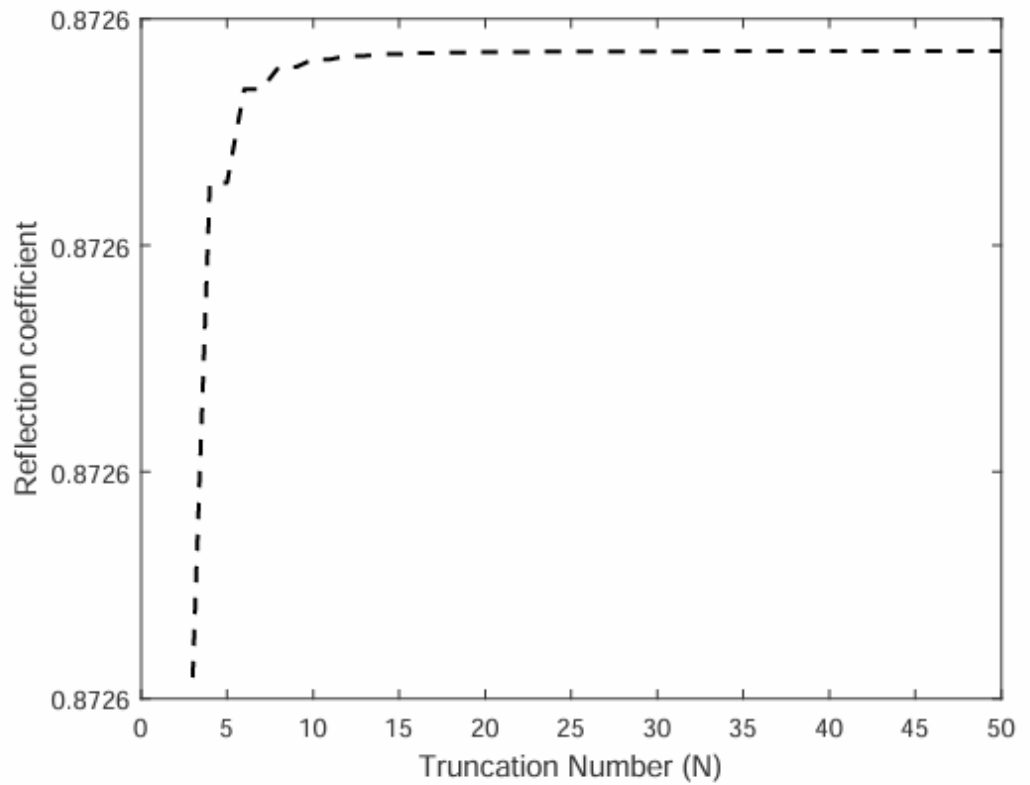


Figure 3.1: Reflection coefficient with respect to truncation number (N).

The Figure 3.2 illustrates the variation of the reflection coefficient with respect to the truncation number N at a frequency of 1 GHz. Around $N=10$, the curve stabilizes and reaches a nearly constant value, indicating numerical convergence. Just like in the previous case (100 MHz), increasing the truncation number beyond this point does not significantly affect the result.

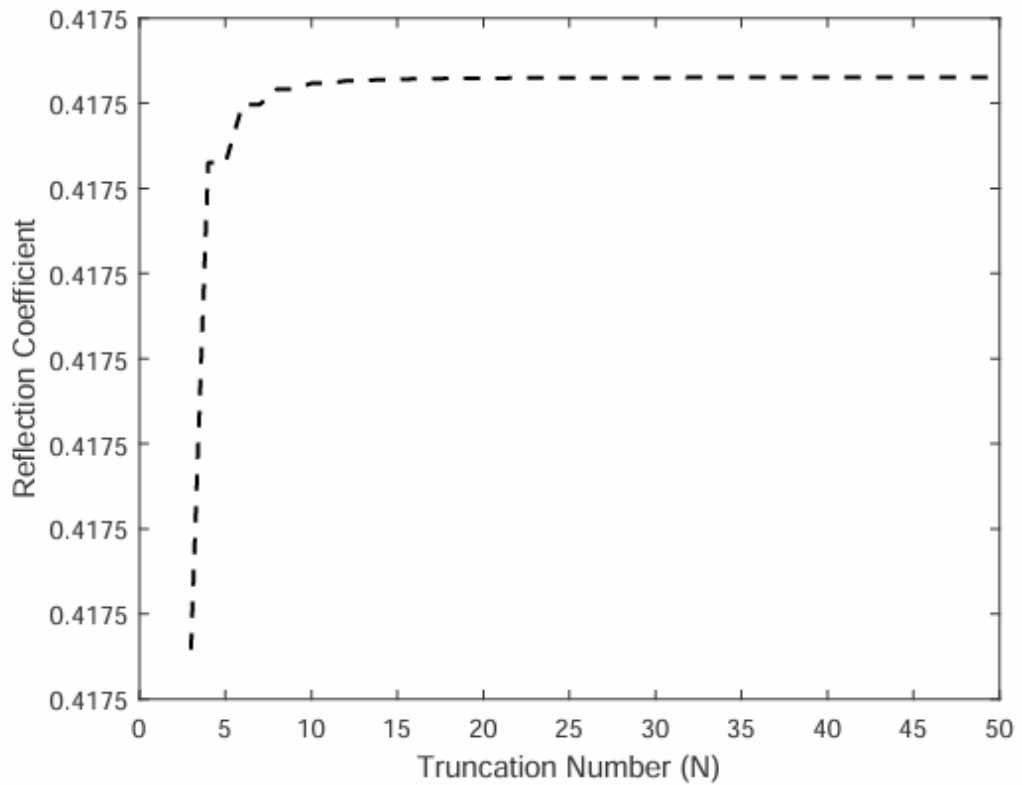


Figure 3.2: Reflection coefficient with respect to truncation number (N).

The Figure 3.4 illustrates the variation of the transmission coefficient with respect to the truncation number (N). As seen, the transmission coefficient converges as the truncation number increases. Initially, for low values of N , there is a noticeable change in the coefficient. However, after approximately $N=10$, the transmission coefficient stabilizes and exhibits negligible variation, indicating that the solution has effectively converged.

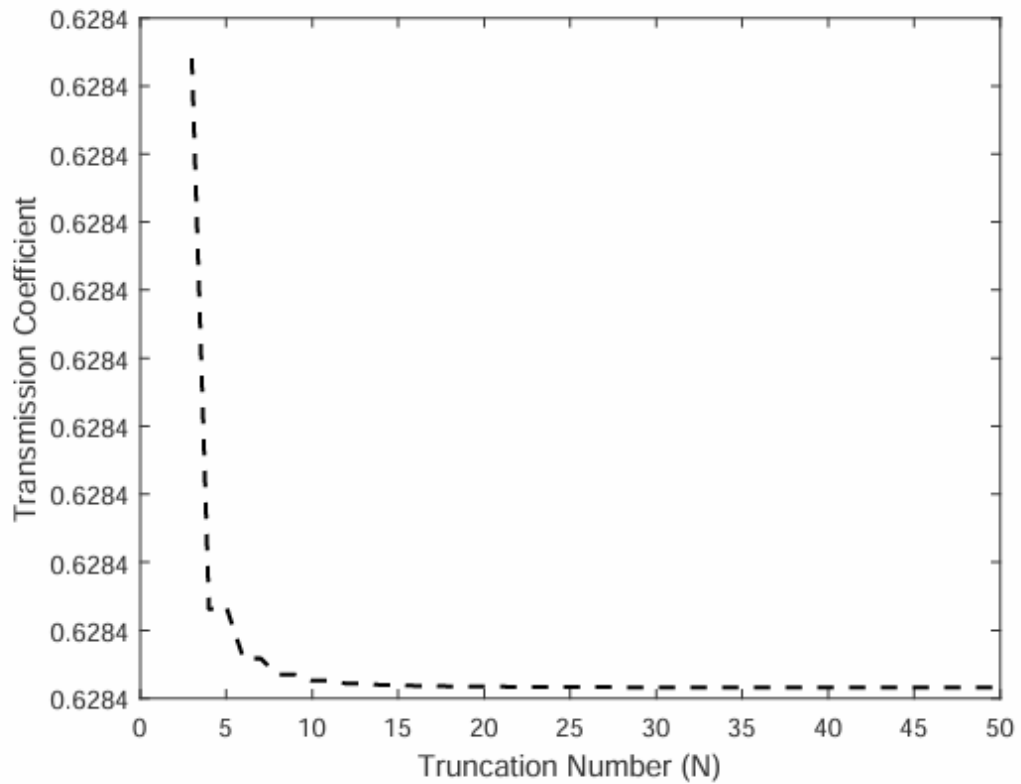


Figure 3.4: Transmission coefficient with respect to truncation number (N).

The analysis of the problem involves solving an infinite system of linear algebraic equations. Based on the observations from Figures 3.1 to 3.4, the truncation number is selected as $N=50$.

The Figure 3.5 depicts the variation of the reflection coefficient amplitude on the geometric parameter ratio b/c . It is observed that the reflection coefficient amplitude decreases with increasing frequency. Moreover, the point corresponding to the largest b/c ratio aligns with the maximum reduction in amplitude.

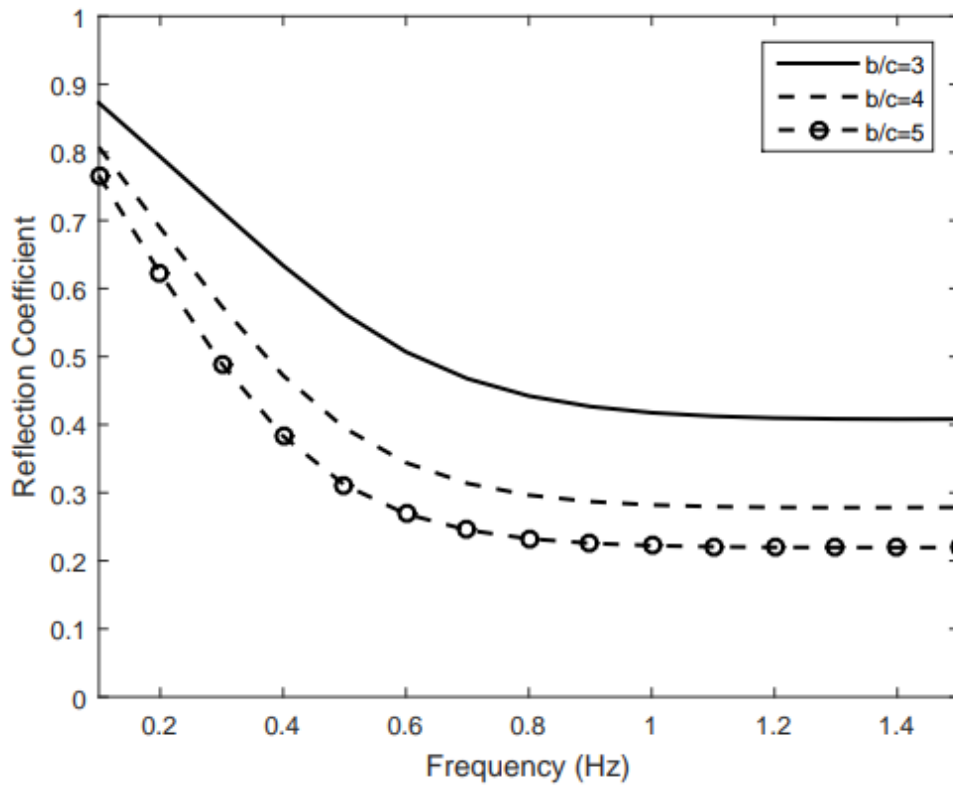


Figure 3.5: Reflection coefficient with respect to frequency for different b/c 's.

The Figure 3.6 presents the variation of the transmission coefficient magnitude with respect to the wave number for different values of the b/c ratio. The data reveal that as the frequency increases, the transmission coefficient amplitude correspondingly rises. The maximum amplitude peak is observed at the highest b/c ratio, underscoring the significant impact of geometrical parameters on the transmission behavior of the coaxial waveguide.

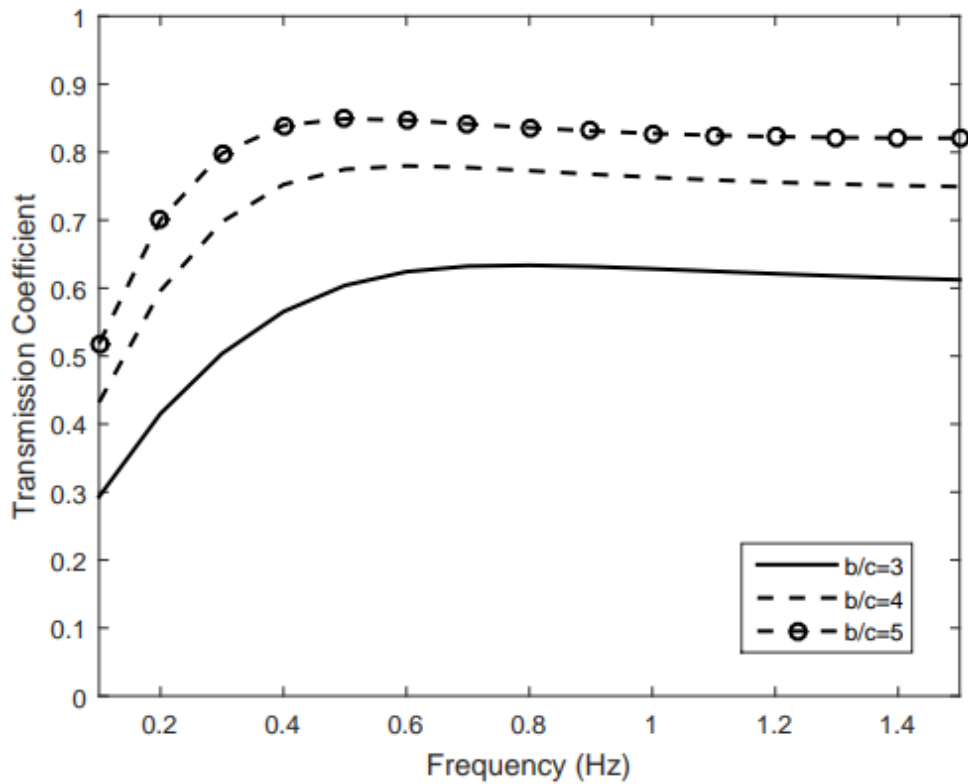


Figure 3.6: Transmission coefficient with respect to frequency for different b/c 's.

When evaluated together, these two figures demonstrate that the impedance coating provides an efficient transition medium by simultaneously reducing reflection and enhancing transmission.

The Figure 3.7 illustrates the variation of the reflection coefficient with respect to frequency for different complex impedance values. As the frequency increases, the reflection coefficient decreases. Similarly, as the value of η increases, the reflection coefficient also decreases. In particular, it can be stated that increasing the real part of η causes the waves to be reflected less.

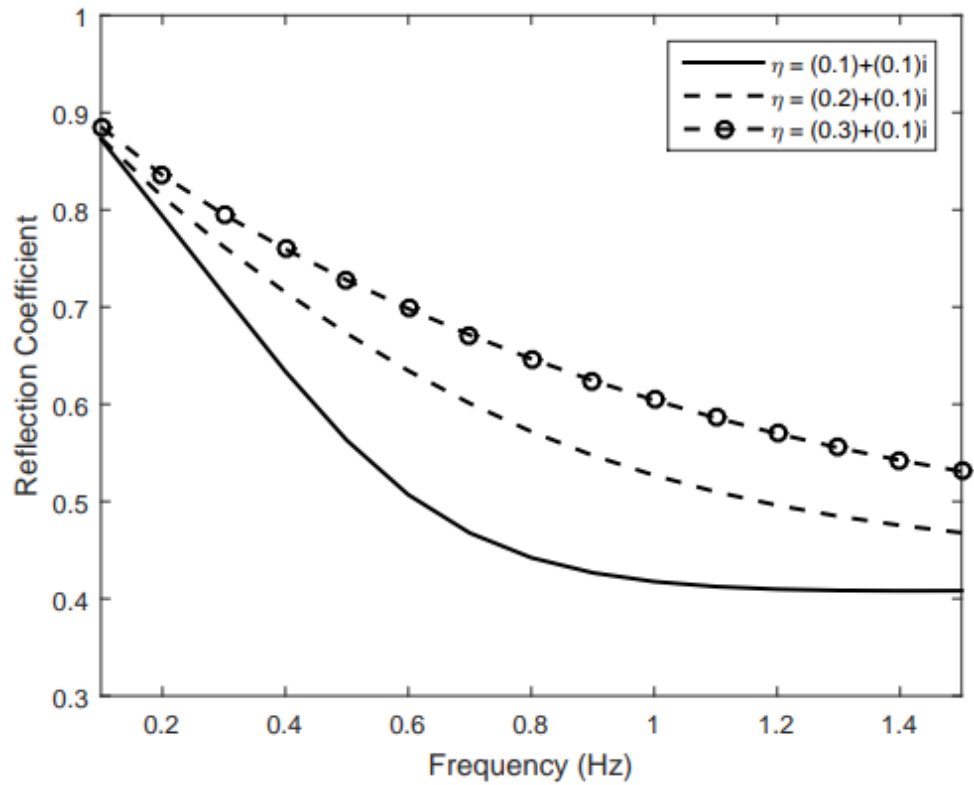


Figure 3.7: Reflection coefficient with respect to frequency for different η 's.

The Figure 3.8 shows the variation of the transmission coefficient with respect to frequency for different complex impedance values. As the frequency increases, the transmission coefficient increases. However, as the value of η increases, the transmission decreases slightly. It is observed that increasing the real part of η reduces the amount of transmitted energy.

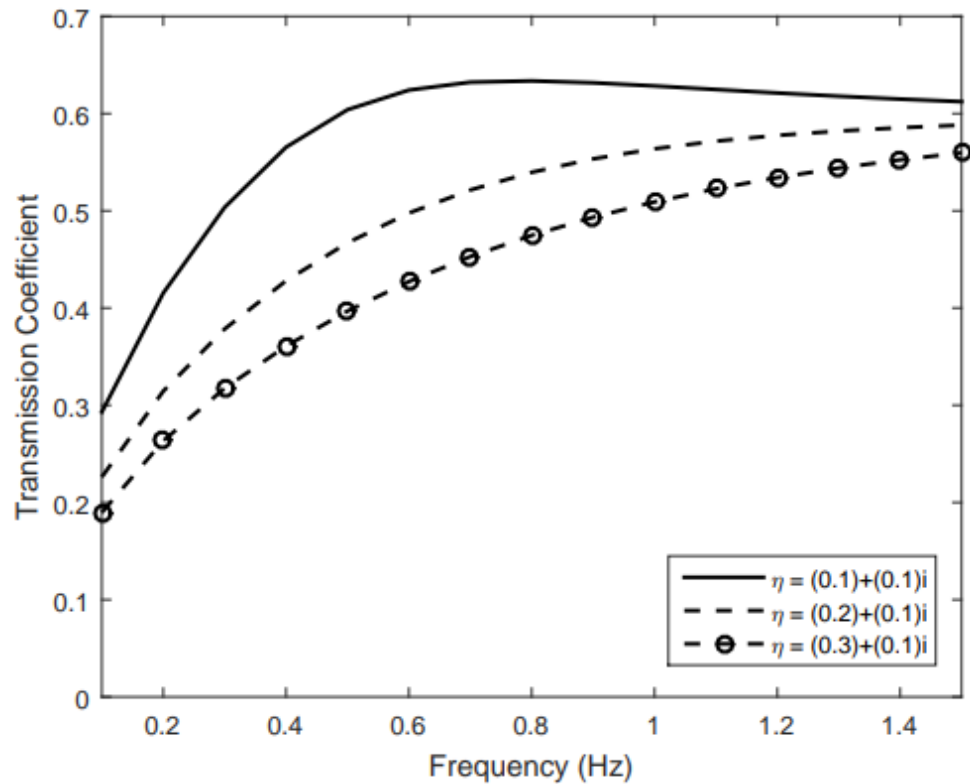


Figure 3.8: Transmission coefficient with respect to frequency for different η 's.

When evaluated together with the previous figure, it becomes evident that surface impedance affects not only the reflection but also significantly influences the transmission performance of the system.

Afterwards, the Figure 3.9 compares the reflection coefficient versus frequency obtained via the Mode-Matching and Wiener–Hopf methods. It is evident that there is a strong agreement between the results obtained from the two approaches. Both techniques consistently show a decreasing trend in reflection with increasing frequency. The close agreement between the results validates the accuracy and reliability of the analytical approaches employed in this study.

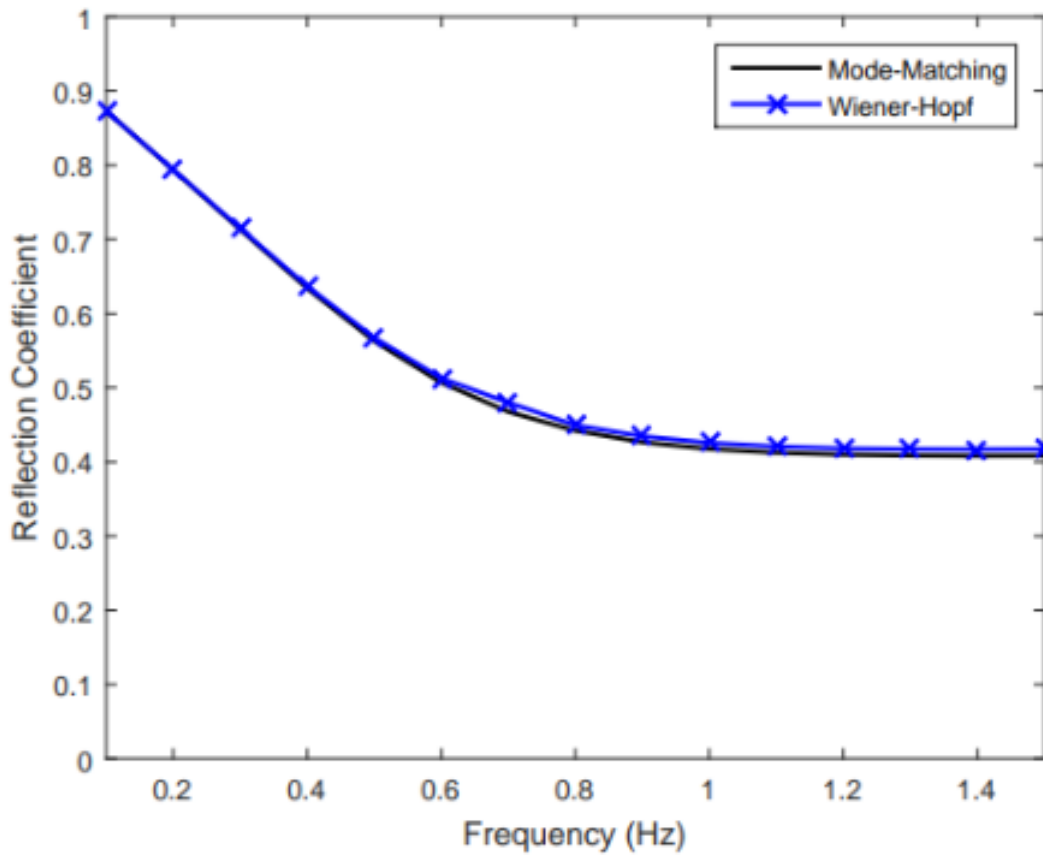


Figure 3.9: Comparison of reflection coefficient with respect to frequency.

Together, these graphical findings provide a clear understanding of the physical mechanisms governing wave behavior in impedance coated coaxial structures and lay the groundwork for further investigation of more complex discontinuities and boundary conditions.

4. CONCLUSIONS

The thesis focuses on the propagation of TEM waves in a coaxial waveguide that includes a step discontinuity with an impedance coating on the inner conductor. The analysis is carried out using the Mode-Matching and Wiener-Hopf methods, known for their high accuracy in frequency-domain solutions.

At the initial stage of the analysis, the step discontinuity with an impedance coating on the inner surface is reduced to a two-dimensional formulation by taking advantage of the axial symmetry of the structure. Subsequently, the Mode-Matching technique is applied to examine the geometry. The method results in a set of equations that need to be solved concurrently, which involves truncating the equations using a specified truncation number denoted as $\mathcal{S}$ and $\mathcal{T}$. Finally, explicit expressions for each equation are derived at the conclusion of the analysis.

In the second stage, the Wiener-Hopf technique is applied by formulating the problem in the Fourier transform domain, taking into account the Helmholtz equation, the associated boundary conditions, and the continuity requirements, ultimately leading to the derivation of a Modified Wiener-Hopf Equation (MWHE). By employing this method, the influence of an impedance-coated groove located on the inner surface of the coaxial waveguide is analyzed. In the final stage of the study, the resulting algebraic system is tackled using a numerical scheme to approximate the scattering coefficients.

In the third stage, the computational aspect of the study is addressed by developing a MATLAB code based on the approximate expressions and governing equations specific to the problem and each applied method. Subsequently, numerical results are visualized through the generation of related plots. Considering the results presented in Figures 3.1-3.4, it can be concluded that increasing the truncation number beyond a certain threshold ($N = 50$ in this study) yields negligible improvement in accuracy, while variations in complex surface impedance significantly influence the reflection behavior, particularly at lower frequencies. In the Figures 3.5-3.9, the combined effect

of geometry and surface impedance enables the design of optimized waveguide structures with controlled reflection and transmission characteristics, which is critical for applications in microwave, radar, and communication systems. These figures clearly demonstrate a high level of consistency between the Wiener–Hopf and Mode-Matching techniques, confirming the validity of both approaches.

These results establish a solid understanding of the impact of a step discontinuity on both the inner and outer walls of a coaxial waveguide on the propagation of TEM waves. Moreover, the comparison between the Wiener–Hopf and Mode-Matching techniques offers valuable insight into their applicability for future studies involving more complex geometries. The investigation of different scattering mechanisms using these methods will be the subject of the author’s forthcoming research.

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BIOGRAPHY

Zeynep SAZAK got Bsc degree as the second highest-ranking student from the Department of Mathematics, Faculty of Basic Sciences, Gebze Technical University, in 2022. She started her M.Sc in the Department of Mathematics at Gebze Technical University Graduate School of Natural and Applied Sciences in the spring semester of 2022-2023. Since 2019, she has been working as a Mathematics Teacher.



PUBLICATIONS WITHIN the SCOPE of THESIS STUDY

Sazak Z., Öztürk H., (2024), "Study Of Electromagnetic Wave Scattering From An Impedance Coated Step In A Coaxial Waveguide", 5th International Conference on Mathematics and its Applications in Science and Engineering, Coimbra, Portugal

