

T.R.
GEBZE INSTITUTE of TECHNOLOGY
GRADUATE SCHOOL of ENGINEERING and SCIENCES

ENDOMORPHISM RINGS OF LIFTING MODULES

HATİCE İNANKİL
A THESIS SUBMITTED for THE DEGREE of
DOCTOR of PHILOSOPHY
DEPARTMENT of MATHEMATICS

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ASSOC. PROF. DR. MUHAMMET TAMER KOŞAN

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**GEBZE YÜKSEK
TEKNOLOJİ ENSTİTÜSÜ**

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İMZA/MÜHÜR

SUMMARY

This thesis consists of four chapters. In the second chapter, main definitions and some properties that we use are given. In the third chapter, we establish a result dual to that of J. Zelmanowitz, in terms of coclosed submodules. Finally in fourth chapter, we apply our result to a particular Galois connection for modules, previously pointed out by [Albu and Nastasescu, 1984], in order to deduce the following consequence: if M is a finitely generated quasi-projective left R -module with $S = \text{End}_R(M)$ and N is an M -generated left R -module, then there exists an order-preserving bijective correspondence between the sets of coclosed left R -submodules of N and coclosed left S -submodules of $\text{Hom}_R(M, N)$. We also relate the dual Goldie dimensions as well as the supplemented properties and the lifting properties of the left R -module N and the left S -module $\text{Hom}_R(M, N)$.

Key Words: Coclosed submodule, Galois connection, Lattice, Lifting module, Quasi-projective module.

ÖZET

Bu tez çalışması dört bölümden oluşmaktadır. İkinci bölümde kullanılan temel tanım ve özellikler verilmiştir. Üçüncü bölümde J. Zelmanowitz'in sonucu eş kapalı alt modüller kullanılarak dualleştirilmiştir Son olarak dördüncü bölümde, elde ettiğimiz sonuç [Albu and Nastasescu, 1984] çalışmasında incelenen modüller için Galois bağlantılarına uygulandı. Bir N sol R –modülünün eş-kapalı altmodüller kümesi ile $Hom_R(M, N)$ nin eş-kapalı sol S - altmodülleri arasında birebir ve örten sırayı koruyan bir eşlemenin varlığını gösteren sonucu elde ettik.

Anahtar Kelimeler: Eş-kapalı Altmodül, Galois Bağlantısı, Latis, Yükselen Modül, Yarı-projektif modül.

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LIST of ABBREVIATIONS and ACRONYMS

<u>Abbreviations</u>	<u>Explanations</u>
<u>and Acronyms</u>	
R	: Ring
$\mathbb{Z}$	: The set of integers
$\mathbb{Q}$	: The set of rational numbers
$\mathbb{N}$	: The set of positive integers
$R\text{-MOD}$	: Category of left R -modules
$Gen(M)$	: The set of M -generated modules which are element in $R\text{-MOD}$
$\sigma[M]$	: The full subcategory of $R\text{-MOD}$ subgenerated by M
$\overset{cs}{K \hookrightarrow L}$	: $K \subseteq L$ is cosmall in M
$\overset{M}{\text{Hom}}_R(M, N)$	: The class of R -module homomorphisms from M to N
$End_R(M)$	: The set of R --module homomorphisms from M to M
$Rad(M)$	: Radical of the module M
$N \ll M$	: N is a small submodule of M
$\mathcal{C}_A$	: The sets of coclosed elements in A
$Tr_M(X)$	: The greatest submodule of X which is M -generated.
$[a, b]$	: The set of elements of a bounded lattice A such that $a \leq x \leq b$

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1. INTRODUCTION

An important general problem in module theory is to relate properties of a module with properties of its endomorphism ring; or more generally, for a left R -module M with $S = \text{End}_R(M)$, to relate properties of a left R -module N with properties of the left S -module $\text{Hom}_R(M, N)$. Within the vast literature dealing with this problem, we point out the work [Zelmanowitz, 1996], which is closely related to our topic and motivates our study. He showed that if M is a left R -module with $S = \text{End}_R(M)$ and N is an M -faithful left R -module, then there exists an order-preserving bijective correspondence between the sets of closed left R -submodules of N and closed left S -submodules of $\text{Hom}_R(M, N)$.

In the third chapter, we establish a result dual to that of J. Zelmanowitz, in terms of coclosed submodules. While closed submodules coincide with complement submodules and every module has a complement, in general coclosed submodules are different of supplement submodules and not every submodule has a supplement [Clark et al. 2006]. These are some of the main obstacles in dualizing results on topics related to coclosed submodules; for instance compare the theories of lifting modules and extending modules. In order to overcome these problems, we need to find the suitable setting for our results and to use a more general approach. We shall first consider the context of bounded lattices and we shall make use of Galois connections between them. The reason for doing that is twofold: first, the notions involved are lattice-theoretic, and secondly, our approach clarifies the exposition and gives a more general explanation for certain conditions we have to impose on our modules. Our main theorem shows that if $(A, \wedge, \vee, 0, 1)$ and $(B, \wedge, \vee, 0, 1)$ are two bounded lattices, and (α, β) is a cosmall Galois connection (in a sense that will be explained in chapter 3), where $\alpha: A \rightarrow B, \alpha(1) = 1$ and $\beta: B \rightarrow A$ preserves finite suprema, then α and β restrict to order-preserving mutually inverse bijections between the sets of coclosed Galois elements in A and coclosed (Galois) elements in B . Also, the main Theorem 1.2 in [Zelmanowitz, 1996] is dualized.

Finially in four chapter. we apply our result to a particular Galois connection for modules, previously pointed out by [Albu and Nastasescu, 1984], in order to deduce the following consequence: if M is a finitely generated quasi-projective left R -module with $S = \text{End}_R(M)$ and N is an M -generated left R -module, then there

exists an order-preserving bijective correspondence between the sets of coclosed left R -submodules of N and coclosed left S -submodules of $\text{Hom}_R(M, N)$. We also relate the dual Goldie dimensions as well as the supplemented properties and the lifting properties of the left R -module N and the left S -module $\text{Hom}_R(M, N)$.

2. BASIC NOTIONS AND DEFINITIONS

In this section we want to introduce terminology and notations we will use throughout investigations. An account of these concepts can be found in the texts by, [Clark et al. 2006], [Rotman, 2008], [Wisbauer,1991].

Throughout, all rings are associative with unity, and all modules are unitary left R -modules.

Definition 2.1: A left R -module, where R is a ring, is an additive abelian group M having a scalar multiplication $R \times M \rightarrow M$, denoted by $(r, m) \rightarrow rm$, such that, for all $m, m' \in M$ and $r, r' \in R$,

- i) $r(m + m') = rm + rm'$,
- ii) $(r + r')m = rm + r'm$,
- iii) $(rr')m = r(r'm)$,
- iv) $1m = m$.

- We often write ${}_R M$ to denote M being a left R -module.

Example 2.2: We present some well known examples for a module.

- i) Every vector space over a field k is a left k -module.
- ii) Every abelian group is a left $\mathbb{Z}$ -module.
- iii) Every ring R is a left module over itself if we define scalar multiplication $R \times R \rightarrow R$ to be the given multiplication of elements of R . More generally, every left ideal in R is a left R -module.

Definition 2.3: A category $\mathcal{C}$ consists of :

- i) a class $Ob \mathcal{C}$, of objects of $\mathcal{C}$;
- ii) for each pair (A, B) of objects of $\mathcal{C}$, a set $Hom_{\mathcal{C}}(A, B)$ of morphisms $f: A \rightarrow B$ from A to B ;

iii) for each triple (A, B, C) of objects of $\mathcal{C}$, a mapping

$$\circ: \text{Hom}_{\mathcal{C}}(B, C) \times \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{C}}(A, C) \quad (2.1)$$

called composition. Here, the composition of morphisms $f: A \rightarrow B$ and $g: B \rightarrow C$ is denoted by gf or $g \circ f$, for which the following axioms hold:

- i) If A, B, C, D are objects of $\mathcal{C}$ and $f: A \rightarrow B$, $g: B \rightarrow C$, $h: C \rightarrow D$ are morphisms, then $(hg)f = h(gf)$ (associativity of composition).
- ii) For every $A \in \text{Ob } \mathcal{C}$, there exists an element 1_A of $\text{Hom}_{\mathcal{C}}(A, A)$ such that $f \circ 1_A = f$ and $1_A \circ g = g$ for every $f \in \text{Hom}_{\mathcal{C}}(A, B)$ and $g \in \text{Hom}_{\mathcal{C}}(B, A)$.

The category of left R -modules, denoted by $R - \text{Mod}$, is the category whose objects are unital left R -modules. The morphisms between left R -modules M and N , are all module homomorphism from M to N , in which $1_M: M \rightarrow M$ is the identity homomorphism defined by $1_M(m) = m$ for every $m \in M$.

Definition 2.4: An R -module N is called M -generated in case there is an epimorphism $M^{(A)} \rightarrow N \rightarrow 0$ for some set A .

Definition 2.5: An R -module N is said to be subgenerated by M if N is isomorphic to a submodule of an M -generated module. We denote by $\sigma[M]$ the full subcategory of $R - \text{Mod}$ whose objects are all R -modules subgenerated by M . A module N is called a subgenerator in $\sigma[M]$ if $\sigma[M] = \sigma[N]$.

- $\sigma[M]$ is closed under direct sums (coproducts), kernels and cokernels.

Definition 2.6: An R -module P is called M -projective if every diagram in $R - \text{Mod}$ with exact row

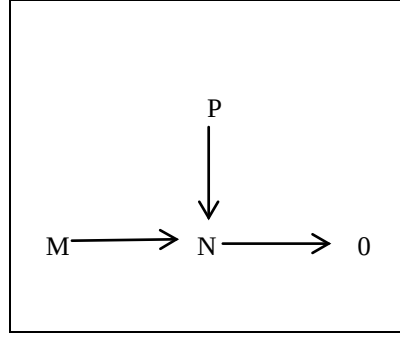


Figure 2.1: Projectivity diagram

can be extended commutatively by some morphism $P \rightarrow M$. If P is P -projective, then P is also called quasi-(or self-) projective.

Lemma 2.7: Let P be an M -projective module. Then,

- i) If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact, then P is M' - and M'' -projective.*
- ii) P is K -projective for submodules K of finitely M -generated modules.*
- iii) If P is finitely generated, then P is K -projective for any $K \in \sigma[M]$.*

Lemma 2.8: Let M be an R -module.

- i) For every family $\{P_i\}_I$ of R -modules, $\bigoplus_I P_i$ is M -projective if and only if P_i is M -projective, for every $i \in I$.*
- ii) For every finite family $\{M_i\}_{i=1}^n$ of R -modules, P is $\bigoplus_{i=1}^n M_i$ -projective if and only if P is M_i -projective for $i = 1, 2, \dots, n$.*

Note that Lemma 2.8 (ii) is not true for infinite families. For example, if $\mathbb{R} = \mathbb{Z}$, then $\mathbb{Q}$ is trivially $\mathbb{Z}$ -projective but is not $\mathbb{Z}^{(\mathbb{N})}$ -projective.

Definition 2.9: A submodule $K \subseteq M$ is called small or superfluous in M , written $K \ll M$, if for every submodule $L \subset M$, the equality $K + L = M$ implies $L = M$. An epimorphism $f : M \rightarrow N$ is said to be small or superfluous if $\text{Ker } f \ll M$; such an f is also called a small cover.

Lemma 2.10: Let K, L, N and M be R -modules.

- i) An epimorphism $f : M \rightarrow N$ in $R\text{-Mod}$ is small if and only if every (mono)morphism $h : L \rightarrow M$ for which $h \circ f$ is an epimorphism is itself an epimorphism.*
- ii) If $f : M \rightarrow N$ and $g : N \rightarrow L$ are two epimorphisms, then $f \circ g$ is small if and only if both f and g are small.*
- iii) If $K \subset L \subset M$, then $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.*
- iv) If $K_1, \dots, K_n$ are small submodules of M , then $K_1 + \dots + K_n$ is also small in M .*
- v) For any $K \ll M$ and $f : M \rightarrow N$, $f(K) \ll N$. The converse is true if f is a small epimorphism.*
- vi) If $K \subset L \subset M$ and L is a direct summand of M , then $K \ll M$ if and only if $K \ll L$.*
- vii) If $K \ll M$, then M is finitely generated if and only if M/K is finitely generated.*

As an example, one can observe that the $\mathbb{Z}$ -module $\mathbb{Z}$ has no non-zero small submodules and $\mathbb{Z}$ is a small submodule in the rationals $\mathbb{Q}$.

Definition 2.11: Given submodules $K \subset L \subset M$, the inclusion $K \subset L$ is called
cosmall in M if $L/K \ll M/K$; we denote this by $K \overset{cs}{\hookrightarrow}_M L$.

Note that this is equivalent to saying that M/K is a small cover of M/L and that
trivially, $L \ll M$ if and only if $0 \overset{cs}{\hookrightarrow}_M L$.

Lemma 2.12: Let $U \subset K \subset L \subset M$ and $G \subset H \subset M$ be any submodules.

i) The followings are equivalent:

- $K \subset L$ is cosmall in M .*
- For any submodule $X \subset M$, $L + X = M$ implies $K + X = M$.*

ii) $U \overset{cs}{\hookrightarrow}_M L$ if and only if $U \overset{cs}{\hookrightarrow}_M K$ and $K \overset{cs}{\hookrightarrow}_M L$.

iii) If $K \overset{CS}{\hookrightarrow} L$ and $K + X = M$ for a submodule $X \subset M$, then $K \cap X \overset{CS}{\hookrightarrow} L \cap X$.

iv) If $G \overset{CS}{\hookrightarrow} H$, $K \overset{CS}{\hookrightarrow} L$ and $G + K = M$, then $K \cap G \overset{CS}{\hookrightarrow} L \cap H$.

v) If $K \subset L \subset M$ and $L = K + S$, where $S \ll M$, then $K \overset{CS}{\hookrightarrow} L$.

vi) If $M = K + L$, $K \subseteq C \subset M$ and $C \cap L \ll M$, then $K \overset{CS}{\hookrightarrow} C$.

vii) Let $f : M \rightarrow N$ be an epimorphism. Then

- If $K \subset L \subset M$ and $K \overset{CS}{\hookrightarrow} L$, then $f(K) \overset{CS}{\hookrightarrow} f(L)$.
- If $C \subseteq D \subseteq N$, then $C \overset{CS}{\hookrightarrow} D$ if and only if $f^{-1}(C) \overset{CS}{\hookrightarrow} f^{-1}(D)$.

Definition 2.13: A submodule $L \subset M$ is called coclosed in M and we write $L \overset{CC}{\hookrightarrow} M$, if L has no proper submodule K for which $K \subset L$ is cosmall in M , that is, $K \overset{CC}{\hookrightarrow} L$ implies $K = L$. Thus L is coclosed in M if and only if, for any proper submodule $K \subset L$, there is a submodule N of M such that $L + N = M$ but $K + N \neq M$.

- Obviously, any direct summand L of M is coclosed in M .

Lemma 2.14: Let $K \subset L \subset M$ be submodules.

- If L is coclosed in M , then L/K is coclosed in M/K .
- If $K \ll L$ and L/K is coclosed in M/K , then L is coclosed in M .
- If $L \subset M$ is coclosed, then $K \ll M$ implies $K \ll L$; hence $\text{Rad}(L) = L \cap \text{Rad}(M)$.
- If L is hollow, then either $L \subset M$ is coclosed in M or $L \ll M$.
- If $f : M \rightarrow N$ is a small epimorphism and L is coclosed in M , then $f(L)$ is coclosed in N .
- If K is coclosed in M , then K is coclosed in L and the converse is true if L is coclosed in M .

Lemma 2.15: Let $L \subset M$ be a submodule. A submodule $K \subset L$ is called a coclosure of L in M if $K \subset L$ is cosmall in M and $K \subset M$ is coclosed; that is, $L/K \ll M/K$, and for $U \subset K$, $K/U \ll M/U$ implies $U = K$. Thus coclosures are characterised by a minimality property with respect to cosmall inclusion.

Notice that in general coclosures need not exist. For example, the $\mathbb{Z}$ -submodule $2\mathbb{Z} \subset \mathbb{Z}$ has no coclosure in $\mathbb{Z}$. One of our concerns will be to find conditions which guarantee the existence of coclosures. Let $K, N \subset M$ be submodules where K is coclosed in M . If $N \subset K$, then K/N is coclosed in M/N . In general $(K + N)/N$ need not be coclosed in M/N .

In this section we give definition of a *dimension* for a modules which is dual to the uniform (or Goldie) dimension.

Definition 2.16: A family $\{N_\lambda\}_\Lambda$ of proper submodules of M is called coindependent if, for any $\lambda \in \Lambda$ and any finite subset $F \subseteq \Lambda \setminus \{\lambda\}$, $N_\lambda + \bigcap_{i \in F} N_i = M$.

Clearly $\{N_\lambda\}_\Lambda$ is coindependent if and only if each of its finite subfamilies is coindependent. Any family $\{N\}$ of one submodule is trivially coindependent.

Definition 2.17: A non-zero R -module M is called hollow if every proper submodule is small in M .

Lemma 2.18: Let M be a non-zero R -module. Then M is hollow if and only if every non-zero factor module of M is indecomposable.

Lemma 2.19: For an R -module M the following properties are equivalent:

- i) M does not contain an infinite coindependent family of submodules.
- ii) For any descending chain $N_1 \supset N_2 \supset \dots$ of submodules of M there exists j such that for all $k \geq j$, $N_k \subseteq N_j$ is cosmall in M .
- iii) $\sup\{k \in \mathbb{N} \mid M \text{ has } k \text{ coindependent submodules}\} = n$, for some $n \in \mathbb{N}$.
- iv) M has coindependent submodules $N_1, \dots, N_n$ such that $\bigcap_{i=1}^n N_i \ll M$ and M/N_i is hollow for every $1 \leq i \leq n$.

v) For some $n \in \mathbb{N}$, there exists a small epimorphism from M to a direct sum of n hollow modules.

vi) For some $n \in \mathbb{N}$, there exists an epimorphism from M to a direct sum of n non-zero modules but no epimorphism from M to a direct sum of more than n non-zero modules.

The natural numbers n in (iii–vi) of Lemma 2.19 are all equal and if M satisfies these conditions we say that M has *hollow dimension* n , denoting this by $hdim(M) = n$. If $M = 0$ we put $hdim(M) = 0$ and if M does not have finite hollow dimension we set $hdim(M) = \infty$.

Notice that M is hollow if and only if it has no nontrivial coindependent family of submodules. Hence a module M is hollow if and only if $hdim(M) = 1$.

Lemma 2.20: Let N be a submodule of M .

i) If $M = M_1 \oplus \dots \oplus M_k$, then $hdim(M) = hdim(M_1) + \dots + hdim(M_k)$.

ii) If $N \ll M$, then $hdim(M) = hdim(M/N)$; conversely, if $hdim(M)$ is finite and equal to $hdim(M/N)$, then $N \ll M$.

3. COSMALL GALOIS CONNECTIONS

Definition 3.1: Let $(A, \leq)$ and $(B, \leq)$ be lattices. A Galois connection between them consists of a pair (α, β) of two order-preserving functions $\alpha: A \rightarrow B$ and $\beta: B \rightarrow A$ such that for $a \in A$ and $b \in B$, we have $\alpha(a) \leq b$, if and only if $a \leq \beta(b)$. Equivalently, (α, β) is a Galois connection if and only if for all $a \in A$, $a \leq \beta\alpha(a)$ and for all $b \in B$, $a\beta(b) \leq b$. An element $a \in A$ (respectively $b \in B$) is called a Galois element if $\beta\alpha(a) = a$ (respectively $\alpha\beta(b) = b$).

Lemma 3.2: Let $(A, \leq)$ and $(B, \leq)$ be lattices, and (α, β) be a Galois connection, where $\alpha: A \rightarrow B$ and $\beta: B \rightarrow A$. Then:

- i) $\alpha\beta\alpha = \alpha$ and $\beta\alpha\beta = \beta$.
- ii) α preserves all suprema in A and β preserves all infimama in B .
- iii) If A and B are bounded, then $\alpha(0) = 0$ and $\beta(1) = 1$.
- iv) The restrictions of α and β to the corresponding sets of Galois elements are mutually inverse bijections [Erne et al. 1993].

Definition 3.3: Let $(A, \wedge, \vee, 0, 1)$ be a bounded lattice.

- i) An element a is called a complement in A if there exists $a' \in A$ such that $a \wedge a' = 0$ and $a \vee a' = 1$.
- ii) Let $a, a' \in A$ such that $a \leq a'$. Then a is called cosmall in $[a, 1]$ if for any $x \in A$, $1 = a' \vee x$ implies $1 = a \vee x$.
- iii) An element $a' \in A$ is called coclosed in A if for any $a \in A$, a' cosmall in $[a, 1]$ implies $a = a'$.
- iv) An element $a' \in A$ is called a coclosure of $a \in A$ in A if a is cosmall in $[a', 1]$ and a' is coclosed in A .

The following lemma is well-known for submodule lattices, and its proof is straightforward.

Lemma 3.4: Let $(A, \wedge, \vee, 0, 1)$ be a bounded lattice.

- i) Let $a, b, c \in A$ such that $a \leq b \leq c$. Then c is cosmall in $[a, 1]$ if and only if b is cosmall in $[a, 1]$ and c is cosmall in $[b, 1]$.*
- ii) If A is modular, then every complement in A is coclosed.*

Note that a coclosure of a given element might not exist. For instance, the subgroup $2\mathbb{Z}$ of the abelian group $\mathbb{Z}$ has no coclosure in the subgroup lattice of $\mathbb{Z}$ [Clark et al. , 2006]. We continue with an easy, but useful lemma.

Lemma 3.5: Let $(A, \wedge, \vee, 0, 1)$ and $(B, \wedge, \vee, 0, 1)$ be bounded lattices, and (α, β) be a Galois connection, where $\alpha: A \rightarrow B, \alpha(1) = 1$ and $\beta: B \rightarrow A$ preserves finite suprema. Then

- i) For all $b \in B$, b is cosmall in $[\alpha\beta(b), 1]$.*
- ii) Every coclosed element of B is Galois.*

Proof 3.5: Let $(A, \wedge, \vee, 0, 1)$ and $(B, \wedge, \vee, 0, 1)$ be bounded lattices, and (α, β) be a Galois connection, where $\alpha: A \rightarrow B, \alpha(1) = 1$ and $\beta: B \rightarrow A$ preserves finite suprema.

- i) Let $b \in B$ and let $1 = b \vee b'$ for some $b' \in B$. Then*

$$1 = \alpha\beta(1) = \alpha\beta(b) \vee \alpha\beta(b') \leq \alpha\beta(b) \vee b' \quad (3.1)$$

and so $1 = \alpha\beta(b) \vee b'$. Hence b is cosmall in $[\alpha\beta(b), 1]$.

- iii) Clear by i).*

Note that if (α, β) is a Galois connection between two bounded lattices and α, β are lattice homomorphisms, then $\alpha(1) = 1$ and β preserves finite suprema. We introduce two special types of Galois connection, which will be useful .

Definition 3.6: Let $(A, \wedge, \vee, 0, 1)$ and $(B, \wedge, \vee, 0, 1)$ be bounded lattices, and (α, β) be a Galois connection, where $\alpha: A \rightarrow B, \alpha(1) = 1$ and $\beta: B \rightarrow A$. We say that (α, β) is cosmall if for all $a \in A$, $\beta\alpha(a)$ is cosmall in $[a, 1]$. We say that a cosmall Galois connection (α, β) is UCC if for every coclosed element $a \in A$, a is the unique coclosure of $\beta\alpha(a)$ in A .

- We present some examples to illustrate the above theory.

Example 3.7: Consider the abelian groups $G = \mathbb{Z}_p \times \mathbb{Z}_{q^2}$ for some primes p and q with $p \neq q$, and $G' = \mathbb{Z}_2 \times \mathbb{Z}_4$, where $\mathbb{Z}_n$ denotes the cyclic group of order $n \in \mathbb{N}$. Their subgroup lattices $S(G)$ and $S(G')$ have the following forms respectively:

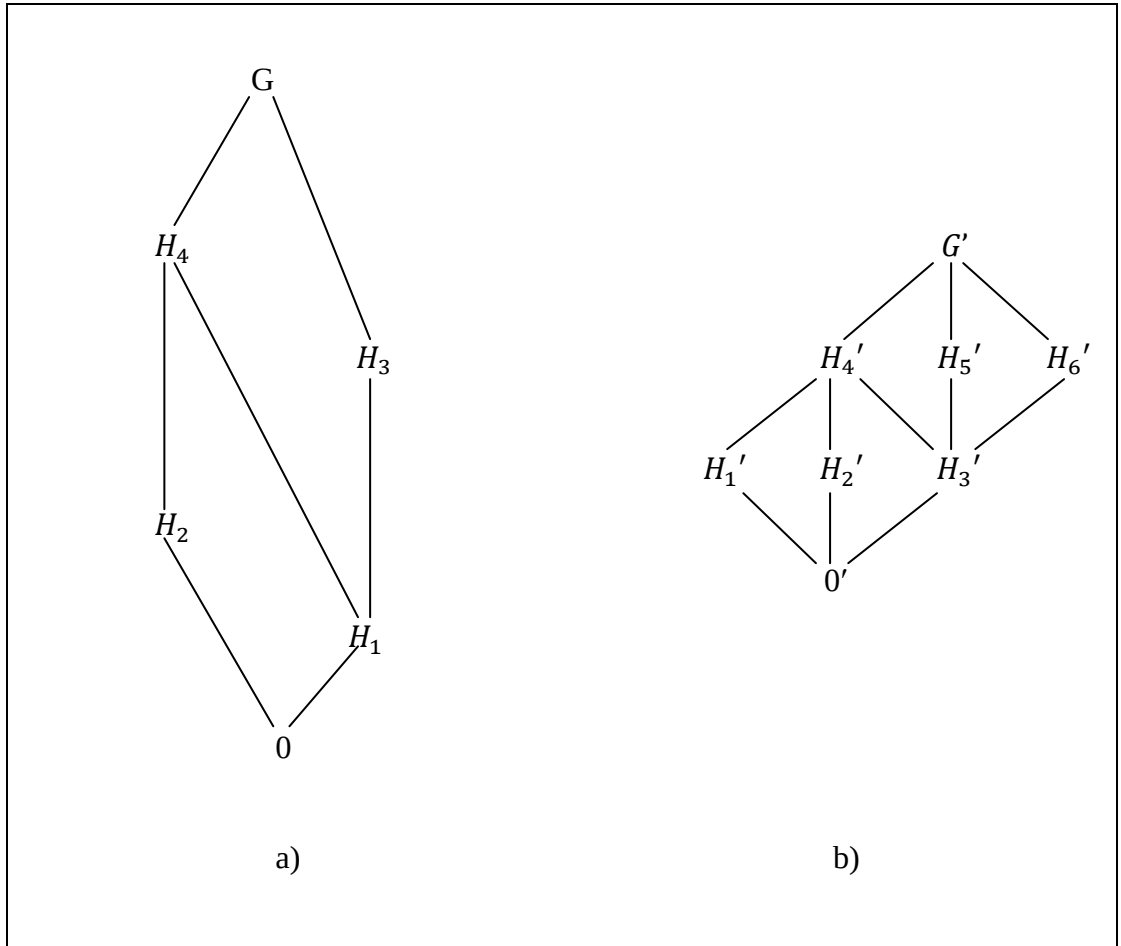


Figure 3.1: a) Subgroup lattice of group G b) Subgroup lattice of group G'

It is easy to check that H_1 is cosmall in $[0, G]$, H_4 is cosmall in $[H_2, G]$, H_1 and H_4 are the only subgroups of G which are not coclosed, 0 is a coclosure of H_1 in G , and H_2 is a coclosure of H_4 in G . Also, H_3' is cosmall in $[0', G']$, H_4' is cosmall in $[H_1', G']$ and $[H_2', G']$, H_3' and H_4' are the only subgroups of G' which are not coclosed, $0'$ is a coclosure of H_3' in G' , and H_1', H_2' are coclosures of H_4' in G' . For the properties of subgroups with unique (co)closure and cosmall subgroups of abelian groups the reader is referred to [Crevei and Olteanu, 2008], [Crevei and Şuteu Szöllosi, 2007].

- Consider the functions $\alpha: S(G) \rightarrow S(G)$ defined by $\alpha(0) = \alpha(H_1) = 0$, $\alpha(H_2) = \alpha(H_4) = H_3$, $\alpha(H_3) = H_2$ and $\alpha(G) = G$, $\beta: S(G) \rightarrow S(G)$ defined by $\beta(0) = \beta(H_1) = H_1$, $\beta(H_2) = \beta(H_4) = H_3$, $\beta(H_3) = H_4$, $\beta(G) = G$. Then (α, β) is a Galois connection from the lattice $(S(G), \subseteq)$ to itself. For every $H \in S(G) \setminus \{H_1, H_4\}$ we have $\alpha\beta(H) = H$. Hence 0 , H_2 , H_3 and G are Galois elements in the domain $B = S(G)$ of α . Also H_1 is cosmall in $[\alpha\beta(H_1), G] = [0, G]$ and H_4 is cosmall in $[\alpha\beta(H_4), G] = [H_2, G]$. On the other hand, for every $H \in S(G) \setminus \{0, H_2\}$ we have $\beta\alpha(H) = H$. Hence H_1 , H_3 , H_4 and G are Galois elements in the domain $A = S(G)$ of α . Also, $\beta\alpha(0) = H_1$ is cosmall in $[0, G]$ and $\beta\alpha(H_2) = H_4$ is cosmall in $[H_2, G]$. Moreover, for every coclosed $H \in S(G)$, H is the unique coclosure of $\beta\alpha(H)$ in $S(G)$. Hence (α, β) is a UCC cosmall Galois connection. Note that H_2 is a coclosed element, but not a Galois element in the domain $A = S(G)$ of α . Hence not every coclosed element of A is Galois under the hypothesis of Lemma 3.5.

- Consider the functions $\alpha: S(G) \rightarrow S(G)$ defined by $\alpha(H_2) = H_4$, $\alpha(H_3) = G$, and $\alpha(H) = H$ for every $H \in S(G) \setminus \{H_2, H_3\}$, and $\beta: S(G) \rightarrow S(G)$ defined by $\beta(H_2) = 0$, $\beta(H_3) = H_1$ and $\beta(H) = H$ for every $H \in S(G) \setminus \{H_2, H_3\}$. Then (α, β) is a Galois connection from the lattice $(S(G), \subseteq)$ to itself. But (α, β) is not a cosmall Galois connection, because for instance $\beta\alpha(H_3) = G$ is not cosmall in $[H_3, G]$. Using the same setting, let us also show that the hypothesis on β to preserve finite suprema in Lemma 3.5 can not be removed. Note that we have $\beta(H_3 + H_4) = G \neq \beta(H_3) + \beta(H_4)$. Then H_2 is not cosmall in $[\alpha\beta(H_2), G] = [0, G]$. Also, H_4 is a Galois element in the codomain $B = S(G)$ of α , but not a coclosed element.

• Consider the functions $\alpha': S(G') \rightarrow S(G')$ defined by $\alpha'(H'_3) = 0$, $\alpha'(H'_4) = H'_1$ and $\alpha(H') = H'$ for every $H' \in S(G') \setminus \{H'_3, H'_4\}$, and $\beta': S(G') \rightarrow S(G')$ defined by $\beta'(0') = H'_3$, $\beta'(H') = H'_4$ and $\beta(H') = H'$ for every $H' \in S(G') \setminus \{0', H'_1\}$. Then (α', β') is a Galois connection from the lattice $(S(G'), \subseteq)$ to itself. For every $H' \in S(G') \setminus \{H'_3, H'_4\}$ we have $\alpha\beta(H') = H'$. Also H'_3 is cosmall in $[\alpha'\beta'(H'_3), G'] = [0', G']$ and H'_4 is cosmall in $[\alpha'\beta'(H'_4), G'] = [H'_1, G']$. On the other hand, for every $H' \in S(G') \setminus \{0', H'_1\}$ we have $\beta'\alpha'(H') = H'$. Also $\beta'\alpha'(0') = H'_3$ is cosmall in $[0', G']$ and $\beta'\alpha'(H'_1) = H'_4$ is cosmall in $[H'_1, G']$. Hence (α', β') is a cosmall Galois connection. But it is not UCC, because for the coclosed subgroup H'_1 , $\beta'\alpha'(H'_1)$ has two coclosures in G' , namely H'_1 and H'_2 .

We finish this section with some results which show that the cosmall property and the dual Goldie dimension may be transferred through cosmall Galois connections.

Lemma 3.8: Let $(A, \leq)$ and $(B, \leq)$ be bounded lattices, and (α, β) be a cosmall Galois connection, where $\alpha: A \rightarrow B$, $\alpha(1) = 1$ and $\beta: B \rightarrow A$ preserves finite suprema. Then

- i) *Let $a, a' \in A$. Then a' is cosmall in $[a, 1]$ if and only if $\alpha(a')$ is cosmall in $[\alpha(a), 1]$.*
- ii) *Let $b, b' \in B$. Then $b' \in B$ is cosmall in $[b, 1]$ if and only if $\beta(b')$ is cosmall in $[\beta(b), 1]$.*

Proof 3.8: Let $(A, \leq)$ and $(B, \leq)$ be bounded lattices, and (α, β) be a cosmall Galois connection, where $\alpha: A \rightarrow B$, $\alpha(1) = 1$ and $\beta: B \rightarrow A$ preserves finite suprema.

- i) *Assume that a' is cosmall in $[a, 1]$. Let $1 = \alpha(a') \vee b'$ for some $b' \in B$. Then $1 = \beta(1) = \beta\alpha(a') \vee \beta(b')$. Since (α, β) is a cosmall Galois connection, it follows that $1 = a' \vee \beta(b')$. Now by the hypothesis, we must have $1 = a \vee \beta(b')$. Then $1 = \alpha(1) = \alpha(a) \vee \alpha\beta(b') \leq \alpha(a) \vee b'$, and so $1 = \alpha(a) \vee b'$. Hence $\alpha(a')$ is cosmall in $[\alpha(a), 1]$. Conversely, assume that $\alpha(a')$ is cosmall in $[\alpha(a), 1]$.*

Let $1 = a' \vee a''$ for some $a'' \in A$. Then $1 = \alpha(1) = \alpha(a') \vee \alpha(a'')$, whence $1 = \alpha(a) \vee \alpha(a'')$, by the hypothesis. Then $1 = \beta(1) = \beta\alpha(a) \vee \beta\alpha(a'')$. Since (α, β) is a cosmall Galois connection, it follows that $1 = a \vee a''$. Hence a' is cosmall in $[a, 1]$.

ii) Assume that b' is cosmall in $[b, 1]$. Let $1 = \beta(b') \vee a'$ for some $a' \in A$. Then $1 = \alpha(1) = \alpha\beta(b') \vee \alpha(a') \leq b' \vee \alpha(a')$, hence $1 = b' \vee \alpha(a')$, and by hypothesis $1 = b \vee \alpha(a')$. Then $1 = \beta(1) = \beta(b) \vee \beta\alpha(a')$. Since (α, β) is a cosmall Galois connection, it follows that $1 = \beta(b) \vee a'$. Hence $\beta(b')$ is cosmall in $[\beta(b), 1]$. Conversely, assume that $\beta(b')$ is cosmall in $[\beta(b), 1]$. Let $1 = b' \vee b''$ for some $b' \in B$. Then $1 = \beta(1) = \beta(b') \vee \beta(b'')$, whence $1 = \beta(b) \vee \beta(b'')$ by the hypothesis. Then $1 = \alpha(1) = \alpha\beta(b) \vee \alpha\beta(b'') \leq b \vee b''$, and so $1 = b \vee b''$. Hence b' is cosmall in $[b, 1]$.

Let $(X, \wedge, \vee, 0, 1)$ be a bounded modular lattice. Recall that a subset Y of $X \setminus \{1\}$ is called meet-independent if $(Y_1 \wedge \cdots \wedge Y_n) \vee x$ for every finite subset $\{y_1, \dots, y_n\}$ of Y and every $x \in Y \setminus \{y_1, \dots, y_n\}$. If there is a finite supremum d of all numbers k such that X has a meet-independent subset with k elements, then X has dual Goldie dimension (or hollow dimension) d ; otherwise X has infinite dual Goldie dimension [Grzeszczuk and Puczyłowski, 1984]. We denote the dual Goldie dimension of X by $hdim(X)$.

Corollary 3.9: Let $(A, \wedge, \vee, 0, 1)$ and $(B, \wedge, \vee, 0, 1)$ be bounded lattices, and (α, β) a Galois connection, where $\alpha: A \rightarrow B, \alpha(1) = 1$ and $\beta: B \rightarrow A$ preserves finite suprema. Then

i) $hdim(A) \leq hdim(B)$.

ii) If every element of A is Galois, then $hdim(A) = hdim(B)$.

Proof 3.9: Let $(A, \wedge, \vee, 0, 1)$ and $(B, \wedge, \vee, 0, 1)$ be bounded lattices, and (α, β) a Galois connection, where $\alpha: A \rightarrow B, \alpha(1) = 1$ and $\beta: B \rightarrow A$ preserves finite suprema.

i) We prove that if A has a finite meet-independent subset with m elements, then so has B . This will imply that $\text{hdim}(A) \leq \text{hdim}(B)$ in both finite and infinite cases for $\text{hdim}(A)$. To this end, let $\{a_1, \dots, a_m\}$ be a meet-independent subset of A . We claim that $Y = \{\alpha(a_1), \dots, \alpha(a_m)\}$ is a meet-independent subset of B with m elements. First, we point out that $\alpha(a_i) \neq 1$ for every $i \in \{1, \dots, m\}$. Indeed, if there is $i \in \{1, \dots, m\}$ such that $\alpha(a_i) = 1$, then $\beta\alpha(a_i) = 1$. Since $\beta\alpha(a_i)$ is cosmall in $[a_i, 1]$, we must have $a_i = 1$ which is a contradiction. Secondly, suppose that $\alpha(a_i) = \alpha(a_j)$ for some $i, j \in \{1, \dots, m\}$ with $i \neq j$. Since $a_i \vee a_j = 1$, we have $\alpha(a_i) \vee \alpha(a_j) = \alpha(a_i \vee a_j) = \alpha(1) = 1$. We conclude that $\alpha(a_i) = \alpha(a_j) = 1$, which contradicts the previous point. Hence Y has m elements. Finally, the meet-independence of $\{a_1, \dots, a_m\}$ is known to be equivalent to the condition $(a_1 \wedge \dots \wedge a_{k-1}) \vee a_k = 1$ for every $k \in \{2, \dots, m\}$. Let $k \in \{2, \dots, m\}$. Then $\alpha(a_1 \wedge \dots \wedge a_{k-1}) \vee \alpha(a_k) = 1$. Since $\alpha(a_1 \wedge \dots \wedge a_{k-1}) \leq \alpha(a_1) \wedge \dots \wedge \alpha(a_{k-1})$, it follows that $(\alpha(a_1) \wedge \dots \wedge \alpha(a_{k-1})) \vee \alpha(a_k) = 1$. This shows that Y is meet-independent.

ii) Suppose that every element of A is Galois. When A has infinite dual Goldie dimension, the conclusion is clear by (i). Assume that $\text{hdim}(A) = m$. By Theorem 9 in [Grzeszczuk and Puczyłowski, 1984], A has a meet-independent subset $\{a_1, \dots, a_m\}$ such that $a_1 \wedge \dots \wedge a_m$ is cosmall in $[0, 1]$. Now assume that there is a meet-independent subset Y of B with more than m elements, possibly infinite. Then there is a meet-independent subset $\{b_1, \dots, b_{m+1}\} \subseteq Y$ of B .

$$\{\alpha(a_1), \dots, \alpha(a_m), b_{m+1}\} \quad (3.2)$$

is also a meet-independent subset of B . Then we have

$$(\alpha(a_1) \wedge \dots \wedge \alpha(a_m)) \vee b_{m+1} = 1 \quad (3.3)$$

Since every element of A is Galois, it follows that

$$\begin{aligned} (a_1 \wedge \dots \wedge a_m) \vee \beta(b_{m+1}) &= (\beta\alpha(a_1) \wedge \dots \wedge \beta\alpha(a_m)) \vee \beta(b_{m+1}) = \\ &= \beta((\alpha(a_1) \wedge \dots \wedge \alpha(a_m)) \vee b_{m+1}) = 1 \end{aligned} \quad (3.4)$$

Since $a_1 \wedge \dots \wedge a_m$ is cosmall in $[0,1]$, we deduce that $\beta(b_{m+1}) = 1$. Then $1 = \alpha\beta(b_{m+1}) \leq b_{m+1}$, and so $b_{m+1}=1$, which is a contradiction. Therefore, by (i) and the above, $\text{hdim}(B) = m = \text{hdim}(A)$.

4.ENDOMORPHISM RINGS OF LIFTING MODULES

4.1. Correspondences of Coclosed Submodules

Let $(A, \wedge, \vee, 0, 1)$ and $(B, \wedge, \vee, 0, 1)$ be bounded lattices, and (α, β) be a Galois connection, where $\alpha: A \rightarrow B$ and $\beta: B \rightarrow A$. We have seen in Lemma 3.5 and Example 3.7 that the set of coclosed elements in B is in general strictly included in the set of Galois elements in B . Also, we have already seen that the restrictions of α and β to the corresponding sets of Galois elements are mutually inverse bijections. We shall show that, under certain conditions, these bijections restrict to ones between the corresponding sets of complement Galois elements, or between the corresponding sets of coclosed Galois elements.

Lemma 4.1.1: Let $(A, \wedge, \vee, 0, 1)$ and $(B, \wedge, \vee, 0, 1)$ be bounded lattices, and (α, β) be a Galois connection, where $\alpha: A \rightarrow B, \alpha(1) = 1$ and $\beta: B \rightarrow A$ preserves finite suprema. Assume that all complements in A are Galois elements. Then

- i) If β is injective, then α preserves complement Galois elements.*
- ii) β preserves complement Galois elements.*
- iii) If β is injective, then α and β restrict to order-preserving mutually inverse bijections between the sets of complement Galois elements in A and B .*

Proof. 4.1.1: First note that, since $0 \in A$ is a Galois element, we have $\beta\alpha(0) = 0$, and so $\beta(0) = 0$.

i) Assume that β is injective. Let a be a complement (Galois) element in A . Then there is $a' \in A$ such that $a \wedge a' = 0$ and $a \vee a' = 1$. Hence $\alpha(a) \vee \alpha(a') = \alpha(1) = 1$. Also, we have

$$0 = a \wedge a' = \beta\alpha(a) \wedge \beta\alpha(a') = \beta(\alpha(a) \wedge \alpha(a')) \quad (4.1)$$

whence $\alpha(a) \wedge \alpha(a') = 0$ by the injectivity of β . Thus $\alpha(a)$ is a complement in B . By Lemma 3.2, $\alpha(a)$ is a Galois element in B .

ii) Let b be a complement Galois element in B . Then there is $b' \in B$ such that $b \wedge b' = 0$ and $b \vee b' = 1$. Then $\beta(b) \wedge \beta(b') = \beta(0) = 0$ and $\beta(b) \vee \beta(b') = \beta(1) = 1$. Thus $\beta(b)$ is a complement in A . By Lemma 3.2, $\beta(b)$ is a Galois element in A .

iii) Clear by i), ii) and Lemma 3.2.

Next we need to recall the following notions, which are the lattice-theoretic versions of the corresponding ones for modules .

Definition 4.1.2: Let $(A, \wedge, \vee, 0, 1)$ be a bounded lattice.

i) *An element $a \in A$ is called a supplement if it is a supplement of some $a' \in A$, that is, a is minimal in A with the property that $1 = a \vee a'$.*

ii) *A is called supplemented if every $a \in A$ has a supplement in A .*

iii) *A is called amply supplemented if for every $a \in A$ there exists a supplement $x \in A$ such that a is cosmall in $[x, 1]$.*

iv) *A is called UCC if every $a \in A$ has a unique coclosure in A , which will be denoted by $\bar{a}$.*

v) *A is called lifting if for every $a \in A$ there exists a complement $x \in A$ such that a is cosmall in $[x, 1]$.*

The following lemma has a similar proof as for modules. You can see Corollary 3.8 in [Ganesan and Vanaja, 2002], Lemma 1.1 and Proposition 1.5 in [Keskin, 2000] and Chapter 41 in [Wisbauer, 1991].

Lemma 4.1.3: Let $(A, \wedge, \vee, 0, 1)$ be a bounded modular lattice.

i) *If $a \in A$ is a supplement, then a is coclosed.*

ii) *If A is supplemented and $a \in A$ is coclosed, then a is a supplement.*

iii) *If A is amply supplemented, then every element $a \in A$ has a coclosure in A .*

iv) *If A is amply supplemented UCC and $a_1, a_2 \in A$ are such that $a_1 \leq a_2$, then their coclosures satisfy $\bar{a}_1 \leq \bar{a}_2$.*

Now we may relate supplemented and lifting type properties of some bounded modular lattices.

Theorem 4.1.4: Let $(A, \wedge, \vee, 0, 1)$ and $(B, \wedge, \vee, 0, 1)$ be bounded lattices, and (α, β) a Galois connection, where $\alpha: A \rightarrow B, \alpha(1) = 1$ and $\beta: B \rightarrow A$ preserves finite suprema. Assume that every element of A is Galois. Then:

i) A is supplemented if and only if B is supplemented.

ii) If B is lifting, then A is lifting. Conversely, if β is injective and A is lifting, then B is lifting.

Proof 4.1.4: Let $(A, \wedge, \vee, 0, 1)$ and $(B, \wedge, \vee, 0, 1)$ be bounded lattices, and (α, β) a Galois connection, where $\alpha: A \rightarrow B, \alpha(1) = 1$ and $\beta: B \rightarrow A$ preserves finite suprema. Assume that every element of A is Galois.

i) First assume that A is supplemented. Let $b \in B$. Then $\beta(b)$ has a supplement x in A . Hence $1 = \beta(b) \vee x$, which implies that

$$1 = \alpha(1) = \alpha\beta(b) \vee \alpha(x) \leq b \vee \alpha(x) \quad (4.2)$$

and so $1 = b \vee \alpha(x)$. Now let $1 = b \vee y'$ for some $y' \in B$ with $y' \leq \alpha(x)$. Then

$$1 = \beta(1) = \beta(b) \vee \beta(y') \quad (4.3)$$

Since x is Galois, we have $\beta(y') \leq \beta\alpha(x) = x$, whence $\beta(y') = x$ by the minimality of x . It follows that $\alpha(x) = \alpha\beta(y') \leq y'$, and so $y' = \alpha(x)$. This shows the minimality of the supplement $\alpha(x)$ of b in B . Thus B is supplemented. Conversely, assume that B is supplemented, and let $a \in A$. Then $\alpha(a)$ has a supplement y in B , and so $1 = \alpha(a) \vee y$. By the hypothesis it follows that

$$1 = \beta(1) = \beta\alpha(a) \vee \beta(y) = a \vee \beta(y) \quad (4.4)$$

Now let $1 = a \vee x'$ for some $x' \in A$ with $x' \leq \beta(y)$. Then $1 = \alpha(1) = \alpha(a) \vee \alpha(x')$ and $\alpha(x') \leq \alpha\beta(y) \leq y$. By the minimality of y , $\alpha(x') = y$. Since x' is Galois, we have $x' = \beta\alpha(x') = \beta(y)$. This shows the minimality of the supplement $\beta(y)$ of a in A . Thus A is supplemented.

ii) Assume that B is lifting. Let $a \in A$. Then there exists a complement $b' \in B$ such that $\alpha(a)$ is cosmall in $[b', 1]$. Since a is Galois, $a = \beta\alpha(a)$ is cosmall in $[\beta(b'), 1]$ by Lemma 3.8. By Lemmas 3.4 and 3.5, b' is a Galois element in B . Then by Lemma 4.1.1, $\beta(b')$ is a complement in A . It follows that A is lifting. Now assume that β is injective and A is lifting. Let $b \in B$. Then there exists a complement $a' \in A$ such that $\beta(b)$ is cosmall in $[a', 1]$. By Lemma 3.8, $\alpha\beta(b)$ is cosmall in $[\alpha(a'), 1]$. By Lemma 3.5, b is cosmall in $[\alpha\beta(b), 1]$. Hence by Lemma 3.4, b is cosmall in $[\alpha(a'), 1]$. By Lemma 4.1.1, $\alpha(a')$ is a complement in B . It follows that B is lifting.

Next we establish our general theorem on a bijective correspondence between sets of coclosed elements induced by some special Galois connections. In order to obtain it, it is natural to try to define some maps by means of unique coclosures of elements, when they do exist. We see in the following theorem that the Galois connection and the condition that these maps are well-defined create a slightly asymmetric situation.

Theorem 4.1.5: Let $(A, \wedge, \vee, 0, 1)$ and $(B, \wedge, \vee, 0, 1)$ be bounded lattices, and (α, β) be a UCC Galois connection, where $\alpha: A \rightarrow B$, $\alpha(1) = 1$ and $\beta: B \rightarrow A$ preserves finite suprema. For $x \in A \cup B$ denote by $\bar{x}$ the unique coclosure of x , when it does exists.

i) Denote

$$\begin{aligned} A &= \{a \in A: a \text{ is coclosed in } A \text{ and } \alpha(a) \text{ has a unique coclosure in } B\} \\ B &= \{b \in B: b \text{ is coclosed in } B \text{ and } \beta(b) \text{ has a unique coclosure in } A\} \end{aligned} \quad (4.5)$$

Consider $\varphi: \mathcal{A} \rightarrow \mathcal{B}$, defined by $\varphi(a) = \overline{\alpha(a)}$ for every $a \in \mathcal{A}$, and $\psi: \mathcal{B} \rightarrow \mathcal{A}$ defined by $\psi(b) = \overline{\beta(b)}$ for every $b \in \mathcal{B}$. Then φ is a well-defined map if and only

if $\mathcal{A}$ coincides with the set of coclosed elements $a \in A$ such that $\alpha(a)$ is coclosed in B . Also, ψ is a well-defined map.

ii) Denote

$$\begin{aligned}\mathcal{A} &= \{a \in A: a \text{ is coclosed in } A \text{ and } \alpha(a) \text{ is coclosed in } B\} \\ \mathcal{B} &= \{b \in B: b \text{ is coclosed in } B \text{ and } \beta(b) \text{ has a unique coclosure in } A\}\end{aligned}\quad (4.6)$$

Then the maps $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ defined by $\varphi(a) = \alpha(a)$ for every $a \in \mathcal{A}$, and $\psi: \mathcal{B} \rightarrow \mathcal{A}$ defined by $\psi(b) = \overline{\beta(b)}$ for every $b \in \mathcal{B}$, are mutually inverse bijections.

Proof 4.1.5: Let $(A, \wedge, \vee, 0, 1)$ and $(B, \wedge, \vee, 0, 1)$ be bounded lattices, and (α, β) be a UCC Galois connection, where $\alpha: A \rightarrow B, \alpha(1) = 1$ and $\beta: B \rightarrow A$ preserves finite suprema.

i) First assume that φ is well-defined. Let $a \in \mathcal{A}$. Then $\varphi(a) = \overline{\alpha(a)} \in \mathcal{B}$, and so $\beta(\overline{\alpha(a)})$ has a unique coclosure, say a_0 . Then $\beta(\overline{\alpha(a)})$ is cosmall in $[a_0, 1]$. Since $\alpha(a)$ is cosmall in $[\overline{\alpha(a)}, 1]$, $\beta\alpha(a)$ is cosmall in $[\beta(\overline{\alpha(a)}), 1]$ by Lemma 3.8. Then by Lemma 3.4 $\beta\alpha(a)$ is cosmall in $[a_0, 1]$. Since (α, β) is UCC cosmall, a is the unique coclosure of $\beta\alpha(a)$, hence $a_0 = a$. Then $\beta(\overline{\alpha(a)})$ is cosmall in $[a, 1]$. By Lemmas 3.5 and 3.8, $\overline{\alpha(a)} = \alpha\beta(\overline{\alpha(a)})$ is cosmall in $[\alpha(a), 1]$. Hence $\overline{\alpha(a)} = \alpha(a)$, and so $\alpha(a)$ is coclosed in B . Now assume that $\mathcal{A}$ coincides with the set of coclosed elements $a \in A$ such that $\alpha(a)$ is coclosed in B . Then $\varphi(a) = \overline{\alpha(a)} = \alpha(a)$ for every $a \in \mathcal{A}$. Let $a \in \mathcal{A}$. Then $\alpha(a)$ is coclosed in B by hypothesis. Since (α, β) is UCC cosmall, $\beta\alpha(a)$ has a unique coclosure in A . Hence $\varphi(a) = \alpha(a) \in \mathcal{B}$, and so φ is well-defined. In order to show that ψ is well-defined, let $b \in \mathcal{B}$. Then $\overline{\beta(b)}$ is coclosed in A . Since $\beta(b)$ is cosmall in $(\overline{\beta(b)}, 1]$, $\alpha\beta(b)$ is cosmall in $[\alpha(\overline{\beta(b)}), 1]$ by Lemma 3.8. Since $b \in B$ is coclosed, we have $\alpha\beta(b) = b$ by Lemma 3.5. Then b is cosmall in $[\alpha(\overline{\beta(b)}), 1]$. It follows that $[\alpha(\overline{\beta(b)})] = b$, because $b \in B$ is coclosed. Hence $\alpha(\overline{\beta(b)})$ is coclosed in B . Thus $\psi(b) = \overline{\beta(b)} \in \mathcal{A}$. Note that ψ is also well-defined if the codomain is the set of coclosed elements $a \in A$ such that $\alpha(a)$ is coclosed in B .

ii) The maps φ and ψ are well-defined by i). If $a \in \mathcal{A}$, then a is the unique coclosure of $\beta\alpha(a)$ in A because (α, β) is UCC cosmall, and we have $\psi\varphi(a) = \beta\alpha(a) = a$. If $b \in \mathcal{B}$, then we have $\varphi\psi(b) = \alpha(\overline{\beta(b)}) = b$ as above. Therefore, φ and ψ are mutually inverse bijections.

Theorem 4.1.6: Let $(A, \wedge, \vee, 0, 1)$ and $(B, \wedge, \vee, 0, 1)$ be bounded lattices, and (α, β) be a Galois connection, where $\alpha: A \rightarrow B, \alpha(1) = 1$ and $\beta: B \rightarrow A$ preserves finite suprema. Then there are mutually inverse bijections between the sets $\mathcal{C}_A$ of coclosed elements in A and $\mathcal{C}_B$ of coclosed elements in B provided one of the following conditions holds:

- i) Every element in A is Galois.
- ii) A is amply supplemented modular and (α, β) is UCC cosmall. If either every element in A is Galois, or A is amply supplemented modular UCC and (α, β) is cosmall, then the above bijections are order-preserving.

Proof 4.1.6: In both cases we use the notation from Theorem 4.1.5, and prove that $\mathcal{A} = \mathcal{C}_A$ and $\mathcal{B} = \mathcal{C}_B$. Then by Theorem 4.1.5 the required mutually inverse bijections will be given by φ and ψ . Clearly, that $\mathcal{A} \subseteq \mathcal{C}_A$ and $\mathcal{B} \subseteq \mathcal{C}_B$.

i) Assume that every element in A is Galois. Then (α, β) is clearly UCC cosmall. Let $a \in \mathcal{C}_A$. By the hypothesis, we have $\alpha = \beta\alpha(a)$. We claim that $\alpha(a)$ is coclosed in B . To this end, let $b' \in B$ be such that $\alpha(a)$ is cosmall in $[b', 1]$. By Lemma 3.8, $\alpha = \beta\alpha(a)$ is cosmall in $[\beta(b'), 1]$. Since a is coclosed in A , we have $\beta(b') = a$. It follows that $\alpha(a) = \alpha\beta(b') \leq b'$, whence $b' = \alpha(a)$. Thus $\alpha(a)$ is coclosed in B , and so $a \in \mathcal{A}$. Now let $b \in \mathcal{C}_B$. We claim that $\beta(b)$ is coclosed in A . To this end, let $a' \in A$ such that $\beta(b)$ is cosmall in $[a', 1]$. By Lemma 3.8, $\alpha\beta(b)$ is cosmall in $[\alpha(a'), 1]$. By Lemma 3.5, every coclosed element in B is a Galois element, hence $b = \alpha\beta(b)$, so b is cosmall in $[\alpha(a'), 1]$. Since b is coclosed in B , it follows that $\alpha(a') = b$. Now by the hypothesis we have $a' = \beta\alpha(a') = \beta(b)$. Thus $\beta(b)$ is coclosed in A , and so $b \in \mathcal{B}$. It follows that $\mathcal{A} = \mathcal{C}_A$ and $\mathcal{B} = \mathcal{C}_B$. By Theorem 4.1.5, the maps $\varphi = \alpha$ and $\psi = \beta$ are mutually inverse bijections between that $\mathcal{C}_A$ and $\mathcal{C}_B$.

ii) Assume that A is amply supplemented modular and (α, β) is UCC cosmall. Let $a \in \mathcal{C}_A$. We claim that $\alpha(a)$ is coclosed in B . To this end, let $b' \in B$ be such that $\alpha(a)$ is cosmall in $[b', 1]$. By Lemma 3.8, $\beta\alpha(a)$ is cosmall in $[\beta(b'), 1]$. Since A is amply supplemented modular, Lemma 4.1.3 yields a coclosure of $\beta(b')$ in A , say a_0 . Then $\beta(b')$ is cosmall in $[a_0, 1]$, whence $\beta\alpha(a)$ is cosmall in $[a_0, 1]$ by Lemma 3.4. Hence a_0 is a coclosure of $\beta\alpha(a)$ in A . Since $(\alpha; \beta)$ is UCC cosmall, a is the unique coclosure of $\beta\alpha(a)$ in A , hence $a_0 = a$. Then $a \leq \beta(b')$, which implies $\alpha(a) \leq b'$ because $(\alpha; \beta)$ is a Galois connection. Hence $\alpha(a) = b'$, and so $\alpha(a)$ is coclosed in B . Thus $a \in \mathcal{A}$. Now let $b \in \mathcal{C}_B$. We claim that $\beta(b)$ has a unique coclosure in A . The existence of a coclosure of $\beta(b)$ in A follows by Lemma 4.1.3, because A is amply supplemented modular. Now assume that a_1 and a_2 are two coclosures of $\beta(b)$ in A . Then $\beta(b)$ is coclosed both in $[a_1, 1]$ and in $[a_2, 1]$. By Lemma 3.8, $\alpha\beta(b)$ is cosmall both in $[\alpha(a_1), 1]$ and in $[\alpha(a_2), 1]$. Since b is coclosed, we have $\alpha\beta(b) = b$ by Lemma 3.5. Hence b is cosmall both in $[\alpha(a_1), 1]$ and in $[\alpha(a_2), 1]$. But b is coclosed in B , hence we must have $\alpha(a_1) = \alpha(a_2) = b$. Since $(\alpha; \beta)$ is a cosmall Galois connection, $\beta\alpha(a_1) = \beta\alpha(a_2)$ is cosmall in $[a_1, 1]$. Then a_1 is a coclosure of $\beta\alpha(a_2)$ in A . Since $(\alpha; \beta)$ is UCC cosmall, a_2 is the unique coclosure of $\beta\alpha(a_2)$ in A , hence $a_1 = a_2$. Then $\beta(b)$ has a unique coclosure in A . Thus $b \in \mathcal{B}$. It follows that $\mathcal{A} = \mathcal{C}_A$ and $\mathcal{B} = \mathcal{C}_B$. By Theorem 4.5, the maps φ and ψ are mutually inverse bijections between $\mathcal{C}_A$ and $\mathcal{C}_B$. Finally, if every element in A is Galois, then φ and ψ are order-preserving, because so are α and β . Also, if A is amply supplemented modular UCC and $(\alpha; \beta)$ is cosmall, then φ and ψ are order-preserving by hypothesis and Lemma 4.1.3.

Corollary 4.1.7: Let $(A, \wedge, \vee, 0, 1)$ and $(B, \wedge, \vee, 0, 1)$ be bounded lattices, and (α, β) a UCC cosmall Galois connection, where $\alpha: A \rightarrow B, \alpha(1) = 1$ and $\beta: B \rightarrow A$ preserves finite suprema, and A is amply supplemented modular. Then the following are equivalent:

i) The mutually inverse bijections from Theorem 4.1.6 are the restrictions of α and β to the sets $\mathcal{C}_A$ of coclosed elements in A and $\mathcal{C}_B$ of coclosed elements in B respectively.

ii) Every coclosed element in A is Galois.

Proof 4.1.7: Note that (i) is equivalent to the condition: $\beta(b) = \overline{\beta(b)}$ for every coclosed element $b \in B$. First assume (i). Let $a \in A$ be a coclosed element. By Theorems 4.1.5 and 4.1.6 (ii) we have $a = \overline{\beta\alpha(a)}$. By the hypothesis we have $\overline{\beta\alpha(a)} = \beta\alpha(a)$. Hence $a = \beta\alpha(a)$, and so a is Galois. Conversely, assume (ii). Let $b \in B$ be a coclosed element. By hypothesis the coclosed element $\overline{\beta(b)} \in A$ is Galois, hence $\overline{\beta(b)} = \beta\alpha(\overline{\beta(b)})$. By Theorems 4.1.5 and 4.1.6 (ii) we have $\alpha(\overline{\beta(b)}) = b$. Hence $\beta(b) = \overline{\beta(b)}$.

Remark 4.1.8:

i) Under the hypotheses of Theorem 4.1.6, every coclosed element in B is Galois by Lemma 3.5. In case every element in A is Galois, Theorem 4.1.6 establishes in fact a bijection between the sets of coclosed Galois elements in A and B . We point out that this is not true in general. Indeed, let $(\alpha; \beta)$ be the UCC cosmall Galois connection from Example 3.7. Then $\alpha(G) = G$ and β preserves finite suprema. The coclosed Galois elements in the domain $A = S(G)$ of α are H_3 and G , whereas the coclosed Galois elements in the codomain $B = S(G)$ of α are $0, H_2, H_3$ and G . Hence there is no bijection between the two sets of coclosed Galois elements. Note that there are elements in $A = S(G)$ which are not Galois, for instance H_2 .

ii) Consider again the UCC cosmall Galois connection from Example 3.7. Then $\alpha(G) = G$ and β preserves finite suprema. Also, $A = S(G)$ is amply supplemented modular, as any subgroup lattice of a finite group. But condition (i) in Corollary 4.1.7 does not hold, because for instance H_3 is coclosed in $S(G)$, but $\beta(H_3) = H_4$ is not coclosed in $S(G)$. Obviously, neither condition (ii) in Corollary 4.1.7 holds, because for instance H_2 is a coclosed element in $A = S(G)$ which is not Galois.

4.2. Endomorphism Rings of Lifting Modules

In this section we apply the above results to submodule lattices of suitable modules, which are clearly bounded modular lattices. Let us first identify some Galois connection between submodule lattices. Our theory of cosmall Galois connections is clearly dualizable to one of some naturally defined essential Galois

connections. As a consequence, one may obtain a lattice theoretic version of Zelmanowitz's results from [Zelmanowitz, 1996].

For modules M and X , we denote by $Gen(M)$ the set of M -generated modules, and by $Tr_M(X) = MHom_R(M, X)$ the greatest submodule of X which is M -generated.

Let M and N be left R -modules, and let $S = End_R(M)$. Consider the following functions between the complete modular lattices $\mathcal{S}_R(N)$ and $\mathcal{S}_S(Hom_R(M, N))$ of left R -submodules of N and left S -submodules of $Hom_R(M, N)$ respectively:

$$\begin{aligned} \alpha: \mathcal{S}_S(Hom_R(M, N)) &\longrightarrow \mathcal{S}_R(N), & \alpha(Y) &= MY, \\ \beta: \mathcal{S}_R(N) &\longrightarrow \mathcal{S}_S(Hom_R(M, N)), & \beta(X) &= Hom_R(M, X) \end{aligned} \quad (4.7)$$

Then $(\alpha; \beta)$ is a Galois connection by Proposition 3.4 in [Albu and Nastasescu, 1984]. The coclosed elements in $\mathcal{S}_R(N)$ are exactly the coclosed submodules of N . A submodule X of N is a Galois element in $\mathcal{S}_R(N)$ if and only if $X \in Gen(M)$ ([Albu and Nastasescu, 1984], Proposition 4.5). Now assume that M is finitely generated quasi-projective and $N \in Gen(M)$. Then $\alpha(Hom_R(M, N)) = N$, β preserves finite suprema and $Y = Hom_R(M, MY) = \beta\alpha(Y)$ for every submodule Y of $Hom_R(M, N)$ by Proposition 4.9 in [Albu and Nastasescu, 1984] or 18.4 in [Wisbauer, 1991]. Hence every submodule of $Hom_R(M, N)$ is Galois.

Lemma 4.2.1: Let M be an R -module and $S = End_R(M)$.

- i) *Every projective module in $\sigma[M]$ is a direct summand of a direct sum of finitely generated submodules of $M^{(\mathbb{N})}$.*
- ii) *Every projective module in $\sigma[M]$ is isomorphic to a submodule of a direct sum $M^{(A)}$.*
- iii) *Let M be self-projective and N in $R\text{-MOD}$:*

- *For every finitely generated S -submodule $I \subset Hom_R(M, N)$, we have*

$$I = Hom_R(M, MI); \quad (4.8)$$

- If M is finitely generated, then equatin 4.8 holds for every S -submodule

$$I = \text{Hom}_R(M, MI) \quad (4.9)$$

vi) Let M be projective in $\sigma[M]$ and $N \in \sigma[M]$. Then, for any submodules $L_1, L_2 \subset N$,

$$\text{Hom}(M, L_1 + L_2) = \text{Hom}(M, L_1) + \text{Hom}(M, L_2) \quad (4.10)$$

Theorem 4.2.2: Assume M to be a finitely generated, self-projective R -module and $S = \text{End}_R(M)$. Then an M -generated module N is supplemented if and only if $\text{Hom}_R(M, N)$ is supplemented as a left S -module.

Let $\sigma[M]$ be the full subcategory of the category of left R -modules consisting of all submodules of M -generated modules. Recall that $N \in \sigma[M]$ is called M -faithful if for every $0 \neq g \in \text{Hom}_R(X, N)$ with $X \in \sigma[M]$, $\text{Hom}_R(M, Xg) \neq 0$. Note that if N is M -faithful, then the above function β is clearly injective. When applied to the bounded modular submodule lattice of a module, Definition 4.1.2 yields the notions of supplemented and lifting modules. Now Corollary 3.9 and Theorem 4.1.4 give the following corollary, whose first two properties are given Theorem 4.2 in [Harmandez and Pardo, 1997] and 43.7 in [Wisbauer, 1991] respectively.

Corollary 4.2.3. Let M be a finitely generated quasi-projective left R -module with $S = \text{End}_R(M)$ and $N \in \text{Gen}(M)$. Then:

- The left R -module N and the left S -module $\text{Hom}_R(M, N)$ have the same dual Goldie dimension.*
- N is supplemented if and only if $\text{Hom}_R(M, N)$ is supplemented.*
- If N is lifting, then $\text{Hom}_R(M, N)$ is lifting. If N is M -faithful and $\text{Hom}_R(M, N)$ is lifting, then N is lifting.*

Finally, Theorem 4.1.6 yields the following consequence.

Corollary 4.2.4: Let M be a finitely generated quasi-projective left R -module with $S = \text{End}_R(M)$ and $N \in \text{Gen}(M)$, and let α, β be the above functions. Then α and β restrict to order-preserving mutually inverse bijections between the sets of coclosed left S –submodules of $\text{Hom}_R(M, N)$ and coclosed left R –submodules of N .

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BIOGRAPHY

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