

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

BINOMIAL EDGE IDEALS

MUHAMMED ERGEN
A THESIS SUBMITTED FOR THE DEGREE OF
MASTER OF SCIENCE
DEPARTMENT OF MATHEMATICS

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SUMMARY

In this thesis, we surveyed the literature on the algebraic properties and graded free resolutions of binomial edge ideals. Firstly, the fundamental properties of closed graphs that emerged with binomial edge ideals are given in terms of combinatorial terms. Also, we precisely stated for which graphs the binomial edge ideals have a quadratic Gröbner basis and gave an easy way to find the reduced Gröbner basis of J_G for any graph G . Then, we examined when J_G is a prime ideal and determined its minimal prime ideals. In addition, we investigated the graded free resolutions of binomial edge ideals and stated for which type of graphs J_G has a linear or pure resolution. Moreover, we studied some lower and upper bounds for the regularity of S/J_G in terms of combinatorial terms.

Keywords: Binomial Edge Ideals, Closed Graphs, Gröbner Bases, Graded Free Resolutions, Regularity.

ÖZET

Bu tezde, iki terimli kenar ideallerinin cebirsel özellikleri ve dereceli serbest çözümlmeleri hakkındaki literatürü inceledik. İlk olarak, iki terimli kenar idealleri ile ortaya çıkan kapalı grafların temel özellikleri kombinatoryal terimler cinsinden verilmiştir. Ayrıca, hangi türden graflar için iki terimli kenar ideallerinin ikinci dereceden bir Gröbner bazına sahip olduğunu kesin bir şekilde ifade ettik ve herhangi bir G grafi için J_G 'nin indirgenmiş Gröbner bazını bulmanın kolay bir yolunu verdik. Sonra, J_G nin ne zaman bir asal ideal olduğunu inceledik ve onun minimal asal ideallerini belirledik. Buna ek olarak, iki terimli kenar ideallerinin dereceli serbest çözümlmeleri üzerine konuştuk ve hangi türden graflar için J_G 'nin lineer veya pür çözümlmeye sahip olduğunu ifade ettik. Dahası, S/J_G nin düzenliliği için kombinatoryal terimler cinsinden bazı alt ve üst sınırları inceledik.

Anahtar Kelimeler: İki Terimli Kenar İdealleri, Kapalı Graflar, Gröbner Bazları, Dereceli Serbest Çözümlmeler, Düzenlilik.

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LIST of ABBREVIATIONS and ACRONYMS

<u>Abbreviations</u>	<u>Explanations</u>
<u>and Acronyms</u>	
$ A $	: Number of elements of a set A .
$\sqrt{I}$	: Radical of an ideal I .
$(I:J)$	: Ideal quotient of two ideals I and J .
$\text{Spec}(R)$	: The set of all prime ideals of a ring R .
$\text{Min}(I)$	: The set of all minimal prime ideals of an ideal I .
$\text{height } I$	: The height of an ideal I .
$\dim R$	: Krull dimension of a ring R .
$H \cong H'$	: H and H' are isomorphic.
$\text{Ker}(f)$	: Kernel of a homomorphism f .
$\text{Im}(f)$	: Image of a homomorphism f .
$R[x_1, \dots, x_n]$	: Polynomial ring over a ring R in the indeterminates $x_1, \dots, x_n$.
$\deg(f)$	: Degree of a polynomial f .
$K[x_1^{\pm}, \dots, x_n^{\pm}]$	: Laurent polynomial ring over a field K in the indeterminates $x_1, \dots, x_n$.
I_A	: Toric ideal of a matrix A .
$\text{rank } A$	: Rank of a matrix A .
$\text{nullity } A$	: Nullity of a matrix A .
$\mathbf{f}_{\mathbf{b}}$	: Binomial corresponding to a vector $\mathbf{b}$.
$<$	: A monomial order.
$<$	: Lexicographic order.
$\text{in}_{<}(f)$	: Initial monomial of a polynomial f with respect to $<$.
$\text{in}_{<}(I)$	: Initial ideal of an ideal I with respect to $<$.
$S(f, g)$	: S -polynomial of two polynomials f and g .
lcm	: Least common multiple.
gcd	: Greatest common divisor.
$V(G)$	: Vertex set of a graph G .
$E(G)$	: Edge set of a graph G .

$N(v)$	: The set of all neighbours of a vertex v .
$[n]$	: The set of all natural numbers from 1 to n .
$\deg v$	: Degree of a vertex v .
P_{l+1}	: A path of length l .
C_m	: A cycle of length m .
$G[T]$	: Induced subgraph of a graph G on the vertex set $T \subseteq V(G)$.
$G - W$	: Induced subgraph of a graph G on the vertex set $V(G) \setminus W$.
K_n	: Complete graph with n vertices.
$K_{r,s}$	: Complete bipartite graph with $r + s$ vertices.
$\dim \Delta$	: Dimension of a simplicial complex Δ .
$\operatorname{coker}(f)$	: Cokernel of a homomorphism f .
$\oplus$	: Direct sum.
$\otimes_R$	: Tensor product over a ring R .
$(\mathbb{F})_i$	: i -th module of a chain complex $\mathbb{F}$.
$\mathbb{F}[-d]$	: Homologically shifted d degrees of a chain complex $\mathbb{F}$.
$\mathcal{H}_i(\mathbb{F})$	: i -th homology of a chain complex $\mathbb{F}$.
Tor	: Tor functor.
M_i	: The set of all homogeneous elements of a graded module M with degree i .
$M(-d)$	: The module obtained by shifting the degrees of a graded module M by d .
$\beta_{i,j}(U)$	: Graded Betti number of a finitely generated graded module U .
$\dim_K V$	: Dimension of a vector space V over a field K .
$\operatorname{projdim}(U)$	: Projective dimension of a finitely generated graded module U .
$\operatorname{reg}(U)$	: Regularity of a finitely generated graded module U .
$P(U, t)$	: Poincaré series of a finitely generated graded module U .
$\mathbb{F} \otimes_K \mathbb{G}$	: Tensor product of two chain complexes $\mathbb{F}$ and $\mathbb{G}$ over a field K .
$\wedge$	: Exterior product.
$M_{\mathbf{a}}$	: The set of all multihomogeneous elements of a multigraded module M with multidegree $\mathbf{a}$.
$S(-\mathbf{a})$	: Polynomial ring obtained by shifting the degrees of the multigraded polynomial ring S by $\mathbf{a}$.

J_G	: Binomial edge ideal of a graph G .
f_{ij}	: A generator of J_G corresponding to the edge $\{i, j\} \in E(G)$.
$\Delta(G)$	: Clique complex of a graph G .
$G_{\prec}$	: Directed graph of a graph G with respect to $\prec$.
u_π	: The monomial corresponding to an admissible path π .
$c(W)$	: Number of the connected compenents of $G - W$.
$\tilde{G}$	: Complete graph on the vertex set $V(G)$.
$P_W(G)$	: Prime ideal of J_G corresponding to a subset $W \subseteq V(G)$.
$\text{supp}(\mathbf{a})$	: The set of indices i such that a_i is nonzero.
$\mathcal{P}(G)$	: The set of all weakly admissible paths in a graph G .
m_P	: The monomial corresponding to a weakly admissible path P .
$\text{mult}(w)$	: The set of indices k such that $x_k y_k$ divides the monomial w .
$\mathcal{M}$	: The set of all monomials corresponding to weakly admissible paths.
$G - e$	: Subgraph of G obtained by removing an edge e from G .
$G \cup e$	: Graph obtained by including a non-edge e to the graph G .
G_e	: Graph obtained by making simplicial vertex the end points of an edge e that does not belong to G .
f_e	: Binomial corresponding to an edge e that does not belong to G .

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1. INTRODUCTION

For any simple graph G with $V(G) = \{1, \dots, n\}$, the binomial edge ideal of G is denoted by J_G and defined as the ideal generated by the binomials $f_{ij} = x_i y_j - x_j y_i$ in the polynomial ring $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ over a field K where $i < j$ and $\{i, j\} \in E(G)$. Binomial edge ideals were introduced independently by Herzog et al. in [1] in 2010 and by Ohtani in [2] in 2011 and since then, it has become the focus of attention of many researchers. In recent years, many algebraic properties of binomial edge ideals, from Hilbert series to graded free resolutions and from Koszulness to toricness, have been examined and the relations between these algebraic properties and the combinatorial properties of the graph G have been investigated. In addition, articles were published in which Cohen-Macaulay and Gorenstein binomial edge ideals were characterized and their homological properties were examined. Although much research has been done on binomial edge ideals, there are still many open problems related to J_G that remain to be resolved.

In the second chapter of this thesis, we have given some fundamental definitions and theorems about rings, Gröbner bases, graphs, modules and graded free resolutions. These definitions and theorems are quite important as they provide a mathematical basis for the topics which will be mentioned in Chapter 3 and Chapter 4.

Chapter 3 is mainly based on some results from the articles [1], [3] and [4]. In particular, we studied the algebraic properties of J_G such as primary decomposition, minimal primes and Gröbner bases. The authors of [1] gave the fundamental algebraic properties of J_G and found a method for calculating the reduced Gröbner basis of J_G with respect to the lexicographic order via admissible paths. Also, they defined the closed graphs that are important to binomial edge ideals and gave the fundamental properties of them in terms of the terms in graph theory. In addition, Ene et al. showed some combinatorial properties of closed graphs in [3]. In [4], Crupi and Rinaldo proved that the necessary and sufficient condition for J_G to have a quadratic Gröbner basis is that G is a closed graph.

In the last chapter, we examined the graded free resolutions and regularity of binomial edge ideals. Matsuda and Murai examined the regularity of J_G in [5] and found lower and upper bounds for it, that is, $l \leq \text{reg}(S/J_G) \leq n - 1$ where l is

the length of the longest induced path in G . Later, Ene et al. in [6] found a better upper bound for the regularity, that is, if $\Delta(G)$ is the clique complex of a graph G , then $\text{reg}(S/J_G) \leq n - \dim \Delta(G)$. Also, we presented another upper bound due to [7] where it was proven that $\text{reg}(S/J_G) \leq r$ where r is the number of the maximal cliques of G . Moreover, we considered a result of Madani and Kiani in [8] where it was shown that, binomial edge ideal of a graph G has a linear resolution if and only if G is complete graph. Lastly, we studied a more general form of this result of the same authors. In particular, we studied the results from [9] and presented their proof that the necessary and sufficient condition for J_G to have a pure resolution is that G is complete or complete bipartite or disjoint union of some paths.



2. PRELIMINARIES

2.1. Ring Theory

In this section, some basic definitions and theorems about rings, ideals and polynomial rings in several variables will be given. In addition to these, monomial ideals, binomial ideals and toric ideals will be introduced, and their algebraic properties will be mentioned.

Definition 2.1: [10] Let R be a nonempty set and let $+$ and $\cdot$ be two binary operations on R . Then, R is said to be a ring if the following properties are satisfied for all $a, b, c \in R$:

$$R1) (a + b) + c = a + (b + c).$$

$$R2) a + b = b + a.$$

$$R3) \text{ There exists } 0 \in R \text{ such that } a + 0 = a.$$

$$R4) \text{ There exists } -a \in R \text{ such that } a + (-a) = 0.$$

$$R5) (a \cdot b) \cdot c = a \cdot (b \cdot c).$$

$$R6) a \cdot (b + c) = a \cdot b + a \cdot c.$$

$$R7) (a + b) \cdot c = a \cdot c + b \cdot c.$$

The operations $+$ and $\cdot$ are called addition and multiplication on R . Also, the element $0 \in R$ mentioned in the property R3) is called the zero element of R .

We will assume that multiplication performs before addition. For simplicity, we will write $a - b$ and ab instead of $a + (-b)$ and $a \cdot b$, respectively.

Definition 2.2: [10] Let R be a ring. If $ab = ba$ for all $a, b \in R$, then R is said to be commutative. Otherwise, R is said to be noncommutative.

Definition 2.3: [10] Let R be a ring. If there exists $1 \in R$ such that $1 \cdot a = a = a \cdot 1$ for all $a \in R$, then we say that R is a ring with identity (or a ring with 1). In this case, the element 1 is called the identity element of the ring R .

Example 2.1: Clearly, $R = \{0\}$ satisfies the conditions for being a ring. This ring is called the trivial ring. Let us consider the sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ together with the usual addition $+$, and multiplication $\cdot$. These sets are commutative rings with identity.

Definition 2.4: [I10] Suppose that R is a ring with identity and u is an element of R . Then, u is said to be a unit (or an invertible) element of R if there exists $v \in R$ such that $uv = vu = 1$. In this case, v is called the inverse of u .

Definition 2.5: [I11] Let R be a nontrivial commutative ring with identity. Then, R is called a field if every nonzero element is unit in R .

Example 2.2: Obviously, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are fields with respect to the usual addition and multiplication. However, the ring $\mathbb{Z}$ is not a field because the only invertible elements in $\mathbb{Z}$ are 1 and -1 .

Definition 2.6: [I10] Let R be a ring and $0 \neq a \in R$. If there exists an element $0 \neq b \in R$ such that $ab = 0$ or $ba = 0$, then a is called a zero divisor in R .

Definition 2.7: [I10] Let R be a commutative ring with identity. If R has no zero divisors, then R is said to be an integral domain.

Remark 2.1: Note that every field is an integral domain.

Definition 2.8: [I11] Let R be a commutative ring and $\emptyset \neq I \subseteq R$. Then, I is called an ideal of R if $a + b \in I$ and $ar \in I$ for all $a, b \in I$ and $r \in R$.

Definition 2.9: [I10] Let R be a commutative ring and $\emptyset \neq A \subset R$. Then, the ideal (A) is said to be the ideal generated by the set A and defined as the intersection of all ideals of R which contain A .

Proposition 2.1: [I10] Let R be a commutative ring with 1 and $\emptyset \neq A \subset R$. Then, $(A) = \{\sum_{i=1}^n r_i a_i : r_i \in R, a_i \in A, 1 \leq i \leq n, n \in \mathbb{N}\}$.

Remark 2.2: [I10] If $A = \{a_1, \dots, a_n\}$, then the ideal (A) is denoted by $(a_1, \dots, a_n)$. In this case, $(A) = (a_1, \dots, a_n)$ is said to be a finitely generated ideal. Then, we can write $(a_1, \dots, a_n) = \{r_1 a_1 + \dots + r_n a_n : r_1, \dots, r_n \in R\}$.

Definition 2.10: [I12] Let R be a ring and I, J be two ideals of R . Then, the sum of the ideals I and J is defined as the smallest ideal in R containing I and J , that is, $I + J = \{a + b : a \in I, b \in J\}$.

Definition 2.11: [12] Let R be a commutative ring and I be an ideal of R . Then, the radical of I is defined as $\sqrt{I} = \{a \in R : a^n \in I \text{ for some } n \in \mathbb{N}\}$. Also, I is called a radical ideal if $I = \sqrt{I}$.

Definition 2.12: [11] Let I and J be two ideals of a commutative ring R . The ideal quotient of I and J is denoted by $(I : J)$ and defined as $(I : J) = \{r \in R : rJ \subseteq I\}$. In particular, if $J = (a)$ where $a \in R$, then the expression $(I : (a))$ can be written more simply as $(I : a)$. In this case, $(I : a) = \{r \in R : ra \in I\}$.

Definition 2.13: [10] Let R be a commutative ring and I be an ideal of R . Then, the set $R/I = \{r + I : r \in R\}$ forms a ring together with the operations $+$ and $\cdot$ defined as $(a + I) + (b + I) = (a + b) + I$ and $(a + I) \cdot (b + I) = ab + I$ for all $a + I, b + I \in R/I$. Then, the ring R/I is called the quotient ring of R by I .

Definition 2.14: [11] An ideal P of a commutative ring R is called a prime ideal of R if $P \neq R$ and $ab \in P$ implies either $a \in P$ or $b \in P$. The set of all prime ideals of R is denoted by $\text{Spec}(R)$.

Proposition 2.2: [10] Suppose that R is a commutative ring with identity and P is an ideal of R with $P \neq R$. Then, P is a prime ideal of R if and only if R/P is an integral domain.

Definition 2.15: [13] Let I be an ideal of a commutative ring R . A prime ideal P of R is called a minimal prime ideal of I if $I \subseteq P$ and there is no prime ideal P' in R such that $I \subseteq P' \subsetneq P$. The set of all minimal prime ideals of I is denoted by $\text{Min}(I)$.

Proposition 2.3: [11] Let R be a commutative ring and I be an ideal of R such that $I \neq R$. Then, we have $\sqrt{I} = \bigcap_{P \in \text{Min}(I)} P$. More specifically, if I is a radical ideal of R , then we have $I = \bigcap_{P \in \text{Min}(I)} P$.

Definition 2.16: [14] The height of a prime ideal P of a commutative ring R is defined as $\text{height } P = \sup\{n : P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = P, P_0, \dots, P_{n-1} \in \text{Spec}(R)\}$. Also, we define the Krull dimension of R as $\dim R = \sup\{\text{height } P : P \in \text{Spec}(R)\}$. If I is any ideal of R , we define height of I as $\text{height } I = \min\{\text{height } P : P \in \text{Min}(I)\}$.

Definition 2.17: [11] Let R and R' be two commutative rings with identity and f be a function from R into R' . Then, f is called a ring homomorphism if it satisfies these properties:

- i) $f(a + b) = f(a) + f(b)$ for all $a, b \in R$;
- ii) $f(a \cdot b) = f(a) \cdot f(b)$ for all $a, b \in R$;
- iii) $f(1) = 1$.

If f is one-to-one and onto, then f is called an isomorphism.

Definition 2.18: [10] Let R and R' be two commutative rings. If there exists an isomorphism from R to R' , then R and R' are called isomorphic and it is denoted by $R \cong R'$.

Definition 2.19: [10] Let R and R' be two commutative rings and f be a homomorphism from R to R' . Then, the set $\text{Ker}(f) = \{r \in R : f(r) = 0\}$ is called the kernel of f .

Remark 2.3: For any ring homomorphism $f : R \rightarrow R'$, $\text{Ker}(f)$ is an ideal of R .

Theorem 2.1: [10] (First Isomorphism Theorem) Let $f : R \rightarrow R'$ be a ring homomorphism. Then $R/\text{Ker}(f) \cong \text{Im}(f)$ where $\text{Im}(f)$ is the image of f .

Theorem 2.2: [10] Let R be a commutative ring and I, J be two ideals of R with $I \subset J$. Then, $(R/I)/(J/I) \cong R/J$.

Definition 2.20: [15] Let R be a ring. The ring $R[x_1, \dots, x_n]$ is called the polynomial ring (or ring of polynomials) over R in the indeterminates $x_1, \dots, x_n$. Each polynomial in $R[x_1, \dots, x_n]$ is in the form $f(x_1, \dots, x_n) = \sum a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n}$ where each $(i_1, \dots, i_n) \in \mathbb{N}^n$, $a_{i_1 \dots i_n} \in R$ and the sum has finitely many terms. The ring operations of $R[x_1, \dots, x_n]$ are as follows:

$$f + g = \sum (a_{i_1 \dots i_n} + b_{i_1 \dots i_n}) x_1^{i_1} \dots x_n^{i_n} \quad (2.1)$$

for all $f = \sum a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n}$, $g = \sum b_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n} \in R[x_1, \dots, x_n]$ and

$$fg = \sum \sum_{\substack{i_t + j_t = k_t \\ \text{for all } t}} (a_{i_1 \dots i_n} b_{j_1 \dots j_n}) x_1^{k_1} \dots x_n^{k_n} \quad (2.2)$$

for all $f = \sum a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n}$, $g = \sum b_{j_1 \dots j_n} x_1^{j_1} \dots x_n^{j_n} \in R[x_1, \dots, x_n]$.

Definition 2.21: [I5] Suppose that R is a ring and $p = (p_1, \dots, p_n) \in \mathbb{N}^n$. A polynomial in the form $x^p = x_1^{p_1} \dots x_n^{p_n}$ is called a monomial in the polynomial ring $R[x_1, \dots, x_n]$. The term ax^p is called a monomial term where $0 \neq a \in R$. The degree of ax^p is defined as $\deg(ax^p) = p_1 + \dots + p_n$. The degree of any polynomial $0 \neq f \in R[x_1, \dots, x_n]$ is denoted by $\deg(f)$ and defined to be the maximum degree of the monomials of f with nonzero coefficients. Also, f is called homogeneous if the degree of every monomial which has nonzero coefficient in f is the same.

Proposition 2.4: [I5] If R is an integral domain, then $R[x_1, \dots, x_n]$ is also an integral domain.

Throughout this chapter, we will assume K is a field and S is the polynomial ring over K in the indeterminates $x_1, \dots, x_n$ unless otherwise stated.

Proposition 2.5: [I4] If $I \subsetneq S$ is an ideal, then $\dim S/I = \dim S - \text{height } I$.

Definition 2.22: [I3] Let I be an ideal in S . Then, I is called a monomial ideal if I can be generated by monomials.

Definition 2.23: [I2] A monomial $x^p = x_1^{p_1} \dots x_n^{p_n}$ in S is said to be squarefree if $0 \leq p_i \leq 1$ for each $i = 1, \dots, n$. Also, an ideal I of S is said to be squarefree monomial ideal if it can be generated by squarefree monomials.

Proposition 2.6: [I2] Suppose that $I \subseteq S$ is a monomial ideal which can be generated by squarefree monomials. Then, I is a radical ideal.

Definition 2.24: [I2] A polynomial in the form $u - v$ is called a binomial of S where u and v are monomials in S . Also, if an ideal I of S can be generated by binomials, it is called a binomial ideal of S .

Definition 2.25: [I4] The ring $K[x_1^\pm, \dots, x_n^\pm]$ is called the Laurent polynomial ring over K in the indeterminates $x_1, \dots, x_n$. Each element of $K[x_1^\pm, \dots, x_n^\pm]$ has the form $f(x_1, \dots, x_n) = \sum a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n}$ where each $(i_1, \dots, i_n) \in \mathbb{Z}^n$, $a_{i_1 \dots i_n} \in K$ and the sum has finitely many terms. The ring operations of $K[x_1^\pm, \dots, x_n^\pm]$ can be written similarly to the ring operations of the polynomial ring $R[x_1, \dots, x_n]$ mentioned in the equations (2.1) and (2.2).

Remark 2.4: Note that $K[x_1^\pm, \dots, x_n^\pm]$ is an integral domain for any field K .

Construction 2.1: [U2] Let us say $T = K[t_1^\pm, \dots, t_d^\pm]$ and let us consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1} & a_{d2} & \cdots & a_{dn} \end{bmatrix}_{d \times n} \quad (2.3)$$

in $\mathbb{Z}^{d \times n}$ where $\mathbb{Z}^{d \times n}$ is the set of $d \times n$ matrices whose entries are integers. The map $\pi : S \rightarrow T$ defined by $x_j \rightarrow t_1^{a_{1j}} \dots t_d^{a_{dj}}$ is a ring homomorphism. We represent the ideal $\text{Ker}(\pi)$ with I_A and call it the toric ideal of the matrix A . As T is an integral domain, $\text{Im}(\pi)$ is also an integral domain and I_A is a prime ideal from Theorem 2.1 and Proposition 2.2

Given any column vector

$$\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \quad (2.4)$$

in $\mathbb{Z}^n$, we associate the binomial

$$\mathbf{f}_{\mathbf{b}} = \prod_{b_i > 0} x_i^{b_i} - \prod_{b_j < 0} x_j^{-b_j}. \quad (2.5)$$

In the equation (2.5), if there is no entries in $\mathbf{b}$ satisfying $b_i > 0$ or $b_j < 0$, then we write 1 instead of $\prod_{b_i > 0} x_i^{b_i}$ or $\prod_{b_j < 0} x_j^{-b_j}$.

Theorem 2.3: [U2] Let I_A be the toric ideal of a matrix $A \in \mathbb{Z}^{d \times n}$. Then, the set $\{\mathbf{f}_{\mathbf{b}} : \mathbf{b} \in \mathbb{Z}^n, A\mathbf{b} = 0\}$ generates I_A .

Remark 2.5: [U2] An ideal I is a toric ideal of S if and only if I is a binomial and prime ideal of S .

2.2. Gröbner Bases

In this section, we will give some fundamental information about polynomials and Gröbner bases. Firstly, we will investigate monomial orders, initial monomial, initial ideal and division algorithm. Also, we will investigate some basic information about Gröbner bases and show how the Gröbner basis of an ideal can be calculated by giving an example.

A nonzero polynomial f in S can have more than one monomial having maximum degree. Then, the degree of f is not useful for ordering monomials in f . So, we need new concepts to ordering the monomials of f , and the following definition answers this need.

Definition 2.26: [12] A monomial order $\prec$ on S is an order of the elements of the set of monomials in S satisfying these conditions:

- i) $u \prec v$ or $v \prec u$ for all monomials u, v in S with $u \neq v$;
- ii) $1 \prec u$ for any monomial u in S ;
- iii) If $u \prec v$ for some monomials u, v in S , then $uw \prec vw$ for any monomial w in S .

Example 2.3: [12] Let $u = x_1^{a_1} \dots x_n^{a_n}, v = x_1^{b_1} \dots x_n^{b_n} \in S$. We write $u < v$ if $\deg(u) < \deg(v)$ or $\deg(u) = \deg(v)$ and $a_k < b_k$ where k is the smallest integer t with $a_t \neq b_t$. Clearly, $<$ satisfies the conditions for being a monomial order and it is said to be the lexicographic order on S induced by $x_1 > \dots > x_n$.

Definition 2.27: [12] The initial monomial of a nonzero polynomial f in S with respect to a monomial order $\prec$ on S is denoted by $\text{in}_\prec(f)$ and defined as the largest of the monomials with a nonzero coefficient in f with respect to $\prec$.

Theorem 2.4: [12] (The Division Algorithm) Suppose that $\prec$ is a monomial order on S and $0 \neq f \in S$. Also, let $\mathcal{G} = \{f_1, \dots, f_t\}$ be an ordered set of nonzero polynomials in S . Then, there exist $g_1, \dots, g_t, r \in S$ which satisfy

$$f = g_1 f_1 + \dots + g_t f_t + r \quad (2.6)$$

where $r = 0$ or $r \neq 0$ and none of the monomials appearing in r is divisible by any of $\text{in}_{\prec}(f_1), \dots, \text{in}_{\prec}(f_t)$ and where $\text{in}_{\prec}(g_i f_i) \preceq \text{in}_{\prec}(f)$ for all $1 \leq i \leq t$ with $g_i \neq 0$.

Definition 2.28: [U2] The polynomial r mentioned in Theorem 2.4 is called the remainder of f on division by $\mathcal{G}$. Also, equation (2.6) is said to be the standard expression of f with respect to the set $\mathcal{G}$.

Note that Theorem 2.4 does not say anything about uniqueness of remainder. If we interchange the elements of $\mathcal{G}$, we can find different remainders. Also, if $r = 0$ in the standard expression (2.6) of f , then f is said to be reduced to 0 with respect to the set $\mathcal{G}$.

Definition 2.29: [U2] Let $\prec$ be a monomial order on S and let $(0) \neq I$ be an ideal in S . The initial ideal of I with respect to $\prec$ is denoted by $\text{in}_{\prec}(I)$ and it is defined as $\text{in}_{\prec}(I) = (\{\text{in}_{\prec}(f) : 0 \neq f \in I\})$.

Definition 2.30: [U2] Let $\prec$ be a monomial order on S and let I be a nonzero ideal in S . Then, a finite subset $\mathcal{G} = \{g_1, \dots, g_t\}$ of nonzero polynomials of I is said to be a Gröbner basis with respect to $\prec$ if $\text{in}_{\prec}(I) = (\text{in}_{\prec}(g_1), \dots, \text{in}_{\prec}(g_t))$.

Theorem 2.5: [U2] Let $(0) \neq I \subseteq S$ be an ideal. If $\mathcal{G} = \{g_1, \dots, g_t\}$ is a Gröbner basis of I with respect to some monomial orders on S , then $I = (g_1, \dots, g_t)$.

Proposition 2.7: [U2] Let $(0) \neq I \subseteq S$ be an ideal and $\mathcal{G}$ be a Gröbner basis of I with respect to some monomial orders on S . Then, the remainder of dividing any $0 \neq f \in S$ by $\mathcal{G}$ is uniquely determined.

Note that if $\mathcal{G}$ is a Gröbner basis of an ideal $(0) \neq I \subseteq S$, then the necessary and sufficient condition for a nonzero polynomial $f \in S$ to be an element of I is that f reduces to 0 with respect to $\mathcal{G}$.

Definition 2.31: [U2] Let $\prec$ be a monomial order on S and $(0) \neq I$ be an ideal in S . A Gröbner basis $\mathcal{G} = \{g_1, \dots, g_t\}$ of I is said to be reduced if $\mathcal{G}$ satisfies these two conditions:

- i) $\text{in}_{\prec}(g_i)$ has the coefficient 1 in g_i for all $1 \leq i \leq t$;
- ii) None of the monomials with nonzero coefficient in g_i is divisible by $\text{in}_{\prec}(g_j)$ whenever $i \neq j$.

Theorem 2.6: [I2] Suppose that I is a nonzero ideal in S . Then, there exists a unique reduced Gröbner basis $\mathcal{G}$ for I .

Definition 2.32: [I2] Let $\prec$ be a monomial order on S . The S -polynomial of two nonzero polynomials $f, g \in S$ with respect to $\prec$ is defined as

$$S(f, g) = \frac{\text{lcm}(\text{in}_{\prec}(f), \text{in}_{\prec}(g))}{a_f \text{in}_{\prec}(f)} f - \frac{\text{lcm}(\text{in}_{\prec}(f), \text{in}_{\prec}(g))}{a_g \text{in}_{\prec}(g)} g \quad (2.7)$$

where a_f and a_g are the coefficients of $\text{in}_{\prec}(f)$ and $\text{in}_{\prec}(g)$ in f and g , respectively.

Clearly, $S(f, g) \in I$ for all nonzero polynomials $f, g \in I$.

Proposition 2.8: [I2] Suppose that $\prec$ is a monomial order on S . If the initial monomials of two nonzero polynomials $f, g \in S$ with respect to $\prec$ are relatively prime, then $S(f, g)$ reduces to 0 with respect to f and g .

Theorem 2.7: [I2] (Buchberger's Criterion) Suppose that $I = (g_1, \dots, g_t)$ is an ideal of S . Then, the set $\mathcal{G} = \{g_1, \dots, g_t\}$ is a Gröbner basis of I if and only if $S(g_i, g_j)$ reduces to 0 with respect to $\mathcal{G}$ for each pair $i \neq j$.

Example 2.4: Calculate the reduced Gröbner basis of $I = (f_{12}, f_{14}, f_{23}, f_{24}, f_{34}, f_{35})$ in $K[x_1, \dots, x_n, y_1, \dots, y_n]$ with respect to the lexicographic order induced by the ordering $x_1 > \dots > x_n > y_1 > \dots > y_n$ where $f_{ij} = x_i y_j - x_j y_i$.

Solution : Let us define $\mathcal{G}_1 = \{f_{12}, f_{14}, f_{23}, f_{24}, f_{34}, f_{35}\}$. Then,

$$S(f_{12}, f_{14}) = \frac{x_1 y_2 y_4}{x_1 y_2} f_{12} - \frac{x_1 y_2 y_4}{x_1 y_4} f_{14} = y_4 f_{12} - y_2 f_{14} = -y_1 f_{24}. \quad (2.8)$$

Thus, $S(f_{12}, f_{14})$ reduces to 0 with respect to $\mathcal{G}_1$. Additionally, $S(f_{12}, f_{23})$, $S(f_{12}, f_{24})$, $S(f_{12}, f_{34})$, $S(f_{12}, f_{35})$ and $S(f_{14}, f_{23})$ reduce to 0 by Proposition 2.8. In addition,

$$S(f_{14}, f_{24}) = \frac{x_1 x_2 y_4}{x_1 y_4} f_{14} - \frac{x_1 x_2 y_4}{x_2 y_4} f_{24} = x_2 f_{14} - x_1 f_{24} = x_4 f_{12}. \quad (2.9)$$

Hence, $S(f_{14}, f_{24})$ reduces to 0. Also,

$$S(f_{14}, f_{34}) = \frac{x_1 x_3 y_4}{x_1 y_4} f_{14} - \frac{x_1 x_3 y_4}{x_3 y_4} f_{34} = x_3 f_{14} - x_1 f_{34} = x_4 f_{13}. \quad (2.10)$$

Here, $\text{in}_<(x_4 f_{13})$ is not divisible by the initial monomial of any of the polynomials in $\mathcal{G}_1$. Hence, $S(f_{14}, f_{34})$ does not reduce to 0 with respect to $\mathcal{G}_1$. Then, let us say $\mathcal{G}_2 = \{f_{12}, f_{14}, f_{23}, f_{24}, f_{34}, f_{35}, x_4 f_{13}\}$. We can write

$$S(f_{12}, x_4 f_{13}) = \frac{x_1 x_4 y_2 y_3}{x_1 y_2} f_{12} - \frac{x_1 x_4 y_2 y_3}{x_4 x_1 y_3} x_4 f_{13} = -x_4 y_1 f_{23}. \quad (2.11)$$

Thus, $S(f_{12}, x_4 f_{13})$ reduces to 0 with respect to $\mathcal{G}_2$. Also, $S(f_{14}, f_{35})$ reduces to 0 by Proposition 2.8. In addition to that,

$$S(f_{14}, x_4 f_{13}) = \frac{x_1 x_4 y_3 y_4}{x_1 y_4} f_{14} - \frac{x_1 x_4 y_3 y_4}{x_4 x_1 y_3} x_4 f_{13} = x_4 y_1 f_{34}. \quad (2.12)$$

So, $S(f_{14}, x_4 f_{13})$ reduces to 0 with respect to $\mathcal{G}_2$. Similarly,

$$S(f_{23}, f_{24}) = \frac{x_2 y_3 y_4}{x_2 y_3} f_{23} - \frac{x_2 y_3 y_4}{x_2 y_4} f_{24} = y_4 f_{23} - y_3 f_{24} = -y_2 f_{34}. \quad (2.13)$$

Hence, $S(f_{23}, f_{24})$ reduces to 0. By Proposition 2.8, $S(f_{23}, f_{34})$ and $S(f_{23}, f_{35})$ reduce to 0. Also,

$$S(f_{23}, x_4 f_{13}) = \frac{x_1 x_2 x_4 y_3}{x_2 y_3} f_{23} - \frac{x_1 x_2 x_4 y_3}{x_4 x_1 y_3} x_4 f_{13} = -x_3 x_4 f_{12}. \quad (2.14)$$

Thus, $S(f_{23}, x_4 f_{13})$ reduces to 0. In addition,

$$S(f_{24}, f_{34}) = \frac{x_2 x_3 y_4}{x_2 y_4} f_{24} - \frac{x_2 x_3 y_4}{x_3 y_4} f_{34} = x_3 f_{24} - x_2 f_{34} = x_4 f_{23}. \quad (2.15)$$

So, $S(f_{24}, f_{34})$ reduces to 0 with respect to $\mathcal{G}_2$. Moreover, $S(f_{24}, f_{35})$ and $S(f_{24}, x_4 f_{13})$ reduces to 0 by Proposition 2.8. Additionally,

$$S(f_{34}, f_{35}) = \frac{x_3 y_4 y_5}{x_3 y_4} f_{34} - \frac{x_3 y_4 y_5}{x_3 y_5} f_{35} = y_5 f_{34} - y_4 f_{35} = -y_3 f_{45}. \quad (2.16)$$

However, $\text{in}_{<}(y_3f_{45})$ is not divisible by the initial monomial of any of the polynomials in $\mathcal{G}_2$. It means $S(f_{34}, f_{35})$ does not reduce to 0 with respect to $\mathcal{G}_2$. Now, let us say $\mathcal{G} = \{f_{12}, f_{14}, f_{23}, f_{24}, f_{34}, f_{35}, x_4f_{13}, y_3f_{45}\}$. One can easily check that all S -pairs of $\mathcal{G}$ reduces to 0. It means, $\mathcal{G}$ is a Gröbner basis for the ideal I . Clearly, the coefficients of the initial monomials of all elements in $\mathcal{G}$ are 1. Furthermore, the initial monomial of any polynomial in $\mathcal{G}$ does not divide any monomial of other polynomials in $\mathcal{G}$. Therefore, $\mathcal{G}$ is the reduced Gröbner basis for I .

2.3. Graph Theory and Combinatorics

In this section, we will present some definitions and theorems about graph theory and combinatorics. Firstly, we will introduce graphs, simple graphs, induced subgraphs, and directed graphs. In addition, we will mention some graph classes that have an important place for simple graphs such as paths, cycles, chordal graphs and give some basic information about simplicial complexes.

Definition 2.33: [16] A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$ which is a set of paired vertices. Each element of $V(G)$ and $E(G)$ is called a vertex and an edge, respectively. If $e = \{u, v\} \in E(G)$, then u and v are called the endpoints of the edge e .

Definition 2.34: [16] Let G be a graph. If there exists an edge $e = \{u, v\} \in E(G)$, then u and v are called adjacent or neighbours in G . The set of all neighbours of a vertex v is denoted by $N(v)$. An edge of the form $e = \{v, v\} \in E(G)$ is called a loop. Also, G is said to have multiple edges if there are more than one edge joining two vertices in G . If a graph G does not have any loops and multiple edges, then G is called a simple graph.

Definition 2.35: [16] Let G be a graph. If the sets $V(G)$ and $E(G)$ have finite number of elements, then G is said to be a finite graph.

Throughout this thesis, we will only deal with finite simple graphs. Usually, we will choose the set $[n] = \{1, \dots, n\}$ as the vertex set of a graph G and simply say G is a graph on $[n]$.

Example 2.5: Let us consider the graph G shown in Figure 2.1. Clearly, we have

$$V(G) = [5] = \{1, 2, 3, 4, 5\} \quad (2.17)$$

and

$$E(G) = \{\{1, 2\}, \{1, 5\}, \{2, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}. \quad (2.18)$$

Here, we can easily see that G is finite and simple graph.

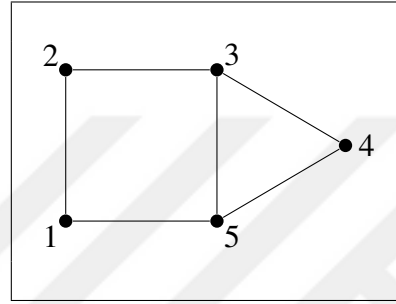


Figure 2.1: A graph G

Definition 2.36: [16] Let G be a graph. The degree of a vertex v is denoted by $\deg v$ and defined as the number of the neighbours of v in G .

Definition 2.37: [16] Let v be a vertex of a graph G . If $\deg v = 0$, then v is called an isolated vertex.

Definition 2.38: [16] Let v be a vertex of a graph G . Then, v is called a simplicial vertex of G if $\{u, u'\} \in E(G)$ for all $u, u' \in N(v)$.

Definition 2.39: [16] A sequence $P : v_0, v_1, \dots, v_l$ is said to be a path of length l in a graph G if $v_0, v_1, \dots, v_l$ are distinct elements of $V(G)$ and $\{v_{i-1}, v_i\} \in E(G)$ for all $1 \leq i \leq l$. We denote by P_{l+1} the path graph of length l . A cycle C of length m in G is a sequence $C : u_1, \dots, u_m, u_1$ where $u_1, \dots, u_m$ are distinct vertices of G and $\{u_1, u_2\}, \dots, \{u_{m-1}, u_m\}, \{u_1, u_m\} \in E(G)$. A cycle graph of length m is usually denoted by C_m . If m is an odd number (or even number), C_m is said to be an odd cycle (or an even cycle).

Definition 2.40: [16] A bipartite graph is a graph whose vertex set can be decomposed into two disjoint sets such that no two vertices within the same set are adjacent.

Proposition 2.9: [I6] A graph G is bipartite if and only if G has no odd cycle.

Definition 2.41: [I6] A graph G is said to be connected if there exists a path between every pair of vertices of G ; otherwise G is called disconnected graph.

Definition 2.42: [I6] A graph G that has no cycles is called acyclic. Also, G is said to be a tree if G is connected and acyclic.

Definition 2.43: [I6] A graph H is said to be a subgraph of a graph G if the vertex set and edge set of G contains those of H and $u, v \in V(H)$ for each $e = \{u, v\} \in E(H)$.

Definition 2.44: [I6] Let G be a graph. A subgraph H of G is called a connected component of G if H is a maximal connected subgraph of G .

Definition 2.45: [I6] Let G be a graph and $T \subset V(G)$. Then, the induced subgraph $G[T]$ of G is the graph whose vertex set is T and whose edge set consists of all edges in $E(G)$ that have both endpoints in T .

Now, let us introduce some notations about induced subgraphs. If $W \subset V(G)$, we write $G - W$ for the induced subgraph on the vertex set $V(G) \setminus W$. If $W = \{v\}$ where $v \in V(G)$, we write simply $G - v$ instead of $G - \{v\}$.

Definition 2.46: [I6] A graph G is called chordal if G does not contain C_m as an induced subgraph for all $m \geq 4$.

Definition 2.47: [I6] A graph G is said to be H -free if G does not contain a graph H as an induced subgraph.

Definition 2.48: [I6] A complete graph K_n is a simple graph with n vertices such that every pair of vertices are adjacent. A complete bipartite graph is a simple bipartite graph such that each pair of vertices in a different partition sets are adjacent. We denote the complete bipartite graph by $K_{r,s}$ where r and s are the number of elements of partition sets.

Definition 2.49: [I6] A directed graph G consists of a set of vertices $V(G)$ and an edge set $E(G)$ which is a family of ordered pairs of the elements of $V(G)$.

Definition 2.50: [I6] A directed cycle C of length m in a directed graph G is a sequence $C : v_1, \dots, v_m, v_1$ where $v_1, \dots, v_m$ are distinct elements of $V(G)$ and $(v_1, v_2), \dots, (v_{m-1}, v_m), (v_m, v_1) \in E(G)$. Also, a directed graph that has no directed cycle is called acyclic.

Example 2.6: Let G be the directed graph displayed in Figure 2.2. Clearly, we have $V(G) = [5]$ and

$$E(G) = \{(1, 2), (2, 3), (3, 4), (3, 5), (4, 5), (5, 1)\}. \quad (2.19)$$

Also, we can easily say that $C : 1, 2, 3, 5, 1$ is a directed cycle in G . On the other hand, $C' : 3, 4, 5, 3$ is not a directed cycle in G because of $(5, 3) \notin E(G)$.

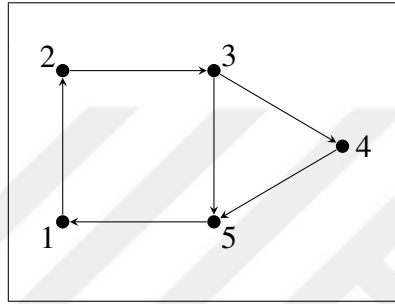


Figure 2.2: A directed graph G

Definition 2.51: [17] Let G be a directed graph. An ordering $v_1, \dots, v_n$ of the vertices of G is called an *acyclic ordering* if we have $i < j$ for each $(v_i, v_j) \in E(G)$.

Proposition 2.10: [17] Let G be an acyclic directed graph. Then, G has an acyclic ordering of its vertices.

Example 2.7: The directed graph G shown in Figure 2.3 is acyclic. Then, there exists an acyclic ordering of the vertices of G by Proposition 2.10. Let us label the vertices a, b, c, d and e with 5, 1, 2, 3 and 4, respectively. Then, one can easily see that $i < j$ is satisfied for all edges (i, j) in G .

Definition 2.52: [14] A family Δ consisting of subsets of a finite set V is said to be a *simplicial complex* on V if $\{v\} \in \Delta$ for each $v \in V$ and $F \in \Delta$ implies $F' \in \Delta$ for all $F' \subset F$. Any element of Δ is called a *face* of Δ . Also, a maximal face of Δ according to inclusion is called a *facet* of Δ . The dimension of Δ is defined as $\dim \Delta = \max\{|F| - 1 : F \in \Delta\}$.

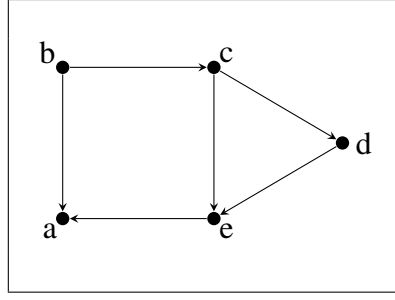


Figure 2.3: A directed graph G

Definition 2.53: [I2] Let F be a facet of a simplicial complex Δ . If there is a facet $F \neq G \in \Delta$ satisfying $H \cap F \subset G \cap F$ for any facet $F \neq H \in \Delta$, then F is called a leaf of Δ and in this case, G is said to be a branch of F . Also, Δ is said to be a quasi-forest if there exists an ordering $F_1, \dots, F_r$ of the facets of Δ such that F_i is a leaf of Δ_i for all $2 \leq i \leq r$ where Δ_i is the simplicial complex with facets $F_1, \dots, F_i$. A leaf order is an ordering of the facets of Δ , which makes Δ a quasi-forest. A connected graph G is said to be a quasi-tree if G is quasi-forest.

2.4. Module Theory

In this section, we will present some basic concepts about modules and vector spaces, and give some properties of direct sum and tensor product. Also, we will give a brief summary of some concepts from homological algebra, such as chain complexes, exact sequences, free resolutions and Tor functor.

Definition 2.54: [I1] Let R be a commutative ring with 1 and let $(M, +)$ be an abelian group. Then, M is called an R -module together with a function $\cdot : R \times M \rightarrow M$ called scalar multiplication that satisfies these properties:

$$M1) (r + r') \cdot m = r \cdot m + r' \cdot m \text{ for all } r, r' \in R, m \in M;$$

$$M2) r \cdot (m + m') = r \cdot m + r \cdot m' \text{ for all } r \in R, m, m' \in M;$$

$$M3) (rr') \cdot m = r \cdot (r' \cdot m) \text{ for all } r, r' \in R, m \in M;$$

$$M4) 1 \cdot m = m \text{ for all } m \in M.$$

We usually write rm instead of $r \cdot m$. Also, note that if R is a field, then M is said to be a vector space over R .

Definition 2.55: [LL] Let M be an R -module and $\emptyset \neq N \subseteq M$. Then, N is called a submodule of M if N is a subgroup of M and $rn \in N$ for all $r \in R, n \in N$.

Definition 2.56: [LL] An R -module M is said to be finitely generated if there exists a subset $\{m_1, \dots, m_t\} \subset M$ such that $M = \{r_1m_1 + \dots + r_tm_t : r_1, \dots, r_t \in R\}$.

Definition 2.57: [LL] Let M be an R -module. Then, M is called a free module if there exists a subset $\{m_\lambda : \lambda \in \Lambda\} \subset M$ such that each $m \in M$ can be uniquely written as $m = \sum_{\lambda \in \Lambda} r_\lambda m_\lambda$ where only finitely many of $r_\lambda \in R$ are nonzero.

Note that every vector space is a free module.

Definition 2.58: Let M be an R -module and N be a submodule of M . Then, the set $M/N = \{m+N : m \in M\}$ forms an R -module structure together with the operations $+$ and $\cdot$ defined as $(m+N) + (m'+N) = (m+m') + N$ and $r \cdot (m+N) = rm + N$ for all $m+N, m'+N \in M/N, r \in R$. Then, the R -module M/N is called the quotient module of M by N .

Definition 2.59: Let M and N be two R -modules. A function $f : M \longrightarrow N$ is called an R -module homomorphism from M into N if the following properties are satisfied:

- i) $f(m + m') = f(m) + f(m')$ for all $m, m' \in M$;
- ii) $f(rm) = rf(m)$ for all $m \in M, r \in R$.

If f is one-to-one and onto, then f is called an isomorphism. In this case, M and N are said to be isomorphic and it is denoted by $M \cong N$.

Definition 2.60: Let $f : M \longrightarrow N$ be an R -module homomorphism. The kernel and cokernel of the homomorphism f are defined as $\text{Ker}(f) = \{m \in M : f(m) = 0\}$ and $\text{coker}(f) = N/\text{Im}(f)$, respectively.

Theorem 2.8: (First Isomorphism Theorem) Let $f : M \longrightarrow N$ be an R -module homomorphism. Then, $M/\text{Ker}(f) \cong \text{Im}(f)$.

Theorem 2.9: Suppose that N and N' are two submodules of an R -module M such that $N' \subset N$. Then, $(M/N')/(N/N') \cong M/N$.

Proposition 2.11: [18] Let M, N and K be R -modules. Also, let $\oplus$ and $\otimes_R$ be the direct sum and tensor product over R , respectively. Then, we have

- i) $(M \otimes_R N) \otimes_R K \cong M \otimes_R (N \otimes_R K)$;
- ii) $(M \oplus N) \otimes_R K \cong (M \otimes_R K) \oplus (N \otimes_R K)$.

Proposition 2.12: [14] Let I and J be ideals of $K[x_1, \dots, x_n]$ and $K[y_1, \dots, y_m]$, respectively where $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ are disjoint sets of indeterminates. Then, $K[x_1, \dots, x_n]/I \otimes_K K[y_1, \dots, y_m]/J \cong K[x_1, \dots, x_n, y_1, \dots, y_m]/(I + J)$.

Definition 2.61: [19] Let $\{F_i\}_{i \in \mathbb{Z}}$ be a family of R -modules and let $\{d_i\}_{i \in \mathbb{Z}}$ be a family of R -module homomorphisms where $d_i : F_i \longrightarrow F_{i-1}$ for each $i \in \mathbb{Z}$. Then, the chain

$$\mathbb{F} : \dots \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \dots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow \dots \quad (2.20)$$

of R -modules is called a chain complex if $\text{Im}(d_{i+1}) \subseteq \text{Ker}(d_i)$ for each $i \in \mathbb{Z}$. Here, the set of homomorphisms $d = \{d_i\}_{i \in \mathbb{Z}}$ is said to be the differential of the chain complex $\mathbb{F}$. Usually, we denote this chain complex by $(\mathbb{F}, d)$. Also, $\mathbb{F}$ is called a left complex if $F_i = 0$ for all $i < 0$. We denote F_i , the i -th module of the chain complex $\mathbb{F}$ with $(\mathbb{F})_i$. Also, for any $t \in \mathbb{Z}$, we denote the homologically shifted t degrees of the chain complex $\mathbb{F}$ by $\mathbb{F}[-t]$ and it is defined as $(\mathbb{F}[-t])_i = F_{i-t}$ for each i .

Definition 2.62: [19] The set $\mathcal{H}_i(\mathbb{F}) = \text{Ker}(d_i) / \text{Im}(d_{i+1})$ is called the i -th homology of a chain complex $(\mathbb{F}, d)$. Also, $\mathbb{F}$ is said to be exact at F_i if $\mathcal{H}_i(\mathbb{F}) = 0$. Furthermore, $\mathbb{F}$ is called an exact sequence if $\mathbb{F}$ is exact at F_i for all $i \in \mathbb{Z}$.

Definition 2.63: [19] A chain complex $(\mathbb{G}, \partial)$ is called a subcomplex of a chain complex $(\mathbb{F}, d)$ if $G_i \subseteq F_i$ and $d_i|_{G_i} = \partial_i$ for all $i \in \mathbb{Z}$.

Definition 2.64: [19] Let U be a finitely generated R -module and

$$\mathbb{F} : \dots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{d_1} F_0 \quad (2.21)$$

be an exact sequence of finitely generated free R -modules. If $\text{coker}(d_1) \cong U$, then $\mathbb{F}$ is called a free R -resolution of U .

Definition 2.65: [19] Let W be a finitely generated R -module. For a free resolution $(\mathbb{F}, d)$ of a finitely generated R -module U , we define the Tor functor $\text{Tor}_i(U, W)$ as the i -th homology of the chain complex

$$\mathbb{F} \otimes_R W : \cdots \longrightarrow F_i \otimes_R W \xrightarrow{d_i \otimes \text{id}} F_{i-1} \otimes_R W \longrightarrow \cdots \xrightarrow{d_1 \otimes \text{id}} F_0 \otimes_R W \longrightarrow 0 \quad (2.22)$$

that is, $\text{Tor}_i(U, W) = \mathcal{H}_i(\mathbb{F} \otimes_R W)$ for all $i > 0$. Also, $\text{Tor}_0(U, W) = U \otimes_R W$.

Definition 2.66: Let M be an R -module. Then, M is said to be a flat module over R if $\mathbb{F} \otimes_R M$ is an exact sequence for any exact sequence $\mathbb{F}$ of R -modules.

Proposition 2.13: [18] Every free module is flat.

Note that every vector space is flat since every vector space is free module.

Proposition 2.14: Let B be a submodule of an R -module A . For any flat module C over R , we have $(A \otimes_R C)/(B \otimes_R C) \cong (A/B) \otimes_R C$.

Proof : Let us consider the exact sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow A/B \longrightarrow 0. \quad (2.23)$$

Since C is flat over R , we can write the exact sequence

$$0 \longrightarrow B \otimes_R C \longrightarrow A \otimes_R C \longrightarrow (A/B) \otimes_R C \longrightarrow 0. \quad (2.24)$$

Hence, we can see $(A \otimes_R C)/(B \otimes_R C) \cong (A/B) \otimes_R C$.

2.5. Graded Free Resolutions

We will start this section by defining graded polynomial rings, graded ideals and graded modules. In addition, we will investigate the graded chain complexes and examine the graded and multigraded free resolutions of finitely generated graded modules. Also, we will show by giving an example how the minimal graded and multigraded free resolutions can be calculated for a finitely generated graded S -module.

Moreover, we will introduce linear and pure resolutions, which are important concepts regarding minimal graded free resolutions. Furthermore, we will examine projective dimension and regularity, which are important in characterizing minimal graded free resolutions, and we will give the properties of regularity.

Let us start by introducing the grading of S . Let S_i be the set of all homogeneous polynomials of degree i in S for all $i \in \mathbb{N}$. Clearly, any $f \in S$ can be uniquely written as $f = h_0 + h_1 + \cdots + h_d$ for some d where $h_i \in S_i$ for each $i = 0, 1, \dots, d$. The polynomials $h_0, h_1, \dots, h_d$ are called homogeneous components of f . By this notation, S can be considered as a vector space over K and written as $S = \bigoplus_{i \in \mathbb{N}} S_i$. By this way, S is said to be $\mathbb{N}$ -graded or standard graded.

Definition 2.67: [19] Let I be an ideal of S with $I \neq S$. Then, I is called a graded ideal if $I = \bigoplus_{i \in \mathbb{N}} I_i$ where $I_i = I \cap S_i$ for all $i \in \mathbb{N}$.

Note that an ideal I of S is a graded ideal if and only if there is a generating set of homogeneous polynomials for I . If I is a graded ideal of S , then we have $S_i I_j \subseteq I_{i+j}$ for every $i, j \in \mathbb{N}$. Also, S/I is graded by $(S/I)_i = S_i/I_i$ for all $i \in \mathbb{N}$.

Let M be an S -module. Then, M is said to be $\mathbb{N}$ -graded if $M = \bigoplus_{i \in \mathbb{N}} M_i$ where each M_i is a vector space over K for all $i \in \mathbb{N}$ and $S_i M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{N}$. Also, any element $m \in M_i$ is called homogeneous element of degree i and the degree is denoted by $\deg(m) = i$ for all $i \in \mathbb{N}$. Note that if M is an $\mathbb{N}$ -graded S -module, then M can be generated by homogeneous elements in M .

Definition 2.68: [19] Let M be an $\mathbb{N}$ -graded S -module and N be a submodule of M . Then, N is called an $\mathbb{N}$ -graded submodule of M if $N = \bigoplus_{i \in \mathbb{N}} N_i$ where $N_i = N \cap M_i$ for all $i \in \mathbb{N}$.

Note that if N is a graded submodule of an $\mathbb{N}$ -graded S -module M , then the quotient module M/N is $\mathbb{N}$ -graded by $(M/N)_i = M_i/N_i$ for all $i \in \mathbb{N}$.

Let M be an $\mathbb{N}$ -graded S -module and $d \in \mathbb{N}$. The module obtained by shifting the degrees of M by d is denoted by $M(-d)$ and it is defined as $M(-d)_i = M_{i-d}$ for all $i \in \mathbb{N}$. Here, the degree of any homogeneous element in $M(-d)$ is d more than in M . In particular, $S(-d)$ is finitely generated graded S -module and its generator $1 \in S(-d)$ has degree d in $S(-d)$. Here, the element $1 \in S(-d)$ is said to be the 1-generator of the module $S(-d)$.

Definition 2.69: [19] Let M, N be two $\mathbb{N}$ -graded S -modules and $\varphi : M \rightarrow N$ be a homomorphism. Then, φ is said to have degree i if $\deg(\varphi(m)) = \deg(m) + i$ for any homogeneous element m of M . In particular, we say that φ has degree 0 if $\deg(\varphi(m)) = \deg(m)$ for any homogeneous element m of M . Also, we say that φ is graded homomorphism if φ has degree i for some $i \geq 0$.

Example 2.8: Let $\varphi : S(-4) \oplus S(-6) \rightarrow S(-2) \oplus S(-3)$ be the S -module homomorphism defined by the matrix

$$A = \begin{bmatrix} x_2 x_n & x_1^3 x_n \\ x_1 & x_2 x_n^2 \end{bmatrix}. \quad (2.25)$$

One can easily see that φ is a graded homomorphism having degree 0. Usually, we write

$$S(-4) \oplus S(-6) \xrightarrow{\begin{bmatrix} x_2 x_n & x_1^3 x_n \\ x_1 & x_2 x_n^2 \end{bmatrix}} S(-2) \oplus S(-3) \quad (2.26)$$

instead of

$$S(-4) \oplus S(-6) \xrightarrow{\varphi} S(-2) \oplus S(-3). \quad (2.27)$$

Definition 2.70: [19] A chain complex

$$\mathbb{F} : \cdots \rightarrow F_i \xrightarrow{d_i} F_{i-1} \rightarrow \cdots \rightarrow F_1 \xrightarrow{d_1} F_0 \rightarrow \cdots \quad (2.28)$$

over S is said to be graded if each F_i is graded S -module and d_i is a graded homomorphism of degree 0 for all i . For each F_i , it can be seen that $F_i = \bigoplus_{j \in \mathbb{N}} F_{i,j}$. Here, we say that any element of $F_{i,j}$ has homological degree i and internal degree j .

Note that if all the modules in $\mathbb{F}$ are finitely generated graded free S -modules, then $F_i \cong \bigoplus_{j \in \mathbb{N}} S(-j)^{b_{i,j}}$ for each $i \geq 0$ where only finite number of the numbers $b_{i,j}$ are nonzero. Then, we can write the graded complex $\mathbb{F}$ in (2.28) as

$$\mathbb{F} : \cdots \longrightarrow \bigoplus_{j \in \mathbb{N}} S(-j)^{b_{i,j}} \xrightarrow{d_i} \bigoplus_{j \in \mathbb{N}} S(-j)^{b_{i-1,j}} \longrightarrow \cdots . \quad (2.29)$$

The numbers $b_{i,j}$ are called the graded Betti numbers of the graded complex $\mathbb{F}$.

Definition 2.71: [19] Let U be a finitely generated graded S -module and

$$\mathbb{F} : \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \quad (2.30)$$

be a free resolution of U . If $\mathbb{F}$ is graded and the isomorphism $\text{coker}(d_1) \cong U$ has degree 0, then $\mathbb{F}$ is called graded free S -resolution of U .

Note that for each F_i in Definition 2.71, a basis consisting of homogeneous elements can be found. In this way, each d_i can be represented by a matrix D_i with homogeneous entries in S . The matrices D_i are said to be differential matrices.

Definition 2.72: [19] Let U be a finitely generated graded S -module and $\mathbb{F}$ be a graded free resolution of U . Then, $\mathbb{F}$ is called minimal graded free resolution of U if $\text{Im}(d_{i+1}) \subseteq (x_1, \dots, x_n)F_i$ for all $i \geq 0$.

Theorem 2.10: [19] Let U be a finitely generated graded S -module. Then, we have the followings:

- i) Minimal graded free resolution of U exists;
- ii) Minimal graded free resolutions of U are isomorphic;
- iii) If the numbers $\beta_{i,j}$ are the graded Betti numbers of the minimal graded free resolution of U , then we have $\beta_{i,j} \leq b_{i,j}$ where the numbers $b_{i,j}$ are the graded Betti numbers of any graded free resolution of U .

Usually, we denote the graded Betti numbers of the minimal graded free resolution of a finitely generated graded S -module U by $\beta_{i,j}(U)$.

Theorem 2.11: [20] Suppose that $\mathbb{F}$ is the minimal graded free resolution of a graded ideal I in S and $\mathcal{G} = \{f_1, \dots, f_t\}$ is a Gröbner basis of I . Also, suppose that

$$S(f_i, f_j) = g_{ij,1}f_1 + g_{ij,2}f_2 + \cdots + g_{ij,t}f_t \quad (2.31)$$

is the standard expression of $S(f_i, f_j)$ with respect to $\mathcal{G}$ for each pair $i < j$. Then, the set of the relations

$$r_{ij} = S(f_i, f_j) - g_{ij,1}f_1 - g_{ij,2}f_2 - \cdots - g_{ij,t}f_t = 0 \quad (2.32)$$

is a system of generators for F_1 .

Let U be a finitely generated graded S -module and let

$$\mathbb{F} : \cdots \longrightarrow F_2 \xrightarrow{D_2} F_1 \xrightarrow{D_1} F_0 \xrightarrow{D_0} U \longrightarrow 0 \quad (2.33)$$

be the minimal graded free resolution of U where $D_0, D_1, D_2, \dots$ are the differential matrices corresponding to the differential maps $d_0, d_1, d_2, \dots$. Then the differential matrices can be written in this way:

0. The columns of D_0 are a minimal system of homogeneous generators for U .
1. The column vectors of D_1 are a minimal system of homogeneous relations on the minimal generators of U .
2. The column vectors of D_2 are a minimal system of homogeneous relations on the relations in d_1 .
- $\vdots$

By the minimality of the relations encoded in d_i , we mean a minimal system of homogeneous generators of the kernel of the differential map.

Example 2.9: Calculate the minimal graded free resolution of A/B if $A = K[x, y, z]$ and $B = (x^2, xy, xz, y^2)$.

Solution : Let us say $m_1 = x^2$, $m_2 = xy$, $m_3 = xz$ and $m_4 = y^2$. Clearly, the set $\{m_1, m_2, m_3, m_4\}$ forms a Gröbner basis for B . Because of Theorem [2.11](#) we must examine the following relations to find the minimal relations between m_1, m_2, m_3, m_4 and the relation vectors.

$$S(m_1, m_2) = ym_1 - xm_2 = 0, \quad \alpha_1 = (y, -x, 0, 0) \quad (2.34)$$

$$S(m_1, m_3) = zm_1 - xm_3 = 0, \quad \alpha_2 = (z, 0, -x, 0) \quad (2.35)$$

$$S(m_1, m_4) = y^2m_1 - x^2m_4 = 0, \quad \alpha_3 = (y^2, 0, 0, -x^2) \quad (2.36)$$

$$S(m_2, m_3) = zm_2 - ym_3 = 0, \quad \alpha_4 = (0, z, -y, 0) \quad (2.37)$$

$$S(m_2, m_4) = ym_2 - xm_4 = 0, \quad \alpha_5 = (0, y, 0, -x) \quad (2.38)$$

$$S(m_3, m_4) = y^2m_3 - xzm_4 = 0, \quad \alpha_6 = (0, 0, y^2, -xz). \quad (2.39)$$

Observe that the vectors α_3 and α_6 can be expressed in terms of $\alpha_1, \alpha_2, \alpha_4$ and α_5 . On the other hand, $\alpha_1, \alpha_2, \alpha_4$ and α_5 can not be expressed in terms of the other vectors. Let us say r_1, r_2, r_3 and r_4 to the relations in equations (2.34), (2.35), (2.37) and (2.38), respectively. Then, all relations between r_1, r_2, r_3, r_4 and the relation vectors can be written as

$$zr_1 - yr_2 + xr_3 + 0.r_4 = 0, \quad \gamma = (z, -y, x, 0) \quad (2.40)$$

Therefore, we can write the minimal graded free resolution as

$$\begin{aligned} \mathbb{F} : 0 \longrightarrow A(-4) \xrightarrow{\begin{bmatrix} z \\ -y \\ x \\ 0 \end{bmatrix}} A(-3)^4 \xrightarrow{\begin{bmatrix} y & z & 0 & 0 \\ -x & 0 & z & y \\ 0 & -x & -y & 0 \\ 0 & 0 & 0 & -x \end{bmatrix}} A(-2)^4 \xrightarrow{\begin{bmatrix} x^2 & xy & xz & y^2 \end{bmatrix}} A \xrightarrow{\varphi} A/B \longrightarrow 0 \end{aligned} \quad (2.41)$$

where $\varphi(f) = f + B$ for all $f \in A$.

Theorem 2.12: [19] Suppose that U is a finitely generated graded S -module. For any $i, j \in \mathbb{N}$, we have $\beta_{i,j}(U) = \dim_K \text{Tor}_i(U, k)_j$ where $\text{Tor}_i(U, k)_j$ is the set of all homogeneous elements of degree j in $\text{Tor}_i(U, k)$.

Definition 2.73: [12] The projective dimension of a finitely generated graded S -module U is defined as $\text{projdim}(U) = \max\{i : \beta_{i,j}(U) \neq 0, j \geq 0\}$. The Castelnuovo-

Mumford regularity (or simply regularity) of U is denoted by $\text{reg}(U)$ and it is defined as $\text{reg}(U) = \max\{j : \beta_{i,i+j}(U) \neq 0 \text{ for some } i\}$.

Let U be a finitely generated graded S -module and let $\text{projdim}(U) = p$ and $\text{reg}(U) = r$. Then, the Betti diagram of U has the form shown in Table 2.1.

Table 2.1: Betti diagram of U

0	0	1	$\dots$	i	$\dots$	p
0	$\beta_{0,0}$	$\beta_{1,1}$	$\dots$	$\beta_{i,i}$	$\dots$	$\beta_{p,p}$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\dots$	$\beta_{i,i+1}$	$\dots$	$\beta_{p,p+1}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\dots$
j	$\beta_{0,j}$	$\beta_{1,1+j}$	$\dots$	$\beta_{i,i+j}$	$\dots$	$\beta_{p,p+j}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
r	$\beta_{0,r}$	$\beta_{1,1+r}$	$\dots$	$\beta_{i,i+r}$	$\dots$	$\beta_{p,p+r}$

Example 2.10: Betti diagram of A/B , for which we found the graded free resolution in Example 2.9 can be given as displayed in Table 2.2. Hence, we can see $\text{projdim}(A/B) = 3$ and $\text{reg}(A/B) = 1$.

Table 2.2: Betti diagram of A/B

	0	1	2	3
0	1	0	0	0
1	0	4	4	1

Proposition 2.15: [I2] (Hilbert's Syzygy Theorem) Suppose that U is a finitely generated graded S -module. Then, $\text{projdim}(U) \leq n$, i.e., $\beta_{i,j}(U) = 0$ for all $i > n, j \geq 0$.

Proposition 2.16: [I9] Suppose that $\prec$ is a monomial order on S and I is a graded ideal in S . Then, $\beta_{i,j}(S/I) \leq \beta_{i,j}(S/\text{in}_{\prec}(I))$ for all $i \geq 0$ and $j \geq 0$. In addition, $\text{reg}(S/I) \leq \text{reg}(S/\text{in}_{\prec}(I))$.

Proposition 2.17: [179] Let $0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0$ be an exact sequence of finitely generated graded S -modules and let the homomorphisms of this exact sequence have degree 0. Then, we have

- i) $\text{reg}(M'') \leq \max\{\text{reg}(M'), \text{reg}(M) - 1\}$;*
- ii) $\text{reg}(M') \leq \max\{\text{reg}(M), \text{reg}(M'')\}$;*
- iii) $\text{reg}(M) \leq \max\{\text{reg}(M'), \text{reg}(M'') + 1\}$.*

Definition 2.74: [179] Let U be a finitely generated graded S -module. The graded Poincaré series of U is denoted by $P(U, t)$ and it is defined as

$$P(U, t) = \sum_{i=0}^n \sum_{j \in \mathbb{N}} \beta_{i,j}(U) x^j t^i. \quad (2.42)$$

Definition 2.75: [121] Let $(\mathbb{F}, d), (\mathbb{G}, d')$ be two chain complexes. The tensor product of the chain complexes $\mathbb{F}$ and $\mathbb{G}$ over K is denoted by $(\mathbb{F} \otimes_K \mathbb{G}, \partial)$ and defined as

$$(\mathbb{F} \otimes_K \mathbb{G})_i = \bigoplus_{i_1+i_2=i} (F_{i_1} \otimes_K G_{i_2}), \quad (2.43)$$

$$\partial_i = \bigoplus_{i_1+i_2=i} (d_{i_1} \otimes_K \text{id}_{G_{i_2}} + (-1)^{i_1} \text{id}_{F_{i_1}} \otimes_K d'_{i_2}) \quad (2.44)$$

for all $i \in \mathbb{Z}$.

Remark 2.6: If we choose $I = (0)$ and $J = (0)$ in Proposition 2.12 then we have $K[x_1, \dots, x_n] \otimes_K K[y_1, \dots, y_m] \cong K[x_1, \dots, x_n, y_1, \dots, y_m]$. Additionally, if we call $R = K[x_1, \dots, x_n]$, $T = K[y_1, \dots, y_m]$ and $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$, then $R(-i) \otimes_K T(-j) \cong S(-i-j)$ for all $i, j \in \mathbb{N}$.

Proposition 2.18: [12] Let I, J be finitely generated graded ideals of the polynomial rings $R = K[x_1, \dots, x_n]$, $T = K[y_1, \dots, y_m]$, respectively where $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ are disjoint sets of indeterminates. Suppose that $\mathbb{F}$ is the minimal graded free R -resolution of R/I and $\mathbb{G}$ is the minimal graded free T -resolution of T/J . Then, $\mathbb{F} \otimes_K \mathbb{G}$ is the minimal graded free S -resolution of $S/(I + J)$ where

$S = K[x_1, \dots, x_n, y_1, \dots, y_m]$. Moreover, we can write the graded Betti numbers of $S/(I + J)$ as

$$\beta_{i,j}(S/(I + J)) = \sum_{\substack{i_1+i_2=i \\ j_1+j_2=j}} \beta_{i_1,j_1}(R/I) \beta_{i_2,j_2}(T/J) \quad (2.45)$$

for all $i \geq 0, j \geq 0$.

Proof : Let us assume the minimal graded free resolutions of R/I and T/J are given as

$$\mathbb{F} : \dots \longrightarrow \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{1,j}(R/I)} \xrightarrow{d_1} \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{0,j}(R/I)} \xrightarrow{d_0} R/I \quad (2.46)$$

and

$$\mathbb{G} : \dots \longrightarrow \bigoplus_{j \in \mathbb{N}} T(-j)^{\beta_{1,j}(T/J)} \xrightarrow{d'_1} \bigoplus_{j \in \mathbb{N}} T(-j)^{\beta_{0,j}(T/J)} \xrightarrow{d'_0} T/J. \quad (2.47)$$

Let us show that the tensor product $\mathbb{F} \otimes_K \mathbb{G}$ of chain complexes $\mathbb{F}$ and $\mathbb{G}$ is the minimal graded free resolution of $R/I \otimes_K T/J \cong S/(I + J)$ over $R \otimes_K T \cong S$. Observe that the module T/J is flat over K . Now, let us consider the i -th homology of the chain complex $\mathbb{F} \otimes_K T/J$. Then,

$$\mathcal{H}_i(\mathbb{F} \otimes_K T/J) = \text{Ker}(d_i \otimes_K \text{id}) / \text{Im}(d_{i+1} \otimes_K \text{id}) \quad (2.48)$$

$$= \text{Ker}(d_i) \otimes_K (T/J) / \text{Im}(d_{i+1}) \otimes_K (T/J) \quad (2.49)$$

$$= (\text{Ker}(d_i) / \text{Im}(d_{i+1})) \otimes_K T/J \quad (2.50)$$

$$= \mathcal{H}_i(\mathbb{F}) \otimes_K T/J. \quad (2.51)$$

Here, equation (2.51) implies $\text{Tor}_0(R/I, T/J) = R/I \otimes_K T/J \cong S/(I + J)$ and $\text{Tor}_i(R/I, T/J) = 0$ for all $i > 0$ since $\mathbb{F}$ is exact and T/J is flat module over K . As $\mathbb{F}$ and $\mathbb{G}$ are minimal, we have $\text{Im}(d_i) \subset (x_1, \dots, x_n)$ and $\text{Im}(d'_i) \subset (y_1, \dots, y_m)$ for all d_i, d'_i . Thus, we can write $\text{Im}(\partial_i) \subset (x_1, \dots, x_n, y_1, \dots, y_m)$ for each differential map

of $\mathbb{F} \otimes_K \mathbb{G}$ defined by $\partial_i = \bigoplus_{i_1+i_2=i} (d_{i_1} \otimes_K \text{id}_{G_{i_2}} + (-1)^{i_1} \text{id}_{F_{i_1}} \otimes_K d'_{i_1})$. Therefore, $\mathbb{F} \otimes_K \mathbb{G}$ is the minimal graded free S -resolution of $S/(I + J)$. Also,

$$(\mathbb{F} \otimes_K \mathbb{G})_i = \bigoplus_{i_1+i_2=i} \left(\left(\bigoplus_{j_1 \in \mathbb{N}} R(-j_1)^{\beta_{i_1,j_1}(R/I)} \right) \otimes_K \left(\bigoplus_{j_2 \in \mathbb{N}} T(-j_2)^{\beta_{i_2,j_2}(T/J)} \right) \right) \quad (2.52)$$

$$= \bigoplus_{i_1+i_2=i} \bigoplus_{j_1+j_2=j} (R(-j_1)^{\beta_{i_1,j_1}(R/I)} \otimes_K T(-j_2)^{\beta_{i_2,j_2}(T/J)}) \quad (2.53)$$

$$= \bigoplus_{i_1+i_2=i} \bigoplus_{j_1+j_2=j} S(-j)^{\beta_{i_1,j_1}(R/I) \beta_{i_2,j_2}(T/J)}. \quad (2.54)$$

Hence, we can write

$$\beta_{i,j}(S/(I + J)) = \sum_{\substack{i_1+i_2=i \\ j_1+j_2=j}} \beta_{i_1,j_1}(R/I) \beta_{i_2,j_2}(T/J) \quad (2.55)$$

from equation (2.54).

Definition 2.76: [I9] Let U be a finitely generated graded S -module. Then, we say that U has a pure resolution if its minimal graded free resolution $\mathbb{F}$ has the form

$$\mathbb{F} : 0 \longrightarrow S(-d_t)^{\beta_t, d_t} \longrightarrow \cdots \longrightarrow S(-d_1)^{\beta_1, d_1} \longrightarrow S(-d_0)^{\beta_0, d_0}. \quad (2.56)$$

Additionally, the pure resolution $\mathbb{F}$ of U is said to be d -linear if $d_0 = d$ and $d_i = d + i$ for each $i = 1, \dots, t$.

Proposition 2.19: [I2] Suppose that I is a graded ideal of S . If $S/\text{in}_{\prec}(I)$ has linear resolution for some monomial orders $\prec$ on S , then S/I has too.

Definition 2.77: [I3] Let I be a graded ideal of S . Then, we say that I has linear quotients if there exists homogeneous polynomials $f_1, f_2, \dots, f_t$ in S such that $I = (f_1, \dots, f_t)$ and the ideal $((f_1, \dots, f_{i-1}) : f_i)$ can be generated by homogeneous polynomials of degree 1 for each $i = 2, \dots, t$.

Proposition 2.20: [I3] Let $I \subset S$ be a graded ideal generated by the elements of degree d . If I has linear quotients, then I has d -linear resolution.

Definition 2.78: [I9] Let U be a finitely generated S -module. An element $f \in S \setminus K$ is said to be U -regular (or nonzero divisor on U) if $fu \neq 0$ for all $0 \neq u \in U$. A sequence $f_1, \dots, f_t$ of elements of S is called a U -regular sequence if $(f_1, \dots, f_t)U \neq U$ and f_i is a nonzero divisor on $U/(f_1, \dots, f_{i-1})$ for all $1 \leq i \leq t$.

Proposition 2.21: [I9] Suppose that U is a finitely generated graded S -module. If a homogeneous element $r \in S \setminus K$ is U -regular, then $\text{reg}(U/rU) = \text{reg}(U) + \deg(r) - 1$.

Proposition 2.22: [I4] Let J be a graded ideal of S such that $J \subset (x_1, \dots, x_{n-1})$. By setting $I = J + (x_n)$, we have $\text{reg}(S/I) = \text{reg}(S/J)$.

Proof : Clearly, x_n is a regular element on S/J . Also, we can write

$$S/J \Big/_{x_n \cdot S/J} = S/J \Big/_{((x_n) + J)/J} = S/J \Big/_{I/J} \cong S/I. \quad (2.57)$$

Then, we can write $\text{reg}(S/I) = \text{reg}(S/J) + \deg(x_n) - 1$ from equation (2.57) and Proposition 2.21. Hence, $\text{reg}(S/I) = \text{reg}(S/J)$.

Construction 2.2: [I2] Let the set $\{e_1, \dots, e_m\}$ be a basis of a free S -module F and let $\mathbf{f} = \{f_1, \dots, f_t\}$ be a sequence of polynomials in S . Also, let $K_0 = S$ and K_i denotes the i -th exterior power of F , more precisely, $K_i = \bigwedge_{j=1}^i F$ for all $1 \leq i \leq m$. Clearly, $B_i = \{e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_i} : 1 \leq j_1 < j_2 < \dots < j_i \leq m\}$ is a basis of K_i for all $1 \leq i \leq m$. Then, we have $\text{rank } K_i = \binom{m}{i}$. Let us consider

$$\mathbf{K}(\mathbf{f}) : 0 \longrightarrow K_m \xrightarrow{d_m} \dots \longrightarrow K_i \xrightarrow{d_i} K_{i-1} \longrightarrow \dots \longrightarrow K_1 \xrightarrow{d_1} K_0 \longrightarrow 0 \quad (2.58)$$

where

$$d_i(e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_i}) = \sum_{k=1}^i (-1)^{k+1} f_{j_k} e_{j_1} \wedge \dots \wedge e_{j_{k-1}} \wedge e_{j_{k+1}} \wedge \dots \wedge e_{j_i} \quad (2.59)$$

for each $i = 1, \dots, m$. One can easily check that $\text{Im}(d_i) \subseteq \text{Ker}(d_{i-1})$ for all i . It means $\mathbf{K}(\mathbf{f})$ is a chain complex. In addition, $\mathbf{K}(\mathbf{f})$ is graded by $\deg(e_i) = 1$ and $\deg(e_{j_1} \wedge \dots \wedge e_{j_i}) = i$ for each $1 \leq i \leq m$. Here, $\mathbf{K}(\mathbf{f})$ is called the Koszul complex attached to the sequence $\mathbf{f}$. If U is a finitely generated graded S -module, we denote

the chain complex $\mathbf{K}(\mathbf{f}) \otimes_S U$ by $\mathbf{K}(\mathbf{f}; U)$. The chain complex $\mathbf{K}(\mathbf{f}; U)$ is called the Koszul complex of U attached to $\mathbf{f}$. With $\mathcal{H}_i(\mathbf{f}; U)$, we denote the i -th homology of the chain complex $\mathbf{K}(\mathbf{f}; U)$ for each i .

Theorem 2.13: [19] Suppose that $\mathbf{f} = \{f_1, \dots, f_t\}$ is a sequence of homogeneous polynomials of S with $\deg(f_j) > 0$ for $1 \leq j \leq t$ and U is a nonzero finitely generated graded S -module. Then, $\mathbf{f}$ is a U -regular sequence if and only if $\mathcal{H}_0(\mathbf{f}; U) = U/(\mathbf{f})U$ and $\mathcal{H}_i(\mathbf{f}; U) = 0$ for all $i > 0$.

Construction 2.3: [19] Let $(\mathbb{U}, d)$ and $(\mathbb{U}', d')$ be the minimal graded free resolutions of the finitely generated graded S -modules V and V' , respectively. Also, let $\varphi : (\mathbb{U}, d) \rightarrow (\mathbb{U}', d')$ be a map of complexes and $\phi : V \rightarrow V'$ be a one-to-one module homomorphism. Then, we have a diagram in the form

$$\begin{array}{ccccccc} \mathbb{U} : \cdots & \longrightarrow & U_i & \xrightarrow{d_i} & U_{i-1} & \longrightarrow & \cdots \longrightarrow U_1 \xrightarrow{d_1} U_0 \\ & & \downarrow \varphi_i & & \downarrow \varphi_{i-1} & & \downarrow \varphi_1 \downarrow \varphi_0 \\ \mathbb{U}' : \cdots & \longrightarrow & U'_i & \xrightarrow{d'_i} & U'_{i-1} & \longrightarrow & \cdots \longrightarrow U'_1 \xrightarrow{d'_1} U'_0 \end{array} \quad (2.60)$$

The chain complex $(\mathbb{W}, \partial)$ is called the mapping cone of φ where $\mathbb{W} = \mathbb{U}[-1] \oplus \mathbb{U}'$, that is, $(\mathbb{W})_i = W_i = U_{i-1} \oplus U'_i$ for each $i \geq 0$ and each $\partial_i : W_i \rightarrow W_{i-1}$ defined as $\partial_i|_{U_{i-1}} = \varphi_{i-1} - d_{i-1}$ and $\partial_i|_{U'_i} = d'_i$. Note that $\mathbb{W}$ is a graded free resolution for $V'/\varphi(V)$. Although $\mathbb{U}$ and $\mathbb{U}'$ are minimal graded free resolutions for V and V' , respectively, $\mathbb{W}$ may not be the minimal graded free resolution of $V'/\varphi(V)$.

Let us start by introducing the multigrading of the polynomial ring S . If $\epsilon_i \in \mathbb{N}^n$ is the i -th unit vector, we can see S is $\mathbb{N}^n$ -graded by $\deg(x_i) = \epsilon_i$. For any vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$, we can associate the monomial $x^{\mathbf{a}} = x_1^{a_1} \dots x_n^{a_n}$ in S . Then, we have $S = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S_{\mathbf{a}}$ where $S_{\mathbf{a}}$ is the vector space over K whose basis is $\{x^{\mathbf{a}}\}$ for each $\mathbf{a} \in \mathbb{N}^n$. Sometimes, we say that S is multigraded instead of $\mathbb{N}^n$ -graded.

Suppose that M is an S -module. Then, we say that M is $\mathbb{N}^n$ -graded (or multigraded) if $M = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} M_{\mathbf{a}}$ where $M_{\mathbf{a}}$ is a vector space over K for all $\mathbf{a} \in \mathbb{N}^n$ and $S_{\mathbf{a}} M_{\mathbf{b}} \subseteq M_{\mathbf{a}+\mathbf{b}}$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$. Each element of $M_{\mathbf{a}}$ is said to be multihomogeneous of degree $\mathbf{a}$.

Note that we denote by $S(-\mathbf{a})$ the $\mathbb{N}^n$ -graded free S -module where the generator $1 \in S(-\mathbf{a})$ has multidegree $\mathbf{a}$ in $S(-\mathbf{a})$. Sometimes, we write $S(x^{\mathbf{a}})$ instead of $S(-\mathbf{a})$.

Let U be a finitely generated $\mathbb{N}^n$ -graded S -module and

$$\mathbb{F} : \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \quad (2.61)$$

be a free resolution of U . Then, $\mathbb{F}$ is called $\mathbb{N}^n$ -graded free S -resolution if each F_i is multigraded S -module and each d_i and the isomorphism $\text{coker}(d_1) \cong U$ has degree 0. Also, $\mathbb{F}$ is called minimal if $\text{Im}(d_{i+1}) \subseteq (x_1, \dots, x_n)F_i$ for each $i \geq 0$. Additionally, $\mathbb{F}$ can be written in the form

$$\mathbb{F} : \cdots \longrightarrow \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}} \longrightarrow \cdots \longrightarrow \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S(-\mathbf{a})^{\beta_{1,\mathbf{a}}} \longrightarrow \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S(-\mathbf{a})^{\beta_{0,\mathbf{a}}}. \quad (2.62)$$

Here, the numbers $\beta_{i,\mathbf{a}}$ are said to be the multigraded Betti numbers and they are usually denoted by $\beta_{i,\mathbf{a}}(U)$ for the module U . Sometimes, we write $\beta_{i,x^{\mathbf{a}}}(U)$ instead of $\beta_{i,\mathbf{a}}(U)$. Note that we have $\beta_{i,\mathbf{a}}(U) = \dim_K \text{Tor}_i(U, k)_{\mathbf{a}}$ for all $\mathbf{a} \in \mathbb{N}^n$ and all $i \geq 0$.

Example 2.11: The minimal multigraded free resolution for the module A/B , which we considered in Example [2.9](#) can be written as

$$\begin{aligned} \mathbb{F} : 0 \longrightarrow A(x^2yz) &\xrightarrow{\begin{bmatrix} z \\ -y \\ x \\ 0 \end{bmatrix}} A(x^2y) \oplus A(x^2z) \oplus A(xyz) \oplus A(xy^2) \\ &\xrightarrow{\begin{bmatrix} y & z & 0 & 0 \\ -x & 0 & z & y \\ 0 & -x & -y & 0 \\ 0 & 0 & 0 & -x \end{bmatrix}} A(x^2) \oplus A(xy) \oplus A(xz) \oplus A(y^2) \\ &\xrightarrow{\begin{bmatrix} x^2 & xy & xz & y^2 \end{bmatrix}} A \xrightarrow{\varphi} A/B \longrightarrow 0 \end{aligned} \quad (2.63)$$

where $\varphi(f) = f + B$ for all $f \in A$.

Let us consider the $\mathbb{N}^n$ -graded chain complex

$$\mathbb{F} : \cdots \longrightarrow \bigoplus_{\mathbf{a} \in \mathbb{N}} F_{i,\mathbf{a}} \longrightarrow \cdots \longrightarrow \bigoplus_{\mathbf{a} \in \mathbb{N}} F_{1,\mathbf{a}} \longrightarrow \bigoplus_{\mathbf{a} \in \mathbb{N}} F_{0,\mathbf{a}} \longrightarrow \cdots . \quad (2.64)$$

The $\mathbf{a}$ -th (*multigraded*) component of $\mathbb{F}$ is denoted by $\mathbb{F}_{\mathbf{a}}$ and it is defined as

$$\mathbb{F}_{\mathbf{a}} : \cdots \longrightarrow F_{i,\mathbf{a}} \longrightarrow \cdots \longrightarrow F_{1,\mathbf{a}} \longrightarrow F_{0,\mathbf{a}} \longrightarrow \cdots . \quad (2.65)$$

Proposition 2.23: [I9] Suppose that $\mathbb{F}$ is an $\mathbb{N}^n$ -graded complex. Then, $\mathbb{F}$ is exact if and only if $\mathbb{F}_{\mathbf{a}}$ is exact for all $\mathbf{a} \in \mathbb{N}^n$.

Proposition 2.24: [I9] Suppose that I is a squarefree monomial ideal of S . If u is not a squarefree monomial, then $\beta_{i,u}(S/I) = 0$ for all $i \geq 0$.

Definition 2.79: [I9] Let U be a finitely generated multigraded S -module. Then, the multigraded Poincaré series of U is denoted by $P(U, t)$ and it is defined as

$$P(U, t) = \sum_{i=0}^n \sum_{\mathbf{a} \in \mathbb{N}^n} \beta_{i,\mathbf{a}}(U) x^{\mathbf{a}} t^i. \quad (2.66)$$

3. BINOMIAL EDGE IDEALS

We start this chapter by defining the binomial edge ideal which is the main object of this thesis.

Definition 3.1: [1] Let G be a graph on the vertex set $V(G) = [n] = \{1, \dots, n\}$. Also, let $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ where K is a field. The binomial edge ideal of G is defined as $J_G = (\{f_{ij} \in S : i < j \text{ and } \{i, j\} \in E(G)\})$ where the polynomials f_{ij} are of the form $f_{ij} = x_i y_j - x_j y_i$.

Throughout the remainder of the thesis, K and S will be considered as a field and the polynomial ring over K in the indeterminates $x_1, \dots, x_n, y_1, \dots, y_n$, respectively, unless otherwise stated.

Example 3.1: Let us consider the graph G shown in Figure 3.1. We can write the binomial edge ideal of G as

$$J_G = (f_{13}, f_{23}, f_{34}, f_{45}, f_{46}) \quad (3.1)$$

$$= (x_1 y_3 - x_3 y_1, x_2 y_3 - x_3 y_2, x_3 y_4 - x_4 y_3, x_4 y_5 - x_5 y_4, x_4 y_6 - x_6 y_4). \quad (3.2)$$

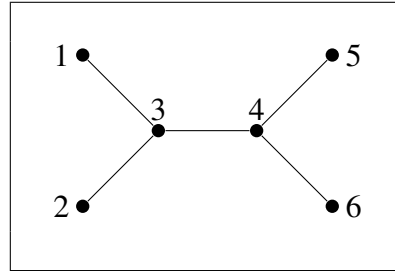


Figure 3.1: A graph G

3.1. Closed Graphs

In this section, closed graphs, which are very important for binomial edge ideals, will be introduced and examples will be given. Also, their basic properties will be given in terms of the terms in graph theory. In addition, some details about the clique complex of closed graphs will be shared.

Definition 3.2: [7] Let G be a graph on $[n]$. Then, G is called closed with respect to the given labeling of vertices if $\{i, k\} \in E(G)$ whenever $\{i, j\}, \{j, k\} \in E(G)$ with the property that either $i < j > k$ or $i > j < k$. Also, G is said to be closed if there exists a labeling of the vertices of G such that G closed with respect to this labeling.

Example 3.2: One can easily observe that the graph G shown in Figure 3.1 is not closed with respect to the given labeling of vertices since $\{4, 5\}, \{4, 6\} \in E(G)$ and $\{5, 6\} \notin E(G)$. The graph $K_{1,3}$ shown in Figure 3.2 is called a claw. Now, let us show that a claw is not a closed graph, i.e., $K_{1,3}$ is not closed with respect to any labeling of the vertices. If we label the vertex b with 1, then we will have $1 < a$ and $1 < c$ but $\{a, c\} \notin E(K_{1,3})$. So, we can not label the vertex b with 1. The similar argument applies to the case where the vertex b is labeled with 4. Then, we can label the vertex a and c with 1 and 4, respectively (other cases are similar). Then, $\{3, 4\} \notin E(K_{1,3})$ if $b = 2$ and $d = 3$. Also, $\{1, 2\} \notin E(H)$ if $b = 3$ and $d = 2$. Therefore, a claw is not a closed graph.

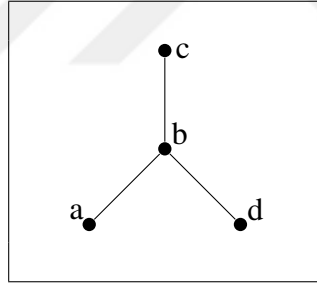


Figure 3.2: A claw

Proposition 3.1: [7] Let G be a closed graph. Then, G is chordal and claw-free.

Proof : Firstly, let us show that G is chordal graph. Assume the contrary. Then, there exists an induced cycle $C : i_1, i_2, \dots, i_t, i_1$ of length $t \geq 4$ in G with the convention $i_{t+1} = i_1$. Let $i_k = \min\{i_1, i_2, \dots, i_t\}$. Then, $\{i_{k-1}, i_{k+1}\} \in E(G)$ as G is closed and $\{i_k, i_{k+1}\}, \{i_k, i_{k-1}\}$ are edges of G . However, this contradicts C being an induced cycle of G . Hence, G must be chordal.

Let us show now that G is claw-free. Observe that all of induced subgraphs of G are closed because G is closed. From Example 3.2 a claw is not closed. Therefore, G does not contain an induced claw, so G is claw-free.

Example 3.3: The converse of Proposition 3.1 is not true. To show this fact, let us consider the graph G , called tent, shown in Figure 3.3a. Clearly, G is chordal and claw-free. We claim that G is not closed. If we label the vertex b by 1, then we will have $1 < c$ and $1 < d$ but $\{c, d\} \notin E(G)$. For similar reasons, it is not possible to label the vertices c and e with 1 or 6. Then, we can label the vertices a and d with 1 and 6, respectively (other cases are similar). Also, $\{c, d\} \notin E(G)$ if $b = 2$ or $e = 2$ and $\{b, f\} \notin E(G)$ if $c = 2$. So, we have to write $f = 2$. Now, $\{a, e\} \notin E(G)$ if $b = 5$ or $c = 5$ and $\{b, f\} \notin E(G)$ if $e = 5$. So, we do not have any vertices that we can label with 5. Hence, a tent is not closed. Similarly, one can easily observe that a net shown in Figure 3.3b is also chordal and claw-free, but not closed.

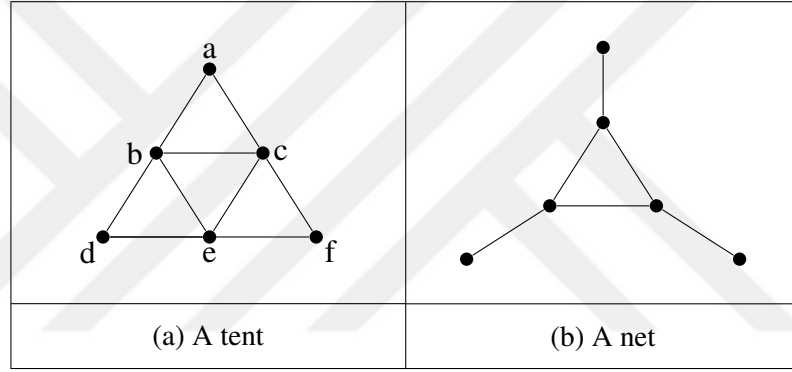


Figure 3.3: A tent and a net

Closed graphs can be expressed completely in terms of the terms in graph theory as follows.

Theorem 3.1: [U2] A graph G is closed if and only if it is chordal and has no induced claw, tent and net.

Proposition 3.2: [U] Suppose that G is a connected bipartite graph on $[n]$. Then, G is closed if and only if G is a path.

Proof : ($\Rightarrow$) Suppose that G is closed. Then, G is chordal by Proposition 3.1. It means G has either a cycle of length 3 or no cycle at all. Since G is bipartite, it has no odd cycle according to Proposition 2.9. Hence, G has no cycle, i.e., G is a tree. Thus, G must be a path since G has no induced claw by Proposition 3.1.

($\Leftarrow$) Suppose that G is a path. If the vertices of G from one end of the path to the other end are labeled as $1, 2, \dots, n$, respectively, then G will be a closed graph with

this labeling.

Proposition 3.3: [U] Suppose that G is a graph on $[n]$. Then, G is closed with respect to the given labeling of vertices if and only if for each $i, j \in V(G)$ with $i < j$, the vertices of the shortest path $i = i_1, i_2, \dots, i_t = j$ from i to j satisfies $i_1 < i_2 < \dots < i_t$.

Proof : ($\Rightarrow$) Suppose that i and j are two vertices of the graph G with $i < j$ and $P : i = i_1, i_2, \dots, i_t = j$ is the shortest path between i and j . Assume that there is a vertex i_k in P such that $i_{k-1} < i_k$ and $i_{k+1} < i_k$ or $i_k < i_{k-1}$ and $i_k < i_{k+1}$. Then, $\{i_{k-1}, i_{k+1}\} \in E(G)$ as G is closed. Hence, $P' : i = i_1, i_2, \dots, i_{k-1}, i_{k+1}, \dots, i_t = j$ is a path from i to j but this contradicts P being the shortest path from i to j . Therefore, we have $i_1 < i_2 < \dots < i_t$.

($\Leftarrow$) Assume that we have $i_1 < i_2 < \dots < i_t$ for each $i, j \in V(G)$ and any shortest path $i = i_1, i_2, \dots, i_t = j$ between i and j where $i < j$. We must examine these two cases:

Case 1: If $\{i, j\}, \{i, k\}$ are two edges in G and $i < j < k$, then the path j, i, k can not be the shortest path from j to k by our assumption. Thus, we say that there exists a shorter path, which is only possible in the case $\{j, k\} \in E(G)$.

Case 2: If $\{i, j\}, \{k, j\}$ are two edges in G and $i < k < j$, then the path i, j, k is not the shortest path from i to k according to the assumption. Hence, $\{i, k\} \in E(G)$.

These two cases give us the desired result.

Definition 3.3: [U2] Let G be a graph on $[n]$. A subset F of $[n]$ is called a clique of G if $\{i, j\} \in E(G)$ for all $i, j \in F$. The family of all such subsets is a simplicial complex whose facets are maximal cliques. This simplicial complex is called the clique complex of G and denoted by $\Delta(G)$.

Theorem 3.2: [3] Let G be a connected graph on $[n]$. Then, G is a closed graph if and only if there exists a labeling of the vertices of G such that the facets of the clique complex $\Delta(G)$ are intervals of $[n]$. Also, in this case $\Delta(G)$ is a quasi-tree.

Proof : ($\Leftarrow$) Let us show that the graph G is closed with respect to the labeling that makes the facets of $\Delta(G)$ to be the intervals of $[n]$. Let $\{i, j\}, \{i, k\} \in E(G)$ with $i < j < k$. Here, $\{i, k\} \in E(G)$ implies that i and k belong to the same facet of $\Delta(G)$, say F . Since F is an interval of $[n]$ containing $[i, k]$, then $j, k \in F$ and $\{j, k\} \in E(G)$.

Similarly, if $\{i, k\}, \{j, k\}$ are edges of G with $i < j < k$, we can see $\{i, j\} \in E(G)$. Hence, G is closed.

($\Rightarrow$) Suppose that G is a closed graph. We proceed by induction on the number of the vertices of G . If $n = 2$, then $G = K_2$ and the only facet of $\Delta(G)$ is $\{1, 2\} = [1, 2]$. So, it is an interval. Now, assume that $n > 2$ and the facets of $\Delta(G')$ are intervals for any graph G' with less than n vertices. Since G is a closed graph, there exists a labeling of the vertices of G such that G is closed with respect to this labeling of vertices. Let $F_1 = \{i : \{1, i\} \in E(G)\} \cup \{1\}$ and $l = \max F_1$. As G is connected, there is a path from $j - 1$ to j for any $j \in [n]$ with $j \geq 2$. Also, the shortest path between $j - 1$ and j is $P : j - 1, j$ by Proposition 3.3. Thus, $\{j - 1, j\} \in E(G)$ for all $j \geq 2$. Hence, if $j \in F_1$, then $\{1, j\}, \{j - 1, j\} \in E(G)$ implies $\{1, j - 1\} \in E(G)$ and $j - 1 \in F_1$ since G is closed. Then, $F_1 = \{1, 2, \dots, l\} = [1, l] \subset [n]$ is an interval. Clearly, F_1 is a clique of G because $\{1, i\}, \{1, j\} \in E(G)$ implies $\{i, j\} \in E(G)$ for any $i, j \in F_1$. Also, if there is a facet H in $\Delta(G)$ containing F_1 , then we can find a vertex $j \in H \setminus F_1$ with $\{1, j\} \in E(G)$. However, this contradicts the definition of the set F_1 . It means F_1 is a facet of $\Delta(G)$.

Now, let F_2 be another facet of $\Delta(G)$ such that $F_1 \cap F_2 \neq \emptyset$. Also, let us say $k = \min(F_1 \cap F_2)$. We will show that $F_1 \cap F_2 = [k, l]$. If $k = l$, then $F_1 \cap F_2 = \{l\}$. So, let us assume $k < l$. If $k < t \leq l$ and $s \in F_2 \setminus F_1$, then $\{k, t\}, \{k, s\} \in E(G)$ and $\{t, s\} \in E(G)$ because G is closed. Thus, it can be seen $t \in F_2$ and $t \in F_1 \cap F_2$. Hence, we can write $F_1 \cap F_2 = [k, l]$.

Now, consider the graph $G' = G - [k - 1]$. Clearly, $|V(G')| < n$ and G' is closed. We can say that the facets $F_2, \dots, F_r$ of $\Delta(G')$ are intervals by the induction hypothesis. Also, note that these facets can be ordered in such a way as to satisfy $\min F_2 < \dots < \min F_r$. If F is a facet of $\Delta(G)$, then $F = F_1$ or $F = F_j$ for some $2 \leq j \leq r$. Therefore, all facets of $\Delta(G)$ are intervals of $[n]$.

Lastly, we can see that $\Delta(G)$ is a quasi-tree. Indeed, if the facets of $\Delta(G)$ are ordered $F_1, F_2, \dots, F_r$ such that $\min F_1 < \min F_2 < \dots < \min F_r$, then $F_1, F_2, \dots, F_r$ is a leaf order since F_i is a leaf of $\langle F_1, \dots, F_i \rangle$.

3.2. Gröbner Bases of Binomial Edge Ideals

In this section, we will investigate Gröbner bases of binomial edge ideals. We will state in Theorem 3.4 that J_G has a quadratic Gröbner basis if and only if G is a closed graph. Also, we will show in Theorem 3.5 how the reduced Gröbner basis of J_G for any graph G can be calculated with respect to lexicographic order. Also, we will illustrate this with an example.

Theorem 3.3: [7] Suppose that G is a graph on $[n]$ and $<$ is the lexicographic order on the polynomial ring S where the indeterminates $x_1, \dots, x_n, y_1, \dots, y_n$ are ordered by $x_1 > \dots > x_n > y_1 > \dots > y_n$. Then, the following statements are equivalent:

- i) *The set $\{f_{ij} : i < j \text{ and } \{i, j\} \in E(G)\}$ of generators is a quadratic Gröbner basis for J_G ;*
- ii) *G is closed with respect to the given labeling of vertices.*

Proof : i) $\Rightarrow$ ii). Suppose that $\{f_{ij} : i < j, \{i, j\} \in E(G)\}$ is a quadratic Gröbner basis for J_G . Assume the contrary of ii). Then, we must consider these two cases:

Case 1: If $\{i, j\}, \{i, l\} \in E(G)$ with $i < j < l$ and $\{j, l\} \notin E(G)$, then $S(f_{ij}, f_{il}) = y_l f_{ij} - y_j f_{il} = -y_i f_{jl} \in J_G$. Also, the initial monomial of any generator of J_G does not divide $\text{in}_<(y_i f_{jl}) = x_j y_i y_l$ because of $\{j, l\} \notin E(G)$.

Case 2: If $\{i, l\}, \{k, l\} \in E(G)$ with $i < k < l$ and $\{i, k\} \notin E(G)$, then $S(f_{il}, f_{kl}) = x_k f_{il} - x_i f_{kl} = x_l f_{ik} \in J_G$. Also, the initial monomial of any generator of J_G does not divide $\text{in}_<(x_l f_{ik}) = x_l x_i y_k$ because of $\{i, k\} \notin E(G)$.

However, both cases contradict that $\{f_{ij} : i < j \text{ and } \{i, j\} \in E(G)\}$ is a Gröbner basis.

ii) $\Rightarrow$ i). We will show that $S(f_{ij}, f_{kl})$ reduces to 0 for all $\{i, j\}, \{k, l\} \in E(G)$. To show this, let us consider these three cases:

Case 1: Let us look at the case $i \neq k, j \neq l$. In this situation, we can say that $S(f_{ij}, f_{kl})$ reduces to 0 according to Proposition 2.8

Case 2: If $i = k$, we can assume $i < j < l$. Then, it can be seen that $\{j, l\}$ is an edge in G . Hence, $S(f_{ij}, f_{il}) = -y_i f_{jl}$ reduces to 0.

Case 3: If $j = l$, we can assume $i < k < l$. Then, it can be seen that $\{i, k\}$ is an edge in G . Hence, $S(f_{il}, f_{kl}) = x_l f_{ik}$ reduces to 0.

Therefore, the generator set of J_G is a quadratic Gröbner basis by these three cases and Buchberger's criterion.

Definition 3.4: [4] Let G be a graph on the vertex set $[n]$. Also, let $\prec$ be a monomial order on S . We define a directed graph $G_{\prec}$ with the same vertex set and edge set $E(G_{\prec}) = \{(i, j) : x_i y_j \in \text{in}_{\prec}(J_G)\}$.

Proposition 3.4: [4] Suppose that G is a graph on $[n]$. Then, $G_{\prec}$ is acyclic for any monomial order $\prec$ on S .

Proof : Let $i_1, i_2, \dots, i_t$ be the vertices of a cycle in G . We will show that these vertices do not form a cycle in $G_{\prec}$. We can assume $(i_1, i_2), (i_2, i_3), \dots, (i_{t-1}, i_t) \in E(G_{\prec})$. If $(i_t, i_1) \in E(G_{\prec})$, then $G_{\prec}$ has a cycle, otherwise it does not. So, we have

$$x_{i_1} y_{i_2} \succ x_{i_2} y_{i_1}, \quad x_{i_2} y_{i_3} \succ x_{i_3} y_{i_2} \quad (3.3)$$

because of Definition 3.4. Thus, we can write

$$y_{i_3}(x_{i_1} y_{i_2}) \succ y_{i_3}(x_{i_2} y_{i_1}), \quad y_{i_1}(x_{i_2} y_{i_3}) \succ y_{i_1}(x_{i_3} y_{i_2}). \quad (3.4)$$

Hence, we see $y_{i_3}(x_{i_1} y_{i_2}) \succ y_{i_1}(x_{i_3} y_{i_2})$ and $x_{i_1} y_{i_3} \succ x_{i_3} y_{i_1}$. Continuing in a similar way, it can be seen that $x_{i_1} y_{i_4} \succ x_{i_4} y_{i_1}$, $x_{i_1} y_{i_5} \succ x_{i_5} y_{i_1}$, $\dots$ and finally $x_{i_1} y_{i_t} \succ x_{i_t} y_{i_1}$. It means $(i_1, i_t) \in E(G_{\prec})$ and $(i_t, i_1) \notin E(G_{\prec})$. Therefore, $G_{\prec}$ is acyclic.

Theorem 3.4: [4] Let G be a graph on $[n]$ Then, the following statements are equivalent:

- i) There exists a monomial order $\prec$ on S such that the set of generators of J_G forms a quadratic Gröbner basis for J_G with respect to $\prec$;
- ii) G is a closed graph.

Proof : i) $\Rightarrow$ ii). Suppose that the set of generators of J_G is a quadratic Gröbner basis with respect to a monomial order $\prec$ on S . We know from Proposition 3.4 that $G_{\prec}$ is acyclic. Then, $G_{\prec}$ has an acyclic ordering of its vertices by Proposition 2.10. Hence, we can guarantee that there exists a labeling $\omega : V(G_{\prec}) \rightarrow [n]$ of vertices of $G_{\prec}$ such that $\omega(i) < \omega(j)$ for each $(i, j) \in E(G_{\prec})$. We claim that G is closed with respect to the labeling ω .

Let $\{i_1, i_2\}, \{i_1, i_3\} \in E(G)$ such that $\omega(i_1) = i$, $\omega(i_2) = j$ and $\omega(i_3) = k$. Then, $\{i, j\}$ and $\{i, k\}$ are two edges of G with respect to the labeling ω . Let us examine these cases:

Case 1: Let us assume $i < j$ and $i < k$. Then, we have

$$x_{i_1}y_{i_2} \succ x_{i_2}y_{i_1}, \quad x_{i_1}y_{i_3} \succ x_{i_3}y_{i_1} \quad (3.5)$$

by Definition 3.4. Since the generators form a Gröbner basis for J_G with respect to $\prec$, we write $S(f_{i_1i_2}, f_{i_1i_3}) = -y_{i_1}f_{i_2i_3} = -y_{i_1}(x_{i_2}y_{i_3} - x_{i_3}y_{i_2})$ reduces to 0. For this, J_G must have a generator $f_{i_p i_q} = x_{i_p}y_{i_q} - x_{i_q}y_{i_p}$ such that $\text{in}_{\prec}(f_{i_p i_q})$ divides $\text{in}_{\prec}(y_{i_1}f_{i_2i_3})$. However, $\text{in}_{\prec}(f_{i_1i_2}) \neq x_{i_2}y_{i_1}$ and $\text{in}_{\prec}(f_{i_1i_3}) \neq x_{i_3}y_{i_1}$ because of the inequality (3.5). It means $\text{in}_{\prec}(f_{i_1i_2})$ and $\text{in}_{\prec}(f_{i_1i_3})$ do not divide $\text{in}_{\prec}(y_{i_1}f_{i_2i_3})$. So, the only possible case for $\text{in}_{\prec}(y_{i_1}f_{i_2i_3})$ to be divisible by $\text{in}_{\prec}(f_{i_p i_q})$ is $f_{i_p i_q} = f_{i_2i_3}$. Hence, $f_{i_2i_3} \in J_G$ and $\{j, k\} \in E(G)$.

Case 2: If we assume $i > j$ and $i > k$, then it can be seen that $\{j, k\} \in E(G)$ by using the similar arguments mentioned in Case 1.

Therefore, G is a closed with respect to ω .

ii) $\Rightarrow$ i). It is trivial by Theorem 3.3

Definition 3.5: [1] *Let i, j be two vertices of a graph G on the vertex set $[n]$ and let $i < j$. A path $\pi : i = i_0, i_1, \dots, i_t = j$ between i and j is said to be admissible if it satisfies these conditions:*

- i) If $k \neq l$, then $i_k \neq i_l$, i.e., all vertices of the path π are different from each other.*
- ii) $i_k < i$ or $i_k > j$ for all $1 \leq k \leq t - 1$, i.e., no i_k takes a value between i and j .*
- iii) The sequence $i, j_1, \dots, j_s, j$ is not a path for any $\{j_1, \dots, j_s\} \subsetneq \{i_1, \dots, i_{t-1}\}$,*

i.e., no pair of the vertices of the path π are adjacent in G , except for the consecutive vertices.

If $i < j$ and $\pi : i = i_0, i_1, \dots, i_t = j$ is an admissible path, we define the monomial

$$u_\pi = \left(\prod_{i_k > j} x_{i_k} \right) \left(\prod_{i_l < i} y_{i_l} \right). \quad (3.6)$$

In the equation (3.6), if there is no vertices in π satisfying $i_k > j$ or $i_l < i$, then we write 1 instead of $\prod_{i_k > j} x_{i_k}$ or $\prod_{i_l < i} y_{i_l}$.

Example 3.4: Let us consider the graph G displayed in Figure 3.4 and the paths

$$\pi : 4, 1, 7, 8, 6 \quad \sigma : 3, 1, 7, 8, 6 \quad \tau : 3, 1, 7, 2, 6. \quad (3.7)$$

Clearly, π is not an admissible path since $\{7, 8\} \subset \{1, 7, 8\}$ and $\pi' : 4, 7, 8, 6$ is a path from 4 to 6. However, σ and τ are admissible paths from 3 to 6. Also, the monomials associated with these admissible paths can be written as $u_\sigma = y_1 x_7 x_8 = x_7 x_8 y_1$ and $u_\tau = y_1 x_7 y_2 = x_7 y_1 y_2$.

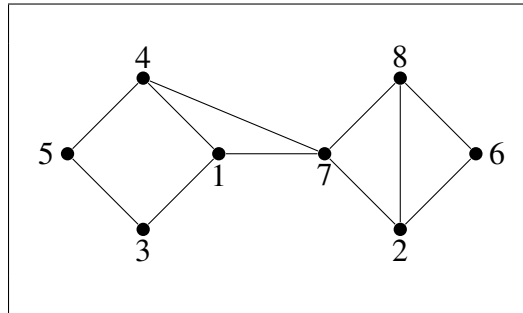


Figure 3.4: A graph G

Lemma 3.1: [12] Suppose that G is a graph on $[n]$ and $<$ is the lexicographic order on S . Also, suppose $\mathcal{G} = \{u_\pi f_{ij} : i < j, \pi \text{ is an admissible path between } i \text{ and } j\}$. Let π be an admissible path between i and j with $i < j$.

i) If x_k divides u_π , then $\frac{u_\pi}{x_k} y_k f_{ij}$ reduces to 0 with respect to the set $\mathcal{G}$ and order $<$;

ii) If y_k divides u_π , then $\frac{u_\pi}{y_k}x_k f_{ij}$ reduces to 0 with respect to the set $\mathcal{G}$ and order $<$.

Proof : i) It can be proven similarly to the proof of ii).

ii) Let us examine these two cases:

Case 1: Let us assume there is no vertices t in π satisfying $k < t < i$. Let us call the parts of the path π from k to i and from k to j as π_1 and π_2 , respectively. Clearly, π_1 and π_2 are admissible paths and we have $u_\pi = y_k u_{\pi_1} u_{\pi_2}$. Thus, we can write

$$\frac{u_\pi}{y_k}x_k f_{ij} = x_i u_{\pi_1}(u_{\pi_2} f_{kj}) - x_j u_{\pi_2}(u_{\pi_1} f_{ki}). \quad (3.8)$$

Hence, $\frac{u_\pi}{y_k}x_k f_{ij}$ reduces to 0 with respect to $\mathcal{G}$.

Case 2: Let us assume there are some vertices t in π satisfying $k < t < i$. Then, we can choose a vertex $k < t < i$ such that the subpath σ of π between k and t does not contain any vertices s such that $k < s < t$. Clearly, σ is an admissible path. If we divide $\frac{u_\pi}{y_k}x_k f_{ij}$ by $u_\sigma f_{kt}$, we can write

$$\frac{u_\pi}{y_k}x_k f_{ij} = \frac{u_\pi}{u_\sigma y_k y_t} f_{ij}(u_\sigma f_{kt}) + \frac{u_\pi}{y_t} x_t f_{ij}. \quad (3.9)$$

So, we have the remainder $\frac{u_\pi}{y_t}x_t f_{ij}$. If similar operations are performed for $\frac{u_\pi}{y_t}x_t f_{ij}$, after finite steps, we obtain a remainder having the form $\frac{u_\pi}{y_{k'}}x_{k'} f_{ij}$ such that there is no t' satisfying $k' < t' < i$. Then, Case 1 gives us the desired result.

Theorem 3.5: [U] Suppose that G is a graph on $[n]$ and $<$ is the lexicographic order on S induced by $x_1 > \dots > x_n > y_1 > \dots > y_n$. Then, the set

$$\mathcal{G} = \{u_\pi f_{ij} : i < j, \pi \text{ is an admissible path between } i \text{ and } j\}$$

forms a reduced Gröbner basis for J_G with respect to $<$.

Proof : Firstly, we must show $\mathcal{G} \subset J_G$, i.e., $u_\pi f_{ij} \in J_G$ for any admissible path $\pi : i = i_0, i_1, \dots, i_{t-1}, i_t = j$ between i and j where $i, j \in V(G)$ and $i < j$. To show this, we can use induction method on the length of admissible paths. If $t = 1$, it is clear that $\{i, j\}$ is an edge in G and $u_\pi f_{ij} = f_{ij} \in J_G$. Now, let $t > 1$, $A = \{i_k : i_k < i\}$ and $B = \{i_l : i_l > j\}$. Then, at least one of the sets A and B is nonempty. Assume

that A is nonempty and $i_{k_0} = \max A$. Observe that $\pi_1 : i_{k_0}, i_{k_0-1}, \dots, i_1, i_0 = i$ and $\pi_2 : i_{k_0}, i_{k_0+1}, \dots, i_{t-1}, i_t = j$ are admissible paths from i_{k_0} to i and from i_{k_0} to j , respectively. Since the lengths of the admissible paths π_1 and π_2 are smaller than t , we have $u_{\pi_1} f_{i_{k_0}i}, u_{\pi_2} f_{i_{k_0}j} \in J_G$ by induction hypothesis. Hence, we can write

$$S(u_{\pi_1} f_{i_{k_0}i}, u_{\pi_2} f_{i_{k_0}j}) = \frac{\text{lcm}(u_{\pi_1} x_{i_{k_0}} y_i, u_{\pi_2} x_{i_{k_0}} y_j)}{u_{\pi_1} x_{i_{k_0}} y_i} u_{\pi_1} f_{i_{k_0}i} - \frac{\text{lcm}(u_{\pi_1} x_{i_{k_0}} y_i, u_{\pi_2} x_{i_{k_0}} y_j)}{u_{\pi_2} x_{i_{k_0}} y_j} u_{\pi_2} f_{i_{k_0}j} \quad (3.10)$$

$$= \frac{u_{\pi_1} u_{\pi_2} x_{i_{k_0}} y_i y_j}{u_{\pi_1} x_{i_{k_0}} y_i} u_{\pi_1} f_{i_{k_0}i} - \frac{u_{\pi_1} u_{\pi_2} x_{i_{k_0}} y_i y_j}{u_{\pi_2} x_{i_{k_0}} y_j} u_{\pi_2} f_{i_{k_0}j} \quad (3.11)$$

$$= u_{\pi_1} u_{\pi_2} y_j (x_{i_{k_0}} y_i - x_i y_{i_{k_0}}) - u_{\pi_1} u_{\pi_2} y_i (x_{i_{k_0}} y_j - x_j y_{i_{k_0}}) \quad (3.12)$$

$$= -u_{\pi_1} u_{\pi_2} y_{i_{k_0}} x_i y_j + u_{\pi_1} u_{\pi_2} y_{i_{k_0}} x_j y_i \quad (3.13)$$

$$= -(u_{\pi_1} y_{i_{k_0}} u_{\pi_2}) (x_i y_j - x_j y_i) \quad (3.14)$$

$$= -u_{\pi} f_{ij}. \quad (3.15)$$

So, $u_{\pi} f_{ij} \in J_G$. Similarly, $u_{\pi} f_{ij} \in J_G$ if $B \neq \emptyset$. Hence, we conclude $\mathcal{G} \subset J_G$.

Now, we will prove that $\mathcal{G}$ is a Gröbner basis for J_G , i.e., $S(u_{\pi} f_{ij}, u_{\sigma} f_{kl})$ reduces to 0 for all $u_{\pi} f_{ij}, u_{\sigma} f_{kl} \in \mathcal{G}$. To show this, let us examine these four cases:

Case 1: Let us look at the case $i = k$ and $j = l$, then we can write

$$S(u_{\pi} f_{ij}, u_{\sigma} f_{ij}) = \frac{\text{lcm}(u_{\pi} x_i y_j, u_{\sigma} x_i y_j)}{u_{\pi} x_i y_j} u_{\pi} f_{ij} - \frac{\text{lcm}(u_{\pi} x_i y_j, u_{\sigma} x_i y_j)}{u_{\sigma} x_i y_j} u_{\sigma} f_{ij} \quad (3.16)$$

$$= \text{lcm}(u_{\pi}, u_{\sigma}) f_{ij} - \text{lcm}(u_{\pi}, u_{\sigma}) f_{ij} \quad (3.17)$$

$$= 0. \quad (3.18)$$

Case 2: If $\text{in}_{<}(f_{ij})$ and $\text{in}_{<}(f_{kl})$ have no common factors, then we can assume $i < k$ and we need to examine the following four situations:

i) $i < j < k < l$;

ii) $i < k < l < j$;

iii) $i < k < j < l$;

iv) $i < j = k < l$.

In the situations ii), iii) and iv), neither x_k nor y_l divides u_π and neither x_i nor y_j divides u_σ . Then, we can write

$$S(u_\pi f_{ij}, u_\sigma f_{kl}) = \frac{\text{lcm}(u_\pi x_i y_j, u_\sigma x_k y_l)}{u_\pi x_i y_j} u_\pi f_{ij} - \frac{\text{lcm}(u_\pi x_i y_j, u_\sigma x_k y_l)}{u_\sigma x_k y_l} u_\sigma f_{kl} \quad (3.19)$$

$$= w x_k y_l f_{ij} - w x_i y_j f_{kl} \quad (3.20)$$

$$= w x_l y_k f_{ij} - w x_j y_i f_{kl} \quad (3.21)$$

$$= \frac{w x_l y_k}{u_\pi} u_\pi f_{ij} - \frac{w x_j y_i}{u_\sigma} u_\sigma f_{kl}. \quad (3.22)$$

where $w = \text{lcm}(u_\pi, u_\sigma)$. Hence, (3.22) is the standard expression for $S(u_\pi f_{ij}, u_\sigma f_{kl})$. It means $S(u_\pi f_{ij}, u_\sigma f_{kl})$ reduces to 0.

So, we must focus on only the situation i). In this situation, x_k may divide u_π or y_j may divide u_σ .

If x_k does not divide u_π and y_j does not divide u_σ , then we can write an equality similar to equality (3.22) and it gives us the desired result.

If x_k divides u_π and y_j does not divide u_σ , then we can see

$$S(u_\pi f_{ij}, u_\sigma f_{kl}) = w(x_i y_j x_l y_k - x_j y_i x_k y_l) \quad (3.23)$$

where $w = \text{lcm}(\frac{u_\pi}{x_k}, u_\sigma)$. If we divide $S(u_\pi f_{ij}, u_\sigma f_{kl})$ by $u_\sigma f_{kl}$, we can write

$$S(u_\pi f_{ij}, u_\sigma f_{kl}) = -\frac{w}{u_\sigma} x_j y_i (u_\sigma f_{kl}) + w x_l y_k f_{ij} \quad (3.24)$$

$$= -\frac{w}{u_\sigma} x_j y_i (u_\sigma f_{kl}) + \frac{w x_l x_k}{u_\pi} \left(\frac{u_\pi}{x_k} y_k f_{ij} \right). \quad (3.25)$$

From i) of Lemma (3.1), $\frac{u_\pi}{x_k} y_k f_{ij}$ reduces to 0. It means $S(u_\pi f_{ij}, u_\sigma f_{kl})$ reduces to 0 with respect to $\mathcal{G}$.

If x_k does not divide u_π and y_j divides u_σ , then we can see

$$S(u_\pi f_{ij}, u_\sigma f_{kl}) = w(x_i y_j x_l y_k - x_j y_i x_k y_l) \quad (3.26)$$

where $w = \text{lcm}(u_\pi, \frac{u_\sigma}{y_j})$. If we divide $S(u_\pi f_{ij}, u_\sigma f_{kl})$ by $u_\pi f_{ij}$, we can write

$$S(u_\pi f_{ij}, u_\sigma f_{kl}) = \frac{w}{u_\pi} x_l y_k (u_\pi f_{ij}) - w x_j y_i f_{kl} \quad (3.27)$$

$$= \frac{w}{u_\pi} x_l y_k (u_\pi f_{ij}) - \frac{w y_i y_j}{u_\sigma} \left(\frac{u_\sigma}{y_j} x_j f_{kl} \right). \quad (3.28)$$

From ii) of Lemma (3.1), $\frac{u_\sigma}{y_j} x_j f_{kl}$ and $S(u_\pi f_{ij}, u_\sigma f_{kl})$ reduces to 0 with respect to $\mathcal{G}$.

If x_k divides u_π and y_j divides u_σ , then we can see

$$S(u_\pi f_{ij}, u_\sigma f_{kl}) = w(x_i y_j x_l y_k - x_j y_i x_k y_l) \quad (3.29)$$

where $w = \text{lcm}(\frac{u_\pi}{x_k}, \frac{u_\sigma}{y_j})$. If we divide $S(u_\pi f_{ij}, u_\sigma f_{kl})$ by $\frac{u_\pi}{x_k} y_k f_{ij}$, we can write

$$S(u_\pi f_{ij}, u_\sigma f_{kl}) = \frac{w x_l x_k}{u_\pi} \left(\frac{u_\pi}{x_k} y_k f_{ij} \right) - w x_j y_i f_{kl} \quad (3.30)$$

$$= \frac{w x_l x_k}{u_\pi} \left(\frac{u_\pi}{x_k} y_k f_{ij} \right) - \frac{w y_i y_j}{u_\sigma} \left(\frac{u_\sigma}{y_j} x_j f_{kl} \right). \quad (3.31)$$

From Lemma (3.1), $\frac{u_\pi}{x_k} y_k f_{ij}$ and $\frac{u_\sigma}{y_j} x_j f_{kl}$ reduce to 0. Hence, $S(u_\pi f_{ij}, u_\sigma f_{kl})$ reduces to 0 with respect to $\mathcal{G}$.

Case 3: Let us look at the case $i = k$, $j < l$. Let us say $\pi : i = i_0, i_1, \dots, i_r = j$ and $\sigma : i = i'_0, i'_1, \dots, i'_s = l$. We can find $0 \leq a < r$ and $0 \leq b < s$ such that $i_a = i'_b$ and the vertices $i_{a+1}, \dots, i_r, i'_{b+1}, \dots, i'_s$ are all distinct. To be more precise, we choose $i_a = i'_b$ as the vertex closest to j or l among the intersection points of the paths π and σ . Then, we can write the path $\tau : j = i_r, i_{r-1}, \dots, i_{a+1}, i_a = i'_b, i'_{b+1}, \dots, i'_{s-1}, i'_s = l$ between j and l . Now, let us call the vertices of τ as $\tau : j = j_0, j_1, \dots, j_t = l$ to get a simpler view. Let us take some vertices from the path τ by setting $j_{t(0)} = j_0$, $j_{t(p)} = \min\{j_c : j_c > j \text{ and } c = t(p-1) + 1, t(p-1) + 2, \dots, t\}$ for $p = 1, 2, \dots, q$ where $j_{t(q)} = j_t$. These vertices that we choose provide the inequality

$$j_0 = j_{t(0)} < j_{t(1)} < \cdots < j_{t(q-1)} < j_{t(q)} = j_t. \quad (3.32)$$

Also, the path $\tau_c : j_{t(c-1)}, j_{t(c-1)+1}, \dots, j_{t(c)-1}, j_{t(c)}$ is admissible for all $c = 1, \dots, q$ since none of the integers $j_{t(c-1)+1}, \dots, j_{t(c)-1}$ are between the integers $j_{t(c-1)}$ and $j_{t(c)}$. One can easily check that the equality

$$\begin{aligned} S(u_\pi f_{ij}, u_\sigma f_{il}) = & - \left(\frac{wx_l}{x_{j_{t(1)}} u_{\tau_1}} \right) u_{\tau_1} f_{jj_{t(1)}} \\ & - \sum_{c=2}^{q-1} \left[\left(\frac{wx_j x_l}{x_{j_{t(c-1)}} x_{j_{t(c)}} u_{\tau_c}} \right) u_{\tau_c} f_{j_{t(c-1)} j_{t(c)}} \right] \\ & - \left(\frac{wx_j}{x_{j_{t(q-1)}} u_{\tau_q}} \right) u_{\tau_q} f_{j_{t(q-1)} l} \end{aligned} \quad (3.33)$$

holds by expanding the sum symbol and binomials where $w = y_i \text{lcm}(u_\pi, u_\sigma)$. If we substitute $S(u_\pi f_{ij}, u_\sigma f_{il}) = -w f_{jl}$ into (3.33), then (3.33) can be written as

$$\begin{aligned} w(x_j y_l - x_l y_j) = & \frac{wx_l}{x_{j_{t(1)}}} (x_j y_{j_{t(1)}} - x_{j_{t(1)}} y_j) \\ & + \sum_{c=2}^{q-1} \left[\frac{wx_j x_l}{x_{j_{t(c-1)}} x_{j_{t(c)}}} (x_{j_{t(c-1)}} y_{j_{t(c)}} - x_{j_{t(c)}} y_{j_{t(c-1)}}) \right] \\ & + \frac{wx_j}{x_{j_{t(q-1)}}} (x_{j_{t(q-1)}} y_l - x_l y_{j_{t(q-1)}}). \end{aligned} \quad (3.34)$$

We claim that (3.34) is the standard expression of $w(x_j y_l - x_l y_j)$. The inequality (3.32) allows us to write

$$wx_j y_l = \frac{wx_j}{x_{j_{t(q-1)}}} x_{j_{t(q-1)}} y_l > \frac{wx_j x_l}{x_{j_{t(q-2)}} x_{j_{t(q-1)}}} x_{j_{t(q-2)}} y_{j_{t(q-1)}} \quad (3.35)$$

$\vdots$

$$> \frac{wx_j x_l}{x_{j_{t(1)}} x_{j_{t(2)}}} x_{j_{t(1)}} y_{j_{t(2)}} \quad (3.36)$$

$$> \frac{wx_l}{x_{j_{t(1)}}} x_j y_{j_{t(1)}}. \quad (3.37)$$

It means (3.34) is the standard expression of $w(x_j y_l - x_l y_j)$ with respect to $<$. So, $S(u_\pi f_{ij}, u_\sigma f_{il})$ reduces to 0.

Case 4: If $j = l$ and $i < k$, then $S(u_\pi f_{il}, u_\sigma f_{kl})$ reduces to 0 by the similar arguments mentioned in Case 3.

Therefore, $\mathcal{G}$ is a Gröbner basis for J_G .

Lastly, we must prove that $\mathcal{G}$ is reduced. Clearly, the coefficients of the initial monomials of all elements in $\mathcal{G}$ are 1. Suppose that $u_\pi f_{ij}$ and $u_\sigma f_{kl}$ are two distinct polynomials of $\mathcal{G}$. Also, suppose $\pi : i = i_0, i_1, \dots, i_r = j$, $\sigma : k = k_0, k_1, \dots, k_s = l$. Let us assume $\text{in}_<(u_\pi f_{ij}) = u_\pi x_i y_j$ divides $u_\sigma x_k y_l$ or $u_\sigma x_l y_k$. Let us consider these four cases:

Case 1: Firstly, let us look at the case $i = k$, $j = l$. By our assumption, $u_\pi x_i y_j$ divides $u_\sigma x_i y_j$ or $u_\sigma x_j y_i$. Hence, $\{i_1, \dots, i_{r-1}\} \subsetneq \{k_0, k_1, \dots, k_s\}$. However, $k, i_1, \dots, i_{r-1}, l$ is a path and this contradicts with the last condition of being an admissible path in Definition 3.5 for σ .

Case 2: Now, let us look at the case $i = k$, $j \neq l$. Then, $u_\pi x_i y_j$ divides $u_\sigma x_i y_l$ or $u_\sigma x_l y_i$. Hence, y_j divides u_σ . It means, $j < k = i$ but this is a contradiction.

Case 3: If $j = l$ and $i \neq k$, then $u_\pi x_i y_j$ divides $u_\sigma x_k y_j$ or $u_\sigma x_j y_k$. Hence, x_i divides u_σ . It means, $i > l = j$ but this is a contradiction.

Case 4: If i, j, k, l are all distinct, $u_\pi x_i y_j$ divides $u_\sigma x_k y_l$ or $u_\sigma x_l y_k$. Thus, we say that $x_i y_j$ divides u_σ . It means $i > l$ and $j < k$ but they imply the inequality $i < j < k < l < i$ which is impossible.

As a result, the four situations mentioned above indicate that $\text{in}_<(u_\pi f_{ij})$ does not divide any monomial of $u_\sigma f_{kl}$. So, $\mathcal{G}$ is the reduced Gröbner basis for J_G .

Example 3.5: Let J_G be the binomial edge ideal of the graph G displayed in Figure 3.5. One can easily check that the only admissible paths in G with length greater than 1 are $\pi : 1, 4, 3$ and $\sigma : 4, 3, 5$. The associated monomials of π and σ are $u_\pi = x_4$ and $u_\sigma = y_3$, respectively. By Theorem 3.5 the set

$$\mathcal{G} = \{f_{12}, x_4 f_{13}, f_{14}, f_{23}, f_{24}, f_{34}, f_{35}, y_3 f_{45}\} \quad (3.38)$$

is the reduced Gröbner basis of J_G . So, thanks to Theorem 3.5 we can see that we have found the same result easier than we did in Example 2.4

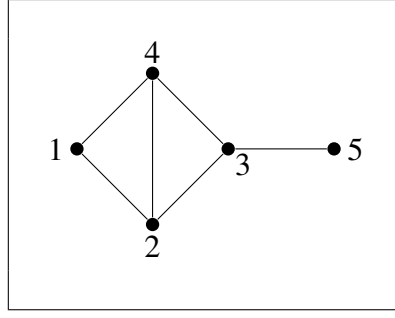


Figure 3.5: A graph G

3.3. Algebraic Properties of J_G

In this section, we will study the algebraic properties of binomial edge ideals. The results that appear in this section are taken from the articles [1] and [12]. Firstly, we will state in Corollary 3.2 that the necessary and sufficient condition for J_G to be a prime ideal is that G is a disjoint union of complete graphs. Also, we will show that for any graph G , any minimal prime ideal of binomial edge ideal of G has the form $P_W(G)$ and J_G can be written as the intersection of the ideals $P_W(G)$ satisfying the condition $W \subset [n]$. Finally, we will give a condition in Theorem 3.8 that allows us to determine completely all minimal primes of J_G .

Proposition 3.5: [12] J_G is a radical ideal for any graph G .

Proof : We will only show that $\sqrt{J_G} \subseteq J_G$ because $J_G \subseteq \sqrt{J_G}$ is always provided. We know that $\mathcal{G} = \{u_\pi f_{ij} : i < j, \pi \text{ is an admissible path between } i \text{ and } j\}$ forms a reduced Gröbner basis for J_G because of Theorem 3.5. Hence, it can be seen $\text{in}_<(J_G) = (\{u_\pi x_i y_j : i < j, \pi \text{ is an admissible path between } i \text{ and } j\})$. Clearly, the monomial $u_\pi x_i y_j$ is squarefree for any $i < j$ and any admissible path π between i and j . Then, $\text{in}_<(J_G)$ is radical ideal from Proposition 2.6. Let $f \in \sqrt{J_G}$, i.e., there exists $k \in \mathbb{N}$ such that $f^k \in J_G$. Thus, we can write $\text{in}_<(f^k) = \text{in}_<(f)^k \in \text{in}_<(J_G)$ and $\text{in}_<(f) \in \text{in}_<(J_G)$ since $\text{in}_<(J_G)$ is a radical ideal. Then, there exists $g_1 \in J_G$ and $a_1 \in K$ such that $\text{in}_<(f) = \text{in}_<(g_1)$ and $\text{in}_<(f) > \text{in}_<(f - a_1 g_1)$. Also, we can say $(f - a_1 g_1)^k \in J_G$ because of $f^k, g_1 \in J_G$. By continuing in a similar way, we can find $g_2, \dots, g_t \in J_G$ and $a_2, \dots, a_t$ satisfying

$$\text{in}_<(f - a_1 g_1) > \text{in}_<(f - a_1 g_1 - a_2 g_2) > \dots > \text{in}_<(f - a_1 g_1 - \dots - a_t g_t) \quad (3.39)$$

and $f - a_1g_1 - \dots - a_tg_t = 0 \in J_G$. Since $g_1, \dots, g_t \in J_G$, we have $f \in J_G$ and $\sqrt{J_G} \subseteq J_G$. Therefore, $J_G = \sqrt{J_G}$.

Theorem 3.6: [12] Let G be a connected graph on $[n]$. Then, J_G is a prime ideal if and only if G is a complete graph.

Proof : ($\Rightarrow$) Suppose that J_G is a prime ideal and $i, j \in V(G)$. As G is connected, there is a path $i = i_1, i_2, \dots, i_t = j$ between i and j . If $t = 2$, then we can see $\{i, j\} \in E(G)$. If $t > 2$, then $f_{i_1i_2}, f_{i_2i_3} \in J_G$. Without loss of generality, we can write $y_{i_3}f_{i_1i_2} + y_{i_1}f_{i_2i_3} = y_{i_2}f_{i_1i_3} \in J_G$ and $y_{i_2} \in J_G$ or $f_{i_1i_3} \in J_G$. Since all generators of J_G have degree 2, we say $y_{i_2} \notin J_G$ and $f_{i_1i_3} \in J_G$. It means $\{i_1, i_3\} \in E(G)$. Thus, $i = i_1, i_3, i_4, \dots, i_t = j$ is a path. By using similar arguments, we can write $\{i_1, i_3\}, \dots, \{i_1, i_{t-1}\} \in E(G)$ and lastly $\{i_1, i_t\} = \{i, j\} \in E(G)$. Therefore, we conclude that G is complete.

($\Leftarrow$) Assume that G is complete. Let us show J_G is a toric ideal. To do this, we will consider the block matrix

$$A = \begin{bmatrix} I_n & I_n \\ X & Y \end{bmatrix} \quad (3.40)$$

where $X = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \end{bmatrix}_{1 \times n}$, $Y = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \end{bmatrix}_{1 \times n}$ and I_n is the identity matrix with size $n \times n$. More clearly,

$$A = \left[\begin{array}{ccccc|ccccc} 1 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 1 \\ \hline 0 & 0 & \dots & 0 & 0 & 1 & 1 & \dots & 1 & 1 \end{array} \right]_{(n+1) \times 2n} \quad (3.41)$$

One can easily show that the vectors

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}_{2n \times 1}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}_{2n \times 1}, \quad \dots, \quad \mathbf{b}_{n-1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 1 \end{bmatrix}_{2n \times 1} \quad (3.42)$$

are the elements of the nullspace of the matrix A where $\mathbf{b}_k = \mathbf{e}_k - \mathbf{e}_n - \mathbf{e}_{n+k} + \mathbf{e}_{2n}$ for all $1 \leq k \leq n-1$. Here, we mean the l -th unit vector of $\mathbb{N}^{2n}$ by $\mathbf{e}_l$. It is clear that the set $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_{n-1}\}$ is linearly independent and $\text{rank } A = n+1$ since the echelon form of the matrix A is itself. Rank-Nullity Theorem implies nullity $A = n-1$ and the set $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_{n-1}\}$ is a basis for the null space of A . One can easily see that $\mathbf{f}_{\mathbf{b}_1} = f_{1n}, \mathbf{f}_{\mathbf{b}_2} = f_{2n}, \dots, \mathbf{f}_{\mathbf{b}_{n-1}} = f_{(n-1)n}$ and $\mathbf{f}_{\mathbf{b}_i - \mathbf{b}_j} = f_{ij}$ for all $1 \leq i < j \leq n-1$. It means $J_G \subseteq I_A$. Now, let us show $I_A \subseteq J_G$. We can see that

$$\mathbf{f}_{c\mathbf{b}_i} = x_i^c y_n^c - x_n^c y_i^c = (x_i y_n - x_n y_i) [(x_i y_n)^{c-1} + (x_i y_n)^{c-2} (x_n y_i) + \dots + (x_n y_i)^{c-1}] \quad (3.43)$$

for any $c \in \mathbb{Z}^+$. A similar equation can be written for negative integers c . So, we have $\mathbf{f}_{c\mathbf{b}_i} \in J_G$ for any $c \in \mathbb{Z}$ and $1 \leq i \leq n-1$. We must show $\mathbf{f}_{c_1 \mathbf{b}_1 + \dots + c_{n-1} \mathbf{b}_{n-1}} \in J_G$ for all $c_1, \dots, c_{n-1} \in \mathbb{Z}$. To do this, we will show, $\mathbf{f}_{c_1 \mathbf{b}_1 + \dots + c_t \mathbf{b}_t} \in J_G$ for all $1 \leq t \leq n-1$ by using induction on t . If $t = 1$, then $\mathbf{f}_{c_1 \mathbf{b}_1} \in J_G$ by (3.43). Now, assume that $t > 1$ and $\mathbf{f}_{c_1 \mathbf{b}_1 + \dots + c_s \mathbf{b}_s} \in J_G$ for any positive s less than t . Without loss of generality, we can write

$$\begin{aligned} \mathbf{f}_{c_1 \mathbf{b}_1 + \dots + c_{t-1} \mathbf{b}_{t-1}} \mathbf{f}_{c_t \mathbf{b}_t} &= (x_1^{c_1} \dots x_{t-1}^{c_{t-1}} y_n^{c_1 + \dots + c_{t-1}} - x_n^{c_1 + \dots + c_{t-1}} y_1^{c_1} \dots y_{t-1}^{c_{t-1}}) \\ &\quad \times (x_t^{c_t} y_n^{c_t} - x_n^{c_t} y_t^{c_t}) \end{aligned} \quad (3.44)$$

$$= x_1^{c_1} \dots x_{t-1}^{c_{t-1}} x_t^{c_t} y_n^{c_1 + \dots + c_{t-1} + c_t} \quad (3.45)$$

$$\begin{aligned} &- (\mathbf{f}_{c_1 \mathbf{b}_1 + \dots + c_{t-1} \mathbf{b}_{t-1}} + x_n^{c_1 + \dots + c_{t-1}} y_1^{c_1} \dots y_{t-1}^{c_{t-1}}) x_n^{c_t} y_t^{c_t} \\ &- (\mathbf{f}_{c_t \mathbf{b}_t} + x_n^{c_t} y_t^{c_t}) x_n^{c_1 + \dots + c_{t-1}} y_1^{c_1} \dots y_{t-1}^{c_{t-1}} \\ &+ x_n^{c_1 + \dots + c_{t-1} + c_t} y_1^{c_1} \dots y_{t-1}^{c_{t-1}} y_t^{c_t} \\ &= x_1^{c_1} \dots x_t^{c_t} y_n^{c_1 + \dots + c_t} - x_n^{c_1 + \dots + c_t} y_1^{c_1} \dots y_t^{c_t} \\ &\quad - x_n^{c_t} y_t^{c_t} \mathbf{f}_{c_1 \mathbf{b}_1 + \dots + c_{t-1} \mathbf{b}_{t-1}} - x_n^{c_1 + \dots + c_t} y_1^{c_1} \dots y_t^{c_t} \mathbf{f}_{c_t \mathbf{b}_t}. \end{aligned} \quad (3.46)$$

Hence, we can easily see that

$$\mathbf{f}_{c_1 \mathbf{b}_1 + \dots + c_t \mathbf{b}_t} = \mathbf{f}_{c_1 \mathbf{b}_1 + \dots + c_{t-1} \mathbf{b}_{t-1}} \mathbf{f}_{c_t \mathbf{b}_t} + x_n^{c_t} y_t^{c_t} \mathbf{f}_{c_1 \mathbf{b}_1 + \dots + c_{t-1} \mathbf{b}_{t-1}} + x_n^{c_1 + \dots + c_t} y_1^{c_1} \dots y_t^{c_t} \mathbf{f}_{c_t \mathbf{b}_t} \quad (3.47)$$

and $\mathbf{f}_{c_1 \mathbf{b}_1 + \dots + c_t \mathbf{b}_t} \in J_G$ by induction hypothesis. Therefore, $\mathbf{f}_{c_1 \mathbf{b}_1 + \dots + c_{n-1} \mathbf{b}_{n-1}} \in J_G$ for all $c_1, \dots, c_{n-1} \in \mathbb{Z}$. It means $\mathbf{f}_{\mathbf{b}} \in J_G$ for all $\mathbf{f}_{\mathbf{b}} \in I_A$ and $I_A \subseteq J_G$. Thus, $J_G = I_A$ and J_G is a toric ideal. So, J_G is prime.

Corollary 3.1: [U2] Suppose that G is complete graph on $[n]$ and R is an integral domain. Then, $R[x_1, \dots, x_n, y_1, \dots, y_n]/J_G$ is an integral domain.

Definition 3.6: [U] Let G be a graph on $[n]$ and $W \subset [n]$. Also, let $G_1, \dots, G_{c(W)}$ be the connected components of the induced subgraph $G - W$ and $\tilde{G}_1, \dots, \tilde{G}_{c(W)}$ be the complete graphs on the vertex sets $V(G_1), \dots, V(G_{c(W)})$, respectively. Then, we define

$$P_W(G) = (\{x_i, y_i : i \in W\}) + J_{\tilde{G}_1} + \dots + J_{\tilde{G}_{c(W)}}. \quad (3.48)$$

Example 3.6: Let us consider the graph G on $[7]$ displayed in Figure 3.6a. If we choose $W = \{2, 5\}$, we say that the induced subgraph $G - W$ of G looks like in Figure 3.6b. Clearly, we have $c(W) = 2$. We call G_1 and G_2 the connected components of $G - W$ and the corresponding complete graphs are $\tilde{G}_1$ and $\tilde{G}_2$, respectively. If we replace G_1 and G_2 with $\tilde{G}_1$ and $\tilde{G}_2$, respectively, we will have the graph shown in Figure 3.6c

Hence, we can write $P_W(G)$ as

$$P_W(G) = (x_2, y_2, x_5, y_5) + (f_{13}, f_{14}, f_{34}) + (f_{67}) = (x_2, y_2, x_5, y_5, f_{13}, f_{14}, f_{34}, f_{67}). \quad (3.49)$$

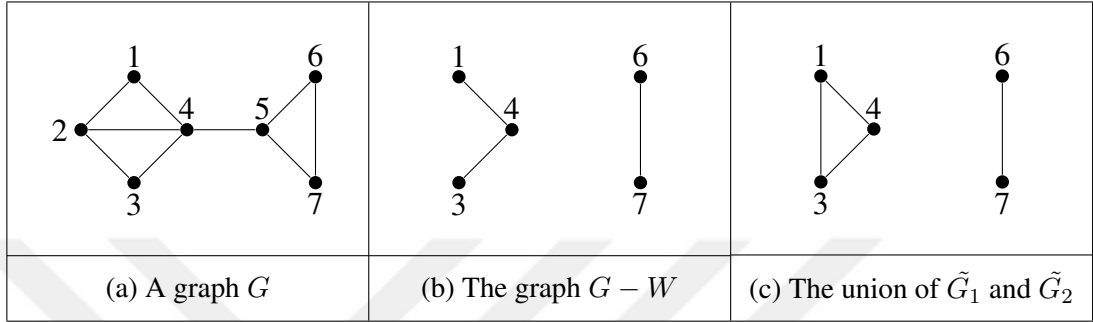


Figure 3.6: A graph G , the graph $G - W$ and the union of $\tilde{G}_1$ and $\tilde{G}_2$

Note that, if all connected components of G are complete, i.e., G is a union of disjoint complete graphs, then $J_G = P_\emptyset(G)$ by Definition 3.6. Also, $J_G \subseteq P_W(G)$ for any $W \subset [n]$.

Lemma 3.2: [12] Suppose that G is a graph on $[n]$. Then, $P_W(G)$ is a prime ideal for any subset $W \subset [n]$.

Proof : Let us prove that $S/P_W(G)$ is an integral domain. Observe that,

$$S/P_W(G) = S/((\{x_i, y_j : i \in W\}) + J_{\tilde{G}_1} + \dots + J_{\tilde{G}_{c(W)}}) \cong S'/(J_{\tilde{G}_1} + \dots + J_{\tilde{G}_{c(W)}}) \quad (3.50)$$

where S' is the polynomial ring over K whose the indeterminates are the elements of the set $\{x_j, y_j : j \notin W\}$. So, it is enough to show that $S'/(J_{\tilde{G}_1} + \dots + J_{\tilde{G}_{c(W)}})$ is an integral domain. To do this, we will prove that $S_i/(J_{\tilde{G}_1} + \dots + J_{\tilde{G}_i})$ is an integral domain by induction on i where S_i is the polynomial ring over K whose the indeterminates are the elements of the set $\{x_j, y_j : j \in \bigcup_{k=1}^i V(\tilde{G}_k)\}$. If $i = 1$, then $S_1/J_{\tilde{G}_1}$ is an integral domain from Corollary 3.1. Let us assume $R = S_{i-1}/(J_{\tilde{G}_1} + \dots + J_{\tilde{G}_{i-1}})$. Then, we can write

$$S_i/(J_{\tilde{G}_1} + \dots + J_{\tilde{G}_i}) \cong R[x_{i_1}, \dots, x_{i_t}, y_{i_1}, \dots, y_{i_t}]/J_{\tilde{G}_i} \quad (3.51)$$

where $V(\tilde{G}_i) = \{i_1, \dots, i_t\}$. So, $S_i/(J_{\tilde{G}_1} + \dots + J_{\tilde{G}_i})$ is an integral domain by Corollary 3.1. Therefore, $S/P_W(G)$ is an integral domain and $P_W(G)$ is a prime ideal.

Corollary 3.2: [U] Let G be a graph on $[n]$. Then, J_G is a prime ideal if and only if G is disjoint union of some complete graphs.

Proof : $(\Rightarrow)$ Let H be a connected component of G . Then, it can be shown that H is complete by repeating the arguments used in the first part of the proof of Theorem 3.6. Hence, G is a disjoint union of complete graphs.

$(\Leftarrow)$ Suppose that G is a union of disjoint complete graphs. Then, we have $J_G = P_\emptyset(G)$. From Lemma 3.2, J_G is prime ideal of S .

Lemma 3.3: [U] Suppose that G is a graph on $[n]$ and W is a subset of $[n]$. Then, $\text{height } P_W(G) = n + |W| - c(W)$.

Proof : Let $G_1, \dots, G_{c(W)}$ be the connected components of $G - W$. One can easily observe that $\text{height } J_{\tilde{G}_j} = n_j - 1$ where $n_j = |V(\tilde{G}_j)|$ for all $1 \leq j \leq c(W)$ by Corollary 3.2. Hence, we can write

$$\text{height } P_W(G) = \text{height} \left((\{x_i, y_i : i \in W\}) + J_{\tilde{G}_1} + \dots + J_{\tilde{G}_{c(W)}} \right) \quad (3.52)$$

$$= \text{height}(\{x_i, y_i : i \in W\}) + \text{height } J_{\tilde{G}_1} + \dots + \text{height } J_{\tilde{G}_{c(W)}} \quad (3.53)$$

$$= 2|W| + (n_1 - 1) + \dots + (n_{c(W)} - 1) \quad (3.54)$$

$$= (|W| + n_1 + \dots + n_{c(W)}) + |W| - c(W) \quad (3.55)$$

$$= n + |W| - c(W). \quad (3.56)$$

Theorem 3.7: [U] Suppose that G is a graph on $[n]$. Then, $J_G = \bigcap_{W \subset [n]} P_W(G)$.

Proof : We know that J_G is a radical ideal from Proposition 3.5. So, J_G can be written as an intersection of its minimal primes by Proposition 2.3. Then, it is enough to show that every minimal prime ideal of J_G has the form $P_W(G)$ for some $W \subset [n]$. To do this, we will first take G as a connected graph and use induction method on $[n]$. If

$n = 2$, then $J_G = (f_{12})$. By Corollary 3.2, J_G is a prime ideal and $J_G = P_\emptyset(G)$. Now, let $n > 2$. Suppose that P is a minimal prime of J_G .

We claim that $x_i \in P$ if and only if $y_i \in P$. To prove this claim, we will show that $T = \{x_i : x_i \in P \text{ and } y_i \notin P\}$ is an empty set. Let us examine the following two cases to see that $T = \{x_1, \dots, x_n\}$ is not possible.

Case 1: If G is complete, then J_G is prime by Corollary 3.2 and its minimal prime ideal is itself. Hence, we have $J_G \subsetneq (x_1, \dots, x_n) \subseteq P$. However, this contradicts our assumption.

Case 2: Suppose that G is not complete. Then, $J_G \subsetneq J_{\tilde{G}}$ and $J_{\tilde{G}}$ is prime from Corollary 3.2. Hence, we have $J_G \subsetneq J_{\tilde{G}} \subsetneq (x_1, \dots, x_n) \subseteq P$. However, this contradicts our assumption.

Hence, $T \neq \{x_1, \dots, x_n\}$. Now, let us assume $T \neq \emptyset$. Then, we have at least an edge $\{i, j\}$ of G satisfying $x_i \in T$ and $x_j \notin T$ since G is connected, otherwise it would be $T = \{x_1, \dots, x_n\}$. Then, $f_{ij} = x_i y_j - x_j y_i \in J_G \subset P$ implies $x_j y_i \in P$. Thus, $x_j \in P$ or $y_i \in P$. We must have $x_j \in P$ because of $x_i \in T$. Also, we see $y_j \in P$ since $x_j \notin T$. Now, let us consider the induced subgraph $G' = G - j$. Then, we can easily see that $J_G + (x_j, y_j) = J_{G'} + (x_j, y_j) \subset P$. Hence, $P/(x_j, y_j)$ is a minimal prime of $J_{G'}/(x_j, y_j)$ in the ring $S/(x_j, y_j)$. Since $|V(G')| < n$, induction hypothesis implies that $P/(x_j, y_j)$ has the form $P_W(G')/(x_j, y_j)$ for some $W \subset [n] \setminus \{j\}$. However, this is impossible since $x_i + (x_j, y_j) \in P/(x_j, y_j)$ and $y_i + (x_j, y_j) \notin P/(x_j, y_j)$. So, $T = \emptyset$.

Similarly, if we set $T' = \{y_i : y_i \in P \text{ and } x_i \notin P\}$, then $T' = \emptyset$. Therefore, we conclude that $x_i \in P$ if and only if $y_i \in P$.

Hence, there exists $W \subset [n]$ such that $P = (\{x_i, y_i : i \in W\}) + \bar{P}$ where $\bar{P}$ is a prime ideal and does not contain any element of the set $\{x_i, y_i : i \in W\}$. Suppose that $G_1, \dots, G_{c(W)}$ are connected components of the graph $G - W$. Now, let us prove $\bar{P} = J_{\tilde{G}_1} + \dots + J_{\tilde{G}_{c(W)}}$.

Clearly, $(\{x_i, y_i : i \in W\}) + J_{\tilde{G}_1} + \dots + J_{\tilde{G}_{c(W)}}$ is a prime ideal and it contains J_G . Since P is a minimal prime ideal of J_G , we write $\bar{P} \subseteq J_{\tilde{G}_1} + \dots + J_{\tilde{G}_{c(W)}}$. Additionally, $J_{\tilde{G}_1} + \dots + J_{\tilde{G}_{c(W)}} \subseteq \bar{P}$ can be shown similarly to the first part of the proof of Theorem 3.6. Hence, we can easily see $\bar{P} = J_{\tilde{G}_1} + \dots + J_{\tilde{G}_{c(W)}}$. It means $P = (\{x_i, y_i : i \in W\}) + J_{\tilde{G}_1} + \dots + J_{\tilde{G}_{c(W)}} = P_W(G)$.

Now, let us assume that G is not connected and $G_1, \dots, G_c$ are the connected

components of G . Then, for any $1 \leq j \leq c$, each minimal prime of J_{G_j} is in the form $(\{x_i, y_i : i \in W_j\}) + J_{\tilde{G}_{j1}} + \cdots + J_{\tilde{G}_{jc_j}}$ where $W_j \subset V(G_j)$ and $G_{j1}, \dots, G_{jc_j}$ are the connected components of $G_j - W_j$. Then,

$$\sum_{j=1}^c \left[(\{x_i, y_i : i \in W_j\}) + J_{\tilde{G}_{j1}} + \cdots + J_{\tilde{G}_{jc_j}} \right] = (\{x_i, y_i : i \in W\}) + \sum_{j=1}^c \sum_{k=1}^{c_j} J_{\tilde{G}_{jk}} \quad (3.57)$$

$$= (\{x_i, y_i : i \in W\}) \quad (3.58)$$

$$+ J_{\tilde{H}_1} + \cdots + J_{\tilde{H}_{c(W)}} \\ = P_W(G) \quad (3.59)$$

is a minimal prime of $J_G = J_{G_1} + \cdots + J_{G_c}$ where $W = \bigcup_{j=1}^c W_j$ and $H_1, \dots, H_{c(W)}$ are the connected components of $G - W$.

Corollary 3.3: [1] $\dim S/J_G = \max\{n - |W| + c(W) : W \subset [n]\}$ for any graph G on $[n]$. Also, if c is the number of the connected components of G , then $\dim S/J_G \geq n + c$.

Proof : It is obvious by Lemma 3.3, Theorem 3.7 and Proposition 2.5.

Lemma 3.4: [1] Suppose that G is a graph on $[n]$ and $W, T \subset [n]$. Also, suppose $G_1, \dots, G_{c(W)}$ and $H_1, \dots, H_{c(T)}$ are connected components of the graphs $G - W$ and $G - T$, respectively. Then, the followings are equivalent:

- i) $P_T(G) \subset P_W(G)$;
- ii) $T \subset W$ and for each $1 \leq i \leq c(T)$, there exists $1 \leq j \leq c(W)$ such that $V(H_i) \setminus W \subset V(G_j)$.

Proof : i) $\Rightarrow$ ii). We can write $P_T(G) \subset P_W(G)$ more clearly as

$$(\{x_i, y_i : i \in T\}) + J_{\tilde{H}_1} + \cdots + J_{\tilde{H}_{c(T)}} \subset (\{x_j, y_j : j \in W\}) + J_{\tilde{G}_1} + \cdots + J_{\tilde{G}_{c(W)}}. \quad (3.60)$$

Obviously, we have $T \subset W$ and

$$(\{x_j, y_j : j \in W\}) + J_{\tilde{H}_1} + \cdots + J_{\tilde{H}_{c(T)}} \subset (\{x_j, y_j : j \in W\}) + J_{\tilde{G}_1} + \cdots + J_{\tilde{G}_{c(W)}}. \quad (3.61)$$

Also, one can easily see that

$$(\{x_j, y_j : j \in W\}) + J_{\tilde{H}_1} + \cdots + J_{\tilde{H}_{c(T)}} = (\{x_j, y_j : j \in W\}) + J_{\tilde{H}'_1} + \cdots + J_{\tilde{H}'_{c(T)}} \quad (3.62)$$

where $H'_i = H_i - W$. Hence, we have $J_{\tilde{H}'_1} + \cdots + J_{\tilde{H}'_{c(T)}} \subset J_{\tilde{G}_1} + \cdots + J_{\tilde{G}_{c(W)}}$. Then, for each $1 \leq i \leq c(T)$, there exists $1 \leq j \leq c(W)$ such that $J_{\tilde{H}'_i} \subset J_{\tilde{G}_j}$. Therefore, we can see $V(H'_i) = V(H_i) \setminus W \subset V(G_j)$.

ii) $\Rightarrow$ i). Trivial.

Theorem 3.8: [U] Suppose that G is a connected graph on $[n]$ and $W \subset [n]$. Then, $P_W(G) \in \text{Min}(J_G)$ if and only if $W = \emptyset$ or $W \neq \emptyset$ and $c(W) > c(W \setminus \{i\})$ for all $i \in W$.

Proof : ($\Rightarrow$) We know from Theorem 3.7 that each minimal prime ideal of J_G is in the form $P_W(G)$ for some $W \subset [n]$. If $W = \emptyset$, then $P_W(G) \subset P_T(G)$ for all $T \subset [n]$ with $T \neq \emptyset$. In this situation, $P_W(G)$ is a minimal prime ideal of J_G . Now, let W be a nonempty subset of $[n]$ and $i \in W$. Also, let $G_1, \dots, G_{c(W)}$ be the connected components of $G - W$. We must consider these three cases:

Case 1: If i is not connected to any of the connected components $G_1, \dots, G_{c(W)}$ by an edge in G , then $G_1, \dots, G_{c(W)}$ and G_0 are the connected components of $G - T$ where $T = W \setminus \{i\}$ and G_0 is the graph consisting of the isolated vertex i . Here, we have $c(T) = c(W \setminus \{i\}) = c(W) + 1$ and $c(W \setminus \{i\}) > c(W)$. Also, it can be seen $T \subset W$ and $V(G_j) \setminus W \subset V(G_j)$ for any $1 \leq j \leq c(W)$. Then, Lemma 3.4 implies $P_T(G) \subset P_W(G)$ but this contradicts the fact that $P_W(G)$ is a minimal prime of J_G .

Case 2: Assume that i is connected to only one of the connected components of $G - W$ by an edge in G , say G_1 . Then, $G_2, \dots, G_{c(W)}$ and G'_1 are the connected components of $G - T$ where $T = W \setminus \{i\}$ and $V(G'_1) = V(G_1) \cup \{i\}$. Here, we have $c(T) = c(W \setminus \{i\}) = c(W)$. Also, we can see $T \subset W$, $V(G'_1) \setminus W \subset V(G_1)$ and $V(G_j) \setminus W \subset V(G_j)$ for any $2 \leq j \leq c(W)$. Then, Lemma 3.4 allows us to write $P_T(W) \subset P_W(G)$ but this is a contradiction again.

Case 3: Suppose that i is connected to at least two connected components of $G - W$, say $G_1, \dots, G_t$ ($t \geq 2$). Hence, $G'_1, G_{t+1}, \dots, G_{c(W)}$ are the connected components of $G - T$ where $T = W \setminus \{i\}$ and $V(G'_1) = V(G_1) \cup \dots \cup V(G_t) \cup \{i\}$. Here, we must have $c(T) = c(W \setminus \{i\}) = c(W) - t + 1 < c(W)$ and $c(W) > c(W \setminus \{i\})$ because of $t \geq 2$.

To sum up, if $P_W(G)$ is a minimal prime ideal of J_G , only the last case of the three cases we have examined above is possible. Therefore, $c(W) > c(W \setminus \{i\})$ is provided for each $i \in W$.

($\Leftarrow$) If $W = \emptyset$, then $P_W(G) \in \text{Min}(J_G)$. Assume that $W \neq \emptyset$ and $P_W(G)$ is not a minimal prime of J_G . Then, there is a subset $T \subsetneq W$ satisfying $P_T(G) \subsetneq P_W(G)$. Let $G_1, \dots, G_{c(W)}$ be the connected components of $G - W$ and $i \in W$ with $i \notin T$. Also let $G_1, \dots, G_t$ be the connected components connected to i in G . Hence, the connected components of $G - W'$ are $G'_1, G_{t+1}, \dots, G_{c(W)}$ where $W' = W \setminus \{i\}$ and $V(G'_1) = V(G_1) \cup \dots \cup V(G_t) \cup \{i\}$. We have $t \geq 2$ since $c(W) > c(W \setminus \{i\}) = c(W')$. By $T \subset W'$, there exists a connected component H of $G - T$ such that G'_1 is a subgraph of H . One can easily observe that $V(G_1), \dots, V(G_t) \subset V(H) \setminus W$. Hence, we have $T \subset W$ but $V(H) \setminus W \not\subset V(G_j)$ for all $1 \leq j \leq c(W)$. Then, we have $P_T(G) \not\subset P_W(G)$ from Lemma 3.4 but this contradicts our assumption. Therefore, $P_W(G) \in \text{Min}(J_G)$.

4. REGULARITY OF BINOMIAL EDGE IDEALS

4.1. Regularity Bounds

We will start this section by introducing the multigrading of S and J_G . Then, we will study the relations between the graded Betti numbers of binomial edge ideals of two graphs G and H where H is an induced subgraph of G . Also, we will give some basic properties of the regularity of S/J_G . In particular, we will show that the regularity of the binomial edge ideal of a path of length l is equal to $l + 1$ which was proven by Matsuda and Murai in [5]. Also, for any graph G , we will give lower and upper bounds for $\text{reg}(S/J_G)$ which were proven in [5], that is, $l \leq \text{reg}(S/J_G) \leq n - 1$ where l is the length of the longest induced path in G .

Firstly, let us examine the multigrading of S and J_G . If $\epsilon_i \in \mathbb{N}^n$ is the i -th unit vector, then it is obvious that S and J_G are $\mathbb{N}^n$ -graded by $\deg x_i = \deg y_i = \epsilon_i$. For any $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$, we define $\text{supp}(\mathbf{a}) = \{i \in [n] : a_i \neq 0\}$.

Additionally, if $\mathbf{e}_k \in \mathbb{N}^{2n}$ is the k -th unit vector, then S is $\mathbb{N}^{2n}$ -graded by $\deg x_i = \mathbf{e}_i$ and $\deg y_i = \mathbf{e}_{n+i}$. However, binomial edge ideals are not $\mathbb{N}^{2n}$ -graded because for any generator $f_{ij} = x_i y_j - x_j y_i$ of J_G , we have $\deg(x_i y_j) = \mathbf{e}_i + \mathbf{e}_{n+j}$ and $\deg(x_j y_i) = \mathbf{e}_j + \mathbf{e}_{n+i}$. Hence, f_{ij} is not homogeneous with respect to this grading and J_G is not $\mathbb{N}^{2n}$ -graded. For any multidegree $(\mathbf{a}, \mathbf{b}) = (a_1, \dots, a_n, b_1, \dots, b_n) \in \mathbb{N}^{2n}$, we associate the monomial $x^{\mathbf{a}} y^{\mathbf{b}} = x_1^{a_1} \dots x_n^{a_n} y_1^{b_1} \dots y_n^{b_n}$. If U is a finitely generated $\mathbb{N}^{2n}$ -graded S -module, we write the Poincaré series of U as

$$P(U, t) = \sum_{k=0}^{2n} \sum_{(\mathbf{a}, \mathbf{b}) \in \mathbb{N}^{2n}} \beta_{k, (\mathbf{a}, \mathbf{b})}(U) x^{\mathbf{a}} y^{\mathbf{b}} t^k. \quad (4.1)$$

Proposition 4.1: [5] Suppose that G is a graph on $[n]$ and T is a subset of $[n]$. Then, we have $\beta_{i, \mathbf{a}}(J_G) = \beta_{i, \mathbf{a}}(J_{G[T]})$ for all $i \geq 0$ and for all $\mathbf{a} \in \mathbb{N}^n$ with $\text{supp}(\mathbf{a}) \subset T$.

Proof : Suppose that the minimal multigraded free resolution of S/J_G is given as

$$\mathbb{F} : 0 \longrightarrow \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S(-\mathbf{a})^{\beta_{p, \mathbf{a}}(J_G)} \longrightarrow \dots \longrightarrow \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S(-\mathbf{a})^{\beta_{0, \mathbf{a}}(J_G)} \xrightarrow{\phi} S. \quad (4.2)$$

where $p = \text{projdim}(J_G)$. Let us consider the subcomplex

$$\mathbb{F}' : 0 \longrightarrow \bigoplus_{\substack{\mathbf{a} \in \mathbb{N}^n \\ \text{supp}(\mathbf{a}) \subset T}} S(-\mathbf{a})^{\beta_{p,\mathbf{a}}(J_G)} \longrightarrow \dots \longrightarrow \bigoplus_{\substack{\mathbf{a} \in \mathbb{N}^n \\ \text{supp}(\mathbf{a}) \subset T}} S(-\mathbf{a})^{\beta_{0,\mathbf{a}}(J_G)} \xrightarrow{\phi'} S \quad (4.3)$$

of $\mathbb{F}$ where ϕ' is the restriction map of ϕ . Let us prove that $\mathbb{F}'$ is minimal $\mathbb{N}^n$ -graded resolution of $S/J_{G[T]}$. We can easily see that $\text{Im}(\phi') = J_{G[T]}$ because of $\text{supp}(\mathbf{a}) \subset T$ and hence, $\text{coker}(\phi') = S/J_{G[T]}$. Also, for any $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ with $\text{supp}(\mathbf{a}) \subset T$, we say that $S(-\mathbf{b})_{\mathbf{a}} = S(\mathbf{a} - \mathbf{b})$ is nonzero if and only if each component of the vector $\mathbf{a} - \mathbf{b}$ is nonnegative. For any $\mathbf{a} \in \mathbb{N}^n$, we have $\mathbb{F}_{\mathbf{a}} = \mathbb{F}'_{\mathbf{a}}$ and Proposition 2.23 implies that $\mathbb{F}'_{\mathbf{a}}$ is exact since $\mathbb{F}$ is exact. Since $\mathbb{F}'_{\mathbf{a}}$ is exact for each $\mathbf{a} \in \mathbb{N}^n$ with $\text{supp}(\mathbf{a}) \subset T$, $\mathbb{F}'$ is exact by Proposition 2.23. Additionally, $\mathbb{F}'$ is minimal free resolution because $\mathbb{F}$ is minimal resolution. Therefore, $\beta_{i,\mathbf{a}}(J_G) = \beta_{i,\mathbf{a}}(J_{G[T]})$ for all $i \geq 0$ and $\mathbf{a} \in \mathbb{N}^n$ with $\text{supp}(\mathbf{a}) \subset T$.

Corollary 4.1: [5] Suppose that G is a graph on $[n]$ and H is an induced subgraph of G . Then, $\beta_{i,j}(J_G) \geq \beta_{i,j}(J_H)$ for all $i \geq 0$ and $j \geq 0$.

Example 4.1: Let us consider the complete bipartite graph $K_{1,3}$. The Betti diagram of $S/J_{K_{1,3}}$ can be written as shown in Table 4.1 with the help of Macaulay 2 (see [22]).

Table 4.1: Betti diagram of $S/J_{K_{1,3}}$

	0	1	2	3
0	1	0	0	0
1	0	3	0	0
2	0	0	4	2

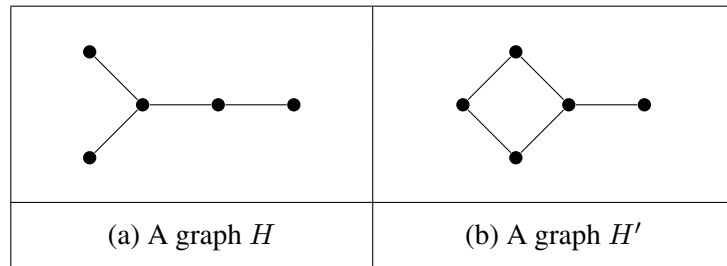


Figure 4.1: A graph H and a graph H'

Hence, we can see $\beta_{3,5}(S/J_{K_{1,3}}) = 2$. If we call H and H' the graphs shown in Figure

[4.1a](#) and Figure [4.1b](#) respectively, then we have $\beta_{3,5}(S/J_H) \geq 2$ and $\beta_{3,5}(S/J_{H'}) \geq 2$ since H and H' contains $K_{1,3}$ as an induced subgraph.

Proposition 4.2: [\[5\]](#) Let P be a path of length l with edges $\{1, 2\}, \{2, 3\}, \dots, \{l, l+1\}$. Then, $\text{reg}(S/J_P) = l$.

Proof : Clearly, we have $J_P = (f_{12}, f_{23}, \dots, f_{l(l+1)})$. We claim that the sequence $f_{12}, f_{23}, \dots, f_{l(l+1)}$ is S -regular. Obviously, we have $(f_{12}, f_{23}, \dots, f_{l(l+1)}) \neq S$. Now, let us show that $f_{k(k+1)}$ is a nonzero divisor on the module $S/(f_{12}, \dots, f_{(k-1)k})$ for every $1 \leq k \leq l$. Assume the contrary, that is, $f_{k(k+1)}$ is not a nonzero divisor on $S/(f_{12}, \dots, f_{(k-1)k})$ for some k . Then, there exists a polynomial $g \in S$ such that $g \notin (f_{12}, \dots, f_{(k-1)k})$ and $f_{k(k+1)}g \in (f_{12}, \dots, f_{(k-1)k})$. Since P is a path, it is closed with respect to given labeling of vertices by [Proposition 3.2](#). So, the set of generators of J_P forms a Gröbner basis for J_P by [Theorem 3.3](#). Similarly, the generator set of $(f_{12}, \dots, f_{(k-1)k})$ is its Gröbner basis. Then, there are unique polynomials $h \in (f_{12}, \dots, f_{(k-1)k})$ and $r \notin (f_{12}, \dots, f_{(k-1)k})$ such that $g = h + r$ according to the division algorithm. Here, no term of r is divisible by any of $\text{in}_<(f_{12}), \dots, \text{in}_<(f_{(k-1)k})$. Thus, we have $f_{k(k+1)}g = f_{k(k+1)}h + f_{k(k+1)}r$. If we say $\mathcal{G} = \{f_{12}, \dots, f_{(k-1)k}\}$, then the polynomial $f_{k(k+1)}r = x_k y_{k+1} r - x_{k+1} y_k r$ does not reduce to 0 with respect to $\mathcal{G}$. This contradicts our assumption. Hence, $f_{k(k+1)}$ is a nonzero divisor on $S/(f_{12}, \dots, f_{(k-1)k})$ for each $1 \leq k \leq l$. So, $f_{12}, f_{23}, \dots, f_{l(l+1)}$ is S -regular sequence. For any $1 \leq k \leq l$, we observe

$$S/(f_{12}, \dots, f_{(k-1)k}) \Big/ (f_{12}, \dots, f_{k(k+1)})/(f_{12}, \dots, f_{(k-1)k}) \cong S/(f_{12}, \dots, f_{k(k+1)}). \quad (4.4)$$

Hence, we can write the equalities

$$\text{reg}(S/(f_{12})) = \text{reg}(S) + \deg(f_{12}) - 1 \quad (4.5)$$

$$\text{reg}(S/(f_{12}, f_{23})) = \text{reg}(S/(f_{12})) + \deg(f_{23}) - 1 \quad (4.6)$$

$\vdots$

$$\text{reg}(S/(f_{12}, \dots, f_{l(l+1)})) = \text{reg}(S/(f_{12}, \dots, f_{(l-1)l})) + \deg(f_{l(l+1)}) - 1 \quad (4.7)$$

by Proposition 2.21. Since $\text{reg}(S) = 0$ and $\deg(f_{12}) = \dots = \deg(f_{l(l+1)}) = 2$, the equations (4.5), (4.6), $\dots$, (4.7) implies $\text{reg}(S/J_P) = \text{reg}(S/(f_{12}, \dots, f_{l(l+1)})) = l$.

Lemma 4.1: [5] Suppose that I is a monomial ideal in S and $\{m_1, \dots, m_g\}$ is a minimal set of generators for I . Then, the inequality

$$P(S/I, t) \leq 1 + \sum_{j=1}^g P(S/((m_1, \dots, m_{j-1}) : m_j), t) m_j t \quad (4.8)$$

is provided coefficient-wise.

Proof : Let us say $I_j = (m_1, \dots, m_j)$ for all $1 \leq j \leq g$. Clearly, we have $I_g = I$. Also, we will assume $I_0 = (0)$. For each $j = 1, \dots, g$, the chain

$$0 \longrightarrow S/(I_{j-1} : m_j) \xrightarrow{\varphi} S/I_{j-1} \xrightarrow{\phi} S/I_j \longrightarrow 0 \quad (4.9)$$

is an exact sequence where φ and ϕ are defined as $\varphi(f + (I_{j-1} : m_j)) = fm_j + I_{j-1}$ and $\phi(f + I_{j-1}) = f + I_j$ for all $f \in S$. If we say $\deg(m_j) = \mathbf{a}$, then the exact sequence in (4.9) can be rewritten in graded form as

$$0 \longrightarrow S/(I_{j-1} : m_j)(-\mathbf{a}) \xrightarrow{\varphi} S/I_{j-1} \xrightarrow{\phi} S/I_j \longrightarrow 0. \quad (4.10)$$

If we denote the minimal graded free resolutions of $\varphi(S/(I_{j-1} : m_j))$ and S/I_j by $\mathbb{U}$ and $\mathbb{U}'$, respectively, then $\mathbb{W} = \mathbb{U}' \oplus \mathbb{U}[-1]$ will be a graded free resolution of $S/I_{j-1} \big/ \varphi(S/(I_{j-1} : m_j)) \cong S/I_j$ by mapping cones. However, $\mathbb{W}$ may not be the minimal graded free resolution of S/I_j . Hence, we can write the inequality

$$P(S/I_j, t) \leq P(S/(I_{j-1} : m_j), t) m_j t + P(S/I_{j-1}, t) \quad (4.11)$$

coefficient-wise for each $j = 1, \dots, g$. Hence, we have

$$\sum_{j=1}^g [P(S/I_j, t) - P(S/I_{j-1}, t)] \leq \sum_{j=1}^g P(S/(I_{j-1} : m_j), t) m_j t. \quad (4.12)$$

Observe that, if we expand the first sum symbol in inequality (4.12), we see

$$P(S/I, t) - P(S, t) \leq \sum_{j=1}^g P(S/(I_{j-1} : m_j), t) m_j t. \quad (4.13)$$

As a final step, if we use $P(S, t) = 1$, then we find the coefficient-wise inequality

$$P(S/I, t) \leq 1 + \sum_{j=1}^g P(S/(I_{j-1} : m_j), t) m_j t. \quad (4.14)$$

Definition 4.1: [12] Let i, j be two vertices of a graph G on the vertex set $[n]$ and let $i < j$. A path $P : i = i_0, i_1, \dots, i_t = j$ between i and j is said to be weakly admissible or w -admissible if $i_k < i$ or $i_k > j$ for all $1 \leq k \leq t - 1$. Also, i, j are said to be the ends of P and $i_1, \dots, i_{t-1}$ the inner vertices of P . The set of all weakly admissible paths of G is denoted by $\mathcal{P}(G)$.

If $i < j$ and $P : i = i_0, i_1, \dots, i_t = j$ is a weakly admissible path, then we define the monomial

$$m_P = x_i \left(\prod_{i_k > j} x_{i_k} \right) \left(\prod_{i_l < i} y_{i_l} \right) y_j. \quad (4.15)$$

Proposition 4.3: [5] Suppose that G is a graph on $[n]$ and $<$ is the lexicographic order on S . Then, $\text{in}_<(J_G) = (\{m_P : P \in \mathcal{P}(G)\})$.

Proof : It is obvious by Definition 4.1 and Theorem 3.5.

Lemma 4.2: [5] Let i, j be two vertices of a graph G on the vertex set $[n]$ and let $i < j$. For any weakly admissible path $P : i = i_0, i_1, \dots, i_t = j$ between i and j , then the following properties hold:

- i) If $i_k < i$ for some k , then there exists $l > k$ such that $P' : i_k, i_{k+1}, \dots, i_l$ is a weakly admissible path and $m_{P'}$ divides $x_{i_k} m_P$;
- ii) If $i_k > j$ for some k , then there exists $l < k$ such that $P' : i_l, i_{l+1}, \dots, i_k$ is a weakly admissible path and $m_{P'}$ divides $y_{i_k} m_P$.

Proof : i) Let us choose l as the smallest number satisfying $k < l$ and $i_k < i_l \leq j$. Then, $P' : i_k, i_{k+1}, \dots, i_l$ is a weakly admissible path because none of the numbers $i_{k+1}, \dots, i_{l-1}$ are between i_k and i_l . If the monomial $m_{P'}$ contains x_s as a multiple for some $k+1 \leq s \leq l-1$, then we must have $s > i_l$. Also, $s > j$ and x_s is a multiple of m_P by the choice of l . If $m_{P'}$ contains y_s as a multiple for some $k+1 \leq s \leq l-1$, then we must have $s < i_k < i$. So, y_s is a multiple of m_P . Thus, $m_{P'}$ divides $x_{i_k} m_P$.

ii) It can be proven similarly to the proof of i).

The path P' providing one of the conditions of Lemma 4.2 is said to be a wedge of P at the vertex i_k . If we say $|\mathcal{P}(G)| = g$, We can sort the weakly admissible paths in G from short to long as $P_1, P_2, \dots, P_g$. In other words, if the length of P_i is less than the length of P_j , then we have $i < j$. However, $i < j$ does not mean that P_i is shorter than P_j because the paths P_i and P_j may have the same length. Also, we can write m_k instead of m_{P_k} for each $k = 1, \dots, g$, i.e., we will use the notation $m_k = m_{P_k}$ for all $k = 1, \dots, g$.

Lemma 4.3: [5] Suppose that G is a graph on $[n]$ and $P_k : i = i_0, i_1, \dots, i_t = j$ is a weakly admissible path where $1 < k \leq g = |\mathcal{P}(G)|$. Also, suppose $1 \leq s \leq t-1$.

i) If $i_s < i$, then $x_{i_s} \in ((m_1, \dots, m_{k-1}) : m_k)$;

ii) If $i_s > j$, then $y_{i_s} \in ((m_1, \dots, m_{k-1}) : m_k)$.

Proof : i) Suppose that $i_s < i$. Then, we can find a wedge P' of P_k at i_s such that $m_{P'}$ divides $x_{i_s} m_{P_k}$ from Lemma 4.2. Observe that, the path P' is shorter than the path P_k and it means $m_{P'} = m_c$ for some $1 \leq c \leq k-1$. Thus, we can write $x_{i_s} m_k \in (m_1, \dots, m_{k-1})$ since m_c divides $x_{i_s} m_k$. So, $x_{i_s} \in ((m_1, \dots, m_{k-1}) : m_k)$.

ii) It is similar to the proof of i).

Definition 4.2: [5] Let w be a monomial in the polynomial ring S . Then, we define $\text{mult}(w) = \{k : x_k y_k \text{ divides } w\}$.

Definition 4.3: [5] Let $\mathcal{M} = \{m_1, m_2, \dots, m_g\}$ where g is the number of the weakly admissible paths in a graph G on $[n]$. A subset $F = \{m_{i_1}, m_{i_2}, \dots, m_{i_k}\}$ of $\mathcal{M}$ with $i_1 < i_2 < \dots < i_k$ is said to be Lyubeznik if none of the monomials $m_1, m_2, \dots, m_{i_j-1}$ divides $\text{lcm}(m_{i_j}, m_{i_{j+1}}, \dots, m_{i_k})$ for all $1 \leq j \leq k$.

Lemma 4.4: [5] Suppose that $F = \{m_{i_1}, m_{i_2}, \dots, m_{i_k}\}$ is a Lyubeznik subset. Then, we have the followings:

- i) None of the inner vertices of P_{i_1} belong to $\text{mult}(\text{lcm}(F))$.
- ii) If none of the inner vertices of any of the paths $P_{i_2}, P_{i_3}, \dots, P_{i_k}$ belong to the set $\text{mult}(\text{lcm}(F))$, then we have $|\text{mult}(\text{lcm}(F))| \leq k - 1$.

Proof : i) Assume that an inner vertex v of the path $P_{i_1} : p_0, p_1, \dots, p_t$ is contained by $\text{mult}(\text{lcm}(F))$. If $v < p_0$, then we can find a wedge P_j of P at v such that m_j divides $x_v m_{i_1} = x_v y_v (m_{i_1}/y_v)$ from Lemma 4.2. So, m_j divides $\text{lcm}(F)$ since v belongs to $\text{mult}(\text{lcm}(F))$. However, this contradicts that F is a Lyubeznik subset of $\mathcal{M}$. Similary, if $v > p_t$, then we can see a similar contradiction occur again.

ii) Let p_j and q_j are the end vertices of P_{i_j} with $p_j < q_j$ for $j = 1, 2, \dots, k$. As no inner vertices of any of $P_{i_1}, \dots, P_{i_k}$ are contained by $\text{mult}(\text{lcm}(F))$, we have

$$|\text{mult}(\text{lcm}(F))| = |\text{mult}(\text{lcm}(m_{i_1}, \dots, m_{i_k}))| = |\text{mult}(\text{lcm}(x_{p_1} y_{q_1}, \dots, x_{p_k} y_{q_k}))|. \quad (4.16)$$

Also, $p_1 < q_1, \dots, p_k < q_k$ implies $|\text{mult}(\text{lcm}(x_{p_1} y_{q_1}, \dots, x_{p_k} y_{q_k}))| \leq k - 1$ and hence, $|\text{mult}(\text{lcm}(F))| \leq k - 1$.

Lemma 4.5: [5] Suppose that $F = \{m_{i_1}, \dots, m_{i_k}\}$ is a Lyubeznik subset of $\mathcal{M}$. Also, let w be a monomial in S and p be a positive integer. Assume that the following statements are satisfied:

- i) $\beta_{p,w}(S/((m_1, \dots, m_{i_1-1}) : m_{i_1} \dots m_{i_k})) \neq 0$;
- ii) None of the inner vertices of any of $P_{i_2}, \dots, P_{i_k}$ belong to $\text{mult}(w \text{lcm}(F))$.

Then, there is a Lyubeznik subset $\tilde{F} = \{m_{j_1}, \dots, m_{j_l}\}$ and a monomial $\tilde{w}$ in S such that the following properties hold:

$$i') \beta_{p-1, \tilde{w}}(S/((m_1, \dots, m_{j_1-1}) : m_{j_1} \dots m_{j_l})) \neq 0;$$

- ii') None of the inner vertices of any of $P_{j_2}, \dots, P_{j_i}$ belong to $\text{mult}(\tilde{w} \text{lcm}(\tilde{F}))$.
- iii') $|\text{mult}(\tilde{w} \text{lcm}(\tilde{F}))| - |\tilde{F}| = |\text{mult}(w \text{lcm}(F))| - |F| - 1$.

Proof : Let us consider the following cases:

Case 1: Let us assume that an inner vertex v of P_{i_1} is contained by the set $\text{mult}(w \text{lcm}(F))$. If x_v divides m_{i_1} , then x_v divides $\text{lcm}(F)$. So, we see that y_v does not divide $\text{lcm}(F)$ from i) of Lemma 4.4. Additionally, y_v divides w . In addition, Lemma 4.3 implies $y_v \in ((m_1, \dots, m_{i_1-1}) : m_{i_1})$ and $y_v \in ((m_1, \dots, m_{i_1-1}) : m_{i_1} \dots m_{i_k})$. Then, the condition i) implies $\beta_{p-1, w/y_v}(S/((m_1, \dots, m_{i_1-1}) : m_{i_1} \dots m_{i_k})) \neq 0$. If we choose $\tilde{F} = F$ and $\tilde{w} = w/y_v$, then clearly the statements i') and ii') are hold. Also, we have $|\text{mult}(\tilde{w} \text{lcm}(\tilde{F}))| = |\text{mult}(w \text{lcm}(F))| - 1$ as v does not belong to $\text{mult}(\tilde{w} \text{lcm}(\tilde{F}))$. Hence, the statement iii') holds. If y_v divides m_{i_1} , it can be shown using similar arguments that i'), ii') and iii') are satisfied.

Case 2: Suppose that no inner vertex of P_{i_1} belongs to $\text{mult}(w \text{lcm}(F))$. We can write $(\overline{m}_1, \dots, \overline{m}_{i_1-1}) = ((m_1, \dots, m_{i_1-1}) : m_{i_1} \dots m_{i_k})$ where

$$\overline{m}_j = \frac{m_j}{\gcd(m_j, m_{i_1} \dots m_{i_k})}, \quad 1 \leq j \leq i_1 - 1. \quad (4.17)$$

Because of $\beta_{p,w}(S/((m_1, \dots, m_{i_1-1}) : m_{i_1} \dots m_{i_k})) = \beta_{p,w}(S/(\overline{m}_1, \dots, \overline{m}_{i_1-1})) \neq 0$, there exists $1 \leq s \leq i_1 - 1$ such that $\beta_{p-1, w/\overline{m}_s}(S/((\overline{m}_1, \dots, \overline{m}_{s-1}) : \overline{m}_s)) \neq 0$ and $\overline{m}_s \notin (\overline{m}_1, \dots, \overline{m}_{s-1})$. Now, let us choose $\tilde{F} = \{m_s, m_{i_1}, \dots, m_{i_k}\}$ and $\tilde{w} = w/\overline{m}_s$. Let us show that $\tilde{F}$ is Lyubeznik subset. For each $1 \leq l \leq s - 1$,

$$\overline{m}_l = \frac{m_l}{\gcd(m_l, m_{i_1} \dots m_{i_k})} = \frac{\text{lcm}(m_l, m_{i_1} \dots m_{i_k})}{m_{i_1} \dots m_{i_k}} \quad (4.18)$$

divides

$$\overline{m}_s = \frac{m_s}{\gcd(m_s, m_{i_1} \dots m_{i_k})} = \frac{\text{lcm}(m_s, m_{i_1} \dots m_{i_k})}{m_{i_1} \dots m_{i_k}} \quad (4.19)$$

if and only if m_l divides $\text{lcm}(m_s, m_{i_1}, \dots, m_{i_k})$. Here, $\overline{m}_l$ does not divide $\overline{m}_s$ from $\overline{m}_s \notin (\overline{m}_1, \dots, \overline{m}_{s-1})$. Additionally, none of the monomials $m_1, m_2, \dots, m_{i_j-1}$ divides

$\text{lcm}(m_{i_j}, m_{i_{j+1}}, \dots, m_{i_k})$ for all $1 \leq j \leq k$ as F is a Lyubeznik subset. Hence, $\tilde{F}$ is a Lyubeznik subset. Also, $((\bar{m}_1, \dots, \bar{m}_{s-1}) : \bar{m}_s) = ((m_1, \dots, m_{s-1}) : m_s m_{i_1} \dots m_{i_k})$ and $w \text{lcm}(F) = \tilde{w} \text{lcm}(\tilde{F})$. Hence, $\tilde{F}$ and $\tilde{w}$ satisfy the statements i'), ii') and iii').

Note that, the statements i') and ii') in Lemma 4.5 are very similar to the statements i) and ii). So, it allows us to use Lemma 4.5 again and again.

Proposition 4.4: [5] Suppose that G is a graph on $[n]$. Also, suppose that $w \in S$ is a monomial and p is a positive integer. If $|\text{mult}(w)| \geq p$, then $\beta_{p,w}(S/\text{in}_{<}(J_G)) = 0$.

Proof : Equivalently, we will show that if $\beta_{p,w}(S/\text{in}_{<}(J_G)) \neq 0$ for a monomial $w \in S$ and a positive integer p , then $|\text{mult}(w)| < p$. From Proposition 4.3 we can write $\text{in}_{<}(J_G) = (m_1, \dots, m_g)$. Additionally, there exists $m_j \in \mathcal{M}$ such that $m_j \notin (m_1, \dots, m_{j-1})$ and $\beta_{p-1, w/m_j}(S/((m_1, \dots, m_{j-1}) : m_j)) \neq 0$ because of Lemma 4.1. Clearly, the set $F = \{m_j\}$ is a Lyubeznik subset. If $p = 1$, then we have $w = m_j$. Hence, it can be seen $p = 1 > 0 = |\text{mult}(\text{lcm}(F))| = |\text{mult}(w)|$. If $p \geq 2$, the set $F = \{m_j\}$ and the monomial w/m_j satisfy the statements i) and ii) of Lemma 4.5. If we apply Lemma 4.5 repeatedly $p - 1$ times, we can find the Lyubeznik subsets $F_1, F_2, \dots, F_{p-2}, F_{p-1}$, and the monomials $w_1, w_2, \dots, w_{p-2}, w_{p-1}$ satisfying the equations

$$|\text{mult}(w_1 \text{lcm}(F_1))| - |F_1| = \left| \text{mult} \left(\frac{w}{m_j} \text{lcm}(F) \right) \right| - |F| - 1 \quad (4.20)$$

$$|\text{mult}(w_2 \text{lcm}(F_2))| - |F_2| = |\text{mult}(w_1 \text{lcm}(F_1))| - |F_1| - 1 \quad (4.21)$$

⋮

$$|\text{mult}(w_{p-1} \text{lcm}(F_{p-1}))| - |F_{p-1}| = |\text{mult}(w_{p-2} \text{lcm}(F_{p-2}))| - |F_{p-2}| - 1 \quad (4.22)$$

and $\beta_{0, w_{p-1}}(S/((m_1, \dots, m_{i_1-1}) : m_{i_1} \dots m_{i_k})) \neq 0$ where $F_{p-1} = \{m_{i_1}, \dots, m_{i_k}\}$. The equations (4.20), (4.21), ..., (4.22) imply the equality

$$|\text{mult}(w_{p-1} \text{lcm}(F_{p-1}))| - |F_{p-1}| = |\text{mult}(w)| - p. \quad (4.23)$$

Also, $\beta_{0, w_{p-1}}(S/((m_1, \dots, m_{i_1-1}) : m_{i_1} \dots m_{i_k})) \neq 0$ implies $w_{p-1} = x^0 y^0$ where $0 = (0, \dots, 0) \in \mathbb{N}^n$. Hence, we have $|\text{mult}(w_{p-1} \text{lcm}(F_{p-1}))| = |\text{mult}(\text{lcm}(F_{p-1}))|$.

Then, equation (4.23) and ii) of Lemma 4.4 allow us to write $|\text{mult}(w)| \leq p - 1$ and $|\text{mult}(w)| < p$.

Theorem 4.1: [5] Let G be a graph on $[n]$. Then, $l \leq \text{reg}(S/J_G) \leq n - 1$ where l is the length of the longest induced path in G .

Proof : Suppose that l be the length of the longest induced path in G . Then, we can see $l \leq \text{reg}(S/J_G)$ by Proposition 4.2 and Corollary 4.1.

If $\beta_{p,w}(S/\text{in}_<(J_G)) \neq 0$ for some $0 \leq p \leq 2n$ and a monomial w in S , then w is squarefree by Proposition 2.24. Hence, we can observe $\deg(w) \leq n + |\text{mult}(w)|$. Then, $\deg(w) - p \leq n + |\text{mult}(w)| - p \leq n - 1$ because of Proposition 4.4. So, we have $\text{reg}(S/\text{in}_<(J_G)) \leq n - 1$. From Proposition 2.16, we find $\text{reg}(S/J_G) \leq n - 1$. To sum up, $l \leq \text{reg}(S/J_G) \leq n - 1$.

Theorem 4.2: [23] Let G be a graph on $[n]$ with $G \neq P_n$. Then, $\text{reg}(S/J_G) \leq n - 2$.

Corollary 4.2: [5] Let G be a graph on $[n]$. $G = P_n$ if and only if $\text{reg}(S/J_G) = n - 1$.

Proof : It is obvious from Proposition 4.2 and Theorem 4.2.

4.2. Binomial Edge Ideals with Linear and Pure Resolutions

We will start this section by exploring the graded Betti numbers of binomial edge ideals. Then, we will state that the necessary and sufficient condition for J_G to have a linear resolution is that G is a complete graph which was proven by Madani and Kiani in [8]. More generally, we will show in Theorem 4.5 that J_G has a pure resolution if and only if G is complete or complete bipartite or disjoint union of some paths which was proven by the same authors in [9].

Lemma 4.6: [8] Let G is a graph on $[n]$. Then, the following properties hold:

- i) $\beta_{2,3}(S/J_G) = 2K_3(G)$ where $K_3(G)$ is the number of K_3 's contained in G as induced subgraph.
- ii) If G is noncomplete and connected, then $\beta_{2,4}(S/J_G) \neq 0$.

Proof : i) Let the minimal graded free resolution of S/J_G be given as

$$\mathbb{F} : 0 \longrightarrow \cdots \longrightarrow \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{2,j}(S/J_G)} \longrightarrow S(-2)^{|E(G)|} \xrightarrow{\phi} S \longrightarrow S/J_G \quad (4.24)$$

where $\phi(e_{ij}) = f_{ij}$ for each $\{i, j\} \in E(G)$ with $i < j$. Since $\mathbb{F}$ is graded, we have $\deg(e_{ij}) = \deg(f_{ij}) = \epsilon_i + \epsilon_j$. Here, we mean the k -th unit vector of $\mathbb{N}^n$ by e_k . Let us find a third degree relation between the generators of J_G . Suppose that $r = \sum g_{ij}e_{ij}$ is a minimal relation in $S(-3)^{\beta_{2,3}(S/J_G)}$. Here, $S(-3)^{\beta_{2,3}(S/J_G)}$ is $\mathbb{N}^n$ -graded because J_G is $\mathbb{N}^n$ -graded. It means $S(-3)^{\beta_{2,3}(S/J_G)}$ is generated by multihomogeneous elements. Then, we can say r is multihomogeneous and $\deg(r) = \deg(g_{ij}f_{ij}) = \mathbf{a} \in \mathbb{N}^n$ for each $\{i, j\} \in E(G)$ with $i < j$. Hence, $\mathbf{a} = \deg(g_{ij}) + \epsilon_i + \epsilon_j$. Since $\mathbf{a}$ is of degree 3, we observe $\deg(g_{ij}) = \epsilon_k$ for some k . One can easily see that the cases $k = i$ or $k = j$ are impossible because r can not be a relation of degree 3 in these cases. Thus, we have $k \neq i$ and $k \neq j$. Then, some of the followings should be provided for r .

$$r = x_k e_{ij} - x_j e_{ik} + x_i e_{jk} \quad (4.25)$$

$$r = y_k e_{ij} - y_j e_{ik} + y_i e_{jk} \quad (4.26)$$

$$r = (x_k + y_k) e_{ij} - (x_j + y_j) e_{ik} + (x_i + y_i) e_{jk} \quad (4.27)$$

$$r = (x_k - y_k) e_{ij} - (x_j - y_j) e_{ik} + (x_i - y_i) e_{jk}. \quad (4.28)$$

Here, we can see that (4.27) and (4.28) can be expressed as the sum and difference of (4.25) and (4.26). So, minimal relations are (4.25) and (4.26). Hence, each induced subgraph K_3 of G gives us 2 relations of degree 3 between the generators of J_G . Therefore, $\beta_{2,3}(S/J_G) = 2K_3(G)$.

ii) Clearly, G contains an induced path $P : i, j, k$ of length 2 as G is connected and noncomplete. Let us assume $i < j < k$. We claim that $r = f_{jk}e_{ij} - f_{ij}e_{jk}$ is a minimal relation in $S(-4)^{\beta_{2,4}(S/J_P)}$. Here, r is not reduced by the elements in $S(-3)^{\beta_{2,3}(S/J_P)}$. Actually, the minimal relations in $S(-3)^{\beta_{2,3}(S/J_P)}$ can be as in the equations (4.25) and (4.26). However, these relations can not be written as (4.25) and (4.26) because of $\{i, k\} \notin E(G)$. Hence, r can not be reduced and it is a minimal relation in $S(-4)^{\beta_{2,4}(S/J_P)}$. Therefore, we have $\beta_{2,4}(S/J_P) \neq 0$ and $\beta_{2,4}(S/J_G) \neq 0$ by Corollary 4.1.

Theorem 4.3: [8] Suppose that G is a graph on $[n]$ and $<$ is the lexicographic order

on S . Then, the followings are equivalent.

- i) The minimal relations between the generators of J_G are linear;
- ii) G is complete;
- iii) $\text{in}_<(J_G)$ can be generated by elements of degree 2 and it has linear quotients;
- iv) $\text{in}_<(J_G)$ has a linear resolution;
- v) J_G has a linear resolution.

Proof : i) $\Rightarrow$ ii). If G is not connected, there must be a minimal relation between the generators of J_G at least degree 4. However, since this would contradict our hypothesis, we observe that G is a connected graph. Now, let us assume that G is not complete. Then, we have $\beta_{2,4}(S/J_G) \neq 0$ from Lemma 4.6. However, this contradicts that the minimal relations between the generators of J_G are linear. Hence, we conclude that G is complete.

ii) $\Rightarrow$ iii). Suppose that G is complete. One can easily see that G is closed and $\text{in}_<(J_G) = (\{x_i y_j : 1 \leq i < j \leq n\})$ by Theorem 3.3. So, $\text{in}_<(J_G)$ can be generated by elements of degree 2. Now, let us show that $\text{in}_<(J_G)$ has linear quotients with respect to the ordering

$$x_1 y_2 > x_1 y_3 > \cdots > x_1 y_n > x_2 y_3 > x_2 y_4 > \cdots > x_2 y_n > \cdots > x_{n-1} y_n \quad (4.29)$$

of its generators. Let us call the generators in (4.29) $u_1, u_2, \dots, u_{\binom{n}{2}}$, respectively. For each $2 \leq t \leq \binom{n}{2}$, we can easily see that

$$((u_1, u_2, \dots, u_{t-1}) : u_t) = \left(\frac{u_1}{\gcd(u_1, u_t)}, \dots, \frac{u_{t-1}}{\gcd(u_{t-1}, u_t)} \right). \quad (4.30)$$

If $1 \leq j \leq n-2$, then we can write

$$((x_1 y_2, \dots, x_1 y_n, x_2 y_3, \dots, x_2 y_n, \dots, x_j y_n) : x_{j+1} y_{j+2}) = (x_1, x_2, \dots, x_j) \quad (4.31)$$

because of the equality (4.30) and $\frac{x_i y_{j+2}}{\gcd(x_i y_{j+2}, x_{j+1} y_{j+2})} = x_i$ for all $1 \leq i \leq j$. If $1 \leq i \leq n-2$ and $i < j < n$, then we can write

$$((x_1 y_2, \dots, x_1 y_n, x_2 y_3, \dots, x_2 y_n, \dots, x_i y_j) : x_i y_{j+1}) = (x_1, \dots, x_{i-1}, y_{i+1}, \dots, y_j). \quad (4.32)$$

from equality (4.30), $\frac{x_k y_{j+1}}{\gcd(x_k y_{j+1}, x_i y_{j+1})} = x_k$ and $\frac{x_i y_l}{\gcd(x_i y_l, x_i y_{j+1})} = y_l$ for any $1 \leq k < i$ and $i < l \leq j$, respectively. Hence, we conclude that $((u_1, u_2, \dots, u_{t-1}) : u_t)$ can be generated by variables for all $2 \leq t \leq \binom{n}{2}$ and $\text{in}_<(J_G)$ has linear quotients.

iii) $\Rightarrow$ iv). It is obvious by Proposition 2.20.

iv) $\Rightarrow$ v). It follows from Proposition 2.19.

v) $\Rightarrow$ i). It is clear by the definition of linear resolution.

Theorem 4.4: [24] Suppose that $K_{m,n}$ is the complete bipartite graph on $[n+m]$ and $n \leq m$. Then, the Betti diagram of $S/J_{K_{m,n}}$ has the form displayed in Table 4.2 where if $n = 1$, then $p = m$ and if $n > 1$, then $p = 2m + n - 2$.

Table 4.2: Betti diagram of $S/J_{K_{m,n}}$

	0	1	2	$\dots$	p
0	1	0	0	$\dots$	0
1	0	mn	0	$\dots$	0
2	0	0	$\beta_{2,4}$	$\dots$	$\beta_{p,p+2}$

Proposition 4.5: [25] Suppose that C_n is a cycle of length $n \geq 5$. Then, the Betti diagram of S/J_{C_n} is given as displayed in Table 4.3.

Lemma 4.7: [9] Let $I \subset R = K[x_1, \dots, x_n]$ and $J \subset T = K[y_1, \dots, y_m]$ be graded ideals and $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$ where $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ are disjoint sets of indeterminates. Also suppose that R/I and T/J has pure resolutions $\mathbb{F}$ and $\mathbb{G}$, respectively given as

$$\mathbb{F} : 0 \longrightarrow R(-d_p)^{\beta_{p,d_p}(R/I)} \longrightarrow \dots \longrightarrow R(-d_1)^{\beta_{1,d_1}(R/I)} \longrightarrow R \longrightarrow R/I \longrightarrow 0 \quad (4.33)$$

Table 4.3: Betti diagram of S/J_{C_n}

	0	1	2	$\dots$	$n-3$	$n-2$	$n-1$	n
0	$\binom{n}{0}$	0	0	$\dots$	0	0	0	0
1	0	$\binom{n}{1}$	0	$\dots$	0	0	0	0
2	0	0	$\binom{n}{2}$	$\dots$	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-3$	0	0	0	$\dots$	$\binom{n}{n-3}$	0	0	0
$n-2$	0	0	$(n-1)\binom{n}{0}$	$\dots$	$4\binom{n}{n-5}$	$\binom{n}{2} + 3\binom{n}{n-4}$	$2\binom{n}{n-3}$	$\binom{n-1}{2} - 1$

and

$$\mathbb{G} : 0 \longrightarrow T(-e_q)^{\beta_{q,e_q}(T/J)} \longrightarrow \dots \longrightarrow T(-e_1)^{\beta_{1,e_1}(T/J)} \longrightarrow T \longrightarrow T/J \longrightarrow 0. \quad (4.34)$$

Then, minimal graded free resolution of $S/(I+J)$ is pure if and only if $d_1 = e_1$, $e_i = ie_1$ and $d_j = jd_1$ for each $2 \leq i \leq p$ and for each $2 \leq j \leq q$.

Proof : Firstly, we know that $\mathbb{F} \otimes_K \mathbb{G}$ is the minimal graded free S -resolution of $S/(I+J)$ from Proposition 2.18. Then we have, $(\mathbb{F} \otimes_K \mathbb{G})_0 = R \otimes_K T = S$ and

$$(\mathbb{F} \otimes_K \mathbb{G})_1 = (R(-d_1)^{\beta_{1,d_1}(R/I)} \otimes_K T) \oplus (R \otimes_K T(-e_1)^{\beta_{1,e_1}(T/J)}) \quad (4.35)$$

$$= S(-d_1)^{\beta_{1,d_1}(R/I)} \oplus S(-e_1)^{\beta_{1,e_1}(T/J)}. \quad (4.36)$$

Also, we can write

$$(\mathbb{F} \otimes_K \mathbb{G})_i = \bigoplus_{i_1+i_2=i} \left(R(-d_{i_1})^{\beta_{i_1,d_{i_1}}(R/I)} \otimes_K T(-e_{i_2})^{\beta_{i_2,e_{i_2}}(T/J)} \right) \quad (4.37)$$

$$= \bigoplus_{i_1+i_2=i} S(-d_{i_1} - e_{i_2})^{\beta_{i_1,d_{i_1}}(R/I)\beta_{i_2,e_{i_2}}(T/J)} \quad (4.38)$$

for each $i > 0$ by assuming $R = R(-d_0)^{\beta_{0,d_0}(R/I)}$ and $T = T(-e_0)^{\beta_{0,e_0}(T/J)}$. Hence, equations (4.36) and (4.38) implies that $S/(I+J)$ has pure resolution if and only if $d_1 = e_1$ and $d_0 + e_k = d_1 + e_{k-1} = \dots = d_{k-1} + e_1 = d_k + e_0$ for all $2 \leq k \leq \max\{p, q\}$

if and only if $d_1 = e_1$, $d_i = id_1$ and $e_j = je_1$ for each $2 \leq i \leq p$ and $2 \leq j \leq q$.

Theorem 4.5: [9] Suppose that G is a graph on $[n]$ with no isolated vertices. Then, S/J_G has a pure resolution if and only if G satisfies one of the followings:

- i) G is complete;
- ii) G is complete bipartite;
- iii) G is a disjoint union of some paths.

Proof : ($\Leftarrow$) If G is complete graph, then S/J_G has a linear (and also pure) resolution from Theorem 4.3. If G is complete bipartite graph, then it has a pure resolution by Theorem 4.4. We know that the vertices of a path P of length l can be labeled such that the sequence of generators of J_P is S -regular from the proof of Proposition 4.2. Also, Theorem 2.13 implies $\beta_{i,j}(S/J_P) = \binom{l}{i}$ if $j = 2i$ and $0 \leq i \leq l$, otherwise $\beta_{i,j}(S/J_P) = 0$. It means S/J_P has a pure resolution. If G is disjoint union of $P_1, \dots, P_t$ where $P_1, \dots, P_t$ are paths, then the minimal graded free resolution of S/J_G is the tensor product of the minimal graded free resolutions of $S_1/J_{P_1}, \dots, S_t/J_{P_t}$ where S_i is the polynomial ring over K whose the indeterminates are the elements of the set $\{x_j, y_j : j \in V(P_i)\}$ for each i . So, S/J_G has a pure resolution by Lemma 4.7.

($\Rightarrow$) If the pure resolution of S/J_G is linear, then G is complete by Theorem 4.3. If the pure resolution of S/J_G is not linear, then G is not complete. Then, let us consider these two cases:

Case 1: Let us assume G is connected. By Lemma 4.6 we have $\beta_{2,4}(S/J_G) \neq 0$. As the minimal graded free resolution of S/J_G is pure, we have $\beta_{2,3}(S/J_G) = 0$. Thus, G does not contain C_3 as an induced subgraph from Lemma 4.6. Also, G does not contain C_m as an induced subgraph for $m \geq 5$ because of Proposition 4.5. So, G contains no induced odd cycles and hence we conclude that it contains no odd cycle. Then, we can say that G is bipartite graph from Proposition 2.9. If G is a tree, then $G = P_n$ or $G = K_{1,n-1}$ or G contains the graph H shown in Figure 4.1a as an induced subgraph. If $G = P_n$ or $G = K_{1,n-1}$, then G is a path or complete bipartite graph. If G contains H as an induced subgraph, then Example 4.1 and Corollary 4.1 implies that $\beta_{3,6}(S/J_G) \neq 0$ and $\beta_{3,5}(S/J_G) \neq 0$ since H contains P_4 and $K_{1,3}$ as induced

subgraphs. However, this contradicts that S/J_G has a pure resolution. Then, let us examine the situation where G is not a tree and V, W are the partition sets of $V(G)$ with $|V| \leq |W|$. We will show that G is complete bipartite graph by induction on n . If $n = 4$, then $G = C_4$ is complete bipartite graph. Assume that every nontree connected graph G' with less than n vertices whose binomial edge ideal has a pure resolution is complete bipartite. Clearly, we have $|V| \geq 2$, $|W| \geq 2$ and G has an induced cycle $C : v, w, v', w', v$ of length 4 where $v, v' \in V$ and $w, w' \in W$. By Corollary 4.1 $G - v$ has a pure resolution. Also, $G - v$ is bipartite. If $G - v$ is disconnected, then G contains the graph H' displayed in Figure 4.1b. Then, Example 4.1 and Corollary 4.1 implies $\beta_{3,6}(S/J_G) \neq 0$ and $\beta_{3,5}(S/J_G) \neq 0$ as H' contains P_4 and $K_{1,3}$ as induced subgraphs but this contradicts that S/J_G has a pure resolution. Hence, $G - v$ is connected graph. If $G - v$ is a tree, then $G - v = P_{n-1}$ or $G - v = K_{1,n-2}$. If $G - v = P_{n-1}$, then G contains H' which is a contradiction. If $G - v = K_{1,n-2}$, then $G = K_{2,n-2}$, otherwise G contains H' as an induced subgraph which is a contradiction. If $G - v$ is nontree, then $G - v$ is complete bipartite graph by induction hypothesis. Thus, G is complete bipartite. Actually, if G is not complete bipartite, G contains H' as induced subgraph but this contradicts the fact that J_G has pure resolution. To sum up, we conclude that $G = P_n$ or G is a complete bipartite.

Case 2: Suppose that G is not connected graph and $G_1, \dots, G_t$ are the connected components of G . Then, S/J_{G_i} has a pure resolution for all $1 \leq i \leq t$ from Corollary 4.1. Then, we can say that G_i is complete or complete bipartite or path for each $i = 1, \dots, t$. Also, we observe that $S/J_{G_i \cup G_j}$ has a pure resolution by Corollary 4.1 whenever $i \neq j$. If at least one of G_i or G_j is complete, then Lemma 4.7 implies that $S/J_{G_i \cup G_j}$ has not a pure resolution since at least one of S/J_{G_i} or S/J_{G_j} has a linear resolution. If G_i or G_j is complete bipartite, then $S/J_{G_i \cup G_j}$ has not a pure resolution because of Lemma 4.7 and Theorem 4.4. Lastly, let us consider the case that both of G_i and G_j are paths. Then, $S/J_{G_i \cup G_j}$ has a pure resolution by Lemma 4.7. Therefore, G is a disjoint union of some paths.

4.3. Combinatorial Upper Bounds for Regularity

In this section, we will give some combinatorial upper bounds for $\text{reg}(S/J_G)$. Firstly, we will introduce some notations that are useful for graphs. Also, in The-

orem 4.7, we will study a result of Ene, Rinaldo and Terai in [6] which states that $\text{reg}(S/J_G) \leq n - \dim \Delta(G)$ for any connected graph G on $[n]$ where $\Delta(G)$ is the clique complex of G . In addition, we will mention of a result of Malayeri, Madani and Kiani in [7] which states that $\text{reg}(S/J_G)$ is less than or equal to the number of maximal cliques of G .

Firstly, let us introduce some notations. Let G be a graph on $[n]$. With $G - e$, we denote the subgraph of G whose vertex set is $V(G)$ and whose edge set is $E(G) \setminus \{e\}$ for any $e \in E(G)$. If $e \notin E(G)$, we represent the graph obtained by including the edge e to the graph G as $G \cup e$. In addition, if $e = \{u, v\}$ is not an edge of G , then G_e represents the graph with the vertex set $V(G_e) = V(G)$ and the edge set $E(G_e) = E(G) \cup \{\{i, j\} : i, j \in N(u)\} \cup \{\{k, l\} : k, l \in N(v)\}$ (see [6]).

Proposition 4.6: [23] Suppose that H is a graph on $[n]$ and $e = \{i, j\}$ is an edge of H . By setting $f_e = f_{ij}$, the following inequalities are satisfied:

- i) $\text{reg}(J_H) \leq \max\{\text{reg}(J_{H-e}), \text{reg}((J_{H-e} : f_e)) + 1\};$
- ii) $\text{reg}(J_{H-e}) \leq \max\{\text{reg}(J_H), \text{reg}((J_{H-e} : f_e)) + 2\};$
- iii) $\text{reg}((J_{H-e} : f_e)) + 2 \leq \max\{\text{reg}(J_{H-e}), \text{reg}(J_H) + 1\}.$

Proof : Let us consider the exact sequence

$$\mathbb{F} : 0 \longrightarrow S/(J_{H-e} : f_e)(-2) \xrightarrow{\phi} S/J_{H-e} \xrightarrow{\varphi} S/J_H \longrightarrow 0 \quad (4.39)$$

where $\phi(g + (J_{H-e} : f_e)) = gf_e + J_{H-e}$ and $\varphi(g + J_{H-e}) = g + J_H$ for any $g \in S$. Clearly, ϕ and φ have degree 0. Then, the desired inequalities follow from Proposition 2.17 and the equation $\text{reg}((J_{H-e} : f_e)(-2)) = \text{reg}((J_{H-e} : f_e)) + 2$.

Theorem 4.6: [26] Suppose that G is a graph on $[n]$ and $e = \{i, j\} \notin E(G)$. Then,

$$(J_G : f_e) = J_{G_e} + (\{g_{P,t} : P : i, i_1, \dots, i_s, j \text{ is a path in } G \text{ and } 0 \leq t \leq s\}) \quad (4.40)$$

where $g_{P,t} = y_{i_1} \dots y_{i_t} x_{i_{t+1}} \dots x_{i_s}$ for each $t = 0, 1, \dots, s$.

Definition 4.4: [6] Suppose that G is a graph on $[n]$. For any $v \in V(G)$, we define $\alpha_G(v) = \binom{\deg v}{2} - |E(G[N(v)])|$. Also, we set $\alpha_G(W) = \min\{\alpha_G(v) : v \in V(G) \setminus W\}$ for any $W \subset [n]$.

Note that if $\deg v = 0$ or $\deg v = 1$, then we assume $\binom{\deg v}{2} = 0$. Also, we can say that $\alpha_G(v) = 0$ if and only if v is a simplicial vertex of G . Similarly, $\alpha_G(W) = 0$ if and only if there exists $v \in V(G) \setminus W$ such that v is a simplicial vertex in G .

Theorem 4.7: [6] Suppose that G is a connected graph on $[n]$. Then,

$$\text{reg}(S/J_G) \leq n - \dim \Delta(G) \quad (4.41)$$

where $\Delta(G)$ is the clique complex of G .

Proof : To prove this theorem, we will apply double induction. We will show that the inequality

$$\text{reg}(J_G) \leq n - |W| + 2 \quad (4.42)$$

is provided for all $W \in \Delta(G)$. We can prove that inequality (4.42) is provided by using induction on $n - |W|$. Clearly, if $n - |W| = 0$, then $n = |W|$ and G is complete graph on $[n]$. Then, Theorem 4.3 implies $\text{reg}(J_G) = 2$ and hence, inequality (4.42) is satisfied. Let us assume $n - |W| > 0$.

Now, we will work with induction on $\alpha_G(W)$. If $\alpha_G(W) = 0$, then $V(G) \setminus W$ has a simplicial vertex v in G .

If $\deg v = 1$, v has only one neighbour v' in G . Let us call $e = \{v, v'\} \in E(G)$. Then, v is an isolated vertex in $G - e$ and $\text{reg}(J_{G-e}) = \text{reg}(J_{(G-e)-v})$. Here, we observe $\text{reg}(J_{(G-e)-v}) \leq (n-1) - |W| + 2 = n - |W| + 1$ from induction hypothesis. Hence, we have

$$\text{reg}(J_{G-e}) \leq n - |W| + 1. \quad (4.43)$$

Also, Theorem 4.6 allows us to write $(J_{G-e} : f_e) = J_{(G-e)_e}$ since there is no path between v and v' in $G - e$. As v is an isolated vertex in $(G - e)_e$, we can write

$\text{reg}((J_{G-e} : f_e)) = \text{reg}(J_{(G-e)_e}) = \text{reg}(J_{(G-e)_e-v})$. Then, induction hypothesis implies $\text{reg}(J_{(G-e)_e-v}) \leq n - |W| + 1$. So, we have

$$\text{reg}((J_{G-e} : f_e)) \leq n - |W| + 1. \quad (4.44)$$

Then, inequalities (4.43) and (4.44) implies $\text{reg}(J_G) \leq n - |W| + 2$ from i) of Proposition 4.6

If $\deg v \geq 2$, we claim that the inequality

$$\text{reg}((J_{G-e} : f_e)) \leq n - |W| + 1 \quad (4.45)$$

is provided for any edge e whose endpoint is v . Let $v_1, \dots, v_t$ be all neighbours of v in G and let $e_i = \{v, v_i\}$ for each $i = 1, \dots, t$. Let us call $e = e_t = \{v, v_t\}$. Since v is a simplicial vertex, $P_i : v, v_i, v_t$ is a path between v and v_t for each $i = 1, \dots, t-1$ and every path between v and v_t must pass through at least one of the v_i 's. Hence, we can write $(J_{G-e} : f_e) = J_{(G-e)_e} + (x_{v_1}, y_{v_1}, \dots, x_{v_{t-1}}, y_{v_{t-1}})$ from Theorem 4.6. Hence, we observe $(J_{G-e} : f_e) = J_{(G-e)_e - \{v_1, \dots, v_{t-1}\}} + (x_{v_1}, y_{v_1}, \dots, x_{v_{t-1}}, y_{v_{t-1}})$. Here, v is an isolated vertex in the graph $(G-e)_e - \{v_1, \dots, v_{t-1}\}$. Thus, we must have $(J_{G-e} : f_e) = J_{(G-e)_e - \{v, v_1, \dots, v_{t-1}\}} + (x_{v_1}, y_{v_1}, \dots, x_{v_{t-1}}, y_{v_{t-1}})$. Clearly, the graph $(G-e)_e - \{v, v_1, \dots, v_{t-1}\}$ has $n - t$ vertices and the set $W \setminus \{v, v_1, \dots, v_{t-1}\}$ may have at least $|W| - (t-1)$ elements because of $v \notin W$. Proposition 2.22 and induction hypothesis allows us to write

$$\text{reg}((J_{G-e} : f_e)) = \text{reg}(J_{(G-e)_e - \{v, v_1, \dots, v_{t-1}\}}) \quad (4.46)$$

$$\leq (n - t) - |W \setminus \{v, v_1, \dots, v_{t-1}\}| + 2 \quad (4.47)$$

$$\leq (n - t) - (|W| - t + 1) + 2 \quad (4.48)$$

$$= n - |W| + 1. \quad (4.49)$$

Hence, the inequality (4.45) is provided.

If we apply i) of Proposition 4.6 and the inequality (4.45) to the graph G , we can write

$$\text{reg}(J_G) \leq \max\{\text{reg}(J_{G-e_1}), n - |W| + 2\}. \quad (4.50)$$

By using similar argument for the graphs $G - e_1, G - \{e_1, e_2\}, \dots, G - \{e_1, \dots, e_{t-2}\}$, we can write the equations

$$\text{reg}(J_{G-e_1}) \leq \max\{\text{reg}(J_{G-\{e_1, e_2\}}), n - |W| + 2\} \quad (4.51)$$

$$\text{reg}(J_{G-\{e_1, e_2\}}) \leq \max\{\text{reg}(J_{G-\{e_1, e_2, e_3\}}), n - |W| + 2\} \quad (4.52)$$

$\vdots$

$$\text{reg}(J_{G-\{e_1, \dots, e_{t-2}\}}) \leq \max\{\text{reg}(J_{G-\{e_1, \dots, e_{t-1}\}}), n - |W| + 2\}. \quad (4.53)$$

The inequalities (4.50), (4.51), (4.52), $\dots$, (4.53) allow us to write

$$\text{reg}(J_G) \leq \max\{\text{reg}(J_{G-\{e_1, \dots, e_{t-1}\}}), n - |W| + 2\}. \quad (4.54)$$

Obviously, v has only one neighbour in $G - \{e_1, \dots, e_{t-1}\}$. Hence, we can write $\text{reg}(J_{G-\{e_1, \dots, e_{t-1}\}}) \leq n - |W| + 2$ as in case of $\deg v = 1$. From inequality (4.54), we deduce $\text{reg}(J_G) \leq n - |W| + 2$.

Now, let $\alpha_G(W) > 0$. Then, no element of $V(G) \setminus W$ is simplicial in G . Let us choose a vertex v in $V(G) \setminus W$ satisfying $\alpha_G(v) = \alpha_G(W)$. As v is not a simplicial vertex, there exist $v_1, v_2 \in N(v)$ such that $\{v_1, v_2\} \notin E(G)$. Let us call $e = \{v_1, v_2\}$ and consider the graph $G \cup e$. Then,

$$\alpha_{G \cup e}(v) = \binom{\deg v}{2} - |E((G \cup e)[N(v)])| \quad (4.55)$$

$$= \binom{\deg v}{2} - |E(G[N(v)])| - 1 \quad (4.56)$$

$$= \alpha_G(v) - 1. \quad (4.57)$$

So, $\alpha_{G \cup e}(W) \leq \alpha_{G \cup e}(v) < \alpha_G(W)$ and induction hypothesis implies

$$\text{reg}(J_{G \cup e}) \leq n - |W| + 2. \quad (4.58)$$

Apart from these, we can write

$$(J_G : f_e) = J_{G_e} + (\{g_{P,t} : P \text{ is a path from } v_1 \text{ to } v_2 \text{ in } G\}) \quad (4.59)$$

from Theorem 4.6. Clearly, $(\{g_{P,t} : P \text{ is a path from } v_1 \text{ to } v_2 \text{ in } G\})$ contains the variables x_v and y_v because $P' : v_1, v, v_2$ is a path in G . Hence, we observe

$$(J_G : f_e) = J_{(G-v)_e} + (\{g_{P,t} : P \text{ is a path from } v_1 \text{ to } v_2 \text{ in } G - v\}) + (x_v, y_v) \quad (4.60)$$

and if we choose $H = G - v$ in Theorem 4.6 we see

$$(J_{G-v} : f_e) = J_{(G-v)_e} + (\{g_{P,t} : P \text{ is a path from } v_1 \text{ to } v_2 \text{ in } G - v\}). \quad (4.61)$$

Equations (4.60), (4.61) and Proposition 2.22 give

$$\text{reg}(J_G : f_e) = \text{reg}(J_{G-v} : f_e) \quad (4.62)$$

Note that induction hypothesis implies

$$\text{reg}(J_{G-v}) \leq (n - 1) - |W| + 2 = n - |W| + 1 \quad (4.63)$$

and

$$\text{reg}(J_{(G-v) \cup e}) \leq (n - 1) - |W| + 2 = n - |W| + 1. \quad (4.64)$$

If we choose $H = (G - v) \cup e$ in iii) of Proposition 4.6 we find

$$\text{reg}(J_{G-v} : f_e) + 2 \leq n - |W| + 2 \quad (4.65)$$

because of equations (4.63) and (4.64). Also, equations (4.62) and (4.65) give

$$\text{reg}(J_G : f_e) \leq n - |W|. \quad (4.66)$$

Finally, if we choose $H = G \cup e$ in ii) of Proposition 4.6, then inequalities (4.58) and (4.66) imply $\text{reg}(J_G) \leq n - |W| + 2$.

Corollary 4.3: [6] Suppose that G is a graph on $[n]$ and $G_1, G_2, \dots, G_t$ are the connected components of G . Then,

$$\text{reg}(S/J_G) \leq n - \dim \Delta(G_1) - \dim \Delta(G_2) - \dots - \dim \Delta(G_t). \quad (4.67)$$

Proof : Suppose that S_i is the polynomial ring over K whose the indeterminates are the elements of the set $\{x_j, y_j : j \in V(G_i)\}$ and suppose $|V(G_i)| = n_i$ for each $i = 1, 2, \dots, t$. We can write

$$S/J_G \cong S_1/J_{G_1} \otimes_K S_2/J_{G_2} \otimes_K \dots \otimes_K S_t/J_{G_t}. \quad (4.68)$$

from Proposition 2.12. Then, Theorem 4.7 allows us to write

$$\text{reg}(S/J_G) = \text{reg}(S_1/J_{G_1}) + \text{reg}(S_2/J_{G_2}) + \dots + \text{reg}(S_t/J_{G_t}) \quad (4.69)$$

$$\leq (n_1 - \dim \Delta(G_1)) + (n_2 - \dim \Delta(G_2)) + \dots + (n_t - \dim \Delta(G_t)) \quad (4.70)$$

$$= (n_1 + n_2 + \dots + n_t) - \dim \Delta(G_1) - \dim \Delta(G_2) - \dots - \dim \Delta(G_t) \quad (4.71)$$

$$= n - \dim \Delta(G_1) - \dim \Delta(G_2) - \dots - \dim \Delta(G_t) \quad (4.72)$$

Theorem 4.8: [7] Let G be a graph on $[n]$. Then, $\text{reg}(S/J_G) \leq r$ where r is the number of maximal cliques of G .

5. CONCLUSION

In this thesis, we studied the algebraic and homological properties of binomial edge ideals through several articles in the literature. In Chapter 3, we have given the combinatorial properties of closed graphs, which have an important place for binomial edge ideals. Then, we examined when a binomial edge ideal has a quadratic Gröbner basis and gave a method for calculating the reduced Gröbner basis of J_G for any graph G . Also, we presented the algebraic properties of J_G and determined its minimal prime ideals. In the next chapter, we investigated the properties provided by the graded Betti numbers and regularity of binomial edge ideals. Then, we gave lower and upper bounds for the regularity of S/J_G , that is, $l \leq \text{reg}(S/J_G) \leq n - 1$ where l is the longest induced path in G . In addition, we gave necessary and sufficient conditions for binomial edge ideals to have linear or pure resolutions. Moreover, we showed that $n - \dim \Delta(G)$ is an upper bound for $\text{reg}(S/J_G)$, and we mentioned that the number of maximal cliques of G is another upper bound. To sum up, this thesis has gathered a lot of background information that would be of use to those who are interested in binomial edge ideals. In the future, minimal graded free resolutions and regularities of binomial edge ideals corresponding to some special graph classes can be researched such as trees, complements of cycles and regular graphs.

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