

T.R.
GEBZE INSTITUTE of TECHNOLOGY
GRADUATE SCHOOL of ENGINEERING and SCIENCES

RIGOROUS ANALYSIS and SIMULATION of TM
FIELDS in WAVEGUIDES with ARBITRARY CROSS-SECTION
SHAPE by MEANS of EVOLUTIONARY EQUATIONS of
TIME-DOMAIN ELECTROMAGNETIC THEORY

ÖMER AKTAŞ
A THESIS SUBMITTED for THE DEGREE of
MASTER of PHILOSOPHY
ELECTRONICS ENGINEERING DEPARTMENT

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LECTURER DR. OLGA SUVOROVA

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2014



**GEBZE YÜKSEK
TEKNOLOJİ ENSTİTÜSÜ**

YÜKSEK LİSANS JÜRİ ONAY FORMU

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İMZA/MÜHÜR

SUMMARY

Analytical Regularization Method is used to solve the problems for waveguides with arbitrary cross-section. Method is powerful in mathematical meaning and efficient in numerical meaning. With the help of regularization method, the initial boundary value problem equivalently reduced to the system of algebraic equations of the first kind $Ax=b$ to the $(I+H)x=b$ second kind equation in the l_2 space of square summable sequences.

Algorithm to solve the problem starts with the parameterization of the contour. Helmholtz' equation for the boundary value problem is solved next. By means of Green's functions, integral presentation of scattering field is obtained. Substitution of this integral representation into the boundary condition gives the relevant integral equation of first kind in space l_2 . Then, the kernel is split into regular and singular parts, and expands every part into its Fourier series, obtaining finally an infinite algebraic system of the first kind. Then by means of double sided regularization algebraic system of the first kind can be reduced to algebraic system of the second kind. With truncation procedure the final equation system can be solved with any desired accuracy.

Numerical results shows, presented method is mathematically strong, and numerically accurate and efficient not only for canonic shapes but also for non-canonic shapes of waveguides.

Key Words: Analytical Regularization, Waveguides with Arbitrary Cross-Section, Time Domain Electromagnetic Problem, Klein-Gordon Equations.

ÖZET

Keyfi şekilli dalga kılavuzlarının modellemesi için matematiksel olarak güçlü, sayısal olarak etkili analitik regülarizasyon yöntemi kullanılacaktır. Regülerleştirme işleminin sonucunda $Ax=b$ tipinde birinci türden denklem sistemi, $(I+H)x=b$ tipinde karesi toplanabilir dizilerin uzayı l_2 ' de ikinci türden sonsuz lineer cebir denklem sistemine indirgenebilir. Bu denklem sistemi sayısal olarak kesme yöntemi ile istenilen doğrulukta çözülebilir.

Problemin çözümü için kullanılan yöntem dalga kılavuzunun çevresinin parametrize edilmesi ile başlar. Helmholtz denklemi için sınır değer problemi çözülür. Green fonksiyonları yardımıyla saçılan alan için integral gösterimi elde edilir. Sınır koşullarının integral eşitliğine işlenmesi ile l_2 uzayında birinci türden bir denklem sistemi elde edilir. Bu denklem sistemi logaritmik tekillik içermektedir. Daha sonra, logaritmik tekillik düzenli ve tekil kısımlarına ayrılır. Daha sonra her parça Fourier açılımı olarak tanımlanır ve böylelikle ikinci türden sonsuz lineer cebirsel sistem elde edilir. Daha sonra çift taraflı regülerleştirici yardımı ile birinci türden lineer cebir sistemi ikinci türden lineer cebir sistemini çevrilir. Problemin çözümü ile elde edilen sistem istenilen N boyutunda kesilerek sonuca ulaşılır.

Sunulan yöntemin, kararlı, hem kanonik hem de kanonik olmayan dalga kılavuzları için etkili ve verimli olduğu daha önceki çalışmalarda ispatlanmıştır[1]. Bu yüzden elde edilen kesim frekansı ve kesimin dalga sayısı sonuçları keyfi şekiller için doğru olarak kabul edilip zaman domeninde elektromanyetik probleme evrimsel yaklaşım metodu ile simülasyonu yapılmıştır.

Anahtar Kelimeler: Analitik Regülarizasyon, Keyfi Kesitli Dalga Kılavuzları, Zaman Domeni Elektromanyetik Problemi, Klein-Gordon Denklemleri.

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LIST of ABBREVIATIONS and ACRONYMS

<u>Abbreviations</u> <u>and Acronyms</u>	<u>Explanations</u>
∇	: Nabla Operator
μ_0	: Magnetic permeability of free space
ε_0	: Dielectric constant of free space
μ	: Permeability of the medium within the waveguide
ε	: Relative permittivity
k	: Wavenumber
k_c	: Cut-off wavenumber
β	: Propagation constant
α	: Lossy parameter
γ	: Dielectric loss
k_0	: Free space wave number
L_2	: Hilbert Space
ϱ	: Dimensionless lossy parameter
ϖ	: Dimensionless frequency
$\mathbf{z}$	: Unit vector along the waveguide axis, Oz
$\mathbf{l}$	: Unit vector tangential to the contour, L
$\mathbf{n}$	: Outer normal unit vector to the domain S
∂_t	: Partial derivative with respect to t
∂_z	: Partial derivative with respect to z
σ	: Conductivity constant of lossy media
$\nabla_{\perp}$	: Nabla operator in transverse coordinate
$\mathcal{E}$	: Electric field strength vector
$\mathcal{H}$	: Magnetic field strength vector
$\mathcal{J}$	: Induced current density
ρ	: Charge density
$\mathbf{E}$	: Transverse components of electric field vector

H	: Transverse components of magnetic field vector
E_z	: z - component of electric field
H_z	: z - component of magnetic field
C	: Velocity of light in vacuum
τ	: Dimensionless time
ξ	: Dimensionless coordinate
BC	: Botanic Curve
CO	: Cassini Oval
TE	: Transverse Electric
TM	: Transverse Magnetic

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1. INTRODUCTION

Waveguides are structures that direct electromagnetic energy along a desired path, such as transmission lines. Maxwell's equations predict that electromagnetic waves can also be guided through hollow metallic tubes.

Waveguides play a fundamental role in the area of microwave engineering as transmission systems or being integrated into other microwave and optical devices such as filters, power dividers, polarizers, and so. The propagation characteristics, such as cutoff wave number and propagation constant can be obtained by means of classical technique of separation of variables, (SVM), if the shape of the waveguide is canonical (rectangular, circular, elliptical, or parabolic. For rectangular and circular case, it is easy to use such technique. For elliptical case it needs rigorous mathematical analysis and usage of Mathieu functions [1].

Nevertheless, it is clear that for investigating problem of waveguides with arbitrary cross section, there is no paper shows usage of SVM. In modern microwave systems, non-canonical, arbitrarily shaped waveguides are widely utilized so accurate numerical methods are required to solve waveguide problems with rather arbitrary cross section. Several techniques have been developed in this context during the last few decades, resulting in a very extensive literature [1].

The first technique is an improved finite-difference procedure for Electromagnetic eigenvalue problems. An improved implementation of finite-differences technique is presented based on well-known theories from [2]. Neumann and Dirichlet boundary condition for homogenous Helmholtz's equation and second order Lagrange interpolation are used to determine the propagation constant, cutoff frequencies, modal field distributions (by eigen vectors) for finite differences technique. Nevertheless, error of calculating cut-off frequencies is not convergent to error of calculating eigen vector since there is no such parameterization for this calculation. Also error of calculating cut-off frequency is changing for each mode which doesn't show any convergence either [1].

Secondly, a new method for increasing the accuracy and decreasing the difficulty of the finite element method is presented in this paper [3]. Authors of the paper states that to improve the accuracy of such method, there are two ways. One way is to divide region into finer segments and the other one is to increase the order

of the elements. From another hand to increase the order of elements is very difficult, and needs a lot of mathematical work. Previous papers focused on this method, shows that difficulty arrives from the related shape functions which are not properly described [1]. In this paper [3], rectangular waveguide is taken as an example. The geometry is divided into a triangle mesh. Some 20 formula for basic functions are presented to show how easy the method is. But as a rule this doesn't necessarily mean that the method is easy, since for more complicated shapes, more than few dozen formulas are needed. Although with usage of complex calculation the accuracy of the method could be increased, author avoided this hard mathematical work and decreased the accuracy of his own method [1].

Thirdly, a new method of using boundary mode matching by dividing the boundary counter as planar part and circular arc is presented in paper [4]. The TE and TM modes are expressed in terms of circular waves whose cutoff wave number and amplitudes are the unknowns of the problem. Plane wave expression is used for the planar part of the boundary because Helmholtz's equation is automatically satisfied. As for the arcs, two different weighting functions are used since the center of the arc might be the same with the center of the circular wave or the different. Usage of fast Fourier transforms cancels the need to use quadratic functions and accelerates the conversion between the different field representations [1].

Lastly, the Analytical Regularization Method (ARM) developed and strongly mathematically formulated in [5]-[13] for numerically effective solutions of mathematical physics problem, including ones of wave diffraction by conductive and dielectric bodies of revolution of arbitrary shapes. ARM for two-dimensional boundary value diffraction problem by arbitrary shaped smooth infinitely thin and perfectly conducting cylindrical screens was suggested in papers [14], [15].

1.1. Aim, Content and Addition of the Thesis

The complexity of great many non-canonical resonant obstacles and cavities creates hard numerical problems of instability of conventional numerical methods. In this thesis two arbitrary shapes (Cassini Oval, Botanic Curve) are used with seven kinds, it is solved with mentioned method Analytical Regularization Method (ARM) and results are illustrated.

The chapters are organized as follows. In Chapter 2, Maxwell's Equations are given and wave equation is obtained. Afterwards General Characteristics of Waveguide, TE and TM modes are given in detail. Rectangular and Circular waveguide geometries are shown and solved in classical theory.

In Chapter 3 brief detail of Analytical Regularization Method (ARM) is given and system of first and second kind explained in detail. Two-Dimensional Diffraction Problem for arbitrary shape is solved.

In Chapter 4, Formulas of the shapes (Cassini Oval, Botanic Curve) are given and seven kind graphics are plotted. Calculated results are illustrated and each graphics are explained.

In Chapter 5, Evolutionary approach, readers can see [16], [17] to the waveguide problem is given and the results (wavenumbers and cut-off frequencies) found in Chapter 4 are illustrated. In conclusion the results are discussed.

2. GENERAL WAVEGUIDE THEORY

2.1. Maxwell Equations

In general view the Maxwell Equations are;

- $\nabla \times H = \frac{\partial D}{\partial t} + J$
- $\nabla \times E = -\frac{\partial B}{\partial t}$
- $\nabla \cdot D = \rho,$
- $\nabla \cdot B = 0$
- $\nabla \cdot J = \frac{\partial \rho}{\partial t}$

We can write wave equations with knowledge of the Maxwell equations of current and charge as

$$\nabla \times \nabla \times E = -\frac{\partial}{\partial t} \nabla \times B = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \quad (2.1)$$

Then we can use this vector characteristic $\nabla \times \nabla \times V = \nabla (\nabla \cdot V) - \nabla^2 V$ with time dependent harmonic $e^{j\omega t}$,

$$\nabla^2 E + k^2 E = 0 \quad (2.2)$$

where $k = \omega \sqrt{\mu_0 \epsilon_0}$ and called as wavenumber.

2.2. General Characteristics of Waveguide

Waveguide is regarded as perfect conductivity and infinite length and homogenous as shown in figure 2.1. We will find the result for normal wave form which depended time as $e^{j\omega t}$ and along with z axis is $e^{j\beta z}$. Here is β known as propagation constant and k_c is cut-off wave number shown as $k_c = \sqrt{k^2 - \beta^2}$

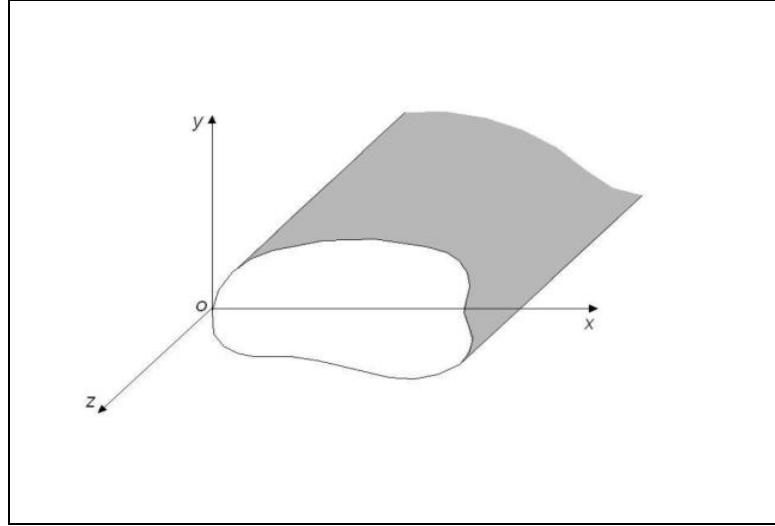


Figure 2.1: Arbitrary cross section shape.

From equation (2.2) we can understand that along with z axis and under $k > \beta$ consideration normal wave is propagating. When $k < \beta$ it is not propagating but it is decreasing exponentially along with z axis. Because of dependency on z axis we will think the problem as a 2 dimensional which is;

$$E(x, y, z) = [e_x(x, y) + ze_z(x, y)] e^{-j\beta z} \quad (2.3)$$

$$H(x, y, z) = [h_x(x, y) + zh_z(x, y)] e^{-j\beta z} \quad (2.4)$$

where $e_x(x, y)$ and $h_x(x, y)$ are transverse electric and magnetic field components on $x0y$, e_z and h_z are longitude electric and magnetic field components on z axis.

In this formulas we accepted that waves are propagating on +z direction it can be changed to -z direction with substitution of $\beta \rightarrow -\beta$. However, if there is a dielectric loss in waveguide, $j\beta$ need to be changed as $\gamma = \alpha + j\beta$. Accepting waveguide has no charges Maxwell equations can be written as;

$$\nabla \times E = -j\omega\epsilon H \quad (2.5)$$

$$\nabla \times H = j\omega\epsilon E \quad (2.6)$$

We can have the three components of each formula without $e^{-j\beta z}$ as

$$\frac{\partial E_z}{\partial y} + j\beta E_y = -j\omega\mu H_x \quad (2.7)$$

$$-\frac{\partial E_z}{\partial x} + j\beta E_x = -j\omega\mu H_y \quad (2.8)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \quad (2.9)$$

$$\frac{\partial H_z}{\partial y} + j\beta H_y = j\omega\varepsilon E_x \quad (2.10)$$

$$-\frac{\partial H_z}{\partial x} - j\beta H_x = j\omega\varepsilon E_y \quad (2.11)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\mu E_z \quad (2.12)$$

After this formulas we can obtain transverse field components depends on E_z , H_z

$$H_x = \frac{j}{k_c^2} \omega\varepsilon \frac{\partial E_z}{\partial y} + \beta \frac{\partial H_z}{\partial x} \quad (2.13)$$

$$H_y = \frac{-j}{k_c^2} \omega\varepsilon \frac{\partial E_z}{\partial x} + \beta \frac{\partial H_z}{\partial y} \quad (2.14)$$

$$E_x = \frac{j}{k_c^2} -\beta \frac{\partial E_z}{\partial y} + \omega\mu \frac{\partial H_z}{\partial x} \quad (2.15)$$

$$E_y = \frac{-j}{k_c^2} \beta \frac{\partial E_z}{\partial x} + \omega\mu \frac{\partial H_z}{\partial y} \quad (2.16)$$

2.2.1. TM Modes

Transverse magnetic fields can be found under $E_z \neq 0, H_z = 0$ condition. From equation (2.2) we can understand that E_z is satisfying two dimensional scalar homegenous Helmholtz equation.

$$\Delta + k_c^2 E_z = 0 \quad (2.17)$$

Other electromagnetic field components can be found under $H_z = 0$ conditon via equations (2.13), (2.14), (2.15) and (2.16). With this condition we can rewrite the formulas in cartesian coordinate system as;

$$H_x = \frac{j\omega\varepsilon}{k_c^2} \frac{\partial E_z}{\partial y} \quad (2.18)$$

$$H_y = \frac{-j\omega\varepsilon}{k_c^2} \frac{\partial E_z}{\partial x} \quad (2.19)$$

$$E_x = \frac{-j\beta}{k_c^2} \frac{\partial E_z}{\partial y} \quad (2.20)$$

$$E_y = \frac{-j\beta}{k_c^2} \frac{\partial E_z}{\partial x} \quad (2.21)$$

When we omit h_z part we have $E_{x,y,z} = [e_{x,y}] e^{-j\beta z}$ then with help of helmholtz wave equation and $k_c^2 = k^2 - \beta^2$ we find;

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2 e_z = 0 \quad (2.22)$$

Where $k = k_0 \sqrt{\varepsilon\mu}$ and k_0 is free space wave number. Equation (2.22) must be solved with given boundary conditions.

2.2.2. TE Modes

Transverse magnetic fields can be found under $H_z \neq 0$, $E_z = 0$ condition. From equation 2.2 we can understand that H_z is satisfying two dimensional scalar homegenous Helmholtz equation.

$$\Delta + k_c^2 H_z = 0 \quad (2.23)$$

Other electromagnetic field components can be found under $E_z = 0$ conditon via equations (2.13), (2.14), (2.15) and (2.16).

$$H_x = \frac{j\beta}{k_c^2} \frac{\partial H_z}{\partial x} \quad (2.24)$$

$$H_y = \frac{-j\beta}{k_c^2} \frac{\partial H_z}{\partial y} \quad (2.25)$$

$$E_x = \frac{j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial x} \quad (2.26)$$

$$E_y = \frac{-j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial y} \quad (2.27)$$

Same as, when we omit e_z part we have $H(x, y, z) = [h_z(x, y)] e^{-j\beta z}$ then with help of helmholtz wave equation and $k_c^2 = k^2 - \beta^2$ we find;

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2 h_z = 0 \quad (2.28)$$

where $k = k_0 \sqrt{\epsilon\mu}$ and k_0 is free space wave number. Equation (2.28) must be solved with given boundary conditions.

Traditional approach does not make possible calculate cut-off wavenumber and cut-off frequency for non-canonical (rectangular, circular or ellipse) shapes. Furthermore, even for canonical waveguides numerical simulation is preferable that solving by separation of variable method, since the last one needs much more mathematical work, especially, exact calculations of special functions. So, using of computational methods for canonical as well as non canonical waveguide is the common approach in modern Electromagnetics.

3. ANALYTICAL REGULARIZATION METHOD

3.1. General Information

Analytical Regularization Method is used to solve the waveguides problems with arbitrary cross-section. The mentioned method has strong mathematical background and effective in numerical analysis. The process of regularization leads us the initial boundary value problem equivalently reduced to the system of algebraic equations of the first kind $Ax = b$ to the $(I + H)x = b$ second kind equation in the L_2 space of square sum able sequences. This equation system with numerical cut-off method could solve in any ways [18].

Algorithm to solve the problem starts with the parameterization of the contour. Helmholtz' equation for the boundary value problem is solved next. By means of Green's functions, integral presentation of scattering field is obtained. Substitution of this integral representation into the boundary condition gives the relevant integral equation of first kind in space L_2 . Then, the kernel is split into regular and singular parts, and expands every part into its Fourier series, obtaining finally an infinite algebraic system of the first kind. Then by means of double sided regularizator algebraic system of the first kind can be reduced to algebraic system of the second kind. With truncation procedure the final equation system can be solved with any desired accuracy [18].

Numerical results shows, presented method is mathematically strong, numerically accurate and efficient not only for canonic shapes but also for non-canonic shapes of waveguides.

As we have mentioned above, several numerical techniques have been developed in this context during the last few decades, resulting in a very extensive literature. All these methods can be roughly separated in two main families: the Finite Difference Method and Finite Element Method. It is noteworthy, that most of them are based on solving of finite system of the first kind, which always creates unstable numerical process, when increasing of dimension of corresponding algebraic system leads to the destruction of the correct solution by round off errors. In diffraction theory there is a direction, referred in general as "Analytical

Regularization Method” (ARM), which reduces boundary value problem to a functional equation of the second kind. For details readers are referred to [19]-[21].

3.2. System of First Kind

A numerical study of boundary value problems of static theory of electromagnetic scattering from different scatters is reduced to the necessary treatment of a finite algebraic system of the kind $A_N x^N = b^N$ with the matrix operator A_N of dimension N . This system is explicitly arrived at by the truncation of the corresponding infinite system $Ax = b$ of the first kind. Conceptually, this may be regarded as “algebraization” of the functional equation of the first kind. In this step, it is doubtful whether the x^N solution of the system $A_N x^N = b^N$ converges to the x^∞ solution when $N \rightarrow \infty$, where x^∞ is the solution of system $Ax = b$. If so, then in what metric is this? Is there a numerical catastrophe when $|x^N - x^\infty| / |x^\infty| > 1$ for sufficiently large N ? Here, an approximate solution x^N of the system $A_N x^N = b^N$ is calculated on a computer which has a word length containing only a finite number m_c of bits.

The second question arises only if $x^N \rightarrow x^\infty$ in the relevant metric agreed upon the metric of the functional space where the initial boundary value problem solution is sought.

It has been known, as N increases that the x^N solution of a system of the first kind does not converge to x^∞ in any metric. This is typical of most parts of boundary value diffraction problems [22]. Nevertheless, do suppose that owing to the special properties of the operator, this convergence takes place and therewith in the wanted metric. Define by the standard way [23], [24] the condition number of the operator A_N can be written as

$$\nu_N = \|A_N\|_2 \|A_N^{-1}\|_2 \quad (3.1)$$

where the operator norm $\|\cdot\|_2$ is created by the Euclidean metric of vectors on N -dimensional space. The systems of the first kind $Ax = b$ are characterized by $\|A_N\|_2 \rightarrow \infty$ or $\|A_N^{-1}\|_2 \rightarrow \infty$ when $N \rightarrow \infty$; i.e., the operators A or A^{-1} are unbounded on the operator norm constructed from the l_2 space vector norm.

Consequently, $v_N \rightarrow \infty$ or $N \rightarrow \infty$. A number of exact binary digits in the x^N solution do not exceed the value [24],

$$m_r = m_c \log_2 v_n \quad (3.2)$$

where m_r describes a number of significant digits in the x^N vector component which has the largest modulus. Therefore, when the elements of the x^∞ vector differ greatly in modulus (for one, decay very fast as their indexes increase with some of these small elements required to be calculated with a high accuracy), the algorithm has the problems of renormalization of the unknown x^∞ vector and balancing of the initial system matrix [24]. Hence, if (3.2) indicates $m_r < 0$ the solution contains no significant digits to be true. In this case, the imbalance $\delta^N \stackrel{\text{def}}{=} A_N x^N - b^N$ will be of the order [24] $\delta^N_2 = 2^{m_c N} x^N_2$, that is, generally speaking, overly small (the exact solution x^N , rounded to m_c , of binary digits solution x^N of the system $A_N x^N - b^N$ gives imbalance of the same value order).

We emphasize that the only practical way to reveal the numerical catastrophe onset is direct calculation of m_r and v_n . Various circumstantial criteria, as energy balance, stabilization of the solution x^N with increasing N , etc., may, as a rule, only give illusions of the initial boundary value problem solution correctness: the reason is that the energy conservation law may be obeyed with a high accuracy by virtue of $\delta^N_2 = 2^{m_c N} x^N_2$ although $m_r < 0$ and none of the x^N significant digits is true; next, on reaching the inequality $m_r < 0$, the solution x^N with increasing N may drift very slowly, being prescribed by the rule of the arithmetical result rounding off in the computer, etc.

Note that the choice of the norm $\| \cdot \|_2$ in equation (3.1) and reasoning about the A and A^{-1} unboundedness on the operator norm just in the l_2 space are not causal. Although, as known [25], [26], all the norms in a finite-dimension space are equivalent, in some important sense associated with N -independence of corresponding estimates of the solution error and the condition number invariance just in the form equation (3.1) for the matrix A_N unitary transforms (as reducible to the sequence of such transforms are known the Gauss method for the system $A_N x^N = b^N$ solution, the method of triangular decomposition of the A_N matrix. [22], [23], [27]. In the “limit” $N \rightarrow \infty$, this is the l_2 metric.

That is why “paper tigers” of mathematical proofs of the existence and boundedness of the operators A and A^{-1} on the corresponding pair of functional spaces, the proof of the x^N convergence to x^∞ in the metric of these spaces, etc., have no effect on a computer that demonstrates (exactly in the equation (3.1)) all the evidence of numerical catastrophe.

It has been known that the choice of a proper basis in the moment method, special quadrature formulae for the solution of certain classes of integral equations, and other means of this kind enable us to reduce the rate of v_n growth at $N \rightarrow \infty$ [28] and, consequently, shift the numerical catastrophe onset towards larger N . For the same purpose, one can recourse to the aid of a computer and the corresponding program realization of the involved arithmetical operations at larger m_c [29]. But against all the expedients, the fundamental properties of the systems of the first kind are unaffected: some time or other numerical catastrophe ensues. This prohibits us from having the problem solution within a preassigned accuracy and greatly restricts the possible range of the problem parameters.

It is an incomplete understanding of the aforestated burdens, first of all, of the behavior of the arithmetical operations involved in computer calculations, and an underestimation of difficulties in stable numerical realization of many theoretically correct (in exact arithmetic) computational algorithms that give rise a great many of recent publications about novel and novel numerical methods aimed at conversion of boundary value diffraction problems to equations of the first kind. The computational stability of these methods, the condition numbers of algebraic systems dealt with, etc., frequently are not even mentioned.

3.3. System of the Second Kind

An initial boundary value problem is assumed to be equivalently reduced in the l_2 to the infinite system of algebraic equations of the form $(I + H)x = b$ with the compact in l_2 the operator H . If well posed, an initial boundary value problem possesses, as a rule, a unique solution. Then, in view of the said equivalence, there exists the bounded in the l_2 operator $(I + H)^{-1}$ and, correspondingly, the value

$$v_\infty \stackrel{\text{def}}{=} \|I + H\|_{l_2} \|I + H\|_{l_2}^{-1} < \infty \quad (3.3)$$

is correctly defined. Likewise, as before, consider the truncated systems

$$(I + H_N) x^N = b^N \quad (3.4)$$

by the H compactness, the sequence of the finite-dimension matrix operators H_N may be selected such that,

$$\|H - H_N\|_{l_2} \rightarrow 0 \quad \text{when} \quad N \rightarrow \infty \quad (3.5)$$

It evidently follows from equation (3.3), (3.5) that the condition numbers of systems equation (3.4) are,

$$\nu_N \stackrel{\text{def}}{=} \|I + H_N\|_2 \|I + H_N\|_2^{-1} \rightarrow \nu_\infty \quad \text{when} \quad N \rightarrow \infty \quad (3.6)$$

and consequently, are uniformly bounded for sufficiently large N . In all practical situations known to us occurring in the solution of boundary value diffraction problems, the value of ν_N for corresponding equations of the second kind varies within a few tens of units (at most, a few hundreds), and consequently, for many contemporary computers $\nu_N \ll 2^{m_c}$.

3.4. Comparison of both kinds

In electromagnetic problems there is various wave guide geometry which could not be solved by classical methods. For this reason, numerical methods are improved for such geometries. For example, such methods like Moment Method, Finite Difference (FT) and Finite Element (FE) as follows, solve problems with reducing to first kind algebraic system.

$$A_N X^N = B^N \quad (3.7)$$

where A_N is N-dimensional matrix operator, X^N is unknown and B^N known function values. System defined in equation (3.7) explicitly or implicitly is obtained by truncation of infinite system of the first kind.

$$AX = B \quad (3.8)$$

Solution calculated for X^N in equations (3.7) and (3.8) when $N \rightarrow \infty$ the system whether converges X^∞ is significant in terms of numerical calculations. An approximate solution X^N of the system $A_N x^N = b^N$ is calculated on a computer which has a word length containing only a finite number m_c , of bits. Generally, as N increases that the x^N solution of a system of the first kind does not converge to x^∞ in any metric. Define by the standard way the condition number of the operator A_N

$$v_N = \|A_N\|_2 \|A_N^{-1}\|_2 \quad (3.9)$$

where the operator norm $\| \cdot \|_2$ is created by the Euclidean metric of vectors on N-dimensional space. But the systems of the first kind $Ax = b$ are characterized by $\|A_N\|_2 \rightarrow \infty$ or $\|A_N^{-1}\|_2 \rightarrow \infty$ when $N \rightarrow \infty$; i.e., the operators A or A^{-1} are unbounded on the operator norm constructed from the l_2 space vector norm. Consequently, $v_N \rightarrow \infty$ or $N \rightarrow \infty$. If we get m_r as a digit value the solution X^N do not exceed the value.

$$m_r = m_c \log_2 v_n \quad (3.10)$$

where m_r describes a number of significant digits in the X^N vector component which has the largest modulus. If equation (3.10) indicates $m_r < 0$ the solution contains no significant digits to be true.

When we think about an initial boundary value problem in l_2 space to the infinite system of algebraic equations we have the form

$$(I + H)x = b \quad (3.11)$$

with the compact in l_2 the operator H . If the boundary value problem defined well it has a unique solution. Then there exists the bounded in the l_2 operator $(I + H)^{-1}$ and correspondingly, the value is correctly defined

$$v_\infty \stackrel{\text{def}}{=} \|I + H\|_{l_2} \|I + H\|_{l_2}^{-1} < \infty \quad (3.12)$$

$$(I + H_N) x^N = b^N \quad (3.13)$$

By the H compactness, the sequence of the finite-dimension matrix operators H_N may be selected such that

$$\|H - H_N\|_{l_2} \rightarrow 0 \quad (3.14)$$

It evidently follows from equation (3.12) - (3.14) that the condition numbers of systems equation (3.13) are,

$$v_N \stackrel{\text{def}}{=} \|I + H_N\|_{l_2} \|I + H_N\|_{l_2}^{-1} \rightarrow v_\infty \quad (3.15)$$

when $N \rightarrow \infty$, uniformly bounded for sufficiently large N is excepted, for many contemporary computers $v_N \ll 2^{m_c}$. So, by contrast to equations of the first kind, equations of the second kind are free from the mentioned principal disadvantages which interfere with their efficient solving. Thus, the equivalent reducing of a boundary value problem to the equation of the second kind in the l_2 provides a good basis for devising efficient algorithms for numerical solution of these problems in a wide range of their parameters.

3.5. Two-Dimensional Diffraction Problems

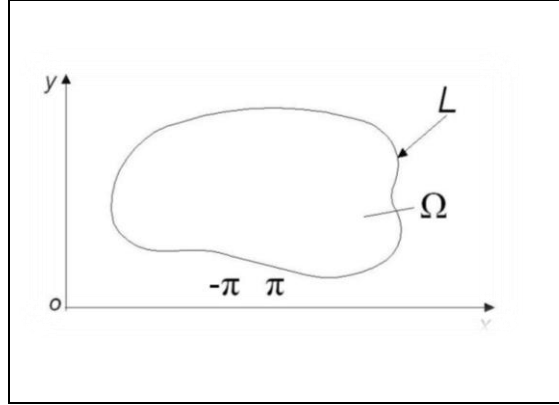


Figure 3.1: Arbitrary cross-section of diffraction problem.

3.5.1. Dirichlet Problem

We will begin with a problem of wave diffraction by an infinite and cylindrical screen homogeneous in z ; its cross-sectional view in the xOy plane is a smooth, one-connected and unclosed contour L . There is no loss of generality in assuming that L is a portion of a smooth, closed, non-self-crossing contour S , (maybe $L=S$). The considered diffraction problem for E-polarized and z -independent electromagnetic wave $u^0(p)$, $p \in R^2$ indecent on the given perfectly conducting screen is converted to the boundary value diffraction problem with the scattered field $u^s(q)$ subject to the Dirichlet boundary condition on the both sides of the L contour. Green's formulae technique allows us to show that $u^s(q)$ permits the integral representation

$$u^s(q) = \int_L E_2(k|q-p|) Z_D(p) dl_p, \quad q \in R^2 \quad (3.16)$$

where $Z_D(p)$ is a complex valued function given on the L contour $E_2(t)$ is Green's function of two-dimensional free space i.e., the fundamental solution of Helmholtz equation,

$$E_2(t) = -\frac{i}{4} H_0^{(1)}(t) \quad (3.17)$$

$H_0^{-1}(t)$ is the 0th-order Hankel function of the first kind. When subject to the Dirichlet boundary condition, equation (3.16) yields the integral equation in $Z_D(p)$,

$$\int_L E_2(k|q-p|) Z_D(p) dl_p = -u_0(q), q \in L \quad (3.18)$$

Once the solution of equation (3.18) has been constructed, the solution of the considered boundary value diffraction problem is given by formula equation (3.16). Now let a smooth parameterization of the S contour be specified by the function $\eta(\theta) \equiv (x(\theta), y(\theta))$, $\theta \in [-\pi, \pi]$ that smoothly parameterizes the contour S by point of the segment $[-\pi, \pi]$ with the identified ends $\theta = \pm\pi$ (i.e., by points lying on a unit circle); the functions $x(\theta), y(\theta)$ are the like Cartesian coordinates of the points $\eta(\theta) \in S$ which are one-to-one with the values of the parameter $\theta \in [-\pi, \pi]$. With any loss of generality, assume that the function $\eta(\theta) \in [-d, d]$ parameterizes the contour L where $d \in (0, \pi]$ is some fixed value. Owing to the one-to-one correspondence of the given parameterization, the function

$$l(\theta) \stackrel{\text{def}}{=} \{x'(\theta)^2 + y'(\theta)^2\}^{1/2} > 0, \theta \in [-\pi, \pi] \quad (3.19)$$

is strictly positive. With the aid of the parameterization $\eta(\theta)$, equation (3.18) is equivalently reduced to the equation

$$\int_{-d}^d Z_D(\tau) E_2(kR(\theta, \tau)) d\tau = G(\theta), \theta \in [-d, d] \quad (3.20)$$

$$Z_D(\theta) = l(\theta) \theta Z_D(\eta(\theta)) \quad (3.21)$$

$$R(\theta, \tau) = \{x(\theta) - x(\tau)^2 + y(\theta) - y(\tau)^2\}^{1/2} \quad (3.22)$$

$$G(\theta) = -u^0(\eta(\theta)) \quad (3.23)$$

Preparatory to solving equation of the first kind equation (3.18), we will first discuss the character of the singularities of this equation kernel-the function $E_2(kR(\theta, \tau))$. Of common knowledge is that this function possesses singularity of logarithmic type. For any integer $M \geq 2$, it can be verified that the following local singular expansion:

$$E_2 \ kR \ \theta, \tau = \frac{1}{2\pi} \ln \theta - r \sum_{m=2}^M A_m \theta \ (\tau - \theta)^2 + F_M^D \ \theta, \tau \quad (3.24)$$

takes place in the domain $\omega = \{ \theta, \tau : \theta - r < 2\pi \quad \theta, \tau \in [-\pi, \pi] \}$ where $F_M^D \ \theta, \tau$ are some function and $F_M^D \in C^M \ \omega$; $A_m \ \theta$ are some infinitely differentiable functions independent of M. In parallel with local singular expansion equation (3.24), one can also build the global singular expansions which describe exhaustively the differential properties of the function $E_2 \ kR \ \theta, \tau$ and, particularly important for our aims, which predetermine the asymptotes of these function Fourier coefficients. With equation (3.24) and the corresponding global singular expansion, define the function $H_0 \ \theta, \tau$ from the equality

$$E_2 \ kR \ \theta, \tau = \frac{1}{2\pi} \{ \ln 2 \sin \frac{\theta - \tau}{2} + H_0 \ \theta, \tau \} \quad \theta, \tau \in [-\pi, \pi] \quad (3.25)$$

From equation (3.24) and the above-mentioned global singular expansion, it follows the corresponding local and global singular expansions of the function $H_0 \ \theta, \tau$ whence, among other things, issues that the function $H_0 \ \theta, \tau$ allows its expansion into the double Fourier series

$$H_0 \ \theta, \tau = \sum_{s,m=-\infty}^{\infty} h_{sm} e^{i(s\theta + m\tau)}; \quad \theta, \tau \in [-\pi, \pi] \quad (3.26)$$

Besides, it is valid that, see [30].

$$\ln 2 \sin \frac{\theta - \tau}{2} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{e^{in(\theta - \tau)}}{|n|} \quad \theta, \tau \in [-\pi, \pi] \quad (3.27)$$

Now define the function $z_D^0 \ \theta$ as

$$z_D^0 \ \theta = \begin{cases} z_D \ \theta, & \theta \in [-d, d] \\ 0, & \theta \in [-\pi, \pi] / [-d, d] \end{cases} \quad (3.28)$$

and write it down as its Fourier series

$$z_D^0 \theta = \sum_{n=-\infty}^{\infty} \xi_n e^{in\theta}, \quad \theta \in [-\pi, \pi] \quad (3.29)$$

with some coefficients ξ_n to be found. From equation (3.20), (3.25), and (3.28) follows

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} z_D^0 \tau \left\{ \ln 2 \sin \frac{\theta - \tau}{2} + H_0(\theta, \tau) \right\} d\tau = G(\theta), \quad \theta \in [-d, d] \quad (3.30)$$

Substitute expansions equation (3.26), (3.27), (3.29) into equation (3.30) and integrate term by term the resulted series on account of the orthogonality property of the functions $e^{in\theta}$ on the interval $[-\pi, \pi]$ to obtain the series equation

$$\sum_{n=-\infty}^{\infty} \frac{\xi_n}{|n|} e^{in\theta} + \sum_{n=-\infty}^{\infty} (W\xi)_n e^{in\theta} = -2G(\theta), \quad \theta \in (-d, d) \quad (3.31)$$

where ξ is the column vector of the unknowns

$$\xi \stackrel{\text{def}}{=} \{\xi_n\}_{n=-\infty}^{\infty} \quad (3.32)$$

and the matrix operator W takes the form

$$W = \{w_{sn}\}_{s,n=-\infty}^{\infty} \quad w_{sn} \stackrel{\text{def}}{=} -2h_{s,-n} \quad (3.33)$$

From equation (3.24) and the bottom equality in equation (3.28), we obtain the other series equation

$$\sum_{n=-\infty}^{\infty} \xi_n e^{in\theta} = 0, \quad \theta \in [-\pi, \pi] \setminus [-d, d] \quad (3.34)$$

It is seen that equations (3.31), (3.34) constitute the dual series equations which are formally of the type equation (3.13), (3.14) once the unknowns ξ_n have been substituted for x_n by the formulae $x_0 = \xi_0$ and $x_n = |n|^{-1} \xi_n, n \neq 0$. It can be proved that equations (3.31), (3.34) do belong to the class of dual series equations of the type equation (3.13), (3.14), whose theory including their regularization procedure

has been built by the authors. In such a manner, we derive the regularization procedure for equations (3.31), (3.34), and so for the initial Dirichlet problem.

The case of $d = \pi$ corresponds to a closed contour $L = S$ and then equation (3.34) vanishes. In this situation, the regularization procedure for dual series equations need not be looked at for it can be derived (nonformally) by elementary means directly from equation (3.31) with $d = \pi$. Multiplying equation (3.31) by $e^{is\theta}$ (with $s = 0, \pm 1, \pm 2, \dots$) and integrating the resulting expression term by term using the orthogonality of the functions $e^{in\theta}$ on $(-\pi, \pi)$, we arrive at the infinite system of linear algebraic equations of the first kind. On multiplying each s^{th} row by $|s|$ at $s \neq 0$, this system is likely to become the system of equations of the second kind (left-sided regularization). The analysis for properties of the matrix operator W equation (3.33) reveals that this way is appropriate only for coordinate problems-for diagonal matrix W . In the general case of non-coordinate problems, another way is followed: it should introduce the new unknowns $\xi = \xi_n |n|^{-1/2}$ at $n \neq 0$ and then each s^{th} equation ($s = 0, \pm 1, \pm 2, \dots$) of the previous algebraic system is multiplied by $|s|^{1/2}$ at $s \neq 0$. The obtained (by the double-sided regularization of the initial algebraic system) infinite algebraic system in the unknowns $\xi = \{\xi_n\}_{n=-\infty}^{\infty}$ is of the form $(I + H)x = b$. With this quite simple particular case for $d = \pi$, the difference between the left-sided regulator $L_0 = \text{diag max } 1, s \text{ }_{n0-\infty}^{\infty}$, $R_0 = I$) and the double-sided regulator $L_0 = R_0 = \text{diag max } 1, s^{1/2} \text{ }_{n0-\infty}^{\infty}$ and distinctions between effects of their application to the initial equation of the first kind are the most noticeable. For the diagonal matrix operator W , this difference is barely perceptible and affects only the right-hand side of the final algebraic system. As well, something of a sort occurs in the general case of non-coordinate problems for $d \neq \pi$.

3.5.2. Neumann Problem

Diffraction of H-polarized wave by the same cylindrical screen, which was considered in the previous point, formulates Neumann boundary value problem. In this case, Green's formulae generate the integral representation of the scattered field other than equation (3.16):

$$u^s(q) = \int_L \frac{\partial E_2(k, q-p)}{\partial n_p} Z_N(p) dl_p, \quad q \in \mathbb{R}^2 \setminus L \quad (3.35)$$

where n_p is the unit outward normal to the contour S at the point $p \in L$; $Z_N(p)$ is some function defined on the contour L . Once equation (3.35) is subject to the Neumann boundary condition at the contour L , we obtain the equation

$$\lim_{h \rightarrow +0} \frac{\partial}{\partial n_q} \int_L \frac{\partial E_2(k, q \pm h n_q - p)}{\partial n_p} Z_N(p) dl_p = - \frac{\partial u^0(q)}{\partial n_q} \quad q \in L \quad (3.36)$$

in $Z_N(p)$. We are permitted neither to allow the derivative within the integral sign at $h = 0$, nor simply put $h = 0$ in the left-hand side in equation (3.36), because the resulting expression diverges. The problem of “calculation” of the limit in equation (3.36) is solved in [31] for a closed contour $L = S$. The employed method depends essentially on a smoothness supposed in [31] degree of the function $Z_N(p)$ smoothness, which is natural in a closed contour but ruled out (in general) for an unclosed one. Therefore, it takes a special consideration of the limiting transition in equation (3.36). We have proved that with corresponding conditions, equation (3.36) with the described parameterization $\eta = \theta$ of the contour S is equivalently reduced to the equation

$$\frac{d^2}{d\theta^2} \int_{-d}^d Z_N(\tau) \ln 2 \sin \frac{\theta - \tau}{2} d\tau + \int_{-d}^d Z_N(\tau) K_0(\theta, \tau) d\tau = F(\theta) \quad \theta \in [-d, d] \quad (3.37)$$

$$Z_N(\theta) = Z_N(\eta = \theta); \quad (3.38)$$

$$F(\theta) = -2\pi l(\theta) \left[\frac{\partial u^0(q)}{\partial n_q} \right]_{q=\eta=\theta} \quad (3.39)$$

and $K_0(\theta, \tau)$ is some function possessing the local singular expansion similar to equation (3.24),

$$K_0(\theta, \tau) = \ln |\theta - \tau| + \sum_{m=0}^M {}^n B_m(\theta) |\theta - \tau|^m + F_M^N(\theta, \tau) \quad (3.40)$$

where ${}^n B_m(\theta)$ are arbitrary, independent of $M = 0, 1, 2, \dots$ infinitely differentiable functions; $F_M^N \in C^M(\omega)$ are some functions (see equation (3.24)). In parallel with

equation (3.39), the corresponding global singular expansion has been constructed. The non-formal scheme of reducing equation (3.37) to dual series equations of the type equation (3.13), (3.14) is quite similar to that described for equation (3.20), [32]. All but on a certain stage, the series being the first sum in equation (3.31) should be twice differentiated term by term with respect to θ . The dual series equations of the type equation (3.13), (3.14) derived by this scheme from (3.37) belong to the class of equations whose theory including the regularization procedure has been constructed in a similar way. Thus, the regularization procedure for the Neumann boundary value problem is obtained. Note that the interchange of the order of summation and integration, term wise integration and differentiation, etc., is allowable only at a non-formal level of description. The justification of the discussed regularization procedures derivation requires in fact that the mathematical treatment of a number of these operations be changed without invalidating the existing result.

It is also noteworthy, how differently in seemingly identical situations (see equation (3.21) and (3.38)) the function $z_0 \theta$ and $z_N \theta$ are derived via $z_D p$ and $z_N p$, respectively. However, as evident from the analysis of equations (3.18) and (3.36), any other way of the evaluation of the function $z_0 \theta$ and $z_N \theta$ gives a radical change in the properties of the kernels of the corresponding analogous of equations (3.20) and (3.37) making them worse catastrophically-the singularity of analogous of the function $H_0 \theta, \tau$ and $K_0 \theta, \tau$ enhances. Therefore, in some sense, formulas equation (3.21) and (3.38) are the best.

4. NUMERICAL RESULTS

4.1. Shapes Used in Problem

4.1.1. Cassini Oval

The first object of our investigation has been the smooth-contoured waveguides of non-canonical cross section. We solved the ARM on Shapes and formulae for them are taken from [33]. Equation of the Cassini Oval is,

$$r^4 - 2a^2r^2 \cos 2\theta - b^4 + a^4 = 0. \quad (4.1)$$

$$x = \cos(\theta) \sqrt{a^4 \cos^2 2\theta - a^4 + b^4 + a^2 \cos 2\theta} \quad (4.2)$$

$$y = \sin(\theta) \sqrt{a^4 \cos^2 2\theta - a^4 + b^4 + a^2 \cos 2\theta} \quad (4.3)$$

with these results, it is plotted for different a and b values with θ from 0 to 2π as it is shown Figures 4.1, 4.2 and 4.4.

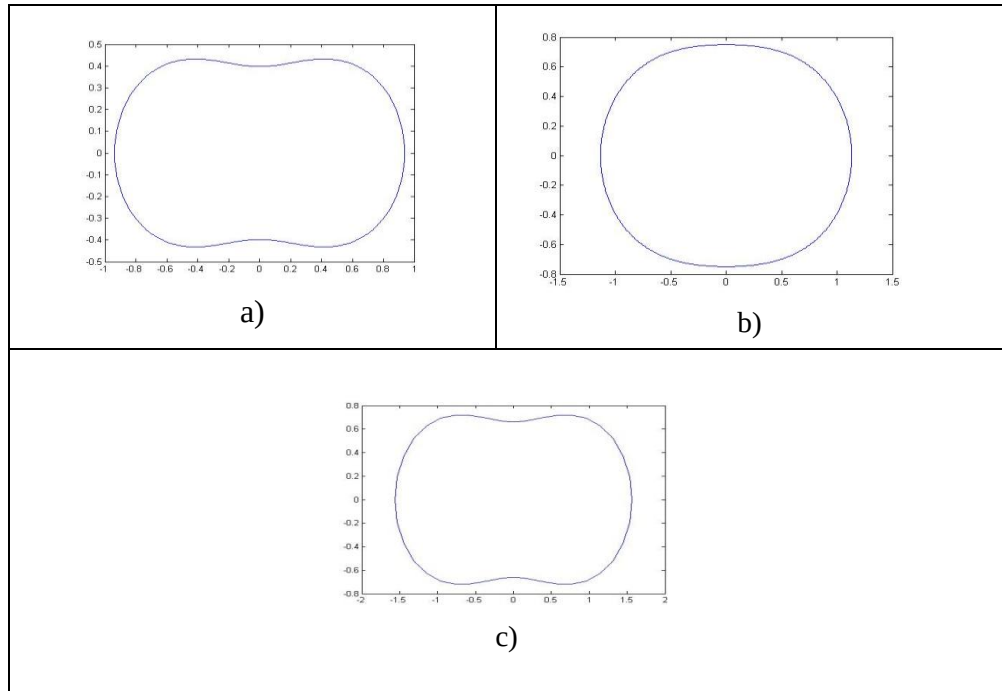


Figure 4.1: Cassini Oval Contour a) $a=0.6$ $b=1.2$ b) $a=0.6$ $b=1.6$ c) $a=1.0$ $b=1.2$.

4.1.2. Botanic Curve

Equation of the botanic curve is shown in equation (4.4), for plotting the graphs the equation can be used as in the equation (4.6) and (4.7);

$$r = 1 + b\sin(a\theta) \quad (4.4)$$

$$x(\theta) = r \cos \theta, \quad y(\theta) = r \sin \theta \quad (4.5)$$

$$x(\theta) = \sin(\theta)(b\sin(a\theta) + 1) \quad (4.6)$$

$$y(\theta) = \cos(\theta)(b\sin(a\theta) + 1) \quad (4.7)$$

with these results, it is plotted for different a and b values with θ from 0 to 2π , as these are shown in figures.

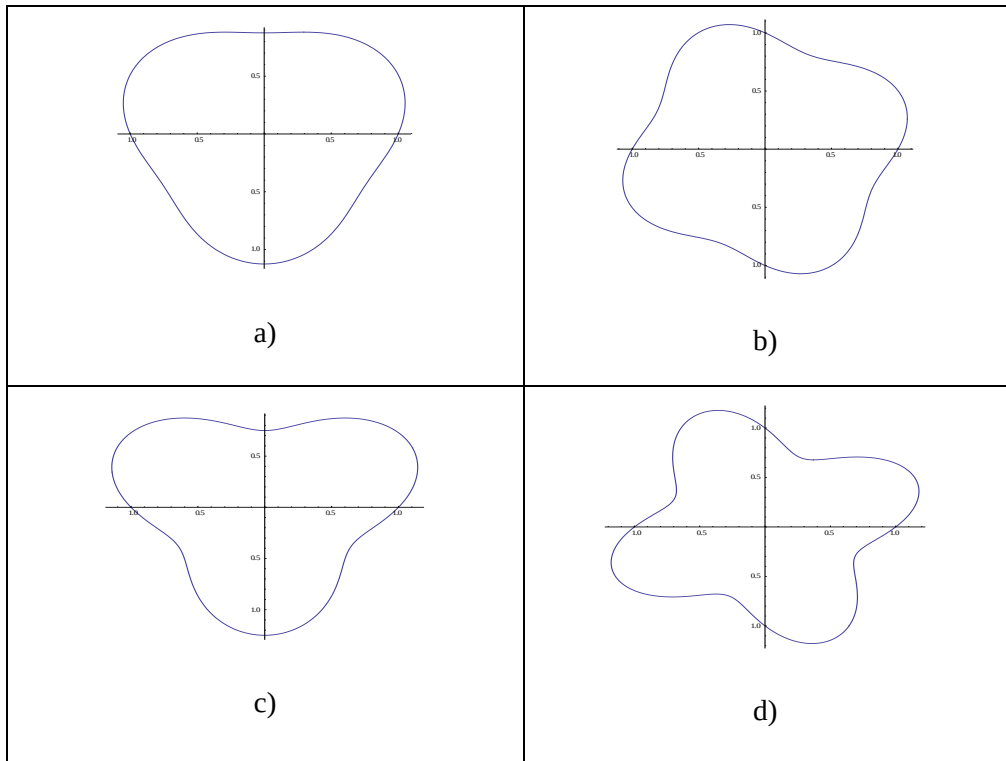


Figure 4.2: Botanic Curve Contour a) $a=3$ $b=0.125$
b) $a=4$ $b=0.125$ c) $a=3$ $b=0.25$ d) $a=4$ $b=0.25$.

4.2. Dirichlet Problem

4.2.1. Calculations for Cassini Oval

One of the most unpleasant problems for evaluation of wavenumber of waveguide of with complicated cross section is typically lack of initial guesses. In Figures 4.3 - 4.5 and figures 4.9 - 4.12 the normalized value of determinant of the system equation (3.3) versus ka is shown. As one can see, functions have very simple and regular behavior with clearly distinguished minima corresponding to the wavenumbers. The position of every minimum gives excellent initial guess for the cutoff wavenumber under search. Results of the system determinant are shown in the figures with changing k values from values 1 to 10 with 0.04 steps.

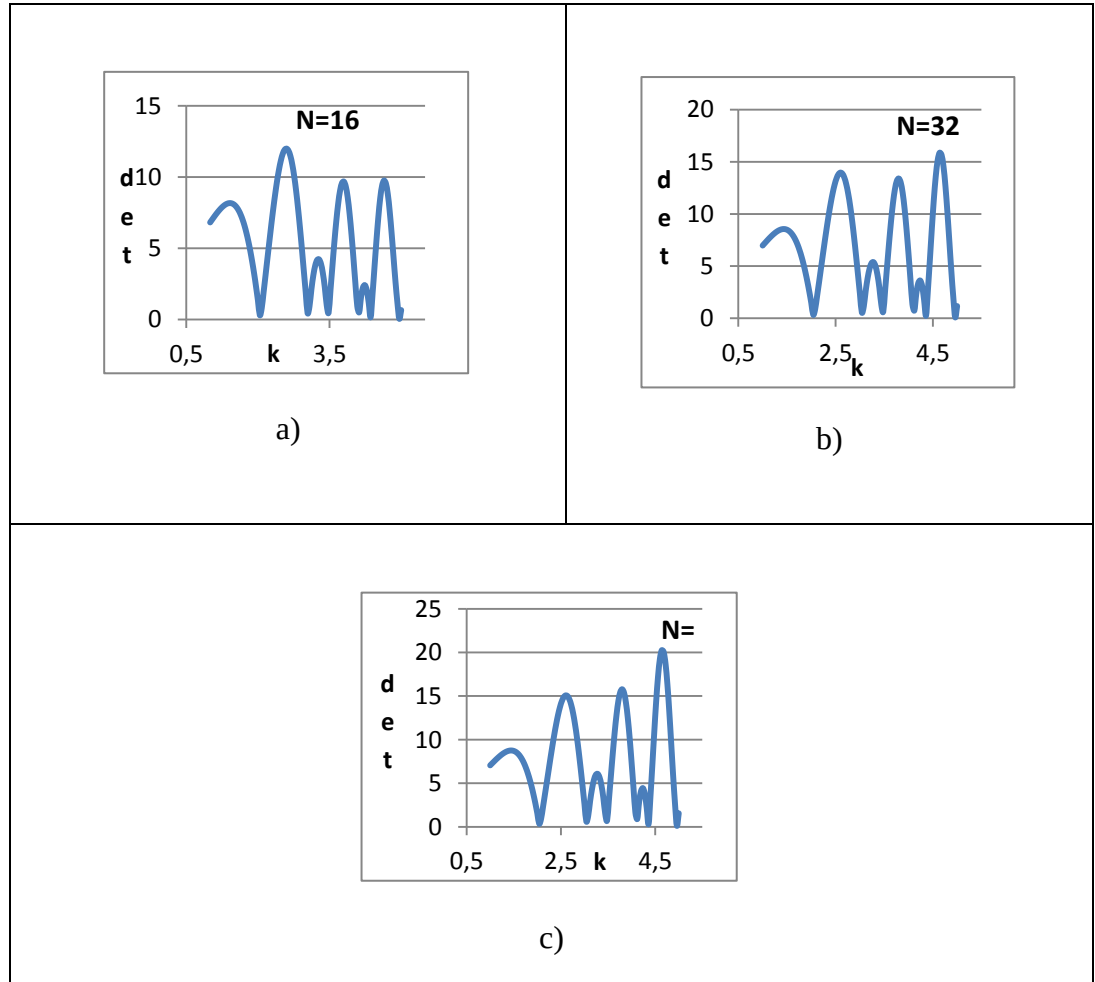
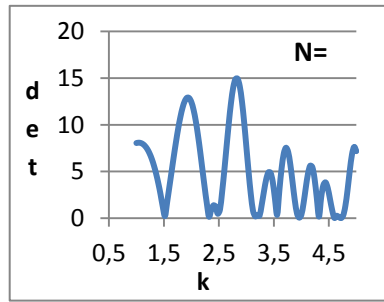
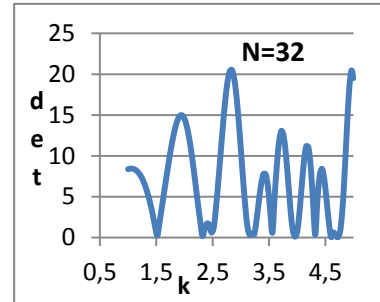


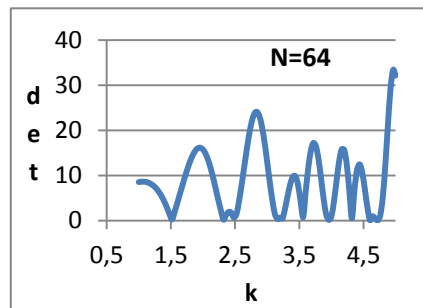
Figure 4.3: In CO 1 relations between cut-off frequency and $\det(I + H_N(k))$.
a) $N=16$ b) $N=32$ c) $N=64$.



a)



b)



c)

Figure 4.4: In CO 2 relations between cut-off frequency and $\det(I + H_N(k))$.

a) $N=16$ b) $N=32$ c) $N=64$.

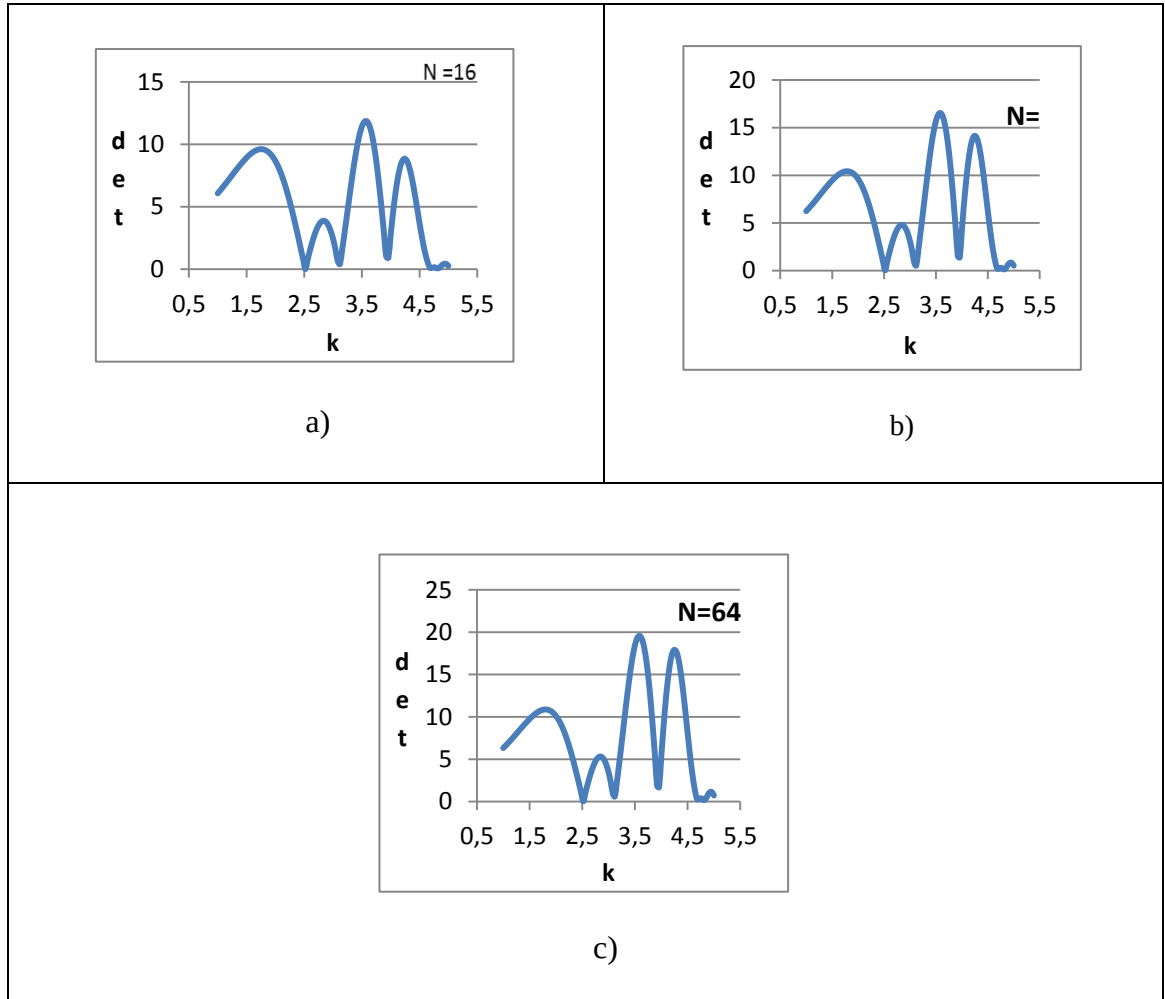


Figure 4.5: In CO 3 relations between cut-off frequency and $\det(I + H_N(k))$.
a) $N=16$ b) $N=32$ c) $N=64$.

We investigated convergence of the first cut-off numbers for Cassini and Botanical waveguides with the following parameters. As numerical experiments show, method has fast convergence by increasing the sizes of the algebraic systems. For the TM mode we get at least 8 significant digits in mantissa when size of the algebraic system is, $N = 16$.

Convergency of the cut-off frequency to system size are shown for changing N and k also figures of the contours are plotted for each contour of Cassini Oval.

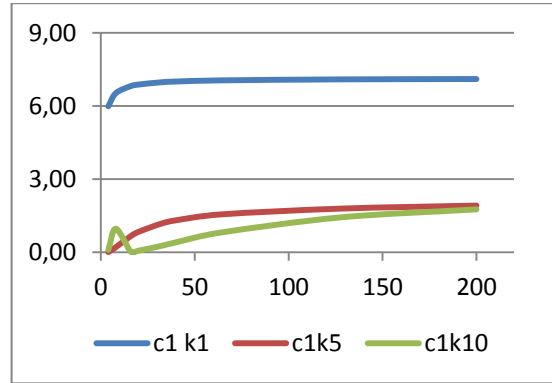


Figure 4.6: Convergency of the cut-off frequency to system size for CO 1.

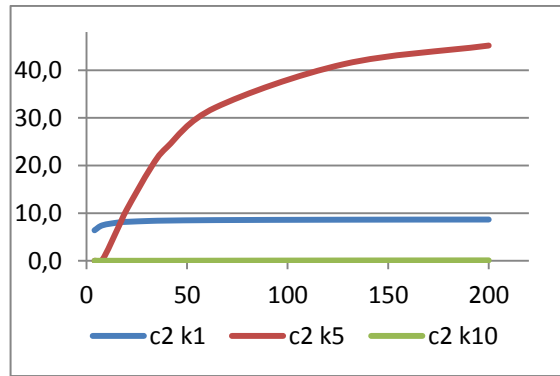


Figure 4.7: Convergency of the cut-off frequency to system size for CO 2.

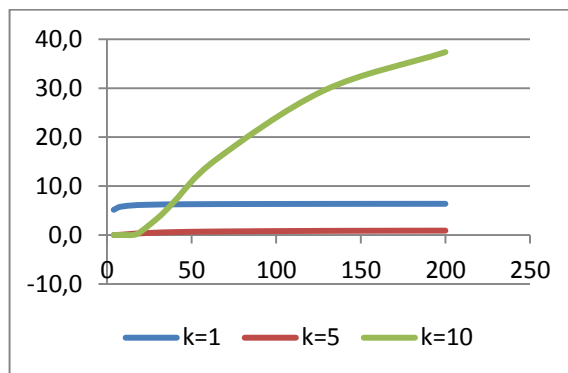


Figure 4.8: Convergency of the cut-off frequency to system size for CO 3.

Convergence of the first and second and third cut-off frequencies (k_c) for different shapes of Cassini Oval is shown.

Table 4.1: Convergence of the cut-off frequencies in CO 1.

Contour 1		a=0.6 b=1.2	
N Truncation Number	1st root	2nd root	3rd root
	2.04	3.04	3.48
4	2.05110720622	3.07526832869	3.46960858385
8	2.05120638826	3.05261882129	3.46766616161
16	2.05120658453	3.05258935376	3.46776258964
32	2.05120658453	3.05258935376	3.46776258965
42	2.05120658453	3.05258935376	3.46776258965

Table 4.2: Convergence of the cut-off frequencies in CO 2.

Contour 2		a=0.6 b=1.6	
N	1st root	2nd root	3rd root
	1.52	2.32	2.48
4	1.51391212152	2.3375411222	2.49850020298
8	1.51389840321	2.32508898808	2.49552205631
16	1.51389841545	2.32507968397	2.49553969955
32	1.51389841545	2.32507968397	2.49553969955
42	1.51389841545	2.32507968397	2.49553969955

Table 4.3: Convergence of the cut-off frequencies in CO 3.

Contour 3		a=1.0 b=1.2	
N	1st root	2nd root	3rd root
	2.52	3.12	3.96
4	2.54517903974	3.20483249727	4.46797986234
8	2.52008660766	3.10743756200	3.95155938823
16	2.51978498694	3.10507311287	3.94108335815
32	2.51978494032	3.10507258791	3.94107854482
42	2.51978494032	3.10507258791	3.94107854482

4.2.2. Calculations for Botanic Curve

Results of the system determinant are shown in the figures with changing k values from values 1 to 10 with 0.04 steps.

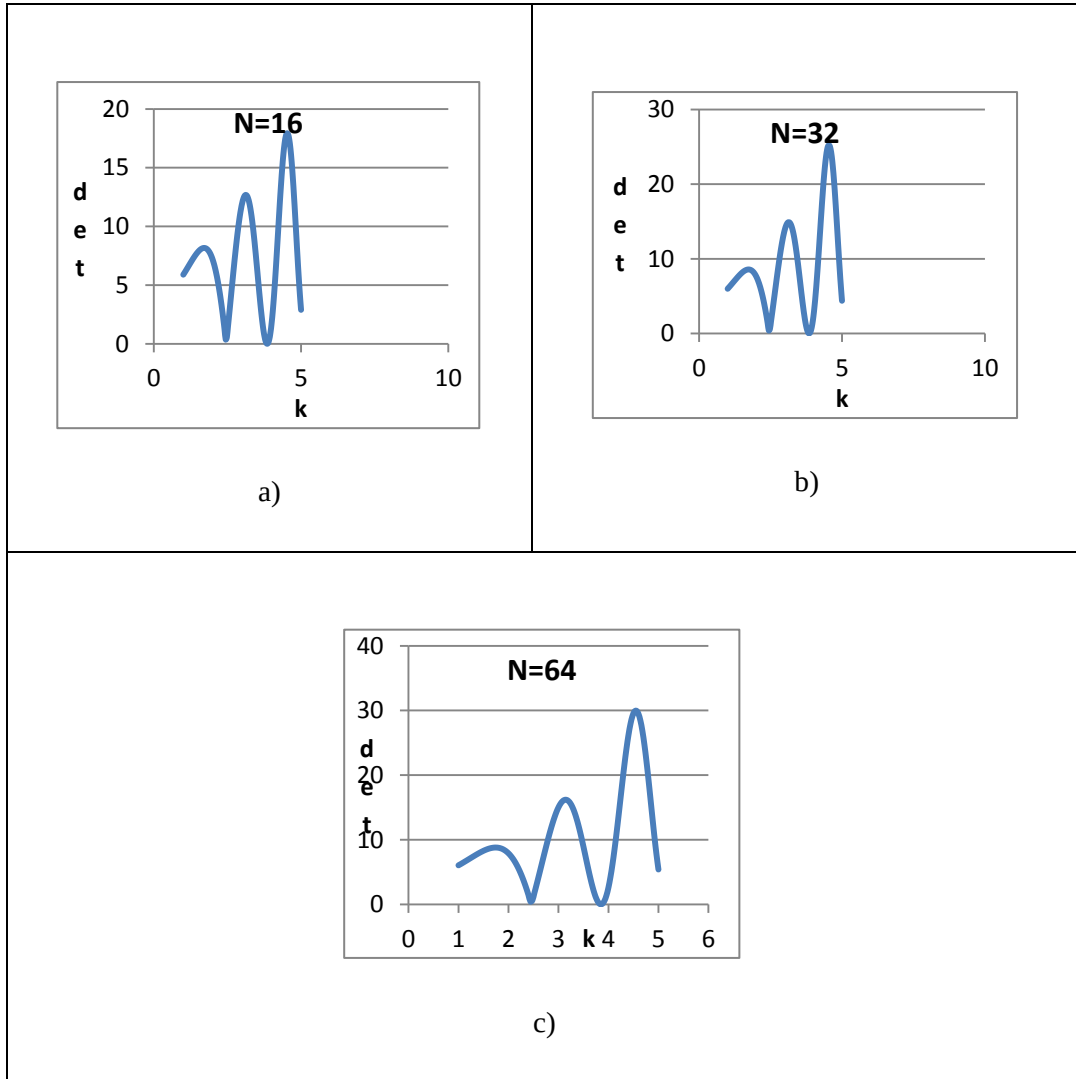


Figure 4.9: In BC 1 relation between cut-off frequency and $\det(I + H_N(k))$.
a) $N=16$ b) $N=32$ c) $N=64$.

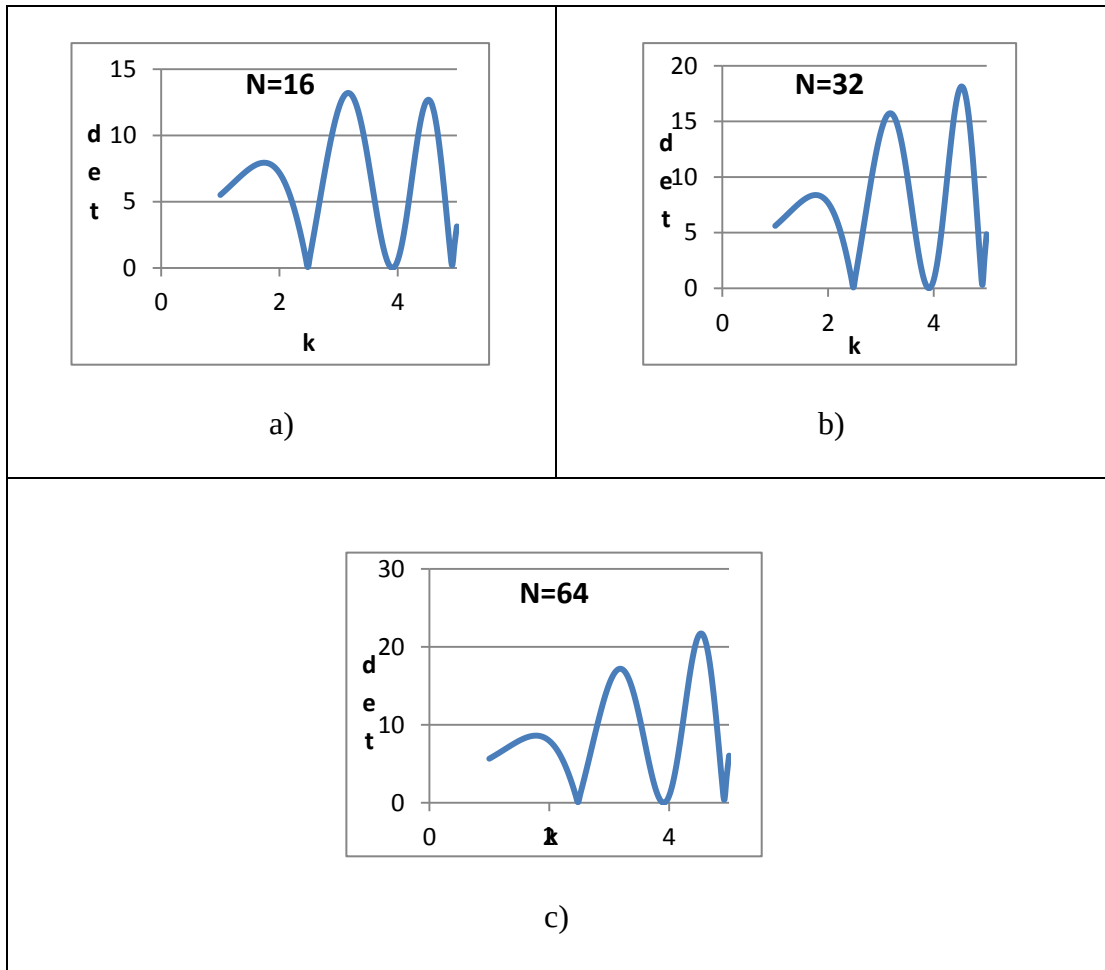


Figure 4.10: In BC 2 relations between cut-off frequency and $\det(I + H_N(k))$.
a) $N=16$ b) $N=32$ c) $N=64$.

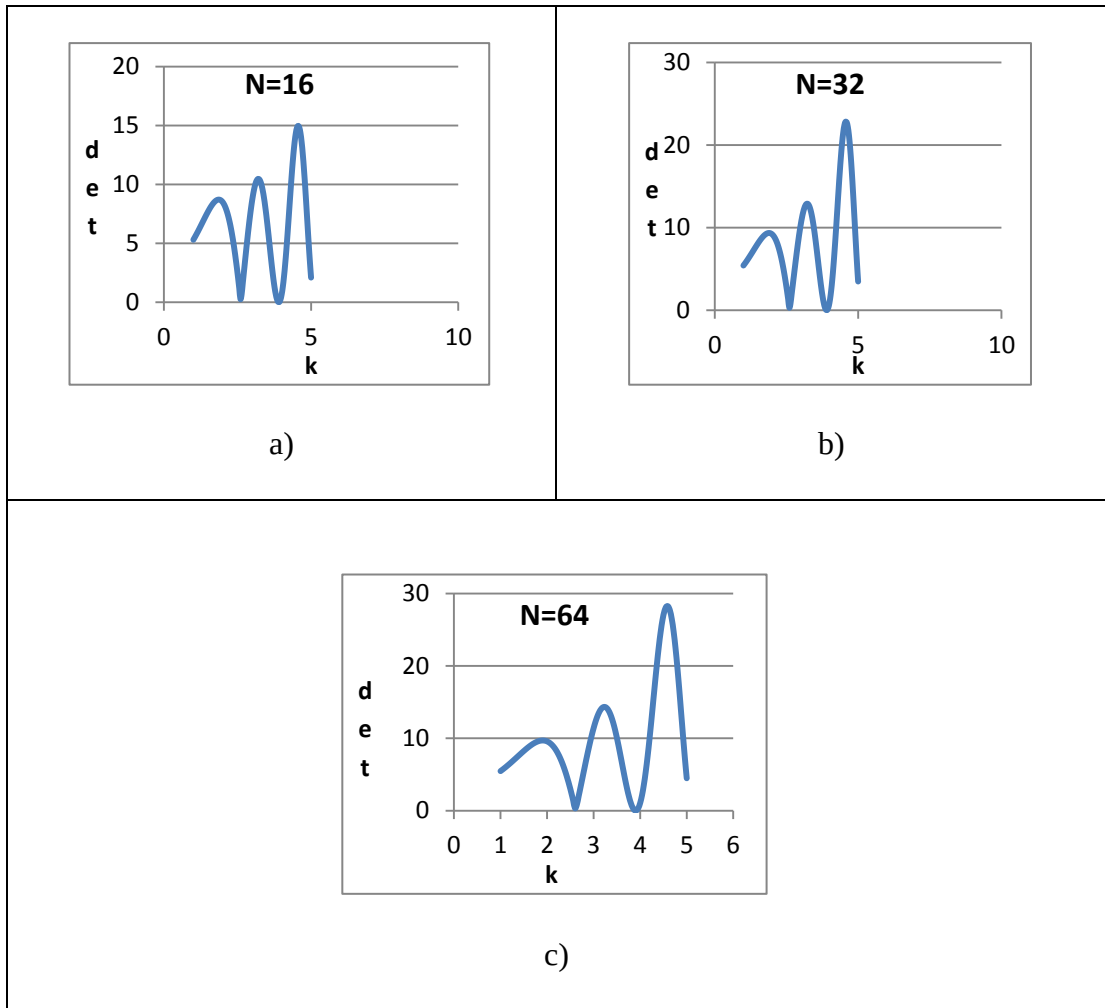


Figure 4.11: In BC 3 relations between cut-off frequency and $\det(I + H_N(k))$.
a) $N=16$ b) $N=32$ c) $N=64$.

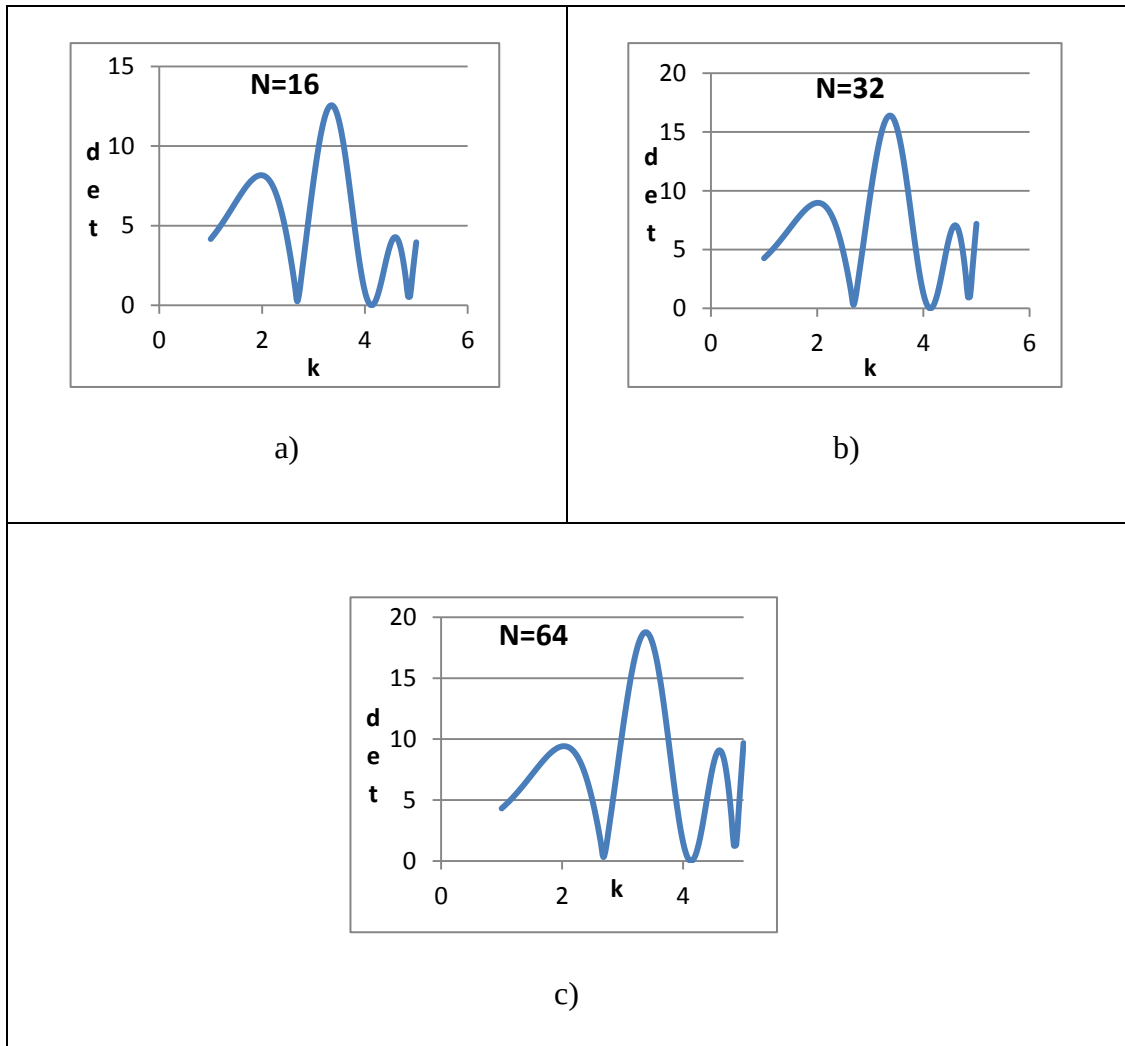


Figure 4.12: In BC 4 relations between cut-off frequency and $\det I + H_N k$.
a) $N=16$ b) $N=32$ c) $N=64$.

Convergences of the cut-off frequency to system size are shown for changing N and k also figures of the contours are plotted for each contour.

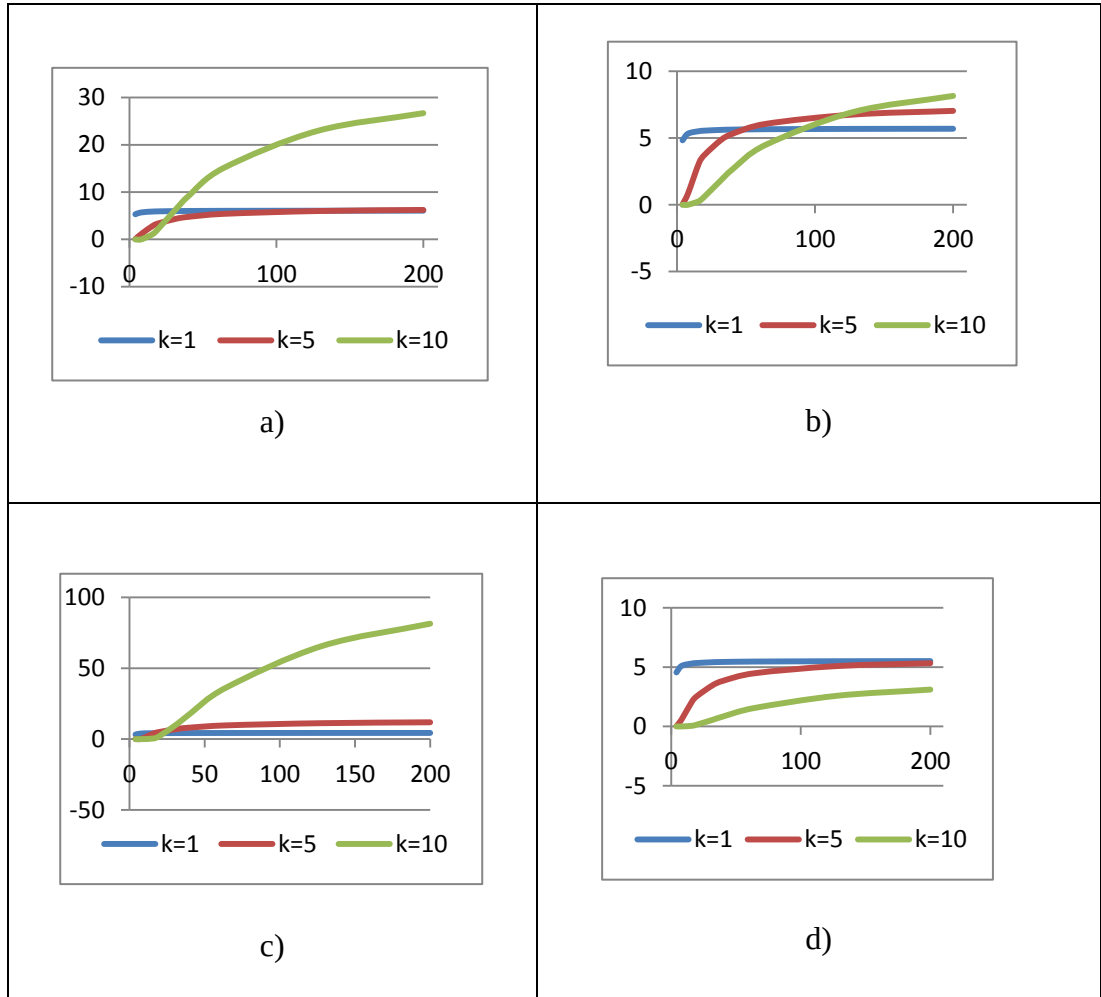


Figure 4.13: Convergency of the cut-off frequency to system size.
for Botanic Curve a) BC 1 b) BC 2 c) BC 3 d) BC 4.

Tables of Convergence of the first and second and third cut-off frequencies (kc) for different shapes of Cassini Oval.

Table 4.4: Convergence of the cut-off frequencies in BC 1.

Contour 1		a=3 b=0.125	
N	1st root	2nd root	3rd root
	2.44	3.84	5.24
4	2.45071250431	3.87561277	5.405956454
8	2.45597183389	3.851139074	5.219227476
16	2.45597453446	3.850861006	5.220105139
32	2.45597453447	3.850861007	5.220105152
42	2.45597453447	3.850861007	5.220105152

Table 4.5: Convergence of the cut-off frequencies in BC 2.

Contour 2		a=4 b=0.125	
N	1st root	2nd root	3rd root
	2.48	3.92	4.92
4	2.49357458802	3.952146336	6.180253573
8	2.47715701746	3.909599195	4.926664016
16	2.47722702715	3.909996838	4.91542297
32	2.47722702720	3.909996866	4.915422347
42	2.47722702720	3.909996866	4.915422347

Table 4.6: Convergence of the cut-off frequencies in BC 3.

Contour 3		a=3 b=0.25	
N	1st root	2nd root	3rd root
	2.6	3.92	5.08
4	2.595068055	3.987816378	5.636736831
8	2.611392542	3.907818788	5.107719961
16	2.611445066	3.903289494	5.081098048
32	2.611445067	3.903289689	5.081097096
42	2.611445067	3.903289689	5.081097096

Table 4.7 Convergence of the cut-off frequencies in BC 4.

Contour 4		a=4 b=0.25	
N	1st root	2nd root	3rd root
	2.68	4.12	4.84
4	2.806683829	5.510827508	5.018288371
8	2.690153195	4.138664127	5.513453615
16	2.691028209	4.131541374	4.859980484
32	2.691028235	4.131537591	4.859842921
42	2.691028235	4.131537591	4.859842921

4.3. Neumann Problem Calculations for Cassini Oval

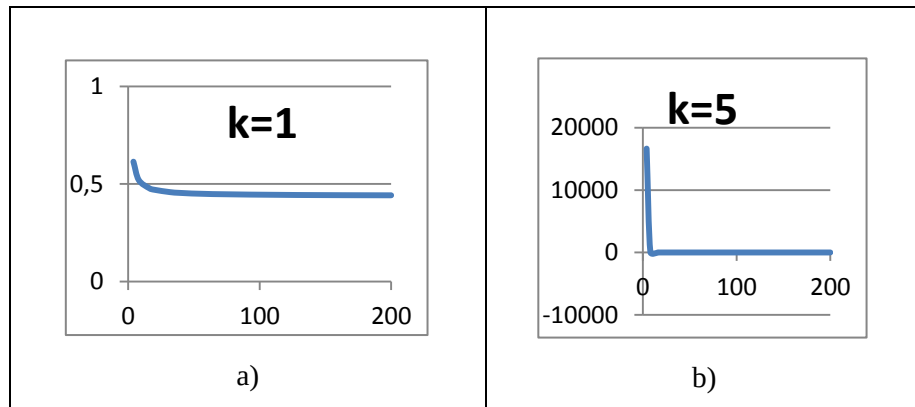


Figure 4.14: Graphs of CO 1 a) $k=1$ b) $k=5$.

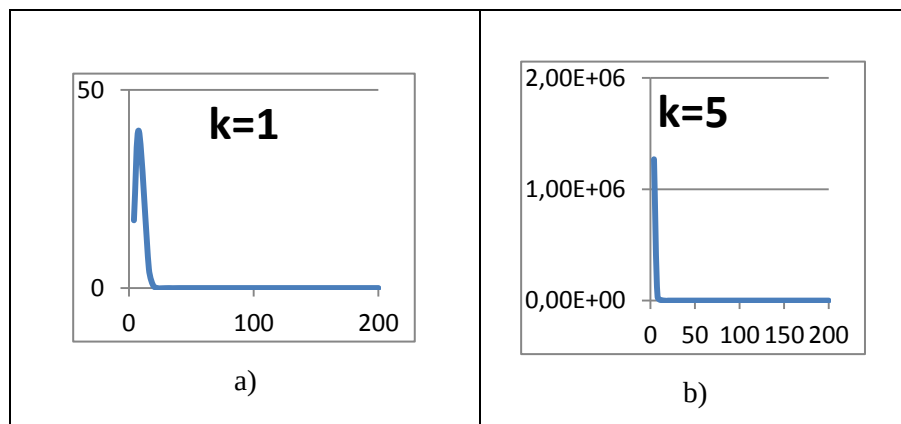


Figure 4.15: Graphs of CO 2 a) $k=1$ b) $k=5$.

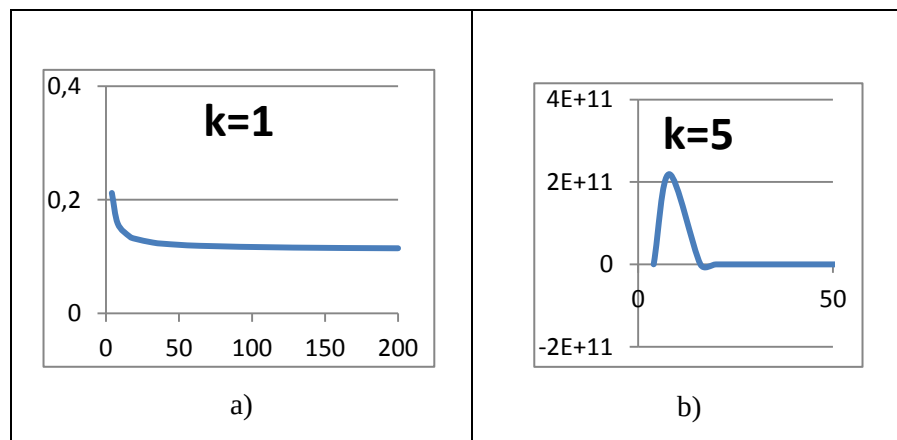


Figure 4.16: Graphs of CO 3 a) $k=1$ b) $k=5$.

5. EVOLUTINARY APPROACH

One of the ways for using of wave numbers which we found in previous section is investigation of propagation of the electromagnetic wave along the waveguides. Let us see how we can use evolutionary approach [17] for that.

5.1. A Description of the Waveguide and the Notations

The waveguide is geometrically homogeneous along its axis, Oz . Its invariable cross-section domain, S , is bounded by a closed singly- connected contour, L . The shape of L may be rather arbitrary provided that none of the possible its inner angles (i.e., measured within S) exceed π . The standard waveguides with the rectangular cross section satisfy this requirement. A right-handed triplet $\{\mathbf{z}, \mathbf{l}, \mathbf{n}\}$ (where $\mathbf{z} \times \mathbf{l} = \mathbf{n}$ and so on) of the mutually orthogonal unit vectors is introduced. The vector $\mathbf{z}$ is oriented along the waveguide axis, Oz ; the vector $\mathbf{l}$ is tangential to the contour, L ; and $\mathbf{n}$ is the outer normal to the domain S . The waveguide surface has the properties of the perfect electric conductor. The waveguide is filled with a lossy medium specified by its conductivity, σ . The relative permittivity and permeability of the medium within the waveguide are taken as $\varepsilon = \mu = 1$ [17].

5.2. Governing Equations

The electromagnetic field quantities are the functions of four independent variables, namely: two transverse coordinates, which are accumulated in the vector $\mathbf{r}$, and the variables (z, t) . In the form (2) of the expected solution, every field component is presented as a product of two functions. In every product, one function depends on (z, t) , only, and the other one depends on $\mathbf{r}$, solely. This trick is known in the partial differential equation theory as an incomplete separation of the variables [34]. In order to apply this method effectively, we should first rearrange Maxwell's equations [17]. Projecting these equations onto the waveguide cross section and the waveguide axis results in

$$\partial_z \mathbf{E} + \mu_0 \partial_t \mathbf{H} \times \mathbf{z} = \nabla_{\perp} E_z \quad (5.1)$$

$$\nabla_{\perp} \cdot \mathbf{z} \times \mathbf{E} = 0 \quad (5.2)$$

$$\partial_z \mathbf{z} \times \mathbf{H} = \epsilon_0 \partial_t \mathbf{E} + \sigma \mathbf{E} \quad (5.3)$$

$$\nabla_{\perp} \cdot \mathbf{H} \times \mathbf{z} = \epsilon_0 \partial_t E_z + \sigma E_z \quad (5.4)$$

$$\nabla_{\perp} \cdot \mathbf{H} = 0 \quad (5.5)$$

$$\nabla_{\perp} \cdot \mathbf{E} + \partial_z E_z = -\frac{\sigma}{\epsilon_0} \int_0^t (\nabla_{\perp} \cdot \mathbf{E} + \partial_z E_z) dt' \quad (5.6)$$

where $\nabla_{\perp}$ is the transverse part of the operator nabla, ∇ . This operator, $\nabla_{\perp}$, acts on the transverse variables, $(\mathbf{r})$, only. Equation (5.1) and (5.2) are the projections of the vector equation, $\nabla \times \mathcal{E} = \mu_0 \partial_t \mathcal{H}$, onto the waveguide cross-section, S , and axis Oz , respectively. Equation (5.3) and (5.4) are the projections of equation $\nabla \times \mathcal{H} = \epsilon_0 \partial_t \mathcal{E} + \sigma \mathcal{E}$ onto the domain S and the axis Oz , respectively. Equation (5.5) is the divergent equation, $\mu_0 \nabla \cdot \mathbf{H} = 0$. Combination of the equation $\epsilon_0 \nabla \cdot \mathcal{E} = \rho$ and the continuity equation, $\nabla \cdot \mathcal{J} = \partial_t \rho$ for the induced current and charge densities, $\mathcal{J} = \sigma \mathcal{E}$ and ρ , results in equation (5.6). A complete set of the Maxwell's differential equation (5.1) - (5.6) should be supplemented with the boundary conditions as;

$$E_z|_L = 0, \mathbf{l} \cdot \mathbf{E}|_L = 0, \mathbf{n} \cdot \mathbf{H}|_L = 0 \quad (5.7)$$

5.3. Solving The Problem

$$\mathbf{E}(\mathbf{r}, z, t) = V(z, t) \epsilon_0^{-1/2} \nabla_{\perp} \phi(\mathbf{r}) \quad (5.8)$$

$$\mathbf{H}(\mathbf{r}, z, t) = I(z, t) \mu_0^{-1/2} \mathbf{z} \times \nabla_{\perp} \phi(\mathbf{r}) \quad (5.9)$$

where the scalar functions $V(z, t)$, $I(z, t)$ and $\phi(\mathbf{r})$ should be found. The free-space constants, ϵ_0 and μ_0 , are installed in equation (5.8) and (5.9) specially in order to provide later the fields $\mathbf{E}$ and $\mathbf{H}$ with their physical dimensions Vm^{-1} (volt per meter) and Am^{-1} (ampere per meter), respectively [17]. The longitudinal field component, E_z , can be resented similarly as

$$E_z(\mathbf{r}, z, t) = E(z, t) \epsilon_0^{-1/2} \phi(\mathbf{r}) \quad (5.10)$$

where $E(z, t)$ is one more unknown functions. When we make substitution as,

$$I(z, t) \nabla_{\perp}^2 \phi(\mathbf{r}) = [c^{-1} \partial_t E(z, t) + 2\gamma c^{-1} E(z, t)] \phi(\mathbf{r}) \quad (5.11)$$

where $c^{-1} = \overline{\epsilon_0 \mu_0}$ is the light speed, and $2\gamma = \sigma/\epsilon_0$ is lossy parameter. The boundary condition $E_z|_L = 0$ results in $\phi(r)|_L = 0$. This fact, jointly with presence of the factor $\nabla_{\perp}^2 \phi(\mathbf{r})$ in equation (5.11), suggests introducing into consideration the well-studied Dirichlet boundary-eigenvalue problem for the transverse Laplacian, $\nabla_{\perp}^2$

$$\nabla_{\perp}^2 \phi(\mathbf{r}) + \kappa_m \phi_m(\mathbf{r}) = 0 \quad \phi_m(r)|_L = 0 \quad (5.12)$$

Let us take the eigenfunctions $\phi_m(\mathbf{r})$ as the potential $\phi(r)$ in formulas equation (5.9), (5.10) and equation (5.11), and scale by the factor $E(z, t)$ as follows: $E(z, t) = \kappa_m e_m(z, t)$. Then equation (5.10) and (5.11) take the form as

$$E_{zm}(\mathbf{r}, z, t) = e_m(z, t) \kappa_m \epsilon_0^{-1/2} \phi_m(\mathbf{r}) \quad (5.13)$$

$$I_m(z, t) = -(\kappa_m c)^{-1} [\partial_t e_m(z, t) + 2\gamma e_m(z, t)] \quad (5.14)$$

where $m = 1, 2$. Equation (5.6) yields a relationship between V_m and e_m as

$$V_m(z, t) - \kappa_m^{-1} \partial_z e_m(z, t) + 2\gamma \int_0^t V_m(z, t') - \kappa_m^{-1} \partial_z e_m(z, t') dt' = 0 \quad (5.15)$$

Factually, this relationship is much simpler. If we introduce a function as

$$f(z, t) = \int_0^t V_m(z, t') - \kappa_m^{-1} \partial_z e_m(z, t') dt' \quad (5.16)$$

and notice that the first pair in equation (5.15) is $\partial_t f(z, t)$, then equation (5.15) yields an initial-value problem for the function $f(z, t)$ as

$$\partial_t f(z, t) + 2\gamma f(z, t) = 0 \quad \text{and} \quad f(z, t)|_{t=0} = 0 \quad (5.17)$$

where the initial condition follows from the mean-value theorem applied to the integral in equation (5.16). Solving this problem results in $f(z, t) \equiv 0$. Hence,

$$V_m(z, t) = \kappa_m^{-1} \partial_z e_m(z, t) \quad (5.18)$$

We shall operate henceforward with the field components taken as

$$\mathbf{H}_m(\mathbf{r}, z, t) = I_m(z, t) \mu_0^{-\frac{1}{2}} [\mathbf{z} \times \nabla_\perp \phi_m(\mathbf{r})] \quad (5.19)$$

$$\mathbf{E}_m(\mathbf{r}, z, t) = V_m(z, t) \epsilon_0^{-1/2} \nabla_\perp \phi_m(\mathbf{r}) \quad (5.20)$$

$$E_{zm}(\mathbf{r}, z, t) = e_m(z, t) \kappa_m \epsilon_0^{-1/2} \phi_m(\mathbf{r}) \quad (5.21)$$

Substitution of the equation (5.18), (5.19) and (5.20) to equation (5.1), it yields

$$\partial_z V_m(z, t) + c^{-1} \partial_t I_m(z, t) - \kappa_m e_m(z, t) = 0 \quad (5.22)$$

Finally, the equation (5.13), (5.14) and (5.18), being substituted to equation (5.22), it results in the well-known Telegraph equation [35].

$$(\kappa_m c)^{-2} \partial_t^2 e_m + 2\gamma (\kappa_m c)^{-2} \partial_t e_m - \kappa_m^{-2} \partial_z^2 e_m + e_m = 0 \quad (5.23)$$

Substitution of the fields' equation (5.19) - (5.21) to equation (5.3) it yields identity, $0 = 0$. The TM-modal wave problem starts with solving the Dirichlet boundary-value problem. Supplement that with the normalization condition, which we choose

$$\nabla_\perp^2 \phi_m(\mathbf{r}) + \kappa_m^2 \phi_m(\mathbf{r}), \quad \phi_m(\mathbf{r})|_{r \in L} = 0, \quad \frac{\mathcal{N}_m}{S} \int_S \phi_m^2(\mathbf{r}) dS = N \quad (5.24)$$

where $\mathcal{N}_m$ is the normalization constant, and N is the nominated number 1. That is, we take this number, 1, and assign the force unit $N = kgms^{-2}$ newton. Evidently, that the constant, $\mathcal{N}_m$, has the physical dimension $N^{\frac{1}{2}}$ provided that potential $\phi_m(\mathbf{r})$ is a dimensionless quantity.

Complete set of the normalized solutions to the problem equation (5.24) generates a complete set of elements of a modal basis as

$$\mathbb{H}_m \mathbf{r} = \mathbf{z} \times \nabla_{\perp} \phi_m r [\mu_0^{-\frac{1}{2}} \mathcal{N}_m] \quad (5.25)$$

$$\mathbb{E}_m \mathbf{r} = \nabla_{\perp} \phi_m r [\epsilon_0^{-\frac{1}{2}} \mathcal{N}_m] \quad (5.26)$$

$$\mathbb{Z}_m \mathbf{r} = \mathbf{z} \kappa_m \nabla_{\perp} \phi_m r [\epsilon_0^{-\frac{1}{2}} \mathcal{N}_m] \quad (5.27)$$

where $\mathcal{N}_m$ is proportional to $N^{\frac{1}{2}}$. Products $[\mu_0^{-\frac{1}{2}} N^{\frac{1}{2}}]$ and $[\epsilon_0^{-\frac{1}{2}} N^{\frac{1}{2}}]$ have physical dimensions A (ampere) and V = kgm²A⁻¹s⁻³ (volt), respectively. Square root of the eigenvalues, $\overline{\kappa_m^2} = \kappa_m$ and the operator $\nabla_{\perp}$, have dimension m⁻¹ (inverse meter). Thus, the elements $\mathbb{H}_m$ have dimension Am⁻¹ (ampere per meter), and $\mathbb{E}_m$ and $\mathbb{Z}_m$, both, have dimension Vm⁻¹ (volt per meter). Hence, the modal amplitudes in equations (5.19) - (2.21) are the dimensionless quantities.

5.4. Waveguide Evolutionary Equations

We will introduce two variables which we scale by z and t .

$$\xi = \kappa_m z \text{ and } \tau = \omega_m t \quad (5.28)$$

where κ_m and $\omega_m = \kappa_m c$ are the cut-off wave number and cut-off frequency. Then TM-waves are presentable as

$$\mathcal{H}_m = I_m \xi, \tau \mathbb{H}_m \mathbf{r} \text{ and } \mathcal{E}_m = V_m \xi, \tau \mathbb{E}_m \mathbf{r} + e_m \xi, \tau \mathbb{Z}_m(\mathbf{r}) \quad (5.29)$$

The modal amplitudes dependent on ξ, τ are dimensionless quantities. The amplitude $e_m \xi, \tau$ should be found by solving equation,

$$\partial_{\tau}^2 e_m \xi, \tau + 2\rho \partial_{\tau} e_m \xi, \tau - \partial_{\xi}^2 e_m \xi, \tau + e_m \xi, \tau = 0 \quad (5.30)$$

where $\varrho = \gamma/\omega_m$ is a dimensionless lossy parameter. The other amplitudes are,

$$V_m \xi, \tau = \partial_\xi e_m \xi, \tau \quad (5.31)$$

$$I_m \xi, \tau = -\partial_\tau e_m \xi, \tau - 2\varrho \partial_\tau e_m \xi, \tau \quad (5.32)$$

5.5. Solving Klein Gordon Equation

Time derivative of the first order, ∂_τ , can be eliminated from equation (5.30) by applying a substitution for the expected solution, $e_m \xi, \tau$, as follows:

$$e_m \xi, \tau = e^{-\varrho\tau} e_m \xi, \tau \quad (5.33)$$

where $e_m \xi, \tau$ is a new unknown function. Simple manipulations with equation (5.33) and (5.31) result in canonical *Klein-Gordon Equation (KGE)* as

$$\partial_\tau^2 e_m \xi, \tau - \partial_\xi^2 e_m \xi, \tau + \eta^2 e_m \xi, \tau = 0 \quad (5.34)$$

$$V_m \xi, \tau = e^{-\varrho\tau} \partial_\xi e_m \xi, \tau \quad (5.35)$$

$$I_m \xi, \tau = -e^{-\varrho\tau} [\partial_\tau e_m \xi, \tau - \varrho \partial_\tau e_m \xi, \tau] \quad (5.36)$$

where $\eta = \sqrt{1 - \varrho^2} \geq 0$, $\varrho = \gamma/\omega_m$, $\gamma = \sigma/2\epsilon_0$ is the lossy parameter. The expected solution to equation (5.34) be in form of,

$$e_m \xi, \tau = A_m \sin[\varpi\tau - \xi\Gamma_m + \varphi_m] \quad (5.37)$$

where $\varpi = \omega/\omega_m$, is dimensional frequency, ω is frequency parameter, $\omega_m = \kappa_m c$, κ_m is square root of an eigenvalue from Dirichlet problem c is the light speed. The constants, A_m and φ_m , are real-valued free numerical parameters. Substitution of the function $e_m \xi, \tau$ to equation (5.34) yields $\Gamma_m = \pm \sqrt{\varpi^2 - \eta^2}$ provided that $\varpi^2 - \eta^2 \geq 0$. In order to simplify notations, we now introduce two “phase functions” as,

$$\Phi_m \xi, \tau = \varpi\tau - \xi\Gamma_m + \varphi_m \equiv \omega t - (z/c) \sqrt{\omega^2 - \omega_m^2 + \gamma^2} + \varphi_m \quad (5.38)$$

$$\Phi_m \xi, \tau = \Phi_m \xi, \tau - \vartheta_m \quad (5.39)$$

where $\vartheta_m = \sin^{-1}(\varrho / \sqrt{\varpi^2 + \varrho^2})$ is lossy phase shift. If $\varrho = \gamma / \omega_m$, (i.e., $\gamma = 0$, the waveguide is lossless) then $\vartheta_m = 0$, and $\Phi_m \xi, \tau = \Phi_m \xi, \tau$. When we use upper sign of $\pm \Gamma_m$ modal amplitudes of TM waves shown as

$$e_m^{\varpi} = A_m e^{-\varrho \tau} \sin \Phi_m \xi, \tau \quad (5.40)$$

$$V_m^{\varpi} = A_m \sqrt{\varpi^2 - \eta^2} e^{-2\varrho \tau} \cos \Phi_m \xi, \tau \quad (5.41)$$

$$I_m^{\varpi} = A_m \sqrt{\varpi^2 + \varrho^2} e^{-2\varrho \tau} \cos^2[\Phi_m \xi, \tau] \quad (5.42)$$

Condition $\Gamma_m = 0$ yields an equation for the frequency parameter, $\varpi_{cut-off}$. Solving this equation specifies so-called “cut-off” frequencies as

$$\varpi_{cut-off} = \eta = \sqrt{1 - \varrho^2} \quad (5.43)$$

where $\varrho = \frac{\gamma}{\omega_m}$, $\omega_m = \kappa_m c$. Nominally admissible value of the parameter $\gamma = \kappa_m c$ yields $\varpi_{cut-off} = 0$. The values of κ_m , $m = 1; 2 \dots$, are called as the “cut-off wave numbers” of the lossless waveguides.

5.6. The Time-harmonic Energetic Waveguide Waves

Exact explicit solutions' equations (5.40) - (5.42) make possible for a fresh look at the energetic field properties in the time-domain. For the time being, we ignore the phase shift, ϑ_m , in equation (5.39). This results in $\Phi_m \xi, \tau = \Phi_m \xi, \tau$ in equation (5.40). Observation of equation (5.41), (5.42) under this supposition suggests introducing a new energetic quantity as

$$S_M^{\varpi} \xi, \tau = \frac{1}{2} (I_m^{\varpi^2} - V_m^{\varpi^2}) = \frac{1}{2} A_m e^{-2\varrho \tau} \cos^2[\Phi_m \xi, \tau] \quad (5.44)$$

$$w_M^{\varpi} \xi, \tau = \frac{1}{2} e_m^{\varpi^2} = \frac{1}{2} A_m e^{-2\varrho \tau} \sin^2[\Phi_m \xi, \tau] \quad (5.45)$$

Notice that the quantity $S_M^{\varpi}(\xi, \tau)$ has the exact physical sense provided that $\varrho = 0$, because $\partial_m \varrho=0 = 0$. That quantity species difference of the energy density stored in the transverse components of the magnetic and electric fields. In the meanwhile, $w_M^{\varpi}(\xi, \tau)$ is the energy density stored in the longitudinal component of the electric field. So, propagation of the TM-modal electromagnetic wave with its amplitudes equation (5.40), (5.42) is accompanied with propagation of two antiphase energetic waves, equation (5.44) and (5.45). This means, physically, that a periodical exchange by energy occurs between $S_M^{\varpi}(\xi, \tau)$ and $w_M^{\varpi}(\xi, \tau)$. Averaging these quantities over the period of oscillations, $T = 2\pi/\omega$, yields

$$\frac{1}{T} \int_{t_0}^{T+t_0} S_M^{\varpi}(\xi, \tau) dt = \frac{1}{T} \int_{t_0}^{T+t_0} w_M^{\varpi}(\xi, \tau) dt = \frac{1}{4} A_m^2 \quad (5.46)$$

So, the averaged values depend neither on time t_0 nor on coordinate ξ . Drop that supposition, $\Phi_m(\xi, \tau) = \Phi_m(\xi, \tau)$, and for the general case introduce a new energetic quantity as

$$S_M^{\varpi}(\xi, \tau) = \frac{1}{2} (I_m^{\varpi^2} - V_m^{\varpi^2}) \quad (5.47)$$

where I_m^{ϖ} and V_m^{ϖ} are specified in equation (5.41), (5.42). It seems natural to name this quantity, $S_M^{\varpi}(\xi, \tau)$ "surplus" of the energy density stored in the transverse field components of a modal wave.

Let us put $A_m = 1$ in equation (5.45) (for the sake of simplicity) and do the same in equation (5.40), (5.42) and (5.47). One can then study the variations in time dependence of the energetic quantities $S_M^{\varpi}(\xi, \tau)$ and $w_M^{\varpi}(\xi, \tau)$ in any waveguide cross section by specifying a chosen coordinate ξ .

In Fig. 1, time dependence of the quantities $S_M^{\varpi}(0, \tau)$ and $w_M^{\varpi}(0, \tau)$ are presented. It is evident that these quantities accomplish antiphased oscillations. Hence, propagation of the TM-modal wave with its amplitudes of equation (5.40) - (5.42) is accompanied by the energetic wave process where exchange by energy between $S_M^{\varpi}(\xi, \tau)$ and $w_M^{\varpi}(\xi, \tau)$ occurs. For different cut-off frequencies the plotted graphics are shown as.

In figure 5.1, the modal energetic waves, which propagate accompany with the electromagnetic wave, are discovered and illustrated graphically. It is evident that these quantities accomplish antiphase oscillations. Hence, propagation of the *TM* - modal wave with its amplitudes is accompanied by the energetic wave process where exchange.

Figure 5.2 exhibits oscillations of the electric and magnetic field energy stored in the waveguide, which completely corresponds with theory of electromagnetic wave propagation.

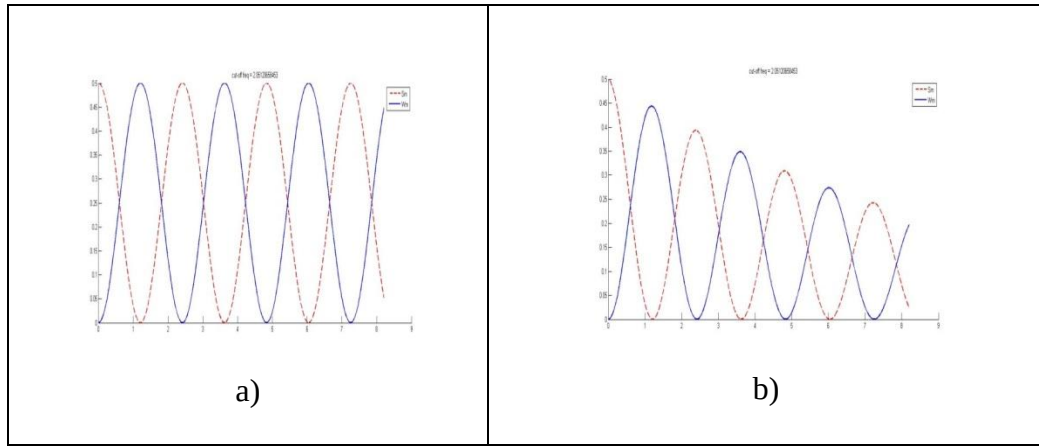


Figure 5.1: Exchange by energy between the surplus of energy
 $Am = 1$, $\xi = 0$, $\varpi = 1.3$, $w_m = 2.051$ a) $\varrho = 0$, b) $\varrho = 0.05$.

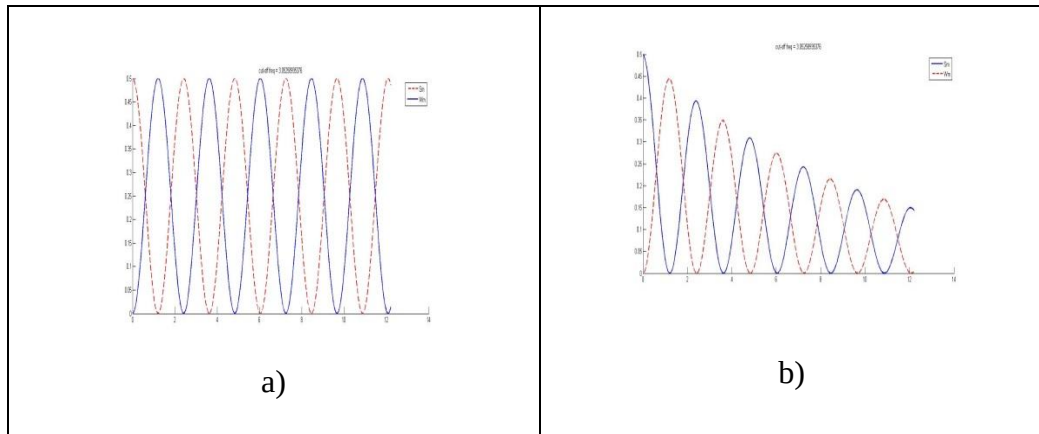


Figure 5.2: Exchange by energy between the surplus of energy
 $Am = 1$, $\xi = 0$, $\varpi = 1.3$, $w_m = 3.052$ a) $\varrho = 0$, b) $\varrho = 0.05$.

6. CONCLUSION

A mathematically rigorous and numerically effective method for solving the boundary value problem on arbitrary shaped contours is used for cutoff wave numbers calculation. The method is based on the ideas of Analytical Regularization Method. As a result of the suggested regularization procedure, the initial boundary value problem is equivalently reduced to the infinite system of the linear algebraic equations of the second kind, i.e., to an equation of the type $(I + H)x = b$ in the space of l_2 . This equation can be solved numerically by means of the truncation method with, in principle, any required accuracy.

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BIOGRAPHY

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