

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL

ON (DE)COUPLINGS OF LIE TYPE STRUCTURES

BEGÜM ATEŞLİ

A THESIS OF DOCTORATE
DEPARTMENT OF MATHEMATICS

ADVISOR: PROF. DR. OĞUL ESEN

JUNE 2024

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DANIŞMAN: PROF. DR. OĞUL ESEN
II. DANIŞMAN: PROF. DR. SERKAN SÜTLÜ

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JURY

MEMBER

(THESIS ADVISOR) : Prof. Dr. Oğul ESEN

MEMBER : Prof Dr. Mansur İSGENDEROĞLU

MEMBER : Prof. Dr. Atabey KAYGUN

MEMBER : Prof. Dr. Coşkun YAKAR

MEMBER : Prof Dr. Aybike ÖZER

APPROVAL

Gebze Technical University Graduate Administration Board

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SIGNATURE/SEAL

ABSTRACT

The aim of this thesis is to merge two vector bundles over a common base manifold to construct a $(n-)$ Lie algebroid over the same manifold. Particular interest, on the other hand, will be on the coupling of two vector spaces into a $(n-)$ Lie algebra. These goals will be achieved in the most general setting that we call the *bicocycle double cross product*. This novel construction subsumes the unified products, double cross products (matched pairs), semi-direct products, and cocycle extensions as particular instances. Understanding the coupling sheds light on understanding the decoupling. We shall present both coupling and decoupling from a purely algebraic standpoint, deriving compatibility conditions for the former and stating conceptual results on the latter.

After exploring Lie algebroid (de)coupling constructions, we shall focus on possible (de)couplings of Lagrangian dynamics, resulting in dynamical formulations on bicocycle double cross product Lie algebroids. This reads the collective motion of two Lagrangian dynamics interacting mutually. On the dual bundle of a bicocycle double cross product, we shall present bicocycle double cross product Hamiltonian dynamics. Considering the line bundle extensions, we shall further examine the irreversible Lagrangian and Hamiltonian dynamics from a (de)coupling perspective.

Additionally, we shall address the classification of n -Lie algebras from the point of view of the generalized derivations.

We then plan to tackle the (de)coupling problem on 3-Lie algebroids, using once more the bicocycle double cross product approach. Within this context, we shall examine the Nambu dynamics on the dual spaces of the coupled algebra(oid)s. Here, the reference to the order 3 in (de)couplings and dynamics is motivated from the relation between the 3-Nambu dynamics and bi-Hamiltonian formulations (through complete integrability).

Finally, in order to provide more physical insight into the algebraic/geometric constructions, we shall present a geometric Hamilton-Jacobi theory within the framework of locally conformally cosymplectic manifolds that possess a Lie algebroid structure. The theory, in this case, offers a geometric pathway to the solution of the Hamilton's equations.

In short, the thesis offers pure algebraic constructions utilizing geometric analysis and dynamical formulations of (de)coupling procedure, thereby serving as a foundation for further analysis and exploration across algebraic, geometric, and physical perspectives.

Keywords: Lie algebroids; Bicocycle double cross product; Hamiltonian dynamics; Lagrangian Dynamics; n -Lie algebroids.

ÖZET

Bu tezin temel amacı, aynı taban katman üzerinde tanımlı iki vektör demetini birleştirerek bir $(n-)$ Lie cebroidi inşa etmektir. Özel olarak n -Lie cebirleri için ise bu durum iki vektör uzayının birleştirilerek bir n -Lie cebiri oluşturması şeklinde tezahür etmektedir. Bu, ikili eşçevrim çift çarpım (bicocycle double cross product) stratejisi olarak adlandırılan ve olası en genel durumu betimleyen bir yaklaşımla yapılmıştır. Bu yaklaşım, birleşik çarpım (unified product), çift çarpım (double cross product), yarı-direkt çarpım ve eşçevrim genişlemelerinin tamamını özel durum olarak içerir. Demetleri birleştirmeyi anlamak, onları ayrıştırmayı anlamakla eşdeğerdir. Bu nedenle, sadece cebirsel bir bakış açısından, birleştirmeye ilişkin uyumluluk koşullarını elde ederken ayrıştırma için teoremler sunulmuştur.

Birleştirilerek elde edilen eşçevrim çift çarpım Lie cebroidi yapıları üzerinde Lagrange dinamikleri yazılmıştır. Bunlar iki Lagrange dinamiğin birbirleriyle etkileşimini betimleyen ve en genel formdaki dinamik formülasyonlardır. Dual demet üzerinde ise aynı yaklaşım devam ettirilerek eşçevrim çift çarpım Hamilton dinamikleri elde edilmiştir. Tersinir olmayan dinamikleri elde etmek için ise çizgi demetleri ile Lie cebroidleri genişletilmiş ve bu durum için hem Lagrange hem de Hamilton denklemleri inşa edilmiştir.

Ek olarak, n -Lie cebirlerinin sınıflandırılması, genelleştirilmiş türevler vasıtasıyla elde edilmiştir.

Ayrıca, ikili eşçevrim çift çarpım stratejisi kullanarak 3-Lie cebroidlerindeki ayrıştırma/birleştirme problemi çözülmüştür. Bu bağlamda, birleştirilmiş cebroidlerin dual uzayları üzerinde Nambu dinamikleri elde edilmiştir. Burada derece olarak 3 değerinin seçimi Nambu dinamikleri ve iki-Hamilton yapıları arasındaki ilişki nedeniyledir. Bu durum tam integral edilebilmeye karşılık gelir.

Son olarak, cebirsel yapılara daha fiziksel bir örnek için Lie cebroidi yapısına sahip yerel konformal kosimplektik katmanlar üzerine geometrik Hamilton-Jacobi teorisi sunulmuştur. Teori, Hamilton denklemlerini çözenin geometrik bir yolunu sunmaktadır.

Özetle, tez saf cebirsel/geometrik yapılar sunmakta ve de bu yapılar üzerinde dinamik denklemlerinin açık ifadelerini içermektedir. Bu yaklaşımın cebirsel, geometrik ve fiziksel bakış açıların her biri için daha ileri analiz ve çalışmalara temel oluşturması beklenmektedir.

Anahtar Kelimeler: Lie cebroidleri; ikili eşçevrim çift çarpım; Hamilton dinamiği; Lagrange dinamiği; n -Lie cebroidleri.

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TABLE OF CONTENTS

	<u>Page</u>
ABSTRACT	v
ÖZET	vi
ACKNOWLEDGEMENT	vii
TABLE OF CONTENTS	viii
LIST OF ABBREVIATIONS AND ACRONYMS	x
 1. INTRODUCTION	 1
2. PRELIMINARIES	12
2.1. Symplectic Manifolds	12
2.2. Hamiltonian Dynamics and Geometric HJ Theory	14
2.3. Poisson and Jacobi Manifolds	17
2.4. Cosymplectic Manifolds	19
2.5. Contact Manifolds	24
3. LIE ALGEBRA(OID)S AND COUPLINGS	28
3.1. Lie Structures	28
3.1.1. Lie Algebras	28
3.1.2. Leibniz Algebras	31
3.1.3. Lie Algebroids	33
3.1.4. Real Line Extensions	37
3.2. Couplings of Algebra(oid)s	39
3.2.1. Unified Product	39
3.2.2. Bicocycle Double Cross Product	44
3.2.3. MP, Cocycle Extensions, and Semi-direct Products	48
3.2.4. Illustrations	53
3.3. Couplings of Dynamics	58
3.3.1. Dynamics on Lie Algebra(oid)s.	58
3.3.2. Couplings of Reversible Dynamics	61
3.3.3. Couplings of Irreversible Dynamics	67
4. n -LIE ALGEBRA(OID)S AND COUPLINGS	73
4.1. n -Lie Algebras and Cohomology	73
4.1.1. n -Lie Algebras	73
4.1.2. Generalized Derivations and Classifications	78
4.1.3. Nambu and Bi-Hamiltonian Dynamics	85
4.2. n -Lie Algebroids, Dynamics, Couplings	89
4.2.1. n -Lie Algebroids	89
4.2.2. BDCP 3-Lie Algebroids	93
4.2.3. BDCP 3-Nambu Dynamics	98
5. ADDENDUM: A CASE STUDY	100

5.1. Locally Conformally Cosymplectic Manifolds	100
5.2. Dynamics on LCC Manifolds	102
5.3. LCC as a Local Lie Algebra and as a Lie Algebroid	103
5.4. Locally Conformally Symplectic Dynamics	109
5.5. Geometric HJ for LCC Dynamics	112
6. CONCLUSIONS	116
REFERENCES	117
BIOGRAPHY	127
PUBLICATIONS AND PRESENTATIONS FROM THE THESIS	128

LIST OF ABBREVIATIONS AND ACRONYMS

GTÜ	:	Gebze Teknik Üniversitesi
DCP	:	Double Cross Product
BDCP	:	Bicycle Double Cross Product
LCS	:	Locally Conformally Symplectic
LCC	:	Locally Conformally Cosymplectic
HJ	:	Hamilton-Jacobi
MP	:	Matched Pair

1. INTRODUCTION

One of the main problems in Classical Mechanics is to achieve the unification of the physical theories. We could argue that this endeavor began with Newton's second law of motion:

$$F(x) = m\ddot{x}, \tag{1.1}$$

where m represents mass and F represents force. This single differential equation addresses the two-body problem, explaining both the Keplerian motion of a planet around the sun and the Galilean motion of a particle on Earth.

In this thesis, we will follow this approach. Specifically, the focus, termed the “(de)coupling problem,” is to derive the equations of motion governing the collective behavior of two systems in a mutual action-reaction relationship. We will provide answers to this problem from both algebraic/geometrical and dynamical perspectives.

When two dynamical systems or two characters of a physical phenomenon (like Magnetohydrodynamics, where one considers both Maxwellian and fluid characters of a continuum) are coupled and brought together, they intervene in each other's independent behaviour. Therefore, achieving the equations that determine the coupled (unified, matched) motion of these two systems/characters cannot be done by simply juxtaposing the equations governing the constitutive systems/characters. Due to mutual interaction, the equations governing the coupled motion will include extra terms compared to those of the equations of the constituted systems/characters. The (de)coupling process can be examined in two steps. Firstly, one needs to determine the extensions of the configuration spaces in a pure algebraic/geometric way. Then, the equations of motion can be coupled by referring to the extensions obtained in the first step.

Symplectic, Poisson, Jacobi and Nambu Manifolds. In contemporary literature, symplectic geometry is one of the primary geometric frameworks used to formulate Classical Mechanics into Hamiltonian formulation [1–5].

Symplectic geometry involves a closed non-degenerate (differential) two-form Ω on a manifold M . The closure of the differential form (that is $d\omega = 0$) reads the integrability of the physical motion whereas the non-degeneracy stands for an isomorphism of smooth functions (up to scalars) and vector fields. This latter aspect establishes a mapping from a Hamiltonian (energy) function $\mathcal{H}$ to the corresponding (Hamiltonian) vector field $X_{\mathcal{H}}$. So, knowing the energy gives the equation of motion. This correspondence is the Hamilton's equation

$$\iota_{X_{\mathcal{H}}}\omega = d\mathcal{H}. \quad (1.2)$$

Here, ι is the contraction operator (interior derivative). The skew-symmetry of the symplectic two-form implies the conservation of the Hamiltonian function throughout the motion, indicating the reversible nature of symplectic Hamiltonian dynamics. It is important to note that Hamilton's equation (1.2) realizes the Newton's second law (1.1) as a particular instance.

The Hamiltonian motion in the symplectic framework consists of an even number of first-order differential equations. So its application area is restricted to even dimensional manifolds. The cotangent bundle T^*Q of a configuration manifold (or in physical terms, momentum phase space) Q is the generic example of a symplectic manifold.

Another approach to Classical Mechanics is the Lagrangian dynamics, which is based on variational principles. In Lagrangian dynamics, the motion is determined by a Lagrangian function defined on the tangent bundle TQ of the configuration manifold. The resulting equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0 \quad (1.3)$$

are known as the Euler-Lagrange equations. Here, L is the Lagrangian function generating the dynamics, (q^i) are (generalized) coordinates on the base manifold Q whereas $(q^i, \dot{q}^i)$ are the induced coordinates on the tangent bundle. In physical terms, $(\dot{q}^i)$ reads the velocities. In its very classical sense, the Lagrangian function is the difference between the kinetic and potential energies. Remarkably, Newton's second second law can be understood as a particular instance of the Euler-Lagrange equations.

Lagrangian and Hamiltonian dynamics are expected to be connected through the Legendre transformation. This connection is straightforward when the so-called regularity (Hessian) condition is satisfied. However, for non-regular cases, the transformation is not immediately evident.

To perform Hamiltonian or Lagrangian analysis of non-canonical dynamical systems, such as those that are not amenable to examination on a tangent/cotangent bundle or odd-dimensional models, a feasible approach is to employ a symmetry reduction, utilizing Lie group actions. For the foundations of the reduction theory we cite [6–8], see also [9] for a more recent work. If, in particular, a Lie group, say G , acts on its tangent bundle TG , the reduction process gives Euler-Poincaré dynamics

$$\frac{d}{dt} \frac{\delta l}{\delta \xi} = ad_{\xi}^* \frac{\delta l}{\delta \xi} \quad (1.4)$$

on the Lie algebra $\mathfrak{g}$. Here, ξ is an element of the Lie algebra, while ad^* refers to the coadjoint action of the Lie algebra $\mathfrak{g}$ on its dual space $\mathfrak{g}^*$, and l is the reduced Lagrangian on $\mathfrak{g}$.

In the Hamiltonian picture, a Lie group action on its cotangent bundle T^*G generates the Lie-Poisson dynamics

$$\frac{d\mu}{dt} = \pm ad_{\frac{\delta h}{\delta \mu}}^* \mu \quad (1.5)$$

on the dual space $\mathfrak{g}^*$ of the Lie algebra. Here, μ is an element of the dual space, and h is the reduced Hamiltonian function on the dual space $\mathfrak{g}^*$. The plus/minus notation is reflecting the left/right reduction. While the non-degeneracy condition is lost on the entire dual space in this scenario, the coadjoint orbits remain symplectic. This reduced Lagrangian/Hamiltonian framework facilitates the examination of various physical theories, including rigid body dynamics, ideal fluid, and plasma theories.

The reduction process leads us into the realm of Poisson geometry generalizing the symplectic geometry which is achieved by relaxing the non-degeneracy condition. A Poisson manifold P is determined by a skew-symmetric algebra [10–14]

$$\{\bullet, \bullet\} : C^\infty(P) \times C^\infty(P) \longrightarrow C^\infty(P) \quad (1.6)$$

on the space of smooth functions satisfying both the Leibniz and the Jacobi identities. The Leibniz identity reads the bracket as a differential operator (mimicking the role of the Hamiltonian vector field), whereas the Jacobi identity is to ensure the integrability corresponding to the closure of the symplectic two-form. In fact, the loss of non-degeneracy brings the advantage of casting the integrals of the motion in the geometry. We call them Casimir functions. One may determine a Poisson bracket by a Poisson bivector field Λ (a bivector commuting with itself under the Schouten bracket). In terms of the Poisson bracket and in terms of the Poisson bivector, the Hamiltonian dynamics is written as

$$X_{\mathcal{H}}(x) = \{x, \mathcal{H}\} = \sharp_{\Lambda}(d\mathcal{H}), \quad (1.7)$$

where $\sharp_{\Lambda}$ is the musical mapping obtained from the Poisson bivector field Λ . Due to the skew-symmetry present in the Poisson algebra, energy conservation persists throughout the motion, making Poisson Hamiltonian motion reversible.

To establish a (linear) Poisson structure in the most general setting, one employs Lie algebroids, which generalize both Lie algebras and tangent bundles. The dual of Lie algebroid admits a Poisson structure. Further, there is Lagrangian dynamics on Lie algebroids generalizing both the Euler-Lagrange equations and Euler-Poincaré equations and there is Hamiltonian dynamics on the dual bundle generalizing both the Hamilton's equations and Lie-Poisson equations. In this thesis, our main focus will be on Lie algebroids, Lagrangian dynamics on Lie algebroids, the dual spaces, and Hamiltonian dynamics on dual spaces.

A generalization of the Poisson geometry can be achieved by relaxing the Leibniz condition within the algebra, leading to what is termed the Jacobi geometry [15–18]. In a Jacobi manifold, a bracket still operates within the function space, but instead of satisfying the Leibniz identity, the algebra reads the first-order differentiability condition. This results in the Hamiltonian vector field, which generates motion, being augmented by the so-called Reeb vector field $\mathcal{E}$:

$$X_{\mathcal{H}} = \sharp_{\Lambda}(d\mathcal{H}) + \mathcal{H}\mathcal{E}. \quad (1.8)$$

Consequently, the Hamiltonian function is not conserved along the motion. This approach facilitates the analysis of irreversible dynamics, such as dissipative

systems, and allows for a thermodynamic interpretation within a geometric framework. Jacobi manifolds involve two important subcases: contact manifolds and locally conformally symplectic manifolds. A Jacobi manifold establishes a Lie algebroid structure as well, which is known as the Jacobi Lie algebroid.

Jacobi manifolds serve as generalizations of Poisson (and symplectic) manifolds, deviating from the conservation of energy principle. On the other hands, Nambu manifolds represent generalizations of Poisson manifolds, allowing for the accommodation of more than one conserved quantity, rendering them suitable for complete and even super-integrability. In an attempt to generalize the classical Hamiltonian mechanics to a theory which accommodates multiple Hamiltonians, Nambu proposed the use of ternary Lie algebras [19]. In [20], the concept of n -Lie algebras is introduced, further elucidating the theoretical framework. This line has since been advanced and refined in [21]. Nambu manifolds carry an n -algebra satisfying the Leibniz identity within the space of functions:

$$\{\bullet, \dots, \bullet\} : C^\infty(P) \times \dots \times C^\infty(P) \longrightarrow C^\infty(P). \quad (1.9)$$

The dynamics is generated by $n - 1$ number of Hamiltonian functions:

$$X_{\mathcal{H}_1, \dots, \mathcal{H}_{n-1}} = \{x, \mathcal{H}_1, \dots, \mathcal{H}_{n-1}\}. \quad (1.10)$$

In the case where $n = 3$, which is of particular interest to us, two Hamiltonian functions are required, giving rise to what is known as bi-Hamiltonian dynamics. A bi-Hamiltonian system *reads* complete integrability [22, 23]. While the dual space of Lie algebroids offers the most general framework for understanding (linear) Poisson manifolds, the same cannot be said for Nambu manifolds. It is not always the case that the dual of an n -Lie algebroid yields a Nambu manifold. However, our interest lies specifically in those frameworks where this property holds true.

Goal: Cohomology and Generalized derivation of n -Lie algebras. In the pioneering work of [24], an important development emerged: the establishment of a cohomology theory in conjunction with n -ary Lie algebras, aimed at examining their formal deformations. This pivotal step marked the inception of a significant line of inquiry. Subsequent to this milestone, in [25], the aforementioned cohomology

theory was subject to further scrutiny, closely intertwined with the exploration of Leibniz cohomology. Noteworthy advancements ensued in recent years. In particular, in [26, 27], this cohomology theory underwent expansion, accommodating arbitrary coefficients, notably in the context where $n = 3$. Furthermore, in [28], it underwent further refinement, transitioning into a cohomology theory intricately linked with an n -Hom-Lie algebra, complemented by its corresponding representation space.

Conversely, cohomology theories, especially those focusing on the simpler aspects known as low-dimensional cohomology groups, play a crucial role in understanding how algebraic and geometric objects can be classified, as discussed in [29, 30]. For instance, in [31], researchers examined how certain types of extensions, known as derivation extensions, relate to 3-Lie algebras. They found that simply adding a derivation to a 3-Lie algebra does not quite fit to create a new 3-Lie algebra structure on the resulting 1-dimensional extension. To address this, they introduced the idea of a generalized derivation tailored for 3-Lie algebras, a concept later broadened to encompass n -Hom-Lie algebras in [28].

Our goal here is to present a new cohomology theory for n -Lie algebras via which we realize a generalized derivation on an n -Lie algebra as a one-cocycle, with coefficients in the space of endomorphisms of the n -Lie algebra on which the generalized derivation acts.

It is widely acknowledged that derivations on structures like n -Lie algebras and Leibniz algebras are closely linked to the first cohomology group of that particular algebraic object, using itself as coefficients. However, as noted in [31] regarding 3-Lie algebras, this connection does not quite hold for generalized derivations. In this thesis, as a byproduct of our research on the n -Lie algebras, it is aimed to bridge this gap by introducing a new cohomology theory for n -Lie algebras. Through this theory, a generalized derivation of an n -Lie algebra is realized as a one-cocycle. It is important to note that the cohomology theory presented in the thesis differs from the existing one for n -Lie algebras for any $n > 3$, while the two theories coincide for $n = 3$.

An Incomplete List on Various Couplings of Lie Structures. The Lie algebraic foundations of Classical Mechanics, including Lie algebras, Lie algebroids, n -Lie algebras, and n -Lie algebroids as well as their dual spaces will be our main tools. We will use these tools to introduce solutions to (de)coupling problems at various levels. Before delving into the novel contributions of this thesis, let us briefly outline the existing work in the field.

In the existing literature, several works have addressed the (de)coupling problem. While investigating the algebraic/geometric foundations of certain physical systems, such as Maxwell-Vlasov dynamics, magnetohydrodynamics, and compressible fluid dynamics, we see that the underlying configuration spaces are semi-direct product Lie groups [32–35]. In these physical systems, the reduced Lagrangian (Euler-Poincaré) dynamics is available on the semi-direct product Lie algebra whereas the reduced Hamiltonian (Lie-Poisson) dynamics is on the dual space. The semi-direct product theory is an example of a Lie group/algebra extension admitting only a one-sided representation.

There are generalizations of the semi-direct product theory. Double cross product (matched pair) theory is one such extension, involving the mutual interactions of two Lie groups/algebras [36, 37]. This theory has also applications in physics. The matched pair Lagrangian and Hamiltonian dynamics studied in [38, 39], the higher order Lagrangian dynamics in [40], the dissipative systems in [41] and the discrete dynamics in [42]. These studies quickly found applications in electromagnetic theory [43], and the kinetic moments of the plasma dynamics. The matched pair strategy has further been used to establish the algebraic relationship between the plasma and the fluid [44, 45], as well as the moments in chemical kinetics [46].

Another notable general extension theory is the unified product theory [47–50]. In a coupling point of view, this framework provides a general way to compose a Lie algebra(oid) with a vector space (bundle). This is achieved by the presence of a *cocycle* term. We write representation and cocycle with italic notations to manifest when they do not satisfy the requirements of classical conditions of these structures. In some literature, the term "twisted" describes this phenomenon.

From a decomposition perspective, this can be understood as examining algebraic properties of the quotient space of an algebra(oid) with one of its subalgebra(oid)s. The literature lacks discussions on the applications of these algebraic structures to Lagrangian and Hamiltonian dynamics as well as an interconnection with the physical phenomena. This will be one of our focus in the present manuscript.

The most general coupling of two geometric structures is achieved through the bicycle double cross product geometry [51]. In this case, one couples two vector bundles to arrive at an algebroid on the direct sum bundle. However, the literature lacks such a geometry, and addressing this gap constitutes another goal for the present work. From the decomposition perspective, this corresponds to analyzing the algebraic inheritance of the constitutive vector bundles (not necessarily a subalgebroid) in any direct sum decomposition of an algebroid. The nonclosure of the brackets on the constitutive bundles is accommodated by two *cocycle* terms, giving the structure its name.

This thesis explores the geometrical/algebraic framework of $(n-)$ Lie algebra(oid)s, with a specific focus on the decoupling and coupling phenomena within the bicycle double cross product realization. The bicycle double cross product theory serves as the most general method for (de)coupling an algebroid into the direct sum of two vector bundles in the presence of mutual *representations*, along with two *cocycle* terms. Consequently, it encompasses unified product, double cross product (matched pair), semi-direct product, and *cocycle* extension frameworks as particular instances. In addition to algebraic constructions, the research extends to both reversible and irreversible Lagrangian and Hamiltonian dynamics on (de)coupled Lie algebroids, as well as Euler-Poincaré-(Herglotz) and Lie-Poisson-(Herglotz) dynamics on (de)coupled Lie algebras, providing insights into potential physical applications.

Hierarchy of Lie Algebroid (De)couplings in Bicycle Double Cross Product Strategy. In the main body of the thesis, we shall provide more concrete definitions of all the geometries and fundamentals of dynamical equations. However, for now, let us gain clarity on the algebraic constructions by examining the bicycle double cross

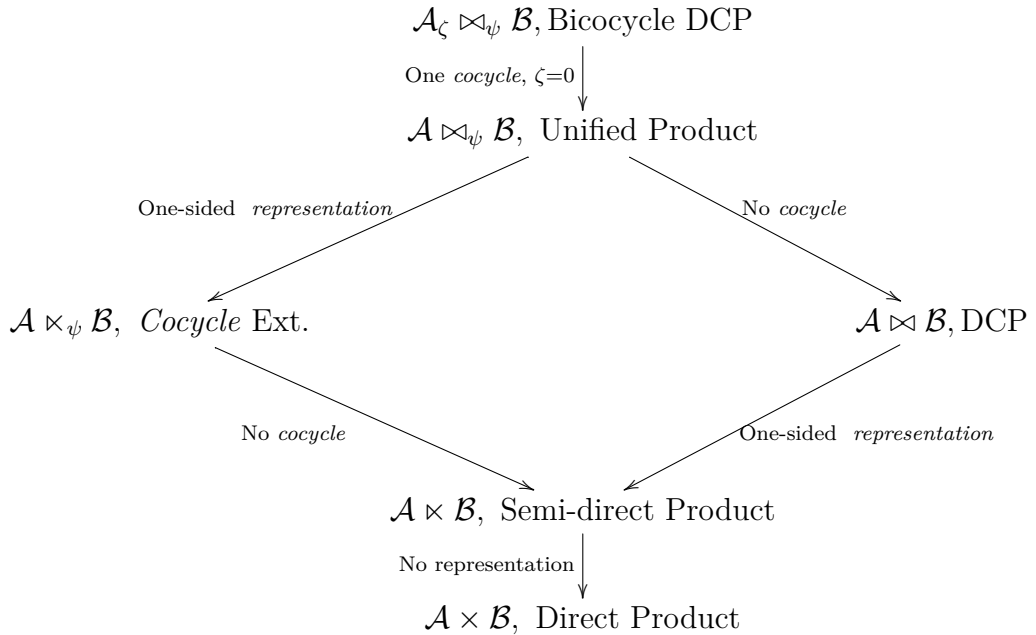
product strategy from a decomposition point of view. Consider an algebra(oid) (which refers to one of the Lie structures among Lie algebras, Lie algebroids, n -Lie algebras, and n -Lie algebroids) denoted by $\mathcal{M}$, and a subalgebra(oid) denoted by $\mathcal{A}$ of $\mathcal{M}$. In the most general scenario, the bracket defined on $\mathcal{M}$ induces a bracket on the quotient space $\mathcal{B} := \mathcal{M}/\mathcal{A}$. The induced bracket on $\mathcal{B}$ does not have to be closed. Failure in the closure determines a *cocycle* term reading how two elements in $\mathcal{B}$ give an element in $\mathcal{A}$. On the other hand, the cross bracket of elements (those brackets in form $[\mathcal{B}, \mathcal{A}]$), that is one element in $\mathcal{B}$ and one element in $\mathcal{A}$, would read mutual *representations*. By considering the Whitney sum $\mathcal{M} = \mathcal{A} \times \mathcal{B}$. We note that for all the bundles fibrated over the same base manifold we denote the Whitney sum by $\times$, and we shall follow this policy for semi-direct products, *cocycle* extensions, and double cross products as well. Now, let us list all possible situations from the simplest to the most complicated ones. Note that $\mathcal{A}$ and $\mathcal{B}$ represent vector spaces for algebras, while they serve as vector bundles for algebroids.

- In the case of the direct product $\mathcal{A} \times \mathcal{B}$, the induced brackets on both $\mathcal{A}$ and the quotient space $\mathcal{B}$ are closed, leaving no room for a *cocycle* term, and the bracket of the mixed elements are identically zero, indicating trivial representation.
- In the theory of the semi-direct product, one still has no *cocycle* terms as a result of the closure of the induced brackets on the individual spaces. The bracket of mixed elements allows only a one-sided representation of $\mathcal{A}$ on $\mathcal{B}$ (or a one-sided representation of $\mathcal{B}$ on $\mathcal{A}$). We denote this case as $\mathcal{A} \ltimes \mathcal{B}$ (resp., $\mathcal{A} \rtimes \mathcal{B}$).
- In *cocycle* extension, a term, called *cocycle* and denoted as ψ , exists in $\mathcal{A}$ if the bracket of two elements in $\mathcal{B}$ is computed. On the other hand, the bracket of the mixed elements results in a one-sided *representation* of $\mathcal{A}$ on $\mathcal{B}$. We denote a *cocycle* extension by $\mathcal{A} \ltimes_{\psi} \mathcal{B}$.
- In the double cross product (matched pair) construction, the bracket of two elements in $\mathcal{B}$ remains in $\mathcal{B}$, indicating the absence of any *cocycle*. Consequently, the quotient space $\mathcal{B}$ is closed under the bracket, and $\mathcal{B}$

becomes an algebra(oid) with the induced bracket. On the other hand, when the bracket of cross elements is computed, terms appear in both $\mathcal{A}$ and $\mathcal{B}$, representing mutual representations. We denote a double cross product by $\mathcal{A} \bowtie \mathcal{B}$.

- In the unified product theory, a *cocycle* term exists, since the bracket of two elements in $\mathcal{B}$ is allowed to appear in $\mathcal{A}$. Additionally, terms appear in both $\mathcal{A}$ and $\mathcal{B}$ when the brackets of the mixed elements are computed. We denote a unified product by $\mathcal{A} \bowtie_{\psi} \mathcal{B}$.
- The most general picture is obtained by relaxing the requirement of $\mathcal{A}$ being a subalgebra(oid). We call this case a bicocycle double cross product Lie algebra(oid) since there exists two *cocycle* terms, namely ψ and ζ , additional to the double cross product (matched pair) structure involving two *representations*. The *cocycle* terms are the manifestations of the bracket of two elements in $\mathcal{B}$ having a component in $\mathcal{A}$, and the bracket of two elements in $\mathcal{A}$ having a component in $\mathcal{B}$. We denote a bicocycle double cross product by $\mathcal{A}_{\zeta} \bowtie_{\psi} \mathcal{B}$.

Here is a diagram to demonstrate the discussion, wherein we abbreviate the term “Double Cross Product” as DCP.



Goal: Bicocycle Double Cross Product (n-)Lie Algebra(oid)s. Another aim in this work is to provide all of the ingredients of the (de)couplings given in this

diagram by determining compatibility conditions and in light of universal lemmas. Knowing how to couple two systems geometrically or algebraically also provides an answer to the problem of dividing a single system into subsystems while preserving the relationships between its constitutive subsystems/subcharacters. In short, if one knows how to couple, then one knows how to decouple, and vice versa. The second objective of this work is to present a comprehensive framework for Hamiltonian and Lagrangian dynamics by leveraging bicycle double cross product outlined in the previous paragraphs. These novel equations will provide a comprehensive description of the collective motion of two dynamical systems, capturing all possible interactions between them. Additionally, we will introduce bicycle double cross product generalizations of dissipative Hamiltonian and Lagrangian dynamics on the extended spaces. It is worth noting once more that unified product dynamics, double cross product (matched pair) dynamics, semi-direct product dynamics, and *cocycle* extension dynamics serve as specific instances of the dynamical constructions presented in this thesis.

Outline. The thesis consists of four main chapters. In the following one, the basics of geometric mechanics are reviewed, including symplectic, cosymplectic, contact manifolds, and dynamics on these geometries. Chapter 3 starts with Lie algebras and Lie algebroids, then delves into the various couplings of these systems, all grounded in a unique approach known as the bicycle double cross product. The dynamical equations on these novel geometries are explored in detail. In Chapter 4, we extend the discussion to 3-Lie algebras(oids). We present generalized derivations and classification of them. Then, employing once more the bicycle double cross product strategy, we merge two 3-Lie algebras(oids). This novel geometry admits the coupling of two Nambu dynamics in the most general form. We present an Addendum to exhibit how Lie algebroids are closely related to concrete physical phenomena. This is achieved on locally conformally cosymplectic manifolds and the geometric Hamilton-Jacobi theory of the Hamiltonian motion.

2. PRELIMINARIES

2.1. Symplectic Manifolds

A manifold M equipped with a non-degenerate two-form Ω is called an almost symplectic manifold [1–5]. The non-degeneracy of the two-form manifests that M is necessarily even-dimensional. The musical mapping

$$\Omega^\flat : \mathfrak{X}(M) \longrightarrow \Gamma^1(M), \quad X \mapsto \iota_X \Omega, \quad (2.1)$$

induced from the two-form Ω is an isomorphism with the inverse mapping

$$\Omega^\sharp : \Gamma^1(M) \longrightarrow \mathfrak{X}(M). \quad (2.2)$$

Here, ι_X denotes the interior derivative. If Ω is both almost symplectic and closed, the manifold is called symplectic.

Let (M, Ω) represent a symplectic manifold, with N as a submanifold of M . N is called Lagrangian if its dimension is half of the dimension of M and the symplectic two-form vanishes on N .

Suppose we have two symplectic manifolds (M_1, Ω_1) and (M_2, Ω_2) , along with a diffeomorphism $\psi : M_1 \rightarrow M_2$. This ψ is referred to as a symplectic diffeomorphism, or a symplectomorphism, if it satisfies

$$\psi^* \Omega_2 = \Omega_1. \quad (2.3)$$

Under a symplectomorphism, the image of a Lagrangian submanifold is Lagrangian. Indeed, it is clear that the composition of two symplectic mappings remains symplectic. Consequently, the set $Diff_{can}(M)$ comprising all symplectic diffeomorphisms on a symplectic manifold M constitutes an (infinite-dimensional) Lie group. The group operation in this context is the composition of mappings.

Cotangent Bundle. The cotangent bundle T^*Q of a manifold Q is a canonically symplectic manifold. The presence of a symplectic structure on a cotangent bundle T^*Q can be derived from the double vector bundle structure inherent

in the tangent bundle of the cotangent bundle, TT^*Q . Considering TT^*Q , we observe the tangent fibration:

$$\tau_{T^*Q} : TT^*Q \rightarrow T^*Q,$$

alongside the tangent mapping of our cotangent fibration π_Q , denoted as:

$$T\pi_Q : TT^*Q \rightarrow TQ.$$

We collect these two fibrations in the following commutative diagram:

$$\begin{array}{ccc} & TT^*Q & \\ T\pi_Q \swarrow & & \searrow \tau_{T^*Q} \\ TQ & & T^*Q \\ \tau_Q \searrow & & \swarrow \pi_Q \\ & Q & \end{array} \quad (2.4)$$

Accordingly, we define the canonical (Liouville) one-form θ_Q on T^*Q as

$$\theta_Q(\xi) = \langle \tau_{T^*Q}(\xi), T\pi_Q(\xi) \rangle, \quad (2.5)$$

for a vector field ξ on T^*Q and the pairing on the right hand side is the duality between T^*Q and TQ . Minus of the exterior derivative of the canonical one-form

$$\Omega_Q = -d\theta_Q \quad (2.6)$$

is non-degenerate and closed. So the two-tuple (T^*Q, Ω_Q) becomes a symplectic manifold. In literature, Ω_Q is called canonical symplectic two-form. According to Darboux's theorem, locally all symplectic manifolds look like a cotangent bundle.

Darboux' Coordinates. Locally, there exists a local coordinate system (q^i, p_i) , called Darboux' coordinates, on cotangent bundle T^*Q such that the canonical one form turns out to be

$$\theta_Q = p_i dq^i, \quad (2.7)$$

whereas the canonical symplectic two-form in (2.6) becomes

$$\Omega_Q = dq^i \wedge dp_i. \quad (2.8)$$

The induced coordinates on the tangent bundle TT^*Q is then given by $(q^i, p_i, \dot{q}^i, \dot{p}_i)$. In terms of the Darboux' coordinates and for the case of the cotangent bundle, the musical mapping (2.1) turns out to be

$$\Omega_Q^\flat : TT^*Q \longrightarrow T^*T^*Q, \quad (q^i, p_i, \dot{q}^i, \dot{p}_i) \mapsto (q^i, p_i, -\dot{p}_i, \dot{q}^i). \quad (2.9)$$

Let α be a one-form on Q , and θ_Q be the canonical one-form then

$$\alpha^*\theta_Q = \alpha. \quad (2.10)$$

An immediate manifestation of this assertion is to establish that the image of a closed one-form α on Q is a Lagrangian submanifold of T^*Q . Particularly, the image spaces of the exterior derivatives of smooth functions on Q are Lagrangian submanifolds of T^*Q . In terms of the Darboux' coordinates, we have that

$$N = \left\{ (q^i, p_j) \in T^*Q : p_i = \frac{\partial f}{\partial q^i} \right\}. \quad (2.11)$$

Symplectic Volume. A symplectic manifold (M, Ω) of dimension $2n$ is a volume manifold with the symplectic volume defined to be n -times exterior product of Ω that is,

$$\text{Vol} = \Omega^n := \Omega \wedge \cdots \wedge \Omega. \quad (2.12)$$

For a given volume form the divergence $\text{div}_{\text{Vol}}(X)$ of a vector field X is defined to be

$$\mathcal{L}_X \text{Vol} = \text{div}_{\text{Vol}}(X) \text{Vol}, \quad (2.13)$$

where $\mathcal{L}$ is the Lie derivative.

2.2. Hamiltonian Dynamics and Geometric HJ Theory

A Hamiltonian vector field on a symplectic manifold (M, Ω) is characterized by the unique vector field $X_{\mathcal{H}}$ that satisfies the equation:

$$\iota_{X_{\mathcal{H}}} \Omega = d\mathcal{H}, \quad (2.14)$$

where $\mathcal{H}$ is the Hamiltonian function on M . Expressing Hamilton's equation (2.14) in terms of the musical mapping defined in (2.2), we have:

$$\Omega^\sharp(d\mathcal{H}) = X_{\mathcal{H}}. \quad (2.15)$$

Since $\Omega^\sharp$ is an isomorphism, a Hamiltonian vector field $X_{\mathcal{H}}$ can always be found for a given exact one-form $d\mathcal{H}$.

In Darboux' coordinates (q^i, p_i) , Hamiltonian vector field on M for a Hamiltonian function $\mathcal{H}$ in $C^\infty(M)$ is computed to be

$$X_{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial \mathcal{H}}{\partial q^i} \frac{\partial}{\partial p_i} \quad (2.16)$$

so that the dynamics prescribed by the Hamiltonian vector field is

$$\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i}. \quad (2.17)$$

For the case of symplectic manifolds, a Hamiltonian vector field, even if it is local, is divergence free with respect to the symplectic volume. To see this, we compute

$$\mathcal{L}_X \text{Vol} = \mathcal{L}_X(\Omega^n) = (\mathcal{L}_X \Omega)^n = (d\iota_X \Omega + \iota_X d\Omega)^n = 0. \quad (2.18)$$

Canonical Poisson Algebra. Without the loss of generality, let us focus on canonical symplectic manifold (T^*Q, Ω_Q) . Here, we define the canonical Poisson bracket on the space of smooth functions $C^\infty(T^*Q)$ as:

$$\{F, \mathcal{H}\} = \Omega_Q(X_F, X_{\mathcal{H}}), \quad (2.19)$$

where X_F and $X_{\mathcal{H}}$ denote Hamiltonian vector fields as defined in (2.14). This bracket is termed the canonical Poisson bracket. Additionally, the Poisson bracket can be defined as:

$$X_{\mathcal{H}}(F) = \{F, \mathcal{H}\}. \quad (2.20)$$

The pair $(C^\infty(T^*Q), \{\bullet, \bullet\})$ forms a Lie algebra due to its skew-symmetry and satisfaction of the Jacobi identity. The skew-symmetry of the symplectic two-form Ω_Q reflects in the skew-symmetry of the canonical Poisson bracket:

$$\{F, \mathcal{H}\} = -\{\mathcal{H}, F\}. \quad (2.21)$$

The closure of Ω_Q showcases the Jacobi identity for the canonical Poisson bracket:

$$\{F, \{\mathcal{H}, G\}\} + \{\mathcal{H}, \{G, F\}\} + \{G, \{F, \mathcal{H}\}\} = 0. \quad (2.22)$$

Furthermore, it can be demonstrated that the canonical Poisson bracket satisfies the Leibniz identity:

$$\{FG, \mathcal{H}\} = \{F, \mathcal{H}\}G + F\{G, \mathcal{H}\}. \quad (2.23)$$

Consider the Darboux' coordinates (q^i, p_i) on the cotangent bundle T^*Q . Then a direct calculation proves that the canonical Poisson bracket (2.19) is

$$\{F, \mathcal{H}\} = \frac{\partial F}{\partial q^i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q^i}. \quad (2.24)$$

The subsequent identity reveals that the space of Hamiltonian vector fields $\mathfrak{X}_{\text{ham}}(T^*Q)$ is closed under the Jacobi Lie bracket:

$$[X_{\mathcal{H}}, X_F] = -X_{\{\mathcal{H}, F\}}. \quad (2.25)$$

Geometric Hamilton-Jacobi Theory. Recall the Hamiltonian vector field $X_{\mathcal{H}}$ defined on the symplectic manifold T^*Q . Then define a projection of this Hamiltonian vector field using a one-form section ϱ on Q in the following manner:

$$X_{\mathcal{H}}^{\varrho} = T\pi_Q \circ X_{\mathcal{H}} \circ \varrho. \quad (2.26)$$

Note that $X_{\mathcal{H}}^{\varrho}$ represents a vector field on Q . Let us present the following diagram to visually depict this projection:

$$\begin{array}{ccc} T^*Q & \xrightarrow{X_{\mathcal{H}}} & TT^*Q \\ \varrho \curvearrowright \downarrow \pi & & \downarrow T\pi \\ Q & \xrightarrow{X_{\mathcal{H}}^{\varrho}} & TQ \end{array} \quad (2.27)$$

The geometric Hamilton-Jacobi (abbreviated as HJ) theory relates the projected vector field $X_{\mathcal{H}}^{\varrho}$ and the Hamiltonian vector field $X_{\mathcal{H}}$. We refer to [52] for further explanations.

Theorem 2.1. *If ϱ is a closed one-form, then the following conditions are equivalent:*

1. *The vector fields $X_{\mathcal{H}}$ and $X_{\mathcal{H}}^{\varrho}$ are ϱ -related that is $T\varrho \circ X_{\mathcal{H}}^{\varrho} = X_{\mathcal{H}} \circ \varrho$.*
2. *The equation $d(\mathcal{H} \circ \varrho) = 0$ holds.*

The ϱ -relatedness stated in the first item gives that if $q(t)$ is an integral curve of the projected field $X_{\mathcal{H}}^{\varrho}$, then $\varrho \circ q(t)$ is an integral curve of the Hamiltonian vector field $X_{\mathcal{H}}$. Thanks to the Poincaré lemma, the second condition gives that, there locally exists a function S on Q satisfying $dS = \varrho$ so that

$$\mathcal{H} \left(q^i, \frac{\partial S}{\partial q^i} \right) = \epsilon, \quad (2.28)$$

where ϵ remains constant. This represents the classical formulation of the time-independent Hamilton-Jacobi equation [2, 53]. For additional details regarding the ongoing discussion, we suggest consulting a recent survey on Hamilton-Jacobi theory [54].

2.3. Poisson and Jacobi Manifolds

A manifold P is called an almost Poisson manifold if it is equipped with a bilinear, skew-symmetric bracket $\{\bullet, \bullet\}$ on the space of smooth functions $\mathcal{C}^\infty(P)$, [13, 14, 55–58]. This framework should satisfy the Leibniz identity, that is, for any arbitrarily chosen functions F_i ,

$$\{F_1, F_2 F_3\} = \{F_1, F_2\} F_3 + F_2 \{F_1, F_3\}. \quad (2.29)$$

On an almost Poisson manifold $(P, \{\bullet, \bullet\})$, consider the Jacobiator:

$$\mathfrak{J} : \wedge^3 \mathcal{C}^\infty(P) \longrightarrow \mathcal{C}^\infty(P), \quad (F_1, F_2, F_3) \mapsto \circlearrowleft \{F_1, \{F_2, F_3\}\}, \quad (2.30)$$

where $\circlearrowleft$ reads the cyclic sum. If the Jacobiator is vanishing for an almost Poisson manifold P , called that the bracket is satisfying the Jacobi identity, then we call it a Poisson manifold. In this case, $(P, \{\bullet, \bullet\})$ is referred to as a Poisson manifold.

For a Hamiltonian function $\mathcal{H}$ defined on a Poisson manifold P , the Hamiltonian vector field $X_{\mathcal{H}}$ is

$$X_{\mathcal{H}}(F) = \{F, \mathcal{H}\}. \quad (2.31)$$

If there exists a non-constant function C such that for all F :

$$\{F, C\} = 0 \quad (2.32)$$

then the almost Poisson framework is called degenerate. Such a function C is called a Casimir function. It should be noted that a Casimir function cannot cause any dynamics since the Hamiltonian vector field is identically zero for a Casimir function.

In the realm of two Poisson manifolds $(P_1, \{\bullet, \bullet\}_1)$ and $(P_2, \{\bullet, \bullet\}_2)$, a differentiable mapping ϕ from P_1 to P_2 is called a Poisson map if it respects the Poisson brackets:

$$\{F, \mathcal{H}\}_2 \circ \phi = \{F \circ \phi, \mathcal{H} \circ \phi\}_1 \quad (2.33)$$

for all F and $\mathcal{H}$ in $C^\infty(P_2)$.

A Poisson framework can be represented by a bivector field Λ as

$$\Lambda(dF_1, dF_2) = \{F_1, F_2\}. \quad (2.34)$$

Thus, alternatively, a Poisson manifold can be represented by a pair (P, Λ) consisting of a manifold and a bivector field. There exists a Schouten-Nijenhuis algebra on multivector fields. In this representation, ensuring the Jacobi identity leads to the bivector field Λ commuting with itself with respect to the Schouten-Nijenhuis bracket, that is

$$[\Lambda, \Lambda] = 0. \quad (2.35)$$

In this case, the characteristic distribution of the Poisson manifold P , meaning the image space of all Hamiltonian vector fields, is integrable:

$$X_{\{F_1, F_2\}} + [X_{F_1}, X_{F_2}] = 0. \quad (2.36)$$

This implies also that the manifold P has a foliation by symplectic leaves. This is a consequence of the non-degeneracy of the Poisson framework on each leaf.

In the realm of almost Poisson manifolds (P, Λ) , the induced mapping $\sharp_\Lambda$ is defined as:

$$\sharp_\Lambda : \Gamma^1(P) \longrightarrow \mathfrak{X}(P), \quad \langle \beta, \sharp_\Lambda(\alpha) \rangle = \Lambda(\alpha, \beta), \quad (2.37)$$

where the pairing on the left-hand side reads the space $\Gamma^1(P)$ of one-form sections with the space $\mathfrak{X}(P)$ of vector fields.

Jacobi Manifolds. A manifold M equipped with a vector field $\mathcal{E}$ and a bivector field Λ reads a Jacobi manifold if the conditions

$$[\Lambda, \Lambda] = 2\mathcal{E} \wedge \Lambda, \quad [\mathcal{E}, \Lambda] = 0, \quad (2.38)$$

are satisfied [16–18, 57]. Here, $[\bullet, \bullet]$ denotes the Schouten-Nijenhuis bracket. A Jacobi manifold is represented by a triple $(M, \Lambda, \mathcal{E})$. From this framework, one can establish an antisymmetric bilinear bracket:

$$\{F_1, F_2\} = \Lambda(dF_1, dF_2) + F_1\mathcal{E}(F_2) - F_2\mathcal{E}(F_1), \quad (2.39)$$

which adheres to the Jacobi identity and the weak Leibniz rule:

$$\text{supp}(\{F_1, F_2\}) \subseteq \text{supp}(F_1) \cap \text{supp}(F_2). \quad (2.40)$$

This indicates that the algebra reads a local Lie algebra in Kirillov's sense [59]. Conversely, a local Lie algebra characterizes a Jacobi structure. Examples of

Jacobi structures are locally conformally symplectic structures [60–63], locally conformally cosymplectic structures [64, 65], contact manifolds [66], and cocontact manifolds [67–70].

In the context of a Jacobi manifold denoted by the triplet $(M, \Lambda, \mathcal{E})$, we introduce a homomorphism from the space of one-forms to the space of vector fields:

$$\sharp_\Lambda : \Gamma^1(M) \longrightarrow \mathfrak{X}(M), \quad \langle \sharp_\Lambda(\mu), \nu \rangle = \Lambda(\mu, \nu), \quad (2.41)$$

for all μ and ν in $\Gamma^1(M)$. For a Hamiltonian function $\mathcal{H}$, the Hamiltonian vector field $X_{\mathcal{H}}$ is written as:

$$X_{\mathcal{H}} = \sharp_\Lambda(d\mathcal{H}) + \mathcal{H}\mathcal{E}. \quad (2.42)$$

It is noteworthy that for the constant function $\mathcal{H} = 1$, the Hamiltonian vector field reduces to $\mathcal{E}$. The mapping that links a Hamiltonian function $\mathcal{H}$ with the Hamiltonian vector field $X_{\mathcal{H}}$ represents a Lie algebra homomorphism:

$$[X_{F_1}, X_{F_2}] = X_{\{F_1, F_2\}}. \quad (2.43)$$

2.4. Cosymplectic Manifolds

A triplet $(\bar{M}, \tau, \Omega)$ called an almost cosymplectic manifold involves a $(2n + 1)$ -dimensional manifold $\bar{M}$, along with a one-form τ and a two-form Ω such that

$$\tau \wedge \Omega^n \neq 0. \quad (2.44)$$

If τ and Ω are closed, $(\bar{M}, \tau, \Omega)$ reads a cosymplectic manifold. On an almost cosymplectic manifold, (cosymplectic) Reeb vector field exists, satisfying

$$\iota_{\mathcal{R}}\tau = 1, \quad \iota_{\mathcal{R}}\Omega = 0. \quad (2.45)$$

We introduce the cosymplectic volume form:

$$\overline{\text{Vol}} = \tau \wedge \text{Vol} = \tau \wedge \Omega^n. \quad (2.46)$$

Extended symplectic manifold $\mathbb{R} \times M$ is a generic example of cosymplectic manifolds. Here, we construct a two-form Ω by pulling back the symplectic two-form on M , and a one-form τ by pulling back the differential form dt on $\mathbb{R}$. This process defines the cosymplectization $(\bar{M}, \tau, \Omega)$ of the symplectic manifold (M, Ω) ,

where $\bar{M} = \mathbb{R} \times M$. Generic examples of cosymplectic manifolds include extended cotangent bundles $\mathbb{R} \times T^*Q$ as well. This geometric perspective suggests the use of Darboux coordinates (t, q^i, p_i) on the cosymplectic manifold such that:

$$\Omega = dq^i \wedge dp_i, \quad \tau = dt. \quad (2.47)$$

In this local setting, the Reeb vector field is expressed simply as:

$$\mathcal{R}^\tau = \frac{\partial}{\partial t}. \quad (2.48)$$

For a cosymplectic manifold, we establish an isomorphism denoted as $\flat^{cs}$:

$$\flat^{cs} : \mathfrak{X}(\bar{M}) \longrightarrow \Gamma^1(\bar{M}), \quad X \mapsto \flat^{cs}(X) = \iota_X \Omega + \langle \tau, X \rangle \tau. \quad (2.49)$$

Its inverse is denoted as $\sharp^{cs}$, where the one-form τ maps to the Reeb field $\mathcal{R}^\tau$. This determines a Poisson bivector field:

$$\Lambda^{cs}(\alpha, \beta) = \Omega(\sharp^{cs}(\alpha), \sharp^{cs}(\beta)). \quad (2.50)$$

In this setup, we have:

$$\sharp_{\Lambda^{cs}}(\alpha) := \Lambda^{cs}(\bullet, \alpha) = \sharp^{cs}(\alpha) - \langle \alpha, \mathcal{R}^\tau \rangle \mathcal{R}^\tau. \quad (2.51)$$

The Poisson bracket is now taking the form

$$\{F, \mathcal{H}\}^{cs} = \Omega(\sharp^{cs}(dF), \sharp^{cs}(d\mathcal{H})). \quad (2.52)$$

The following equalities determine a cosymplectic Hamiltonian vector field:

$$\iota_{X_{\mathcal{H}}^{cs}} \Omega = d\mathcal{H} - \langle d\mathcal{H}, \mathcal{R}^\tau \rangle \tau, \quad \iota_{X_{\mathcal{H}}^{cs}} \tau = 0. \quad (2.53)$$

The Jacobi-Lie bracket of two cosymplectic Hamiltonian vector fields is again a cosymplectic Hamiltonian vector field and satisfies:

$$[X_F^{cs}, X_{\mathcal{H}}^{cs}] = -X_{\{F, \mathcal{H}\}^{cs}}. \quad (2.54)$$

Thus, the space of cosymplectic Hamiltonian vector fields forms a Lie algebra. Furthermore, the algebraic equation (2.54) enables us to define a Lie algebra homomorphism from the space $C^\infty(\bar{M})$ of smooth functions on the cosymplectic manifold $\bar{M}$ to the space of cosymplectic Hamiltonian vector fields:

$$\bar{\Phi} : C^\infty(\bar{M}) \longrightarrow \mathfrak{X}_{\text{cs-ham}}(\bar{M}), \quad \mathcal{H} \mapsto X_{\mathcal{H}}^{cs}. \quad (2.55)$$

In terms of Darboux coordinates (t, q^i, p_i) , the musical mappings $\sharp^{cs}$ and $\sharp_{\Lambda^{cs}}$ are computed as:

$$\begin{aligned}\sharp^{cs} : \alpha_i dq^i + \alpha^i dp_i + u dt &\mapsto \alpha^i \frac{\partial}{\partial q^i} - \alpha_i \frac{\partial}{\partial p_i} + u \frac{\partial}{\partial t} \\ \sharp_{\Lambda^{cs}} : \alpha_i dq^i + \alpha^i dp_i + u dt &\mapsto \alpha^i \frac{\partial}{\partial q^i} - \alpha_i \frac{\partial}{\partial p_i}.\end{aligned}\quad (2.56)$$

The bivector Λ^{cs} and the Poisson bracket are computed as:

$$\Lambda^{cs} = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}, \quad \{F, \mathcal{H}\}^{cs}(t, q, p) = \frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial F}{\partial q^i} - \frac{\partial \mathcal{H}}{\partial q^i} \frac{\partial F}{\partial p_i}. \quad (2.57)$$

Although the cosymplectic Poisson bracket $\{\bullet, \bullet\}^{cs}$ looks like the symplectic Poisson bracket, they are distinct. The bracket in (2.19) is non-degenerate, while that in (2.52) is degenerate, allowing non-constant Casimir functions. Functions dependent only on t are Casimir functions for the algebra (2.52). Notice that, if the real variable t is identified with the time (we will be more clear about this discussion in the upcoming paragraphs where we present evolution vector fields) then the dynamics determined by the cosymplectic Hamiltonian vector field represents time dependent Hamiltonian dynamics

$$\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i}(t, q, p), \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i}(t, q, p). \quad (2.58)$$

For a Hamiltonian function $\mathcal{H}$, we define an evolution vector field $E_{\mathcal{H}}$ using the following conditions:

$$\iota_{E_{\mathcal{H}}} \Omega = d\mathcal{H} - \langle d\mathcal{H}, \mathcal{R}^\tau \rangle \tau, \quad \iota_{E_{\mathcal{H}}} \tau = 1. \quad (2.59)$$

Notice that, expressed in terms of the musical isomorphism $\sharp^{cs}$ and the Reeb field, the evolution vector field is:

$$E_{\mathcal{H}} = \sharp^{cs}(d\mathcal{H}) + \mathcal{R}^\tau - \langle d\mathcal{H}, \mathcal{R}^\tau \rangle \mathcal{R}^\tau. \quad (2.60)$$

In this formulation, the first and third terms compose the Hamiltonian vector field $X_{\mathcal{H}}^{cs}$. In the Darboux coordinates (t, q^i, p_i) , we have:

$$E_{\mathcal{H}}(t, q, p) = \frac{\partial}{\partial t} + \frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial \mathcal{H}}{\partial q^i} \frac{\partial}{\partial p_i} \quad (2.61)$$

and the dynamics generated by the evolution vector field $E_{\mathcal{H}}$ is precisely:

$$\dot{t} = 1, \quad \dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i}(t, q, p), \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i}(t, q, p). \quad (2.62)$$

Since $\dot{t} = 1$, we can consider t as a time-parameter up to an affine term. Thus, we can argue that cosymplectic Hamiltonian dynamics provides a convenient geometric formulation for time-dependent Hamiltonians.

Hamilton-Poincaré Realization. Consider the cotangent bundle T^*Q with the Liouville one-form θ_Q and the symplectic two-form Ω_Q . Additionally, recall the cosymplectic pair (τ, Ω) in (2.47) defined on the extended cotangent bundle $\mathbb{R} \times T^*Q$. By means of a Hamiltonian function $\mathcal{H}$, an alternative cosymplectic structure on the extended cotangent bundle is determined through the Hamilton-Poincaré forms:

$$\theta_{\mathcal{H}} = \theta_Q - \mathcal{H}dt, \quad \Omega_{\mathcal{H}} = \Omega_Q + d\mathcal{H} \wedge dt, \quad (2.63)$$

respectively.

Noting $\tau = dt$, it is straightforward to observe that $(\mathbb{R} \times T^*Q, \tau, \Omega_{\mathcal{H}})$ forms a cosymplectic manifold. For this second cosymplectic structure, the Reeb field $\mathcal{R}_{\mathcal{H}}^{\tau}$ is defined as:

$$\iota_{\mathcal{R}_{\mathcal{H}}^{\tau}} dt = 1, \quad \iota_{\mathcal{R}_{\mathcal{H}}^{\tau}} \Omega_{\mathcal{H}} = 0. \quad (2.64)$$

Note that the evolution vector field $E_{\mathcal{H}}$ and $\mathcal{R}_{\mathcal{H}}^{\tau}$ are equivalent.

HJ Theorem for Cosymplectic Hamiltonian Dynamics. The Hamilton-Jacobi problem involves determining a smooth function $S = S(q^i, t)$ defined on the extended space $Q \times \mathbb{R}$, satisfying the equation:

$$\frac{\partial S}{\partial t} + H\left(t, q^i, \frac{\partial S(q^i, t)}{\partial t}\right) = f(t). \quad (2.65)$$

Here, $f(t)$ is an arbitrary real-valued function. In the classical realization of time-dependent Hamilton-Jacobi theory, the function f is often taken to be zero.

We determine mappings:

$$\begin{aligned} \tilde{\gamma} : Q \times \mathbb{R} &\longrightarrow T^*Q \times T^*\mathbb{R}, & (q, t) &\mapsto (\bar{\gamma}(q, t), t, \bar{\bar{\gamma}}(q, t)), \\ \gamma : Q \times \mathbb{R} &\longrightarrow T^*Q \times \mathbb{R}, & (q, t) &\mapsto (\bar{\gamma}(q, t), t). \end{aligned} \quad (2.66)$$

Considering $T^*\mathbb{R} = \mathbb{R} \times \mathbb{R}$, where each fiber $T_t^*\mathbb{R}$ is isomorphic to $\mathbb{R}$ for $t \in \mathbb{R}$: $\bar{\gamma}(q, t)$ lies in T^*Q , while $\bar{\bar{\gamma}}(q, t)$ resides in $\mathbb{R}$. It is notable that under this notation, the equation $\gamma = \pi \circ \tilde{\gamma}$ holds true.

Consider now a time-dependent Hamiltonian function $\mathcal{H}$ on the cosymplectic manifold $\mathbb{R} \times T^*Q$. Notice that, the evolution flow $E_{\mathcal{H}}$ generated by the Hamiltonian function $\mathcal{H}$ is defined by the cosymplectic formulation in (2.59). Referring to the projection π , we pull-back $\mathcal{H}$ and define the extended Hamiltonian

$$\mathcal{H}^s(z, t, s) = \pi^*\mathcal{H}(z, t) + s \quad (2.67)$$

on the cotangent bundle $T^*(Q \times \mathbb{R})$. The flow $X_{\mathcal{H}^s}$ generated by the Hamiltonian function $\mathcal{H}^s$ is determined through the symplectic relation $\iota_{X_{\mathcal{H}^s}}\bar{\Omega} = d\mathcal{H}^s$. Here, $\bar{\Omega}$ is the (lifted) symplectic two-form given locally as $\bar{\Omega} = dq^i \wedge dp_i + dt \wedge ds$. In terms of Darboux coordinates, these vector fields are computed to be

$$X_{\mathcal{H}^s} = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} + \frac{\partial}{\partial t} - \frac{\partial H}{\partial t} \frac{\partial}{\partial s}. \quad (2.68)$$

It is evident from the local observation that $X_{\mathcal{H}^s}$ and $E_{\mathcal{H}}$ are π -related [71], that is,

$$E_{\mathcal{H}} \circ \pi = T\pi \circ X_{\mathcal{H}^s}. \quad (2.69)$$

We project the vector fields $X_{\mathcal{H}^s}$ and $E_{\mathcal{H}}$ to the extended configuration space $Q \times \mathbb{R}$ by means of the sections $\tilde{\gamma}$ and γ . Accordingly, we arrive at the following vector fields

$$X_{\mathcal{H}^s}^{\tilde{\gamma}} := T\pi_{Q \times \mathbb{R}} \circ X_{\mathcal{H}^s} \circ \tilde{\gamma}, \quad E_{\mathcal{H}}^{\gamma} := T\tau \circ X_{\mathcal{H}^s} \circ \gamma, \quad (2.70)$$

respectively. The projected vector fields $X_{\mathcal{H}^s}^{\tilde{\gamma}}$ and $E_{\mathcal{H}}^{\gamma}$ coincide. By choosing $\bar{\gamma} = \bar{\gamma}_i(q, t) dq^i$ we have that

$$X_{\mathcal{H}^s}^{\tilde{\gamma}} = E_{\mathcal{H}}^{\gamma} = \frac{\partial}{\partial t} + \frac{\partial \mathcal{H}}{\partial p_i} \Big|_{p_i = \bar{\gamma}_i(q, t)} \frac{\partial}{\partial q^i}. \quad (2.71)$$

We state the following geometric Hamilton-Jacobi theorem for cosymplectic manifolds [71].

Theorem 2.2. *Suppose $\tilde{\gamma}$ is closed. Then, the following conditions are equivalent:*

1. $X_{\mathcal{H}^s}^{\tilde{\gamma}}$ and $E_{\mathcal{H}}$ are connected as

$$T\pi \circ T\tilde{\gamma} \circ X^{\tilde{\gamma}}\mathcal{H}^s = E_{\mathcal{H}} \circ \pi \circ \tilde{\gamma}. \quad (2.72)$$

2. This equation holds true:

$$d(\mathcal{H}^s \circ \tilde{\gamma}) \in \langle \tau \rangle. \quad (2.73)$$

Here, $\langle \tau \rangle$ means the space spanned by τ .

2.5. Contact Manifolds

Hamiltonian dynamics is defined on Poisson (more specifically symplectic) manifolds that offer a reversible motion where the Hamiltonian function is preserved. A generalization of the classical Hamiltonian dynamics offering irreversible dynamics is achieved on contact manifolds. In this subsection, we are recalling this formalism.

Let N be an odd-dimensional manifold of dimension $(2n + 1)$. A contact structure on N is a maximally non-integrable smooth distribution of codimension one, defined locally by a one-form η with the non-integrability condition

$$d\eta^n \wedge \eta \neq 0. \quad (2.74)$$

Such a one-form η is called a (local) contact form [5, 66]. We consider the existence of a global contact one-form and denote a contact manifold by (N, η) . Given a contact one-form η , the vector field $\mathcal{R}^\eta$ satisfying

$$\iota_{\mathcal{R}^\eta} \eta = 1, \quad \iota_{\mathcal{R}^\eta} d\eta = 0 \quad (2.75)$$

is unique, and it is called the (contact) Reeb vector field. For a contact manifold (N, η) , there is a musical isomorphism

$$\flat : \mathfrak{X}(N) \longrightarrow \Gamma^1(N), \quad \xi \mapsto \iota_\xi d\eta + \langle \eta, \xi \rangle \eta. \quad (2.76)$$

We denote the inverse of (2.76) by $\sharp$. Referring to this, we define a bivector field Λ on N as

$$\Lambda(\alpha, \beta) = d\eta(\sharp\alpha, \sharp\beta). \quad (2.77)$$

The pair $(\Lambda, \mathcal{R}^\eta)$ gives a Jacobi structure [16, 17], since they satisfy (2.38). This realization permits us to define a Jacobi (contact) bracket

$$\{F, \mathcal{H}\}^c = \Lambda(dF, d\mathcal{H}) + F\mathcal{R}^\eta(\mathcal{H}) - \mathcal{H}\mathcal{R}^\eta(F). \quad (2.78)$$

Notice that this bracket satisfies the Jacobi identity but not the Leibniz identity.

Dissipation on Extended Manifold. Consider a contact manifold (N, η) . We define the contact Hamiltonian vector field $\xi_{\mathcal{H}}$ as [72–76]

$$\iota_{\xi_{\mathcal{H}}} d\eta = d\mathcal{H} - \mathcal{R}^\eta(\mathcal{H})\eta, \quad \iota_{\xi_{\mathcal{H}}} \eta = -\mathcal{H}. \quad (2.79)$$

The relationship between the Hamiltonian vector field and the contact bracket is given by the equation

$$\xi_{\mathcal{H}}(F) = \{F, \mathcal{H}\}^c - F\mathcal{R}^\eta(\mathcal{H}). \quad (2.80)$$

For a given Hamiltonian function $\mathcal{H}$, the contact Hamiltonian vector field is a strict contact vector field if and only if $\mathcal{R}^\eta(\mathcal{H}) = 0$. Accordingly, one has strict contact Hamiltonian vector field as

$$\iota_{\xi_{\mathcal{H}}}d\eta = d\mathcal{H}, \quad \iota_{\xi_{\mathcal{H}}}\eta = -\mathcal{H}. \quad (2.81)$$

In order to observe the dissipative behaviour of this dynamics we exhibit two calculations:

$$\mathcal{L}_{\xi_{\mathcal{H}}}\mathcal{H} = -\mathcal{R}^\eta(\mathcal{H})\mathcal{H}, \quad \mathcal{L}_{\xi_{\mathcal{H}}}(\eta \wedge (d\eta)^n) = -(n+1)\mathcal{R}^\eta(\mathcal{H})\eta \wedge (d\eta)^n, \quad (2.82)$$

where the former one is showing that the Hamiltonian function is not preserved under the Hamiltonian function whereas the latter gives the dissipation of the volume form $\eta \wedge (d\eta)^n$ (given for $2n+1$ -dimensional contact manifold). Accordingly we compute the divergence of the contact Hamiltonian vector field as

$$\text{div}(\xi_{\mathcal{H}}) = -(n+1)\mathcal{R}^\eta(\mathcal{H}). \quad (2.83)$$

A generic example of a contact manifold is the extended symplectic manifold $N = T^*Q \times \mathbb{R}$. It is to write the contact one-form referring to the canonical one-form θ_Q and the differential one-form dz defined on $\mathbb{R}$. More concretely, by the abuse of the notation (we denote pullbacks and pushforwards of the tensor fields by the same notation not to cause notation inflation), we have

$$\eta = dz - \theta_Q. \quad (2.84)$$

In the Darboux coordinates (q^i, p_i, z) on N , the contact one-form and the Reeb field are computed to be

$$\eta = dz - p_i dq^i, \quad \mathcal{R}^\eta = \frac{\partial}{\partial z}. \quad (2.85)$$

Furthermore, in the Darboux coordinates, the bivector $\Lambda_{T^*Q \times \mathbb{R}}$ is computed to be

$$\Lambda_{T^*Q \times \mathbb{R}} = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + p_i \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial p_i}. \quad (2.86)$$

Here, the first term is due to the canonical Poisson bivector field Λ_{T^*Q} on T^*Q obtained from the symplectic two-form Ω_Q . By defining the Liouville vector field

$$\Delta^*(q^i, p_i, z) = p_i \frac{\partial}{\partial p_i} \quad (2.87)$$

on the extended cotangent bundle, we can write the contact bivector field as

$$\Lambda_{T^*Q \times \mathbb{R}} = \Lambda_{T^*Q} + \mathcal{R}^\eta \wedge \Delta^*. \quad (2.88)$$

In the local picture, the contact bracket (2.78) is written as

$$\{F, \mathcal{H}\}^c = \frac{\partial F}{\partial q^i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q^i} + \left(F - p_i \frac{\partial F}{\partial p_i}\right) \frac{\partial \mathcal{H}}{\partial z} - \left(\mathcal{H} - p_i \frac{\partial \mathcal{H}}{\partial p_i}\right) \frac{\partial F}{\partial z}. \quad (2.89)$$

Accordingly, the contact Hamiltonian vector field for a given Hamiltonian function $\mathcal{H} = \mathcal{H}(q^i, p_i, z)$ becomes

$$\xi_{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial}{\partial q^i} - \left(\frac{\partial \mathcal{H}}{\partial q^i} + \frac{\partial \mathcal{H}}{\partial z} p_i\right) \frac{\partial}{\partial p_i} + \left(p_i \frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{H}\right) \frac{\partial}{\partial z}. \quad (2.90)$$

Thus, we obtain the contact Hamilton's equations for $\mathcal{H}$ as

$$\frac{dq^i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q^i} - p_i \frac{\partial \mathcal{H}}{\partial z}, \quad \frac{dz}{dt} = p_i \frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{H}. \quad (2.91)$$

Lagrangian Dynamics. To add dissipation to the Lagrangian dynamics on the tangent bundle TQ , we extend it by a real line and get the extended tangent bundle $TQ \times \mathbb{R}$ equipped with the induced coordinates $(q^i, \dot{q}^i, z)$. Hence, dissipative dynamics is generated by a Lagrangian function $L = L(q^i, \dot{q}^i, z)$ on $TM \times \mathbb{R}$. To derive the dynamical equations governed by such a Lagrangian function, the application of the Herglotz principle is necessary. This principle is defined by an action functional, as described in [77–80], with a more comprehensive coverage in [81] for singular Lagrangians. In the present work, we present the discussion for regular Lagrangians. For a regular Lagrangian function L , the fiber derivative determines a diffeomorphism from the extended tangent bundle $TQ \times \mathbb{R}$ to the extended cotangent bundle $T^*Q \times \mathbb{R}$ as

$$\mathbb{F}L : TQ \times \mathbb{R} \longrightarrow T^*Q \times \mathbb{R}, \quad (q^i, \dot{q}^i, z) \mapsto (q^i, \frac{\partial L}{\partial \dot{q}^i}, z). \quad (2.92)$$

We pull the contact one-form η on $T^*Q \times \mathbb{R}$ and write it as η^T . This reads a contact manifold $(TQ \times \mathbb{R}, \eta^T)$. We define the energy function

$$E_L(q^i, \dot{q}^i, z) = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L(q^i, \dot{q}^i, z), \quad (2.93)$$

and determine the contact motion on the extended tangent bundle $TQ \times \mathbb{R}$. On the dual picture the energy function corresponds to the Hamiltonian function

$$\mathcal{H}(q^i, p_i, z) = \dot{q}^i p_i - L(q^i, \dot{q}^i, z). \quad (2.94)$$

Hence, from the contact dynamics (2.81) given in coordinate-free manner, assuming $\mathcal{H} := E_L$ as the contact Hamiltonian function and $\eta := \eta^T$ as the contact one-form, we arrive at the Herglotz equations (also known as the generalized Euler–Lagrange equations)

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial \dot{q}^i} \frac{\partial L}{\partial z}. \quad (2.95)$$

See that this generalizes the classical Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0 \quad (2.96)$$

for a Lagrangian function $L = L(q^i, \dot{q}^i)$ defined on the tangent bundle TQ .

3. LIE ALGEBRA(OID)S AND COUPLINGS

3.1. Lie Structures

3.1.1. Lie Algebras

Let $\mathfrak{g}$ be a vector space over the real numbers $\mathbb{R}$. It is called Lie algebra if it admits a bilinear operation

$$[\bullet, \bullet] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}, \quad (3.1)$$

which satisfies, for all ξ, ζ , and ε in $\mathfrak{g}$,

- the skew-symmetry

$$[\xi, \zeta] = -[\zeta, \xi], \quad (3.2)$$

- the Jacobi identity

$$[[\xi, \zeta], \varepsilon] + [[\varepsilon, \xi], \zeta] + [[\zeta, \varepsilon], \xi] = 0. \quad (3.3)$$

Here, the bracket in (3.1) is called Lie bracket. We denote a Lie algebra by a two-tuple $(\mathfrak{g}, [\bullet, \bullet])$ and sometimes by only $\mathfrak{g}$ if the content is clear.

In finite dimensions, assume that a Lie algebra $\mathfrak{g}$ admits a basis $\{e_i\}$. Computation of the Lie brackets of two basis elements

$$[e_i, e_j] = C_{ij}^k e_k \quad (3.4)$$

determines a set of scalars C_{ij}^k in $\mathbb{R}$. These scalars are called structure constants of the Lie algebra. The skew-symmetry of the Lie bracket reads that the constants are skew-symmetric under a reordering of the subindices that is

$$C_{ij}^k = -C_{ji}^k. \quad (3.5)$$

Further, the Jacobi identity asks that the following equality

$$C_{ij}^k C_{kl}^m + C_{li}^k C_{kj}^m + C_{jl}^k C_{ki}^m = 0 \quad (3.6)$$

must be satisfied. It is immediate to see that a basis admitting a set of scalars defines a Lie algebra if and only if the set satisfies (3.5) and (3.6).

A vector subspace of a Lie algebra $\mathfrak{g}$ is called a Lie subalgebra if it is closed under the Lie bracket. An ideal $\mathfrak{i} \subseteq \mathfrak{g}$ is a subalgebra satisfying the stronger condition

$$[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}. \quad (3.7)$$

A linear map from a Lie algebra to another Lie algebra is called Lie algebra homomorphism if it respects Lie brackets. That is, a linear mapping

$$\varphi : (\mathfrak{g}_1, [\bullet, \bullet]_1) \longrightarrow (\mathfrak{g}_2, [\bullet, \bullet]_2) \quad (3.8)$$

is a Lie algebra homomorphism if

$$\varphi([\xi, \xi']_1) = [\varphi(\xi), \varphi(\xi')]_2 \quad (3.9)$$

for all ξ and ξ' in $\mathfrak{g}_1$. A Lie algebra homomorphism is called a Lie algebra isomorphism if it is also an isomorphism (that is bijective).

A derivation on the Lie algebra $\mathfrak{g}$ is a linear map $D : \mathfrak{g} \mapsto \mathfrak{g}$ that obeys the Leibniz rule

$$D([\xi, \xi']) = [D(\xi), \xi'] + [\xi, D(\xi')]. \quad (3.10)$$

The inner derivation associated to any $\xi \in \mathfrak{g}$ is the adjoint mapping

$$ad_\xi : \mathfrak{g} \longrightarrow \mathfrak{g}, \quad ad_\xi \xi' = [\xi, \xi']. \quad (3.11)$$

This is a derivation as a consequence of the Jacobi identity. The derivations form a vector space which is a Lie algebra under the commutator bracket

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1 \quad (3.12)$$

where $\circ$ denotes the composition of mappings.

Lie algebra is a vector space, and its dual space is denoted by $\mathfrak{g}^*$. The left coadjoint action of the Lie algebra $\mathfrak{g}$ on its dual space $\mathfrak{g}^*$ is defined by

$$ad^* : \mathfrak{g} \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*, \quad \langle ad_\xi^* \mu, \tilde{\xi} \rangle = -\langle \mu, ad_\xi \tilde{\xi} \rangle. \quad (3.13)$$

Lie-Poisson Dynamics. We start with a Lie algebra $(\mathfrak{g}, [\bullet, \bullet])$. The dual of a Lie algebra $\mathfrak{g}^*$ admits a Poisson bracket, called Lie-Poisson bracket. In this case, for

two function(al)s f and h on the dual space $\mathfrak{g}^*$, the (plus) Lie-Poisson bracket is determined by

$$\{f, h\}(\mu) = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right\rangle, \quad \mu \in \mathfrak{g}^*, \quad (3.14)$$

where $\delta f / \delta \mu$ is the partial (for infinite cases, the Fréchet) derivative of the function(al) f , the bracket on the right hand side is the Lie algebra bracket on $\mathfrak{g}$, and the pairing is the duality between $\mathfrak{g}^*$ and $\mathfrak{g}$. Note that, to make this definition holds for all function(al)s, one needs to assume the reflexivity condition on $\mathfrak{g}$, that is the double dual $\mathfrak{g}^{**}$ is in $\mathfrak{g}$. In this case, the derivative $\delta f / \delta \mu$ becomes an element of the Lie algebra $\mathfrak{g}$ and this makes the definition in (3.14) valid.

The dynamics of an observable f , governed by a Hamiltonian function(al) h , is then computed to be

$$\begin{aligned} \dot{f} &= \{f, h\}(\mu) = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right\rangle \\ &= \left\langle \mu, -ad_{\delta h / \delta \mu} \frac{\delta f}{\delta \mu} \right\rangle \\ &= \left\langle ad_{\delta h / \delta \mu}^* \mu, \frac{\delta f}{\delta \mu} \right\rangle. \end{aligned} \quad (3.15)$$

Then, we obtain the equation of motion governed by a Hamiltonian function(al) h that is the Lie-Poisson equation

$$\dot{\mu} = ad_{\delta h / \delta \mu}^* \mu. \quad (3.16)$$

Euler-Poincaré Dynamics. Consider a real-valued Lagrangian function l defined on a Lie algebra $\mathfrak{g}$. Take the variation of the action integral:

$$\delta \int_a^b l(\xi) dt = \int_a^b \left\langle \frac{\delta l}{\delta \xi}, \delta \xi \right\rangle dt. \quad (3.17)$$

The extremum of this action integral reads the Euler-Poincaré equation

$$\frac{d}{dt} \frac{\delta l}{\delta \xi} = -ad_{\xi}^* \frac{\delta l}{\delta \xi}. \quad (3.18)$$

The critical step in the variation is to use the the principle of reduced variation given by

$$\delta \xi = \dot{\eta} + [\xi, \eta], \quad (3.19)$$

see [82, 83].

Coordinate realizations. Assume an N -dimensional Lie algebra $\mathfrak{g}$ admitting a basis $\{e_i\} = \{e_1, \dots, e_N\}$. One has the dual basis $\{e^i\} = \{e^1, \dots, e^N\}$ on the dual space $\mathfrak{g}^*$ defined to be

$$\langle e_i, e^j \rangle = \delta_i^j. \quad (3.20)$$

Here δ_i^j is the Kronecker delta which gives 1 if the indices i and j are the same and gives zero otherwise. We denote an element of $\mathfrak{g}^*$ by $\mu = \mu_i e^i$ where the coordinates $\{\mu_1, \dots, \mu_N\}$ are being real numbers. The Lie-Poisson bracket (3.14) can be computed in this coordinate picture as

$$\{f, h\} = C_{ij}^k \mu_k \frac{\partial f}{\partial \mu_i} \frac{\partial h}{\partial \mu_j}, \quad (3.21)$$

where C_{ij}^k are being the structure constants on $\mathfrak{g}$. The calculation in (3.21) reads that the coefficients Λ_{ij} of the Poisson bivector linearly as follows

$$\Lambda_{ij} = C_{ij}^k \mu_k. \quad (3.22)$$

In this case, the Lie-Poisson equations (3.16) are computed to be

$$\dot{\mu}_i = C_{ij}^k \mu_k \frac{\partial h}{\partial \mu_j}. \quad (3.23)$$

3.1.2. Leibniz Algebras

Let us recall the concept of a (left) Leibniz algebra as outlined by [31], with supplementary insights from [84, 85]. A (left) Leibniz algebra, denoted as $\mathfrak{L}$, is a structured vector space endowed with a bilinear skew-symmetric operation $[\bullet, \bullet]$ on $\mathfrak{L}$. This map adheres to a fundamental property known as the Leibniz identity, expressed by

$$[\xi, [\zeta, \epsilon]] = [[\xi, \zeta], \epsilon] + [\zeta, [\xi, \epsilon]] \quad (3.24)$$

for all elements ξ, ζ , and ϵ belonging to $\mathfrak{L}$. We denote a Leibniz algebra by a pair $(\mathfrak{L}, [\bullet, \bullet])$.

A representation of a Leibniz algebra $(\mathfrak{L}, [\bullet, \bullet])$ on a vector space V consisting of the following linear mappings

$$\lambda : \mathfrak{L} \rightarrow gl(V), \quad \rho : \mathfrak{L} \rightarrow gl(V), \quad (3.25)$$

whereby

$$\lambda([\xi, \zeta]) = \lambda(\xi) \circ \lambda(\zeta) - \lambda(\zeta) \circ \lambda(\xi), \quad (3.26)$$

$$\rho([\xi, \zeta]) = \lambda(\xi) \circ \rho(\zeta) - \rho(\zeta) \circ \lambda(\xi), \quad (3.27)$$

$$\rho(\zeta) \circ \rho(\xi) = -\rho(\zeta) \circ \lambda(\xi), \quad (3.28)$$

for every pair $\xi, \zeta \in \mathfrak{L}$. Here, $g\ell(V)$ denotes the space of general linear operators on the vector space V . It is worth noting here that we may also use the term “representation” for (V, λ, ρ) , or just for V when we want to refer to the vector space on which the algebra has a representation.

In another perspective, referring to the left and right actions, we may view λ as a left action

$$\triangleright : \mathfrak{L} \otimes V \longrightarrow V, \quad \xi \triangleright v := \lambda(\xi)(v), \quad (3.29)$$

while we can view ρ as a right action

$$\triangleleft : V \otimes \mathfrak{L} \longrightarrow V, \quad v \triangleleft \xi := \rho(\xi)(v), \quad (3.30)$$

for any ξ and ζ in $\mathfrak{L}$, and any v in V . Accordingly, the conditions (3.26)-(3.28) may be rewritten as

$$[\xi, \zeta] \triangleright v = \xi \triangleright (\zeta \triangleright v) - \zeta \triangleright (\xi \triangleright v), \quad (3.31)$$

$$v \triangleleft [\xi, \zeta] = \xi \triangleright (v \triangleleft \zeta) - (\xi \triangleright v) \triangleleft \zeta, \quad (3.32)$$

$$(v \triangleleft \xi) \triangleleft \zeta = -(\xi \triangleright v) \triangleleft \zeta. \quad (3.33)$$

Suppose we have a Leibniz algebra $\mathfrak{L}$ and a representation V of $\mathfrak{L}$. It can be readily confirmed that $V \rtimes \mathfrak{L} := V \oplus \mathfrak{L}$ forms another Leibniz algebra through the operation

$$\left[(v, \xi), (w, \zeta) \right] := \left(\xi \triangleright w + v \triangleleft \zeta, [\xi, \zeta] \right), \quad (3.34)$$

known as the semi-direct sum Leibniz algebra.

For a (left) Leibniz algebra $\mathfrak{L}$, a linear mapping D referred as a derivation if it satisfies

$$D([\xi, \zeta]) = [D(\xi), \zeta] + [\xi, D(\zeta)] \quad (3.35)$$

for all $\xi, \zeta \in \mathfrak{L}$. Given a Leibniz algebra $\mathfrak{L}$, the (left) adjoint action

$$ad_{\xi}^L : \mathfrak{L} \longrightarrow \mathfrak{L}, \quad ad_{\xi}^L(\zeta) := [\xi, \zeta] \quad (3.36)$$

behaves as a derivation for any ξ in $\mathfrak{L}$. This conclusion directly follows from the properties of (left) Leibniz algebras and the behaviour of the differential operator defined in terms of the Leibniz bracket.

Let $\mathfrak{L}$ be a Leibniz algebra, and suppose V is a representation space of $\mathfrak{L}$. Then, we define a structure $CL(\mathfrak{L}, V)$ as the direct sum of spaces $CL^n(\mathfrak{L}, V)$ for all non-negative integers n , where $CL^n(\mathfrak{L}, V)$ consists of linear maps from $\mathfrak{L}$ raised to the n -th tensor power to V :

$$CL(\mathfrak{L}, V) := \bigoplus_{n \geq 0} CL^n(\mathfrak{L}, V), \quad CL^n(\mathfrak{L}, V) := \text{Hom}(\mathfrak{L}^{\otimes n}, V). \quad (3.37)$$

This structure is equipped with a differential operator d , which acts on $CL^n(\mathfrak{L}, V)$ and is given by

$$\begin{aligned} df(\xi_1, \dots, \xi_{n+1}) &= \sum_{1 \leq i < j \leq n+1} (-1)^i f(\xi_1, \dots, \widehat{\xi_i}, \dots, \xi_{j-1}, [\xi_i, \xi_j], \dots, \xi_{n+1}) \\ &+ \sum_{k=1}^n (-1)^{k+1} \xi_k \triangleright f(\xi_1, \dots, \widehat{\xi_k}, \dots, \xi_{n+1}) + (-1)^{n+1} f(\xi_1, \dots, \xi_n) \triangleleft \xi_{n+1}. \end{aligned} \quad (3.38)$$

The operator d takes a function f from $CL^n(\mathfrak{L}, V)$ and produces another function involving the Leibniz bracket and actions of elements of $\mathfrak{L}$ on f . This construction forms what we call a differential complex. The homology of this complex, denoted $H(CL(\mathfrak{L}, V), d)$, is termed the Leibniz algebra cohomology of $\mathfrak{L}$ with coefficients in V , represented as $HL^*(\mathfrak{L}, V)$.

3.1.3. Lie Algebroids

Given a manifold M , a Lie algebroid $\mathcal{A}$ over the base manifold M is a vector bundle $\tau : \mathcal{A} \mapsto M$, together with a map

$$\mathfrak{a}_{\mathcal{A}} : \mathcal{A} \longrightarrow TM \quad (3.39)$$

of vector bundles, called the anchor map, and a skew-symmetric bilinear bracket $[\bullet, \bullet]_{\mathcal{A}}$ on the space $\Gamma(\mathcal{A})$ of sections of $\mathcal{A}$ so that

$$\begin{aligned} (i) \quad & \mathfrak{a}_{\mathcal{A}}([X_1, X_2]_{\mathcal{A}}) = [\mathfrak{a}_{\mathcal{A}}(X_1), \mathfrak{a}_{\mathcal{A}}(X_2)], \\ (ii) \quad & [X_1, fX_2]_{\mathcal{A}} = f[X_1, X_2]_{\mathcal{A}} + \mathcal{L}_{\mathfrak{a}_{\mathcal{A}}(X_1)}(f)X_2, \\ (iii) \quad & \circlearrowleft [X_1, [X_2, X_3]_{\mathcal{A}}]_{\mathcal{A}} = 0, \end{aligned} \quad (3.40)$$

for any X_1, X_2 and X_3 in $\Gamma(\mathcal{A})$, and any f in $\mathcal{C}^\infty(M)$ [86–88]. The condition (iii) is the Jacobi identity. A Lie algebroid is denoted by a quintuple

$$(\mathcal{A}, \tau, M, \mathfrak{a}_{\mathcal{A}}, [\bullet, \bullet]_{\mathcal{A}}), \quad (3.41)$$

or in short by $\mathcal{A}$, or presented by a commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathfrak{a}_{\mathcal{A}}} & TM \\ & \searrow \tau & \swarrow \tau_M \\ & M & \end{array} \quad (3.42)$$

where τ_M is the tangent bundle projection.

In a finite-dimensional setting, consider a local coordinate system (x^i) on the base manifold M and a local coordinate system (x^i, y^α) on the total space $\mathcal{A}$. Simultaneously, select the basis set (e_α) for the section space $\Gamma(\mathcal{A})$. In this setup, we can represent the anchor map $\mathfrak{a}_{\mathcal{A}}$ and the structure functions for the Lie algebroid frame as

$$\mathfrak{a}_{\mathcal{A}}(e_\alpha) = (\mathfrak{a}_{\mathcal{A}})_\alpha^i \frac{\partial}{\partial x^i}, \quad [e_\alpha, e_\beta]_{\mathcal{A}} = \mathfrak{H}_{\alpha\beta}^\gamma e_\gamma. \quad (3.43)$$

Here, $(\mathfrak{a}_{\mathcal{A}})_\alpha^i$ represents the components of the anchor map matrix, and $\mathfrak{H}_{\alpha\beta}^\gamma$ denote the structure functions corresponding to the Lie algebroid frame.

The representation of a Lie algebroid $(\mathcal{A}, \tau, M, \mathfrak{a}_{\mathcal{A}}, [\bullet, \bullet]_{\mathcal{A}})$ on a vector bundle $(\mathcal{B}, \kappa, M)$ is defined to be a map

$$\sigma : \Gamma(\mathcal{A}) \times \Gamma(\mathcal{B}) \longrightarrow \Gamma(\mathcal{B}), \quad (X, Y) \mapsto \sigma(X, Y)$$

such that for arbitrary sections X, X_1, X_2 in $\Gamma(\mathcal{A})$ of the algebroid, for any section Y in $\Gamma(\mathcal{B})$ of the vector bundle, and any function f in $\mathcal{C}^\infty(M)$ defined on the base manifold the following conditions

$$\begin{aligned} \sigma(fX, Y) &= f\sigma(X, Y), \\ \sigma(X, fY) &= f\sigma(X, Y) + \mathcal{L}_{\mathfrak{a}_{\mathcal{A}}(X)}(f)Y, \\ \sigma([X_1, X_2]_{\mathcal{A}}, Y) &= \sigma(X_1, \sigma(X_2, Y)) - \sigma(X_2, \sigma(X_1, Y)) \end{aligned} \quad (3.44)$$

are satisfied respecting the $\mathcal{C}^\infty(M)$ -module structure of the sections.

Let $(\mathcal{A}, \tau, M, \mathfrak{a}_{\mathcal{A}}, [\bullet, \bullet]_{\mathcal{A}})$ denote a Lie algebroid. We express its dual bundle as $(\mathcal{A}^*, \pi, M)$. Here, π is the projection dual to the projection τ . The fibers of

$\mathcal{A}^*$ over each point x in the base M , denoted as $\mathcal{A}_x^* = \pi^{-1}(x)$, are the linear algebraic dual vector spaces of the fibers of $\mathcal{A}$ corresponding to $\mathcal{A}_x = \tau^{-1}(x)$. For fundamentals on Lie algebroid geometry we refer [89–91].

The space of functions on the dual space $\mathcal{A}^*$ is spanned by two classes of functions. The first class consists of constant functions on fibers, i.e., functions on the base M . The second class consists of linear functions on fibers, i.e., the sections of the projection τ . Let us analyze the first class. For a function f on M , we can define a function $\widehat{f}$ on $\mathcal{A}^*$ as follows:

$$\widehat{f} : \mathcal{A}^* \longrightarrow \mathbb{R}, \quad \widehat{f}(u) = f \circ \pi(u). \quad (3.45)$$

The second class will be obtained from the sections of the Lie algebroid $\mathcal{A}$. For this, let us take a section X . The linear function $\widehat{X}$ on $\mathcal{A}^*$ for every u in $\mathcal{A}^*$ is defined as:

$$\widehat{X} : \mathcal{A}^* \longrightarrow \mathbb{R}, \quad \widehat{X}(u) = \langle u, X \circ \pi(u) \rangle. \quad (3.46)$$

For these functions, a Poisson bracket on $\mathcal{A}^*$ is defined as:

$$\{\widehat{f}, \widehat{g}\}_{\mathcal{A}^*} = 0, \quad \{\widehat{f}, \widehat{X}\}_{\mathcal{A}^*} = \widehat{\mathcal{L}_{\mathfrak{a}_{\mathcal{A}}(X)} f}, \quad \{\widehat{X}_1, \widehat{X}_2\}_{\mathcal{A}^*} = [\widehat{X}_1, \widehat{X}_2]_{\mathcal{A}}. \quad (3.47)$$

Here, the value of the second bracket on the right-hand side is the directional derivative of the function f in the direction of the vector field $\mathfrak{a}_{\mathcal{A}}(X)$.

In finite dimensions, let (x^i) be a local coordinate system on M and (x^i, y^α) be a local coordinate system on $\mathcal{A}$. In parallel, choose the basis set (e_α) for the section space $\Gamma(\mathcal{A})$. Let us choose the dual basis (e^α) for the section space of $\mathcal{A}^*$. Assuming (x^i) , (x^i, y^α) , and (x^i, y_α) are local coordinates on M , $\mathcal{A}$, and $\mathcal{A}^*$ respectively, the Poisson brackets in (3.47) take the form [91–93]:

$$\{x^i, x^j\}_{\mathcal{A}^*} = 0, \quad \{x^i, y_\alpha\}_{\mathcal{A}^*} = (\mathfrak{a}_{\mathcal{A}})^i_\alpha, \quad \{y_\alpha, y_\beta\}_{\mathcal{A}^*} = \mathfrak{H}^\gamma_{\alpha\beta} y_\gamma, \quad (3.48)$$

where $(\mathfrak{a}_{\mathcal{A}})^i_\alpha$ represents the components of the anchor map in the chosen bases, and $\mathfrak{H}^\gamma_{\alpha\beta}$ are the structure functions determined in equation (3.43). In light of these local representations, the Poisson bivector corresponding to the Poisson bracket is computed to be

$$\Lambda_{\mathcal{A}^*} = \frac{1}{2} \mathfrak{H}^\gamma_{\alpha\beta} y_\gamma \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta} + (\mathfrak{a}_{\mathcal{A}})^i_\alpha \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y_\alpha}. \quad (3.49)$$

For a Hamiltonian function $\mathcal{H}$ defined on the total space of the dual bundle $\mathcal{A}^*$ the Hamilton's equation is

$$\frac{du}{dt} = X_{\mathcal{H}}(u) = \{u, \mathcal{H}\}_{\mathcal{A}^*}. \quad (3.50)$$

Accordingly, in finite dimensions, the Hamiltonian vector field $X_{\mathcal{H}}$ is computed to be

$$X_{\mathcal{H}} = (\mathfrak{a}_{\mathcal{A}})_{\alpha}^i \frac{\partial \mathcal{H}}{\partial y_{\alpha}} \frac{\partial}{\partial x^i} - \left(\mathfrak{H}_{\alpha\beta}^{\gamma} y_{\gamma} \frac{\partial \mathcal{H}}{\partial y_{\beta}} + (\mathfrak{a}_{\mathcal{A}})_{\alpha}^i \frac{\partial \mathcal{H}}{\partial x^i} \right) \frac{\partial}{\partial y_{\alpha}}. \quad (3.51)$$

The integral curves of the section given by this vector field satisfy Hamilton's equations in coordinates:

$$\frac{dx^i}{dt} = (\mathfrak{a}_{\mathcal{A}})_{\alpha}^i \frac{\partial \mathcal{H}}{\partial y_{\alpha}}, \quad \frac{dy_{\alpha}}{dt} = -\mathfrak{H}_{\alpha\beta}^{\gamma} y_{\gamma} \frac{\partial \mathcal{H}}{\partial y_{\beta}} - (\mathfrak{a}_{\mathcal{A}})_{\alpha}^i \frac{\partial \mathcal{H}}{\partial x^i}. \quad (3.52)$$

Consider a Lagrangian function $L = L(x^i, y^{\alpha})$ on the Lie algebroid $\mathcal{A}$. To study the Lagrangian dynamics in a coordinate-free approach without referring to the Poisson structure on the dual bundle, see [92, 94]. The Legendre transformation (fiber derivative) is a map from the Lie algebroid to the dual bundle as

$$\mathbb{F}L : \mathcal{A} \longrightarrow \mathcal{A}^*, \quad (x^i, y^{\alpha}) \mapsto \left(x^i, y_{\alpha} = \frac{\partial L}{\partial y^{\alpha}} \right) \quad (3.53)$$

Assuming that the Lagrangian is regular (non-degenerate) we argue that the Legendre transformation is a (at least local) diffeomorphism. Referring to the Legendre transformation, we pull the Poisson structure on the dual bundle $\mathcal{A}^*$ to the Lie algebroid. This reads a Poisson structure, denoted by $\{\bullet, \bullet\}_{\mathcal{A}}$, on the Lie algebroid as well. Define the energy function determined by L as

$$E_L : \mathcal{A} \longrightarrow \mathbb{R}, \quad E_L(x^i, y^{\alpha}) = \frac{\partial L}{\partial y^{\alpha}} y^{\alpha} - L(x^i, y^{\alpha}). \quad (3.54)$$

On the space $\mathcal{A}$, the dynamics is computed to be

$$\frac{dx^i}{dt} = \{x^i, E_L\}_{\mathcal{A}}, \quad \frac{dy^{\alpha}}{dt} = \{y^{\alpha}, E_L\}_{\mathcal{A}}. \quad (3.55)$$

Explicitly computing we arrive at the following Euler-Lagrange equations on the Lie algebroid as [93]:

$$\frac{d}{dt} \frac{\partial L}{\partial y^{\alpha}} = (\mathfrak{a}_{\mathcal{A}})_{\alpha}^i \frac{\partial L}{\partial x^i} + \mathfrak{H}_{\alpha\beta}^{\gamma} y^{\beta} \frac{\partial L}{\partial y^{\gamma}}, \quad \frac{dx^i}{dt} = (\mathfrak{a}_{\mathcal{A}})_{\alpha}^i y^{\alpha}. \quad (3.56)$$

3.1.4. Real Line Extensions

In this subsection, we mainly follow [95–97]. Suppose that $(\mathcal{A}, \tau, M, \mathfrak{a}_{\mathcal{A}}, [\bullet, \bullet]_{\mathcal{A}})$ is a Lie algebroid and $(\mathcal{A}^*, \pi, M)$ be its dual vector bundle equipped with the associated Poisson structure $\Lambda_{\mathcal{A}^*}$ defined globally in (3.47) and given locally in (3.49). As we have arrived at the dissipative dynamics starting from the canonical Poisson structure by the real line $\mathbb{R}$ extension, we extend the Poisson manifold $\mathcal{A}^*$ by $\mathbb{R}$ to define contactization of the Poisson structure $\Lambda_{\mathcal{A}^*}$. This will lead us a Jacobi manifold structure on $\mathcal{A}^* \times \mathbb{R}$. Accordingly taking local coordinates (x^i, y_{α}, z) on the extended bundle $\mathcal{A}^* \times \mathbb{R}$, and referring to (2.88), we define the following bivector field

$$\Lambda_{\mathcal{A}^* \times \mathbb{R}} = \Lambda_{\mathcal{A}^*} + \mathcal{R} \wedge \Delta^* = \frac{1}{2} \mathfrak{H}_{\alpha\beta}^{\gamma} y_{\gamma} \frac{\partial}{\partial y_{\alpha}} \wedge \frac{\partial}{\partial y_{\beta}} + \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y_{\alpha}} + y_{\alpha} \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial y_{\alpha}}, \quad (3.57)$$

where the Reeb field and the Liouville vector field are given as

$$\mathcal{R} = \frac{\partial}{\partial z}, \quad \Delta^* = y_{\alpha} \frac{\partial}{\partial y_{\alpha}}, \quad (3.58)$$

respectively. It is possible to see that the pair $(\Lambda_{\mathcal{A}^* \times \mathbb{R}}, \mathcal{R})$ determines a Jacobi structure. Hence we can define a Jacobi bracket on $\mathcal{A}^* \times \mathbb{R}$ as follows:

$$\{F, \mathcal{H}\}_{\mathcal{A}^* \times \mathbb{R}} = \Lambda_{\mathcal{A}^* \times \mathbb{R}}(dF, d\mathcal{H}) + F\mathcal{R}(\mathcal{H}) - \mathcal{H}\mathcal{R}(F). \quad (3.59)$$

Assuming the local coordinates (x^i, y_{α}, z) on the extended space, we compute the local realization of this bracket as

$$\begin{aligned} \{x^i, x^j\}_{\mathcal{A}^* \times \mathbb{R}} &= 0, \\ \{z, x^i\}_{\mathcal{A}^* \times \mathbb{R}} &= 0, \\ \{z, y_{\alpha}\}_{\mathcal{A}^* \times \mathbb{R}} &= y_{\alpha}, \\ \{x^i, y_{\alpha}\}_{\mathcal{A}^* \times \mathbb{R}} &= (\mathfrak{a}_{\mathcal{A}})^i_{\alpha}, \\ \{y_{\alpha}, y_{\beta}\}_{\mathcal{A}^* \times \mathbb{R}} &= \mathfrak{H}_{\alpha\beta}^{\gamma} y_{\gamma}, \end{aligned} \quad (3.60)$$

where $(\mathfrak{a}_{\mathcal{A}})^i_{\alpha}$ is the anchor map whereas $\mathfrak{H}_{\alpha\beta}^{\gamma}$ are the structure functions of the Lie algebroid.

For a given Hamiltonian function $\mathcal{H} = \mathcal{H}(x^i, y_{\alpha}, z)$ defined on the extended dual space $\mathcal{A}^* \times \mathbb{R}$, the Hamiltonian vector field is defined to be

$$\xi_{\mathcal{H}}(F) = \{F, \mathcal{H}\}_{\mathcal{A}^* \times \mathbb{R}} - F\mathcal{R}(\mathcal{H}). \quad (3.61)$$

So, in the local coordinates (x^i, y_α, z) on $\mathcal{A}^* \times \mathbb{R}$, the Hamiltonian vector field is computed to be

$$\begin{aligned} \xi_{\mathcal{H}} = & (\mathfrak{a}_{\mathcal{A}})_\alpha^i \frac{\partial \mathcal{H}}{\partial y_\alpha} \frac{\partial}{\partial x^i} - \left((\mathfrak{a}_{\mathcal{A}})_\alpha^i \frac{\partial \mathcal{H}}{\partial x^i} + \mathfrak{H}_{\alpha\beta}^\gamma y_\gamma \frac{\partial \mathcal{H}}{\partial y_\beta} + y_\alpha \frac{\partial \mathcal{H}}{\partial z} \right) \frac{\partial}{\partial y_\alpha} \\ & + \left(y_\alpha \frac{\partial \mathcal{H}}{\partial y_\alpha} - \mathcal{H} \right) \frac{\partial}{\partial z}. \end{aligned} \quad (3.62)$$

In accordance with, we have the dissipative Hamiltonian dynamics on the extended dual space $\mathcal{A}^* \times \mathbb{R}$ of the Lie algebroid $\mathcal{A}$ as

$$\begin{aligned} \frac{dx^i}{dt} &= (\mathfrak{a}_{\mathcal{A}})_\alpha^i \frac{\partial \mathcal{H}}{\partial y_\alpha}, \\ \frac{dy_\alpha}{dt} &= -(\mathfrak{a}_{\mathcal{A}})_\alpha^i \frac{\partial \mathcal{H}}{\partial x^i} - \mathfrak{H}_{\alpha\beta}^\gamma y_\gamma \frac{\partial \mathcal{H}}{\partial y_\beta} - y_\alpha \frac{\partial \mathcal{H}}{\partial z}, \\ \frac{dz}{dt} &= y_\alpha \frac{\partial \mathcal{H}}{\partial y_\alpha} - \mathcal{H}. \end{aligned} \quad (3.63)$$

A particular instance is to choose the Lie algebroid $\mathcal{A} = TM$ and the anchor map as the identity mapping. This gives the dual space as $\mathcal{A}^* = T^*M$ as the cotangent bundle. In this case, the Hamiltonian vector field reduces to the contact Hamiltonian vector field (2.90) and the dynamical equations turn out to be the contact Hamilton's equations (2.91).

Let us obtain the Lagrangian picture of the real line extension permitting dissipative terms. Even though, in general, it is possible to have this for any (even singular) Lagrangian functions, we show this construction for regular Lagrangian functions. Consider a Lagrangian function $L = L(x^i, y^\alpha, z)$ defined on the extended Lie algebroid $\mathcal{A} \times \mathbb{R}$ with coordinates (x^i, y^α, z) . We then define the fiber derivative (or the Legendre transformation)

$$\mathbb{F}L : \mathcal{A} \times \mathbb{R} \longrightarrow \mathcal{A}^* \times \mathbb{R}, \quad (x^i, y^\alpha, z) \mapsto (x^i, \frac{\partial L}{\partial y^\alpha}, z). \quad (3.64)$$

The regularity of the Lagrangian function reads that the fiber derivative is at least a local diffeomorphism. This provides us to pull the Jacobi structure $(\Lambda_{\mathcal{A}^* \times \mathbb{R}}, \mathcal{R})$ back to the extended Lie algebroid. We denote the Jacobi structure on $\mathcal{A} \times \mathbb{R}$ by the pair $(\Lambda_{\mathcal{A} \times \mathbb{R}}, \mathcal{R})$. This pair determines a Jacobi bracket $\{\bullet, \bullet\}_{\mathcal{A} \times \mathbb{R}}$ on the extended space. Defining the energy function

$$E_L(x^i, y^\alpha, z) = y^\alpha \frac{\partial L}{\partial y^\alpha} - L(x^i, y^\alpha, z) \quad (3.65)$$

as the Hamiltonian function, we can determine the equations of motion as in the form of Herglotz equations

$$\begin{aligned}\frac{d}{dt} \frac{\partial L}{\partial y^\alpha} - (\mathfrak{a}_{\mathcal{A}})^i_\alpha \frac{\partial L}{\partial x^i} &= \mathfrak{H}_{\alpha\beta}^\gamma y^\beta \frac{\partial L}{\partial y^\gamma} + \frac{\partial L}{\partial z} \frac{\partial L}{\partial y^\alpha}, \\ \frac{dx^i}{dt} &= (\mathfrak{a}_{\mathcal{A}})^i_\alpha y^\alpha, \\ \frac{dz}{dt} &= L(x^i, y^\alpha, z).\end{aligned}\tag{3.66}$$

Let us examine a particular case $\mathcal{A} = TM$, where the Lie algebroid is the tangent bundle. Note that in the case, equations (3.66) are equivalent to the Herglotz equations of the classical form in (2.95).

Another case is obtained by taking $\mathcal{A}$ as a Lie algebra. In this case, we arrive at the Jacobi bracket on the extended dual space $\mathfrak{g}^* \times \mathbb{R}$ equipped with the coordinates (y_α, z) as

$$\begin{aligned}\{z, y_\alpha\}_{\mathfrak{g}^* \times \mathbb{R}} &= y_\alpha, \\ \{y_\alpha, y_\beta\}_{\mathfrak{g}^* \times \mathbb{R}} &= \mathfrak{H}_{\alpha\beta}^\gamma y_\gamma.\end{aligned}\tag{3.67}$$

Then for a Hamiltonian function $h = h(y_\alpha, z)$ on $\mathfrak{g}^* \times \mathbb{R}$ we have the Lie-Poisson-Herglotz equations:

$$\begin{aligned}\frac{dy_\alpha}{dt} &= -\mathfrak{H}_{\alpha\beta}^\gamma y_\gamma \frac{\partial h}{\partial y_\beta} - y_\alpha \frac{\partial h}{\partial z}, \\ \frac{dz}{dt} &= y_\alpha \frac{\partial h}{\partial y_\alpha} - h.\end{aligned}\tag{3.68}$$

In this case, the equations (3.66) reduces to the Euler-Poincaré-Herglotz equations

$$\begin{aligned}\frac{d}{dt} \frac{\partial L}{\partial y^\alpha} + \mathfrak{H}_{\alpha\beta}^\gamma y^\beta \frac{\partial L}{\partial y^\gamma} &= \frac{\partial L}{\partial z} \frac{\partial L}{\partial y^\alpha}, \\ \frac{dz}{dt} &= L(x^i, y^\alpha, z).\end{aligned}\tag{3.69}$$

3.2. Couplings of Algebra(oid)s

3.2.1. Unified Product

Let $(\mathcal{A}, \tau, M, \mathfrak{a}_{\mathcal{A}}, [\bullet, \bullet]_{\mathcal{A}})$ be a Lie algebroid and $(\mathcal{B}, \kappa, M)$ be an arbitrary vector bundle together with a vector bundle map $\mathfrak{a}_{\mathcal{B}} : \mathcal{B} \rightarrow TM$. Our goal is to couple these two bundles respecting their algebraic and geometric properties. To this end, we first define the Whitney (direct) sum vector bundle $\mathcal{A} \times \mathcal{B}$ over the base

manifold M . For now, this is only a vector bundle. We shall now determine a Lie algebroid structure on this space. Start by assuming the existence of the following skew-symmetric bilinear maps

$$\psi : \Gamma(\mathcal{B}) \wedge \Gamma(\mathcal{B}) \longrightarrow \Gamma(\mathcal{A}), \quad \theta : \Gamma(\mathcal{B}) \wedge \Gamma(\mathcal{B}) \longrightarrow \Gamma(\mathcal{B}) \quad (3.70)$$

on the sections of the vector bundle. In the sequel, the role of these mappings will be more clear. For now, referring to these maps, we write a skew-symmetric *bracket* of two elements Y_1, Y_2 in $\Gamma(\mathcal{B})$ taking values in the direct sum of the section spaces $\Gamma(\mathcal{A})$ and $\Gamma(\mathcal{B})$ as follows

$$\begin{aligned} [\bullet, \bullet]_{\mathcal{B}} : \Gamma(\mathcal{B}) \wedge \Gamma(\mathcal{B}) &\longrightarrow \Gamma(\mathcal{A}) \oplus \Gamma(\mathcal{B}), \\ [Y_1, Y_2]_{\mathcal{B}} &= \psi(Y_1, Y_2) + \theta(Y_1, Y_2). \end{aligned} \quad (3.71)$$

Here, $\Gamma(\mathcal{A}) \oplus \Gamma(\mathcal{B})$ denotes the direct sum of the section spaces. See that, this fails to be closed on $\Gamma(\mathcal{B})$ due to the presence of ψ called *cocycle* term. We continue our construction by defining two mappings

$$\rho : \Gamma(\mathcal{B}) \times \Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{A}), \quad \sigma : \Gamma(\mathcal{A}) \times \Gamma(\mathcal{B}) \longrightarrow \Gamma(\mathcal{B}). \quad (3.72)$$

Referring to all these operations, on the sections of the bundle $\mathcal{A} \times \mathcal{B}$, we define the following bracket

$$\begin{aligned} \Gamma(\mathcal{A} \times \mathcal{B}) \wedge \Gamma(\mathcal{A} \times \mathcal{B}) &\longrightarrow \Gamma(\mathcal{A} \times \mathcal{B}), \\ [(X_1, Y_1), (X_2, Y_2)]_{\bowtie_{\psi}} &= \left([X_1, X_2]_{\mathcal{A}} - \rho(Y_2, X_1) + \rho(Y_1, X_2) + \psi(Y_1, Y_2), \right. \\ &\quad \left. \theta(Y_1, Y_2) - \sigma(X_2, Y_1) + \sigma(X_1, Y_2) \right). \end{aligned} \quad (3.73)$$

This bracket is skew-symmetric but it does not necessarily satisfy the Jacobi identity. We shall determine the conditions on the mappings ψ and θ in (3.70), ρ and σ in (3.72) to ensure the Jacobi identity is satisfied. Before listing these compatibility conditions let us exhibit a bundle map

$$\mathfrak{a}_{\bowtie_{\psi}} := \mathcal{A} \times \mathcal{B} \longrightarrow TM, \quad (X, Y) \mapsto \mathfrak{a}_{\mathcal{A}}(X) + \mathfrak{a}_{\mathcal{B}}(Y) \quad (3.74)$$

which will play the role of an anchor map.

After a direct computation, we obtain a list of conditions on ψ , θ , ρ and σ for the vector bundle $\mathcal{A} \times \mathcal{B}$ to be a Lie algebroid with the bracket $[\bullet, \bullet]_{\bowtie_{\psi}}$ given in (3.73)

and the anchor map $\mathbf{a}_{\bowtie_\psi}$ in (3.74). We present these conditions as follows. First of all, we ask that σ must be a Lie algebroid representation on the vector bundle $\mathcal{B}$. then we obtain the following cyclic identities relating the terms ψ , θ , ρ and σ as

$$\begin{aligned} \circlearrowleft \psi(\theta(Y_1, Y_2), Y_3) - \circlearrowleft \rho(Y_3, \psi(Y_1, Y_2)) &= 0, \\ \circlearrowleft \theta(\theta(Y_1, Y_2), Y_3) + \circlearrowleft \sigma(\psi(Y_1, Y_2), Y_3) &= 0, \end{aligned} \quad (3.75)$$

for X_i in $\Gamma(\mathcal{A})$, and Y_i in $\Gamma(\mathcal{B})$ with $i = 1, 2, 3$. Here, $\circlearrowleft$ reads the cyclic sum. These conditions are obtained by employing the Jacobi identity directly to the product bracket. Further, we have the following conditions:

$$\begin{aligned} &[\rho(Y_3, X_2), X_1]_{\mathcal{A}} - [\rho(Y_3, X_1), X_2]_{\mathcal{A}} \\ &= \rho(\sigma(X_2, Y_3), X_1) - \rho(Y_3, [X_1, X_2]_{\mathcal{A}}) - \rho(\sigma(X_1, Y_3), X_2), \\ &\rho(Y_3, \rho(Y_2, X_1)) + \rho(\theta(Y_2, Y_3), X_1) - \rho(Y_2, \rho(Y_3, X_1)) \\ &= -\psi(\sigma(X_1, Y_2), Y_3) - [\psi(Y_2, Y_3), X_1]_{\mathcal{A}} - \psi(Y_2, \sigma(X_1, Y_3)), \\ &- \sigma(\rho(Y_2, X_1), Y_3) - \sigma(X_1, \theta(Y_2, Y_3)) + \sigma(\rho(Y_3, X_1), Y_2) \\ &= -\theta(\sigma(X_1, Y_2), Y_3) + \theta(\sigma(X_1, Y_3), Y_2). \end{aligned} \quad (3.76)$$

Then we compute the Leibniz rule for this picture which results in

$$\begin{aligned} \psi(Y_1, fY_2) &= f\psi(Y_1, Y_2), \\ \theta(Y_1, fY_2) &= f\theta(Y_1, Y_2) + \mathcal{L}_{\mathbf{a}_{\mathcal{B}}(Y_1)}(f)Y_2, \\ \rho(fY, X) &= f\rho(Y, X), \\ \rho(Y, fX) &= f\rho(Y, X) + \mathcal{L}_{\mathbf{a}_{\mathcal{B}}(Y)}(f)X. \end{aligned} \quad (3.77)$$

We conclude the compatibility condition list for the unified Lie algebroid setting by those related with anchor maps:

$$\begin{aligned} [\mathbf{a}_{\mathcal{B}}(Y), \mathbf{a}_{\mathcal{A}}(X)] &= \mathbf{a}_{\mathcal{A}}(\rho(Y, X)) - \mathbf{a}_{\mathcal{B}}(\sigma(X, Y)), \\ [\mathbf{a}_{\mathcal{B}}(Y_1), \mathbf{a}_{\mathcal{B}}(Y_2)] &= \mathbf{a}_{\mathcal{A}}(\psi(Y_1, Y_2)) + \mathbf{a}_{\mathcal{B}}(\theta(Y_1, Y_2)). \end{aligned} \quad (3.78)$$

If all the conditions in (3.75)-(3.78) are satisfied then we write $\mathcal{A} \bowtie_\psi \mathcal{B}$ instead of a direct sum notation to exhibit the unified product Lie algebroid character of the algebraic structure. We record all of these observations in the following theorem.

Theorem 3.1. *A Lie algebroid $(\mathcal{A}, \tau, M, \mathbf{a}_{\mathcal{A}}, [\bullet, \bullet]_{\mathcal{A}})$ and a vector bundle $(\mathcal{B}, \kappa, M)$ admitting mappings (3.70) and (3.72) determines a (unified product) Lie algebroid*

structure

$$(\mathcal{A} \bowtie_\psi \mathcal{B}, \tau_{\bowtie_\psi}, M, \mathfrak{a}_{\bowtie_\psi}, [\bullet, \bullet]_{\bowtie_\psi}) \quad (3.79)$$

with the Lie bracket (3.73) and the anchor map (3.74) if and only if the conditions listed in (3.75)-(3.78) are satisfied.

This theorem establishes coupling a Lie algebroid with a vector bundle due to the presence of mutual *representations* and a *cocycle* term. In the following subsection, where the bicycle double cross product theory is presented, we add another *cocycle* term to facilitate the coupling of two vector bundles by relaxing the Lie algebroid requirement of one of the constitutive spaces.

In finite dimensions, let us establish a local coordinate system (x^i) on M , and denote the module basis of the transitive $\mathcal{C}^\infty(M)$ -module of sections of $\mathcal{A}$ by (e_α) . This induces local coordinates (x^i, y^α) on $\mathcal{A}$. Similarly, let us represent the module basis of the transitive $\mathcal{C}^\infty(M)$ -module of sections of $\mathcal{B}$ as (e_a) . So that we consider the local coordinates (x^i, y^a) on $\mathcal{B}$. Accordingly, by the abuse of notation, we refer the module basis of the direct sum bundle $\mathcal{A} \bowtie_\psi \mathcal{B}$ by the set (e_α, e_a) , and the local coordinates by (x^i, y^α, y^a) . For the anchor map, we have

$$\mathfrak{a}_{\bowtie_\psi}(e_\alpha) = (\mathfrak{a}_\mathcal{A})_\alpha^i \frac{\partial}{\partial x^i}, \quad \mathfrak{a}_{\bowtie_\psi}(e_a) = (\mathfrak{a}_\mathcal{B})_a^i \frac{\partial}{\partial x^i}. \quad (3.80)$$

For the mutual *representations* ρ and σ exhibited in (3.72), we assume

$$\rho(e_a, e_\alpha) = \mathfrak{R}_{a\alpha}^\beta e_\beta, \quad \sigma(e_\alpha, e_a) = -\mathfrak{S}_{a\alpha}^b e_b, \quad (3.81)$$

respectively. For the mappings ψ and θ in (3.70), we record the local coordinates as

$$\psi(e_a, e_b) = \mathfrak{P}_{ab}^\alpha e_\alpha, \quad \theta(e_a, e_b) = \mathfrak{T}_{ab}^d e_d, \quad (3.82)$$

respectively. In light of these characterizations, we compute the unified product Lie brackets as

$$\begin{aligned} [e_\beta, e_\gamma]_{\bowtie_\psi} &= \mathfrak{H}_{\beta\gamma}^\alpha e_\alpha, \\ [e_b, e_d]_{\bowtie_\psi} &= \mathfrak{P}_{bd}^\alpha e_\alpha + \mathfrak{T}_{bd}^a e_a, \\ [e_a, e_\alpha]_{\bowtie_\psi} &= \mathfrak{R}_{a\alpha}^\beta e_\beta + \mathfrak{S}_{a\alpha}^b e_b, \end{aligned} \quad (3.83)$$

where $\mathfrak{H}_{\beta\gamma}^\alpha$ are structure functions of the Lie algebroid $\mathcal{A}$ presented in (3.43).

We present the following theorem, referred to as the universal theorem for unified product Lie algebroids which elucidates the decomposition of a Lie algebroid as a direct sum of one of its Lie subalgebroids and the complementary bundle.

Theorem 3.2. *Let $(\mathcal{M}, v, M, \mathfrak{a}, [\bullet, \bullet]_{\mathcal{M}})$ be a Lie algebroid and*

$$\left(\mathcal{A}, \tau = v|_{\mathcal{A}}, M, \mathfrak{a}_{\mathcal{A}} = \mathfrak{a}|_{\mathcal{A}}, [\bullet, \bullet]_{\mathcal{A}} = [\bullet, \bullet]_{\mathcal{M}}|_{\mathcal{A}} \right) \quad (3.84)$$

be a Lie subalgebroid so that $\mathcal{M} \cong \mathcal{A} \times \mathcal{B}$ as vector bundles, for some complementary vector bundle

$$(\mathcal{B}, \kappa = v|_{\mathcal{B}}, M). \quad (3.85)$$

Then $\mathcal{M}$ is a unified product Lie algebroid that is, $\mathcal{M} = \mathcal{A} \bowtie_{\psi} \mathcal{B}$. In this case, the cocycle term and the mutual representations are computed through

$$[Y_1, Y_2]_{\mathcal{M}} = \psi(Y_1, Y_2) + \theta(Y_1, Y_2), \quad [Y, X]_{\mathcal{M}} = \rho(Y, X) - \sigma(X, Y). \quad (3.86)$$

These mappings are satisfying (3.75)-(3.78).

As a particular instance of the unified product Lie algebroid structure we consider that the base manifold M is a single point. Referring to the previous constructions, this particular choice reads a Lie algebra $\mathfrak{g}$ and a vector space $\mathfrak{h}$. In this case, all the anchor maps turn out to be identically zero. We assume the existence of ψ and θ mappings presented in (3.70) and *representations* ρ and σ in (3.72). The Lie algebra bracket $[\bullet, \bullet]_{\bowtie_{\psi}}$ on the vector space $\mathfrak{g} \oplus \mathfrak{h}$ is structurally the same with (3.73). We write the following theorem for this case.

Theorem 3.3. *A Lie algebra $(\mathfrak{g}, [\bullet, \bullet]_{\mathfrak{g}})$ and a vector space $\mathfrak{h}$ admitting mappings (3.70) and (3.72) (with $\mathfrak{g} = \Gamma(\mathcal{A})$ and $\mathfrak{h} = \Gamma(\mathcal{B})$) determines a unified product Lie algebra structure*

$$(\mathfrak{g} \bowtie_{\psi} \mathfrak{h}, [\bullet, \bullet]_{\bowtie_{\psi}}) \quad (3.87)$$

with the Lie bracket (3.73) if and only if the conditions listed in (3.75) and (3.76) are satisfied. Further, let $(\mathfrak{k}, [\bullet, \bullet]_{\mathfrak{k}})$ be a Lie algebra and

$$\left(\mathfrak{g}, [\bullet, \bullet]_{\mathfrak{g}} = [\bullet, \bullet]_{\mathfrak{k}}|_{\mathfrak{g}} \right) \quad (3.88)$$

be a Lie subalgebra so that $\mathfrak{k} \cong \mathfrak{g} \oplus \mathfrak{h}$ as vector space, for some complementary vector space $\mathfrak{h}$. Then $\mathfrak{k}$ is a unified product Lie algebra $\mathfrak{k} = \mathfrak{g} \bowtie_{\psi} \mathfrak{h}$. In this case, the cocycle term and the mutual actions are computed through (3.86). These mappings are satisfying (3.75) and (3.76).

3.2.2. Bicocycle Double Cross Product

In the previous subsection, we presented the phenomenon of (de)coupling Lie algebroid writing a Lie algebroid as the direct sum of a Lie algebroid and a vector bundle defined on the same base manifold. In this subsection, we now present the most general way of (de)coupling Lie algebroid as the direct sum of two vector bundles admitting all possible interactions and under the existence of two *cocycle* terms.

We start with two vector bundles $(\mathcal{A}, \tau, M)$ and $(\mathcal{B}, \kappa, M)$ admitting bundle maps denoted to be $\mathbf{a}_{\mathcal{A}} : \mathcal{A} \mapsto TM$ and $\mathbf{a}_{\mathcal{B}} : \mathcal{B} \mapsto TM$, respectively. We assume ψ and θ are the mappings exhibited in (3.70), determining a bilinear skew-symmetric operation in (3.71). We further consider two mutual maps ρ and σ given in (3.72). The difference in the present setting is the relaxation of the bracket structure on the section of the vector bundle $\mathcal{A}$. We achieve this by accommodating the bracket with two bilinear skew-symmetric operations

$$\phi : \Gamma(\mathcal{A}) \wedge \Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{A}), \quad \zeta : \Gamma(\mathcal{A}) \wedge \Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{B}) \quad (3.89)$$

on the sections of the vector bundle $\mathcal{A}$. These mappings define a bilinear skew-symmetric operation on two elements X_1, X_2 in $\Gamma(\mathcal{A})$ taking values in the direct sum of the section spaces $\Gamma(\mathcal{A})$ and $\Gamma(\mathcal{B})$ as

$$\begin{aligned} [\bullet, \bullet]_{\mathcal{A}} : \Gamma(\mathcal{A}) \wedge \Gamma(\mathcal{A}) &\longrightarrow \Gamma(\mathcal{A}) \oplus \Gamma(\mathcal{B}), \\ [X_1, X_2]_{\mathcal{A}} &= \phi(X_1, X_2) + \zeta(X_1, X_2). \end{aligned} \quad (3.90)$$

Notice that this operation turns out to be a well-defined bilinear operation on $\Gamma(\mathcal{A})$ if and only if ζ is identically zero. As we shall depict in the sequel, after the integrability conditions are fixed, the vanishing of ζ reads the unified product Lie algebroid picture.

On the sections of the direct sum vector bundle $\mathcal{A} \times \mathcal{B}$, we determine a bilinear skew-symmetric operation denoted by $[\bullet, \bullet]_{\zeta \bowtie \psi}$. Here, the subscripts ζ and ψ are manifesting the *cocycle* terms. Due to the existence of a pair of *cocycle* operators, we call this structure as bicocycle double cross product Lie algebroid structure.

Using the mappings (3.70), (3.72) and (3.89), we write the following bracket

$$\begin{aligned} \Gamma(\mathcal{A} \times \mathcal{B}) \wedge \Gamma(\mathcal{A} \times \mathcal{B}) &\longrightarrow \Gamma(\mathcal{A} \times \mathcal{B}), \\ [(X_1, Y_1), (X_2, Y_2)]_{\zeta \bowtie \psi} &= \left(\phi(X_1, X_2) - \rho(Y_2, X_1) + \rho(Y_1, X_2) \right. \\ &\quad \left. + \psi(Y_1, Y_2), \theta(Y_1, Y_2) - \sigma(X_2, Y_1) + \sigma(X_1, Y_2) + \zeta(X_1, X_2) \right). \end{aligned} \quad (3.91)$$

Evidently, one needs to ask for some compatibility conditions to be sure that this bilinear skew-symmetric bracket satisfies the Jacobi identity. Let us first define the bundle map

$$\mathfrak{a}_{\zeta \bowtie \psi} := \mathcal{A} \times \mathcal{B} \longrightarrow TM, \quad (X, Y) \mapsto \mathfrak{a}_{\mathcal{A}}(X) + \mathfrak{a}_{\mathcal{B}}(Y). \quad (3.92)$$

We start by listing the cyclic sums relating the ingredients of the bracket as

$$\begin{aligned} \circlearrowleft \phi(\phi(X_1, X_2), X_3) + \circlearrowleft \rho(\zeta(X_1, X_2), X_3) &= 0, \\ \circlearrowleft \zeta(\phi(X_1, X_2), X_3) - \circlearrowleft \sigma(X_3, \zeta(X_1, X_2)) &= 0, \\ \circlearrowleft \psi(\theta(Y_1, Y_2), Y_3) - \circlearrowleft \rho(Y_3, \psi(Y_1, Y_2)) &= 0, \\ \circlearrowleft \theta(\theta(Y_1, Y_2), Y_3) + \circlearrowleft \sigma(\psi(Y_1, Y_2), Y_3) &= 0. \end{aligned} \quad (3.93)$$

Then we list the following identities:

$$\begin{aligned} \rho(Y_3, \phi(X_1, X_2)) - \psi(\zeta(X_1, X_2), Y_3) + \phi(\rho(Y_3, X_2), X_1) \\ = \rho(\sigma(X_2, Y_3), X_1) + \phi(\rho(Y_3, X_1), X_2) - \rho(\sigma(X_1, Y_3), X_2), \\ \sigma(\phi(X_1, X_2), Y_3) + \theta(\zeta(X_1, X_2), Y_3) - \zeta(\rho(Y_3, X_2), X_1) \\ = \sigma(X_1, \sigma(X_2, Y_3)) - \zeta(\rho(Y_3, X_1), X_2) - \sigma(X_2, \sigma(X_1, Y_3)), \\ \rho(Y_3, \rho(Y_2, X_1)) + \psi(\sigma(X_1, Y_2), Y_3) + \phi(\psi(Y_2, Y_3), X_1) \\ = -\rho(\theta(Y_2, Y_3), X_1) + \rho(Y_2, \rho(Y_3, X_1)) + \psi(\sigma(X_1, Y_3), Y_2), \\ \sigma(\rho(Y_2, X_1), Y_3) - \theta(\sigma(X_1, Y_2), Y_3) - \zeta(\psi(Y_2, Y_3), X_1) \\ = -\sigma(X_1, \theta(Y_2, Y_3)) + \sigma(\rho(Y_3, X_1), Y_2) - \theta(\sigma(X_1, Y_3), Y_2). \end{aligned} \quad (3.94)$$

We then compute the conditions justifying the Lie homomorphism properties of the bicycle double cross product anchor map $\mathfrak{a}_{\zeta \bowtie \psi}$ in terms of the bundle maps $\mathfrak{a}_{\mathcal{A}}$ and $\mathfrak{a}_{\mathcal{B}}$. This gives the following identities:

$$\begin{aligned} [\mathfrak{a}_{\mathcal{A}}(X_1), \mathfrak{a}_{\mathcal{A}}(X_2)] &= \mathfrak{a}_{\mathcal{A}}(\phi(X_1, X_2)) + \mathfrak{a}_{\mathcal{B}}(\zeta(X_1, X_2)), \\ [\mathfrak{a}_{\mathcal{B}}(Y), \mathfrak{a}_{\mathcal{A}}(X)] &= \mathfrak{a}_{\mathcal{A}}(\rho(Y, X)) - \mathfrak{a}_{\mathcal{B}}(\sigma(X, Y)), \\ [\mathfrak{a}_{\mathcal{B}}(Y_1), \mathfrak{a}_{\mathcal{B}}(Y_2)] &= \mathfrak{a}_{\mathcal{A}}(\psi(Y_1, Y_2)) + \mathfrak{a}_{\mathcal{B}}(\theta(Y_1, Y_2)). \end{aligned} \quad (3.95)$$

Later, we calculate the product rule identities for the bicocycle double cross product bracket, and arrive at the list:

$$\begin{aligned}
\phi(X_1, fX_2) &= f\phi(X_1, X_2) + \mathcal{L}_{\mathfrak{a}_{\mathcal{A}}(X_1)}(f)X_2, \\
\zeta(X_1, fX_2) &= f\zeta(X_1, X_2), \\
\rho(fY, X) &= f\rho(Y, X), \\
\sigma(X, fY) &= f\sigma(X, Y) + \mathcal{L}_{\mathfrak{a}_{\mathcal{A}}(X)}(f)Y, \\
\rho(Y, fX) &= f\rho(Y, X) + \mathcal{L}_{\mathfrak{a}_{\mathcal{B}}(Y)}(f)X, \\
\sigma(fX, Y) &= f\sigma(X, Y), \\
\psi(Y_1, fY_2) &= f\psi(Y_1, Y_2), \\
\theta(Y_1, fY_2) &= f\theta(Y_1, Y_2) + \mathcal{L}_{\mathfrak{a}_{\mathcal{B}}(Y_1)}(f)Y_2.
\end{aligned} \tag{3.96}$$

After computing all these conditions, we are now ready to state the following theorem establishing the bicocycle double cocycle product Lie algebroid structure.

Theorem 3.4. *Two vector bundles $(\mathcal{A}, \tau, M)$ and $(\mathcal{B}, \kappa, M)$ admitting mappings (3.70), (3.72) and (3.89) determine a (bicocycle double cocycle product) Lie algebroid structure*

$$(\mathcal{A}_{\zeta} \bowtie_{\psi} \mathcal{B}, \tau_{\zeta \bowtie_{\psi}}, M, \mathfrak{a}_{\zeta \bowtie_{\psi}}, [\bullet, \bullet]_{\zeta \bowtie_{\psi}}) \tag{3.97}$$

with the Lie bracket (3.91) and the anchor map (3.92) if and only if the conditions listed in (3.93)-(3.96) are satisfied.

This theorem generalizes the unified product Lie algebroid structure to the sum of two vector bundles. More concretely, as a particular instance, if ζ in (3.89) is identically zero, then one arrives at Theorem 3.1 where unified product Lie algebroid structure is exhibited.

In finite dimensions, we consider local coordinate system (x^i) on M , basis of $\Gamma(\mathcal{A})$ and $\Gamma(\mathcal{B})$ as (e_{α}) and (e_a) respectively. The anchor map is computed to be

$$\mathfrak{a}_{\zeta \bowtie_{\psi}}(e_{\alpha}) = (\mathfrak{a}_{\mathcal{A}})_{\alpha}^i \frac{\partial}{\partial x^i}, \quad \mathfrak{a}_{\zeta \bowtie_{\psi}}(e_a) = (\mathfrak{a}_{\mathcal{B}})_a^i \frac{\partial}{\partial x^i}. \tag{3.98}$$

We refer (3.81) for the local realizations of the mutual *representations* ρ and σ in (3.72), recall (3.82) for the mappings ψ and θ . For the mappings ϕ and ζ we assume

$$\phi(e_{\alpha}, e_{\beta}) = \mathfrak{H}_{\alpha\beta}^{\gamma} e_{\gamma}, \quad \zeta(e_{\alpha}, e_{\beta}) = \mathfrak{Z}_{\alpha\beta}^a e_a, \tag{3.99}$$

respectively. Here $\mathfrak{H}_{\beta\gamma}^\alpha$ (while we are denoting with the same letter by the abuse of notation) are now failed to be structure functions due to the existence of the *cocycle* terms $\mathfrak{Z}_{\alpha\beta}^a$. Therefore, we compute the following values for the Lie brackets:

$$\begin{aligned} [e_\beta, e_\gamma]_{\zeta \bowtie_\psi} &= \mathfrak{H}_{\beta\gamma}^\alpha e_\alpha + \mathfrak{Z}_{\beta\gamma}^a e_a, \\ [e_b, e_d]_{\zeta \bowtie_\psi} &= \mathfrak{P}_{bd}^\alpha e_\alpha + \mathfrak{T}_{bd}^a e_a, \\ [e_a, e_\alpha]_{\zeta \bowtie_\psi} &= \mathfrak{R}_{a\alpha}^\beta e_\beta + \mathfrak{S}_{a\alpha}^b e_b. \end{aligned} \tag{3.100}$$

In the following theorem, which we call the universal theorem for bicocycle double cross product Lie algebroids, the compatibility conditions list is directly satisfied if a Lie algebroid is decomposed into the sum of two complementary subbundles. This theorem is generalizing Theorem 3.2 where we ask that in a decomposition of a Lie algebroid, one of the subbundles attains a Lie subalgebroid structure.

Theorem 3.5. *Let $(\mathcal{M}, v, M, [\bullet, \bullet]_{\mathcal{M}}, \mathfrak{a})$ be a Lie algebroid and*

$$(\mathcal{A}, \tau = v|_{\mathcal{A}}, M), \quad (\mathcal{B}, \kappa = v|_{\mathcal{B}}, M) \tag{3.101}$$

be two complementary vector bundles so that $\mathcal{M} \cong \mathcal{A} \times \mathcal{B}$. Then $\mathcal{M}$ is a bicocycle double cross product Lie algebroid that is, $\mathcal{M} = \mathcal{A} \zeta \bowtie_\psi \mathcal{B}$. In this case, the cocycle terms, and the mutual actions are computed for arbitrary sections as

$$\begin{aligned} [X_1, X_2]_{\mathcal{M}} &= \phi(X_1, X_2) + \zeta(X_1, X_2), \\ [Y_1, Y_2]_{\mathcal{M}} &= \psi(Y_1, Y_2) + \theta(Y_1, Y_2), \\ [Y, X]_{\mathcal{M}} &= \rho(Y, X) - \sigma(X, Y). \end{aligned} \tag{3.102}$$

As in the case of the unified product Lie algebra(oid) discussion done in the previous subsection, we provide here the particular instance of bicocycle double cross product Lie algebroid where the base manifold is the unity. So that we first take two vector spaces $\mathfrak{g}$ and $\mathfrak{h}$. Then consider ϕ and ζ mappings in (3.89) defining the skew-symmetric bilinear operation on $\mathfrak{g}$, and ψ and θ mappings presented in (3.70) determining operation on $\mathfrak{h}$. We further assume the mutual *representations* ρ and σ given in (3.72). Observe that in defining these mappings, specifically for this Lie algebra scenario, we set $\mathfrak{g} = \Gamma(\mathcal{A})$ and $\mathfrak{h} = \Gamma(\mathcal{B})$. In Lie algebra case, the anchor maps are identically zero. The Lie algebra bracket $[\bullet, \bullet]_{\zeta \bowtie_\psi}$ on the vector space $\mathfrak{g} \oplus \mathfrak{h}$ is the one in (3.91). We write the following theorem for this case.

Theorem 3.6. *Two vector spaces $\mathfrak{g}$ and $\mathfrak{h}$ admitting mappings (3.70), (3.72) and (3.89) (with $\mathfrak{g} = \Gamma(\mathcal{A})$ and $\mathfrak{h} = \Gamma(\mathcal{B})$) determine a bicocycle double cocycle product Lie algebra structure*

$$(\mathfrak{g}_\zeta \bowtie_\psi \mathfrak{h}, [\bullet, \bullet]_{\zeta \bowtie_\psi}) \quad (3.103)$$

with the Lie bracket (3.91) if and only if the conditions listed in (3.93) and (3.94) are satisfied. Moreover, let $(\mathfrak{k}, [\bullet, \bullet]_{\mathfrak{k}})$ be a Lie algebra realizing $\mathfrak{g}$ and $\mathfrak{h}$ as complementary vector spaces so that $\mathfrak{k} \cong \mathfrak{g} \oplus \mathfrak{h}$. Then $\mathfrak{k}$ is a bicocycle double cross product Lie algebra that is, $\mathfrak{k} = \mathfrak{g}_\zeta \bowtie_\psi \mathfrak{h}$. In this case, the cocycle terms, and the mutual actions are computed through (3.102).

3.2.3. MP, Cocycle Extensions, and Semi-direct Products

In this subsection, we list various specific instances of the unified product (hence bicocycle double cross product) Lie algebroid structures. More concretely, we present double cross product (matched pair, abbreviated as MP) Lie algebroids, semi-direct product Lie algebroids, and *cocycle* extension Lie algebroids, along with the algebraic versions of these frameworks.

Double Cross Product (Matched Pair) Lie Algebroid. In the unified product Lie algebroid coupling presented in Theorem 3.1, there is a Lie algebroid and a vector bundle involved. A specific instance of this construction arises when there are two Lie algebroids. Algebraically, this corresponds to the vanishing of the *cocycle* term ψ in (3.71), along with the fulfillment of the Lie algebroid conditions for the vector bundle $(\mathcal{B}, \kappa, M)$.

Two Lie algebroids, which admit mutual representations of each other and share the same base manifold, form a double cross product (matched pair) Lie algebroid if the direct sum of their total spaces possesses a Lie algebroid structure on the same base. In this case, each individual Lie algebroid serves as a Lie subalgebroid of the direct sum, safeguarding the representation terms. The double cross product Lie algebroid structure was first announced in [87].

To have the double cross product Lie algebroid structure in an explicit form we start with two Lie algebroids

$$(\mathcal{A}, \tau, M, \mathfrak{a}_\mathcal{A}, [\bullet, \bullet]_\mathcal{A}), \quad (\mathcal{B}, \kappa, M, \mathfrak{a}_\mathcal{B}, [\bullet, \bullet]_\mathcal{B}) \quad (3.104)$$

over the same base M with mutual representations

$$\rho : \Gamma(\mathcal{B}) \times \Gamma(\mathcal{A}) \rightarrow \Gamma(\mathcal{A}), \quad \sigma : \Gamma(\mathcal{A}) \times \Gamma(\mathcal{B}) \rightarrow \Gamma(\mathcal{B}).$$

To construct a Lie algebroid structure on the direct sum of the total spaces of Lie algebroids, we need to define two operations, namely anchor map and Lie bracket. We introduce the anchor map as follows:

$$\mathfrak{a}_{\bowtie} := \mathcal{A} \times \mathcal{B} \longrightarrow TM, \quad (X, Y) \mapsto \mathfrak{a}_{\mathcal{A}}(X) + \mathfrak{a}_{\mathcal{B}}(Y), \quad (3.105)$$

and exhibit the Lie bracket

$$\begin{aligned} \Gamma(\mathcal{A} \times \mathcal{B}) \wedge \Gamma(\mathcal{A} \times \mathcal{B}) &\longrightarrow \Gamma(\mathcal{A} \times \mathcal{B}), \\ [(X_1, Y_1), (X_2, Y_2)]_{\bowtie} &= \left([X_1, X_2]_{\mathcal{A}} - \rho(Y_2, X_1) + \rho(Y_1, X_2), \right. \\ &\quad \left. [Y_1, Y_2]_{\mathcal{B}} - \sigma(X_2, Y_1) + \sigma(X_1, Y_2) \right) \end{aligned} \quad (3.106)$$

along with the conditions justifying what is called double cross product Lie algebroid $\mathcal{A} \bowtie \mathcal{B} := \mathcal{A} \times \mathcal{B}$ as follows. The compatibility conditions for the representations are

$$\begin{aligned} \rho(Y, [X_1, X_2]_{\mathcal{A}}) &= [\rho(Y, X_1), X_2]_{\mathcal{A}} + [X_1, \rho(Y, X_2)]_{\mathcal{A}} - \rho(\sigma(X_1, Y), X_2) \\ &\quad + \rho(\sigma(X_2, Y), X_1), \\ \sigma(X, [Y_1, Y_2]_{\mathcal{B}}) &= [\sigma(X, Y_1), Y_2]_{\mathcal{B}} + [Y_1, \sigma(X, Y_2)]_{\mathcal{B}} - \sigma(\rho(Y_1, X), Y_2) \\ &\quad + \sigma(\rho(Y_2, X), Y_1), \end{aligned} \quad (3.107)$$

whereas for the anchor map we compute

$$[\mathfrak{a}_{\mathcal{B}}(Y), \mathfrak{a}_{\mathcal{A}}(X)] = \mathfrak{a}_{\mathcal{A}}(\rho(Y, X)) - \mathfrak{a}_{\mathcal{B}}(\sigma(X, Y)), \quad (3.108)$$

for any X, X_1, X_2 in $\Gamma(\mathcal{A})$, and any Y, Y_1, Y_2 in $\Gamma(\mathcal{B})$.

Theorem 3.7. *Under the assumption that ρ and σ are satisfying the conditions of being Lie algebroid representations, the anchor map (3.105) and the Lie bracket (3.106) give rise to a Lie algebroid structure on the total space if and only if the conditions (3.107) and (3.108) are satisfied.*

In finite dimensions, we are following the notations and letters presented in (3.98) and (3.100). In these local coordinates, for the representations ρ and σ presented in (3.81), we write the double cross bracket as

$$[e_{\beta}, e_{\gamma}]_{\bowtie} = \mathfrak{H}_{\beta\gamma}^{\alpha} e_{\alpha}, \quad [e_b, e_d]_{\bowtie} = \mathfrak{T}_{bd}^a e_a, \quad [e_a, e_{\alpha}]_{\bowtie} = \mathfrak{R}_{a\alpha}^{\beta} e_{\beta} + \mathfrak{S}_{a\alpha}^b e_b. \quad (3.109)$$

See that, in this case, $\mathfrak{H}_{\beta\gamma}^\alpha$ are structure functions of the Lie algebroid $\mathcal{A}$ while $\mathfrak{T}_{bd}^a e_a$ are structure functions of the Lie algebroid $\mathcal{B}$.

There are two interesting subcases of the double cross product Lie algebroids. They are obtained by choosing one of the representations as trivial. These choices read semi-direct product Lie algebroids. We first assume that the representation ρ is trivial. This case reads only a representation of $\mathcal{A}$ on $\mathcal{B}$. We record this semi-direct product case notationally as $\mathcal{A} \ltimes \mathcal{B}$ and the Lie bracket as

$$[(X_1, Y_1), (X_2, Y_2)]_\ltimes = \left([X_1, X_2]_{\mathcal{A}}, [Y_1, Y_2]_{\mathcal{B}} - \sigma(X_2, Y_1) + \sigma(X_1, Y_2) \right). \quad (3.110)$$

In local picture, this is the subcase where the functions $\mathfrak{R}_{a\alpha}^\beta$ are all zero so that

$$[e_\beta, e_\gamma]_\ltimes = \mathfrak{H}_{\beta\gamma}^\alpha e_\alpha, \quad [e_b, e_d]_\ltimes = \mathfrak{T}_{bd}^a e_a, \quad [e_a, e_\alpha]_\ltimes = \mathfrak{S}_{a\alpha}^b e_b. \quad (3.111)$$

On the other hand, instead of ρ , we consider the representation σ be trivial. This permits only a representation of $\mathcal{B}$ on $\mathcal{A}$ so that denoted as $\mathcal{A} \rtimes \mathcal{B}$. In this semi-direct product Lie algebroid, the Lie bracket is

$$[(X_1, Y_1), (X_2, Y_2)]_\rtimes = \left([X_1, X_2]_{\mathcal{A}} - \rho(Y_2, X_1) + \rho(Y_1, X_2), [Y_1, Y_2]_{\mathcal{B}} \right). \quad (3.112)$$

In the local picture, this subcase is the one where the functions $\mathfrak{S}_{a\alpha}^b$ are all zero given that

$$[e_\beta, e_\gamma]_\rtimes = \mathfrak{H}_{\beta\gamma}^\alpha e_\alpha, \quad [e_b, e_d]_\rtimes = \mathfrak{T}_{bd}^a e_a, \quad [e_a, e_\alpha]_\rtimes = \mathfrak{R}_{a\alpha}^\beta e_\beta. \quad (3.113)$$

Theorem 3.7 concerns the double cross product Lie algebroid structure from the coupling point of view. Contrarily, the following one achieves this from the decoupling point of view. We call this the universal theorem for double cross product Lie algebroids.

Theorem 3.8. *Let $(\mathcal{M}, v, M, \mathfrak{a}, [\bullet, \bullet]_{\mathcal{M}})$ be a Lie algebroid, and let for two subalgebroids*

$$\left(\mathcal{A}, \tau = v|_{\mathcal{A}}, M, \mathfrak{a}_{\mathcal{A}} = \mathfrak{a}|_{\mathcal{A}}, [\bullet, \bullet]_{\mathcal{A}} = [\bullet, \bullet]_{\mathcal{M}}|_{\mathcal{A}} \right) \quad (3.114)$$

and

$$\left(\mathcal{B}, \kappa = v|_{\mathcal{B}}, M, \mathfrak{a}_{\mathcal{B}} = \mathfrak{a}|_{\mathcal{B}}, [\bullet, \bullet]_{\mathcal{B}} = [\bullet, \bullet]_{\mathcal{M}}|_{\mathcal{B}} \right), \quad (3.115)$$

$\mathcal{M} \cong \mathcal{A} \times \mathcal{B}$ as vector bundles. Then $\mathcal{M}$ is a double cross product Lie algebroid $\mathcal{M} = \mathcal{A} \bowtie \mathcal{B}$. In this case, the mutual representations are computed through for arbitrary sections as

$$[Y, X]_{\mathcal{M}} = \rho(Y, X) - \sigma(X, Y), \quad (3.116)$$

for arbitrary sections X in $\Gamma(\mathcal{A})$ and Y in $\Gamma(\mathcal{B})$.

Let us record here the double cross product Lie algebra construction by letting the base manifold be a singleton. As in the previous algebra discussions, the anchor map turns out to be the zero map.

Theorem 3.9. *Two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ admitting mutual representations (3.72) determine a double cocycle product (matched pair) Lie algebra*

$$(\mathfrak{g} \bowtie \mathfrak{h}, [\bullet, \bullet]_{\bowtie}) \quad (3.117)$$

with the Lie bracket (3.106) if and only if the conditions listed in (3.107) are satisfied. Moreover, let $(\mathfrak{k}, [\bullet, \bullet]_{\mathfrak{k}})$ be a Lie algebra realizing $\mathfrak{g}$ and $\mathfrak{h}$ as complementary Lie subalgebras so that $\mathfrak{k} \cong \mathfrak{g} \times \mathfrak{h}$. Then $\mathfrak{k}$ is a double cross product Lie algebra that is, $\mathfrak{k} = \mathfrak{g} \bowtie \mathfrak{h}$. In this case, the mutual representations are in (3.86).

Cocycle Extension Lie Algebroid. Let us record the unified Lie algebroid structure studied in Subsection 3.2.4 and make the following particular choices in this realization. We consider a Lie algebroid and a vector bundle given in

$$(\mathcal{A}, \tau, M, \mathfrak{a}_{\mathcal{A}}, [\bullet, \bullet]_{\mathcal{A}}), \quad (\mathcal{B}, \kappa, M) \quad (3.118)$$

over the same base manifold with an anchor map $\mathfrak{a}_{\mathcal{B}}$. We assume only the Lie algebroid action

$$\sigma : \Gamma(\mathcal{A}) \times \Gamma(\mathcal{B}) \longrightarrow \Gamma(\mathcal{B}) \quad (3.119)$$

and the skew-symmetric mapping

$$\psi : \Gamma(\mathcal{B}) \wedge \Gamma(\mathcal{B}) \longrightarrow \Gamma(\mathcal{A}). \quad (3.120)$$

We present the anchor map

$$\mathfrak{a}_{\bowtie_{\psi}} := \mathcal{A} \times \mathcal{B} \longrightarrow TM, \quad (X, Y) \mapsto \mathfrak{a}_{\mathcal{A}}(X) + \mathfrak{a}_{\mathcal{B}}(Y), \quad (3.121)$$

as well as the Lie bracket

$$\begin{aligned} \Gamma(\mathcal{A} \times \mathcal{B}) \wedge \Gamma(\mathcal{A} \times \mathcal{B}) &\longrightarrow \Gamma(\mathcal{A} \times \mathcal{B}), \\ [(X_1, Y_1), (X_2, Y_2)]_{\times_\psi} &= \left([X_1, X_2]_{\mathcal{A}} + \psi(Y_1, Y_2), \sigma(X_1, Y_2) - \sigma(X_2, Y_1) \right). \end{aligned} \quad (3.122)$$

We compute a list of conditions on ψ , and σ for the vector bundle $\mathcal{A} \times \mathcal{B}$ to be a Lie algebroid with the bracket $[\bullet, \bullet]_{\times_\psi}$ in (3.122):

$$\begin{aligned} \circ \sigma(\psi(Y_1, Y_2), Y_3) &= 0, \\ \psi(Y_2, \sigma(X_1, Y_3)) - \psi(\sigma(X_1, Y_2), Y_3) - [\psi(Y_2, Y_3), X_1]_{\mathcal{A}} &= 0, \\ \psi(Y_1, fY_2) &= f\psi(Y_1, Y_2), \\ \rho(Y, fX) &= f\rho(Y, X) + \mathcal{L}_{\mathbf{a}_{\mathcal{B}}(Y)}(f)X, \end{aligned} \quad (3.123)$$

for X_i in $\Gamma(\mathcal{A})$, Y_i in $\Gamma(\mathcal{B})$ with $i = 1, 2, 3$, and smooth function f defined on the base manifold. For the anchor map $\mathbf{a}_{\times_\psi}$ in (3.121) we have

$$\begin{aligned} [\mathbf{a}_{\mathcal{B}}(Y), \mathbf{a}_{\mathcal{A}}(X)] &= -\mathbf{a}_{\mathcal{B}}(\sigma(X, Y)), \\ [\mathbf{a}_{\mathcal{B}}(Y_1), \mathbf{a}_{\mathcal{B}}(Y_2)] &= \mathbf{a}_{\mathcal{A}}(\psi(Y_1, Y_2)). \end{aligned} \quad (3.124)$$

If the compatibility conditions in (3.123) and (3.124) are satisfied then we write $\mathcal{A} \times_\psi \mathcal{B}$ instead of a direct sum notation to exhibit the *cocycle* character of the algebraic structure.

Theorem 3.10. *A Lie algebroid $(\mathcal{A}, \tau, M, \mathbf{a}_{\mathcal{A}}, [\bullet, \bullet]_{\mathcal{A}})$ and a vector bundle $(\mathcal{B}, \kappa, M)$ admitting (3.119) and (3.120) determines a cocycle extension Lie algebroid structure*

$$(\mathcal{A} \times_\psi \mathcal{B}, \tau_{\times_\psi}, M, \mathbf{a}_{\times_\psi}, [\bullet, \bullet]_{\times_\psi}) \quad (3.125)$$

with the Lie bracket (3.122) and the anchor map (3.121) if and only if the conditions listed in (3.123) and (3.124) are satisfied.

We record here that if the base manifold is the unity so that the anchor is identically vanishes, then the Lie bracket (3.122) is a *cocycle* extension Lie bracket. The compatibility condition in this case is only the first two lines on (3.123).

In finite dimensions, the *cocycle* extension Lie algebroid refers to the case where the functions $\mathfrak{Z}_{\alpha\beta}^a$, $\mathfrak{R}_{a\alpha}^\beta$, and $\mathfrak{T}_{bd}^a$ are all zero that is

$$[e_\beta, e_\gamma]_{\times_\psi} = \mathfrak{H}_{\beta\gamma}^\alpha e_\alpha, \quad [e_b, e_d]_{\times_\psi} = \mathfrak{P}_{bd}^a e_a, \quad [e_a, e_\alpha]_{\times_\psi} = \mathfrak{S}_{a\alpha}^b e_b. \quad (3.126)$$

3.2.4. Illustrations

Twisted Poisson Algebroids and Dirac Structures

Consider a Poisson manifold (P, Λ) . The sections of the cotangent bundle $\pi_P : T^*P \rightarrow P$ admit a skew-symmetric bilinear operation

$$[\alpha, \beta] = \mathcal{L}_{\sharp_\Lambda(\alpha)}(\beta) - \mathcal{L}_{\sharp_\Lambda(\beta)}(\alpha) - d(\Lambda(\alpha, \beta)). \quad (3.127)$$

A direct observation reads that this bracket enables us to determine an almost Lie algebroid $(T^*P, \pi_P, P, \sharp_\Lambda, [\bullet, \bullet])$ where the musical mapping $\sharp_\Lambda$ is the anchor map. We denote this algebroid by the following commutative diagram:

$$\begin{array}{ccc} T^*P & \xrightarrow{\sharp_\Lambda} & TP \\ & \searrow \pi_P \quad \swarrow \tau_P & \\ & P & \end{array} \quad (3.128)$$

If Λ is a Poisson bivector field by satisfying the Jacobi identity (2.35) then the Jacobiator for the bracket (3.127) is identically vanishing. That means the bracket (3.127) satisfies the Jacobi identity.

Dirac Bundle Realization. An almost Dirac structure on a manifold M is a subbundle D of the Whitney sum $T^*M \times TM$ such that D is a maximal isotropic subbundle of $T^*M \times TM$ with respect to the pairing

$$\begin{aligned} (T^*M \times TM) \oplus (T^*M \times TM) &\longrightarrow \mathbb{R}, \\ (v_1, u_1) \cdot (v_2, u_2) &= u_1(v_2) + u_2(v_1). \end{aligned} \quad (3.129)$$

In the literature, it is more common to write the Dirac bracket as a subbundle of $TM \times T^*M$, but in this paper, we reverse the order of the constitutive bundles and determine the Dirac structure as a subbundle of Whitney sum $T^*M \times TM$. Therefore we denote a Dirac structure by

$$D \subset T^*M \times TM, \quad D = D^\perp. \quad (3.130)$$

Define a bundle structure $\tau : D \rightarrow M$ by projecting a pair (v, u) to the base point in M . If $pr_2(D)$ has a constant rank, then we say that D is a regular almost Dirac structure, and $pr_2(D)$ defines a regular foliation.

An almost Dirac structure is called a Dirac structure if the space $\Gamma(D)$ of sections for the fibration $D \rightarrow M$ is closed with respect to the (skew-symmetric) Courant

bracket

$$\begin{aligned}
[\bullet, \bullet] : \Gamma(D) \oplus \Gamma(D) &\longrightarrow \Gamma(D), \\
[(\alpha_1, U_1), (\alpha_2, U_2)] &= (\mathcal{L}_{U_1}\alpha_2 - \mathcal{L}_{U_2}\alpha_1 + \frac{1}{2}d(\alpha_1(U_2) - \alpha_2(U_1)), [U_1, U_2]).
\end{aligned} \tag{3.131}$$

A Dirac structure D on a manifold M carries a Lie algebroid structure with the anchor map $pr_2 : D \mapsto TM$ (projection to the second factor in D). We denote this Lie algebroid by

$$\begin{array}{ccc}
D & \xrightarrow{pr_2} & TM \\
& \searrow \tau & \swarrow \tau_M \\
& M &
\end{array} \tag{3.132}$$

We record this Lie algebroid structure by

$$D = (D, \tau, M, pr_2, [\bullet, \bullet]) \tag{3.133}$$

as well.

Let us examine this Lie algebroid in the understanding of unified product point of view. See that we can write the Dirac bundle as the direct sum of two of its subbundles. We define the decomposition simply by

$$(pr_1, pr_2) \circ \Delta : D \longrightarrow D^{T^*} \times D^T \tag{3.134}$$

where Δ is the diagonal map taking an element in D to a two copies of it in $D \times D$. So that D^T is the tangent bundle part whereas D^{T^*} is the cotangent bundle part in D . It is immediate to observe that there is no cocycle term ($\psi = 0$) reading that both D^T and D^{T^*} are Lie subalgebroids of D . So that we are in the realm of double cross product Lie algebroid category. More particularly, the mutual actions are computed through

$$[U, \alpha] = \left(\mathcal{L}_U \alpha - \frac{1}{2}d\alpha(U), 0 \right) \tag{3.135}$$

which gives that

$$\rho(U, \alpha) = \mathcal{L}_U \alpha - \frac{1}{2}d\alpha(U), \quad \sigma(\alpha, U) = 0. \tag{3.136}$$

Evidently this reads a semi-direct product Lie algebroid decomposition

$$D = D^{T^*} \ltimes D^T \tag{3.137}$$

as a particular instance of double cross product Lie algebroid structure where the action term σ is identically zero.

Twisted Category. There is an interesting subcategory of almost Poisson manifolds. In this case, the Jacobi identity (2.35) is still not satisfied but it is equal to the pull-back of a closed three-form [98, 99]. Let us depict this geometry in more detail. It is possible to generalize the musical mapping $\sharp_\Lambda$ in (2.37) to the space $\Gamma^k(P)$ of k -forms on P as well. We use the same notation for this mapping hoping no confusion may arise. We define

$$\begin{aligned} \sharp_\Lambda : \Gamma^k(P) &\longrightarrow \mathfrak{X}^k(P), \\ \langle \alpha_1 \wedge \cdots \wedge \alpha_k, \sharp_\Lambda(\omega) \rangle &= (-1)^k \langle \omega, \sharp_\Lambda(\alpha_1) \wedge \cdots \wedge \sharp_\Lambda(\alpha_k) \rangle \end{aligned} \quad (3.138)$$

where the notation $\sharp_\Lambda$ on the right-hand side of the equality is the musical mapping (2.37) between the first-order tensor fields. By loosening the Jacobi identity (2.35) as

$$\frac{1}{2}[\Lambda, \Lambda] = \sharp_\Lambda(\varphi) \quad (3.139)$$

for a closed three-form φ , one defines twisted Poisson structure. Here, $\sharp_\Lambda$ notation refers to the mapping in (3.138) for $k = 3$. We denote a twisted Poisson manifold by a three-tuple (P, Λ, φ) . In terms of the functions, the condition (3.139) is

$$\circlearrowleft \{F_1, \{F_2, F_3\}\} = \varphi(\sharp_\Lambda(dF_1), \sharp_\Lambda(dF_2), \sharp_\Lambda(dF_3)), \quad (3.140)$$

where $\sharp_\Lambda$ is the musical mapping in (2.37). Finally, we note that

$$X_{\{F_1, F_2\}} + [X_{F_1}, X_{F_2}] = -\sharp_\Lambda(\varphi(X_{F_1}, X_{F_2}, \bullet)). \quad (3.141)$$

Recall the bracket (3.127) defined on the space of one-form sections over an almost Poisson manifold. Evidently, this bracket is not satisfying the Jacobi identity for the case of twisted Poisson manifolds. On the other hand, we can introduce another bracket

$$[\alpha, \beta]_\varphi = [\alpha, \beta] + \iota_{\sharp_\Lambda(\alpha) \wedge \sharp_\Lambda(\beta)}(\varphi). \quad (3.142)$$

This bracket satisfies the Jacobi identity. Further, the quintuple $(T^*P, \pi_P, P, \Lambda^\sharp, [\cdot, \cdot]_\varphi)$ is a Lie algebroid.

Consider an almost Dirac structure D on a manifold M and, φ -twisted skew-symmetric Courant bracket [99, 100]

$$\begin{aligned} [(\alpha_1, U_1), (\alpha_2, U_2)]_\varphi &= [(\alpha_1, U_1), (\alpha_2, U_2)] + \varphi(U_1, U_2, \bullet) \\ &= (\mathcal{L}_{U_1}\alpha_2 - \mathcal{L}_{U_2}\alpha_1 + \frac{1}{2}d(\alpha_1(U_2) - \alpha_2(U_1)) + \varphi(U_1, U_2, \bullet), [U_1, U_2]). \end{aligned} \quad (3.143)$$

An almost Dirac structure D of $T^*M \times TM$ is a φ -twisted Dirac structure if the integrability condition with respect to the φ -twisted skew-symmetric Courant bracket $[\bullet, \bullet]_\varphi$. As in the ordinary case, a twisted Dirac structure D on M induces a Lie algebroid on D given by the anchor map $pr_2|_D$ and the bracket $[\bullet, \bullet]_\varphi$. Therefore $pr_2(D)$ is an integrable distribution on M . If D is a regular almost Dirac structure such that $pr_2(D) \subset TM$ is an integrable distribution on M , then, there exists an exact three-form φ with respect to which D is a φ -twisted Dirac structure [101]. We denote this Lie algebroid by

$$D_\varphi = (D, \tau, M, pr_2, [\bullet, \bullet]_\varphi). \quad (3.144)$$

This uniform product analysis reads a more complicated decomposition of the Dirac bundle. Once again, referring to (3.134), we write $D = D^{T^*} \times D^T$. Notice that D^{T^*} is remaining to be a Lie subalgebroid while D^T is not. Let us examine the interaction between these subbundles step by step. The mutual *actions* of the bundles are exactly the same with those given in (3.136). If we take the bracket of two tangent elements

$$[(0, U_1), (0, U_2)]_\varphi = \left(\varphi(U_1, U_2, \bullet), [U_1, U_2] \right) \quad (3.145)$$

This reads the following interaction term and the *cocycle*

$$\theta(U_1, U_2) = [U_1, U_2] \in \Gamma(D^T), \quad \psi(U_1, U_2) = \varphi(U_1, U_2, \bullet) \in \Gamma(D^{T^*}). \quad (3.146)$$

So we decompose this Dirac bundle as

$$D = D^{T^*} \bowtie_\psi D^T \quad (3.147)$$

since one of the actions is trivial. This cannot be examined in the realm of the *cocycle* extension of the subbundle $D^T \subset TM$ of the tangent bundle with the subbundle $D^{T^*} \subset T^*M$ of the cotangent bundle. Since in this case, the

representation σ is zero but, for *cocycle* extensions, we have ρ is zero. So that the twisted Dirac bundle is an example of a unified product (hence bicocycle double cross product) Lie algebroids.

Decomposition of Jacobi Lie Algebroid

The cotangent bundle of a Poisson manifold admits a Lie algebroid structure [10, 102]. For a Jacobi manifold $(M, \Lambda, \mathcal{E})$, the picture is as follows. Consider the extended cotangent bundle $\mathbb{R} \times T^*M$ as the total space of the first jet prolongation of the fibration $M \mapsto \mathbb{R}$ [103]. The space of sections of $p_M : \mathbb{R} \times T^*M \mapsto M$ is the product space of real valued functions and one-forms $C^\infty(M) \times \Gamma^1(M)$. So it consists of pairs (F, μ) where F be a real valued function and μ is a one-form. A bracket on $C^\infty(M) \times \Gamma^1(M)$ can be defined to be

$$\begin{aligned} & \left\{ (F_1, \mu), (F_2, \nu) \right\}_J \\ &= \left(\Lambda(\nu, \mu) + \sharp_\Lambda(\mu)(F_2) - \sharp_\Lambda(\nu)(F_1) + F_1\mathcal{E}(F_2) - F_2\mathcal{E}(F_1), \right. \\ & \quad \left. \mathcal{L}_{\sharp_\Lambda(\mu)}\nu - \mathcal{L}_{\sharp_\Lambda(\nu)}\mu - d(\Lambda(\mu, \nu)) + F_1\mathcal{L}_\mathcal{E}\nu - F_2\mathcal{L}_\mathcal{E}\mu - \iota_\mathcal{E}(\mu \wedge \nu) \right), \end{aligned} \quad (3.148)$$

where $\sharp_\Lambda$ is the musical mapping induced by the bivector field Λ . Referring to this bracket, the quintuple $(\mathbb{R} \times T^*M, p_M, M, \mathcal{E} + \sharp_\Lambda, \{\bullet, \bullet\}_J)$ realizes a Lie algebroid structure [104]. Here, the anchor map is given by

$$\mathcal{E} + \sharp_\Lambda : C^\infty(M) \times \Gamma^1(M) \longrightarrow \mathfrak{X}(M), \quad (F, \mu) \mapsto \sharp_\Lambda(\mu) + F\mathcal{E}. \quad (3.149)$$

We cite [105–107] for some further analysis on this Lie algebroid. If $\mathcal{E}$ is identically zero, then one can consider the cotangent bundle T^*M as the total space and the bracket (3.148) reduces to the algebra of one-forms on the Poisson manifold hence the Poisson Lie algebroid structure.

Let us examine this Lie algebroid structure in the view of a unified product point of view referring to the Universal Theorem. More precisely, we show Jacobi algebroid fits unified product decoupling. We determine the bracket on the function space by computing the Jacobi Lie algebroid bracket for two functions as

$$\{F_1, F_2\}_J = (F_1\mathcal{E}(F_2) - F_2\mathcal{E}(F_1), 0). \quad (3.150)$$

This reads the space $C^\infty(M)$ of functions as a Lie subalgebroid of the Jacobi algebroid. Furthermore, referring to the previous calculation, we see the Lie

bracket on the induced Lie subalgebroid as

$$\{F_1, F_2\}_{C^\infty(M)} = F_1\mathcal{E}(F_2) - F_2\mathcal{E}(F_1). \quad (3.151)$$

On the other hand, for the space of one-form sections, the situation is different. This space is not closed under the induced Lie bracket:

$$\{\mu, \nu\}_J = \left(\Lambda(\nu, \mu), \mathcal{L}_{\sharp_\Lambda(\mu)}\nu - \mathcal{L}_{\sharp_\Lambda(\nu)}\mu - d(\Lambda(\mu, \nu)) - \iota_{\mathcal{E}}(\mu \wedge \nu) \right), \quad (3.152)$$

since there appears a term in the first slot manifesting the existence of *cocycle* term. Accordingly, in light of (3.86), we write

$$\begin{aligned} \theta(\mu, \nu) &= \mathcal{L}_{\sharp_\Lambda(\mu)}\nu - \mathcal{L}_{\sharp_\Lambda(\nu)}\mu - d(\Lambda(\mu, \nu)) - \iota_{\mathcal{E}}(\mu \wedge \nu), \\ \psi(\mu, \nu) &= \Lambda(\nu, \mu). \end{aligned} \quad (3.153)$$

Notice that we have that the bivector field turns out to be the *cocycle* term. Further, we compute the mutual actions using the bracket of cross terms as

$$\{\mu, F\}_J = \left(\sharp_\Lambda(\mu)(F), -F\mathcal{L}_{\mathcal{E}}\mu \right), \quad (3.154)$$

which gives the following mappings:

$$\rho(\mu, F) = \sharp_\Lambda(\mu)(F), \quad \sigma(F, \mu) = F\mathcal{L}_{\mathcal{E}}\mu. \quad (3.155)$$

Therefore we conclude that Jacobi Lie algebroid is an extension of Poisson Lie algebroid in the realm of unified product Lie algebroid framework.

3.3. Couplings of Dynamics

3.3.1. Dynamics on Lie Algebra(oid)s.

Let us recall the dynamical equations on Lie algebroids once more to be more explicit in (de)coupling procedure in upcoming subsections.

For a given Lie algebroid $\mathcal{A}$, Hamiltonian dynamics exist on the dual bundle $\mathcal{A}^*$, as discussed in [108, 109]. For local coordinates (x^i, y_α) , and for a given Hamiltonian function $\mathcal{H} = \mathcal{H}(x^i, y_\alpha)$, one has Hamilton's equations

$$\frac{dx^i}{dt} = (\mathfrak{a}_{\mathcal{A}})_\alpha^i \frac{\partial \mathcal{H}}{\partial y_\alpha}, \quad \frac{dy_\alpha}{dt} = -\mathfrak{H}_{\alpha\beta}^\gamma y_\gamma \frac{\partial \mathcal{H}}{\partial y_\beta} - (\mathfrak{a}_{\mathcal{A}})_\alpha^i \frac{\partial \mathcal{H}}{\partial x^i}, \quad (3.156)$$

where $(\mathfrak{a}_{\mathcal{A}})^i_\alpha$ is the local realization of the anchor map while $\mathfrak{H}^\gamma_{\alpha\beta}$ denotes the structure functions of the Lie algebroid. If the Lie algebroid is a tangent bundle, (3.156) corresponds to the classical Hamilton's equation on the cotangent bundle, arising from the existence of the canonical symplectic structure. If the Lie algebroid is specifically a Lie algebra $\mathfrak{g}$ (with the base manifold being a singleton, making the anchor map trivial), (3.156) simplifies to the Lie-Poisson equations on the dual space $\mathfrak{g}^*$. Assuming local coordinates (y_α) on $\mathfrak{g}^*$ and considering a Hamiltonian function $h = h(y_\alpha)$, the Lie-Poisson equations are given by [4, 9, 110]

$$\frac{dy_\alpha}{dt} = -\mathfrak{H}^\gamma_{\alpha\beta} y_\gamma \frac{\partial h}{\partial y_\beta}. \quad (3.157)$$

Here, $\mathfrak{H}^\gamma_{\alpha\beta}$ are now the structure constants of the Lie algebra. The Lie-Poisson equations, describing the dynamics on $\mathfrak{g}^*$, can also be derived through the reduction by symmetry method.

On the other hand, within the framework of a Lie algebroid $\mathcal{A}$, Lagrangian dynamics can be formulated. In the coordinates (x^i, y^α) on $\mathcal{A}$, the dynamics for a given Lagrangian function $L = L(x^i, y^\alpha)$ is described by the Euler-Lagrange equations [94]:

$$\frac{dx^i}{dt} = (\mathfrak{a}_{\mathcal{A}})^i_\alpha y^\alpha, \quad \frac{d}{dt} \frac{\partial L}{\partial y^\alpha} = (\mathfrak{a}_{\mathcal{A}})^i_\alpha \frac{\partial L}{\partial x^i} + \mathfrak{H}^\gamma_{\alpha\beta} y^\beta \frac{\partial L}{\partial y^\gamma}. \quad (3.158)$$

A coordinate-independent realization of this formulation is established in [92]. In the special case where the Lie algebroid is a tangent bundle, the equations (3.158) correspond to the classical Euler-Lagrange equations. Alternatively, if the Lie algebroid is a Lie algebra $\mathfrak{g}$ with coordinates (y^α) , (3.158) transforms into the Euler-Poincaré equations:

$$\frac{d}{dt} \frac{\partial l}{\partial y^\alpha} = \mathfrak{H}^\gamma_{\alpha\beta} y^\beta \frac{\partial l}{\partial y^\gamma}, \quad (3.159)$$

for a Lagrangian function $l = l(y^\alpha)$. In regular cases, the connection between Lagrangian and Hamiltonian dynamics is established through the Legendre transformation.

The reversible nature of both (3.156) and (3.158) highlights limitations in their applicability to irreversible or/and dissipative processes. Contact structures emerge as a fitting framework for describing irreversible dynamics, as extensively explored

in the literature [72, 73, 76, 80, 111–114]. Notably, this approach has garnered increased attention in the field of thermodynamics, with recent contributions shaping discussions on topics such as [115–123]. A generic way to arrive at contact manifold is to extend a symplectic manifold, say T^*Q , by a line bundle. In Darboux' coordinates (q^i, p_i, z) on the product bundle $T^*Q \times \mathbb{R}$, for a Hamiltonian function $\mathcal{H} = \mathcal{H}(x^i, p_i, z)$, the contact Hamilton's equations are

$$\frac{dx^i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial x^i} - p_i \frac{\partial \mathcal{H}}{\partial z}, \quad \frac{dz}{dt} = p_i \frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{H}. \quad (3.160)$$

Accordingly, through the extension of the tangent bundle with a line bundle, that is $TQ \times \mathbb{R}$, one can establish a suitable geometric framework for contact Lagrangian dynamics. In induced coordinates $(x^i, \dot{x}^i, z)$ on $TQ \times \mathbb{R}$, where $L = L(x^i, \dot{x}^i, z)$ is the Lagrangian function, the corresponding equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = \frac{\partial L}{\partial \dot{x}^i} \frac{\partial L}{\partial z} \quad (3.161)$$

are known as Herglotz (or generalized Euler-Lagrange) equations, as discussed in [80].

To extend the scope of dynamics on Lie algebroids to encompass dissipative phenomena, a recent approach has emerged, which may be termed the 'contactization' of Lie algebroids. To implement this 'contactization' procedure, one extends both the Lie algebroid and the dual bundle by incorporating a trivial line bundle [95–97]. In this geometric framework, considering the extension $\mathcal{A}^* \times \mathbb{R}$ with coordinates (x^i, y_α, z) on the dual bundle, the resulting dissipative Hamilton's equations, generated by a Hamiltonian function $\mathcal{H} = \mathcal{H}(x^i, y_\alpha, z)$, take the form:

$$\frac{dx^i}{dt} = (\mathfrak{a}_{\mathcal{A}})_\alpha^i \frac{\partial \mathcal{H}}{\partial y_\alpha}, \quad \frac{dy_\alpha}{dt} = -(\mathfrak{a}_{\mathcal{A}})_\alpha^i \frac{\partial \mathcal{H}}{\partial x^i} - \mathfrak{H}_{\alpha\beta}^\gamma y_\gamma \frac{\partial \mathcal{H}}{\partial y_\beta} - y_\alpha \frac{\partial \mathcal{H}}{\partial z}, \quad \frac{dz}{dt} = y_\alpha \frac{\partial \mathcal{H}}{\partial y_\alpha} - \mathcal{H}. \quad (3.162)$$

At this juncture, it is noteworthy that this set of equations seamlessly combines the structure of Hamilton's equations (3.156) on the dual bundle $\mathcal{A}^*$ with the contact Hamilton's equations presented in (3.160). When the Lie algebroid corresponds to the tangent bundle, leading to the extended dual space being the extended cotangent bundle T^*Q , the Hamilton's equations (3.162) naturally yield the contact Hamiltonian dynamics as expressed in (3.160). In the context of the present work, the formulation (3.162) holds particular importance due to

its robust algebraic foundations, which are well-suited for in-depth discussions on the decoupling phenomenon. If the Lie algebroid is a Lie algebra $\mathfrak{g}$, the resulting equations are known as Lie-Poisson-Herglotz equations on the extended dual space $\mathfrak{g}^* \times \mathbb{R}$. In coordinates (y_α, z) on $\mathfrak{g}^* \times \mathbb{R}$, with a Hamiltonian function $h = h(y_\alpha, z)$, the Lie-Poisson-Herglotz equations take the form:

$$\frac{dy_\alpha}{dt} = -\mathfrak{H}_{\alpha\beta}^\gamma y_\gamma \frac{\partial h}{\partial y_\beta} - y_\alpha \frac{\partial h}{\partial z}, \quad \frac{dz}{dt} = y_\alpha \frac{\partial h}{\partial y_\alpha} - h. \quad (3.163)$$

From the Lagrangian perspective, we introduce a Lagrangian function $L = L(x^i, y^\alpha, z)$ defined on the extended Lie algebroid $\mathcal{A} \times \mathbb{R}$ with coordinates (x^i, y^α, z) . This Lagrangian function governs the dissipative Euler-Lagrange equations:

$$\frac{d}{dt} \frac{\partial L}{\partial y^\alpha} - (\mathfrak{a}_{\mathcal{A}})^i_\alpha \frac{\partial L}{\partial x^i} = \mathfrak{H}_{\alpha\beta}^\gamma y^\beta \frac{\partial L}{\partial y^\gamma} + \frac{\partial L}{\partial z} \frac{\partial L}{\partial y^\alpha}, \quad \frac{dx^i}{dt} = (\mathfrak{a}_{\mathcal{A}})^i_\alpha y^\alpha, \quad \frac{dz}{dt} = L(x^i, y^\alpha, z). \quad (3.164)$$

When the Lie algebroid corresponds to the tangent bundle, the result is the formulation of contact Lagrangian dynamics on the extended tangent bundle $TQ \times \mathbb{R}$ as exhibited in (3.161). If the Lie algebroid is specifically a Lie algebra $\mathfrak{g}$, the dynamics described in (3.161) transform into the Euler-Poincaré-Herglotz equations. In this scenario, a Lagrangian function $l = l(y^\alpha, z)$ is defined on $\mathfrak{g} \times \mathbb{R}$ with coordinates (y^α, z) , leading to the Euler-Poincaré-Herglotz equations:

$$\frac{d}{dt} \frac{\partial l}{\partial y^\alpha} = \mathfrak{H}_{\alpha\beta}^\gamma y^\beta \frac{\partial l}{\partial y^\gamma} + \frac{\partial l}{\partial z} \frac{\partial l}{\partial y^\alpha}, \quad \frac{dz}{dt} = l(x^i, y^\alpha, z). \quad (3.165)$$

3.3.2. Couplings of Reversible Dynamics

This section is reserved for the Lagrangian dynamics on bicocycle double cross product Lie algebroids and the Hamiltonian dynamics on the dual space. This will give us both the collective motion of the dynamical systems as well as the decomposition. We start with the bicocycle double cross product Poisson structure and bicocycle double cross product Hamiltonian dynamics.

Bicocycle Double Cross Product Poisson Bracket. We first recall the space of functions defined in (3.45) and (3.46) and carry them to the dual space $(\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*, \pi_{\zeta \bowtie_\psi}, M)$ of the unified product Lie algebroid structure $(\mathcal{A}_\zeta \bowtie_\psi \mathcal{B}, \tau_{\zeta \bowtie_\psi}, M, \mathfrak{a}_{\zeta \bowtie_\psi}, [\bullet, \bullet]_{\zeta \bowtie_\psi})$. The dual product space is $(\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*, \pi_{\zeta \bowtie_\psi}, M)$ to point out its product character. The total space is topologically equivalent to

$\mathcal{A}^* \times \mathcal{B}^*$ but we denote it by The space of functions on the dual space $\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*$ is spanned by three classes of functions. The first class of functions are

$$\widehat{f} : \mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^* \longrightarrow \mathbb{R}, \quad \widehat{f}(u) = f \circ \pi_{\zeta \bowtie_\psi}(u). \quad (3.166)$$

from the functions on the base manifold M . These functions are constants on the fibers. The second and third classes of functions come from the section of the constitutive spaces. Referring to the sections X in $\Gamma(\mathcal{A})$ and Y in $\Gamma(\mathcal{B})$ we determine linear independent sections of $\mathcal{A}_\zeta \bowtie_\psi \mathcal{B}$. We denote these sections as X and Y respectively, in order not to cause notation inflation. That is, we take $X = (X, 0)$ and $Y = (0, Y)$. They are defining functions on the total space as

$$\widehat{Z} : \mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^* \longrightarrow \mathbb{R}, \quad \widehat{Z}(w) = \langle w, Z \circ \pi_{\zeta \bowtie_\psi}(w) \rangle \quad (3.167)$$

where Z is either X or Y . These functions are linear on the fibers. Therefore, in light of the Poisson bracket in (3.47), we present the Poisson bracket on the dual space $\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*$ of the bicocycle double cross product Lie algebroid as

$$\begin{aligned} \{\widehat{f}, \widehat{g}\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*} &= 0, \\ \{\widehat{f}, \widehat{X}\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*} &= \widehat{\mathcal{L}_{\mathbf{a}_\zeta \bowtie_\psi}(X)} f, \\ \{\widehat{f}, \widehat{Y}\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*} &= \widehat{\mathcal{L}_{\mathbf{a}_\zeta \bowtie_\psi}(Y)} f, \\ \{\widehat{X}_1, \widehat{X}_2\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*} &= \phi(\widehat{X_1}, \widehat{X_2}) + \zeta(\widehat{X_1}, \widehat{X_2}) \\ \{\widehat{Y}_1, \widehat{Y}_2\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*} &= \psi(\widehat{Y_1}, \widehat{Y_2}) + \theta(\widehat{Y_1}, \widehat{Y_2}) \\ \{\widehat{Y}, \widehat{X}\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*} &= \rho(\widehat{Y}, \widehat{X}) - \sigma(\widehat{X}, \widehat{Y}), \end{aligned} \quad (3.168)$$

where one can consider the bracket on the fourth line as the result of the bracket of two elements $\widehat{X}_1$ and $\widehat{X}_2$ taking values both in $\mathcal{A}$ and $\mathcal{B}$, whereas the terms on the right side of the fifth line as the result of the bracket of two elements $\widehat{Y}_1$ and $\widehat{Y}_2$ taking values both in $\mathcal{A}$ and $\mathcal{B}$. The sixth line is the manifestation of the *representations*.

Local Picture. Let us turn to the general case of the dual space $\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*$ of the bicocycle double cross product Lie algebroid then examine the Poisson structure listed above. Assume the local charts (x^i) , (x^i, y^α) , and (x^i, y_α) denote local coordinates on the base manifold M , on the Lie algebroid $\mathcal{A}$, and the dual space $\mathcal{A}^*$ respectively. On the vector bundle $\mathcal{B}$ we choose the local coordinates (x^i, y^a) ,

and on the dual space $\mathcal{B}^*$ we take (x^i, y_a) . As a result, on the unified product space $\mathcal{A}_\zeta \bowtie_\psi \mathcal{B}$ we have the induced local coordinates (x^i, y^α, y^a) , and on the dual bundle $\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*$ we have (x^i, y_α, y_a) . In this case, referring to the local characterizations (3.100), the bicocycle double cross product Poisson bracket (3.47) takes the form:

$$\begin{aligned}
\{x^i, x^j\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*} &= 0, \\
\{x^i, y_\alpha\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*} &= (\mathfrak{a}_\mathcal{A})_\alpha^i, \\
\{x^i, y_a\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*} &= (\mathfrak{a}_\mathcal{B})_a^i, \\
\{y_\alpha, y_\beta\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*} &= \mathfrak{H}_{\alpha\beta}^\gamma y_\gamma + \mathfrak{Z}_{\alpha\beta}^a y_a, \\
\{y_a, y_b\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*} &= \mathfrak{P}_{ab}^\gamma y_\gamma + \mathfrak{T}_{ab}^d y_d, \\
\{y_a, y_\beta\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*} &= \mathfrak{R}_{a\beta}^\gamma y_\gamma + \mathfrak{S}_{a\beta}^d y_d
\end{aligned} \tag{3.169}$$

where $\mathfrak{a}_\alpha^i$ represents the components of the anchor map in the chosen bases, and $\mathfrak{C}_{ij}^k$ encodes the structure/*representation/cocycle* functions. See that $\mathfrak{H}_{\alpha\beta}^\gamma$ is standing for the ϕ mapping, $\mathfrak{Z}_{\alpha\beta}^a$ is for the ζ mapping, $\mathfrak{Z}_{\alpha\beta}^a$ is for the ζ mapping, $\mathfrak{P}_{ab}^\gamma$ is due to the ψ , and $\mathfrak{T}_{ab}^d$ represents θ .

Let us examine the Poisson bracket on the particular instances of bicocycle double cross product structure. Notice that if the ζ term is zero then reads the unified product Poisson structure which we denote by $\{\bullet, \bullet\}_{\mathcal{A}^* \bowtie_\psi \mathcal{B}^*}$. If both ζ and ψ are zero then we arrive at the double cross product Poisson structure $\{\bullet, \bullet\}_{\mathcal{A}^* \bowtie \mathcal{B}^*}$. If ζ , θ and σ are all zero then we have *cocycle* extension Poisson bracket $\{\bullet, \bullet\}_{\mathcal{A}^* \ltimes_\psi \mathcal{B}^*}$. The case of the dual space of the double cross product Lie algebra has been examined in [39, 45].

Hamiltonian Dynamics on the Dual of BDCP Lie Algebroids. For a Hamiltonian function $\mathcal{H} = \mathcal{H}(x^i, y_\alpha, y_a)$ defined on the dual bundle $\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*$, reads the Hamiltonian vector field $X_\mathcal{H}$ as

$$\frac{du}{dt} = \{u, \mathcal{H}\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*} \tag{3.170}$$

where the Poisson bracket on the right hand side is the one defined in (3.168). In local coordinates (x^i, y_α, y_a) on $\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*$ and referring to the local Poisson

bracekts in (3.169), we compute the Hamiltonian vector field as

$$\begin{aligned}
X_{\mathcal{H}} = & \left((a_{\mathcal{B}})_a^i \frac{\partial \mathcal{H}}{\partial y_a} + (\mathfrak{a}_{\mathcal{A}})_\alpha^i \frac{\partial \mathcal{H}}{\partial y_\alpha} \right) \frac{\partial}{\partial x^i} \\
& - \left(\mathfrak{Z}_{\alpha\beta}^d y_d \frac{\partial \mathcal{H}}{\partial y_\beta} + \mathfrak{H}_{\alpha\beta}^\gamma y_\gamma \frac{\partial \mathcal{H}}{\partial y_\beta} - \mathfrak{S}_{b\alpha}^d y_d \frac{\partial \mathcal{H}}{\partial y_b} - \mathfrak{K}_{b\alpha}^\gamma y_\gamma \frac{\partial \mathcal{H}}{\partial y_b} \right. \\
& \quad \left. + (\mathfrak{a}_{\mathcal{A}})_\alpha^i \frac{\partial \mathcal{H}}{\partial x^i} \right) \frac{\partial}{\partial y_\alpha} \\
& - \left(\mathfrak{S}_{a\beta}^d y_d \frac{\partial \mathcal{H}}{\partial y_\beta} + \mathfrak{T}_{ab}^d y_d \frac{\partial \mathcal{H}}{\partial y_b} + \mathfrak{P}_{ab}^\gamma y_\gamma \frac{\partial \mathcal{H}}{\partial y_b} + \mathfrak{R}_{a\beta}^\gamma y_\gamma \frac{\partial \mathcal{H}}{\partial y_\beta} \right. \\
& \quad \left. + (\mathfrak{a}_{\mathcal{A}})_a^i \frac{\partial \mathcal{H}}{\partial x^i} \right) \frac{\partial}{\partial y_a}.
\end{aligned} \tag{3.171}$$

In accordance with this, what we call bicocycle double cross product Hamilton's equations are computed to be

$$\begin{aligned}
\frac{dx^i}{dt} &= (\mathfrak{a}_{\mathcal{B}})_a^i \frac{\partial \mathcal{H}}{\partial y_a} + (\mathfrak{a}_{\mathcal{A}})_\alpha^i \frac{\partial \mathcal{H}}{\partial y_\alpha}, \\
\frac{dy_\alpha}{dt} &= -(\mathfrak{a}_{\mathcal{A}})_\alpha^i \frac{\partial \mathcal{H}}{\partial x^i} - \mathfrak{H}_{\alpha\beta}^\gamma y_\gamma \frac{\partial \mathcal{H}}{\partial y_\beta} + \underbrace{\mathfrak{S}_{b\alpha}^d y_d \frac{\partial \mathcal{H}}{\partial y_b}}_{\text{rep. of } \mathcal{A} \text{ on } \mathcal{B}} + \underbrace{\mathfrak{K}_{b\alpha}^\gamma y_\gamma \frac{\partial \mathcal{H}}{\partial y_b}}_{\text{rep. of } \mathcal{B} \text{ on } \mathcal{A}} \\
&\quad - \underbrace{\mathfrak{Z}_{\alpha\beta}^d y_d \frac{\partial \mathcal{H}}{\partial y_\beta}}_{\text{cocycle term } \zeta}, \\
\frac{dy_a}{dt} &= -(\mathfrak{a}_{\mathcal{B}})_a^i \frac{\partial \mathcal{H}}{\partial x^i} - \mathfrak{T}_{ab}^d y_d \frac{\partial \mathcal{H}}{\partial y_b} - \underbrace{\mathfrak{S}_{a\beta}^d y_d \frac{\partial \mathcal{H}}{\partial y_\beta}}_{\text{rep. of } \mathcal{A} \text{ on } \mathcal{B}} - \underbrace{\mathfrak{R}_{a\beta}^\gamma y_\gamma \frac{\partial \mathcal{H}}{\partial y_\beta}}_{\text{rep. of } \mathcal{B} \text{ on } \mathcal{A}} \\
&\quad - \underbrace{\mathfrak{P}_{ab}^\gamma y_\gamma \frac{\partial \mathcal{H}}{\partial y_b}}_{\text{Cocycle term } \psi}.
\end{aligned} \tag{3.172}$$

Lie Algebra Framework. One of the most important particular instance of the bicocycle double cross product Lie algebroid discussion is to write all of the ingredients of Poisson and Hamiltonian dynamics for the case of Lie algebra framework. To have this we take the base manifold as the unity $M = \{e\}$ and the anchor maps as trivial. In this case, one has two vector spaces $\mathfrak{g}$ and $\mathfrak{h}$ and their bicocycle double cross product Lie algebra $\mathfrak{g}_\zeta \bowtie_\psi \mathfrak{h}$ equipped with the Lie bracket $[\bullet, \bullet]_{\zeta \bowtie_\psi}$. In this case, the dual space $\mathfrak{g}_\zeta^* \bowtie_\psi \mathfrak{h}^*$ is endowed with the bicocycle double cross product Lie-Poisson bracket in local coordinates given by

$$\begin{aligned}
\{y_\alpha, y_\beta\}_{\mathfrak{g}_\zeta^* \bowtie_\psi \mathfrak{h}^*} &= \mathfrak{H}_{\alpha\beta}^\gamma y_\gamma + \mathfrak{Z}_{\alpha\beta}^d y_d, \\
\{y_a, y_b\}_{\mathfrak{g}_\zeta^* \bowtie_\psi \mathfrak{h}^*} &= \mathfrak{P}_{ab}^\gamma y_\gamma + \mathfrak{T}_{ab}^d y_d, \\
\{y_a, y_\beta\}_{\mathfrak{g}_\zeta^* \bowtie_\psi \mathfrak{h}^*} &= \mathfrak{R}_{a\beta}^\gamma y_\gamma + \mathfrak{S}_{a\beta}^d y_d,
\end{aligned} \tag{3.173}$$

where the coefficients are those determining representations/cocycles as in the Lie algebroid setting. For a given Hamiltonian function $h = h(y_\alpha, y_a)$ defined on the dual space, the bicocycle double cross product Lie-Poisson equations are computed to be

$$\begin{aligned} \frac{dy_\alpha}{dt} &= -\mathfrak{H}_{\alpha\beta}^\gamma y_\gamma \frac{\partial h}{\partial y_\beta} + \underbrace{\mathfrak{S}_{b\alpha}^d y_d \frac{\partial h}{\partial y_b}}_{\text{rep. of } \mathfrak{g} \text{ on } \mathfrak{h}} + \underbrace{\mathfrak{R}_{b\alpha}^\gamma y_\gamma \frac{\partial h}{\partial y_b}}_{\text{rep. of } \mathfrak{h} \text{ on } \mathfrak{g}} - \underbrace{\mathfrak{Z}_{\alpha\beta}^d y_d \frac{\partial h}{\partial y_\beta}}_{\text{Cocycle term } \zeta}, \\ \frac{dy_a}{dt} &= -\mathfrak{T}_{ab}^d y_d \frac{\partial h}{\partial y_b} - \underbrace{\mathfrak{S}_{a\beta}^d y_d \frac{\partial h}{\partial y_\beta}}_{\text{rep. of } \mathfrak{g} \text{ on } \mathfrak{h}} - \underbrace{\mathfrak{R}_{a\beta}^\gamma y_\gamma \frac{\partial h}{\partial y_\beta}}_{\text{rep. of } \mathfrak{h} \text{ on } \mathfrak{g}} - \underbrace{\mathfrak{P}_{ab}^\gamma y_\gamma \frac{\partial h}{\partial y_b}}_{\text{Cocycle term } \psi}. \end{aligned} \quad (3.174)$$

Hierarchical Reductions of BDCP Hamiltonian Dynamics. According to the hierarchical ordering of the subcases of bicocycle double cross product Lie algebroids (as well as Lie algebras), by taking corresponding terms as zero maps, one arrives at Hamiltonian (and later similarly Lagrangian dynamics) on unified product Lie algebra(oid)s, double cross product Lie algebra(oid)s, as well as *cocycle* extension Lie algebra(oid)s in a straightforward way. Let us examine these reductions in the view of the hierarchical ordering presented in Diagram 1. So we have the followings:

- To have the unified product Poisson bracket on the dual space $\mathcal{A}^* \bowtie_\psi \mathcal{B}^*$ of the unified product Lie algebroid $\mathcal{A} \bowtie_\psi \mathcal{B}$ examined Subsection 3.2.1, we choose the *cocycle* term ζ is equal to zero in the fourth line of the list of Poisson brackets in (3.168). Accordingly, in the local realization in (3.169) (for the case of Lie algebra, consider (3.173)), this case reads the functions $\mathfrak{Z}_{\alpha\beta}^a$ as zero. Following this, to have unified product Hamilton's equations one takes $\mathfrak{Z}_{\alpha\beta}^a = 0$ in (3.172) (for Lie algebras, consider (3.174)).
- To have the double cross product (matched pair) Poisson bracket on the dual space $\mathcal{A}^* \bowtie \mathcal{B}^*$ of the double cross product Lie algebroid $\mathcal{A} \bowtie \mathcal{B}$, we choose both of the *cocycle* terms ζ and ψ are zero in the fourth and fifth lines of (3.168). In the local picture (3.169) (for Lie algebra case, assume (3.174)), this corresponds that $\mathfrak{Z}_{\alpha\beta}^a$ and $\mathfrak{P}_{ab}^\gamma$ are all zero. Following this, to have double cross product (matched pair) Hamilton's equations one takes $\mathfrak{Z}_{\alpha\beta}^a = \mathfrak{P}_{ab}^\gamma = 0$ in (3.172) (for Lie algebras, consider (3.174)). If further one takes one of the representations ρ (or σ) is trivial that is $\mathfrak{R}_{a\beta}^\gamma = 0$

(or reciprocally, $\mathfrak{S}_{a\beta}^d = 0$) we arrive at semi-direct Poisson bracket and semi-direct Hamilton's equations in Lie algebra(oid) framework.

- Another subcase is the *cocycle* extension Poisson bracket and *cocycle* extension Hamilton's equations. To have the *cocycle* extension Poisson bracket, we take the *representation* ρ as trivial, one of the *cocycle* terms, say ζ is trivial, and the θ is zero in (3.168). For the local picture of the *cocycle* extension Poisson bracket and the *cocycle* extension Hamilton's equations, this corresponds to take $\mathfrak{R}_{a\alpha}^\beta = \mathfrak{Z}_{\alpha\beta}^a = \mathfrak{T}_{bd}^a = 0$ in (3.169) and (3.172), respectively. For the Lie algebra case, consider (3.173) and (3.174).

Lagrangian Dynamics on the Bicocycle Double Cross Product Lie Algebroids.

Consider Lie algebroid $\mathcal{A}_\zeta \bowtie_\psi \mathcal{B}$. Let us also consider a local coordinate system on $\mathcal{A}_\zeta \bowtie_\psi \mathcal{B}$ as (x^i, y^α, y^a) . Similar to the Euler-Lagrange equations on a Lie algebroid, the Euler-Lagrange equations generated by a Lagrangian function $L = L(x^i, y^\alpha, y^a)$ on the unified product Lie algebroid $\mathcal{A}_\zeta \bowtie_\psi \mathcal{B}$ (in local coordinates) are two sets of equations. The first set of equations can be directly interpreted as:

$$\frac{dx^i}{dt} = (\mathfrak{a}_\mathcal{A})_\alpha^i y^\alpha + (\mathfrak{a}_\mathcal{B})_a^i y^a. \quad (3.175)$$

The second set of equations can be expressed as:

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial y^\beta} &= (\mathfrak{a}_\mathcal{A})_\beta^i \frac{\partial L}{\partial x^i} + \underbrace{\mathfrak{H}_{\beta\gamma}^\alpha y^\gamma \frac{\partial L}{\partial y^\alpha}}_{\text{rep. of } \mathcal{B} \text{ on } \mathcal{A}} - \underbrace{\mathfrak{R}_{d\beta}^\alpha y^d \frac{\partial L}{\partial y^\alpha}}_{\text{rep. of } \mathcal{A} \text{ on } \mathcal{B}} - \underbrace{\mathfrak{S}_{d\beta}^a y^d \frac{\partial L}{\partial y^a}}_{\text{rep. of } \mathcal{A} \text{ on } \mathcal{B}} \\ &\quad + \underbrace{\mathfrak{Z}_{\beta\gamma}^a y^\gamma \frac{\partial L}{\partial y^a}}_{\text{Cocycle term } \zeta}, \\ \frac{d}{dt} \frac{\partial L}{\partial y^b} &= (\mathfrak{a}_\mathcal{B})_b^i \frac{\partial L}{\partial x^i} + \mathfrak{T}_{bd}^a y^d \frac{\partial L}{\partial y^a} + \underbrace{\mathfrak{S}_{b\gamma}^a y^\gamma \frac{\partial L}{\partial y^a}}_{\text{rep. of } \mathcal{A} \text{ on } \mathcal{B}} + \underbrace{\mathfrak{R}_{b\gamma}^\alpha y^\gamma \frac{\partial L}{\partial y^\alpha}}_{\text{rep. of } \mathcal{B} \text{ on } \mathcal{A}} \\ &\quad + \underbrace{\mathfrak{P}_{bd}^\alpha y^d \frac{\partial L}{\partial y^\alpha}}_{\text{Cocycle term } \psi}. \end{aligned} \quad (3.176)$$

Lie Algebra Framework. Let us examine the Euler-Poincaré dynamics on the bicocycle double cross product Lie algebra $\mathfrak{g}_\zeta \bowtie_\psi \mathfrak{h}$. For this, we choose the anchor maps as zero in (3.176) and the local coordinates (y^α, y^a) . In accordance with, we take a Lagrangian function $l = l(y^\alpha, y^a)$, and write bicocycle double cross product

Euler-Poincaré equations as

$$\begin{aligned}
\frac{d}{dt} \frac{\partial l}{\partial y^\beta} &= \mathfrak{H}_{\beta\gamma}^\alpha y^\gamma \frac{\partial L}{\partial y^\alpha} - \underbrace{\mathfrak{R}_{d\beta}^\alpha y^d \frac{\partial l}{\partial y^\alpha}}_{\text{rep. of } \mathfrak{h} \text{ on } \mathfrak{g}} - \underbrace{\mathfrak{S}_{d\beta}^a y^d \frac{\partial l}{\partial y^a}}_{\text{rep. of } \mathfrak{g} \text{ on } \mathfrak{h}} + \underbrace{\mathfrak{Z}_{\beta\gamma}^a y^\gamma \frac{\partial l}{\partial y^a}}_{\text{Cocycle term } \zeta}, \\
\frac{d}{dt} \frac{\partial l}{\partial y^b} &= \underbrace{\mathfrak{T}_{bd}^a y^d \frac{\partial l}{\partial y^a}}_{\text{rep. of } \mathfrak{g} \text{ on } \mathfrak{h}} + \underbrace{\mathfrak{S}_{b\gamma}^a y^\gamma \frac{\partial l}{\partial y^a}}_{\text{rep. of } \mathfrak{h} \text{ on } \mathfrak{g}} + \underbrace{\mathfrak{R}_{b\gamma}^\alpha y^\gamma \frac{\partial l}{\partial y^\alpha}}_{\text{rep. of } \mathfrak{h} \text{ on } \mathfrak{g}} + \underbrace{\mathfrak{P}_{bd}^\alpha y^d \frac{\partial l}{\partial y^\alpha}}_{\text{Cocycle term } \psi}.
\end{aligned} \tag{3.177}$$

Hierarchical Reductions of BDCP Lagrangian Dynamics. We refer Diagram 1 to reduce the bicocycle double cross product Lagrangian dynamics given in (3.176) for Lie algebroids and (3.177) for lie algebras to particular instances:

- To have the unified product Euler-Lagrange equations on the unified product Lie algebroid $\mathcal{A} \bowtie_\psi \mathcal{B}$ and Lie algebra $\mathfrak{g} \bowtie_\psi \mathfrak{h}$ we choose the *cocycle* term ζ hence $\mathfrak{Z}_{\alpha\beta}^a$ are all zero in (3.176) and (3.177), respectively.
- To have the double cross product (matched pair) Lagrangian dynamics on Lie algebroid $\mathcal{A}^* \bowtie \mathcal{B}^*$ and Lie algebra $\mathfrak{g} \bowtie \mathfrak{h}$, we choose both of the *cocycle* terms ζ and ψ are zero so that that $\mathfrak{Z}_{\alpha\beta}^a$ and $\mathfrak{P}_{ab}^\gamma y_\gamma$ are all zero in (3.176) and (3.177), respectively. Moreover, if one takes one of the representations ρ (or σ) is trivial that is $\mathfrak{R}_{a\beta}^\gamma y_\gamma = 0$ (or reciprocally, $\mathfrak{S}_{a\beta}^d y_d = 0$) we arrive at semi-direct product Lagrangian dynamics.
- For the *cocycle* extension Lagrangian dynamics, we take θ , ρ and ζ trivial. This reads $\mathfrak{R}_{a\alpha}^\beta = \mathfrak{Z}_{\alpha\beta}^a = \mathfrak{T}_{bd}^a = 0$ in (3.176) and (3.177).

3.3.3. Couplings of Irreversible Dynamics

This subsection is reserved for the (de)coupling of the dynamical systems where an irreversible character is endowed. For this, we first recall bicocycle double product Lie algebroid structure in Subsection (3.2.2) denoted as $\mathcal{A}_\zeta \bowtie_\psi \mathcal{B}$ with coordinates (x^i, y^α, y^a) . Then we consider the dual bundle $\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*$ with coordinates (x^i, y_α, y_a) . Next we remark the Poisson structure on $\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^*$ given in Subsection 3.3.2. See that bicocycle double cross product Poisson bracket is given in (3.168) as a coordinate-free way, and in coordinates in (3.169).

We extend the bicocycle double cross product Lie algebroid with the real line $\mathbb{R}$ to arrive at the product space $\mathcal{A}_\zeta \bowtie_\psi \mathcal{B} \times \mathbb{R}$ with local coordinates (x^i, y^α, y^a, z) . The dual of this bundle is $\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^* \times \mathbb{R}$ with local coordinates (x^i, y_α, y_a, z) .

Dissipative Hamiltonian Dynamics on the Dual of BDCP Lie Algebroids. We extend the Poisson brackets in (3.169). More concretely, we merge (3.169) and the Jacobi brackets in (3.60) where the extension with the action is examined. This itinerary map leads us to arrive at the following set of Jacobi brackets

$$\begin{aligned}
\{x^i, x^j\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^* \times \mathbb{R}} &= 0, \\
\{z, x^i\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^* \times \mathbb{R}} &= 0, \\
\{z, y_\alpha\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^* \times \mathbb{R}} &= y_\alpha, \\
\{z, y_a\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^* \times \mathbb{R}} &= y_a, \\
\{x^i, y_\alpha\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^* \times \mathbb{R}} &= (\mathfrak{a}_\mathcal{A})_\alpha^i, \\
\{x^i, y_a\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^* \times \mathbb{R}} &= (\mathfrak{a}_\mathcal{B})_a^i, \\
\{y_\alpha, y_\beta\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^* \times \mathbb{R}} &= \mathfrak{H}_{\alpha\beta}^\gamma y_\gamma + \mathfrak{Z}_{\alpha\beta}^a y_a, \\
\{y_a, y_b\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^* \times \mathbb{R}} &= \mathfrak{P}_{ab}^\gamma y_\gamma + \mathfrak{T}_{ab}^d y_d, \\
\{y_a, y_\beta\}_{\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^* \times \mathbb{R}} &= \mathfrak{R}_{a\beta}^\gamma y_\gamma + \mathfrak{S}_{a\beta}^d y_d
\end{aligned} \tag{3.178}$$

which we call (dissipative) bicycle double cross product Jacobi structure.

Accordingly, for a Hamiltonian function $\mathcal{H} = \mathcal{H}(x^i, y_\alpha, y_a, z)$ defined on the extended bundle $\mathcal{A}_\zeta^* \bowtie_\psi \mathcal{B}^* \times \mathbb{R}$ the Hamilton's equations which we call bicycle double cross product dissipative Hamilton's equations, is the following set of equations:

$$\begin{aligned}
\frac{dx^i}{dt} &= (\mathfrak{a}_\mathcal{A})_\alpha^i \frac{\partial \mathcal{H}}{\partial y_\alpha} + (\mathfrak{a}_\mathcal{B})_a^i \frac{\partial \mathcal{H}}{\partial y_a}, \\
\frac{dy_\alpha}{dt} &= - \left((\mathfrak{a}_\mathcal{A})_\alpha^i \frac{\partial \mathcal{H}}{\partial x^i} + \mathfrak{Z}_{\alpha\beta}^a y_a \frac{\partial \mathcal{H}}{\partial y_\beta} + \mathfrak{H}_{\alpha\beta}^\gamma y_\gamma \frac{\partial \mathcal{H}}{\partial y_\beta} - \mathfrak{R}_{b\alpha}^\gamma y_\gamma \frac{\partial \mathcal{H}}{\partial y_b} \right. \\
&\quad \left. - \mathfrak{S}_{b\alpha}^d y_d \frac{\partial \mathcal{H}}{\partial y_b} + y_\alpha \frac{\partial \mathcal{H}}{\partial z} \right), \\
\frac{dy_a}{dt} &= - \left((\mathfrak{a}_\mathcal{B})_a^i \frac{\partial \mathcal{H}}{\partial x^i} + \mathfrak{R}_{a\beta}^\gamma y_\gamma \frac{\partial \mathcal{H}}{\partial y_\beta} + \mathfrak{S}_{a\beta}^b y_b \frac{\partial \mathcal{H}}{\partial y_\beta} + \mathfrak{P}_{ab}^\gamma y_\gamma \frac{\partial \mathcal{H}}{\partial y_b} \right. \\
&\quad \left. + \mathfrak{T}_{ab}^d y_d \frac{\partial \mathcal{H}}{\partial y_b} + y_a \frac{\partial \mathcal{H}}{\partial z} \right), \\
\frac{dz}{dt} &= y_a \frac{\partial \mathcal{H}}{\partial y_a} + y_\alpha \frac{\partial \mathcal{H}}{\partial y_\alpha} - \mathcal{H}.
\end{aligned} \tag{3.179}$$

BDCP Lie-Poisson-Herglotz Equations. Let us examine the dynamics on Lie algebra setting. To have this, we take the base manifold $M = \{e\}$ then the anchor map turns out to be zero. Then we have the extended dual space $\mathfrak{g}_\zeta^* \bowtie_\psi \mathfrak{h}^* \times \mathbb{R}$

of the bicocycle double cross product space $\mathfrak{g}_\zeta \bowtie_\psi \mathfrak{h}$. The dual space is endowed with coordinates (y_α, y_a, z) and the Jacobi bracket in (3.178) takes the particular form:

$$\begin{aligned} \{z, y_\alpha\}_{\mathfrak{g}_\zeta^* \bowtie_\psi \mathfrak{h}^* \times \mathbb{R}} &= y_\alpha, \\ \{z, y_a\}_{\mathfrak{g}_\zeta^* \bowtie_\psi \mathfrak{h}^* \times \mathbb{R}} &= y_a, \\ \{y_\alpha, y_\beta\}_{\mathfrak{g}_\zeta^* \bowtie_\psi \mathfrak{h}^* \times \mathbb{R}} &= \mathfrak{H}_{\alpha\beta}^\gamma y_\gamma + \mathfrak{Z}_{\alpha\beta}^a y_a, \\ \{y_a, y_b\}_{\mathfrak{g}_\zeta^* \bowtie_\psi \mathfrak{h}^* \times \mathbb{R}} &= \mathfrak{P}_{ab}^\gamma y_\gamma + \mathfrak{T}_{ab}^d y_d, \\ \{y_a, y_\beta\}_{\mathfrak{g}_\zeta^* \bowtie_\psi \mathfrak{h}^* \times \mathbb{R}} &= \mathfrak{R}_{a\beta}^\gamma y_\gamma + \mathfrak{S}_{a\beta}^d y_d. \end{aligned} \quad (3.180)$$

For a Hamiltonian function $h = h(y_\alpha, y_a, z)$ defined on the dual space $\mathfrak{g}_\zeta^* \bowtie_\psi \mathfrak{h}^* \times \mathbb{R}$, we have bicocycle double cross product Lie-Poisson-Herglotz equations given by

$$\begin{aligned} \frac{dy_\alpha}{dt} &= -\left(\mathfrak{Z}_{\alpha\beta}^a y_a \frac{\partial h}{\partial y_\beta} + \mathfrak{H}_{\alpha\beta}^\gamma y_\gamma \frac{\partial h}{\partial y_\beta} - \mathfrak{R}_{b\alpha}^\gamma y_\gamma \frac{\partial h}{\partial y_b} - \mathfrak{S}_{b\alpha}^d y_d \frac{\partial h}{\partial y_b} + y_\alpha \frac{\partial h}{\partial z} \right), \\ \frac{dy_a}{dt} &= -\left(\mathfrak{R}_{a\beta}^\gamma y_\gamma \frac{\partial h}{\partial y_\beta} + \mathfrak{S}_{a\beta}^d y_b \frac{\partial h}{\partial y_\beta} + \mathfrak{P}_{ab}^\gamma y_\gamma \frac{\partial h}{\partial y_b} + \mathfrak{T}_{ab}^d y_d \frac{\partial h}{\partial y_b} + y_a \frac{\partial h}{\partial z} \right), \\ \frac{dz}{dt} &= y_a \frac{\partial h}{\partial y_a} + y_\alpha \frac{\partial h}{\partial y_\alpha} - h. \end{aligned} \quad (3.181)$$

Hierarchical Reductions of BDCP Irreversible Dynamics. We once more refer Diagram 1 where there is a hierarchical orders of couplings given pictorially. Referring to this diagram, we now present unified product, double cross product (matched pair), *cocycle* extension and semi-direct product dissipative Herglotz type Hamiltonian dynamics one by one:

- The unified product Jacobi brackets on the dual space $\mathcal{A}^* \bowtie_\psi \mathcal{B}^* \times \mathbb{R}$ of the extended Lie algebroid and the dual space $\mathfrak{g}^* \bowtie_\psi \mathfrak{h}^* \times \mathbb{R}$ of the extended Lie algebra are the ones where the *cocycle* term ζ is trivial. Locally this gives that the functions $\mathfrak{Z}_{\alpha\beta}^a$ are all zero. For the Jacobi bracket (3.178) this reads the second term on the right hand side of the seventh line is zero for Lie algebroid framework. For the Jacobi bracket (3.180) on Lie algebra level, this case occurs if the the second term on the right hand side of the third line is zero. Accordingly, this results with the disappearing of $\mathfrak{Z}_{\alpha\beta}^a$ terms in Hamiltonian dynamics (3.179) on the Lie algebroid level whereas absence of $\mathfrak{Z}_{\alpha\beta}^a$ terms in Lie-Poisson-Herlogtz equation in (3.181).
- The double cross product (matched pair) Jacobi brackets on the dual spaces $\mathcal{A}^* \bowtie \mathcal{B}^* \times \mathbb{R}$ and $\mathfrak{g}^* \bowtie \mathfrak{h}^* \times \mathbb{R}$ occurs if both of the *cocycle* terms ζ and ψ

are zero. In the local pictures, this corresponds to take $\mathfrak{Z}_{\alpha\beta}^a = \mathfrak{P}_{ab}^\gamma y_\gamma = 0$ in the Jacobi brackets (3.178) and (3.180), respectively. In light of this choices we have irreversible matched pair dynamics on $\mathcal{A}^* \bowtie \mathcal{B}^* \times \mathbb{R}$ if $\mathfrak{Z}_{\alpha\beta}^a = \mathfrak{P}_{ab}^\gamma y_\gamma = 0$ in (3.179). Similarly, matched pair Lie-Poisson-Herlogtz equations are obtained if $\mathfrak{Z}_{\alpha\beta}^a = \mathfrak{P}_{ab}^\gamma = 0$ in (3.181). Additionally, if we take one of the representations ρ (or σ) is trivial lthat is if ocallly $\mathfrak{R}_{a\beta}^\gamma = 0$ (or reciprocally, $\mathfrak{S}_{a\beta}^d = 0$) then we have semi-direct Jacobi brackets and semi-direct irreversible Hamilton's equations. for the Lie algebra setting this results with semi-direct product Lie-Poisson-Herlogtz equations.

- The *cocycle* extension Jacobi bracket and cocycle extension dissipative Hamilton's equations are obtained if the *representation* ρ , the *cocycle* term ζ , and the term θ are zero. For the local picture this corresponds to take $\mathfrak{R}_{a\alpha}^\beta = \mathfrak{Z}_{\alpha\beta}^a = \mathfrak{T}_{bd}^a = 0$ in (3.178) and (3.180) for the Jacobi brackets, and in (3.179) and (3.181) for the equations of motion.

Lagrangian Dynamics on the Bicocycle Double Cross Product Lie Algebroids.

Let us now examine irreversible Lagrangian dynamics on the real line extension $\mathcal{A}_\zeta \bowtie_\psi \mathcal{B} \times \mathbb{R}$ of the bicocycle double cross product Lie algebroid $\mathcal{A}_\zeta \bowtie_\psi \mathcal{B}$. On the extended bundle, We consider the local coordinates (x^i, y^α, y^a, z) where z is the coordinates on $\mathbb{R}$. Accordingly for a Lagrangian function $L = L(x^i, y^\alpha, y^a, z)$, we have two sets of equations as we have in reversible Lagrangian motion. The first set of equations is the one determining the dynamics of the base coordinates given by

$$\frac{dx^i}{dt} = (\mathfrak{a}_\mathcal{A})_\alpha^i y^\alpha + (\mathfrak{a}_\mathcal{B})_a^i y^a. \quad (3.182)$$

As it can be easily observed, this equation is the same with the reversible case. The second set of equations determine the dynamics on the fiber coordinates as

follows:

$$\begin{aligned}
\frac{d}{dt} \frac{\partial L}{\partial y^\beta} &= (\mathfrak{a}_A)_\beta^i \frac{\partial L}{\partial x^i} + \underbrace{\mathfrak{H}_{\beta\gamma}^\alpha y^\gamma \frac{\partial L}{\partial y^\alpha}}_{\text{rep. of } \mathcal{B} \text{ on } \mathcal{A}} - \underbrace{\mathfrak{R}_{d\beta}^\alpha y^d \frac{\partial L}{\partial y^\alpha}}_{\text{rep. of } \mathcal{A} \text{ on } \mathcal{B}} - \underbrace{\mathfrak{S}_{d\beta}^a y^d \frac{\partial L}{\partial y^a}}_{\text{rep. of } \mathcal{A} \text{ on } \mathcal{B}} \\
&\quad + \underbrace{\mathfrak{Z}_{\beta\gamma}^a y^\gamma \frac{\partial L}{\partial y^a}}_{\text{Cocycle term } \zeta} + \underbrace{\frac{\partial L}{\partial y^\beta} \frac{\partial L}{\partial z}}_{\text{dissipative term}}, \\
\frac{d}{dt} \frac{\partial L}{\partial y^b} &= (\mathfrak{a}_B)_b^i \frac{\partial L}{\partial x^i} + \underbrace{\mathfrak{T}_{bd}^a y^d \frac{\partial L}{\partial y^a}}_{\text{rep. of } \mathcal{A} \text{ on } \mathcal{B}} + \underbrace{\mathfrak{S}_{b\gamma}^a y^\gamma \frac{\partial L}{\partial y^a}}_{\text{rep. of } \mathcal{A} \text{ on } \mathcal{B}} + \underbrace{\mathfrak{R}_{b\gamma}^\alpha y^\gamma \frac{\partial L}{\partial y^\alpha}}_{\text{rep. of } \mathcal{B} \text{ on } \mathcal{A}} \\
&\quad + \underbrace{\mathfrak{P}_{bd}^\alpha y^d \frac{\partial L}{\partial y^\alpha}}_{\text{Cocycle term } \psi} + \underbrace{\frac{\partial L}{\partial y^b} \frac{\partial L}{\partial z}}_{\text{dissipative term}}.
\end{aligned} \tag{3.183}$$

Lie Algebra Framework. As a particular instance of the irreversible collective motion on the Lie algebroid setting, we can examine irreversible motion on the Lie algebra level. For this, we take the real line extension $\mathfrak{g}_\zeta \bowtie_\psi \mathfrak{h} \times \mathbb{R}$ of the bicocycle double cross product Lie algebra $\mathfrak{g}_\zeta \bowtie_\psi \mathfrak{h} \times \mathbb{R}$. The irreversible Lagrangian dynamics obtained on the extended space is now bicocycle double cross product Euler-Poincaré-Herglotz equations. For a Lagrangian function $l = l(y^\alpha, y^a, z)$ on $\mathfrak{g}_\zeta \bowtie_\psi \mathfrak{h} \times \mathbb{R}$, we have

$$\begin{aligned}
\frac{d}{dt} \frac{\partial l}{\partial y^\beta} &= \mathfrak{H}_{\beta\gamma}^\alpha y^\gamma \frac{\partial l}{\partial y^\alpha} - \underbrace{\mathfrak{R}_{d\beta}^\alpha y^d \frac{\partial l}{\partial y^\alpha}}_{\text{rep. of } \mathfrak{h} \text{ on } \mathfrak{g}} - \underbrace{\mathfrak{S}_{d\beta}^a y^d \frac{\partial l}{\partial y^a}}_{\text{rep. of } \mathfrak{g} \text{ on } \mathfrak{h}} + \underbrace{\mathfrak{Z}_{\beta\gamma}^a y^\gamma \frac{\partial l}{\partial y^a}}_{\text{Cocycle term } \zeta} \\
&\quad + \underbrace{\frac{\partial l}{\partial y^\beta} \frac{\partial l}{\partial z}}_{\text{dissipative term}}, \\
\frac{d}{dt} \frac{\partial l}{\partial y^b} &= \mathfrak{T}_{bd}^a y^d \frac{\partial l}{\partial y^a} + \underbrace{\mathfrak{S}_{b\gamma}^a y^\gamma \frac{\partial l}{\partial y^a}}_{\text{rep. of } \mathfrak{g} \text{ on } \mathfrak{h}} + \underbrace{\mathfrak{R}_{b\gamma}^\alpha y^\gamma \frac{\partial l}{\partial y^\alpha}}_{\text{rep. of } \mathfrak{h} \text{ on } \mathfrak{g}} + \underbrace{\mathfrak{P}_{bd}^\alpha y^d \frac{\partial l}{\partial y^\alpha}}_{\text{Cocycle term } \psi} \\
&\quad + \underbrace{\frac{\partial l}{\partial y^b} \frac{\partial l}{\partial z}}_{\text{dissipative term}}.
\end{aligned} \tag{3.184}$$

Hierarchical Reductions of Dissipative BDCP Lagrangian Dynamics. We recall Diagram 1 and apply the hierarchical ordering presented in it to the irreversible Lagrangian dynamics both in Lie algebroid category given in (3.182) and (3.183) and in Lie algebra category in (3.184):

- The unified product dissipative Euler-Lagrange equations on the extension $\mathcal{A} \bowtie_\psi \mathcal{B} \times \mathbb{R}$ of the unified product Lie algebroid $\mathcal{A} \bowtie_\psi \mathcal{B}$ is obtained by

taking one of the *cocycle* terms, say ζ , is trivial in bicocycle double cross product constructions. Locally, this is achieved by taking $\mathfrak{Z}_{\alpha\beta}^a = 0$ in the system (3.183) for Lie algebroid geometry. Similarly, we assume $\mathfrak{Z}_{\alpha\beta}^a = 0$ in (3.184) for unified product Euler-Poincaré-Herglotz equations on extended unified product Lie algebra $\mathfrak{g} \bowtie_{\psi} \mathfrak{h} \times \mathbb{R}$.

- The double cross product (matched pair) irreversible Lagrangian dynamics on the extension $\mathcal{A} \bowtie \mathcal{B} \times \mathbb{R}$ of the double cross product Lie algebroid $\mathcal{A} \bowtie \mathcal{B}$, we choose both of the *cocycle* terms ζ and ψ are zero so that that $\mathfrak{Z}_{\alpha\beta}^a$ and $\mathfrak{P}_{ab}^{\gamma}$ are all zero in (3.183). To have matched pair Euler-Poincaré-Herglotz equations on the extended space $\mathfrak{g} \bowtie_{\psi} \mathfrak{h} \times \mathbb{R}$, we take $\mathfrak{Z}_{\alpha\beta}^a = \mathfrak{P}_{ab}^{\gamma} = 0$ in (3.184). To arrive at the case of semi-direct product theories, we further assume one of the representations ρ (or σ) is trivial that is $\mathfrak{R}_{a\beta}^{\gamma} = 0$ (or reciprocally, $\mathfrak{S}_{a\beta}^d = 0$). In the Lie algebra level these reductions result in semi-direct product Euler-Poincaré-Herglotz equations.
- Another particular instance is to get the *cocycle* extension irreversible Lagrangian dynamics. To have this, we take θ , ρ and ζ trivial. This reads $\mathfrak{R}_{a\alpha}^{\beta} = \mathfrak{Z}_{\alpha\beta}^a = \mathfrak{T}_{bd}^a = 0$ in (3.183) for Euler-Lagrange equations defined on the extended Lie algebroid, and in (3.184) for *cocycle* extension of Euler-Poincaré-Herglotz equations.

4. n -LIE ALGEBRA(OID)S AND COUPLINGS

4.1. n -Lie Algebras and Cohomology

4.1.1. n -Lie Algebras

We dive into the fundamental concepts of n -Lie algebras, as discussed in seminal works such as [19, 21, 25]. An n -Lie algebra is defined as a linear space $\mathfrak{G}$ endowed with a linear operator

$$[\bullet, \dots, \bullet] : \mathfrak{G}_n \rightarrow \mathfrak{G},$$

satisfying the fundamental identity expressed by

$$[x_1, \dots, x_{n-1}, [x_n, \dots, x_{2n-1}]] = \sum_{k=0}^{n-1} [x_n, \dots, [x_1, \dots, x_{n-1}, x_{n+k}], \dots, x_{2n-1}], \quad (4.1)$$

for all $x_1, \dots, x_{2n-1}$ in $\mathfrak{G}$. Here $\mathfrak{G}_n$ is defined to be $\wedge^n \mathfrak{G}$. Building upon the insights from [25], within the framework of an n -Lie algebra $\mathfrak{G}$, the space $\mathfrak{G}_{n-1}$ exhibits the structure of a Leibniz algebra, defined by the operation:

$$[x_1, x_2] := \sum_{k=1}^{n-1} x_2^1 \wedge \dots \wedge [x_1^1, \dots, x_1^{n-1}, x_2^k] \wedge \dots \wedge x_2^{n-1} \quad (4.2)$$

for any $x_1 := x_1^1 \wedge \dots \wedge x_1^{n-1}$ and $x_2 := x_2^1 \wedge \dots \wedge x_2^{n-1}$ in $\mathfrak{G}_{n-1}$. Additionally, as demonstrated in [25], $\mathfrak{G}$ can be understood as a representation of the Leibniz algebra $\mathfrak{G}_{n-1}$.

In the realm of n -Lie algebras, a vector space V , together with a linear mapping

$$\mu : \mathfrak{G}_{n-1} \longrightarrow g\ell(V), \quad (4.3)$$

earns the title of a representation of $\mathfrak{G}$ if, as stipulated in [124], the following conditions hold for any $x_1, \dots, x_{2n-2} \in \mathfrak{G}$:

$$\begin{aligned} \mu(x_1, \dots, x_{n-1})\mu(x_n, \dots, x_{2n-2}) - \mu(x_n, \dots, x_{2n-2})\mu(x_1, \dots, x_{n-1}) \\ = \sum_{k=0}^{n-2} \mu(x_n, \dots, [x_1, \dots, x_{n-1}, x_{n+k}], \dots, x_{2n-2}), \end{aligned} \quad (4.4)$$

$$\begin{aligned} \mu([x_1, \dots, x_n], x_{n+1}, \dots, x_{2n-2}) \\ = \sum_{k=1}^n (-1)^k \mu(x_1, \dots, \hat{x}_k, \dots, x_n)\mu(x_{n+1}, \dots, x_{2n-2}, x_k). \end{aligned} \quad (4.5)$$

An immediate and illustrative example of a representation is the adjoint representation of an n -Lie algebra $\mathfrak{G}$ on itself:

$$ad : \mathfrak{G}_{n-1} \longrightarrow gl(\mathfrak{G}), \quad ad(x_1, \dots, x_{n-1})(x_n) := [x_1, \dots, x_n] \quad (4.6)$$

for any $x_1, \dots, x_n$ in $\mathfrak{G}$. To introduce a more substantial example, reminiscent of a dual representation, we document the subsequent generalization of [26] concerning representations of n -Lie algebras [125].

Theorem 4.1. *Given an n -Lie algebra $\mathfrak{G}$, and a representation (V, μ) of $\mathfrak{G}$,*

$$\begin{aligned} \sum_{k=1}^n (-1)^k \mu(x_1, \dots, \widehat{x_k}, \dots, x_n) \mu(x_{n+1}, \dots, x_{2n-2}, x_k) + \\ \sum_{k=1}^n (-1)^k \mu(x_{n+1}, \dots, x_{2n-2}, x_k) \mu(x_1, \dots, \widehat{x_k}, \dots, x_n) = 0 \end{aligned} \quad (4.7)$$

for any $x_1, \dots, x_{2n-2} \in \mathfrak{G}$.

Consider an n -Lie algebra $\mathfrak{G}$ along with a representation (V, η) . Then, the space $\text{Hom}(V, W)$ consisting of linear maps from V to a vector space W also forms a representation of $\mathfrak{G}$ using:

$$\begin{aligned} \mu : \mathfrak{G}_{n-1} \rightarrow gl(\text{Hom}(V, W)), \\ \left(\mu(x_1, \dots, x_{n-1})(T) \right)(v) := -T(\eta(x_1, \dots, x_{n-1})(v)), \end{aligned} \quad (4.8)$$

see also [26, 28].

Let us now visit the cohomology theory for n -Lie algebras, as outlined in [25], with further insights available in [28]. Consider an n -Lie algebra $\mathfrak{G}$, and suppose (V, μ) is a representation of $\mathfrak{G}$. Then,

$$C(\mathfrak{G}, V) := \bigoplus_{m \geq 0} C^m(\mathfrak{G}, V), \quad C^m(\mathfrak{G}, V) := \text{Hom}\left(\mathfrak{G}_{n-1}^{\wedge(m-1)} \wedge \mathfrak{G}, V\right), \quad m \geq 1, \quad (4.9)$$

with $C^0(\mathfrak{G}, V) := \mathfrak{G}_{n-2} \otimes V$, is a differential complex via

$$\begin{aligned} \delta : C^0(\mathfrak{G}, V) &\rightarrow C^1(\mathfrak{G}, V), & \delta(z_1 \wedge \dots \wedge z_{n-2} \otimes v)(y) &:= \mu(z_1, \dots, z_{n-2}, y)(v), \\ \delta : C^m(\mathfrak{G}, V) &\rightarrow C^{m+1}(\mathfrak{G}, V), & m &\geq 1 \end{aligned} \quad (4.10)$$

$$\begin{aligned}
\delta f(x_1, \dots, x_m, y) := & \sum_{1 \leq i < j \leq m} (-1)^i f(x_1, \dots, \widehat{x_i}, \dots, [x_i, x_j], \dots, x_m, y) + \\
& \sum_{i=1}^m (-1)^i f(x_1, \dots, \widehat{x_i}, \dots, x_m, [x_i^1, \dots, x_i^{n-1}, y]) + \\
& \sum_{i=1}^m (-1)^{i+1} \mu(x_i^1, \dots, x_i^{n-1}) \left(f(x_1, \dots, \widehat{x_i}, \dots, x_m, y) \right) + \\
& \sum_{i=1}^{n-1} (-1)^{n-1+m+i} \mu(x_m^1, \dots, \widehat{x_m^i}, \dots, x_m^{n-1}, y) \left(f(x_1, \dots, x_{m-1}, x_m^i) \right),
\end{aligned} \tag{4.11}$$

for any $f \in C^m(\mathfrak{G}, V)$, any $x_i := x_i^1 \wedge \dots \wedge x_i^{n-1} \in \mathfrak{G}_{n-1}$, any $z_1 \wedge \dots \wedge z_{n-2} \in \mathfrak{G}_{n-2}$, and any $y \in \mathfrak{G}$.

The homology $H(C(\mathfrak{G}, V), \delta)$ of the differential complex (4.9) is referred to as the n -Lie algebra cohomology of $\mathfrak{G}$, with coefficients in V , and is given by $H^*(\mathfrak{G}, V)$.

In light of [25], we will now elucidate the connection between the cohomology of an n -Lie algebra $\mathfrak{G}$ and the cohomology of the corresponding associated Leibniz algebra $\mathfrak{G}_{n-1}$.

The following auxiliary result on the representations may be obtained by a routine verification of (3.31), (3.32), and (3.33), and hence the proof is omitted.

Theorem 4.2. *For an n -Lie algebra $\mathfrak{G}$, along with a representation (V, μ) , the linear space $\mathfrak{G}_{n-2} \otimes V$ forms a representation of the Leibniz algebra $\mathfrak{G}_{n-1}$ via*

$$x \triangleright (z_1 \wedge \dots \wedge z_{n-2} \otimes v) \tag{4.12}$$

$$:= \sum_{k=1}^{n-2} z_1 \wedge \dots \wedge [x^1, \dots, x^{n-1}, z_k] \wedge \dots \wedge z_{n-2} \otimes v$$

$$+ z_1 \wedge \dots \wedge z_{n-2} \otimes \mu(x^1, \dots, x^{n-1})(v),$$

$$(z_1 \wedge \dots \wedge z_{n-2} \otimes v) \triangleleft x \tag{4.13}$$

$$:= \sum_{k=1}^{n-1} (-1)^{n+k} x^1 \wedge \dots \wedge \widehat{x^k} \wedge \dots \wedge x^{n-1} \otimes \mu(z_1, \dots, z_{n-2}, x^k)(v).$$

Below, we present the connection between the cohomology of an n -Lie algebra and its associated Leibniz algebra of fundamental objects.

Theorem 4.3. *For an n -Lie algebra $\mathfrak{G}$, along with a representation (V, μ) , the*

diagram

$$\begin{array}{ccc} C^m(\mathfrak{G}, V) & \xrightarrow{\Delta^m} & CL^m(\mathfrak{G}_{n-1}, \mathfrak{G}_{n-2} \otimes V) \\ \delta \downarrow & & \downarrow d \\ C^{m+1}(\mathfrak{G}, V) & \xrightarrow{\Delta^{m+1}} & CL^{m+1}(\mathfrak{G}_{n-1}, \mathfrak{G}_{n-2} \otimes V) \end{array}$$

commutes for any $m \geq 0$, where

$$\Delta^m : C^m(\mathfrak{G}, V) \rightarrow CL^m(\mathfrak{G}_{n-1}, \mathfrak{G}_{n-2} \otimes V), \quad f \mapsto \Delta^m(f),$$

$$\Delta^m(f)(x_1, \dots, x_m)$$

$$:= \sum_{k=1}^{n-1} (-1)^k x_m^1 \wedge \dots \wedge \widehat{x_m^k} \wedge \dots \wedge x_m^{n-1} \otimes f(x_1, \dots, x_{m-1}, x_m^k), \quad m \geq 1,$$

$$\Delta^0 := -Id : \mathfrak{G}_{n-2} \otimes V \rightarrow \mathfrak{G}_{n-2} \otimes V.$$

Proof. We first consider

$$\begin{aligned} & \Delta^{m+1}(\delta f)(x_1, \dots, x_{m+1}) \\ &= \sum_{k=1}^{n-1} (-1)^k x_{m+1}^1 \wedge \dots \wedge \widehat{x_{m+1}^k} \wedge \dots \wedge x_{m+1}^{n-1} \otimes \delta f(x_1, \dots, x_m, x_{m+1}^k) \\ &= \sum_{1 \leq i < j \leq m} \sum_{k=1}^{n-1} (-1)^{i+k} x_{m+1}^1 \wedge \dots \wedge \widehat{x_{m+1}^k} \wedge \dots \wedge x_{m+1}^{n-1} \\ & \quad \otimes f(x_1, \dots, \widehat{x_i}, \dots, [x_i, x_j], \dots, x_m, x_{m+1}^k) \\ &+ \sum_{i=1}^m \sum_{k=1}^{n-1} (-1)^{i+k} x_{m+1}^1 \wedge \dots \wedge \widehat{x_{m+1}^k} \wedge \dots \wedge x_{m+1}^{n-1} \\ & \quad \otimes f(x_1, \dots, \widehat{x_i}, \dots, x_m, [x_i^1, \dots, x_i^{n-1}, x_{m+1}^k]) \\ &+ \sum_{i=1}^m \sum_{k=1}^{n-1} (-1)^{i+1+k} x_{m+1}^1 \wedge \dots \wedge \widehat{x_{m+1}^k} \wedge \dots \wedge x_{m+1}^{n-1} \\ & \quad \otimes \mu(x_i^1, \dots, x_i^{n-1}) \left(f(x_1, \dots, \widehat{x_i}, \dots, x_m, x_{m+1}^k) \right) \\ &+ \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} (-1)^{n-1+m+i+k} x_{m+1}^1 \wedge \dots \wedge \widehat{x_{m+1}^k} \wedge \dots \wedge x_{m+1}^{n-1} \\ & \quad \otimes \mu(x_m^1 \wedge \dots \wedge \widehat{x_m^i} \wedge \dots \wedge x_m^{n-1}, x_{m+1}^k) \left(f(x_1, \dots, x_{m-1}, x_m^i) \right), \end{aligned} \tag{4.14}$$

while at the same time we obtain

$$\begin{aligned}
d(\Delta^m f)(x_1, \dots, x_{m+1}) = & \\
& \sum_{1 \leq i < j \leq m+1} (-1)^i (\Delta^m f)(x_1, \dots, \widehat{x_i}, \dots, [x_i, x_j], \dots, x_{m+1}) + \\
& \sum_{k=1}^m (-1)^{k+1} x_k \triangleright (\Delta^m f)(x_1, \dots, \widehat{x_k}, \dots, x_{m+1}) + \\
& (-1)^{m+1} (\Delta^m f)(x_1, \dots, x_m) \triangleleft x_{m+1} = \\
& \sum_{1 \leq i < j \leq m} (-1)^i (\Delta^m f)(x_1, \dots, \widehat{x_i}, \dots, [x_i, x_j], \dots, x_{m+1}) + \\
& \sum_{1 \leq i \leq m} (-1)^i (\Delta^m f)(x_1, \dots, \widehat{x_i}, \dots, x_m, [x_i, x_{m+1}]) + \\
& \sum_{k=1}^m (-1)^{k+1} x_k \triangleright (\Delta^m f)(x_1, \dots, \widehat{x_k}, \dots, x_{m+1}) + \\
& (-1)^{m+1} (\Delta^m f)(x_1, \dots, x_m) \triangleleft x_{m+1}.
\end{aligned}$$

Then we have

$$\begin{aligned}
d(\Delta^m f)(x_1, \dots, x_{m+1}) = & \\
& \sum_{1 \leq i < j \leq m} \sum_{k=1}^{n-1} (-1)^{i+k} x_{m+1}^1 \wedge \dots \wedge \widehat{x_{m+1}^k} \wedge \dots \wedge x_{m+1}^{n-1} \otimes \\
& f(x_1, \dots, \widehat{x_i}, \dots, [x_i, x_j], \dots, x_{m+1}^k) + \\
& \sum_{p=1}^{n-1} \sum_{\substack{k=1 \\ k \neq p}}^{n-1} \sum_{i=1}^m (-1)^{k+i} x_{m+1}^1 \wedge \dots \wedge \widehat{x_{m+1}^k} \wedge \dots \wedge [x_i^1 \dots, x_i^{n-1}, x_{m+1}^p] \wedge \dots \\
& \dots \wedge x_{m+1}^{n-1} \otimes f(x_1, \dots, \widehat{x_i}, \dots, x_m, x_{m+1}^k) + \\
& \sum_{p=1}^{n-1} \sum_{i=1}^m (-1)^{p+i} x_{m+1}^1 \wedge \dots \wedge \widehat{x_{m+1}^p} \wedge \dots \wedge x_{m+1}^{n-1} \otimes \\
& f(x_1, \dots, \widehat{x_i}, \dots, x_m, [x_i^1 \dots, x_i^{n-1}, x_{m+1}^p]) + \\
& \sum_{i=1}^m (-1)^{i+1} x_i \triangleright (\Delta^m f)(x_1, \dots, \widehat{x_i}, \dots, x_{m+1}) + \\
& (-1)^{m+1} (\Delta^m f)(x_1, \dots, x_m) \triangleleft x_{m+1}.
\end{aligned}$$

Later, we compute

$$\begin{aligned}
& \sum_{i=1}^m (-1)^{i+1} x_i \triangleright (\Delta^m f)(x_1, \dots, \widehat{x}_i, \dots, x_{m+1}) = \\
& \sum_{k=1}^{n-1} \sum_{i=1}^m (-1)^{i+1+k} x_i \triangleright \left(x_{m+1}^1 \wedge \dots \wedge \widehat{x_{m+1}^k} \wedge \dots \wedge x_{m+1}^{n-1} \otimes \right. \\
& \qquad \qquad \qquad \left. f(x_1, \dots, \widehat{x}_i, \dots, x_{m+1}^k) \right) = \\
& \sum_{\substack{p=1 \\ p \neq k}}^{n-1} \sum_{k=1}^{n-1} \sum_{i=1}^m (-1)^{i+1+k} x_{m+1}^1 \wedge \dots \wedge \widehat{x_{m+1}^k} \wedge \dots \wedge [x_i^1, \dots, x_i^{n-1}, x_{m+1}^p] \wedge \dots \\
& \qquad \qquad \qquad \dots \wedge x_{m+1}^{n-1} \otimes f(x_1, \dots, \widehat{x}_i, \dots, x_m, x_{m+1}^k) + \\
& \sum_{k=1}^{n-1} \sum_{i=1}^m (-1)^{i+1+k} x_{m+1}^1 \wedge \dots \wedge \widehat{x_{m+1}^k} \wedge \dots \wedge x_{m+1}^{n-1} \otimes \\
& \qquad \qquad \qquad \mu(x_i^1, \dots, x_i^{n-1}) \left(f(x_1, \dots, \widehat{x}_i, \dots, x_{m+1}^k) \right),
\end{aligned}$$

and that

$$\begin{aligned}
& (-1)^{m+1} (\Delta^m f)(x_1, \dots, x_m) \triangleleft x_{m+1} = \\
& \sum_{k=1}^{n-1} (-1)^{m+1+k} \left(x_m^1 \wedge \dots \wedge \widehat{x_m^k} \wedge \dots \wedge x_m^{n-1} \otimes f(x_1, \dots, x_{m-1}, x_m^k) \right) \triangleleft x_{m+1} = \\
& \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} (-1)^{n+i+m+1+k} x_{m+1}^1 \wedge \dots \wedge \widehat{x_{m+1}^i} \wedge \dots \\
& \qquad \qquad \qquad \dots \wedge x_{m+1}^{n-1} \otimes \mu(x_m^1 \wedge \dots \wedge \widehat{x_m^k} \wedge \dots \wedge x_m^{n-1}, x_{m+1}^i) \left(f(x_1, \dots, x_{m-1}, x_m^k) \right).
\end{aligned}$$

So, we get the required result. $\square$

4.1.2. Generalized Derivations and Classifications

In this section, we aim to elucidate the cohomological classification of generalized derivations within n -Lie algebras. Notably, for the case of $n = 3$, our findings subsume those in [31]. To delve deeper, we will depict a generalized derivation of an n -Lie algebra as a 1-cocycle in a cohomology theory associated with n -Lie algebras, while an inner generalized derivation will be represented as a 1-coboundary.

An n -Lie algebra $\mathfrak{G}$ defines a linear operator $D : \mathfrak{G} \rightarrow \mathfrak{G}$ as a *derivation* if it satisfies

$$D([x_1, \dots, x_n]) = \sum_{k=1}^n [x_1, \dots, D(x_k), \dots, x_n]$$

for any $x_1, \dots, x_n \in \mathfrak{G}$. Notably, the fundamental identity (4.1) immediately implies that

$$ad(x_1, \dots, x_{n-1}) : \mathfrak{G} \rightarrow \mathfrak{G}$$

defined by (4.6) is a derivation for any $x_1, \dots, x_{n-1} \in \mathfrak{G}$. These derivations are known as inner derivations.

A generalized derivation of an n -Lie algebra $\mathfrak{G}$, however, is a linear map $D : \mathfrak{G}_{n-1} \rightarrow \mathfrak{G}$ that fulfills the following conditions:

$$D(x_1, \dots, x_{n-2}, [x_{n-1}, \dots, x_{2n-2}]) = \sum_{k=0}^{n-1} [x_{n-1}, \dots, D(x_1, \dots, x_{n-2}, x_{n-1+k}), \dots, x_{2n-2}], \quad (4.15)$$

$$\sum_{k=0}^{n-2} D(x_n, \dots, [x_1, \dots, x_{n-1}, x_{n+k}], \dots, x_{2n-2}) = [x_1, \dots, x_{n-1}, D(x_n, \dots, x_{2n-2})] + (-1)^n [D(x_1, \dots, x_{n-1}), x_n, \dots, x_{2n-2}] \quad (4.16)$$

$$D(x_1, \dots, x_{n-2}, D(x_{n-1}, \dots, x_{2n-3})) = \sum_{k=0}^{n-2} D(x_{n-1}, \dots, D(x_1, \dots, x_{n-2}, x_{n-1+k}), \dots, x_{2n-3}), \quad (4.17)$$

for any $x_1, \dots, x_{n-1} \in \mathfrak{G}$ [28].

As noted in [28], it is immediately evident that the map $ad_y : \mathfrak{G}_{n-1} \rightarrow \mathfrak{G}$ defined as

$$ad_y(z_1, \dots, z_{n-1}) := [y, z_1, \dots, z_{n-1}] \quad (4.18)$$

qualifies as a generalized derivation for any $y \in \mathfrak{G}$. This type of generalized derivation is termed an inner generalized derivation.

Note that if $D : \mathfrak{G}_{n-1} \rightarrow \mathfrak{G}$ is a generalized derivation, then $\mathfrak{G} \oplus_D k := \mathfrak{G} \oplus k$ forms an n -Lie algebra using the operation:

$$[x_1 + \alpha_1 D, \dots, x_n + \alpha_n D] := [x_1, \dots, x_n] + \sum_{k=1}^n (-1)^{k+1} \alpha_k D(x_1, \dots, \widehat{x_k}, \dots, x_n) \quad (4.19)$$

This applies for any $x_1 + \alpha_1 D, \dots, x_n + \alpha_n D \in \mathfrak{G} \oplus_D k$. Let us discuss the conditions stated in equations (4.15) and (4.17).

In the context of the equation labeled as (4.20), we can easily observe that (4.15) is synonymous with $D^\sharp(x_1, \dots, x_{n-2})$ being a derivation of the n -Lie algebra $\mathfrak{G}$. This

holds true for all $x_1, \dots, x_{n-2}, y$ belonging to $\mathfrak{G}$, where $D^\sharp$ is defined as follows:

$$D^\sharp : \mathfrak{G}_{n-2} \rightarrow g\ell(\mathfrak{G}), \quad D^\sharp(x_1, \dots, x_{n-2})(y) := D(x_1, \dots, x_{n-2}, y). \quad (4.20)$$

The equivalence of condition (4.17) for the n -Lie algebra $\mathfrak{G}$ with the structure of an $(n-1)$ -Lie algebra can be expressed through the mapping defined as

$$D : \mathfrak{G}_{n-1} \rightarrow \mathfrak{G}, \quad x_1 \wedge \dots \wedge x_{n-1} \mapsto D(x_1, \dots, x_{n-1}).$$

Concerning condition (4.16), we can draw an analogy to [31]. Prior to that, it is beneficial to define the semi-direct sum Leibniz algebra $\mathfrak{G} \rtimes \mathfrak{G}_{n-1} := \mathfrak{G} \oplus \mathfrak{G}_{n-1}$ corresponding to an n -Lie algebra $\mathfrak{G}$.

To start, given that $\mathfrak{G}$ is an adjoint representation over itself according to (4.6), it naturally becomes a representation of the Leibniz algebra $\mathfrak{G}_{n-1}$ via the operations:

$$x \triangleright y := [x^1, \dots, x^{n-1}, y], \quad y \triangleleft x := -[x^1, \dots, x^{n-1}, y],$$

for any $y \in \mathfrak{G}$ and any $x := x^1 \wedge \dots \wedge x^{n-1} \in \mathfrak{G}_{n-1}$. Consequently, $\mathfrak{G} \oplus \mathfrak{G}_{n-1}$ forms a Leibniz algebra defined by:

$$[y_1 + x_1, y_2 + x_2]_s := [x_1^1, \dots, x_1^{n-1}, y_2] - [x_2^1, \dots, x_2^{n-1}, y_1] + [x_1, x_2],$$

for any $y_1, y_2 \in \mathfrak{G}$, and any $x_1 := x_1^1 \wedge \dots \wedge x_1^{n-1}, x_2 := x_2^1 \wedge \dots \wedge x_2^{n-1} \in \mathfrak{G}_{n-1}$, where

$$[x_1, x_2] := \sum_{k=1}^{n-1} x_2^1 \wedge \dots \wedge [x_1^1, \dots, x_1^{n-1}, x_2^k] \wedge \dots \wedge x_2^{n-1} \in \mathfrak{G}_{n-1}.$$

As a result, [31] extends directly to the following case.

Theorem 4.4. *Given an n -Lie algebra $\mathfrak{G}$ and a generalized derivation $D : \mathfrak{G}_{n-1} \rightarrow \mathfrak{G}$, define the mapping $\hat{D} : \mathfrak{G} \rtimes \mathfrak{G}_{n-1} \rightarrow \mathfrak{G} \rtimes \mathfrak{G}_{n-1}$ as follows:*

$$\hat{D} : \mathfrak{G} \rtimes \mathfrak{G}_{n-1} \rightarrow \mathfrak{G} \rtimes \mathfrak{G}_{n-1}, \quad \hat{D}(y + x) := D(x),$$

for any $y \in \mathfrak{G}$ and any $x := x^1 \wedge \dots \wedge x^{n-1} \in \mathfrak{G}_{n-1}$. This $\hat{D}$ is then a derivation of Leibniz algebras.

The cohomological classification of generalized derivations. Here we will delve into the main result of this section, introducing a novel cohomology theory tailored for

n -Lie algebras, which characterizes generalized derivations of n -Lie algebras as 1-cocycles. To kick things off, let's lay out the differential complex designed for n -Lie algebras.

Let $\mathfrak{G}$ be an n -Lie algebra, and let (V, μ) be a representation of $\mathfrak{G}$. Let also

$$\mathcal{C}(\mathfrak{G}, V) := \bigoplus_{m \geq 0} \mathcal{C}^m(\mathfrak{G}, V), \quad \mathcal{C}^m(\mathfrak{G}, V) := \text{Hom}\left(\mathfrak{G}_{n-1}^{\wedge(m-1)} \wedge \mathfrak{G}_{n-2}, V\right), \quad m \geq 1, \quad (4.21)$$

with $\mathcal{C}^0(\mathfrak{G}, V) := \mathfrak{G} \otimes V$, and

$$\begin{aligned} \delta : \mathcal{C}^0(\mathfrak{G}, V) &\rightarrow \mathcal{C}^1(\mathfrak{G}, V), & \delta(z \otimes v)(y) &:= \mu(z, y^1, \dots, y^{n-2})(v), \\ \delta : \mathcal{C}^m(\mathfrak{G}, V) &\rightarrow \mathcal{C}^{m+1}(\mathfrak{G}, V), & m &\geq 1 \\ \delta f(x_1, \dots, x_m, y) &:= \\ &\sum_{1 \leq i < j \leq m} (-1)^i f(x_1, \dots, \widehat{x_i}, \dots, [x_i, x_j], \dots, x_m, y) + \\ &\sum_{i=1}^m (-1)^i f(x_1, \dots, \widehat{x_i}, \dots, x_m, [x_i, y]) + \\ &\sum_{i=1}^m (-1)^{i+1} \mu(x_i^1, \dots, x_i^{n-1}) \left(f(x_1, \dots, \widehat{x_i}, \dots, x_m, y) \right) + \\ &\sum_{i=1}^{n-1} (-1)^{m+i+1} \mu(x_m^i, y^1, \dots, y^{n-2}) \left(f(x_1, \dots, x_{m-1}, X_m^i) \right), \end{aligned} \quad (4.22)$$

for any $f \in \mathcal{C}^m(\mathfrak{G}, V)$, any $x_i := x_i^1 \wedge \dots \wedge x_i^{n-1} \in \mathfrak{G}_{n-1}$, and any $y := y^1 \wedge \dots \wedge y^{n-2} \in \mathfrak{G}_{n-2}$, where $X_m^i := x_m^1 \wedge \dots \wedge \widehat{x_m^i} \wedge \dots \wedge x_m^{n-1} \in \mathfrak{G}_{n-2}$,

$$[x_i, x_j] := \sum_{k=1}^{n-1} x_j^1 \wedge \dots \wedge [x_i^1, \dots, x_i^{n-1}, x_j^k] \wedge \dots \wedge x_j^{n-1} \in \mathfrak{G}_{n-1},$$

and

$$[x_i, y] := \sum_{k=1}^{n-2} y^1 \wedge \dots \wedge [x_i^1, \dots, x_i^{n-1}, y^k] \wedge \dots \wedge y^{n-2} \in \mathfrak{G}_{n-2}.$$

Let us verify that (4.22) indeed forms a differential. To begin with, we recall from Proposition 4.2 that $\mathfrak{G} \otimes V$ is a representation via the operations

$$x \triangleright (z \otimes v) := [x^1, \dots, x^{n-1}, z] \otimes v + z \otimes \mu(x^1, \dots, x^{n-1})(v), \quad (4.23)$$

$$(z \otimes v) \triangleleft x := \sum_{k=1}^{n-1} (-1)^k x^k \wedge \mu(z, X^k)(v), \quad (4.24)$$

where $x := x^1 \wedge \dots \wedge x^{n-1} \in \mathfrak{G}_{n-1}$, and $X^k := x^1 \wedge \dots \wedge \widehat{x^k} \wedge \dots \wedge x^{n-1} \in \mathfrak{G}_{n-2}$.

Now, let's establish the following analogue of Proposition 4.3.

Theorem 4.5. For an n -Lie algebra $\mathfrak{G}$, along with a representation (V, μ) , the diagram

$$\begin{array}{ccc} \mathcal{C}^m(\mathfrak{G}, V) & \xrightarrow{\Theta^m} & CL^m(\mathfrak{G}_{n-1}, \mathfrak{G} \otimes V) \\ \delta \downarrow & & \downarrow d \\ \mathcal{C}^{m+1}(\mathfrak{G}, V) & \xrightarrow{\Theta^{m+1}} & CL^{m+1}(\mathfrak{G}_{n-1}, \mathfrak{G} \otimes V) \end{array}$$

commutes for any $m \geq 0$, where

$$\begin{aligned} \Theta^m : \mathcal{C}^m(\mathfrak{G}, V) &\rightarrow CL^m(\mathfrak{G}_{n-1}, \mathfrak{G} \otimes V), & f &\mapsto \Theta^m(f), \\ \Theta^m(f)(x_1, \dots, x_m) &:= \sum_{k=1}^{n-1} (-1)^{k+1} x_m^k \otimes f(x_1, \dots, x_{m-1}, X_m^k), & m &\geq 1, \quad (4.25) \\ \Theta^0 &:= -Id : \mathfrak{G} \otimes V \rightarrow \mathfrak{G} \otimes V, \end{aligned}$$

and $X_m^k := x_m^1 \wedge \dots \wedge \widehat{x_m^k} \wedge \dots \wedge x_m^{n-1} \in \mathfrak{G}_{n-2}$.

Proof. We begin with

$$\begin{aligned} \Theta^{m+1}(\delta f)(x_1, \dots, x_{m+1}) &= \sum_{k=1}^{n-1} (-1)^{k+1} x_{m+1}^k \otimes (\delta f)(x_1, \dots, x_m, X_{m+1}^k) = \\ &\sum_{k=1}^{n-1} \sum_{1 \leq i < j \leq m} (-1)^{i+k+1} x_{m+1}^k \otimes f(x_1, \dots, \widehat{x_i}, \dots, [x_i, x_j], \dots, x_m, X_{m+1}^k) + \\ &\sum_{k=1}^{n-1} \sum_{i=1}^m (-1)^{i+k+1} x_{m+1}^k \otimes f(x_1, \dots, \widehat{x_i}, \dots, x_m, [x_i, X_{m+1}^k]) + \\ &\sum_{k=1}^{n-1} \sum_{i=1}^m (-1)^{i+k} x_{m+1}^k \otimes \mu(x_i^1, \dots, x_i^{n-1}) \left(f(x_1, \dots, \widehat{x_i}, \dots, x_m, X_{m+1}^k) \right) + \\ &\sum_{k=1}^{n-1} \sum_{i=1}^{n-1} (-1)^{i+k+m} x_{m+1}^k \otimes \mu(x_m^i, X_{m+1}^k) \left(f(x_1, \dots, x_m, X_m^i) \right), \end{aligned}$$

meanwhile, we obtain

$$\begin{aligned} d(\Theta^m(f))(x_1, \dots, x_{m+1}) &= \\ &\sum_{1 \leq i < j \leq m+1} (-1)^i (\Theta^m(f))(x_1, \dots, \widehat{x_i}, \dots, [x_i, x_j], \dots, x_{m+1}) + \\ &\sum_{i=1}^m (-1)^{i+1} x_i \triangleright \left((\Theta^m(f))(x_1, \dots, \widehat{x_i}, \dots, x_{m+1}) \right) + \\ &(-1)^{m+1} \left((\Theta^m(f))(x_1, \dots, x_m) \right) \triangleleft x_{m+1}. \end{aligned}$$

It is worth mentioning that

$$\begin{aligned}
& \sum_{1 \leq i < j \leq m+1} (-1)^i (\Theta^m(f))(x_1, \dots, \widehat{x_i}, \dots, [x_i, x_j], \dots, x_{m+1}) = \\
& \sum_{1 \leq i < j \leq m} (-1)^i (\Theta^m(f))(x_1, \dots, \widehat{x_i}, \dots, [x_i, x_j], \dots, x_{m+1}) + \\
& \sum_{i=1}^m (-1)^i (\Theta^m(f))(x_1, \dots, \widehat{x_i}, \dots, [x_i, x_{m+1}]) = \\
& \sum_{k=1}^{n-1} \sum_{1 \leq i < j \leq m} (-1)^{i+k+1} x_{m+1}^k \otimes f(x_1, \dots, \widehat{x_i}, \dots, [x_i, x_j], \dots, x_m, X_{m+1}^k) + \\
& \sum_{k=1}^{n-1} \sum_{i=1}^m (-1)^{i+k+1} x_{m+1}^k \otimes f(x_1, \dots, \widehat{x_i}, \dots, [x_i, X_{m+1}^k]) + \\
& \sum_{k=1}^{n-1} \sum_{i=1}^m (-1)^{i+k+1} [x_i^1, \dots, x_i^{n-1}, x_{m+1}^k] \otimes f(x_1, \dots, \widehat{x_i}, \dots, X_{m+1}^k).
\end{aligned}$$

The conclusion can be derived from (4.23) and (4.24). Specifically,

$$\begin{aligned}
& \sum_{i=1}^m (-1)^{i+1} x_i \triangleright \left((\Theta^m(f))(x_1, \dots, \widehat{x_i}, \dots, x_{m+1}) \right) = \\
& \sum_{k=1}^{n-1} \sum_{i=1}^m (-1)^{i+k} [x_i^1, \dots, x_i^{n-1}, x_{m+1}^k] \otimes f(x_1, \dots, \widehat{x_i}, \dots, X_{m+1}^k) + \\
& \sum_{k=1}^{n-1} \sum_{i=1}^m (-1)^{i+k} x_{m+1}^k \otimes \mu(x_i^1, \dots, x_i^{n-1}) \left(f(x_1, \dots, \widehat{x_i}, \dots, x_m, X_{m+1}^k) \right),
\end{aligned}$$

and

$$\begin{aligned}
& (-1)^{m+1} \left((\Theta^m(f))(x_1, \dots, x_m) \right) \triangleleft x_{m+1} = \\
& (-1)^{m+1} \left(\sum_{k=1}^{n-1} (-1)^{k+1} x_m^k \otimes f(x_1, \dots, x_{m-1}, X_m^k) \right) \triangleleft x_{m+1} \\
& \sum_{k=1}^{n-1} \sum_{i=1}^{n-1} (-1)^{i+k+m} x_{m+1}^i \otimes \mu(x_m^k, X_{m+1}^i) \left(f(x_1, \dots, x_m, X_m^k) \right).
\end{aligned}$$

□

For an n -Lie algebra $\mathfrak{G}$ along with a representation (V, μ) , the map $\delta : \mathcal{C}^n(\mathfrak{G}, V) \rightarrow \mathcal{C}^{n+1}(\mathfrak{G}, V)$ defined in (4.22) for $n \geq 0$ is a differential map, that is, it satisfies $\delta^2 = 0$. This property can be readily inferred from the injectivity of (4.25).

We will use $\mathcal{H}^*(\mathfrak{G}, V)$ to represent the homology $H(\mathcal{C}(\mathfrak{G}, V), \delta)$ of the differential complex (4.21). Following convention, $\mathcal{Z}^m(\mathfrak{G}, V)$ refers to the space of m -cocycles, while $\mathcal{B}^m(\mathfrak{G}, V)$ denotes the space of m -coboundaries.

It is noteworthy that the differentials (4.10) and (4.22), as well as the differential complexes (4.9) and (4.21), coincide only for $n = 3$.

Now, we are set to prove our main result.

Theorem 4.6. *For an n -Lie algebra $\mathfrak{G}$, suppose we have a linear mapping $D : \mathfrak{G}_{n-1} \rightarrow \mathfrak{G}$. Let $D^\sharp : \mathfrak{G}_{n-2} \rightarrow g\ell(\mathfrak{G})$ denote the corresponding mapping defined by (4.20). In this context, $D^\sharp$ belongs to $\mathcal{Z}^1(\mathfrak{G}, g\ell(\mathfrak{G}))$ if condition (4.16) holds true.*

Proof. For any $x = x^1 \wedge \cdots \wedge x^{n-1} \in \mathfrak{G}_{n-1}$, any $y = y^1 \wedge \cdots \wedge y^{n-2} \in \mathfrak{G}_{n-2}$, and any $z \in \mathfrak{G}$, we can express $(\delta D^\sharp)(x, y)(z)$ as

$$\begin{aligned} (\delta D^\sharp)(x, y)(z) &= -D^\sharp([x, y])(z) + \left(\mu(x^1, \dots, x^{n-1})(D^\sharp(y)) \right)(z) + \\ &\quad \sum_{i=1}^{n-1} (-1)^i \left(\mu(x^i, y)(D^\sharp(X^i)) \right)(z), \end{aligned}$$

where $X^i := x^1 \wedge \cdots \wedge \widehat{x^i} \wedge \cdots \wedge x^{n-1} \in \mathfrak{G}_{n-2}$, and

$$[x, y] := \sum_{k=1}^{n-2} y^1 \wedge \cdots \wedge [x^1, \dots, x^{n-1}, y^k] \wedge \cdots \wedge y^{n-2} \in \mathfrak{G}_{n-2}.$$

Expanding further, we have

$$\begin{aligned} (\delta D^\sharp)(x, y)(z) &= -D([x, y], z) - D^\sharp(y)([x, z]) + \sum_{i=1}^{n-1} (-1)^{i+1} D^\sharp(X^i)([x^i, y, z]) = \\ &= -D([x, y], z) - D(y, [x, z]) - \sum_{i=1}^{n-1} D(x^1, \dots, [y, z, x^i], \dots, x^{n-1}). \end{aligned}$$

If condition (4.16) holds, then we simplify to

$$\begin{aligned} (\delta D^\sharp)(x, y)(z) &= \\ &= - \left([x, D(y, z)] + (-1)^n [D(x), y, z] \right) - \left([y, z, D(x)] + (-1)^n [D(y, z), x] \right) = 0. \end{aligned}$$

□

For $n = 3$, assuming that $D : \mathfrak{G}_2 \rightarrow \mathfrak{G}$ satisfies (4.15), we can also deduce that $D^\sharp \in \mathcal{Z}^1(\mathfrak{G}, g\ell(\mathfrak{G}))$ if and only if (4.16) is satisfied.

Given $D : \mathfrak{G}_{n-1} \rightarrow \mathfrak{G}$, it is immediate from the definition of $\delta : \mathcal{C}^0(\mathfrak{G}, V) \rightarrow \mathcal{C}^1(\mathfrak{G}, V)$ that $D^\sharp \in \mathcal{B}^1(\mathfrak{G}, g\ell(\mathfrak{G}))$ if and only if there exists $\xi \otimes f \in \mathfrak{G} \otimes g\ell(\mathfrak{G})$ such that $D^\sharp = \delta(\xi \otimes f)$, or equivalently

$$D(y, z) = D^\sharp(y)(z) = \left(\mu(\xi, y)(f) \right)(z) = -f([\xi, y, z]) = ad_\xi(f)(y, z).$$

Thus, we can state that $D^\sharp \in \mathcal{B}^1(\mathfrak{G}, g\ell(\mathfrak{G}))$ if and only if $D = ad_\xi(f)$ for some $\xi \in \mathfrak{G}$ and $f \in g\ell(\mathfrak{G})$. This characterization is analogous to the definition of an

inner generalized derivation in [28] and [31], where inner generalized derivations $D : \mathfrak{G}_{n-1} \rightarrow \mathfrak{G}$ correspond to those satisfying $D^\sharp = \delta(\xi \otimes Id)$ for some $\xi \in \mathfrak{G}$, with $Id \in gl(\mathfrak{G})$ representing the identity transformation.

4.1.3. Nambu and Bi-Hamiltonian Dynamics

Consider a manifold P and the space $C^\infty(P)$ of smooth functions on P equipped with k multilinear mapping

$$\{\bullet, \bullet, \dots, \bullet\} : C^\infty(P) \times \dots \times C^\infty(P) \longrightarrow C^\infty(P). \quad (4.26)$$

We assume that the bracket is skew-symmetric that is

$$\{F_1, \dots, F_k\} = (-1)^{\epsilon(\sigma)} \{F_{\sigma(1)}, \dots, F_{\sigma(k)}\}, \quad (4.27)$$

where $\sigma \in S_k$ -the symmetric group of k elements- and $\epsilon(\sigma)$ is its parity.

The bracket (4.26) is called a Nambu-Poisson bracket of order k if the following axioms hold for all $F_1, F_2, \dots, F_{2k-1}$ in $C^\infty(P)$ are satisfied.

- Leibniz identity:

$$\begin{aligned} \{F_1 F_2, F_3, \dots, F_{k+1}\} &= F_1 \{F_2, F_3, \dots, F_{k+1}\} \\ &+ \{F_1, F_3, \dots, F_{k+1}\} F_2. \end{aligned} \quad (4.28)$$

- Fundamental (also called Fillippov or Takhtajan) identity

$$\begin{aligned} &\{\{F_1, \dots, F_{k-1}, F_k\}, F_{k+1}, \dots, F_{2k-1}\} \\ &+ \{F_k, \{F_1, \dots, F_{k-1}, F_{k+1}\}, F_{k+2}, \dots, F_{2k-1}\} \\ &+ \dots + \{F_k, \dots, F_{2k-2}, \{F_1, \dots, F_{k-1}, F_{2k-1}\}\} \\ &= \{F_1, \dots, F_{k-1}, \{F_k, \dots, F_{2k-1}\}\}. \end{aligned} \quad (4.29)$$

A manifold admitting such a bracket is called a Nambu-Poisson manifold of order k , [19, 21, 25, 126, 127]. Notice that, for $k = 2$ this definition reduces to a Poisson bracket. In this case, (4.29) turns out to be the Jacobi identity in its very classical sense.

Nambu-Poisson Multivector Field. A Nambu-Poisson bracket can be realized by a Nambu-Poisson multivector field $\eta \in \Gamma(\wedge^k TP)$, a section of $\wedge^k TP$, such that

$$\eta(dF_1, \dots, dF_k) := \{F_1, \dots, F_k\}. \quad (4.30)$$

In this case, η is Nambu-Poisson k -multivector field. As announced in [128, 129], η is a Nambu-Poisson k -multivector field if and only if

$$[\iota_\psi \eta, \eta] = (-1)^k (\iota_{d\psi} \eta) \eta \quad (4.31)$$

for any $(k-1)$ -form ψ . Here, $\iota_\psi \eta$ refers to the contraction of the k multivector field η with the differential form of order $(k-1)$. So that $\iota_\psi \eta$ is a vector field. Accordingly, the bracket on the left hand side is the Lie derivative of the multivector field η with the vector field $\iota_\psi \eta$.

Contractions to Poisson Framework. Consider a manifold P equipped with a k -bracket $\{\bullet, \dots, \bullet\}$. It is a Nambu-Poisson manifold of order k if and only if its contraction

$$\eta_F = \iota_{dF} \eta \quad (4.32)$$

is a Nambu-Poisson manifold of order $k-1$ for all smooth F on P . One can iteratively carry this reduction up to bivector fields. So, we can argue that a k -bracket $\{\bullet, \dots, \bullet\}$ is a Nambu-Poisson bracket if its all contractions with $k-2$ functions read a Poisson bivector that is

$$[\eta_{F_1 F_2 \dots F_{k-2}}, \eta_{F_1 F_2 \dots F_{k-2}}] = 0 \quad (4.33)$$

for all smooth functions $F_1, \dots, F_{k-2}$ on P . Here, the bracket $[\bullet, \bullet]$ is Schouten bracket. The contraction is defined to be

$$\eta_{F_1 \dots F_{k-2}} := \iota_{dF_{k-2}} \dots \iota_{dF_2} \iota_{dF_1} \eta \quad (4.34)$$

We record the case for ternary algebras for future reference. A 3-vector field η is Nambu-Poisson if and only if

$$[\eta_F, \eta_F] = 0 \quad (4.35)$$

for any smooth function F in $C^\infty(P)$.

Nambu-Hamiltonian Dynamics. Consider a Nambu-Poisson manifold $(P, \{\bullet, \dots, \bullet\})$. The Nambu-Hamiltonian dynamics on P involves $(k-1)$ -number of Nambu-Hamiltonian functions $(\mathcal{H}_1, \dots, \mathcal{H}_{k-1})$ in $C^\infty(P)$. The dynamics is governed by the Nambu-Hamilton's equation

$$\frac{dx}{dt} = \{x, \mathcal{H}_1, \dots, \mathcal{H}_{k-1}\}. \quad (4.36)$$

The corresponding Hamiltonian vector field is defined to be

$$X_{\mathcal{H}_1, \dots, \mathcal{H}_{k-1}}(F) = \{F, \mathcal{H}_1, \dots, \mathcal{H}_{k-1}\}. \quad (4.37)$$

A solution to the Nambu-Hamilton's equations of motion produces an evolution operator which by virtue of the Takhtajan identity preserves the Nambu bracket structure. According to that, a Nambu-Hamiltonian vector field preserves the Nambu-Poisson k multivector field η that is

$$[X_{\mathcal{H}_1, \dots, \mathcal{H}_{k-1}}, \eta] = 0 \quad (4.38)$$

where the bracket here is the Schouten-Nijenhuis bracket. This reads that the characteristic distribution, generated by all the Hamiltonian vector fields, is involutive

$$[X_{\mathcal{H}_1, \dots, \mathcal{H}_{n-1}}, X_{F_1, \dots, F_{n-1}}] = \sum_{i=1}^{n-1} X_{F_1, \dots, \{\mathcal{H}_1, \dots, \mathcal{H}_{n-1}, F_i\}, \dots, F_{n-1}}. \quad (4.39)$$

Volume Manifold. In the first place, we assume that a manifold M is paracompact and that the dimension of M is equal to the order of η . Recall that any n -vector field η on an n -dimensional manifold becomes a Nambu-Poisson tensor by the result of [24]. Moreover η is assumed to be nowhere vanishing on M . Thus in our case, η is non-vanishing Nambu-Poisson tensor of order n . The volume form Ω corresponding to η , can be defined as follows [3]:

$$X_1 \wedge \dots \wedge X_n = (-1)^{n-1} \Omega(X_1, \dots, X_n) \eta$$

for all $X_1, \dots, X_n \in T_x(M)$.

Bi-Hamiltonian Dynamics. A dynamical system is called bi-Hamiltonian if it admits two different Hamiltonian structures

$$\frac{du}{dt} = \{u, \mathcal{H}_2\}_1 = \{u, \mathcal{H}_1\}_2 \quad (4.40)$$

with the requirement that the Poisson brackets $\{\bullet, \bullet\}_1$ and $\{\bullet, \bullet\}_2$ be compatible, that is

$$\{\bullet, \bullet\}_1 + c\{\bullet, \bullet\}_2 \quad (4.41)$$

must satisfy the Jacobi identity for any constant c , [130, 131]. In three dimensions, an autonomous bi-Hamiltonian system is superintegrable. By fixing the Hamiltonian functions H_1 and H_2 , we can write Nambu-Hamiltonian system (4.36) in the

bi-Hamiltonian form, see, for example [132, 133],

$$\frac{du}{dt} = \{u, \mathcal{H}_1\}^{\mathcal{H}_2} = \{u, \mathcal{H}_2\}^{\mathcal{H}_1} \quad (4.42)$$

where the Poisson brackets $\{\bullet, \bullet\}^{\mathcal{H}_2}$ and $\{\bullet, \bullet\}^{\mathcal{H}_1}$ are defined by

$$\{F, \mathcal{H}\}^{\mathcal{H}_2} = \{F, \mathcal{H}, \mathcal{H}_2\}, \quad \{F, \mathcal{H}\}^{\mathcal{H}_1} = \{F, \mathcal{H}_1, \mathcal{H}\}, \quad (4.43)$$

respectively. Both of the brackets $\{\bullet, \bullet\}^{\mathcal{H}_2}$ and $\{\bullet, \bullet\}^{\mathcal{H}_1}$ satisfy the Jacobi and Leibniz identities.

Nambu Dynamics on the dual of a 3-Lie Algebras. A 3-Lie algebra is a vector space $\mathfrak{g}$ equipped with a skew symmetric ternary algebra

$$[\bullet, \bullet, \bullet] : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}, \quad (4.44)$$

called as the 3-Lie bracket, satisfying the fundamental identity [21]

$$[\xi_1, \xi_2, [\xi_3, \xi_4, \xi_5]] = [[\xi_1, \xi_2, \xi_3], \xi_4, \xi_5] + [\xi_3, [\xi_1, \xi_2, \xi_4], \xi_5] + [\xi_3, \xi_4, [\xi_1, \xi_2, \xi_5]]$$

for all $\xi_1, \dots, \xi_5$ in $\mathfrak{g}$.

For two 3-Lie algebra elements (ξ_1, ξ_2) we define adjoint operator as follows

$$ad_{(\xi_1, \xi_2)} : \mathfrak{g} \longrightarrow \mathfrak{g}, \quad ad_{(\xi_1, \xi_2)}(\xi) = [\xi_1, \xi_2, \xi]. \quad (4.45)$$

Note that $ad_{(\xi_1, \xi_2)}$ is a derivation thanks to the fundamental identity. Let us denote the linear algebraic dual of the 3-Lie algebra $\mathfrak{g}$ by $\mathfrak{g}^*$. Then, we can define the coadjoint operator

$$ad_{(\xi_1, \xi_2)}^* : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*, \quad \langle ad_{(\xi_1, \xi_2)}^*(\mu), \xi \rangle = -\langle \mu, ad_{(\xi_1, \xi_2)}(\xi) \rangle. \quad (4.46)$$

Let $\mathfrak{g}$ be a finite-dimensional 3-Lie algebra spanned by the vectors (e_α) . We determine the structure constants of this algebra as:

$$[e_\alpha, e_\beta, e_\gamma] = \mathfrak{C}_{\alpha\beta\gamma}^\theta e_\theta \quad (4.47)$$

Here, the set $\mathfrak{C}_{\alpha\beta\gamma}^\theta$ determines the structure constants of the 3-Lie subalgebra $\mathfrak{g}$.

The duals of Lie algebras are Poisson spaces called Lie-Poisson manifolds. However, for 3-Lie algebras this is not straightforward. The integrability criterion must be

carefully examine when passing from a 3-Lie algebra to the dual space and vice versa. In any case, assuming that this is done, we define a linear Nambu-Poisson structure as follows [19]:

$$\{F, H, K\}(\mu) = \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu}, \frac{\delta K}{\delta \mu} \right] \right\rangle \quad (4.48)$$

Notice that this linear Nambu-Poisson dynamics is computed in terms of the coadjoint action as follows:

$$\begin{aligned} \left\langle \frac{\delta F}{\delta \mu}, \dot{\mu} \right\rangle &= \dot{F} = \{F, H, K\}(\mu) = \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu}, \frac{\delta K}{\delta \mu} \right] \right\rangle \\ &= \left\langle \mu, \text{ad}_{\left(\frac{\delta H}{\delta \mu}, \frac{\delta K}{\delta \mu} \right)} \left(\frac{\delta F}{\delta \mu} \right) \right\rangle \\ &= - \left\langle \text{ad}_{\left(\frac{\delta H}{\delta \mu}, \frac{\delta K}{\delta \mu} \right)}^*(\mu), \frac{\delta F}{\delta \mu} \right\rangle. \end{aligned} \quad (4.49)$$

Accordingly we have the equation of motion as

$$\dot{\mu} = -\text{ad}_{\left(\frac{\delta H}{\delta \mu}, \frac{\delta K}{\delta \mu} \right)}^*(\mu). \quad (4.50)$$

In coordinates, assuming (μ_α) coordinate system on $\mathfrak{g}^*$, we have that

$$\{\mu_\alpha, \mu_\beta, \mu_\gamma\} = \mathfrak{C}_{\alpha\beta\gamma}^\theta \mu_\theta. \quad (4.51)$$

In terms of the functions this gives that

$$\{F, H, K\}(\mu) = \mathfrak{C}_{\alpha\beta\gamma}^\theta \mu_\theta \frac{\partial F}{\partial \mu_\alpha} \frac{\partial H}{\partial \mu_\beta} \frac{\partial K}{\partial \mu_\gamma}. \quad (4.52)$$

Accordingly, for a Hamiltonian pair (H, K) , we write the Nambu Hamilton's equation as

$$\dot{\mu}_\alpha = \mathfrak{C}_{\alpha\beta\gamma}^\theta \mu_\theta \frac{\partial H}{\partial \mu_\beta} \frac{\partial K}{\partial \mu_\gamma}. \quad (4.53)$$

4.2. n -Lie Algebroids, Dynamics, Couplings

4.2.1. n -Lie Algebroids

An n -Lie (also called Filippov) algebroid $\mathcal{A}$ of order n over the base M is a (real) vector bundle $\tau_{\mathcal{A}} : \mathcal{A} \mapsto M$ admitting a skew-symmetric algebra on the sections [126, 134, 135]

$$[\bullet, \dots, \bullet] : \Gamma(\mathcal{A}) \times \dots \times \Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{A}) \quad (4.54)$$

together with a bundle (so-called anchor) map

$$\mathbf{a}_{\mathcal{A}} : \wedge^{n-1} \mathcal{A} \longrightarrow TM. \quad (4.55)$$

For an n -Lie algebroid the fundamental identity

$$\begin{aligned} & [X_1, \dots, X_{n-1}, [X_n, \dots, X_{2n-1}]] \\ &= \sum_{k=0}^{n-1} [X_n, \dots, [X_1, \dots, X_{n-1}, X_{n+k}], \dots, X_{2n-1}] \end{aligned} \quad (4.56)$$

must hold, for any $X_1, \dots, X_{2n-1}$ and X in $\Gamma(\mathcal{A})$. Further the anchor map and the n -bracket are related by

$$\begin{aligned} & [\mathbf{a}_{\mathcal{A}}(X_1, \dots, X_{n-1}), \mathbf{a}_{\mathcal{A}}(X_{n+1}, \dots, X_{2n-1})] \\ &= \sum_{k=0}^{n-1} \mathbf{a}_{\mathcal{A}}(X_{n+1}, \dots, [X_1, \dots, X_{n-1}, X_{n+k}], \dots, X_{2n-1}), \end{aligned} \quad (4.57)$$

and the product rule

$$[X_1, \dots, X_{n-1}, fX] = f[X_1, \dots, X_{n-1}, X] + \mathcal{L}_{\mathbf{a}_{\mathcal{A}}(X_1, \dots, X_{n-1})}(f)X, \quad (4.58)$$

for all smooth functions f on the base manifold.

Note that when $n = 2$, the conditions above are reduced to the conditions given for Lie algebroids in the previous section. So, a Lie algebroid is actually a Filippov algebroid of order 2.

As previously mentioned, in this thesis our main interest is 3-Lie algebroids. So we wish to state this case explicitly for future reference. A 3-Lie algebroid $\mathcal{A}$ over the base M is a (real) vector bundle $\tau_{\mathcal{A}} : \mathcal{A} \mapsto M$ whose sections admit the 3-Lie bracket

$$[\bullet, \bullet, \bullet]_{\mathcal{A}} : \wedge^3 \mathcal{A} \longrightarrow \mathcal{A}. \quad (4.59)$$

In this case the fundamental identity turns out to be

$$\begin{aligned} [X_1, X_2, [X_3, X_4, X_5]] &= [[X_1, X_2, X_3], X_4, X_5] + [X_3, [X_1, X_2, X_4], X_5] \\ &+ [X_3, X_4, [X_1, X_2, X_5]], \end{aligned} \quad (4.60)$$

for any set of smooth sections X_i and X in $\Gamma(\mathcal{A})$. The anchor map is taking two section of the vector bundle to the tangent bundle and it is written by

$$\mathbf{a}_{\mathcal{A}} : \mathcal{A} \wedge \mathcal{A} \rightarrow TM. \quad (4.61)$$

The conditions in (4.57) and (4.58) now become

$$\begin{aligned} [\mathfrak{a}_{\mathcal{A}}(X_1, X_2), \mathfrak{a}_{\mathcal{A}}(X_3, X_4)] &= \mathfrak{a}_{\mathcal{A}}([X_1, X_2, X_3], X_4) + \mathfrak{a}_{\mathcal{A}}(X_3, [X_1, X_2, X_4]) \\ [X_1, X_2, fX] &= f[X_1, X_2, X] + \mathcal{L}_{\mathfrak{a}_{\mathcal{A}}(X_1, X_2)}(f)X \end{aligned} \quad (4.62)$$

respectively, for any $f \in \mathcal{C}^\infty(M)$. We denote a 3-Lie algebroid by a quintuple

$$(\mathcal{A}, \tau_{\mathcal{A}}, M, \mathfrak{a}_{\mathcal{A}}, [\bullet, \bullet, \bullet]_{\mathcal{A}}). \quad (4.63)$$

Let us define the representation of a 3 Lie algebroid $\mathcal{A}$ on a vector bundle $(\mathcal{B}, \tau_{\mathcal{B}}, M)$.

It is given by a map

$$\sigma : \Gamma(\mathcal{B}) \times \wedge^2 \Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{B}), \quad (Y, X_1, X_2) \mapsto \sigma(Y, X_1, X_2) \quad (4.64)$$

which is skew-symmetric with respect to its last two entries. A representation should satisfy the following commutation relationships with the 3-Lie bracket

$$\begin{aligned} \sigma(Y, X, [X_1, X_2, X_3]) &= \sigma(\sigma(Y, X, X_1), X_2, X_3) + \sigma(\sigma(Y, X, X_2), X_1, X_3) \\ &\quad + \sigma(\sigma(Y, X, X_3), X_1, X_2), \\ \sigma(\sigma(Y, X_3, X_4), X_1, X_2) &= \sigma(Y, [X_1, X_2, X_3], X_4) \\ &\quad + \sigma(\sigma(Y, X_1, X_2), X_3, X_4) + \sigma(Y, X_3, [X_1, X_2, X_4]) \end{aligned} \quad (4.65)$$

for arbitrary sections of the 3-Lie algebroid X_i, X in $\Gamma(\mathcal{A})$ and Y in $\Gamma(\mathcal{B})$. Further, a representation must fulfill the following Leibniz rule identities

$$\begin{aligned} \sigma(Y, fX_1, X_2) &= f\sigma(Y, X_1, X_2) \\ \sigma(Y, X_1, fX_2) &= f\sigma(Y, X_1, X_2), \\ \sigma(fY, X_1, X_2) &= f\sigma(X_1, X_2, Y) + \mathcal{L}_{\mathfrak{a}_{\mathcal{A}}(X_1, X_2)}(f)Y \end{aligned} \quad (4.66)$$

for an arbitrary function f defined on the base manifold M . Here, the second term on the right hand side of the last line is the directional derivative of the function f in the direction of the vector field $\mathfrak{a}_{\mathcal{A}}(X_1, X_2)$.

Linear 3-Nambu Poisson Framework. Being a vector bundle $(\mathcal{A}, \tau_{\mathcal{A}}, M)$ there exists the dual bundle $(\mathcal{A}^*, \pi_{\mathcal{A}}, M)$. The space of functions on the dual space $\mathcal{A}^*$ is spanned by two classes of functions. The first class consists of constant functions on fibers, i.e., functions on the base M . The second class consists of linear functions on fibers, i.e., the sections of the projection $\tau_{\mathcal{A}}$. Let us analyze the first class. For a function f on M , we can define a function $\hat{f}$ on $\mathcal{A}^*$ as follows:

$$\hat{f} : \mathcal{A}^* \longrightarrow \mathbb{R}, \quad \hat{f}(u) = f \circ \pi_{\mathcal{A}}(u). \quad (4.67)$$

The second class will be obtained from the sections of the Lie algebroid $\mathcal{A}$. For this, let us take a section X . The linear function $\widehat{X}$ on $\mathcal{A}^*$ for every u in $\mathcal{A}^*$ is defined as:

$$\widehat{X} : \mathcal{A}^* \longrightarrow \mathbb{R}, \quad \widehat{X}(u) = \langle u, X \circ \pi_{\mathcal{A}}(u) \rangle. \quad (4.68)$$

Similar to the Lie algebroid framework, we define two sets of functions on the total space $\mathcal{A}^*$ of the dual bundle. The first set consists of those that are constant on each fiber and are obtained by the pull-backs of the smooth functions on M . If f is a function on the base manifold, then in accordance with (4.67) we denote its lift to $\mathcal{A}^*$ by $\widehat{f}$. The second set, on the other hand, consists of those functions that are linear on the fibers. These are determined by the pointwise evaluations of the sections of the 3-Lie algebroid. Accordingly, if X is a section of the algebroid then $\widehat{X}$ will be the corresponding linear function as defined in (4.68). Referring to the set of functions $(\widehat{f}, \widehat{X})$ and in light of the 3-Lie algebroid structure, we determine the linear 3-Nambu-Poisson bracket on $\mathcal{A}^*$ as follows

$$\begin{aligned} \{\widehat{f}, \widehat{g}, \widehat{k}\}_{\mathcal{A}^*} &= 0, \\ \{\widehat{f}, \widehat{g}, \widehat{X}\}_{\mathcal{A}^*} &= 0, \\ \{\widehat{f}, \widehat{X}_1, \widehat{X}_2\}_{\mathcal{A}^*} &= \mathcal{L}_{(\mathfrak{a}_{\mathcal{A}}(X_1, X_2))} \widehat{f}, \\ \{\widehat{X}_1, \widehat{X}_2, \widehat{X}_3\}_{\mathcal{A}^*} &= [\widehat{X_1, X_2, X_3}]_{\mathcal{A}}. \end{aligned} \quad (4.69)$$

It is a direct verification to show that the two-tuple $(\mathcal{A}^*, \{\bullet, \bullet, \bullet\}_{\mathcal{A}^*})$ is a 3-Nambu Poisson manifold.

Referring to this Nambu-Poisson structure, we can determine 3-Nambu-Hamiltonian dynamics on the dual space $\mathcal{A}^*$ according to the Nambu-Hamilton equation generated by two Hamiltonians

$$\frac{du}{dt} = \{u, H_1, H_2\}_{\mathcal{A}^*}. \quad (4.70)$$

Referring to the discussion done on bi-Hamiltonian character of 3-Nambu Hamiltonian dynamics we define bi-Hamiltonian realization of linear Nambu dynamics in this framework as well:

$$\frac{du}{dt} = \{u, H_1\}_{\mathcal{A}^*}^{H_2} = \{u, H_2\}_{\mathcal{A}^*}^{H_1}. \quad (4.71)$$

where the Poisson brackets are defined as in (4.43).

In finite dimensions, assume a local coordinate (x^i) on M and (x^i, y^α) on the total space $\mathcal{A}$ of the 3-Lie algebroid. We denote a section of this fibration by (e_α) so that structure functions and the anchor map are computed to be

$$[e_\alpha, e_\beta, e_\gamma] = \mathfrak{H}_{\alpha\beta\gamma}^\theta e_\theta, \quad (\mathbf{a}_\mathcal{A})(e_\alpha, e_\beta) = (\mathbf{a}_\mathcal{A})_{\alpha\beta}^i \frac{\partial}{\partial x^i}, \quad (4.72)$$

respectively. On the total space of the dual bundle the dual coordinates are (x^i, y_α) . In this case, the 3-Nambu-Poisson bracket (4.69) is written by

$$\begin{aligned} \{x^i, x^j, x^k\}_{\mathcal{A}^*} &= 0, \\ \{x^i, x^j, y_\alpha\}_{\mathcal{A}^*} &= 0, \\ \{x^i, y_\alpha, y_\beta\}_{\mathcal{A}^*} &= (\mathbf{a}_\mathcal{A})_{\alpha\beta}^i, \\ \{y_\alpha, y_\beta, y_\gamma\}_{\mathcal{A}^*} &= \mathfrak{H}_{\alpha\beta\gamma}^\theta y_\theta. \end{aligned} \quad (4.73)$$

while the Nambu-Poisson 3-vector field turns out to be

$$\Lambda_{\mathcal{A}^*} = \frac{1}{3!} \mathfrak{H}_{\alpha\beta\gamma}^\theta y_\theta \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta} \wedge \frac{\partial}{\partial y_\gamma} + \frac{1}{2} (\mathbf{a}_\mathcal{A})_{\alpha\beta}^i \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta}. \quad (4.74)$$

Accordingly, the Nambu Hamiltonian dynamics (4.70) generated by a Hamiltonian function pair (H_1, H_2) is obtained as the following system of equations

$$\begin{aligned} \frac{dx^i}{dt} &= \{x^i, H_1, H_2\} = (\mathbf{a}_\mathcal{A})_{\alpha\beta}^i \frac{\partial H_1}{\partial y_\alpha} \frac{\partial H_2}{\partial y_\beta}, \\ \frac{dy_\alpha}{dt} &= \{y_\alpha, H_1, H_2\} = (\mathbf{a}_\mathcal{A})_{\alpha\beta}^i \left(\frac{\partial H_1}{\partial x^i} \frac{\partial H_2}{\partial y_\beta} - \frac{\partial H_2}{\partial x^i} \frac{\partial H_1}{\partial y_\beta} \right) + \mathfrak{H}_{\alpha\beta\gamma}^\theta y_\theta \frac{\partial H_1}{\partial y_\beta} \frac{\partial H_2}{\partial y_\gamma}. \end{aligned} \quad (4.75)$$

4.2.2. BDCP 3-Lie Algebroids

In this section, we endow the bicocycle double cross product strategy to two vector bundles over the same base manifold to arrive at a 3-Lie algebroid structure on the direct sum. More explicitly, given two vector bundles $(\mathcal{A}, \tau, M)$ and $(\mathcal{B}, \kappa, M)$, we construct a 3-Lie algebroid structure on $\mathcal{A} \times \mathcal{B}$. We start to the construction step by step.

Let $(\mathcal{B}, \tau, M)$ be an arbitrary vector bundle together with a vector bundle map $\mathbf{a}_\mathcal{B} : \mathcal{B} \rightarrow TM$. Define

$$\psi : \wedge^3 \Gamma(\mathcal{B}) \rightarrow \Gamma(\mathcal{A}), \quad \theta : \wedge^3 \Gamma(\mathcal{B}) \rightarrow \Gamma(\mathcal{B}) \quad (4.76)$$

so that, for any $Y_1, Y_2, Y_3 \in \Gamma(\mathcal{B})$,

$$\begin{aligned} [\bullet, \bullet, \bullet]_\mathcal{B} : \Gamma(\mathcal{B}) \wedge \Gamma(\mathcal{B}) \wedge \Gamma(\mathcal{B}) &\longrightarrow \Gamma(\mathcal{A}) \oplus \Gamma(\mathcal{B}) \\ [Y_1, Y_2, Y_3]_\mathcal{B} &= \psi(Y_1, Y_2, Y_3) + \theta(Y_1, Y_2, Y_3). \end{aligned} \quad (4.77)$$

See that the existence of non-zero ψ prevents the closure of the bracket $\mathcal{B}$. We call ψ as *cocycle* term.

Define two operations

$$\phi : \Gamma(\mathcal{A}) \wedge \Gamma(\mathcal{A}) \wedge \Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{A}), \quad \zeta : \Gamma(\mathcal{A}) \wedge \Gamma(\mathcal{A}) \wedge \Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{B}), \quad (4.78)$$

so that we can write a bracket on $\Gamma(\mathcal{A})$ as

$$\begin{aligned} [\bullet, \bullet, \bullet]_{\mathcal{A}} : \Gamma(\mathcal{A}) \wedge \Gamma(\mathcal{A}) \wedge \Gamma(\mathcal{A}) &\longrightarrow \Gamma(\mathcal{A}) \oplus \Gamma(\mathcal{B}) \\ [X_1, X_2, X_3]_{\mathcal{A}} &= \phi(X_1, X_2, X_3) + \zeta(X_1, X_2, X_3) \end{aligned} \quad (4.79)$$

for elements X_1, X_2, X_3 in $\Gamma(\mathcal{A})$. Note that the existence of the *cocycle* term ζ prevents this mapping from being a bracket.

Further, we define two mappings

$$\rho : \Gamma(\mathcal{B}) \times \Gamma(\mathcal{B}) \times \Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{A}), \quad \sigma : \Gamma(\mathcal{B}) \times \Gamma(\mathcal{A}) \times \Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{B}). \quad (4.80)$$

Considering these mappings, we can define the following bracket

$$\begin{aligned} [(X_1, Y_1), (X_2, Y_2), (X_3, Y_3)]_{\zeta \bowtie \psi} = & \\ & \left(\phi(X_1, X_2, X_3) + \psi(Y_1, Y_2, Y_3) + \rho(Y_2, Y_3, X_1) - \rho(Y_1, Y_3, X_2) \right. \\ & \quad \left. + \rho(Y_1, Y_2, X_3); \right. \\ & \quad \left. \theta(Y_1, Y_2, Y_3) + \zeta(X_1, X_2, X_3) + \sigma(Y_3, X_1, X_2) - \sigma(Y_2, X_1, X_3) \right. \\ & \quad \left. + \sigma(Y_1, X_2, X_3) \right) \end{aligned} \quad (4.81)$$

on the sections of $\mathcal{A} \times \mathcal{B}$.

We now present an exhaustive list of conditions on all the mappings introduced in this section. The conditions are obtained from the verification of the fundamental

identity. The following are the ones consisting of only X 's and Y 's:

$$\begin{aligned}
& \phi(X_1, X_2, \phi(X_3, X_4, X_5)) = \phi(\phi(X_1, X_2, X_3), X_4, X_5) \\
& \quad + \phi(X_3, \phi(X_1, X_2, X_4), X_5) + \phi(X_3, X_4, \phi(X_1, X_2, X_5)), \\
& \zeta(X_1, X_2, \phi(X_3, X_4, X_5)) + \sigma(\zeta(X_3, X_4, X_5), X_1, X_2) = \\
& \quad \zeta(\phi(X_1, X_2, X_3), X_4, X_5) + \sigma(\zeta(X_1, X_2, X_3), X_4, X_5) + \\
& \quad \zeta(X_3, \phi(X_1, X_2, X_4), X_5) - \sigma(\zeta(X_1, X_2, X_4), X_3, X_5) + \\
& \quad \zeta(X_3, X_4, \phi(X_1, X_2, X_5)) + \sigma(\zeta(X_1, X_2, X_5), X_3, X_4), \\
& \theta(Y_1, Y_2, \theta(Y_3, Y_4, Y_5)) = \theta(\theta(Y_1, Y_2, Y_3), Y_4, Y_5) + \\
& \quad \theta(Y_3, \theta(Y_1, Y_2, Y_4), Y_5) + \theta(Y_3, Y_4, \theta(Y_1, Y_2, Y_5)), \\
& \psi(Y_1, Y_2, \theta(Y_3, Y_4, Y_5)) + \rho(Y_1, Y_2, \psi(Y_3, Y_4, Y_5)) = \\
& \quad \psi(\theta(Y_1, Y_2, Y_3), Y_4, Y_5) + \rho(Y_4, Y_5, \psi(Y_1, Y_2, Y_3)) + \\
& \quad \psi(Y_3, \theta(Y_1, Y_2, Y_4), Y_5) - \rho(Y_3, Y_5, \psi(Y_1, Y_2, Y_4)) + \\
& \quad \psi(Y_3, Y_4, \theta(Y_1, Y_2, Y_5)) + \rho(Y_3, Y_4, \psi(Y_1, Y_2, Y_5)).
\end{aligned} \tag{4.82}$$

Then the following are the ones containing four X 's and only one Y :

$$\begin{aligned}
& \sigma(\sigma(Y_5, X_3, X_4), X_1, X_2) = \sigma(Y_5, \phi(X_1, X_2, X_3), X_4) \\
& \quad + \sigma(Y_5, X_3, \phi(X_1, X_2, X_4)) + \sigma(\sigma(Y_5, X_1, X_2), X_3, X_4) \\
& \rho(\zeta(X_1, X_2, X_3), Y_5, X_4) - \rho(\zeta(X_1, X_2, X_4), Y_5, X_3) = 0, \\
& \sigma(Y_2, X_1, \phi(X_3, X_4, X_5)) = \sigma(\sigma(Y_2, X_1, X_3), X_4, X_5) \\
& \quad - \sigma(\sigma(Y_2, X_1, X_4), X_3, X_5) + \sigma(\sigma(Y_2, X_1, X_5), X_3, X_4) \\
& \rho(Y_2, \zeta(X_3, X_4, X_5), X_1) = 0.
\end{aligned} \tag{4.83}$$

The following are those containing three X 's and two Y 's:

$$\begin{aligned}
& \phi(X_1, X_2, \rho(Y_4, Y_5, X_3)) = \psi(\zeta(X_1, X_2, X_3), Y_4, Y_5) + \rho(Y_4, Y_5, \phi(X_1, X_2, X_3)) \\
& \quad + \rho(\sigma(Y_4, X_1, X_2), Y_5, X_3) + \rho(Y_4, \sigma(Y_5, X_1, X_2), X_3), \\
& \zeta(X_1, X_2, \rho(Y_4, Y_5, X_3)) = \theta(\zeta(X_1, X_2, X_3), Y_4, Y_5), \\
& \rho(Y_2, \sigma(Y_5, X_3, X_4), X_1) = \rho(\sigma(Y_2, X_1, X_3), Y_5, X_4) \\
& \quad - \rho(\sigma(Y_2, X_1, X_4), Y_5, X_3) + \phi(X_3, X_4, \rho(Y_2, Y_5, X_1))
\end{aligned} \tag{4.84}$$

$$\begin{aligned}
\zeta(X_3, X_4, \rho(Y_2, Y_5, X_1)) &= 0, \\
\rho(Y_1, Y_2, \phi(X_3, X_4, X_5)) + \psi(Y_1, Y_2, \zeta(X_3, X_4, X_5)) &= \phi(\rho(Y_1, Y_2, X_3), X_4, X_5) \\
&+ \phi(X_3, \rho(Y_1, Y_2, X_4), X_5) + \phi(X_3, X_4, \rho(Y_1, Y_2, X_5)), \\
\theta(Y_1, Y_2, \zeta(X_3, X_4, X_5)) &= \zeta(\rho(Y_1, Y_2, X_3), X_4, X_5) \\
&+ \zeta(X_3, \rho(Y_1, Y_2, X_4), X_5) + \zeta(X_3, X_4, \rho(Y_1, Y_2, X_5)).
\end{aligned} \tag{4.85}$$

The following are the ones containing two X 's and three Y 's:

$$\begin{aligned}
\phi(X_1, X_2, \psi(Y_3, Y_4, Y_5)) &= \psi(\sigma(Y_3, X_1, X_2), Y_4, Y_5) \\
&+ \psi(Y_3, \sigma(Y_4, X_1, X_2), Y_5) + \psi(Y_3, Y_4, \sigma(Y_5, X_1, X_2)), \\
\zeta(X_1, X_2, \psi(Y_3, Y_4, Y_5)) + \sigma(\theta(Y_3, Y_4, Y_5), X_1, X_2) &= \theta(\sigma(Y_3, X_1, X_2), Y_4, Y_5) \\
&+ \theta(Y_3, \sigma(Y_4, X_1, X_2), Y_5) + \theta(Y_3, Y_4, \sigma(Y_5, X_1, X_2)) \\
\sigma(Y_2, X_1, \rho(Y_4, Y_5, X_3)) &= \theta(\sigma(Y_2, X_1, X_3), Y_4, Y_5) \\
&- \sigma(Y_5, X_3, \rho(Y_2, Y_4, X_1)) + \sigma(Y_4, X_3, \rho(Y_2, Y_5, X_1)), \\
\psi(\sigma(Y_2, X_1, X_3), Y_4, Y_5) &= 0, \\
\psi(Y_1, Y_2, \sigma(Y_5, X_3, X_4)) &= \phi(X_3, X_4, \psi(Y_1, Y_2, Y_5)), \\
\theta(Y_1, Y_2, \sigma(Y_5, X_3, X_4)) &= \sigma(Y_5, \rho(Y_1, Y_2, X_3), X_4) + \sigma(Y_5, X_3, \rho(Y_1, Y_2, X_4)) \\
&+ \zeta(X_3, X_4, \psi(Y_1, Y_2, Y_5)) + \sigma(\theta(Y_1, Y_2, Y_5), X_3, X_4).
\end{aligned} \tag{4.86}$$

Then the following are those containing only one X , and four Y 's:

$$\begin{aligned}
\rho(Y_2, \theta(Y_3, Y_4, Y_5), X_1) &= \rho(Y_4, Y_5, \rho(Y_2, Y_3, X_1)) \\
&- \rho(Y_3, Y_5, \rho(Y_2, Y_4, X_1)) + \rho(Y_3, Y_4, \rho(Y_2, Y_5, X_1)) \\
\sigma(Y_2, \psi(Y_3, Y_4, Y_5), X_1) &= 0 \\
\rho(Y_1, Y_2, \rho(Y_4, Y_5, X_3)) &= \rho(Y_4, Y_5, \rho(Y_1, Y_2, X_3)) \\
&+ \rho(\theta(Y_1, Y_2, Y_4), Y_5, X_3) + \rho(Y_4, \theta(Y_1, Y_2, Y_5), X_3) \\
\sigma(Y_5, X_3, \psi(Y_1, Y_2, Y_4)) - \sigma(Y_4, X_3, \psi(Y_1, Y_2, Y_5)) &= 0.
\end{aligned} \tag{4.87}$$

The compatibility conditions related with the anchor maps can be listed as follows:

$$\begin{aligned}
[\mathbf{a}_{\mathcal{A}}(X_1, X_2), \mathbf{a}_{\mathcal{A}}(X_3, X_4)] &= \mathbf{a}_{\mathcal{A}}(\phi(X_1, X_2, X_3), X_4) + \mathbf{a}_{\mathcal{A}}(X_3, \phi(X_1, X_2, X_4)) \\
\mathbf{a}_{\mathcal{B}}, (\zeta(X_1, X_2, X_3), Y_4) &= 0,
\end{aligned} \tag{4.88}$$

$$\begin{aligned}
[\mathfrak{a}_{\mathcal{A}}(X_1, X_2), \mathfrak{a}_{\mathcal{B}}(Y_3, Y_4)] &= \mathfrak{a}_{\mathcal{B}}(\sigma(Y_3, X_1, X_2), Y_4) + \mathfrak{a}_{\mathcal{B}}(Y_3, \sigma(Y_4, X_1, X_2)), \\
\mathfrak{a}_{\mathcal{B}}(\sigma(Y_2, X_1, X_3), Y_4) - \mathfrak{a}_{\mathcal{A}}(X_3, \rho(Y_2, Y_4, X_1)) &= 0, \\
[\mathfrak{a}_{\mathcal{B}}(Y_1, Y_2), \mathfrak{a}_{\mathcal{A}}(X_3, X_4)] &= \mathfrak{a}_{\mathcal{A}}(\rho(Y_1, Y_2, X_3), X_4) + \mathfrak{a}_{\mathcal{A}}(X_3, \rho(Y_1, Y_2, X_4)), \\
\mathfrak{a}_{\mathcal{A}}(X_3, \psi(Y_1, Y_2, Y_4)) &= 0, \\
[\mathfrak{a}_{\mathcal{B}}(Y_1, Y_2), \mathfrak{a}_{\mathcal{B}}(Y_3, Y_4)] &= \mathfrak{a}_{\mathcal{B}}(\theta(Y_1, Y_2, Y_3), Y_4) + \mathfrak{a}_{\mathcal{A}}(Y_3, \theta(Y_1, Y_2, Y_4)).
\end{aligned} \tag{4.89}$$

We conclude the list of compatibility conditions by those about the Leibniz identity:

$$\begin{aligned}
\phi(X_1, X_2, fX) &= f\phi(X_1, X_2, X) + \mathcal{L}_{\mathfrak{a}_{\mathcal{A}}(X_1, X_2)}(f)X, \\
\zeta(X_1, X_2, fX) &= f\zeta(X_1, X_2, X), \\
\sigma(fY, X_1, X_2) &= f\sigma(Y, X_1, X_2) + \mathcal{L}_{\mathfrak{a}_{\mathcal{A}}(X_1, X_2)}(f)Y, \\
\sigma(Y_2, X_1, fX) &= f\sigma(Y_2, X_1, X), \\
\rho(fY, Y_2, X_1) &= f\rho(Y, Y_2, X_1), \\
\rho(Y_1, Y_2, fX) &= f\rho(Y_1, Y_2, X) + \mathcal{L}_{\mathfrak{a}_{\mathcal{B}}(Y_1, Y_2)}(f)X, \\
\psi(Y_1, Y_2, fY) &= f\psi(Y_1, Y_2, Y), \\
\theta(Y_1, Y_2, fY) &= f\theta(Y_1, Y_2, Y) + \mathcal{L}_{\mathfrak{a}_{\mathcal{B}}(Y_1, Y_2)}(f)Y.
\end{aligned} \tag{4.90}$$

Referring to all these conditions we are now ready to state the coupling theorem of the vector bundles $\mathcal{A}$ and $\mathcal{B}$ in bicocycle double cross product approach.

Theorem 4.7. *Two vector bundles $(\mathcal{A}, \tau, M)$ and $(\mathcal{B}, \kappa, M)$ admitting mappings exhibited in (4.76), (4.78) and (4.80), determine a (bicocycle double cross product) 3-Lie algebroid structure with the 3-bracket (4.79) if and only if the conditions in (4.82)-(4.87), (4.88)-(4.89) and (4.90) are all satisfied.*

We call this construction as bicocycle double cross product 3-Lie algebroid. If one of the *cocycle* terms are zero (that is if ψ or ζ) the one arrives at the unified product 3-Lie algebroid. If $\psi = 0$ the 3-Lie bracket (4.79) turns out to be

$$\begin{aligned}
&[(X_1, Y_1), (X_2, Y_2), (X_3, Y_3)]_{\zeta \bowtie} \\
&= \left(\phi(X_1, X_2, X_3) + \rho(Y_2, Y_3, X_1) - \rho(Y_1, Y_3, X_2) + \rho(Y_1, Y_2, X_3); \right. \\
&\quad \theta(Y_1, Y_2, Y_3) + \zeta(X_1, X_2, X_3) + \sigma(Y_3, X_1, X_2) - \sigma(Y_2, X_1, X_3) \\
&\quad \left. + \sigma(Y_1, X_2, X_3) \right).
\end{aligned} \tag{4.91}$$

If on the other hand $\zeta = 0$ the 3-Lie bracket (4.79) becomes

$$\begin{aligned}
& [(X_1, Y_1), (X_2, Y_2), (X_3, Y_3)]_{\bowtie_\psi} \\
&= \left(\phi(X_1, X_2, X_3) + \psi(Y_1, Y_2, Y_3) + \rho(Y_2, Y_3, X_1) \right. \\
&\quad \left. - \rho(Y_1, Y_3, X_2) + \rho(Y_1, Y_2, X_3); \right. \\
&\quad \left. \theta(Y_1, Y_2, Y_3) + \sigma(Y_3, X_1, X_2) - \sigma(Y_2, X_1, X_3) + \sigma(Y_1, X_2, X_3) \right).
\end{aligned} \tag{4.92}$$

If both *cocycles* are trivial, then we arrive at the double cross product 3-Lie algebroid given by

$$\begin{aligned}
& [(X_1, Y_1), (X_2, Y_2), (X_3, Y_3)]_{\bowtie} \\
&= \left(\phi(X_1, X_2, X_3) + \rho(Y_2, Y_3, X_1) - \rho(Y_1, Y_3, X_2) + \rho(Y_1, Y_2, X_3); \right. \\
&\quad \left. \theta(Y_1, Y_2, Y_3) + \sigma(Y_3, X_1, X_2) - \sigma(Y_2, X_1, X_3) + \sigma(Y_1, X_2, X_3) \right).
\end{aligned} \tag{4.93}$$

In finite dimensions, let us consider the local coordinate system (x^i) on M , along with the bases (e_α) and (e_a) of $\Gamma(\mathcal{A})$ and $\Gamma(\mathcal{B})$ respectively. Then the anchor maps are

$$\mathbf{a}_{\zeta \bowtie_\psi}(e_\alpha, e_\beta) = (\mathbf{a}_{\mathcal{A}})_{\alpha\beta}^i \frac{\partial}{\partial x^i}, \quad \mathbf{a}_{\zeta \bowtie_\psi}(e_a, e_b) = (\mathbf{a}_{\mathcal{B}})_{ab}^i \frac{\partial}{\partial x^i}, \tag{4.94}$$

while we determine the structure functions associated with (4.76), (4.78) and (4.80) by

$$\begin{aligned}
[e_\alpha, e_\beta, e_\gamma]_{\zeta \bowtie_\psi} &= \bar{\mathfrak{C}}_{\alpha\beta\gamma}^\theta e_\theta + \bar{\mathfrak{C}}_{\alpha\beta\gamma}^a e_a, \\
[e_b, e_\beta, e_\gamma]_{\zeta \bowtie_\psi} &= \bar{\mathfrak{C}}_{b\beta\gamma}^\theta e_\theta + \bar{\mathfrak{C}}_{b\beta\gamma}^a e_a, \\
[e_b, e_c, e_\gamma]_{\zeta \bowtie_\psi} &= \bar{\mathfrak{C}}_{bc\gamma}^\theta e_\theta + \bar{\mathfrak{C}}_{bc\gamma}^a e_a, \\
[e_b, e_c, e_d]_{\zeta \bowtie_\psi} &= \bar{\mathfrak{C}}_{bcd}^\theta e_\theta + \bar{\mathfrak{C}}_{bcd}^a e_a.
\end{aligned} \tag{4.95}$$

4.2.3. BDCP 3-Nambu Dynamics

In this section, we follow the procedure given in (4.2.1) to determine the Nambu dynamics on the dual of the bicocycle double cross product 3-Lie algebroid.

Let us assume a local coordinate (x^i) on M and (x^i, μ_α, μ_a) on the dual space $\mathcal{A}^* \times \mathcal{B}^*$ so that we determine the anchor map as $(\mathbf{a}_{\mathcal{A}})_{\alpha\beta}^i$. Then the bicocycle

double cross product Nambu-Poisson bracket is written as

$$\begin{aligned}
\{x^i, x^j, x^k\}_{\mathcal{A}^* \times \mathcal{B}^*} &= 0, \\
\{x^i, x^j, \mu_\alpha\}_{\mathcal{A}^* \times \mathcal{B}^*} &= 0, \\
\{x^i, x^j, \mu_a\}_{\mathcal{A}^* \times \mathcal{B}^*} &= 0, \\
\{x^i, \mu_\alpha, \mu_\beta\}_{\mathcal{A}^* \times \mathcal{B}^*} &= (\mathfrak{a}_{\mathcal{A}})^i_{\alpha\beta}, \\
\{x^i, \mu_a, \mu_\beta\}_{\mathcal{A}^* \times \mathcal{B}^*} &= 0 \\
\{x^i, \mu_a, \mu_b\}_{\mathcal{A}^* \times \mathcal{B}^*} &= (\mathfrak{a}_{\mathcal{A}})^i_{ab}, \\
\{\mu_\alpha, \mu_\beta, \mu_\gamma\}_{\mathcal{A}^* \times \mathcal{B}^*} &= \mathfrak{C}_{\alpha\beta\gamma}^\theta \mu_\theta + \mathfrak{C}_{\alpha\beta\gamma}^d \mu_d, \\
\{\mu_a, \mu_\beta, \mu_\gamma\}_{\mathcal{A}^* \times \mathcal{B}^*} &= \mathfrak{C}_{a\beta\gamma}^\theta \mu_\theta + \mathfrak{C}_{a\beta\gamma}^d \mu_d, \\
\{\mu_a, \mu_b, \mu_\gamma\}_{\mathcal{A}^* \times \mathcal{B}^*} &= \mathfrak{C}_{ab\gamma}^\theta \mu_\theta + \mathfrak{C}_{ab\gamma}^d \mu_d, \\
\{\mu_a, \mu_b, \mu_c\}_{\mathcal{A}^* \times \mathcal{B}^*} &= \mathfrak{C}_{abc}^\theta \mu_\theta + \mathfrak{C}_{abc}^d \mu_d.
\end{aligned} \tag{4.96}$$

The dynamical equations governed by the Hamiltonian pair $(\mathcal{H}_1, \mathcal{H}_2)$ are computed to be

$$\begin{aligned}
\dot{x}^i &= \{x^i, \mathcal{H}_1, \mathcal{H}_2\} = (\mathfrak{a}_{\mathcal{A}})^i_{\alpha\beta} \frac{\partial \mathcal{H}_1}{\partial \mu_\alpha} \frac{\partial \mathcal{H}_2}{\partial \mu_\beta} + (\mathfrak{a}_{\mathcal{A}})^i_{ab} \frac{\partial \mathcal{H}_1}{\partial \mu_a} \frac{\partial \mathcal{H}_2}{\partial \mu_b}, \\
\dot{\mu}_\alpha &= \{\mu_\alpha, \mathcal{H}_1, \mathcal{H}_2\} = \\
&\quad (\mathfrak{a}_{\mathcal{A}})^i_{\alpha\beta} \left(\frac{\partial \mathcal{H}_1}{\partial x^i} \frac{\partial \mathcal{H}_2}{\partial \mu_\beta} - \frac{\partial \mathcal{H}_2}{\partial x^i} \frac{\partial \mathcal{H}_1}{\partial \mu_\beta} \right) + (\mathfrak{C}_{\alpha\beta\gamma}^\theta \mu_\theta + \mathfrak{C}_{\alpha\beta\gamma}^d \mu_d) \frac{\partial \mathcal{H}_1}{\partial \mu_\beta} \frac{\partial \mathcal{H}_2}{\partial \mu_\gamma} + \\
&\quad (\mathfrak{C}_{\alpha ab}^\theta \mu_\theta + \mathfrak{C}_{\alpha ab}^d \mu_d) \frac{\partial \mathcal{H}_1}{\partial \mu_a} \frac{\partial \mathcal{H}_2}{\partial \mu_b} + (\mathfrak{C}_{\alpha\beta b}^\theta \mu_\theta + \mathfrak{C}_{\alpha\beta b}^d \mu_d) \frac{\partial \mathcal{H}_1}{\partial \mu_{[\beta}} \frac{\partial \mathcal{H}_2}{\partial \mu_{b]}}, \\
\dot{\mu}_a &= \{\mu_a, \mathcal{H}_1, \mathcal{H}_2\} = \\
&\quad (\mathfrak{a}_{\mathcal{A}})^i_{ab} \left(\frac{\partial \mathcal{H}_1}{\partial x^i} \frac{\partial \mathcal{H}_2}{\partial \mu_b} - \frac{\partial \mathcal{H}_2}{\partial x^i} \frac{\partial \mathcal{H}_1}{\partial \mu_b} \right) + (\mathfrak{C}_{abc}^d \mu_d + \mathfrak{C}_{abc}^\theta \mu_\theta) \frac{\partial \mathcal{H}_1}{\partial \mu_b} \frac{\partial \mathcal{H}_2}{\partial \mu_c} + \\
&\quad (\mathfrak{C}_{a\beta\gamma}^\theta \mu_\theta + \mathfrak{C}_{a\beta\gamma}^d \mu_d) \frac{\partial \mathcal{H}_1}{\partial \mu_\beta} \frac{\partial \mathcal{H}_2}{\partial \mu_\gamma} + (\mathfrak{C}_{a\beta b}^\theta \mu_\theta + \mathfrak{C}_{a\beta b}^d \mu_d) \frac{\partial \mathcal{H}_1}{\partial \mu_{[\beta}} \frac{\partial \mathcal{H}_2}{\partial \mu_{b]}}.
\end{aligned} \tag{4.97}$$

5. ADDENDUM: A CASE STUDY

5.1. Locally Conformally Cosymplectic Manifolds

Let M be a $(2n + 1)$ -dimensional manifold with a family of local charts $\{U_\alpha\}$. We assume that each local chart admits cosymplectic structure, say $(U_\alpha, \tau_\alpha, \Omega_\alpha)$. Accordingly, we have

$$d\tau_\alpha = 0, \quad d\Omega_\alpha = 0. \quad (5.1)$$

We further assume that on the intersection, $U_\alpha \cap U_\beta$ of two charts U_α and U_β , the local forms are related as

$$e^{\sigma_\alpha} \tau_\alpha = e^{\sigma_\beta} \tau_\beta, \quad e^{2\sigma_\alpha} \Omega_\alpha = e^{2\sigma_\beta} \Omega_\beta \quad (5.2)$$

for a family of real-valued local functions $\{\sigma_\alpha\}$. It is evident from the local relations (5.2) that if we glue up local one-form τ_α and local two-form Ω_α then we arrive at line bundle valued one-form and a line bundle valued two-form on M . To have a real valued one-form and a real valued two-form we introduce the following local forms

$$\tau|_\alpha := e^{\sigma_\alpha} \tau_\alpha, \quad \Omega|_\alpha := e^{2\sigma_\alpha} \Omega_\alpha, \quad (5.3)$$

respectively. We glue up these forms and determine a real-valued one form τ and a real-valued two-form Ω in a respective manner. See that, on the local chart U_α , we denote the local realization of the global one-form τ as $\tau|_\alpha$ while we denote the local realization of Ω as $\Omega|_\alpha$. Due to the existence of local conformal structures, the differential forms τ and Ω are not closed. But, it is easy to see that the non-degeneracy $\tau \wedge \Omega^n \neq 0$ holds true. Therefore we have that the triple (M, τ, Ω) is an almost cosymplectic manifold. Such a manifold structure is locally conformally cosymplectic (abbreviated as LCC) manifold. We also record here that the local Reeb vector field $\mathcal{R}_\alpha$ due to the cosymplectic space $(U_\alpha, \tau_\alpha, \Omega_\alpha)$ and the Reeb field $\mathcal{R}$ for the almost cosymplectic manifold (M, τ, Ω) are related as

$$\mathcal{R}|_\alpha = e^{-\sigma_\alpha} \mathcal{R}_\alpha, \quad (5.4)$$

and a global vector field $\mathcal{R}$ by gluing up all local vector fields $\mathcal{R}|_\alpha$.

Lee form Realization. We consider a LCC manifold admitting local functions $\{\sigma_\alpha\}$ determining the conformal character of the structure. It is possible to see that the exterior derivatives of local functions agree on the intersections that is $d\sigma_\alpha = d\sigma_\beta$. These locally exact one-forms can be merged to a closed one-form Θ on the manifold M . The closed one-form Θ is called the Lee form. Accordingly, by taking the exterior derivatives of the differential forms in (5.3), we see that

$$d\tau = \Theta \wedge \tau, \quad d\Omega = 2\Theta \wedge \Omega. \quad (5.5)$$

One can use this global formulation to define LCC manifolds. An almost cosymplectic manifold (M, τ, Ω) transitions into a Lee conformal cosymplectic (LCC) manifold if it accommodates a closed one-form Θ adhering to (5.5).

Expressed through the Lichnerowicz–de Rham (LdR) differential, an almost cosymplectic manifold $(M, \tau, \Omega, \Theta)$ becomes LCC if and only if the one-form τ and the two-form Ω satisfy

$$d_\Theta \tau = d\tau - \Theta \wedge \tau = 0, \quad d_{2\Theta} \Omega = d\Omega - \Theta \wedge \Omega = 0. \quad (5.6)$$

In this configuration, a LCC manifold is denoted by a quadruple $(M, \tau, \Omega, \Theta)$, underscoring the presence of the Lee form Θ . If Θ is an exact one-form, then $(M, \tau, \Omega, \Theta = d\sigma)$ is termed a globally conformally cosymplectic (GCC) manifold.

Note that in case $\Theta = 0$, the equalities in (5.5) imply that the one-form τ and the two-form Ω are closed. Since the non-degeneracy condition is also satisfied, (M, τ, Ω) itself becomes a cosymplectic manifold. So a cosymplectic manifold can be considered as a particular instance of a LCC manifold with $\Theta = 0$.

Extended Cotangent Bundle as a LCC Manifold. Consider the cotangent bundle T^*Q , equipped with the Liouville one-form θ_Q and the symplectic two-form ω_Q . Pulling back these canonical forms to the extended cotangent bundle $T^*Q \times \mathbb{R}$ is denoted as Θ_Q and Ω_Q , respectively. let us suppose there exists a closed one-form

$$\Psi(q, t) = \psi_j(q, t) dq^j + \zeta(q, t) dt, \quad (5.7)$$

defined on the extended configuration space $Q \times \mathbb{R}$ and pulled back to the extended cotangent bundle $T^*Q \times \mathbb{R}$ via the canonical projection. We denote this pulled-back one-form as Θ . Now, define the following forms:

$$\tau = d_\Theta t = dt - t\Theta, \quad \Omega_{2\Theta} = -d_{2\Theta} \Theta_Q = \Omega_Q + 2\Theta \wedge \Theta_Q. \quad (5.8)$$

It is immediate to verify that the quadruple $(T^*Q \times \mathbb{R}, \tau, \Omega, \Theta)$ forms a LCC manifold. In terms of the Darboux coordinates (q^i, p_i, t) on $T^*Q \times \mathbb{R}$, the local realization of this LCC is computed as follows. Assuming the Lee form is given by

$$\Theta(q, t) = \psi_j(q, t)dq^j + \zeta(q, t)dt, \quad (5.9)$$

the differential forms are computed as

$$\begin{aligned} \Omega &= -d_{2\Theta}\Theta_Q = -d\Theta_Q + 2\Theta \wedge \Theta_Q = dq^i \wedge dp_i + 2p_i\Theta \wedge dq^i, \\ \tau &= d_\Theta t = dt - t\Theta. \end{aligned} \quad (5.10)$$

5.2. Dynamics on LCC Manifolds

Let $(M, \tau, \Omega, \Theta)$ be a $(2n+1)$ -dimensional LCC manifold equipped with a Lee form Θ . We take a chart with an induced cosymplectic structure, denoted as $(U_\alpha, \tau_\alpha, \Omega_\alpha)$, and consider a local Hamiltonian function $\mathcal{H}_\alpha$ on U_α . As a cosymplectic manifold, $(U_\alpha, \tau_\alpha, \Omega_\alpha)$ accommodates a local cosymplectic Hamiltonian vector field $X_{\mathcal{H}_\alpha}$ for $\mathcal{H}_\alpha$ via the following Hamilton's equations [136]:

$$\iota_{X_{\mathcal{H}_\alpha}}\tau_\alpha = 0, \quad \iota_{X_{\mathcal{H}_\alpha}}\Omega_\alpha = d\mathcal{H}_\alpha - \mathcal{R}_\alpha(\mathcal{H}_\alpha)\tau_\alpha. \quad (5.11)$$

The local Hamiltonian functions $\mathcal{H}_\alpha$ on each chart are then collected into a line bundle $L \rightarrow M$ valued Hamiltonian function. Moreover, by gluing the local functions

$$\mathcal{H}|_\alpha = e^{\sigma_\alpha}\mathcal{H}_\alpha$$

defined on U_α , we obtain a real-valued Hamiltonian function $\mathcal{H}$ over the entire M . We glue the vector fields

$$X_{\mathcal{H}}|_\alpha = e^{-\sigma_\alpha}X_{\mathcal{H}_\alpha} \quad (5.12)$$

to obtain the global Hamiltonian vector field $X_{\mathcal{H}}$. We can determine the global Hamilton's equations for a LCC manifold as [65]:

$$\iota_{X_{\mathcal{H}}}\tau = 0, \quad \iota_{X_{\mathcal{H}}}\Omega = d_\Theta\mathcal{H} - \langle d_\Theta\mathcal{H}, \mathcal{R} \rangle \tau. \quad (5.13)$$

Being non-degenerate, a LCC pair (Ω, τ) can determine a musical isomorphism as in (2.49). We denote the inverse mapping with $\sharp$. It is evident that an alternative method to obtain the Hamiltonian vector field $X_{\mathcal{H}}$ is through:

$$X_{\mathcal{H}} = \sharp(d_\Theta\mathcal{H}) - \langle d_\Theta\mathcal{H}, \mathcal{R} \rangle \mathcal{R}. \quad (5.14)$$

Here, $\sharp$ represents the musical isomorphism derived from the non-degenerate pair (Ω, τ) .

Let us take the special case $\Theta = 0$ into consideration. The equations in (5.13) is reduced to

$$\iota_{X_{\mathcal{H}}}\tau = 0, \quad \iota_{X_{\mathcal{H}}}\Omega = dH - \mathcal{R}(H)\tau. \quad (5.15)$$

Hence we obtain the Hamiltonian vector field $X_{\mathcal{H}}$ of the cosymplectic framework [136]. Alternatively, (5.14) with $\Theta = 0$ leads us to write

$$X_{\mathcal{H}} = \sharp(d\mathcal{H}) - \mathcal{R}(\mathcal{H})\mathcal{R}, \quad (5.16)$$

which gives the cosymplectic Hamiltonian vector field in terms of the musical isomorphism and the Reeb field.

Evolution Vector Field for LCC Manifolds. To obtain the equation of motion, we define the local evolution vector field E_{α} as the sum of the Reeb field $\mathcal{R}_{\alpha}$ and the Hamiltonian vector field X_{α} . Utilizing equations (5.4) and (5.12), we can determine a local vector field

$$E_{\mathcal{H}}|_{\alpha} = e^{-\sigma_{\alpha}}E_{\alpha}, \quad (5.17)$$

then by gluing up these vector fields, we define the global evolution vector field $E_{\mathcal{H}}$ as

$$\iota_{E_{\mathcal{H}}}\tau = 1, \quad \iota_{E_{\mathcal{H}}}\Omega = d_{\Theta}H - \langle d_{\Theta}H, \mathcal{R} \rangle \tau. \quad (5.18)$$

Note that when we focus on the almost cosymplectic structure of (M, τ, Ω) , $X_{\mathcal{H}}$ in (5.13) doesn't yield the Hamiltonian vector field corresponding to H unless $\Theta = 0$. Consequently, $E_{\mathcal{H}}$ in (5.18) differs from the evolution vector field of the almost cosymplectic manifold (M, τ, Ω) unless $\Theta = 0$.

5.3. LCC as a Local Lie Algebra and as a Lie Algebroid

For a LCC manifold $(M, \tau, \Omega, \Theta)$, within each local chart U_{α} endowed with the cosymplectic structure $(\tau_{\alpha}, \Omega_{\alpha})$, the local isomorphism $\sharp_{\alpha}$ and the local two-form Ω_{α} yield a Poisson bivector field Λ_{α} on U_{α} . This is expressed as

$$\Lambda_{\alpha}(\gamma_{\alpha}, \rho_{\alpha}) = \Omega_{\alpha}(\sharp_{\alpha}(\gamma_{\alpha}), \sharp_{\alpha}(\rho_{\alpha})). \quad (5.19)$$

Thus, for two local functions F_α and $\mathcal{H}_\alpha$, we define the associated local Poisson bracket as

$$\{F_\alpha, \mathcal{H}_\alpha\}_\alpha = \Lambda_\alpha(dF_\alpha, d\mathcal{H}_\alpha). \quad (5.20)$$

Following the established conventions, let us now examine these local realizations. First, we substitute (5.19) into (5.20) to obtain

$$\begin{aligned} \{F_\alpha, \mathcal{H}_\alpha\}_\alpha &= \Omega_\alpha(\sharp_\alpha(dF_\alpha), \sharp_\alpha(d\mathcal{H}_\alpha)) = \Omega_\alpha(X_{F_\alpha}, X_{\mathcal{H}_\alpha}) \\ &= e^{-2\sigma_\alpha} \Omega|_\alpha(e^{\sigma_\alpha} X_F|_\alpha, e^{\sigma_\alpha} X_{\mathcal{H}}|_\alpha) \\ &= \Omega|_\alpha(X_F|_\alpha, X_{\mathcal{H}}|_\alpha), \end{aligned} \quad (5.21)$$

where we have utilized the local picture of the Hamiltonian vector fields in (5.12).

On the other hand, the definition in (5.20) allows a direct computation:

$$\begin{aligned} \{F_\alpha, \mathcal{H}_\alpha\}_\alpha &= \Lambda_\alpha(dF_\alpha, d\mathcal{H}_\alpha) = \Lambda_\alpha(d(e^{-\sigma_\alpha} F|_\alpha), d(e^{-\sigma_\alpha} \mathcal{H}|_\alpha)) \\ &= \Lambda_\alpha(e^{-\sigma_\alpha} dF|_\alpha - e^{-\sigma_\alpha} F|_\alpha d\sigma_\alpha, e^{-\sigma_\alpha} d\mathcal{H}|_\alpha - e^{-\sigma_\alpha} \mathcal{H}|_\alpha d\sigma_\alpha) \\ &= e^{-2\sigma_\alpha} \Lambda_\alpha(dF|_\alpha - F|_\alpha d\sigma_\alpha, d\mathcal{H}|_\alpha - \mathcal{H}|_\alpha d\sigma_\alpha) \\ &= e^{-2\sigma_\alpha} \Lambda_\alpha(dF|_\alpha, d\mathcal{H}|_\alpha) - e^{-2\sigma_\alpha} \mathcal{H}|_\alpha \Lambda_\alpha(dF|_\alpha, d\sigma_\alpha) \\ &\quad - e^{-2\sigma_\alpha} F|_\alpha \Lambda_\alpha(d\sigma_\alpha, d\mathcal{H}|_\alpha). \end{aligned} \quad (5.22)$$

The left-hand side being global, as evidenced by the calculation in (5.21), suggests the local identification

$$\Lambda|_\alpha = e^{-2\sigma_\alpha} \Lambda_\alpha. \quad (5.23)$$

See that Λ_α is the Poisson bivector on the open neighbourhood U_α whereas $\Lambda|_\alpha$ is local realization of a global bivector field Λ . As we shall examine in the upcoming paragraphs Λ fails to be a Poisson bivector. We define a global vector field V on LCC manifold M as

$$V = \iota_{2\Theta} \Lambda, \quad V|_\alpha = \iota_{2d\sigma_\alpha} \Lambda|_\alpha \quad (5.24)$$

where Θ is the Lee form.

To establish that (M, V, Λ) constitutes a Jacobi manifold or Kirillov sense a local Lie algebra, let us verify the identities in (2.38). We begin with the local Poisson bivector Λ_α . Since it is Poisson, its Schouten-Nijenhuis bracket is zero, that is

$[\Lambda_\alpha, \Lambda_\alpha] = 0$. We use this then compute

$$\begin{aligned}
[\Lambda_\alpha, \Lambda_\alpha] &= [e^{2\sigma_\alpha} \Lambda|_\alpha, e^{2\sigma_\alpha} \Lambda|_\alpha] \\
&= [e^{2\sigma_\alpha} \Lambda|_\alpha, e^{2\sigma_\alpha}] \wedge \Lambda|_\alpha + e^{2\sigma_\alpha} [e^{2\sigma_\alpha} \Lambda|_\alpha, \Lambda|_\alpha] \\
&= e^{2\sigma_\alpha} [\Lambda|_\alpha, e^{2\sigma_\alpha}] \wedge \Lambda|_\alpha + e^{4\sigma_\alpha} [\Lambda|_\alpha, \Lambda|_\alpha] + e^{2\sigma_\alpha} [e^{2\sigma_\alpha}, \Lambda|_\alpha] \wedge \Lambda|_\alpha \\
&= e^{2\sigma_\alpha} (-\iota_{d(e^{2\sigma_\alpha})} \Lambda|_\alpha) \wedge \Lambda|_\alpha + e^{4\sigma_\alpha} [\Lambda|_\alpha, \Lambda|_\alpha] + e^{2\sigma_\alpha} (-\iota_{d(e^{2\sigma_\alpha})} \Lambda|_\alpha) \wedge \Lambda|_\alpha \\
&= 2e^{4\sigma_\alpha} (-\iota_{2d\sigma_\alpha} \Lambda|_\alpha) \wedge \Lambda|_\alpha + e^{4\sigma_\alpha} [\Lambda|_\alpha, \Lambda|_\alpha],
\end{aligned} \tag{5.25}$$

As this expression is zero, we conclude that

$$[\Lambda|_\alpha, \Lambda|_\alpha] = 2\iota_{2d\sigma_\alpha} \Lambda|_\alpha \wedge \Lambda|_\alpha. \tag{5.26}$$

Since all the tensor fields in this expression admit global realizations, we have the global equality:

$$[\Lambda, \Lambda] = 2V \wedge \Lambda \tag{5.27}$$

where we used (5.24). This satisfies the first requirement in definition (2.38). For the second one, we need to recall some properties from [137]. First, let us compute:

$$[V, \Lambda] = [\iota_{2\Theta} \Lambda, \Lambda] = -\frac{1}{2} \iota_{2\Theta} [\Lambda, \Lambda]. \tag{5.28}$$

Here the second equality is the Schouten-Nijenhuis bracket for Θ and Λ . Further, for a closed one-form Θ , the distribution of the interior product over the Schouten-Nijenhuis bracket reads

$$\begin{aligned}
\iota_{2\Theta} [\Lambda, \Lambda] &= 2\iota_{2\Theta} (V \wedge \Lambda) = 2\iota_{2\Theta} V \wedge \Lambda - 2V \wedge \iota_{2\Theta} \Lambda \\
&= 2\iota_{2\Theta} \iota_{2\Theta} \Lambda \wedge \Lambda - 2\iota_{2\Theta} \Lambda \wedge \iota_{2\Theta} \Lambda = 0
\end{aligned} \tag{5.29}$$

where, in the last calculation, both terms vanish identically due to the antisymmetric properties. Merging the calculations in (5.28) and (5.29), we conclude that $[V, \Lambda] = 0$. Thus, we can conclude that (M, V, Λ) is indeed a Jacobi manifold or in Kirillov sense a local Lie algebra.

Se we define a Jacobi bracket on the space $C^\infty(M)$ as

$$\{F, \mathcal{H}H\}^{(LCC)} = \Lambda(dF, d\mathcal{H}) + FV(H) - \mathcal{H}V(F), \tag{5.30}$$

and we can argue

$$[X_F, X_{\mathcal{H}}] = -X_{\{F, \mathcal{H}\}^{(LCC)}}. \tag{5.31}$$

Hence we conclude that the space of LCC Hamiltonian vector fields forms a Lie algebra

$$\mathfrak{X}_{LCC-ham}(M) = \{X_{\mathcal{H}} \in \mathfrak{X}(M) : \iota_{X_{\mathcal{H}}}\tau = 0, \iota_{X_{\mathcal{H}}}\Omega = d_{\Theta}\mathcal{H} - \langle d_{\Theta}\mathcal{H}, \mathcal{R} \rangle \tau\}, \quad (5.32)$$

where the Lie algebra bracket is defined as the minus of the Jacobi-Lie bracket. This gives that the mapping

$$\Psi : C^{\infty}(M) \longrightarrow \mathfrak{X}_{LCC-ham}(M), \quad \mathcal{H} \mapsto X_{\mathcal{H}} \quad (5.33)$$

is a one-to-one Lie algebra homomorphism.

Lie Algebroid Realization. Suppose once more we have a LCC manifold denoted by $(M, \tau, \Omega, \Theta)$, along with the musical isomorphism $\flat$ defined in equation (2.49) and its inverse $\sharp$. These isomorphisms are denoted as:

$$\flat(X) = X^{\flat}, \quad \sharp(\mu) = \mu^{\sharp} \quad (5.34)$$

for any vector field X and any one-form μ . Using the definition of $\flat$, we can express the identity:

$$\Omega(\mu^{\sharp}, X) = \langle \mu, X \rangle - \tau(\mu^{\sharp})\tau(X). \quad (5.35)$$

Now, we define a bracket operation on the space $\Gamma^1(M)$ of one-form sections as follows:

$$\{\bullet, \bullet\}_{\Gamma^1(M)} : \Gamma^1(M) \times \Gamma^1(M) \longrightarrow \Gamma^1(M), \quad (\mu, \nu) \mapsto \flat([\mu^{\sharp}, \nu^{\sharp}]), \quad (5.36)$$

where the bracket inside the musical mapping $\flat$ is the Jacobi-Lie bracket of vector fields on M . This definition implies that the musical mappings $\flat$ and $\sharp$ form Lie algebra isomorphisms. we compute the Lichnerowicz-deRham (LdR) differential

of Ω at $(\mu^\sharp, \nu^\sharp, X)$, yielding:

$$\begin{aligned}
d_{2\Theta}\Omega(\mu^\sharp, \nu^\sharp, X) &= d\Omega(\mu^\sharp, \nu^\sharp, X) - 2\Theta \wedge \Omega(\mu^\sharp, \nu^\sharp, X) \\
&= \mu^\sharp(\Omega(\nu^\sharp, X)) - \nu^\sharp(\Omega(\mu^\sharp, X)) + X(\Omega(\mu^\sharp, \nu^\sharp)) \\
&\quad - \Omega([\mu^\sharp, \nu^\sharp], X) + \Omega([\mu^\sharp, X], \nu^\sharp) - \Omega([\nu^\sharp, X], \mu^\sharp) \\
&\quad - 2\Theta(\mu^\sharp)\Omega(\nu^\sharp, X) + 2\Theta(\nu^\sharp)\Omega(\mu^\sharp, X) - 2\Theta(X)\Omega(\mu^\sharp, \nu^\sharp) \\
&= \mu^\sharp(\langle \nu, X \rangle) - \mu^\sharp(\tau(\nu^\sharp))\tau(X) - \tau(\nu^\sharp)\mu^\sharp(\tau(X)) \\
&\quad - \nu^\sharp(\langle \mu, X \rangle) + \nu^\sharp(\tau(\mu^\sharp))\tau(X) + \tau(\mu^\sharp)\nu^\sharp(\tau(X)) \\
&\quad + \langle d\iota_{\nu^\sharp}\iota_{\mu^\sharp}\Omega, X \rangle - \langle \{\mu, \nu\}_{\Gamma^1}, X \rangle + \tau([\mu^\sharp, \nu^\sharp])\tau(X) \\
&\quad - \langle \nu, [\mu^\sharp, X] \rangle + \tau(\nu^\sharp)\tau([\mu^\sharp, X]) + \langle \mu, [\nu^\sharp, X] \rangle - \tau(\mu^\sharp)\tau([\nu^\sharp, X]) \\
&\quad - 2\Theta(\mu^\sharp)(\langle \nu, X \rangle - \tau(\nu^\sharp)\tau(X)) + 2\Theta(\nu^\sharp)(\langle \mu, X \rangle - \tau(\mu^\sharp)\tau(X)) \\
&\quad - 2\iota_{\nu^\sharp}\iota_{\mu^\sharp}\Omega\langle \Theta, X \rangle.
\end{aligned} \tag{5.37}$$

In this derivation, we utilized the identity (5.35). Furthermore, applying the commutation property of the Lie derivative and the interior derivative, we find:

$$\mu^\sharp(\langle \tau, X \rangle) - \langle \tau, [\mu^\sharp, X] \rangle = \langle \mathcal{L}_{\mu^\sharp}\tau, X \rangle, \quad \nu^\sharp(\langle \tau, X \rangle) - \langle \tau, [\nu^\sharp, X] \rangle = \langle \mathcal{L}_{\nu^\sharp}\tau, X \rangle. \tag{5.38}$$

Theorem 5.1. *Consider a LCC manifold $(M, \tau, \Omega, \Theta)$. The space $\Lambda^1(M)$ of one-form sections forms a Lie algebra equipped with the following Lie bracket:*

$$\begin{aligned}
\{\mu, \nu\}_{\Gamma^1(M)} &= \mathcal{L}_{\mu^\sharp}\nu - 2\Theta(\mu^\sharp)\nu - \mathcal{L}_{\nu^\sharp}\mu + 2\Theta(\nu^\sharp)\mu + d_{2\Theta}\iota_{\nu^\sharp}\iota_{\mu^\sharp}\Omega \\
&\quad + \tau(\mu^\sharp)(\mathcal{L}_{\nu^\sharp}\tau - 2\Theta(\nu^\sharp)\tau) - \tau(\nu^\sharp)(\mathcal{L}_{\mu^\sharp}\tau - 2\Theta(\mu^\sharp)\tau) \\
&\quad + (\tau([\mu^\sharp, \nu^\sharp]) - \mu^\sharp(\tau(\nu^\sharp)) + \nu^\sharp(\tau(\mu^\sharp)))\tau.
\end{aligned} \tag{5.39}$$

When $\Theta = 0$, the expression simplifies to the bracket for the algebra of differential one-forms on a cosymplectic manifold. In this scenario, the bracket reduces to:

$$\begin{aligned}
\{\mu, \nu\}_{\Gamma^1(M)} &= \mathcal{L}_{\mu^\sharp}\nu - \mathcal{L}_{\nu^\sharp}\mu + d\iota_{\nu^\sharp}\iota_{\mu^\sharp}\Omega + \tau(\mu^\sharp)\mathcal{L}_{\nu^\sharp}\tau - \tau(\nu^\sharp)\mathcal{L}_{\mu^\sharp}\tau \\
&\quad + (\tau([\mu^\sharp, \nu^\sharp]) - \mu^\sharp(\tau(\nu^\sharp)) + \nu^\sharp(\tau(\mu^\sharp)))\tau.
\end{aligned} \tag{5.40}$$

For a LCC manifold $(M, \tau, \Omega, \Theta)$, we have established an algebraic structure (5.39) on the space $\Gamma^1(M)$ of one-form sections. This structure allows us to define the Lie algebroid $(T^*M, \pi_M, M, \sharp, \bullet, \bullet_{\Gamma^1(M)})$ through the following commutative diagram,

where the anchor map is given by the musical isomorphism $\sharp$:

$$\begin{array}{ccc}
 (T^*M, \{\bullet, \bullet\}_{\Gamma^1(M)}) & \xrightarrow{\sharp} & TM \\
 \searrow \pi_M & & \swarrow \tau_M \\
 & M &
 \end{array} \tag{5.41}$$

Demonstrating that the bracket (5.39) satisfies the Leibniz identity is straightforward. Let $F \in C^\infty(M)$ and $\mu, \nu \in \Gamma^1(M)$. We have

$$\begin{aligned}
 \{\mu, F\nu\}_{\Gamma^1(M)} &= \mathcal{L}_{\mu^\sharp}(F\nu) - 2\Theta(\mu^\sharp)F\nu - \mathcal{L}_{(F\nu)^\sharp}\mu + 2\Theta((F\nu)^\sharp)\mu + d_{2\Theta}\iota_{(F\nu)^\sharp}\iota_{\mu^\sharp}\Omega \\
 &\quad + \tau(\mu^\sharp)(\mathcal{L}_{(F\nu)^\sharp}\tau - 2\Theta((F\nu)^\sharp)\tau) - \tau((F\nu)^\sharp)(\mathcal{L}_{\mu^\sharp}\tau - 2\Theta(\mu^\sharp)\tau) \\
 &\quad + \left(\tau([\mu^\sharp, (F\nu)^\sharp]) - \mu^\sharp(\tau((F\nu)^\sharp)) + (F\nu)^\sharp(\tau(\mu^\sharp))\right)\tau \\
 &= \mathcal{L}_{\mu^\sharp}(F)\nu + F\mathcal{L}_{\mu^\sharp}(\nu) - 2F\Theta(\mu^\sharp)\nu - F\mathcal{L}_{\nu^\sharp}\mu - dF \wedge \iota_{\nu^\sharp}\mu + 2F\Theta(\nu^\sharp)\mu \\
 &\quad + dF \wedge \iota_{\nu^\sharp}\iota_{\mu^\sharp}\Omega + Fd\iota_{\nu^\sharp}\iota_{\mu^\sharp}\Omega - 2F\Theta \wedge \iota_{\nu^\sharp}\iota_{\mu^\sharp}\Omega \\
 &\quad + \tau(\mu^\sharp)(F\mathcal{L}_{\nu^\sharp}\tau + dF \wedge \iota_{\nu^\sharp}\tau - 2F\Theta(\nu^\sharp)\tau) - F\tau(\nu^\sharp)(\mathcal{L}_{\mu^\sharp}\tau - 2\Theta(\mu^\sharp)\tau) \\
 &\quad + \left(\mu^\sharp(F)\tau(\nu^\sharp) + F\tau([\mu^\sharp, \nu^\sharp]) - \mu^\sharp(F)\tau(\nu^\sharp) \right. \\
 &\quad \left. - F\mu^\sharp(\tau(\nu^\sharp)) + F\nu^\sharp(\tau(\mu^\sharp))\right)\tau \\
 &= \mathcal{L}_{\mu^\sharp}(F)\nu + F\mathcal{L}_{\mu^\sharp}(\nu) - 2F\Theta(\mu^\sharp)\nu - F\mathcal{L}_{\nu^\sharp}\mu + 2F\Theta(\nu^\sharp)\mu + Fd_{2\Theta}\iota_{\nu^\sharp}\iota_{\mu^\sharp}\Omega \\
 &\quad + F\tau(\mu^\sharp)(\mathcal{L}_{\nu^\sharp}\tau - 2\Theta(\nu^\sharp)\tau) - F\tau(\nu^\sharp)(\mathcal{L}_{\mu^\sharp}\tau - 2\Theta(\mu^\sharp)\tau) \\
 &\quad + F\left(\tau([\mu^\sharp, \nu^\sharp]) - \mu^\sharp(\tau(\nu^\sharp)) + \nu^\sharp(\tau(\mu^\sharp))\right)\tau \\
 &\quad - dF \wedge \iota_{\nu^\sharp}\mu + dF \wedge \iota_{\nu^\sharp}\iota_{\mu^\sharp}\Omega + \tau(\mu^\sharp)dF \wedge \iota_{\nu^\sharp}\tau
 \end{aligned} \tag{5.42}$$

where we have made use of the linearity of $\Theta, \sharp, \tau$, as well as the relations

$$\begin{aligned}
 \mathcal{L}_{F\nu^\sharp}\mu &= F\mathcal{L}_{\nu^\sharp}\mu + dF \wedge \iota_{\nu^\sharp}\mu, \\
 \mathcal{L}_{F\nu^\sharp}\tau &= F\mathcal{L}_{\nu^\sharp}\tau + dF \wedge \iota_{\nu^\sharp}\tau, \\
 [\mu^\sharp, F\nu^\sharp] &= \mu^\sharp(F)\nu^\sharp + F[\mu^\sharp, \nu^\sharp].
 \end{aligned} \tag{5.43}$$

Since the definition of $\flat$ implies that $\iota_{\mu^\sharp}\Omega + \tau(\mu^\sharp)\tau = \mu$, we conclude that

$$\{\mu, F\nu\}_{\Gamma^1(M)} = F\{\mu, \nu\}_{\Gamma^1(M)} + \mathcal{L}_{\mu^\sharp}(F)\nu. \tag{5.44}$$

5.4. Locally Conformally Symplectic Dynamics

An almost symplectic manifold (N, ω) is called locally conformally symplectic (abbreviated as LCS) manifold if the two-form ω is closed locally up to a conformal factor [61, 138, 139]. A LCS manifold admits an open covering by a family of open sets U_α with an associated set of functions ρ_α such that

$$d(e^{-\rho_\alpha} \omega|_\alpha) = 0 \quad (5.45)$$

where $\omega|_\alpha$ is the restriction of the almost symplectic structure ω to U_α . This permits us to define a local two-form $\omega_\alpha = e^{-\rho_\alpha} \omega|_\alpha$ which turns out to be symplectic. In the intersection $U_\alpha \cap U_\beta$ of two open neighbourhoods, the local symplectic two-forms are related in the following way

$$\omega_\beta = e^{-(\rho_\beta - \rho_\alpha)} \omega_\alpha, \quad \kappa_{\beta\alpha} = e^{\rho_\alpha} / e^{\rho_\beta} = e^{-(\rho_\beta - \rho_\alpha)}. \quad (5.46)$$

In the intersection of three open neighbourhoods, the scalars satisfy the cocycle condition $\kappa_{\beta\alpha} \kappa_{\alpha\rho} = \kappa_{\beta\rho}$, so that we can glue the local symplectic two-forms ω_α up to a line bundle $L \mapsto N$ valued two-form on N .

It can easily be seen that the exterior derivative of the local functions coincide in the intersection of two neighbourhoods that is $d\rho_\alpha = d\rho_\beta$. By gluing up these exact one-forms, we arrive at a closed one-form, called the Lee one-form and denoted by θ , that takes the local form $d\rho_\alpha$, see [138]. This observation permits us to get an alternative definition of LCS manifold as follows. An almost symplectic manifold (N, ω) is called a LCS manifold if there exists θ such that

$$d\omega = \theta \wedge \omega. \quad (5.47)$$

In terms of the LdR differential, an almost symplectic manifold (N, ω) is a LCS manifold if and only if

$$d_\theta \omega = 0 \quad (5.48)$$

where d_θ denotes the LdR differential. We denote a LCS manifold admitting a Lee-form θ by the triplet (N, ω, θ) . Notice that a LCS manifold turns out to be a symplectic manifold if the Lee-form is identically zero.

LCS Hamiltonian Dynamics. Consider a LCS manifold (N, ω, θ) , and a local symplectic space $(U_\alpha, \omega_\alpha)$. For a local Hamiltonian function $\mathcal{H}_\alpha$ on U_α , we write

the geometric Hamilton's equation by

$$\iota_{X_\alpha} \omega_\alpha = d\mathcal{H}_\alpha, \quad (5.49)$$

where X_α is the local Hamiltonian vector field function associated to this framework. We glue local Hamiltonian functions $\mathcal{H}_\alpha$ to a section of the line bundle $L \mapsto N$. We also glue the following local functions

$$\mathcal{H}|_\alpha = e^{\rho_\alpha} \mathcal{H}_\alpha \quad (5.50)$$

those obeying the intersection law $e^{\rho_\alpha} \mathcal{H}_\alpha = e^{\rho_\beta} \mathcal{H}_\beta$ to a real valued function H on M . In the light of these realizations, we glue the local Hamilton's equation (5.49) to

$$\iota_{X_{\mathcal{H}}} \omega = d_\theta \mathcal{H}, \quad (5.51)$$

where $X_{\mathcal{H}}$ is the vector field obtained by gluing all the vector fields X_α .

LCS Structure on the Cotangent Bundle. Let ψ be a closed one-form on the base manifold Q and we pull it back to T^*Q by means of the cotangent bundle projection π_Q . This determines a closed semi-basic one-form $\theta = \pi_Q^*(\psi)$ on T^*Q . Using the LdR differential, we define

$$\omega_\theta = -d_\theta(\theta_Q) = \omega_Q + \theta \wedge \theta_Q \quad (5.52)$$

on the cotangent bundle T^*Q . Since $d\omega_\theta = \theta \wedge \omega_\theta$ holds, the triple $(T^*Q, \omega_\theta, \theta)$ is a LCS manifold [140–143]. It is important to note that all LCS manifolds admit such a local picture.

LCS vs. LCC. Let $(M, \eta, \Omega; \Theta)$ be a $(2n+1)$ -dimensional LCC manifold equipped with a Lee-form Θ , and then consider a chart U_α with induced cosymplectic structure $(U_\alpha, \eta_\alpha, \Omega_\alpha)$. Referring to the local cosymplectic space $(U_\alpha, \eta_\alpha, \Omega_\alpha)$ we determine a local symplectic space $(U_\alpha \times \mathbb{R}, \overline{\Omega}_\alpha)$ where the symplectic two-form is determined to be

$$\overline{\Omega}_\alpha = \pi^* \Omega_\alpha + \pi^* \eta_\alpha \wedge ds_\alpha \quad (5.53)$$

where s_α is the coordinate on the real line. Considering another patch, we have that

$$\begin{aligned} \overline{\Omega}_\beta &= \pi^* \Omega_\beta + \pi^* \eta_\beta \wedge ds_\beta \\ &= \pi^* \lambda_{\beta\alpha} \Omega_\alpha + \pi^* \kappa_{\beta\alpha} \eta_\alpha \wedge \pi^* \kappa_{\beta\alpha} ds_\alpha \\ &= \pi^* \lambda_{\beta\alpha} \left(\pi^* \Omega_\alpha + \pi^* (\eta_\alpha \wedge ds_\alpha) \right) = \pi^* \lambda_{\beta\alpha} \overline{\Omega}_\alpha, \end{aligned} \quad (5.54)$$

where we have assumed the local identification

$$ds_\beta = \pi^* \kappa_{\beta\alpha} ds_\alpha. \quad (5.55)$$

Let us glue the local extended charts $U_\alpha \times \mathbb{R}$ and obtain the following manifold

$$\overline{M} := \bigsqcup_\alpha (U_\alpha \times \mathbb{R}). \quad (5.56)$$

See that, $\overline{M}$ is a one-dimensional fiber bundle over the LCC manifold M and it is not necessarily trivial. We denote this by $\pi : \overline{M} \mapsto M$. Being the collection of local symplectic spaces, we first examine that whether or not the local symplectic forms can be glued up to a global one on $\overline{M}$. Instead, we notice that, the local character $\overline{\Omega}_\beta = \pi^* \lambda_{\beta\alpha} \overline{\Omega}_\alpha$ provides a global line bundle $\overline{L} \mapsto \overline{M}$ valued two-form on $\overline{M}$. In order to arrive at global two-forms on $\overline{M}$, we introduce the local form

$$\overline{\Omega}|_\alpha := e^{2(\sigma_\alpha \circ \pi)} \overline{\Omega}_\alpha. \quad (5.57)$$

A direct observation gives that these local two-forms coincide in the overlapping spaces, that is, $\overline{\Omega}|_\alpha = \overline{\Omega}|_\beta$. So that they can be glued up to a well-defined global real valued two-form $\overline{\Omega}$ on $\overline{M}$. In the global view, we have that

$$d\overline{\Omega} = 2\pi^* \Theta \wedge \overline{\Omega}, \quad d_{2\pi^* \Theta} \overline{\Omega} = 0. \quad (5.58)$$

This reads that $(\overline{M}, \overline{\Omega})$ is a locally conformally symplectic manifold with Lee-form $2\pi^* \Theta$. Here, $d_{2\pi^* \Theta}$ stands for the LdR differential.

See that, the one-forms defined in (5.55) can be glued up only to a global line bundle $\overline{L} \mapsto \overline{M}$ valued one-form. On the other hand, we can define a family of local forms

$$u|_\alpha = e^{\sigma_\alpha \circ \pi} ds_\alpha \quad (5.59)$$

which, in the light of (5.55), can be glued up to the following one-form

$$u = d_{\pi^* \Theta} s = ds - s\pi^* \Theta. \quad (5.60)$$

See that this one-form is not closed but it is closed under the LdR differential, that is,

$$du = \pi^* \Theta \wedge u, \quad d_{\pi^* \Theta} u = 0. \quad (5.61)$$

Referring to all these global formulations, we state the following theorem.

Theorem 5.2. *Consider a LCC manifold (M, η, Ω) with a Lee-form Θ , and then assume a one-dimensional fibration $\pi : \overline{M} \rightarrow M$ where $\overline{M}$ is defined as in (5.56). Define a two-form on $\overline{M}$ as*

$$\overline{\Omega} = \pi^*\Omega + \pi^*\eta \wedge u, \quad (5.62)$$

where u is the one-form in (5.59). Then, (M, η, Ω) is a LCC manifold with a Lee-form Θ if and only if $(\overline{M}, \overline{\Omega})$ is a LCS manifold with Lee-form $2\pi^*\Theta$.

Proof. Let us consider the two-form $\overline{\Omega}$ in (5.62) and take the exterior derivative, then

$$\begin{aligned} d\overline{\Omega} &= d(\pi^*\Omega + \pi^*\eta \wedge u) = \pi^*d\Omega + \pi^*d\eta \wedge u - \pi^*\eta \wedge du \\ &= \pi^*(2\Theta \wedge \Omega) + \pi^*(\Theta \wedge \eta) \wedge u - \pi^*\eta \wedge \pi^*\Theta \wedge u \\ &= 2\pi^*\Theta \wedge \pi^*\Omega + 2\pi^*\Theta \wedge \pi^*\eta \wedge u \\ &= 2\pi^*\Theta \wedge (\pi^*\Omega + \pi^*\eta \wedge u) \\ &= 2\pi^*\Theta \wedge \overline{\Omega}, \end{aligned} \quad (5.63)$$

where we have used the LCC structure identities (5.5) in the second line. This shows that the two-form $\overline{\Omega}$ is a LCS two-form with Lee-form $2\pi^*\Theta$. Compare this with the display in (5.58) to see that both the local and the global approaches read the same result. ■

5.5. Geometric HJ for LCC Dynamics

Let us completely shift our perspective to the fibrations and sections discussed in Section 2.4. We recontextualize the extended cotangent bundle $T^*Q \times \mathbb{R}$ and the cotangent bundle $T^*(Q \times \mathbb{R})$. Instead of regarding $T^*Q \times \mathbb{R}$ as hosting cosymplectic and symplectic structures, we now treat it as a LCC manifold. This means we consider it equipped with the differential forms $(\tau_\Theta, \Omega_\Theta)$ in (5.8).

Before diving into the computation of a geometric HJ theory for this realization,

let us begin by constructing a commutative diagram

$$\begin{array}{ccccc}
 T^*(Q \times \mathbb{R}) & \xrightarrow{X_{\mathcal{H}^s}} & TT^*(Q \times \mathbb{R}) & & (5.64) \\
 \uparrow \tilde{\gamma} & \searrow \pi & \downarrow T\pi & & \\
 & T^*Q \times \mathbb{R} & \xrightarrow{E_{\mathcal{H}}} & T(T^*Q \times \mathbb{R}) & \\
 \downarrow \pi_{Q \times \mathbb{R}} & \nearrow \gamma & \downarrow T\pi_{Q \times \mathbb{R}} & \nearrow T(\pi \circ \tilde{\gamma}) & \\
 Q \times \mathbb{R} & \xrightarrow{X_{\mathcal{H}^s}^{\tilde{\gamma}}} & T(Q \times \mathbb{R}) & & \\
 & \downarrow E_{\mathcal{H}}^{\gamma} & & &
 \end{array}$$

that corresponds to the fibrations:

$$\begin{aligned}
 \pi : T^*Q \times T^*\mathbb{R} &\longrightarrow T^*Q \times \mathbb{R}, & (z, t, s) &\mapsto (z, t), \\
 \tau : T^*Q \times \mathbb{R} &\longrightarrow Q \times \mathbb{R}, & (z, t) &\mapsto (\pi_Q(z), t).
 \end{aligned} \tag{5.65}$$

In view of Diagram 5.64, we have a section γ of the fibration τ from $T^*Q \times \mathbb{R}$ to $Q \times \mathbb{R}$, and a section $\tilde{\gamma}$ of the cotangent bundle projection $\pi_{Q \times \mathbb{R}}$ defined on $T^*(Q \times \mathbb{R})$. The characterization of these sections are

$$\begin{aligned}
 \tilde{\gamma} : Q \times \mathbb{R} &\longrightarrow T^*Q \times T^*\mathbb{R}, & (q, t) &\mapsto (\tilde{\gamma}(q, t), t, \bar{\gamma}(q, t)), \\
 \gamma : Q \times \mathbb{R} &\longrightarrow T^*Q \times \mathbb{R}, & (q, t) &\mapsto (\tilde{\gamma}(q, t), t).
 \end{aligned} \tag{5.66}$$

The time-dependent Hamiltonian function $\mathcal{H}$ on the extended cotangent bundle $T^*Q \times \mathbb{R}$ gives rise to the LCC evolution vector field $E_{\mathcal{H}}$, as defined by (5.18). This vector field is visualized in Diagram 5.64 as an arrow originating from $T^*Q \times \mathbb{R}$ and pointing to its tangent bundle $T(T^*Q \times \mathbb{R})$.

For the extended Hamiltonian function

$$\mathcal{H}^s(z, t, s) = \pi^*H(z, t) + s \tag{5.67}$$

on the cotangent bundle $T^*(Q \times \mathbb{R})$, the LCS Hamiltonian vector field is determined through (5.51). Here, $\mathcal{H}^s$ is precisely defined as in (5.67). In Diagram 5.64, the Hamiltonian vector field $X_{\mathcal{H}^s}$ is depicted as an arrow originating from $T^*(Q \times \mathbb{R})$ and pointing to its tangent bundle $TT^*(Q \times \mathbb{R})$.

We utilize the projection γ to project the LCC evolution vector field $E_{\mathcal{H}}$ onto the manifold $Q \times \mathbb{R}$, and similarly, we use the projection $\tilde{\gamma}$ to project the LCS Hamiltonian vector field $X_{\mathcal{H}^s}$ onto $Q \times \mathbb{R}$. This yields the expressions:

$$X_{\mathcal{H}^s}^{\tilde{\gamma}} = T\pi_{Q \times \mathbb{R}} \circ X_{\mathcal{H}^s} \circ \tilde{\gamma}, \quad E_H^{\gamma} := T\tau \circ E_{\mathcal{H}} \circ \gamma. \tag{5.68}$$

These projection operations mirror those in (2.70), and a direct computation reveals that the projected vector fields coincide. They are represented graphically by a single arrow originating from $Q \times \mathbb{R}$ and pointing to its tangent bundle.

Theorem 5.3. *Suppose $\tilde{\gamma}$ is a closed LdR one-form on $Q \times \mathbb{R}$, meaning $d_\Theta \tilde{\gamma} = 0$. Employing the notation introduced in this section, the following conditions are equivalent:*

1. $X_{\mathcal{H}^s}^{\tilde{\gamma}}$ and $E_{\mathcal{H}}$ are γ -related, that is,

$$T\pi \circ T\tilde{\gamma} \circ X_{\mathcal{H}^s}^{\tilde{\gamma}} = E_{\mathcal{H}} \circ \pi \circ \tilde{\gamma}. \quad (5.69)$$

- 2.

$$d_\Psi(\mathcal{H}^s \circ \tilde{\gamma}) \in \langle d_\Psi t \rangle, \quad (5.70)$$

where $\langle d_\Psi t \rangle$ denotes the span space of $d_\Psi t$.

Proof. Suppose $X_{\mathcal{H}^s}^{\tilde{\gamma}}$ and $E_{\mathcal{H}}$ are related by γ . So, the pull-back γ^* of γ distributes over the interior derivative, i.e.,

$$\tilde{\gamma}^* \pi^* \iota_{E_{\mathcal{H}}} \Omega = \iota_{X_{\mathcal{H}^s}^{\tilde{\gamma}}} \tilde{\gamma}^* \pi^* \Omega. \quad (5.71)$$

The computation prompted by the left-hand side of (5.70) unfolds as follows:

$$\begin{aligned} d_\Psi(\mathcal{H}^s \circ \tilde{\gamma}) &= d_\Psi \tilde{\gamma}^* \mathcal{H}^s = \tilde{\gamma}^* d_\Theta \mathcal{H}^s \\ &= \tilde{\gamma}^* d_\Theta(\pi^* H + s) \\ &= \tilde{\gamma}^* \pi^* d_\Theta H + d_\Psi \tilde{\gamma}^* s \\ &= \tilde{\gamma}^* \pi^* (\iota_{E_{\mathcal{H}}} \Omega + \langle d_\Theta H, \mathcal{R} \rangle \tau) + d_\Psi \bar{\gamma} \\ &= \iota_{X_{\mathcal{H}^s}^{\tilde{\gamma}}} (\tilde{\gamma}^* \pi^* \Omega) + \iota_{X_{\mathcal{H}^s}^{\tilde{\gamma}}} \tilde{\gamma}^* (\pi^* \tau \wedge u) - \iota_{X_{\mathcal{H}^s}^{\tilde{\gamma}}} \tilde{\gamma}^* (\pi^* \tau \wedge u) + \gamma^* (\langle d_\Theta H, \mathcal{R} \rangle \tau) + d_\Psi \bar{\gamma} \\ &= \iota_{X_{\mathcal{H}^s}^{\tilde{\gamma}}} \tilde{\gamma}^* \bar{\Omega} - (\iota_{X_{\mathcal{H}^s}^{\tilde{\gamma}}} \tilde{\gamma}^* \pi^* \tau) \tilde{\gamma}^* u + (\iota_{X_{\mathcal{H}^s}^{\tilde{\gamma}}} \tilde{\gamma}^* u) \tilde{\gamma}^* \pi^* \tau + \gamma^* (\langle d_\Theta H, \mathcal{R} \rangle \tau) + d_\Psi \bar{\gamma} \\ &= 0 - (\tilde{\gamma}^* \pi^* \iota_{E_{\mathcal{H}}} \tau) d_\Psi \bar{\gamma} + (\iota_{X_{\mathcal{H}^s}^{\tilde{\gamma}}} d_\Psi \bar{\gamma}) \gamma^* \tau + \gamma^* (\langle d_\Theta H, \mathcal{R} \rangle \tau) + d_\Psi \bar{\gamma} \\ &= -d_\Psi \bar{\gamma} + (\iota_{X_{\mathcal{H}^s}^{\tilde{\gamma}}} d_\Psi \bar{\gamma}) \gamma^* \tau + \gamma^* (\langle d_\Theta H, \mathcal{R} \rangle \tau) + d_\Psi \bar{\gamma} \\ &= (\iota_{X_{\mathcal{H}^s}^{\tilde{\gamma}}} d_\Psi \bar{\gamma} + \gamma^* \langle d_\Theta H, \mathcal{R} \rangle) \gamma^* \tau. \end{aligned}$$

In the first line, we capitalize on the commutativity between the pull-back operation and the Lichnerowicz-deRham differential. Moving to the second line, we leverage the definition of the Hamiltonian function $\mathcal{H}^s$ as outlined in (5.67). Upon reaching

the fourth line, we invoke the definitions (5.18) of the evolution vector field $E_{\mathcal{H}}$ and the explicit formulation (5.66) of the section $\tilde{\gamma}$.

As we transition to the fifth line, we employ the identity (5.71), and to facilitate our calculations, we introduce and subsequently subtract the term $\iota_{X_{\mathcal{H}^s}^{\tilde{\gamma}}} \tilde{\gamma}^* \pi^* \tau \wedge u$, aiming to extract the LCS two-form $\overline{\Omega}$ as presented in (5.62). By the sixth line, we exploit the distributive property of the pull-back operation $\tilde{\gamma}^*$ on wedge products, followed by the application of the generalized Leibniz property of the interior derivative.

In the seventh line, appealing to the LdR closedness of $\tilde{\gamma}$, we establish the vanishing of $\tilde{\gamma}^* \overline{\Omega}$ identically. Moreover, we utilize the identity (5.71) and the definition of $\tilde{\gamma}$.

Proceeding to the eighth line, due to the definition of $E_{\mathcal{H}}$, the first and last terms cancel each other. The distributive nature of the pull-back operation γ^* yields the outcome in the last line. Notably, since $\gamma^* \tau$ indeed equals $d_{\theta} t$, we ultimately arrive at the desired result. The converse statement can be demonstrated through direct computation in local coordinates. ■

6. CONCLUSIONS

This thesis comprehensively studies Lie structures and their (de)couplings. It begins with a quick introduction to the basic tools of geometric mechanics in Chapter 2, which includes symplectic, Poisson, Jacobi, cosymplectic, and contact manifolds, as well as dynamical formulations available on these geometries.

In Chapter 3, our emphasis is on Lie and Leibniz algebras, and Lie algebroids. We present algebraic (de)couplings of all these geometries from a bicycle double cross product perspective. This approach provides the most general couplings of two vector bundles to form a Lie algebroid. The implications of this novel construction is studied on dynamical equations both in Lagrangian and Hamiltonian forms.

Chapter 4 begins with a cohomological analysis of n -Lie algebras, with the first result being the generalized derivation and classification of these geometries. Then, (de)couplings of 3-Lie algebroids are explored using the bicycle double cross product strategy. The choice of the order 3 here is due to its close relationship with bi-Hamiltonian dynamics.

In Chapter 5, geometric Hamilton-Jacobi theory is developed for Hamiltonian dynamics on locally conformally cosymplectic manifolds. It is established that locally conformally cosymplectic manifolds are examples of Lie algebroids.

The thesis provides pure algebraic constructions using geometric analysis and dynamical formulations. In this manner, it is proper for further analysis and discovering various applications across algebraic, geometric, and physical points of view.

REFERENCES

- [1] Abraham R., Marsden J. E., (1978), “Foundations of mechanics”, Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass.
- [2] Arnold V. I., (1989), “Mathematical methods of classical mechanics”, Vol. 60 of Graduate Texts in Mathematics, second Edition, Springer-Verlag, New York. Translated from the Russian by K. Vogtmann and A. Weinstein.
- [3] de León M., Rodrigues P. R., (1989), “Methods of differential geometry in analytical mechanics”, Vol. 158 of North-Holland Mathematics Studies, North-Holland Publishing Co., Amsterdam.
- [4] Holm D., Schmah T., Stoica C., (2009), “Geometric mechanics and symmetry”, Vol. 12 of Oxford Texts in Applied and Engineering Mathematics, Oxford University Press, Oxford. From finite to infinite dimensions, With solutions to selected exercises by David C. P. Ellis.
- [5] Libermann P., Marle C.-M., (1987), “Symplectic geometry and analytical mechanics”, Vol. 35 of Mathematics and its Applications, D. Reidel Publishing Co., Dordrecht. Translated from the French by Bertram Eugene Schwarzbach.
- [6] Marsden J. E., Weinstein A., (1974), “Reduction of symplectic manifolds with symmetry”, Rep. Mathematical Phys., 5(1), 121–130.
- [7] Marsden J. E., Ratiu T., (1986), “Reduction of Poisson manifolds”, Lett. Math. Phys., 11(2), 161–169.
- [8] Marsden J. E., Scheurle J., (1993), The reduced Euler-Lagrange equations, in “Dynamics and control of mechanical systems (Waterloo, ON, 1992)”, Vol. 1 of Fields Inst. Commun., Amer. Math. Soc., Providence, RI, 139–164.
- [9] Marsden J. E., Ratiu T. S., (1999), “Introduction to mechanics and symmetry”, Vol. 17 of Texts in Applied Mathematics, second Edition, Springer-Verlag, New York. A basic exposition of classical mechanical systems.
- [10] Bhaskara K. H., Viswanath K., (1988), “Poisson algebras and Poisson manifolds”, Vol. 174 of Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York.
- [11] Laurent-Gengoux C., Pichereau A., Vanhaecke P., (2013), “Poisson structures”, Vol. 347 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer, Heidelberg.
- [12] Vaisman I., (2012), “Lectures on the geometry of Poisson manifolds”, Vol. 118, Birkhäuser.
- [13] Weinstein A., (1983), “The local structure of Poisson manifolds”, J. Differential Geom., 18(3), 523–557.

- [14] Weinstein A., (1998), “Poisson geometry”, *Differential Geom. Appl.*, 9(1-2), 213–238. Symplectic geometry.
- [15] Agore A. L., Militaru G., (2015), “Jacobi and Poisson algebras”, *J. Noncommut. Geom.*, 9(4), 1295–1342.
- [16] Lichnerowicz A., (1978), “Les variétés de Jacobi et leurs algèbres de Lie associées”, *J. Math. Pures Appl.* (9), 57(4), 453–488.
- [17] Marle C., (1991), On Jacobi manifolds and Jacobi bundles, in “Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989)”, Vol. 20 of *Math. Sci. Res. Inst. Publ.*, Springer, New York, 227–246.
- [18] Vaisman I., (2002), “Jacobi manifolds”, *Selected topics in Geometry and Mathematical Physics*, 1, 81–100.
- [19] Nambu Y., (1973), “Generalized hamiltonian dynamics”, *Phys. Rev. D*, 7, 2405–2412.
- [20] Filippov V. T., (1985), “ n -Lie algebras”, *Sibirsk. Mat. Zh.*, 26(6), 126–140, 191.
- [21] Takhtajan L., (1994), “On foundation of the generalized Nambu mechanics”, *Comm. Math. Phys.*, 160(2), 295–315.
- [22] Fokas A., Gel’fand I., (1993), Bi-hamiltonian structures and integrability, in “Important Developments in Soliton Theory”, Springer, 259–282.
- [23] Magri F., Casati P., Falqui G., Pedroni M., (2004), “Eight lectures on integrable systems”, *Integrability of nonlinear systems*, 209–250.
- [24] Gautheron P., (1996), “Some remarks concerning Nambu mechanics”, *Letters in Mathematical Physics*, 37(1), 103–116.
- [25] Daletskii Y. L., Takhtajan L. A., (1997), “Leibniz and Lie algebra structures for Nambu algebra”, *Lett. Math. Phys.*, 39(2), 127–141.
- [26] Bai C., Guo L., Sheng Y., (2019), “Bialgebras, the classical Yang-Baxter equation and Manin triples for 3-Lie algebras”, *Adv. Theor. Math. Phys.*, 23(1), 27–74.
- [27] Zhang T., (2022), “Extending structures for 3-Lie algebras”, *Comm. Algebra*, 50(4), 1469–1497.
- [28] Song L., Tang R., (2019), “Cohomologies, deformations and extensions of n -Hom-Lie algebras”, *J. Geom. Phys.*, 141, 65–78.
- [29] Hochschild G., Serre J.-P., (1953a), “Cohomology of group extensions”, *Trans. Amer. Math. Soc.*, 74, 110–134.
- [30] Hochschild G., Serre J. P., (1953b), “Cohomology of Lie algebras”, *Ann. of Math.* (2), 57, 591–603.
- [31] Song L., Jiang J., (2018), “Generalized derivation extensions of 3-Lie algebras and corresponding Nambu-Poisson structures”, *J. Geom. Phys.*, 124, 74–85.

- [32] Cendra H., Holm D. D., Marsden J. E., Ratiu T. S., (1998), Lagrangian reduction, the Euler-Poincaré equations, and semidirect products, in “Geometry of differential equations”, Vol. 186 of Amer. Math. Soc. Transl. Ser. 2, Amer. Math. Soc., Providence, RI, 1–25.
- [33] Kupershmidt B. A., Ratiu T., (1983), “Canonical maps between semidirect products with applications to elasticity and superfluids”, Comm. Math. Phys., 90(2), 235–250.
- [34] Marsden J. E., Ratiu T., Weinstein A., (1984*a*), Reduction and Hamiltonian structures on duals of semidirect product Lie algebras, in “Fluids and plasmas: geometry and dynamics (Boulder, Colo., 1983)”, Vol. 28 of Contemp. Math., Amer. Math. Soc., Providence, RI, 55–100.
- [35] Marsden J. E., Rațiu T., Weinstein A., (1984*b*), “Semidirect products and reduction in mechanics”, Transactions of the american mathematical society, 281(1), 147–177.
- [36] Majid S., (1990*a*), “Matched pairs of Lie groups associated to solutions of the Yang-Baxter equations”, Pacific Journal of Mathematics, 141(2), 311–332.
- [37] Majid S., (1990*b*), “Physics for algebraists: Non-commutative and non-cocommutative hopf algebras by a bicrossproduct construction”, Journal of Algebra, 130(1), 17–64.
- [38] Esen O., Sütü S., (2017), “Lagrangian dynamics on matched pairs”, Journal of Geometry and Physics, 111, 142–157.
- [39] Esen O., Sütü S., (2016), “Hamiltonian dynamics on matched pairs”, International Journal of Geometric Methods in Modern Physics, 13(10), 1650128.
- [40] Esen O., Kudeyt M., Sütü S., (2021), “Second order Lagrangian dynamics on double cross product groups”, J. Geom. Phys., 159, 103934, 18.
- [41] Esen O., Özcan G., Sütü S., (2022), “On extensions, Lie-Poisson systems, and dissipation”, J. Lie Theory, 32(2), 327–382.
- [42] Esen O., Sütü S., (2021), “Discrete dynamical systems over double cross-product lie groupoids”, International Journal of Geometric Methods in Modern Physics, 18(04), 2150057.
- [43] Esen O., Pavelka M., Grmela M., (2017), “Hamiltonian coupling of electromagnetic field and matter”, Int. J. Adv. Eng. Sci. Appl. Math., 9(1), 3–20.
- [44] Esen O., Grmela M., Gümral H., Pavelka M., (2019), “Lifts of symmetric tensors: fluids, plasma, and Grad hierarchy”, Entropy, 21(9), Paper No. 907, 33.
- [45] Esen O., Sütü S., (2021), “Matched pair analysis of the Vlasov plasma”, J. Geom. Mech., 13(2), 209–246.
- [46] Ajji A., Chaouki J., Esen O., Grmela M., Klika V., Pavelka M., (2023), “On geometry of multiscale mass action law and its fluctuations”, Phys. D, 445, Paper No. 133642, 22.

- [47] Agore A. L., Militaru G., (2011), “Extending structures II: The quantum version”, *J. Algebra*, 336, 321–341.
- [48] Agore A. L., Militaru G., (2014*a*), “Extending structures for Lie algebras”, *Monatsh. Math.*, 174(2), 169–193.
- [49] Agore A. L., Militaru G., (2014*b*), “Extending structures I: the level of groups”, *Algebr. Represent. Theory*, 17(3), 831–848.
- [50] Agore A. L., Militaru G., (2020), “Extending structures fundamentals and applications”, *Monographs and Research Notes in Mathematics*, CRC Press, Boca Raton, FL.
- [51] Esen O., Guha P., Sütülü S., (2023), “Bicocycle double cross constructions”, *J. Algebra Appl.*, 22(12), Paper No. 2350254, 36.
- [52] Cariñena J. F., Gràcia X., Marmo G., Martínez E., Muñoz Lecanda M. C., Román-Roy N., (2006), “Geometric Hamilton-Jacobi theory”, *Int. J. Geom. Methods Mod. Phys.*, 3(7), 1417–1458.
- [53] Goldstein H., (1980), “Classical mechanics”, *Addison-Wesley Series in Physics*, second Edition, Addison-Wesley Publishing Co., Reading, Mass.
- [54] Esen O., de León M., Lainz M., Sardón C., Zajac M., (2022), “Reviewing the geometric Hamilton-Jacobi theory concerning Jacobi and Leibniz identities”, *arXiv preprint arXiv:2202.06896*.
- [55] Bhaskara K. H., Viswanath K., (1988), “Poisson algebras and Poisson manifolds”, Vol. 174 of *Pitman Research Notes in Mathematics Series*, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York.
- [56] Dufour J., Zung N. T., (2005), “Poisson structures and their normal forms”, Vol. 242 of *Progress in Mathematics*, Birkhäuser Verlag, Basel.
- [57] Lichnerowicz A., (1977), “Les variétés de Poisson et leurs algèbres de Lie associées”, *J. Differential Geometry*, 12(2), 253–300.
- [58] Vaisman I., (1994), “Lectures on the geometry of Poisson manifolds”, Vol. 118 of *Progress in Mathematics*, Birkhäuser Verlag, Basel.
- [59] Kirillov A. A., (1976), “Local Lie algebras”, *Uspehi Mat. Nauk*, 31(4(190)), 57–76.
- [60] Banyaga A., (2002), “Some properties of locally conformal symplectic structures”, *Comment. Math. Helv.*, 77(2), 383–398.
- [61] Bazzoni G., (2018), “Locally conformally symplectic and Kähler geometry”, *EMS Surv. Math. Sci.*, 5(1-2), 129–154.
- [62] Esen O., de León M., Sardón C., Zajac M., (2021), “Hamilton-Jacobi formalism on locally conformally symplectic manifolds”, *J. Math. Phys.*, 62(3), Paper No. 033506, 15.
- [63] Vaisman I., (1985), “Locally conformal symplectic manifolds.”, *International Journal of Mathematics and Mathematical Sciences*, 8, 521–536.

- [64] Ateşli B., Esen O., de León M., Sardón C., (2023), “On locally conformally cosymplectic Hamiltonian dynamics and Hamilton-Jacobi theory”, *J. Phys. A*, 56(1), Paper No. 015204, 40.
- [65] Chinea D., de León M., Marrero J. C., (1991), “Locally conformal cosymplectic manifolds and time-dependent Hamiltonian systems”, *Comment. Math. Univ. Carolin.*, 32(2), 383–387.
- [66] Arnold V. I., (1989), “Mathematical methods of classical mechanics”, Vol. 60 of *Graduate Texts in Mathematics*, second Edition, Springer-Verlag, New York. Translated from the Russian by K. Vogtmann and A. Weinstein.
- [67] Ateşli B., Esen O., de León M., Sardón C., (2023), “On non-autonomous hamiltonian dynamics, dual spaces, and kinetic lifts”, arXiv preprint arXiv:2308.10336.
- [68] de León M., Gaset J., Gràcia X., Muñoz Lecanda M. C., Rivas X., (2023), “Time-dependent contact mechanics”, *Monatsh. Math.*, 201(4), 1149–1183.
- [69] Gaset J., López-Gordón A., Rivas X., (2023), “Symmetries, conservation and dissipation in time-dependent contact systems”, *Fortschr. Phys.*, 71(8-9), Paper No. 2300048, 17.
- [70] Rivas X., Torres D., (2023), “Lagrangian-Hamiltonian formalism for cocontact systems”, *J. Geom. Mech.*, 15(1), 1–26.
- [71] de León M., Sardón C., (2017), “Cosymplectic and contact structures for time-dependent and dissipative Hamiltonian systems”, *J. Phys. A*, 50(25), 255205, 23.
- [72] Bravetti A., (2017), “Contact Hamiltonian dynamics: the concept and its use”, *Entropy*, 19(10), Paper No. 535, 12.
- [73] Bravetti A., Cruz H., Tapias D., (2017), “Contact Hamiltonian mechanics”, *Ann. Physics*, 376, 17–39.
- [74] de León M., Lainz M., Muñiz Brea A., (2021), “The Hamilton–Jacobi theory for contact Hamiltonian systems”, *Mathematics*, 9(16), 1993.
- [75] de León M., Lainz Valcázar M., (2019), “Contact Hamiltonian systems”, *J. Math. Phys.*, 60(10), 102902, 18.
- [76] Esen O., Lainz M., de León M., Marrero J. C., (2021), “Contact dynamics: Legendrian and lagrangian submanifolds”, *Mathematics*, 9(21), 2704.
- [77] Guenther R. B., Schwerdtfeger H., Herglotz G., Guenther C., Gottsch J., (1996), “The Herglotz lectures on contact transformations and Hamiltonian systems”, Juliusz Schauder Center for Nonlinear Studies. Nicholas Copernicus University.
- [78] Herglotz G., (1930), “Berührungstransformationen, lectures at the university of göttingen”, University of Göttingen, Göttingen.
- [79] Herglotz G., (1985), “Vorlesungen über die Mechanik der Kontinua”, Vol. 3 of *Teubner-Archiv zur Mathematik [Teubner Archive on Mathematics]*, BSB B. G. Teubner Verlagsgesellschaft, Leipzig. With a preface by H. Beckert,

Edited and with a foreword by Ronald B. Guenther and Hans Schwerdtfeger,
With English, French and Russian summaries.

- [80] de León M., Lainz M., (2019a), “A review on contact Hamiltonian and Lagrangian systems”, *Revista de la Real Academia de Ciencias Canaria* Volume XXXI-Mathematics, 1–46.
- [81] de León M., Lainz M. V., (2019b), “Singular Lagrangians and precontact Hamiltonian systems”, *International Journal of Geometric Methods in Modern Physics*, 16(10), 1950158.
- [82] Cendra H., Marsden J. E., Pekarsky S., Ratiu T. S., (2003), “Variational principles for Lie-Poisson and Hamilton-Poincaré equations”, *Mosc. Math. J.*, 3(3), 833–867, 1197–1198.
- [83] Marsden J. E., Ratiu T. S., (1999), “Introduction to mechanics and symmetry”, Vol. 17 of *Texts in Applied Mathematics*, second Edition, Springer-Verlag, New York. A basic exposition of classical mechanical systems.
- [84] Loday J.-L., Pirashvili T., (1993), “Universal enveloping algebras of Leibniz algebras and (co)homology”, *Math. Ann.*, 296(1), 139–158.
- [85] Feldvoss J., (2019), Leibniz algebras as non-associative algebras, in “Nonassociative mathematics and its applications”, Vol. 721 of *Contemp. Math.*, Amer. Math. Soc., Providence, RI, 115–149.
- [86] Mackenzie K. C. H., (2005), “General theory of Lie groupoids and Lie algebroids”, Vol. 213 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge.
- [87] Mokri T., (1997), “Matched pairs of Lie algebroids”, *Glasgow Math. J.*, 39(2), 167–181.
- [88] Pradines J., (1967), “Théorie de Lie pour les groupoïdes différentiables. Calcul différentiel dans la catégorie des groupoïdes infinitésimaux”, *C. R. Acad. Sci. Paris Sér. A-B*, 264, A245–A248.
- [89] Mackenzie K. C. H., (1987), “Lie groupoids and Lie algebroids in differential geometry”, Vol. 124 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge.
- [90] Mackenzie K. C. H., (1995), “Lie algebroids and Lie pseudoalgebras”, *Bull. London Math. Soc.*, 27(2), 97–147.
- [91] Weinstein A., (2001), Groupoids: unifying internal and external symmetry. A tour through some examples, in “Groupoids in analysis, geometry, and physics (Boulder, CO, 1999)”, Vol. 282 of *Contemp. Math.*, Amer. Math. Soc., Providence, RI, 1–19.
- [92] Martínez E., (2001), “Lagrangian mechanics on Lie algebroids”, *Acta Appl. Math.*, 67(3), 295–320.
- [93] Weinstein A., (1996a), Lagrangian mechanics and groupoids, in “Mechanics day (Waterloo, ON, 1992)”, Vol. 7 of *Fields Inst. Commun.*, Amer. Math. Soc., Providence, RI, 207–231.

- [94] Weinstein A., (1996*b*), “Lagrangian mechanics and groupoids”, Mechanics day (Waterloo, ON, 1992), 207-231, Fields Inst. Commun. 7, Amer. Math. Soc., Providence, RI.
- [95] Anahory Simoes A., Colombo L., de León M., Salgado M., Souto S., (2024), “Euler–Lagrange–Herglotz equations on Lie algebroids”, Anal. Math. Phys., 14(1), Paper No. 3.
- [96] Anahory Simoes A., Colombo L., de León M., Marrero J. C., Martín de Diego D., Padrón E., (2023), “Reduction by symmetries of contact mechanical systems on lie groups”, arXiv e-prints, arXiv-2306.
- [97] Anahory Simoes A., de León M., Lainz M., de Diego D. M., (2020), “Contact geometry for simple thermodynamical systems with friction”, Proc. A., 476(2241), 20200244, 16.
- [98] Klimčík C., Strobl T., (2002), “WZW-Poisson manifolds”, J. Geom. Phys., 43(4), 341–344.
- [99] Ševera P., Weinstein A., (2001), “Poisson geometry with a 3-form background”, Progress of Theoretical Physics Supplement, 144, 145–154.
- [100] Courant T., Weinstein A., (1988), “Beyond poisson structures”, Action hamiltoniennes de groupes. Troisième théorème de Lie (Lyon, 1986), 27, 39–49.
- [101] Balseiro P., García-Naranjo L. C., (2012), “Gauge transformations, twisted Poisson brackets and Hamiltonization of nonholonomic systems”, Arch. Ration. Mech. Anal., 205(1), 267–310.
- [102] Coste A., Dazord P., Weinstein A., (1987), Groupoïdes symplectiques, in “Publications du Département de Mathématiques. Nouvelle Série. A, Vol. 2”, Vol. 87 of Publ. Dép. Math. Nouvelle Sér. A, Univ. Claude-Bernard, Lyon, i–ii, 1–62.
- [103] Saunders D. J., (1989), “The geometry of jet bundles”, Vol. 142 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge.
- [104] Kerbrat Y., Souici-Benhammadi Z., (1993), “Variétés de Jacobi et groupoïdes de contact”, C. R. Acad. Sci. Paris Sér. I Math., 317(1), 81–86.
- [105] de León M., López B., Marrero J. C., Padrón E., (2003), “On the computation of the Lichnerowicz-Jacobi cohomology”, J. Geom. Phys., 44(4), 507–522.
- [106] de León M., Marrero J. C., Padrón E., (1997*a*), “ H -Chevalley-Eilenberg cohomology of a Jacobi manifold and Jacobi-Chern class”, C. R. Acad. Sci. Paris Sér. I Math., 325(4), 405–410.
- [107] de León M., Marrero J. C., Padrón E., (1997*b*), “On the geometric quantization of Jacobi manifolds”, J. Math. Phys., 38(12), 6185–6213.
- [108] de León M., Marrero J. C., Martínez E., (2005), “Lagrangian submanifolds and dynamics on Lie algebroids”, J. Phys. A, 38(24), R241–R308.

- [109] Grabowska K., Urbański P., Grabowski J., (2006), “Geometrical mechanics on algebroids”, *Int. J. Geom. Methods Mod. Phys.*, 3(3), 559–575.
- [110] Abraham R., Marsden J. E., (1978), “Foundations of mechanics”, Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass.
- [111] Cannarsa P., Cheng W., Wang K., Yan J., (2019), “Herglotz’ generalized variational principle and contact type hamilton-jacobi equations”, *Trends in control theory and partial differential equations*, 39–67.
- [112] de León M., Lainz M., (2019), “Contact Hamiltonian systems”, *J. Math. Phys.*, 60(10), 102902, 18.
- [113] de León M., Sardón C., (2016), “Cosymplectic and contact structures to resolve time-dependent and dissipative hamiltonian systems”, *Journal of Physics A: Mathematical and Theoretical*.
- [114] Esen O., Lainz Valcázar M., de León M., Sardón C., (2023), “Implicit contact dynamics and Hamilton-Jacobi theory”, *Differential Geom. Appl.*, 90, Paper No. 102030, 31.
- [115] Bravetti A., (2019*a*), “Contact geometry and thermodynamics”, *Int. J. Geom. Methods Mod. Phys.*, 16(suppl. 1), 1940003, 51.
- [116] Bravetti A., (2019*b*), “Contact geometry and thermodynamics”, *Int. J. Geom. Methods Mod. Phys.*, 16(suppl. 1), 1940003, 51.
- [117] Esen O., Grmela M., Pavelka M., (2022*a*), “On the role of geometry in statistical mechanics and thermodynamics. I. Geometric perspective”, *J. Math. Phys.*, 63(12), Paper No. 122902, 32.
- [118] Esen O., Grmela M., Pavelka M., (2022*b*), “On the role of geometry in statistical mechanics and thermodynamics. II. Thermodynamic perspective”, *J. Math. Phys.*, 63(12), Paper No. 123305, 22.
- [119] Ghosh A., Bhamidipati C., (2019), “Contact geometry and thermodynamics of black holes in ads spacetimes”, *Phys. Rev. D*, 100, 126020.
- [120] Goto S.-i., (2016), “Contact geometric descriptions of vector fields on dually flat spaces and their applications in electric circuit models and nonequilibrium statistical mechanics”, *J. Math. Phys.*, 57(10), 102702, 40.
- [121] Grmela M., (2014), “Contact geometry of mesoscopic thermodynamics and dynamics”, *Entropy*, 16(3), 1652–1686.
- [122] Mrugala R., Nulton J. D., Schön J. C., Salamon P., (1991), “Contact structure in thermodynamic theory”, *Rep. Math. Phys.*, 29(1), 109–121.
- [123] Anahory Simoes A., de León M., Lainz M., de Diego D. M., (2020), “Contact geometry for simple thermodynamical systems with friction”, *Proc. A.*, 476(2241), 20200244, 16.
- [124] Dzhamadil’ daev A. S., (2004), “Representations of vector product n -Lie algebras”, *Comm. Algebra*, 32(9), 3315–3326.

- [125] Ateşli B., Esen O., Sütlü S., (2022), “Cohomologies and generalized derivations of n -Lie algebras”, New York J. Math., 28, 1637–1655.
- [126] Grabowski J., Marmo G., (2000), “On Filippov algebroids and multiplicative Nambu-Poisson structures”, Differential Geom. Appl., 12(1), 35–50.
- [127] Nakanishi N., (1998), “On Nambu-Poisson manifolds”, Rev. Math. Phys., 10(4), 499–510.
- [128] Hagiwara Y., (2002), “Nambu-Dirac manifolds”, J. Phys. A, 35(5), 1263–1281.
- [129] Ibáñez R., de León M., Marrero J. C., Padrón E., (1999), “Leibniz algebroid associated with a Nambu-Poisson structure”, J. Phys. A, 32(46), 8129–8144.
- [130] Magri F., Morosi C., et al. (2008), “A geometrical characterization of hamiltonian systems through the theory of poisson-nijenhuis manifolds”, Quaderni di Matematica.
- [131] Olver P. J., (1986), “Applications of Lie groups to differential equations”, Vol. 107 of Graduate Texts in Mathematics, Springer-Verlag, New York.
- [132] Guha P., (2006), “Quadratic Poisson structures and Nambu mechanics”, Nonlinear Anal., 65(11), 2025–2034.
- [133] Esen O., Ghose Choudhury A., Guha P., (2016), “Bi-Hamiltonian structures of 3D chaotic dynamical systems”, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 26(13), 1650215, 11.
- [134] Mishra S. K., Mukherjee G., Naolekar A., (2022), “Cohomology and deformations of Filippov algebroids”, Proc. Indian Acad. Sci. Math. Sci., 132(1), Paper No. 2, 25.
- [135] Vallejo J. A., (2001), “Nambu-Poisson manifolds and associated n -ary Lie algebroids”, J. Phys. A, 34(13), 2867–2881.
- [136] Cappelletti-Montano B., De Nicola A., Yudin I., (2013), “A survey on cosymplectic geometry”, Rev. Math. Phys., 25(10), 1343002, 55.
- [137] Marle C. M., (1997), “The Schouten-Nijenhuis bracket and interior products”, J. Geom. Phys., 23(3-4), 350–359.
- [138] Lee H.-C., (1943), “A kind of even-dimensional differential geometry and its application to exterior calculus”, American Journal of Mathematics, 65, 433.
- [139] Vaisman I., (1985), “Locally conformal symplectic manifolds”, Internat. J. Math. Math. Sci., 8(3), 521–536.
- [140] Banyaga A., (2002), “Some properties of locally conformal symplectic structures”, Commentarii Mathematici Helvetici, 77, 383–398.
- [141] Chantraine B., Murphy E., (2019), “Conformal symplectic geometry of cotangent bundles”, Journal of Symplectic Geometry, 17(3), 639 – 661.
- [142] Haller S., Rybicki T., (1999), “On the group of diffeomorphisms preserving a locally conformal symplectic structure”, Ann. Global Anal. Geom., 17(5), 475–502.

- [143] Otiman A., Stanciu M., (2017), “Darboux–weinstein theorem for locally conformally symplectic manifolds”, *Journal of Geometry and Physics*, 111, 1–5.

BIOGRAPHY

Begüm ATEŞLİ received her Bachelor's degree from the Department of Mathematics at İ.D. Bilkent University in 2014, and her Master's degree from the same university and department in 2017. As of 2019, she has been working as a Research Assistant at Gebze Technical University and continues to serve in this role.

PUBLICATIONS AND PRESENTATIONS FROM THE THESIS

Here are the publications done during the PhD Thesis:

- [1] Ateşli B., Esen O., de León M., Sardón C., (2023), “On locally conformally cosymplectic Hamiltonian dynamics and Hamilton-Jacobi theory”, J. Phys. A, 56(1), Paper No. 015204, 40.
- [2] Ateşli B., Esen O., Sütlü S., (2022), “Cohomologies and generalized derivations of n-Lie algebras”, New York J. Math., 28, 1637–1655.