

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL

**ON LOCALLY CONFORMAL DYNAMICS AND
KINETIC THEORIES**



AYTEN GEZİCİ

A THESIS OF DOCTORATE
DEPARTMENT OF MATHEMATICS

ADVISOR: PROF. DR. OĞUL ESEN

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ABSTRACT

Jacobi manifolds are integrable structures whose odd-dimensional leaves are contact manifolds, while the even-dimensional leaves are locally conformal symplectic manifolds. In this thesis, we focus on these geometric structures.

Within the locally conformal framework, we present both Hamiltonian and Lagrangian dynamics. As a novel result, we derive the locally conformal Euler–Lagrange equations, which extend the classical Euler–Lagrange equations. Furthermore we formulate kinetic dynamics in this setting and derive the locally conformal momentum–Vlasov equations and the locally conformal Vlasov equations. In addition, we establish a geometric pathway from particle motion to kinetic dynamics in the locally conformal framework, using the notions of lifts and vertical representatives.

From the perspective of discrete dynamics, we derive the locally conformal Lagrangian formulation. We present the locally conformal discrete Euler–Lagrange equations, which generalize the classical discrete Euler–Lagrange equations. By applying the Legendre transformation, we obtain the locally conformal discrete Hamiltonian equations. In addition, we develop higher-order locally conformal discrete Lagrangian dynamics and derive the second- and third-order locally conformal discrete Euler–Lagrange equations.

We also study another framework, known as conformal Hamiltonian dynamics. In this setting, we consider conformal Hamiltonian vector fields, which have non-zero divergence. We work with the Lie algebra of these vector fields and introduce conformal kinetic dynamics in both momentum and density formulations. As a result, we derive conformal kinetic equations for each formulation. In addition, we present a geometric description of conformal kinetic dynamics in terms of momenta, using vertical representatives.

In the contact framework, we present the Hamiltonian dynamics. This structure includes a contact one-form and the associated Reeb vector field. Using the contact one-form, we also define a bivector field. Together, the bivector and the Reeb vector field make the contact manifold a Jacobi manifold. We then formulate kinetic dynamics in terms of momenta and the density function. As a result, we derive the contact momentum–Vlasov equation and the contact Vlasov equation.

Keywords: Jacobi Manifold, Kinetic Dynamics, Locally Conformally Symplectic Manifold, Contact Manifold, Discrete Dynamics

ÖZET

Jacobi katmanları, tek boyutlu altkatmanları kontakt katmanlar, çift boyutlu altkatmanları ise yerel konformal simplektik katmanlar olan integrallenebilir yapılardır. Bu tez, kontakt ve yerel konformal simplektik katmanlar üzerine odaklanmaktadır.

Yerel konformal çerçevede, hem Hamilton hem de Lagrange dinamiği sunulmuştur. Yeni bir sonuç olarak, klasik Euler–Lagrange denklemlerinin bir genellemesi olan yerel konformal Euler–Lagrange denklemleri türetilmiştir. Bununla birlikte, kinetik dinamik formüle edilmiş ve yerel konformal momentum–Vlasov denklemleri ile yerel konformal Vlasov denklemleri elde edilmiştir. Ek olarak, parçacık hareketinden kinetik dinamiğe uzanan geometrik bir geçiş yolu kurulmuştur; bu yapı, lift ve dikey temsilci kavramları kullanılarak inşa edilmiştir.

Yerel konformal çerçevede, kesikli Lagrange yapı elde edilmiştir. Klasik kesikli Euler–Lagrange denklemlerinin bir genellemesi olan yerel konformal kesikli Euler–Lagrange denklemleri sunulmuştur. Legendre dönüşümü uygulanarak yerel konformal kesikli Hamilton denklemler elde edilmiştir. Ayrıca, daha yüksek mertebeden yerel konformal kesikli Lagrange dinamik geliştirilmiş ve ikinci ile üçüncü mertebeden yerel konformal kesikli Euler–Lagrange denklemleri türetilmiştir.

Ayrıca, konformal Hamilton dinamik olarak bilinen başka bir çerçeveyi de incelemekteyiz. Bu bağlamda, diverjansları sıfırdan farklı konformal Hamilton vektör alanları ele alınmıştır. Bu vektör alanlarının Lie cebiri ile çalışılmış, konformal kinetik dinamik momentum ve yoğunluk formülasyonlarında tanıtılmıştır. Sonuç olarak, her bir formülasyon için konformal kinetik denklemler türetilmiştir. Ayrıca, momenta cinsinden konformal kinetik dinamik, dikey temsilciler kullanılarak geometrik olarak da ifade edilmiştir.

Kontakt çerçevesinde, Hamilton dinamiği sunulmuştur. Kontakt katman bir kontakt bir-formu ve ilişkili Reeb vektör alanını içerir. Kontakt bir-formunu kullanarak, ayrıca bir bivektör alanı tanımlanmıştır. Kontakt katman, bivektör ve Reeb vektör alanı birlikte bir Jacobi katmanıdır. Ardından kinetik dinamik, momenta ve yoğunluk fonksiyonu cinsinden formüle edilmiştir. Sonuç olarak kontakt momentum–Vlasov denklemi ve kontakt Vlasov denklemi türetilmiştir.

Anahtar Kelimeler: Jacobi Katman, Kinetik Dinamik, Yerel Konformal Simplektik Katman, Kontakt Katman, Kesikli Dinamik

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TABLE OF CONTENTS

	<u>Page</u>
ABSTRACT	v
ÖZET	vi
ACKNOWLEDGEMENT	vii
TABLE of CONTENTS	viii
LIST OF SYMBOLS AND ABBREVIATIONS	x
 1. INTRODUCTION	 1
2. FUNDAMENTALS OF GEOMETRIC MECHANICS	7
2.1. Symplectic and Poisson Manifolds	7
2.2. Hamiltonian Dynamics	9
2.3. Lagrangian Dynamics	13
2.4. Presymplectic Formulation of Lagrange Dynamics	15
2.5. Lie Groups and Lie Algebras	16
2.6. Lie-Poisson Dynamics and Coadjoint Flow	19
2.7. Jacobi Manifolds	20
2.8. Complete Lift and Jet Decomposition of Vector Fields	21
3. LOCALLY CONFORMAL DYNAMICS	26
3.1. Locally Conformally Symplectic Manifolds	26
3.2. Cotangent Bundle and Coordinate Realizations	27
3.3. Locally Conformally Symplectic Hamiltonian Dynamics	30
3.4. Locally Conformally Symplectic Manifold as a Jacobi Manifold	34
3.5. Locally Conformal Lagrangian Dynamics	36
4. DISCRETE DYNAMICS ON LOCALLY CONFORMAL FRAMEWORK	42
4.1. Discrete Lagrangian and Discrete Hamiltonian Dynamics	42
4.2. Locally Conformal Discrete Lagrangian Dynamics	45
4.3. Locally Conformal Discrete Hamiltonian Dynamics	50
4.4. Second-Order Discrete Locally Conformal Lagrangian Dynamics	53
4.5. Third-Order Discrete Locally Conformal Lagrangian Dynamics	56
5. LOCALLY CONFORMAL KINETIC DYNAMICS	59
5.1. Kinetic Lift of LCS Hamiltonian Dynamics in Momentum Variables	59
5.2. Kinetic Lift of LCS Hamiltonian Dynamics in Densities	63
5.3. A Geometric Pathway to Locally Conformal Kinetic Motion	69
6. CONFORMAL KINETIC DYNAMICS	76
6.1. Conformal Hamiltonian Dynamics	76
6.2. Lie Algebra of Conformal Vector Fields	78
6.3. Conformal Kinetic Dynamics in Momentum Formulation	79

6.4. Conformal Kinetic Dynamics in Density Formulation	82
6.5. From Momentum to Density Formulations	84
6.6. Kinetic Conformal Dynamics as Vertical Representative	86
7. CONTACT KINETIC DYNAMICS	87
7.1. Contact Manifolds	87
7.2. Dynamics on Contact Manifolds	90
7.3. Contact Kinetic Dynamics in Momentum Formulation	93
7.4. Contact Kinetic Dynamics in Density Formulation	98
7.5. A Hierarchy from Contact to Conformal Dynamics	101
8. CONCLUSIONS	104
REFERENCES	105
BIOGRAPHY	112
PUBLICATIONS AND PRESENTATIONS FROM THE THESIS	113
APPENDICES	114

LIST OF SYMBOLS AND ABBREVIATIONS

GTÜ	:	Gebze Teknik Üniversitesi
HE	:	Hamilton's Equation
LCS	:	Locally Conformal Symplectic
LCEL	:	Locally Conformal Euler-Lagrange
dLCS	:	Discrete Locally Conformal Symplectic
dHE	:	Discrete Hamilton's Equation
LdR	:	Lichnerowicz-deRham



1. INTRODUCTION

In its most general form, we consider a continuum as a collection of particles, each of which obeys Newton's second law of motion:

$$m\ddot{x} = F(x), \quad (1.1)$$

where m represents the mass of a particle and F denotes the force acting on it, [1, 2]. To study the dynamics of a continuum, kinetic theory employs statistical methods instead of formulating the Newtonian equations of motion for individual particles and accounting for all possible interactions. Kinetic theory focuses on the evolution of a probability density function that describes the distribution of particles in phase space. Before delving into the details of geometric kinetic theory, let us first provide a brief overview of the contemporary understanding of classical dynamics.

Reversible Hamiltonian Dynamics. A coordinate-free and elegant formulation of Newton's second law (1.1) is provided by symplectic geometry, a foundational framework in classical mechanics that is central to both pure mathematical research and its applications to physical problems via the Hamiltonian formulation. A symplectic manifold is a pair (M, Ω) , where M is a manifold and Ω is a symplectic two-form, which is closed and non-degenerate [3, 4]. In this context, Hamilton's equation is written as

$$\iota_{X_H}\Omega = dH, \quad (1.2)$$

where H is the Hamiltonian function, X_H is the Hamiltonian vector field generating the motion, and ι_{X_H} is the contraction operator. Applying the contraction operator and taking the exterior derivative of both sides of Hamilton's equation, we obtain the conservation laws:

$$\mathcal{L}_{X_H}H = 0, \quad \mathcal{L}_{X_H}\Omega = 0, \quad (1.3)$$

respectively. Here, $\mathcal{L}_{X_H}$ denotes the Lie derivative. The first equation in (1.3) states that the Hamiltonian function, which corresponds to the particle energy, is conserved. The second equation indicates that the symplectic two-form, and

consequently the symplectic volume, is conserved. This reflects the reversible character of Hamiltonian dynamics in symplectic framework.

We note that cotangent bundles are naturally equipped with canonical symplectic two-forms and serve as prototypical examples of symplectic manifolds. If the symplectic manifold is identified as the momentum phase space, specifically the cotangent bundle T^*Q of the configuration space Q , and the Hamiltonian function is defined as the total mechanical energy of a single particle, then the Hamilton's equations (1.2) reduce to the Newton's second law (1.1).

Due to the non-degeneracy condition, the dimension of a symplectic manifold must always be even. Poisson geometry generalizes symplectic geometry by relaxing the non-degeneracy condition, thereby making it applicable in any dimension. A Poisson manifold M admits a skew-symmetric algebra, known as Poisson algebra,

$$\{\bullet, \bullet\}^{(P)} : \mathcal{F}(M) \times \mathcal{F}(M) \longrightarrow \mathcal{F}(M) \quad (1.4)$$

on the space of smooth functions on the manifold, [5]. The bracket is asked to satisfy the Leibniz and Jacobi identities. In the Poisson setting, Hamilton's equation is written as

$$\dot{z} = \{z, H\}^{(P)}. \quad (1.5)$$

After a direct substitution in (1.5), it is immediate to see that the Hamiltonian function is preserved:

$$\dot{H} = \{H, H\}^{(P)} = 0, \quad (1.6)$$

due to the skew-symmetry. This implies that the Hamiltonian dynamics is reversible within the Poisson framework as well.

Reduction Under Symmetry. Although several dynamical systems can be recast in Hamiltonian formulation, either in the symplectic picture as in (1.2) or in the Poisson picture as in (1.5), there are physical systems that require more detailed examination when performing their Hamiltonian analysis. One of the tools for analyzing such systems, which do not fit the canonical pictures, is the reduction method, specifically symplectic reduction [4] and Poisson reduction [3].

Reduction arises as a result of a symmetry possessed by system. This symmetry is encoded in a Lie group action on configuration, momentum, or state spaces. In

geometric mechanics, the reduction method is used to derive the reduced dynamics on the reduced (quotient) manifold.

For a given physical system, symmetry analysis is far from trivial. One exception occurs if the configuration space of a physical system is a Lie group, and the symmetry is due to the same Lie group. The configuration spaces of many physical models in rigid body dynamics, fluid dynamics, and plasma dynamics are Lie groups. Consequently, this reduction holds significant physical interpretations and offers various applications. Let us examine this geometry in a bit more detail.

Lie-Poisson Formalism. Assume that the configuration space of a physical system is a Lie group G . Then, one can compute the lifted actions of the group on its tangent bundle TG and cotangent bundle T^*G . The quotient spaces $\mathfrak{g} \simeq TG/G$ and $\mathfrak{g}^* \simeq T^*G/G$ correspond to the Lie algebra and its dual space, respectively. On the dual space $\mathfrak{g}^*$, there exists a Poisson structure known as the Lie-Poisson bracket [6–8]:

$$\{A, B\}(\rho) = \pm \left\langle \rho, \left[\frac{\delta A}{\delta \rho}, \frac{\delta B}{\delta \rho} \right] \right\rangle, \quad (1.7)$$

where A and B are two function(al)s on $\mathfrak{g}^*$, and the bracket on the right hand side is the Lie algebra bracket. Notice that $\delta A / \delta \rho$ denotes the Fréchet derivative of the functional A , assuming reflexivity. Notice that there is a $\pm$ in front of the Lie-Poisson bracket in equation (1.7). If the symmetry is due to the right action (as in, for example, plasma dynamics), the $+$ bracket is used; on the other hand, if the symmetry is due to the left action (as in, for example, rigid body dynamics), the $-$ bracket is used. Given a Hamiltonian function(al) H , the dynamics is governed by the Lie-Poisson equations computed in terms of the coadjoint action as

$$\dot{\rho} = \mp ad_{\delta H / \delta \rho}^* \rho, \quad (1.8)$$

where ad^* is the coadjoint action of the Lie algebra on the dual space.

Geometric Plasma Motion. The dynamics of a non-relativistic and collisionless plasma resting in $Q \subset \mathbb{R}^3$ is determined by the plasma density function $f = f(q^i, p_i)$, defined on the momentum phase space T^*Q with Darboux coordinates

(q^i, p_i) . Equation of motion then is a coupled integrodifferential system

$$\begin{aligned} \frac{\partial f}{\partial t} + \frac{1}{m} p_i \frac{\partial f}{\partial q^i} - e \frac{\partial \phi}{\partial q^i} \frac{\partial f}{\partial p_i} &= 0 \\ \nabla_q^2 \phi_f(q) &= -e \int f(q, p) d^3 p \end{aligned} \quad (1.9)$$

which are known as the Vlasov–Poisson equations, where e is the charge and ϕ is the potential. Hamiltonian analysis of this system may be recalled from [9, 10], wherein it is well established that the Vlasov–Poisson system (1.9) admits Hamiltonian formulation.

In a series of papers [11, 12], while investigating Lie–Poisson formulation of the Vlasov equation, an intermediate level of description is obtained on the space of one-forms on T^*Q . Let us briefly recall this approach. We assume the Vlasov plasma to be a continuum resting in $Q \subset \mathbb{R}^3$. The momentum phase space is the cotangent bundle T^*Q which is canonically symplectic. The dynamics is determined by the left action of the group $\text{Diff}_{\text{can}}(T^*Q)$ of canonical diffeomorphisms on the cotangent bundle T^*Q . The right action of $\text{Diff}_{\text{can}}(T^*Q)$ on T^*Q defines the symmetry of the motion and is referred to as the particle relabeling. This symmetry induces the symmetry on the tangent bundle $T\text{Diff}_{\text{can}}(T^*Q)$ and cotangent bundle $T^*\text{Diff}_{\text{can}}(T^*Q)$. The reduction of the tangent bundle reads the Lie algebra $\mathfrak{X}_{\text{S-ham}}(T^*Q)$ of Hamiltonian vector fields equipped with minus Jacobi–Lie bracket.

In [11, 12], the dual space of the Hamiltonian vector fields $\mathfrak{X}_{\text{S-ham}}(M)$ is computed to be

$$\mathfrak{X}_{\text{S-ham}}^*(M) = \left\{ \Pi \otimes d\mu \in \Gamma^1(M) \otimes \text{Den}(M) : \text{div} \Omega^\sharp(\Pi) \neq 0 \right\} \cup \{0\} \quad (1.10)$$

and the dual operation of the Lie algebra homomorphism $\hat{\Phi}$ is

$$\hat{\Phi}^* : \mathfrak{X}_{\text{S-ham}}^*(M) \longrightarrow \text{Den}(M), \quad \Pi \otimes d\mu \mapsto f d\mu = (\text{div} \Omega^\sharp(\Pi)) d\mu. \quad (1.11)$$

In this pure symplectic case, the coadjoint flow turns out to be

$$ad_{X_H}^* \Pi = \mathcal{L}_{X_H} \Pi \quad (1.12)$$

while the Lie–Poisson dynamics becomes the momentum–Vlasov equations

$$\dot{\Pi} = -\mathcal{L}_{X_H} \Pi. \quad (1.13)$$

In this case, the dynamics is represented by the evolution of a dual element, more precisely a one-form Π , governed by Hamiltonian vector field X_H through

$$\dot{\Pi} = -\mathcal{L}_{X_H}\Pi \quad (1.14)$$

where $\mathcal{L}_{X_H}$ is the Lie derivative, whereas $H = p^2/2m + e\phi$ is assumed to be the total energy of a single particle. The link between the Vlasov equation (the first line in (1.9)) and the momentum-Vlasov equation (1.14) is determined through the dual mapping of Lie algebra homomorphism $H \mapsto X_H$ which is computed to be

$$f = \text{div}\Omega_Q^\sharp(\Pi). \quad (1.15)$$

Here, Ω_Q is the canonical symplectic two-form on T^*Q , and $\Omega_Q^\sharp$ is the induced musical isomorphism, while div denotes the symplectic divergence. Since (1.15) is determined as the dual of a Lie algebra homomorphism, it is both a Poisson and a momentum map. In this geometrization, the Poisson equation (the second line in (1.9)) is realized as the momentum map due to the gauge invariance of Hamiltonian dynamics.

One of the interesting features of the momentum-Vlasov equation (1.14) is its pure geometric derivation. More precisely, starting from a Hamiltonian vector field X_H and then lifting it to the cotangent bundle and later taking to the vertical representative, one arrives at a generalized vector field $V\widehat{X}_H$ which determines the motion of Π as given in (1.14). Keeping the same line of thought, further analysis has also been carried out on fluid dynamics [13, 14]. Additionally, rich algebraic structure of momentum-Vlasov dynamics is examined in [15], inspired from the moment algebra of Vlasov dynamics, see for instance [16–18].

The Thesis Goals and Content. Our interest in this thesis is to work on the generalization of irreversible Hamiltonian dynamics. In the present thesis, this is done in various ways, including locally conformal dynamics, conformal Hamiltonian systems, and contact dynamics. Locally conformal dynamics and contact dynamics are two aspects of Jacobi dynamics, which are not necessarily reversible. Hence, this thesis generalizes geometric reversible kinetic motion to a broader framework that also encompasses irreversible motions.

The thesis consists of six main chapters. The next chapter introduces fundamental

concepts in geometric mechanics, such as symplectic, Poisson, and Jacobi manifolds. Chapter 3 covers the theory of locally conformal Hamiltonian and Lagrangian dynamics. Chapter 4 presents the discretization of locally conformal dynamics. In Chapter 5, we develop a kinetic theory for a system of particles within the locally conformally symplectic framework. Chapter 6 discusses conformal Hamiltonian dynamics (distinct from locally conformal dynamics) and provides a kinetic theory for this setting. Finally, Chapter 7 formulates a kinetic theory within the framework of contact geometry.



2. FUNDAMENTALS OF GEOMETRIC MECHANICS

2.1. Symplectic and Poisson Manifolds

This subsection reviews the symplectic and Poisson geometries.

Symplectic Manifolds. A two-form Ω on a manifold M is non-degenerate if the following condition holds

$$\Omega(X, Y) = 0, \quad \forall X \in \mathfrak{X}(M) \quad (2.1)$$

implies that $Y = 0$. A two-form Ω on a manifold M is called almost symplectic if it is non-degenerate, [4, 8, 19–22]. An almost symplectic two-form is called symplectic if it is closed that is

$$d\Omega = 0. \quad (2.2)$$

Then the couple (M, Ω) is called a symplectic manifold. The musical mapping

$$\Omega^\flat : \mathfrak{X}(M) \longrightarrow \Gamma^1(M), \quad \Omega^\flat(X)(Y) = (\iota_X \Omega)(Y) := \Omega(X, Y) \quad (2.3)$$

induced from the two-form Ω is an isomorphism with the inverse mapping

$$\Omega^\sharp : \Gamma^1(M) \longrightarrow \mathfrak{X}(M), \quad \Omega^\sharp := (\Omega^\flat)^{-1}. \quad (2.4)$$

Given a one-form γ and a vector field X , we shall represent by

$$\langle \gamma, X \rangle = \Omega(\Omega^\sharp(\gamma), X) \quad (2.5)$$

the relation between the symplectic two-form Ω , and the musical mapping $\Omega^\sharp$. The non-degeneracy of Ω suffices to define a non-vanishing top form on the manifold, called the symplectic volume

$$d\mu = \frac{(-1)^{n(n-1)/2}}{n!} \Omega \wedge \cdots \wedge \Omega. \quad (2.6)$$

Cotangent bundles are generic examples of symplectic manifolds. On the cotangent bundle T^*Q of a manifold Q there exists the canonical one-form whose value on a vector field X is defined to be

$$\Theta_Q(X) := \langle \tau_{T^*Q}(X), T\pi_Q(X) \rangle. \quad (2.7)$$

Here, π_Q is the cotangent bundle projection while $T\pi_Q$ is the tangent mapping, τ_{T^*Q} is the tangent bundle projection from TT^*Q to TQ . The pairing on the right-hand side of (2.7) is the natural pairing between T^*Q and TQ . The minus of the exterior derivative of the canonical one-form is closed and non-degenerate so that it is a symplectic two-form

$$\Omega_Q = -d\Theta_Q \quad (2.8)$$

on T^*Q . It is called canonical symplectic two-form.

In Darboux coordinates (q^i, p_i) on cotangent bundle T^*Q one can express canonical one-form in (2.7) as

$$\Theta_Q = p_i dq^i \quad (2.9)$$

and symplectic two-form in (2.8) as

$$\Omega_Q = dq^i \wedge dp_i. \quad (2.10)$$

Consider the manifold Q with $\dim Q = n$ and the cotangent bundle T^*Q . The volume form $d\mu$ defined in (2.6) on T^*Q can be express in Darboux coordinates (q^i, p_i) as

$$d\mu = dq^1 \wedge \cdots \wedge dq^n \wedge dp_1 \wedge \cdots \wedge dp_n. \quad (2.11)$$

Taking the local coordinates on TT^*Q as $(q^i, p_i, \dot{q}^i, \dot{p}_i)$ express the musical map in (2.3) as

$$\Omega^\flat : TT^*Q \longrightarrow T^*T^*Q, \quad (q^i, p_i, \dot{q}^i, \dot{p}_i) \mapsto (q^i, p_i, -\dot{p}_i, \dot{q}^i). \quad (2.12)$$

Assuming the Darboux coordinates on T^*T^*Q as (q^i, p_i, Π_i, Π^i) , the inverse of the musical map $\Omega^\sharp$ is computed to be

$$\Omega^\sharp : T^*T^*Q \longrightarrow TT^*Q, \quad (q^i, p_i, \Pi_i, \Pi^i) \mapsto (q^i, p_i, \Pi^i, -\Pi_i). \quad (2.13)$$

Poisson Manifolds. Let M be a manifold equipped with a bilinear operation $\{\bullet, \bullet\}^{(AP)}$ on the smooth functions $\mathcal{F}(M)$ which is skew-symmetric that is,

$$\{F, H\}^{(AP)} = -\{H, F\}^{(AP)}, \quad (2.14)$$

and satisfying the Leibniz identities

$$\begin{aligned}\{FK, H\}^{(AP)} &= \{F, H\}^{(AP)}K + F\{K, H\}^{(AP)}, \\ \{F, KH\}^{(AP)} &= K\{F, H\}^{(AP)} + \{F, K\}^{(AP)}H\end{aligned}\tag{2.15}$$

for all F, K and $H \in \mathcal{F}(M)$. In this case, M is called an almost Poisson manifold, and the bracket is called almost Poisson bracket. Consider an almost Poisson manifold $(M, \{\bullet, \bullet\}^{(AP)})$, and define the following map, called as Jacobiator,

$$\mathcal{J} : \mathcal{F}(M) \times \mathcal{F}(M) \times \mathcal{F}(M) \rightarrow \mathcal{F}(M), \quad (F, H, K) \mapsto \circlearrowleft \{F, \{H, K\}^{(AP)}\}^{(AP)},\tag{2.16}$$

where $\circlearrowleft$ denotes the cyclic sum that is

$$\begin{aligned}\circlearrowleft \{F, \{H, K\}^{(AP)}\}^{(AP)} &= \{F, \{H, K\}^{(AP)}\}^{(AP)} + \{H, \{K, F\}^{(AP)}\}^{(AP)} \\ &\quad + \{K, \{F, H\}^{(AP)}\}^{(AP)}.\end{aligned}\tag{2.17}$$

If for an almost Poisson bracket $\{\bullet, \bullet\}^{(AP)}$, if the corresponding Jacobiator (2.17) vanishes identically then, it is said to be that the bracket is satisfying the Jacobi identity that is

$$\circlearrowleft \{F, \{H, K\}^{(P)}\}^{(P)} = 0,\tag{2.18}$$

for all F, H and K in $\mathcal{F}(M)$. An almost Poisson bracket satisfying the Jacobi identity is called a Poisson bracket and is denoted by $\{\bullet, \bullet\}^{(P)}$. The manifold M is then called a Poisson manifold, see [22, 23].

2.2. Hamiltonian Dynamics

This section reviews the Hamiltonian dynamics in symplectic and Poisson geometries individually.

Hamiltonian Dynamics on Symplectic Manifolds. Consider a symplectic manifold (M, Ω) . Given Hamiltonian function H , the Hamiltonian vector field X_H is defined through the Hamilton's equation

$$\iota_{X_H}\Omega = dH.\tag{2.19}$$

Taking the exterior derivative of (both sides of) the equality, we see that the Lie derivative of the symplectic two-form vanishes. As such, the divergence of

a Hamiltonian vector field is zero concerning the symplectic volume (2.6). We record these as

$$\mathcal{L}_{X_H}\Omega = 0, \quad \mathcal{L}_{X_H}d\mu = 0, \quad (2.20)$$

respectively. Next, the integration of (2.20) implies that the integral flows φ_t of the Hamiltonian vector field preserve both the symplectic two-form and the symplectic volume

$$\varphi_t^*(\Omega) = \Omega, \quad \varphi_t^*(d\mu) = d\mu, \quad (2.21)$$

respectively. Furthermore, the calculations

$$\mathcal{L}_{X_H}(H) = X_H(H) = 0, \quad \varphi_t^*(H) = H \quad (2.22)$$

show us that the Hamiltonian function is constant along the motion. This corresponds to the conservation of energy and determines the reversible character of the symplectic Hamiltonian dynamics, see [3, 4, 19].

Using the symplectic two-form, we define the following bracket

$$\{F, H\}^{(S)} := \Omega(X_F, X_H). \quad (2.23)$$

This bracket is skew-symmetric and satisfies the Leibniz identity and Jacobi identity. Therefore, we refer to the bracket $\{\bullet, \bullet\}^{(S)}$ as the Poisson bracket. Since the characteristic distribution is integrable, the space of Hamiltonian vector fields is closed under the Jacobi-Lie bracket. More precisely we have

$$[X_F, X_H] = -X_{\{F, H\}^{(S)}}. \quad (2.24)$$

In other words,

$$\mathfrak{X}_{\text{ham}}(M) := \{X_H \in \mathfrak{X}(M) : \iota_{X_H}\Omega = dH\} \quad (2.25)$$

is a Lie algebra.

In Darboux coordinates (q^i, p_i) , Hamiltonian vector field on M for a Hamiltonian function H is computed to be

$$X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \quad (2.26)$$

so that the dynamics prescribed by the Hamiltonian vector field is

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \quad (2.27)$$

We can express Poisson bracket defined in (2.23) in local coordinates

$$\{F, H\}^{(s)} = \frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q^i} \frac{\partial F}{\partial p_i}. \quad (2.28)$$

Hamiltonian Dynamics on Poisson Manifolds. Consider a Poisson manifold $(M, \{\bullet, \bullet\}^{(P)})$. The Hamiltonian vector field X_H associated with a function H is defined, for all F in $\mathcal{F}(M)$,

$$X_H(F) := \{F, H\}^{(P)}. \quad (2.29)$$

A function is called Casimir function if it commutes with all the other functions. So that a function C is a Casimir function if

$$\{C, F\}^{(P)} = 0 \quad (2.30)$$

for all F in $\mathcal{F}(M)$. Notice that, the Hamiltonian vector field for a Casimir function is the zero field. The almost Hamiltonian dynamics generated by a Hamiltonian function H is then computed to be

$$\dot{z} = \{z, H\}^{(P)} = X_H(z), \quad (2.31)$$

where $z = z(t)$ represents a curve in M . Hence, $\dot{z}$ represents the time derivative of z . The skew-symmetry of the bracket manifests that the Hamiltonian function H is conserved all along the motion. The calculation is as follows

$$\begin{aligned} \dot{H}(z) &= dH(z) \cdot \dot{z} = dH(z) \cdot X_H(z) \\ &= \Omega^\flat(X_H(z)) \cdot X_H(z) \\ &= \Omega(X_H(z), X_H(z)) = 0. \end{aligned} \quad (2.32)$$

Bivector Field Notation. We define a bivector field associated with an almost Poisson bracket $\{\bullet, \bullet\}^{(AP)}$ through the relation

$$\Lambda(dF, dH) := \{F, H\}^{(AP)} \quad (2.33)$$

for all F and H in $\mathcal{F}(M)$. Here, dF and dH are deRham exterior derivatives of the smooth functions F and H , respectively. So that we may represent an almost Poisson manifold by a two-tuple (M, Λ) as well. This geometry enables us to define a musical mapping $\Lambda^\sharp$ induced by the bivector field Λ as

$$\Lambda^\sharp : \Gamma^1(M) \rightarrow \mathfrak{X}(M), \quad \langle \beta, \Lambda^\sharp(\gamma) \rangle = \Lambda(\beta, \gamma) \quad (2.34)$$

where the pairing on the left-hand side is the one between the space $\Gamma^1(M)$ of one-form sections and the space $\mathfrak{X}(M)$ of vector fields. In terms of $\Lambda^\sharp$, the Hamiltonian vector field X_H in (2.29) can be defined as

$$\Lambda^\sharp(dH) = X_H. \quad (2.35)$$

Consider the bivector field obtained via the bracket defined in (2.23)

$$\Lambda(dF, dH) = \{F, H\}^{(S)}. \quad (2.36)$$

In Darboux coordinates it is computed as

$$\Lambda(q, p) = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}. \quad (2.37)$$

Jacobiator and Poisson Manifolds. Consider a Poisson manifold $(M, \{\bullet, \bullet\}^{(P)})$.

Notice that, for a Poisson manifold, we have the identity that

$$[X_H, X_F] = -X_{\{H, F\}^{(P)}} \quad (2.38)$$

where the bracket on the left-hand side is the Lie-Jacobi bracket of vector fields. The identity (2.38) reads that the characteristic distribution, spanned by Hamiltonian vector fields, of a Poisson bracket is integrable since Jacobi-Lie bracket of two Hamiltonian vector field is another Hamiltonian vector field. In the present Poisson case the definition of the Hamilton's equation (2.31) remains the same.

Recall bivector field defined in (2.33). A direct calculation proves the identity that

$$\begin{aligned} & \frac{1}{2}[\Lambda, \Lambda](dF, dH, dK) \\ &= \{F, \{H, K\}^{(AP)}\}^{(AP)} + \{H, \{K, F\}^{(AP)}\}^{(AP)} + \{K, \{F, H\}^{(AP)}\}^{(AP)} \end{aligned} \quad (2.39)$$

where the bracket on the left-hand side is the Schouten-Nijenhuis bracket of multivector fields and it is defined as follows

$$[X, Y] = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_m \wedge Y_1 \wedge \cdots \wedge \hat{Y}_j \wedge \cdots \wedge Y_n \quad (2.40)$$

where $X = X_1 \wedge \cdots \wedge X_m$ and $Y = Y_1 \wedge \cdots \wedge Y_n$ are multivector fields of order m, n respectively.

It is immediate to see from (2.39) that $\{\bullet, \bullet\}^{(AP)}$ is a Poisson bracket if and only if the bivector field Λ commutes with itself under the Schouten-Nijenhuis bracket that is

$$[\Lambda, \Lambda] = 0. \quad (2.41)$$

So that we can define a Poisson bracket by a two-tuple (M, Λ) with Λ is satisfying the Jacobi identity (2.41). A bivector field satisfying (2.41) is called Poisson bivector.

2.3. Lagrangian Dynamics

Consider the tangent bundle TQ of a manifold Q with local coordinates (q^i) . Then $(q^i, \dot{q}^i)$ becomes the induced coordinates on TQ . Define a Lagrangian function

$$L : TQ \rightarrow \mathbb{R}, \quad L = L(q^i, \dot{q}^i). \quad (2.42)$$

Consider a curve $q(t)$ that joins two fixed points $q(a) = q_a$ and $q(b) = q_b$ in Q over interval $[a, b]$. Using the Lagrangian function define action sum in the following

$$S(q) = \int_a^b L(q(t), \dot{q}(t)) dt. \quad (2.43)$$

To find an extreme value of this integral, we begin by introducing a family of curves $q = q(t, \epsilon)$ that smoothly depend on a parameter ϵ , connecting two fixed points $q_a = q(a)$ and $q_b = q(b)$ in Q . In this context, the variation is defined as the derivative of $q(t, \epsilon)$ with respect to ϵ , specifically:

$$\delta q(t) := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} q(t, \epsilon).$$

Since it is assumed that q_a and q_b are fixed, we have that

$$\delta q(a) = \delta q(b) = 0. \quad (2.44)$$

Calculate variation of the action integral in (2.43) as follows

$$\begin{aligned}
\delta \int_a^b L(q^i, \dot{q}^i) dt &= \int_a^b \left(\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i \right) dt \\
&= \int_a^b \left(\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \frac{d}{dt} \delta q^i \right) dt \\
&= \int_a^b \left(\frac{\partial L}{\partial q^i} \delta q^i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta q^i \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) \delta q^i \right) dt \quad (2.45) \\
&= \frac{\partial L}{\partial \dot{q}^i} \delta q^i \Big|_a^b - \int_a^b \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} \right) \delta q^i dt \\
&= - \int_a^b \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} \right) \delta q^i dt.
\end{aligned}$$

The fundamental theorem of calculus of variations provides the following equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0. \quad (2.46)$$

The equation in (2.46) is known as the Euler-Lagrange equation. When written explicitly, it takes the following form:

$$\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \ddot{q}^j + \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} \dot{q}^j - \frac{\partial L}{\partial q^i} = 0. \quad (2.47)$$

To solve for $\ddot{q}^i$ explicitly, it is necessary to ensure that the determinant of the following matrix is nonzero:

$$W = [W_{ij}] = \left[\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right]. \quad (2.48)$$

Here, W represents the Hessian matrix with respect to $\dot{q}^i$. If $\det(W) \neq 0$, the system is locally solvable for $\ddot{q}^i$. Let $W^{-1} = [W^{ij}]$ denote the inverse of the matrix W . Equation (2.47) can be expressed as

$$\ddot{q}^i = W^{ij} \left(\frac{\partial L}{\partial q^j} - \dot{q}^l \frac{\partial^2 L}{\partial q^j \partial \dot{q}^l} \right). \quad (2.49)$$

For the further information about Lagrangian dynamics, we refer to [4, 20]

Legendre Transformation. Let us define Legendre transformation between tangent bundle TQ and cotangent bundle T^*Q

$$\mathbb{F}L : TQ \longrightarrow T^*Q, \quad (q^i, \dot{q}^i) \mapsto \left(q^i, p_i = \frac{\partial L}{\partial \dot{q}^i} \right). \quad (2.50)$$

We can define the Hamiltonian function on the cotangent bundle T^*Q through the Legendre transformation of the Lagrangian function $L = L(q^i, \dot{q}^i)$. Assuming determinant of the Hessian matrix is nonzero, that is

$$\det(W) = \det \left[\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right] \neq 0. \quad (2.51)$$

Then the Hamiltonian is given by

$$H(q, p) = p_i \dot{q}^i(q, p) - L(q, \dot{q}(q, p)), \quad (2.52)$$

where $p_i = \frac{\partial L}{\partial \dot{q}^i}$.

2.4. Presymplectic Formulation of Lagrange Dynamics

Poincaré-Cartan Forms. Let L be a Lagrangian function on TQ . Using the Legendre transformation $\mathbb{F}L$ in (2.50) we can pull back the canonical one-form Θ_Q and the symplectic two-form Ω_Q from the cotangent bundle T^*Q to the tangent bundle TQ . We have the following expressions

$$\theta_L = \mathbb{F}L^* \Theta_Q, \quad \omega_L = \mathbb{F}L^* \Omega_Q, \quad (2.53)$$

where θ_L and ω_L are referred to as the Poincaré-Cartan one-form and two-form, respectively.

In local coordinates Poincaré-Cartan one-form is given by

$$\theta_L = \frac{\partial L}{\partial \dot{q}^i} dq^i. \quad (2.54)$$

Taking the minus exterior derivative of θ_L we get Poincaré-Cartan two-form as follows

$$\begin{aligned} \omega_L &= -d\theta_L = dq^i \wedge d\left(\frac{\partial L}{\partial \dot{q}^i}\right) \\ &= \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} dq^i \wedge d\dot{q}^j. \end{aligned} \quad (2.55)$$

Note that Poincaré-Cartan two-form in (2.55) is closed. Therefore (TQ, ω_L) is referred to as a presymplectic manifold, see [24]. If the determinant of the Hessian matrix is nonzero, meaning that L is regular then ω_L becomes non-degenerate, and (TQ, ω_L) is called a symplectic manifold.

Consider presymplectic manifold (TQ, ω_L) . Define a real-valued function on TQ

$$E_L : TQ \longrightarrow \mathbb{R}, \quad u \mapsto \Delta(L)(u) - L(u), \quad (2.56)$$

called as energy function. Here, Δ is Liouville vector field on TQ , defined in local coordinates as

$$\Delta(q, \dot{q}) = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}. \quad (2.57)$$

The energy function in local coordinates turns out to be

$$E_L(q, \dot{q}) = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i}(q, \dot{q}) - L(q, \dot{q}). \quad (2.58)$$

The following equation

$$\iota_{X_L} \omega_L = dE_L \quad (2.59)$$

gives the presymplectic equations of motion on (TQ, ω_L) . If ω_L is non-degenerate then we have explicit expression of the vector field X_L as follows

$$X_L = \dot{q}^i \frac{\partial}{\partial q^i} + W^{ij} \left(\frac{\partial L}{\partial q^j} - \dot{q}^l \frac{\partial^2 L}{\partial q^j \partial \dot{q}^l} \right) \frac{\partial}{\partial \dot{q}^i}. \quad (2.60)$$

To see pre-symplectic structure in different frameworks, we refer to [25–27].

2.5. Lie Groups and Lie Algebras

Let G be a differentiable manifold equipped with a group structure. If group operations of multiplication

$$m : G \times G \rightarrow G, \quad (g, h) \mapsto gh \quad (2.61)$$

and inversion

$$i : G \rightarrow G, \quad g \mapsto g^{-1} \quad (2.62)$$

are smooth maps, then G is called a Lie group. Let us give an example of the Lie group.

Example 1: Let us consider M_n as the set of all $n \times n$ matrices, and $GL(n)$ as the set of invertible $n \times n$ matrices. This implies that the determinant of matrices in $GL(n)$ is nonzero, that is,

$$GL(n) = \{A \in M_n : \det(A) \neq 0\}. \quad (2.63)$$

We define the following operations on $GL(n)$. Multiplication is given by

$$m : GL(n) \times GL(n) \rightarrow GL(n), \quad (A, B) \mapsto m(A, B) = AB, \quad (2.64)$$

where AB denotes the standard matrix product. Inversion is

$$i : GL(n) \rightarrow GL(n), \quad A \mapsto i(A) = A^{-1}. \quad (2.65)$$

The inverse of a matrix A is explicitly given by

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)}, \quad (2.66)$$

where $\text{adj}(A)$ denotes adjoint of A . With these operations, $GL(n)$ forms a Lie group. For more information on Lie groups, we refer to [28–31].

Now $e \in G$ denoting the identity element of G , the tangent space at the identity element of the group G is called the Lie algebra $\mathfrak{g} := T_e G$ of G . The Lie algebra structure on $\mathfrak{g}$ is determined by a bilinear bracket

$$[\bullet, \bullet] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \quad (2.67)$$

that satisfies the following properties

- Skew-symmetry: $[\xi, \eta] = -[\eta, \xi]$,
- Jacobi identity: $[[\xi, \eta], \varepsilon] + [[\eta, \varepsilon], \xi] + [[\varepsilon, \xi], \eta] = 0$.

The operation $[\bullet, \bullet]$ is called Lie bracket on $\mathfrak{g}$. We refer to [32–36] for the Lie algebras.

Example 2: Let us consider the set of all $n \times n$ matrices, denoted by M_n . This set forms the Lie algebra of the Lie group $GL(n)$ and is denoted by $\mathfrak{gl}(n)$. The algebra is equipped with the following commutator:

$$[\bullet, \bullet] : \mathfrak{gl}(n) \times \mathfrak{gl}(n) \rightarrow \mathfrak{gl}(n), \quad (A, B) \mapsto [A, B] = AB - BA. \quad (2.68)$$

This bracket satisfies the properties of skew-symmetry and Jacobi identity. Therefore, this commutator operation is a Lie bracket.

We define a bilinear map on $\mathfrak{g}$ using the Lie bracket, given by

$$ad : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}, \quad (\xi, \eta) \mapsto ad_\xi \eta := [\xi, \eta]. \quad (2.69)$$

This map, referred to as the adjoint action, can be interpreted as the left action of $\mathfrak{g}$ on itself. Let us denote $\mathfrak{g}^*$ as the dual space of the Lie algebra $\mathfrak{g}$. The action of $\mathfrak{g}$ on its dual $\mathfrak{g}^*$, called the coadjoint action, is defined as follows

$$ad^* : \mathfrak{g} \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*, \quad \langle ad_\xi^* \rho, \eta \rangle = \langle \rho, ad_\xi \eta \rangle. \quad (2.70)$$

Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras and let $\phi : \mathfrak{g} \mapsto \mathfrak{h}$ be a Lie algebra homomorphism, that is,

$$\phi[\xi, \eta] = [\phi(\xi), \phi(\eta)], \quad (2.71)$$

for any ξ and η in $\mathfrak{g}$.

Example 3: Let us consider the set of all diffeomorphisms on M by $\text{Diff}(M)$. The Lie algebra of $\text{Diff}(M)$ is the space of vector fields $\mathfrak{X}(M)$. The Lie algebra bracket is the opposite Jacobi-Lie bracket of vector fields, that is,

$$ad_X Y = [X, Y]_{\mathfrak{X}(M)} = -[X, Y]_{JL} = -\mathcal{L}_X Y, \quad (2.72)$$

with $\mathcal{L}_X$ being the Lie derivative operator.

We define the dual space $\mathfrak{X}^*(M)$ of the Lie algebra as the space of one-form densities $\Gamma^1(M) \otimes \text{Den}(M)$ on M , where the pairing between a vector field X and a dual element $\Pi \otimes d\mu$ is defined to be the L_2 -pairing (simply multiply-and-integrate form)

$$\langle \bullet, \bullet \rangle_{L_2} : \Gamma^1(M) \otimes \text{Den}(M) \times \mathfrak{X}(M) \longrightarrow \mathbb{R}, \quad (\Pi \otimes d\mu, X) \mapsto \int_M \langle \Pi, X \rangle d\mu. \quad (2.73)$$

The pairing inside the integral is that between the one-form Π and the vector field X , and $d\mu$ is a density (a volume form) on M .

To compute the coadjoint action of the Lie algebra onto the dual space, we perform the following calculation

$$\begin{aligned} \langle ad_X^* (\Pi \otimes d\mu), Y \rangle &= \langle \Pi \otimes d\mu, ad_X Y \rangle = - \int_M \langle \Pi, \mathcal{L}_X Y \rangle d\mu \\ &= \int_M \langle \mathcal{L}_X \Pi + \text{div}(X) \Pi, Y \rangle d\mu \end{aligned} \quad (2.74)$$

where $\text{div}(X)$ stands for the divergence of the vector field with respect to the volume form $d\mu$. To write the second line of this calculation, we have used integration by parts. Hence,

$$ad_X^* (\Pi \otimes d\mu) = (\mathcal{L}_X \Pi + \text{div}(X) \Pi) \otimes d\mu, \quad (2.75)$$

where $\text{div}(X)$ denotes the divergence of the vector field X with respect to the volume form $d\mu$. At this point, without loss of generalization, we fix the volume form $d\mu$, so that we particularly consider a dual element as a one-form Π .

2.6. Lie-Poisson Dynamics and Coadjoint Flow

Let $(\mathfrak{g}, [\bullet, \bullet])$ be a Lie algebra. The dual space $\mathfrak{g}^*$ admits a Poisson bracket, called the Lie-Poisson bracket, which is defined to be

$$\{A, B\}(\rho) = \left\langle \rho, \left[\frac{\delta A}{\delta \rho}, \frac{\delta B}{\delta \rho} \right] \right\rangle, \quad (2.76)$$

see, for instance, [6–8], where ρ is an element in the dual space $\mathfrak{g}^*$, A and B are two functionals on $\mathfrak{g}^*$, and the pairing on the right-hand side is the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. Notice also that $\delta A / \delta \rho$ stands for the Fréchet derivative of the functional A . Assuming the reflexivity on vector spaces, we view $\delta A / \delta \rho$ in $\mathfrak{g}$, justifying thus the Lie bracket that appears on the right-hand side of (2.76). We note further that the right-hand side of (2.76) with the opposite sign would still define a Poisson algebra. We shall, however, prefer the above convention and justify our choice in the following paragraph.

Given a Hamiltonian functional H , the dynamics is governed by the Lie-Poisson equations computed in terms of the coadjoint action as

$$\dot{\rho} = -ad_{\delta H / \delta \rho}^* \rho. \quad (2.77)$$

The dual spaces $\mathfrak{g}^*$ and $\mathfrak{h}^*$ are Lie-Poisson spaces and the dual mapping $\phi^* : \mathfrak{h}^* \mapsto \mathfrak{g}^*$ of a Lie algebra homomorphism $\phi : \mathfrak{g} \mapsto \mathfrak{h}$ is a momentum and a Poisson mapping [8]. The relation between the coadjoint actions is computed to be

$$\phi^* \circ ad_{\phi(\xi)}^* = ad_{\xi}^* \circ \phi^* \quad (2.78)$$

for all ξ in $\mathfrak{g}$.

Lie-Poisson Dynamics for Diffeomorphism Group. Now we consider a particle that moves according to the dynamics generated by a vector field X defined on the manifold M . This particle motion can be lifted to the evolution of distribution functions as follows. Consider a linear Hamiltonian functional on the

space of one-form densities $\Gamma^1(M) \otimes \text{Den}(M)$ given by

$$H(\Pi \otimes d\mu) = \int_M \langle \Pi, X \rangle d\mu, \quad (2.79)$$

where $d\mu$ is a volume form. Then $\delta H / \delta \Pi$ being the Fréchet derivative of H with respect to the momenta, the Lie-Poisson equation turns out to be

$$\dot{\Pi} \otimes d\mu = -ad_{\delta H / \delta \Pi}^*(\Pi \otimes d\mu) = -ad_X^*(\Pi \otimes d\mu). \quad (2.80)$$

Now we fix the volume $d\mu$ and recall the coadjoint action given in (2.75), which gives the Lie-Poisson equation

$$\dot{\Pi} = -\mathcal{L}_X \Pi - \text{div}(X)\Pi. \quad (2.81)$$

If the dynamics is generated by a divergence-free vector field (for example, the case of incompressible fluid flow, or Vlasov flow), then the second term on the right-hand side of (2.81) drops, and we obtain

$$\dot{\Pi} = -\mathcal{L}_X \Pi. \quad (2.82)$$

2.7. Jacobi Manifolds

Let M be a manifold, and consider a bivector field Λ and a vector field $\mathcal{E}$ on M . The triple $(M, \Lambda, \mathcal{E})$ is called a Jacobi manifold if the following conditions are satisfied:

$$[\Lambda, \Lambda] = -2\mathcal{E} \wedge \Lambda, \quad [\Lambda, \mathcal{E}] = 0, \quad (2.83)$$

where the brackets denote the Schouten-Nijenhuis bracket in (2.40). For further details on Jacobi manifolds, see [37–39].

On a Jacobi manifold, there is a bilinear operation, referred to as the Jacobi bracket, which is defined as follows:

$$\{F, H\}^{(J)} = \Lambda(dF, dH) + F\mathcal{E}(H) - H\mathcal{E}(F), \quad (2.84)$$

where F and H are smooth functions on the manifold M . We observe the bracket in (2.84) satisfies skew-symmetry and Jacobi identity, therefore it is a Lie bracket. However, because of the presence of the last two terms $F\mathcal{E}(H)$ and $-H\mathcal{E}(F)$ it does not satisfy the Leibniz rule. As a result, it is not a Poisson bracket.

Notice that if the vector field vanishes identically that is if $\mathcal{E} = 0$, then the bracket (2.84) simplifies to $\{F, H\}^{(J)} = \Lambda(dF, dH)$ therefore satisfies the Leibniz rule. Under this condition, the structural equations of a Jacobi manifold, as given in (2.83), simplify. Specifically, the first property in (2.83) reduces to:

$$[\Lambda, \Lambda] = 0. \quad (2.85)$$

This vanishing condition implies that the bivector field Λ defines a Poisson structure, and consequently, the manifold (M, Λ) turns out to be a Poisson manifold.

If H is a real-valued function on the Jacobi manifold M , the Hamiltonian vector field X_H associated with H is defined as:

$$X_H = \Lambda^\sharp(dH) - H\mathcal{E} \quad (2.86)$$

where $\Lambda^\sharp$ is the musical mapping obtaining by the bivector Λ defined in (2.34). The relationship between the Hamiltonian vector field and the Jacobi bracket is computed to be

$$X_H(F) = \{F, H\}^{(J)} - F\mathcal{E}(H). \quad (2.87)$$

The characteristic distribution of a Jacobi manifold is the image space spanned by the Hamiltonian vector fields. We say a Jacobi manifold transitive if the characteristic distribution is onto. The characteristic distribution of a Jacobi manifold is completely integrable. Here, the even-dimensional leaves are locally conformally symplectic while the odd-dimensional leaves are contact, [40, 41]. This observation determines the starting point of the thesis.

2.8. Complete Lift and Jet Decomposition of Vector Fields

We introduce several definitions of lifts for a vector field X defined on M to its cotangent bundle, along with the jet decomposition of a vertical field. We used these definitions to construct irreversible kinetic dynamics, such as locally conformal kinetic and conformal kinetic dynamics. For more information about these geometric objects, we refer to [42–47].

Complete lift. Let M be a manifold and X be a vector field on M . The complete cotangent lift of X is a vector field X^{c*} on the cotangent bundle T^*M and defined through

$$T\pi_M \circ X^{c*} = X \circ \pi_M, \quad (2.88)$$

where $T\pi_M$ is the tangent mapping of the cotangent bundle projection π_M [3, 48, 49]. The complete cotangent lift X^{c*} is a Hamiltonian vector field associated to the Hamiltonian (called also as momentum) function $H^{c*} = \langle \Theta_M, X \rangle$ where Θ_M is the canonical one-form on T^*M . It is possible to see that the complete cotangent lifts are closed under the Jacobi-Lie bracket:

$$[X^{c*}, Y^{c*}] = [X, Y]^{c*}, \quad (2.89)$$

for two vector fields X and Y on M . We record the space of complete cotangent lifted vector fields on T^*M as

$$\mathfrak{X}^{c*}(T^*M) = \{X^{c*} \in \mathfrak{X}(T^*M) : X \in \mathfrak{X}(M)\} \subset \mathfrak{X}(T^*M). \quad (2.90)$$

The identity (2.89) reads the following injective Lie algebra homomorphism taking a vector field X on a manifold M to its complete cotangent lift X^{c*} :

$$^{c*}: \mathfrak{X}(M) \longrightarrow \mathfrak{X}^{c*}(T^*M), \quad X \mapsto X^{c*}. \quad (2.91)$$

If a local coordinate system (x^a) is chosen on M , and the Darboux coordinates (x^a, y_a) on T^*M , then the complete cotangent lift of a vector field $X = X^a \partial / \partial x^a$ is computed to be

$$X^{c*} = X^a \frac{\partial}{\partial x^a} - \frac{\partial X^b}{\partial x^a} y_b \frac{\partial}{\partial y_a}. \quad (2.92)$$

Jet Decomposition of Vector Fields. The goal here is to determine the vertical part of a vector field. Generally, this requires a connection. Instead, we refer to the first jet bundle to achieve this. To be more precise about the construction, we start with the holonomic lift of a vector field.

Holonomic Lift. Start with the cotangent bundle $\pi_M : T^*M \mapsto M$ and consider the first jet bundle $J^1 T^*M$ with the induced coordinates:

$$(x, y, y_x) = (x^a, y_a, \partial y_b / \partial x^a). \quad (2.93)$$

We define the holonomic lift X^{hol} of a vector field X on M to the cotangent bundle T^*M according to the following equation defined in terms of an arbitrary function f on T^*M and a one-form γ on M as [50, 51]

$$X^{\text{hol}}(f) \circ \gamma = \mathcal{L}_X(f \circ \gamma). \quad (2.94)$$

In terms of the local coordinates, for $X = X^a \partial / \partial x^a$, the holonomic lift defined in (2.94) is computed to be

$$X^{\text{hol}} = X^a \frac{\partial}{\partial x^a} + X^a \frac{\partial y_b}{\partial x^a} \frac{\partial}{\partial y_b}. \quad (2.95)$$

Notice that X^{hol} is a generalized vector field since the coefficient functions depend on the first-order jet bundle variables. The algebra of generalized vector fields is determined by the prolonged bracket $[\bullet, \bullet]_{\text{pro}}$. In the following, we present the expression for the prolonged bracket of generalized vector fields.

A generalized vector field on T^*M , then, is of the form

$$\chi = \chi^a(x) \frac{\partial}{\partial x^a} + \chi_b(x, y, y_x) \frac{\partial}{\partial y_b}. \quad (2.96)$$

As such, the first order prolongation $pr^1 \chi$ of χ may be given by

$$pr^1 \chi = \chi + \Delta_{ba} \frac{\partial}{\partial (\partial y_b / \partial x^a)}, \quad \Delta_{ba} = D_{x^a} (\chi_e - \chi^b (\partial y_e / \partial x^b)) + (\partial^2 y_e / \partial x^a \partial x^b) \chi^b, \quad (2.97)$$

where D_{x^a} is the total derivative operator with respect to x^a , and $\partial^2 y_e / \partial x^a \partial x^b$ is an element of the second order jet bundle. Furthermore, the Lie bracket of two first-order generalized vector fields χ and ψ is the unique first-order generalized vector field is given by

$$[\chi, \psi]_{\text{pro}} = (pr^1 \chi(\psi^a) - pr^1 \psi(\chi^a)) \frac{\partial}{\partial x^a} + (pr^1 \chi(\psi_a) - pr^1 \psi(\chi_a)) \frac{\partial}{\partial y_a}. \quad (2.98)$$

We cite [13, 14, 50–52] for generalized vector fields and their algebraic structures.

Proposition 2.1: The holonomic lift operator (2.94) respects prolongation structure that is

$$[X, Y]^{\text{hol}} = [X^{\text{hol}}, Y^{\text{hol}}]_{\text{pro}}, \quad (2.99)$$

where $[\bullet, \bullet]_{\text{pro}}$ is the prolonged bracket.

Holonomic Part of a Projectable Vector Field. Consider a projectable vector field W defined on the cotangent bundle T^*M , the holonomic part $\mathfrak{H}W$ is defined to be the holonomic lift of its projection:

$$\mathfrak{H}W = ((\pi_M)_*W)^{\text{hol}}. \quad (2.100)$$

On the cotangent bundle T^*M , a projectable vector field W has the following expression

$$W = W^a(x) \frac{\partial}{\partial x^a} + W_b(x, y) \frac{\partial}{\partial y_b}. \quad (2.101)$$

According to (2.100), the holonomic part of this vector field is computed to be

$$\mathfrak{H}W = W^a(x) \frac{\partial}{\partial x^a} + W^a \frac{\partial y_b}{\partial x^a} \frac{\partial}{\partial y_b}. \quad (2.102)$$

The operation $\mathfrak{H}$ extracts the holonomic part of a projectable vector field:

$$\mathfrak{H} : \mathfrak{X}^{\text{proj}}(T^*M) \longrightarrow \mathfrak{X}^{\text{gen}}(T^*M), \quad W \mapsto \mathfrak{H}(W), \quad (2.103)$$

where $\mathfrak{X}^{\text{proj}}(T^*M)$ is the space of projectable vector fields whereas the space $\mathfrak{X}^{\text{gen}}(T^*M)$ is the space of generalized vector fields of order one.

Proposition 2.2: The operation $\mathfrak{H}$ in (2.103) is a Lie algebra homomorphism

$$\mathfrak{H}[W, U] = [\mathfrak{H}W, \mathfrak{H}U]_{\text{pro}} \quad (2.104)$$

for two arbitrary generalized vector fields W and U .

Proof. To prove the assertion, we compute

$$\begin{aligned} \mathfrak{H}[W, U] &= ((\pi_M)_*[W, U])^{\text{hol}} \\ &= [(\pi_M)_*W, (\pi_M)_*U]^{\text{hol}} \\ &= [((\pi_M)_*W)^{\text{hol}}, ((\pi_M)_*U)^{\text{hol}}]_{\text{pro}} \\ &= [\mathfrak{H}W, \mathfrak{H}U]_{\text{pro}}, \end{aligned} \quad (2.105)$$

where we have used the commutativity of the push-forward operation and the Jacobi-Lie bracket and Proposition 2.1. $\square$

Vertical Representative. To read just the vertical dynamics for a projectable vector field W on T^*M , one needs to get rid of the holonomic part inside W [50]. We call this to be vertical representative of W and define

$$\mathfrak{V}W = W - \mathfrak{H}W = \left(W_b(x, y) - W^a(x) \frac{\partial y_b}{\partial x^a} \right) \frac{\partial}{\partial y_b}, \quad (2.106)$$

where, for W , the local expression (2.101) is assumed. The vertical representative $\mathfrak{W}W$ of a vector field W is a generalized vector field, as is the holonomic part $\mathfrak{H}W$. Notice that $\mathfrak{W}W$ lies in the kernel of the tangent mapping $T\pi_M$ of the cotangent bundle projection, so it is a vertical vector field. We consider two projectable vector fields W and U on T^*M . A straight forward calculation gives the prolonged bracket of vertical representatives as

$$[\mathfrak{W}W, \mathfrak{W}U]_{\text{pro}} = \mathfrak{V}[W, U]_{\text{pro}} + \mathfrak{B}(W, U), \quad (2.107)$$

where the *curvature* $\mathfrak{B}$ is a vertical-vector valued two-form

$$\mathfrak{B}(W, U) = [\mathfrak{H}U, \mathfrak{W}W]_{\text{pro}} + [\mathfrak{W}U, \mathfrak{H}W]_{\text{pro}}. \quad (2.108)$$

Therefore, we can argue that taking vertical representative is not a Lie algebra homomorphism for arbitrary projectable vector fields.

3. LOCALLY CONFORMAL DYNAMICS

3.1. Locally Conformally Symplectic Manifolds

A manifold M is called a locally conformally symplectic (LCS) manifold if it admits a non-degenerate two-form Ω so that

$$d\Omega = \varphi \wedge \Omega \quad (3.1)$$

for a closed (called as Lee) one-form φ , see [53–56]. We denote an LCS manifold by (M, Ω, φ) . In terms of the Lichnerowicz-deRham differential¹ we can write the equality (3.1) as

$$d_\varphi \Omega = 0. \quad (3.2)$$

If the dimension of an LCS manifold is $2n$ then we determine the LCS volume-form as

$$dv = \frac{(-1)^{n(n-1)/2}}{n!} \Omega \wedge \cdots \wedge \Omega. \quad (3.3)$$

The local picture for an LCS manifold is as follows. On every local chart of the manifold M there exists local symplectic two-form. We record a symplectic chart as $(U_\alpha, \Omega_\alpha)$. On the intersection $U_\alpha \cap U_\beta$ of two local charts, the local symplectic two-forms Ω_α and Ω_β are related with

$$e^{\sigma_\alpha} \Omega_\alpha = e^{\sigma_\beta} \Omega_\beta, \quad (3.4)$$

where the family of real-valued functions $\{\sigma_\alpha\}$ are the conformal factors. It is noteworthy to see that the conformal factors satisfy the cocycle condition. That is the conformal relations determine scalars

$$\lambda_{\beta\alpha} = e^{\sigma_\alpha} / e^{\sigma_\beta} = e^{-(\sigma_\beta - \sigma_\alpha)} \quad (3.5)$$

satisfying the cocycle condition

$$\lambda_{\beta\alpha} \lambda_{\alpha\gamma} = \lambda_{\beta\gamma}. \quad (3.6)$$

¹For a closed one-form B , the Lichnerowicz-deRham differential is defined to be $d_B \Omega = d\Omega - B \wedge \Omega$, see [57].

This allows us to assemble local symplectic two-forms Ω_α into a line bundle-valued two-form on M . We define a real-valued global two-form Ω using the following local characterization:

$$\Omega|_\alpha = e^{\sigma_\alpha} \Omega_\alpha, \quad (3.7)$$

where $\Omega|_\alpha$ is the restriction of the global two-form to U_α . The exterior derivative of this two-form is computed to be

$$d\Omega|_\alpha = d\sigma_\alpha \wedge \Omega|_\alpha, \quad (3.8)$$

which is the same as the LCS two-form condition in (3.1) with the Lee one-form $\varphi|_\alpha = d\sigma_\alpha$ defined as the exterior derivative of the conformal factor.

3.2. Cotangent Bundle and Coordinate Realizations

Assume a closed one-form ϑ on the base manifold Q . We pull back this one-form to the cotangent bundle using the cotangent bundle projection and obtain a closed and semi-basic one-form

$$\varphi = (\pi_Q)^* \vartheta. \quad (3.9)$$

Then we take minus of LdR differential of the canonical one-form Θ_Q defined in (2.7) on the cotangent bundle T^*Q to arrive at the following two-form

$$\Omega_\varphi := -d_\varphi \Theta_Q = -d\Theta_Q + \varphi \wedge \Theta_Q = \Omega_Q + \varphi \wedge \Theta_Q \quad (3.10)$$

which is an LCS two-form.

Proposition 3.1: The three-tuple $(T^*Q, \Omega_\varphi, \varphi)$ where φ is a closed and semi-basic one-form, defined in (3.9), on T^*Q and Ω_φ is a two-form defined as in (3.10) is an LCS manifold.

Proof. For an open covering $\{U_\alpha\}$ of the cotangent bundle T^*Q , there exists a family $\{\sigma_\alpha\}$ of local smooth functions which lead us to define local one-forms

$$\Theta_\alpha = e^{-\sigma_\alpha} \Theta_Q|_\alpha. \quad (3.11)$$

Here, $\Theta_Q|_\alpha$ stands for the restriction of the canonical one-form Θ_Q to the local frame U_α . In view of the identity $\Theta_Q|_\alpha = \Theta_Q|_\beta$, the setting (3.11) reads the equality

$$e^{\sigma_\alpha} \Theta_\alpha = e^{\sigma_\beta} \Theta_\beta \quad (3.12)$$

on the intersection $U_\alpha \cap U_\beta$. So, the set $\{\Theta_\alpha\}$ of local one-forms can only be glued to a line bundle valued one-form on the cotangent bundle T^*Q . To show that (3.12) is indeed resulting with the local relation (3.4), we take the exterior derivative of both sides of (3.11). So, we have

$$\begin{aligned} d\Theta_Q|_\alpha &= d(e^{\sigma_\alpha}\Theta_\alpha) = d(e^{\sigma_\alpha}) \wedge \Theta_\alpha + e^{\sigma_\alpha}d\Theta_\alpha \\ &= e^{\sigma_\alpha}(d\sigma_\alpha \wedge \Theta_\alpha - \Omega_\alpha), \end{aligned} \quad (3.13)$$

where the local two-form $\Omega_\alpha = -d\Theta_\alpha$. This calculation can be repeated for another chart, say U_β , as follows

$$d\Theta_Q|_\beta = e^{\sigma_\beta}(d\sigma_\beta \wedge \Theta_\beta - \Omega_\beta). \quad (3.14)$$

Accordingly, for a possible intersection of two local charts, one has the following equality

$$e^{\sigma_\alpha}\Omega_\alpha = e^{\sigma_\beta}\Omega_\beta, \quad (3.15)$$

which is the same as the locally conformal relation in (3.4). More directly, it is also possible to validate that the exterior derivative of Ω_φ is

$$d\Omega_\varphi = d(\Omega_Q + \varphi \wedge \Theta_Q) = \varphi \wedge \Omega_\varphi, \quad d_\varphi\Omega_\varphi = 0. \quad (3.16)$$

□

The inverse of this assertion is also true. Every LCS manifold looks locally like the one in Proposition 3.1, see [58, 59]. In this setting, since the Lee one-form φ is defined using the cotangent bundle projection, it is determined only by the variables on the base manifold Q . So, in this geometry, one expects that the conformal factors should be independent of the fiber (momentum) variables. This observation will be helpful in upcoming calculations.

LCS Manifold in Coordinates. A local chart of the cotangent bundle T^*Q can be determined as follows. Consider an open cover

$$Q = \bigsqcup_\alpha V_\alpha \quad (3.17)$$

of the base manifold with dimension n . This local characterization can be lifted to the tangent bundle TQ and the cotangent bundle T^*Q as

$$TQ = \bigsqcup_\alpha TV_\alpha = \bigsqcup_\alpha V_\alpha \times \mathbb{R}^n, \quad T^*Q = \bigsqcup_\alpha T^*V_\alpha = \bigsqcup_\alpha V_\alpha \times (\mathbb{R}^n)^*, \quad (3.18)$$

respectively. We can further identify the dual space $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$ so that $T^*V_\alpha = V_\alpha \times \mathbb{R}^n$. The Darboux coordinates (q^i, p_i) on T^*V_α reads the canonical one-form and the symplectic two-form as

$$\Theta_Q|_\alpha = p_i dq^i, \quad \Omega_Q|_\alpha = dq^i \wedge dp_i, \quad (3.19)$$

respectively. In this picture, the local one-form Θ_α defined in (3.11) turns out to be

$$\Theta_\alpha = r_i dq^i, \quad r_i := e^{-\sigma_\alpha} p_i. \quad (3.20)$$

As presented in (3.10), the Lee one-form is a semi-basic one-form on the cotangent bundle framework. So, without a loss of generality, we can assume that the local function depends only on the coordinates (q^i) of the base manifold Q . That is $\sigma_\alpha = \sigma_\alpha(q)$. In accordance, we take the minus of the exterior derivative of the local one-form Θ_α in order to compute the local symplectic two-form Ω_α as

$$\Omega_\alpha = -d\Theta_\alpha = dq^i \wedge dr_i = e^{-\sigma_\alpha} \left(dq^i \wedge dp_i + \frac{1}{2} A_{ij} dq^i \wedge dq^j \right). \quad (3.21)$$

Here, we have defined a skew-symmetric tensor field

$$A_{ij} := \varphi_i p_j - \varphi_j p_i, \quad \varphi_i := \frac{\partial \sigma_\alpha}{\partial q^i}. \quad (3.22)$$

Later, we have a global real-valued two-form Ω whose restriction to the open chart T^*V_α is

$$\Omega_\varphi|_\alpha = e^{\sigma_\alpha} \Omega_\alpha = \Omega_Q|_\alpha + A|_\alpha = dq^i \wedge dp_i + \frac{1}{2} A_{ij} dq^i \wedge dq^j. \quad (3.23)$$

See that the first term on the right-hand side is the canonical symplectic two-form Ω_Q whereas the second one is precisely $\varphi \wedge \Theta_Q$. Therefore, the local realization fits very well with the global definition of Ω in (3.10).

In summary, we have two sets of coordinates on T^*V_α , (q^i, p_i) is the Darboux coordinates for the canonical symplectic two-form $\Omega_Q = dq^i \wedge dp_i$ and the other is (q^i, r_i) which is the Darboux coordinates for the local symplectic two-form

$\Omega_\alpha = dq^i \wedge dr_i$. The following diagram summarizes the situation in one picture:

$$\begin{array}{ccc}
\Omega_Q|_\alpha & & \\
(3.19) & \searrow \text{twist (3.10)} & \\
\begin{array}{c} \uparrow -d \\ \Theta_Q|_\alpha \end{array} & \xrightarrow{-d_\varphi} & \Omega_\varphi|_\alpha \\
(3.19) & & (3.23) \\
\begin{array}{c} \uparrow e^{\sigma_\alpha} \\ \Theta_\alpha \end{array} & \xrightarrow{-d} & \Omega_\alpha \\
(3.20) & & (3.20)
\end{array}$$

We cite [59] for Darboux-Weinstein theorem for LCS manifolds.

3.3. Locally Conformally Symplectic Hamiltonian Dynamics

Consider the LCS manifold (M, Ω, φ) admitting local symplectic structures $(U_\alpha, \Omega_\alpha)$ with conformal factors (σ_α) . For a local Hamiltonian function H_α on U_α , we write local Hamilton's equation as

$$\iota_{X_\alpha} \Omega_\alpha - dH_\alpha = 0, \quad (3.24)$$

where X_α is the local Hamiltonian vector field. Assuming the local equality

$$e^{\sigma_\alpha} H_\alpha = e^{\sigma_\beta} H_\beta \quad (3.25)$$

on the intersection $U_\alpha \cap U_\beta$, we can glue local Hamiltonian functions to a section of the line bundle over the LCS manifold. Instead of line-bundle valued functions, we can glue the following local functions

$$H|_\alpha = e^{\sigma_\alpha} H_\alpha \quad (3.26)$$

those constructing a real-valued function H on T^*Q . Here, $H|_\alpha$ is the restriction of the global Hamiltonian function H to the local chart U_α . This observation permits us to glue the local Hamilton's equation (3.24) to the LCS Hamilton's equation

$$\iota_{\xi_H} \Omega = d_\varphi H. \quad (3.27)$$

Here, the LCS Hamiltonian vector field ξ_H is obtained by gluing together the local Hamiltonian vector fields X_α in (3.24), such that

$$\xi_H|_\alpha = X_\alpha. \quad (3.28)$$

Notice that if the one-form φ encoding the local conformal structure is zero then LCS Hamilton's equation (3.24) reduces to Hamilton's equation (2.19) in the symplectic framework.

LCS Hamiltonian Dynamics in Coordinates. We start with the local Hamiltonian dynamics X_α in (3.24) generated by a local Hamilton function $H_\alpha = H_\alpha(q^i, r_i)$ in terms of (q^i, r_i) coordinates which is the Darboux frame for the local symplectic two-form Ω_α . We shall glue the local systems to arrive at the LCS Hamiltonian dynamics (3.27) in terms of (q^i, p_i) coordinates.

Proposition 3.2: In terms of the Darboux coordinates (q^i, p_i) of the canonical symplectic structure on the cotangent bundle T^*Q , for the LCS manifold $(T^*Q, \Omega_\varphi, \varphi)$, the LCS Hamilton's equation (3.27) is expressed as:

$$\frac{dq^i}{dt} = \frac{\partial H|_\alpha}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H|_\alpha}{\partial q^i} - A_{ij} \frac{\partial H|_\alpha}{\partial p_j} + H|_\alpha \varphi_i, \quad (3.29)$$

where the covariant quantity A_{ij} is as defined in (3.22).

Proof. Consider Hamilton's equations on the cotangent bundle T^*V_α computed to be as

$$\frac{dq^i}{dt} = \frac{\partial H_\alpha}{\partial r_i}, \quad \frac{dr_i}{dt} = -\frac{\partial H_\alpha}{\partial q^i}. \quad (3.30)$$

Next, we write the dynamics in terms of the Darboux coordinates (q^i, p_i) . As the first step, we recall the local realization of the global Hamiltonian function in (3.26) and write this as

$$H|_\alpha(q^i, p_i) = e^{\sigma_\alpha} H_\alpha(q^i, r_i), \quad r_i = e^{-\sigma_\alpha} p_i. \quad (3.31)$$

Referring to this identification, the first system in the local Hamilton's equation in (3.30) reads

$$\begin{aligned} \frac{dq^i}{dt} &= \frac{\partial}{\partial r_i} (e^{-\sigma_\alpha} H|_\alpha(q^i, p_i)) = e^{-\sigma_\alpha} \frac{\partial}{\partial r_i} H|_\alpha(q^i, p_i) \\ &= e^{-\sigma_\alpha} \frac{\partial H|_\alpha}{\partial p_j} \frac{\partial p_j}{\partial r_i} = e^{-\sigma_\alpha} \frac{\partial H|_\alpha}{\partial p_i} e^{\sigma_\alpha} = \frac{\partial H|_\alpha}{\partial p_i}. \end{aligned} \quad (3.32)$$

For the latter equation in (3.30), we examine the left-hand and the right-hand sides one by one. On the left-hand side, we get

$$\frac{dr_i}{dt} = \frac{d}{dt}(e^{-\sigma_\alpha} p_i) = -e^{-\sigma_\alpha} \left(\varphi_j p_i \frac{dq^j}{dt} - \frac{dp_i}{dt} \right) = -e^{-\sigma_\alpha} \left(\varphi_j p_i \frac{\partial H|_\alpha}{\partial p_j} - \frac{dp_i}{dt} \right) \quad (3.33)$$

whereas for the right-hand side we compute

$$\frac{\partial H_\alpha}{\partial q^i} = e^{-\sigma_\alpha} \left(\frac{\partial H|_\alpha}{\partial q^i} - \varphi_i H|_\alpha + \frac{\partial H|_\alpha}{\partial p_j} \varphi_i p_j \right). \quad (3.34)$$

By collecting the calculations done for the right-hand side and the left-hand side, we write the second equation in (3.30) as

$$\frac{dp_i}{dt} = -\frac{\partial H|_\alpha}{\partial q_i} - A_{ij} \frac{\partial H|_\alpha}{\partial p_j} + H|_\alpha \varphi_i. \quad (3.35)$$

Eventually, we arrive at (3.29) recording coordinate formulation of the LCS Hamiltonian dynamics. $\square$

Gluing Up the Local Canonical Poisson Bracket. Consider the LCS manifold $(T^*Q, \Omega_\varphi, \varphi)$ with a local space T^*V_α with coordinates (q^i, r_i) . In this realization, we note the local canonical Poisson bracket as

$$\{F_\alpha, H_\alpha\}_\alpha^{(S)} = \frac{\partial F_\alpha}{\partial q^i} \frac{\partial H_\alpha}{\partial r_i} - \frac{\partial F_\alpha}{\partial r_i} \frac{\partial H_\alpha}{\partial q^i}. \quad (3.36)$$

Recalling the relationship between the global and local functions

$$F|_\alpha(q^i, p_i) = e^{\sigma_\alpha} F_\alpha(q^i, r_i), \quad r_i = e^{-\sigma_\alpha} p_i \quad (3.37)$$

as well as the change of basis

$$\frac{\partial}{\partial q^i} = \frac{\partial}{\partial q^i} + \varphi_i p_j \frac{\partial}{\partial p_j}, \quad \frac{\partial}{\partial r_i} = e^{\sigma_\alpha} \frac{\partial}{\partial p_i} \quad (3.38)$$

we compute

$$\begin{aligned} e^{\sigma_\alpha} \{F_\alpha, H_\alpha\}_\alpha^{(S)} &= \frac{\partial F|_\alpha}{\partial q^i} \frac{\partial H|_\alpha}{\partial p_i} - \frac{\partial F|_\alpha}{\partial p_i} \frac{\partial H|_\alpha}{\partial q^i} + \varphi_i \left(\frac{\partial F|_\alpha}{\partial p_i} H|_\alpha - \frac{\partial H|_\alpha}{\partial p_i} F|_\alpha \right) \\ &\quad + A_{ij} \frac{\partial F|_\alpha}{\partial p_j} \frac{\partial H|_\alpha}{\partial p_i}. \end{aligned} \quad (3.39)$$

We define a global bracket whose restriction to the local chart is defined as

$$\{F, H\}^{(LCS)}|_\alpha := e^{\sigma_\alpha} \{F_\alpha, H_\alpha\}_\alpha^{(S)}. \quad (3.40)$$

On the other hand, consider the LCS Hamiltonian vector field ξ_H defined in (3.27). Referring the LCS Hamiltonian dynamics (3.29), we can express the vector field ξ_H in the local coordinates (q^i, p_i) as

$$\xi_H = \frac{\partial H|_\alpha}{\partial p_i} \frac{\partial}{\partial q^i} + \left(-\frac{\partial H|_\alpha}{\partial q^i} - A_{ij} \frac{\partial H|_\alpha}{\partial p_j} + \varphi_i H|_\alpha \right) \frac{\partial}{\partial p_i}. \quad (3.41)$$

We compute the LCS two-form Ω as

$$\Omega|_\alpha = e^{\sigma_\alpha} \Omega_\alpha = \Omega_Q|_\alpha + A|_\alpha = dq^i \wedge dp_i + \frac{1}{2} A_{ij} dq^i \wedge dq^j. \quad (3.42)$$

For the LCS manifold $(T^*Q, \Omega_\varphi, \varphi)$, the induced Jacobi pair (Λ, Z_φ) , given respectively in (3.49) and (3.50), the coordinate characterizations are computed as follows:

$$\Lambda = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} A_{ij} \frac{\partial}{\partial p_j} \wedge \frac{\partial}{\partial p_i}, \quad Z_\varphi = -\varphi_i \frac{\partial}{\partial p_i}. \quad (3.43)$$

Remark 1: An explicit example of locally but not globally Hamiltonian system was studied earlier in [60], see also [12] and [61]. Suppose a dynamical system in three dimensions represented by a vector field v can be written in terms of some one-forms α, β, γ as

$$\iota_v(*1) = d\alpha = \gamma \wedge \beta. \quad (3.44)$$

On an orientable three-manifold, this means that v is a curl vector with a non-integrable potential α

$$\alpha \wedge d\alpha = \alpha \wedge \beta \wedge \gamma \neq 0 \quad (3.45)$$

because the right-hand side is a volume. This system is locally bi-Hamiltonian if

$$\gamma \wedge d\gamma = \beta \wedge d\beta = 0 \quad (3.46)$$

which are manifestations of Jacobi identity in three dimensions under the identification

$$\beta = \beta_i dx^i, \quad \beta_i = \varepsilon_{ijk} A^{jk} \quad (3.47)$$

of one-forms and Poisson tensors. The integrating factors that turn β and γ into Poisson structures can only be defined locally

$$d(e^{-2 \int \alpha} \beta) = 0, \quad d(e^{2 \int \alpha} \gamma) = 0 \quad (3.48)$$

by choosing the parametrized path to integrate the non-integrable potential α of v .

3.4. Locally Conformally Symplectic Manifold as a Jacobi Manifold

Consider an LCS manifold (M, Ω, φ) . Whether referring to the almost symplectic two-form Ω , or to the local symplectic two-forms Ω_α defined on the local charts, it is possible to show that M is a Jacobi manifold.

Starting with the global definition of LCS manifolds and referring to the inverse mapping $\Omega^\sharp$ of the musical mapping, we introduce the Lee vector field as

$$Z_\varphi = \Omega^\sharp(\varphi). \quad (3.49)$$

Next, we define a bivector Λ on M . For two elements ζ and ν of $\Gamma^1(M)$, we construct the bivector field as follows

$$\Lambda(\zeta, \nu) = \Omega(\Omega^\sharp(\zeta), \Omega^\sharp(\nu)). \quad (3.50)$$

We cite [62] for the proof of the following proposition.

Proposition 3.3: An LCS manifold (M, Ω, φ) is a Jacobi manifold with the pair (Λ, Z_φ) where Λ is in (3.50), and the vector field $\mathcal{E}$ is the Lee vector field Z_φ in (3.49).

Accordingly, an LCS manifold (M, Ω, φ) admits a Jacobi bracket of two functions determined through

$$\{F, H\}^{(LCS)} = \Lambda(dF, dH) + FZ_\varphi(H) - HZ_\varphi(F). \quad (3.51)$$

We can express the restriction of the Jacobi bracket on a local chart U_α as

$$\{F, H\}^{(LCS)}|_\alpha = e^{\sigma_\alpha} \{F_\alpha, H_\alpha\}_\alpha^{(S)} \quad (3.52)$$

where the bracket on the right-hand side is the canonical local Poisson bracket defined using the local symplectic two-form Ω_α . It is possible to see that

$$\Omega(\xi_F, \xi_H) = \{F, H\}^{(LCS)}. \quad (3.53)$$

Local Reversible Motion to Global Irreversible Motion. Let us deduce the LCS (Jacobi) bracket in (3.51) starting with the local canonical Poisson

bracket. This may be a way to see how local reversible motion is related to the global irreversible motion. Assume local symplectic space $(U_\alpha, \Omega_\alpha)$. For two local functions F_α and H_α , we write the local canonical Poisson bivector field as

$$\{F_\alpha, H_\alpha\}_\alpha^{(S)} = \Lambda_\alpha(dF_\alpha, dH_\alpha). \quad (3.54)$$

To have a global picture we further assume that there exists global functions, F and H , and their local characterizations are in form

$$F|_\alpha = e^{\sigma_\alpha} F_\alpha, \quad H|_\alpha = e^{\sigma_\alpha} H_\alpha, \quad (3.55)$$

respectively. The definition in (3.54) brings out the direct calculation

$$\begin{aligned} \{F_\alpha, H_\alpha\}_\alpha^{(S)} &= \Lambda_\alpha(dF_\alpha, dH_\alpha) = \Lambda_\alpha(d(e^{-\sigma_\alpha} F|_\alpha), d(e^{-\sigma_\alpha} H|_\alpha)) \\ &= \Lambda_\alpha(e^{-\sigma_\alpha} dF|_\alpha - e^{-\sigma_\alpha} F|_\alpha d\sigma_\alpha, e^{-\sigma_\alpha} dH|_\alpha - e^{-\sigma_\alpha} H|_\alpha d\sigma_\alpha) \\ &= e^{-2\sigma_\alpha} \Lambda_\alpha(dF|_\alpha - F|_\alpha d\sigma_\alpha, dH|_\alpha - H|_\alpha d\sigma_\alpha) \\ &= e^{-2\sigma_\alpha} \Lambda_\alpha(dF|_\alpha, dH|_\alpha) - e^{-2\sigma_\alpha} H|_\alpha \Lambda_\alpha(dF|_\alpha, d\sigma_\alpha) \\ &\quad - e^{-2\sigma_\alpha} F|_\alpha \Lambda_\alpha(d\sigma_\alpha, dH|_\alpha). \end{aligned} \quad (3.56)$$

The bracket $\{F_\alpha, H_\alpha\}_\alpha^{(S)}$ on the left-hand side of the calculation reads a local function, referring to the policy (3.55), we need to multiply the bracket with e^{σ_α} to make it a local function which can be glued up to a real-valued global function. Accordingly, we rewrite the result of the calculation (3.56) as

$$\begin{aligned} e^{\sigma_\alpha} \{F_\alpha, H_\alpha\}_\alpha^{(S)} &= e^{-\sigma_\alpha} \Lambda_\alpha(dF|_\alpha, dH|_\alpha) - e^{-\sigma_\alpha} H|_\alpha \Lambda_\alpha(dF|_\alpha, d\sigma_\alpha) \\ &\quad - e^{-\sigma_\alpha} F|_\alpha \Lambda_\alpha(d\sigma_\alpha, dH|_\alpha). \end{aligned} \quad (3.57)$$

We introduce the following identifications

$$\Lambda|_\alpha := e^{-\sigma_\alpha} \Lambda_\alpha, \quad Z_\varphi|_\alpha := \iota_{d\sigma_\alpha} \Lambda|_\alpha. \quad (3.58)$$

Substituting these identifications into (3.57) gives exactly the LCS bracket (3.51). Note that the local-global bracket characterization in (3.52) holds true for the present construction as well.

Divergence of LCS Hamiltonian vector field. In terms of the Lee vector field, we can write the LCS Hamiltonian vector field as

$$\xi_H = \Omega^\sharp(dH) - HZ_\varphi. \quad (3.59)$$

A constant Hamiltonian function determines non-trivial dynamics. For example, for the Hamiltonian function $H = -1$, the LCS Hamiltonian vector field is precisely Lee vector field $\xi_{-1} = Z_\varphi$ and it is far from being zero if φ is not zero. Further, a non-constant Hamiltonian function H is not conserved along the locally conformal Hamiltonian motion. One computes

$$\xi_H(H) = -HZ_\varphi(H). \quad (3.60)$$

Additionally, it is important to note that the LCS two-form is not conserved along the LCS Hamiltonian motion. To see this, we compute

$$\begin{aligned} \mathcal{L}_{\xi_H}(\Omega) &= d\iota_{\xi_H}(\Omega) + \iota_{\xi_H}d\Omega = dd_\varphi H + \iota_{\xi_H}(\varphi \wedge \Omega) \\ &= -dH \wedge \varphi + (\iota_{\xi_H}\varphi)\Omega - \varphi \wedge \iota_{\xi_H}\Omega \\ &= -dH \wedge \varphi + (\iota_{\xi_H}\varphi)\Omega - \varphi \wedge (dH - H\varphi) \\ &= -dH \wedge \varphi + (\iota_{\xi_H}\varphi)\Omega - \varphi \wedge dH = \langle \varphi, \xi_H \rangle \Omega. \end{aligned} \quad (3.61)$$

The non-degeneracy of LCS two-form Ω manifests the existence of a volume-form. If the dimension of an LCS manifold is $2n$ then we determine the LCS volume-form in (3.3). Concerning LCS volume-form (3.3), the divergence of the LCS Hamiltonian vector field is computed to be

$$\operatorname{div}(\xi_H) = n\langle \varphi, \xi_H \rangle. \quad (3.62)$$

3.5. Locally Conformal Lagrangian Dynamics

Let us present locally conformal Lagrangian dynamics. To accomplish this, we begin by recalling the classical form of the Euler-Lagrange equations. Additionally, we continue our investigation of the geometrical construction and coordinate system introduced in the previous subsection. Accordingly, we operate within the framework of the local trivialization $TV_\alpha = V_\alpha \times \mathbb{R}^n$ of the tangent bundle TQ .

Euler-Lagrange Equations. We consider a local Lagrangian function $L_\alpha = L_\alpha(q^i, \dot{q}^i)$ on the local tangent bundle TV_α . We define the local fiber derivative from TV_α to T^*V_α given by

$$\mathbb{F}L_\alpha : TV_\alpha \longrightarrow T^*V_\alpha, \quad (q^i, \dot{q}^i) \mapsto (q^i, r_i) = \left(q^i, \frac{\partial L_\alpha}{\partial \dot{q}^i} \right). \quad (3.63)$$

Recall the local one-form $\Theta_\alpha = r_i dq^i$ defined in (3.11) on the trivial cotangent bundle T^*V_α . We pull this one-form back to the local tangent bundle using the local fiber derivative and then arrive at the local Poincaré-Cartan one-form

$$\theta_\alpha = \mathbb{F}L_\alpha^* \Theta_\alpha = \frac{\partial L_\alpha}{\partial \dot{q}^i} dq^i, \quad r_i = \frac{\partial L_\alpha}{\partial \dot{q}^i}. \quad (3.64)$$

Using this local one-form, we can define the local Poincaré-Cartan two-form

$$\omega_\alpha = \mathbb{F}L_\alpha^* \Omega_\alpha = -\mathbb{F}L_\alpha^* d\Theta_\alpha = -d\mathbb{F}L_\alpha^* \Theta_\alpha = -d\theta_\alpha \quad (3.65)$$

on TV_α . In coordinates we have

$$\omega_\alpha = \frac{\partial^2 L_\alpha}{\partial \dot{q}^i \partial \dot{q}^j} dq^i \wedge dq^j + \frac{\partial^2 L_\alpha}{\partial \dot{q}^i \partial q^j} dq^i \wedge d\dot{q}^j. \quad (3.66)$$

This local two-form is closed but not necessarily non-degenerate. It is non-degenerate if and only if the rank of the Hessian matrix $[\partial^2 L_\alpha / \partial \dot{q}^i \partial \dot{q}^j]$ is the maximum. In this case, ω_α turns out to be a local symplectic two-form on TV_α .

The local energy function E_{L_α} for a given local Lagrangian function L_α is defined by means of the Liouville vector field Δ as

$$E_{L_\alpha} = \Delta(L_\alpha) - L_\alpha, \quad \Delta = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}. \quad (3.67)$$

We introduce the following covariance relation

$$\iota_{Y_\alpha} \omega_\alpha = dE_{L_\alpha} \quad (3.68)$$

on the tangent bundle TV_α . This equation reads the local Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L_\alpha}{\partial \dot{q}^i} = \frac{\partial L_\alpha}{\partial q^i}. \quad (3.69)$$

It is noteworthy that, to determine a local vector field that generates the local Euler-Lagrange equations, the non-degeneracy of the local Lagrangian function L_α is necessary.

Locally Conformal Poincaré-Cartan Forms. As we have announced previously, the tangent bundle TQ is assumed to be covered by local charts TV_α , and on each local chart, we have local Lagrangian functions. We assume that, on intersections, say $TV_\alpha \cap TV_\beta$, the local Lagrangian functions satisfy the following conformal relation

$$e^{\sigma_\alpha} L_\alpha = e^{\sigma_\beta} L_\beta \quad (3.70)$$

for a family of smooth functions $\{\sigma_\alpha\}$. This construction reads that the local Lagrangian functions $\{L_\alpha\}$ cannot be glued up to a real-valued Lagrangian function on the tangent bundle TQ . Instead, these local functions define a line-bundle valued function on TQ . On the other hand, it is immediate to see that multiplying the local functions by conformal factors, the ones in (3.70), read a global real-valued function L on TQ . The restriction of L to an open chart TV_α is

$$L|_\alpha = e^{\sigma_\alpha} L_\alpha. \quad (3.71)$$

This local-to-global realization is also true for local Poincaré-Cartan forms. Notice that local Poincaré-Cartan forms defined as (3.64) are related with conformal factors on the intersection $TV_\alpha \cap TV_\beta$. That is

$$e^{\sigma_\alpha} \theta_\alpha = e^{\sigma_\beta} \theta_\beta. \quad (3.72)$$

In light of this observation, we can argue that the local Poincaré-Cartan one-forms can only be glued up to a line bundle valued one-form. Again, following the same argument, we see that multiplying the local Poincaré-Cartan one-form with conformal factors results in a global Poincaré-Cartan one-form θ_L on TQ . To see this, we propose the following

$$\theta_L|_\alpha = \mathbb{F}L|_\alpha^*(\Theta_Q|_\alpha) = \mathbb{F}L|_\alpha^*(e^{\sigma_\alpha} \Theta_\alpha) = e^{\sigma_\alpha} \mathbb{F}L|_\alpha^*(\Theta_\alpha) = e^{\sigma_\alpha} \theta_\alpha, \quad (3.73)$$

where we denote $\exp(\sigma_\alpha) \circ \mathbb{F}L|_\alpha = \exp(\sigma_\alpha)$ by the abuse of notation. In other words, we have the relation

$$\theta_\alpha = e^{-\sigma_\alpha} \theta_L|_\alpha = e^{-\sigma_\alpha} \frac{\partial L|_\alpha}{\partial \dot{q}^i} dq^i, \quad (3.74)$$

where $\theta_L|_\alpha$ is the restriction of the one-form defined on TQ to the chart TV_α . Accordingly, we have that local Poincaré-Cartan two-forms are related as

$$e^{\sigma_\alpha} \omega_\alpha = e^{\sigma_\beta} \omega_\beta, \quad (3.75)$$

and we define local two-forms

$$\omega_\varphi|_\alpha = e^{\sigma_\alpha} \omega_\alpha \quad (3.76)$$

that can be glued up to a real-valued two-form. Note that the minus of the exterior derivative of $\theta_L|_\alpha$ gives the following two-form

$$\omega_L|_\alpha = -d\theta_L|_\alpha = -d(e^{\sigma_\alpha} \theta_\alpha) = -e^{\sigma_\alpha} d\sigma_\alpha \wedge \theta_\alpha - e^{\sigma_\alpha} d\theta_\alpha = -d\sigma_\alpha \wedge e^{\sigma_\alpha} \theta_\alpha + e^{\sigma_\alpha} \omega_\alpha. \quad (3.77)$$

The global realizations of each term give that

$$\omega_\varphi = \omega_L + \varphi \wedge \theta_L \quad (3.78)$$

while in terms of the Lichnerowicz-deRham differential we have

$$\omega_\varphi = -d_\varphi \theta_L. \quad (3.79)$$

This two-form is locally conformal Poincaré-Cartan two-form. We present the discussion in the following commutative diagram.

$$\begin{array}{ccc}
 & \omega_L|_\alpha & \\
 & (3.77) & \\
 & \uparrow -d & \searrow \text{twist (3.78)} \\
 \theta_L|_\alpha & & \omega_\varphi|_\alpha \\
 (3.73) & \xrightarrow{-d_\varphi} & (3.79) \\
 \uparrow e^{\sigma_\alpha} & & \uparrow e^{\sigma_\alpha} \\
 \theta_\alpha & \xrightarrow{-d} & \omega_\alpha \\
 (3.64) & & (3.65)
 \end{array}$$

Let us glue up the Euler-Lagrange equation (3.68). First, by gluing the local energy function, we get

$$E_L|_\alpha = e^{\sigma_\alpha} E_{L_\alpha}. \quad (3.80)$$

Using (3.76) and (3.80) in the local Euler-Lagrange equation (3.68) we arrive at the locally conformal Euler-Lagrange (abbreviated as LCEL) equation

$$\iota_{\xi_L} \omega_\varphi = d_\varphi E_L \quad (3.81)$$

where the vector field ξ_L is the global picture of the vector field Y_α . In the following proposition, we record the local characterization of the locally conformal Euler-Lagrange equation, see also [63].

Proposition 3.4: In the induced coordinates $(q^i, \dot{q}^i)$ on the tangent bundle TQ , the locally conformal Euler-Lagrange equations (3.81) are

$$\frac{d}{dt}\left(\frac{\partial L|_\alpha}{\partial \dot{q}^i}\right) - \frac{\partial L|_\alpha}{\partial q^i} = \varphi_j \dot{q}^j \left(\frac{\partial L|_\alpha}{\partial \dot{q}^i}\right) - \varphi_i L|_\alpha. \quad (3.82)$$

Proof. Referring to the relation (3.71) and assuming that $\sigma_\alpha = \sigma_\alpha(q)$ we compute

$$p_i = e^{\sigma_\alpha} r_i = e^{\sigma_\alpha} \frac{\partial L_\alpha}{\partial \dot{q}^i} = \frac{\partial e^{\sigma_\alpha} L_\alpha}{\partial \dot{q}^i} = \frac{\partial L|_\alpha}{\partial \dot{q}^i}, \quad \frac{\partial L|_\alpha}{\partial q^i} = e^{\sigma_\alpha} \frac{\partial L_\alpha}{\partial q^i}. \quad (3.83)$$

The (inverse) Legendre transformation reads the associated Lagrangian function $L|_\alpha$ for a given Hamiltonian function $H|_\alpha$ as

$$L|_\alpha(q^i, \dot{q}^i) = e^{\sigma_\alpha} L_\alpha(q^i, \dot{q}^i) = \dot{q}^i e^{\sigma_\alpha} r_i(q, \dot{q}) - H|_\alpha(q^i, e^{\sigma_\alpha} r_i(q, \dot{q})) \quad (3.84)$$

then we have the partial derivative as

$$\frac{\partial L|_\alpha}{\partial q^j} = \dot{q}^i e^{\sigma_\alpha} \varphi_j r_i + \dot{q}^i e^{\sigma_\alpha} \frac{\partial r_i}{\partial q^j} - \frac{\partial H|_\alpha}{\partial q^j} - \frac{\partial H|_\alpha}{\partial p_i} \left(e^{\sigma_\alpha} \varphi_j r_i(q, \dot{q}) + e^{\sigma_\alpha} \frac{\partial r_i}{\partial q^j} \right) = - \frac{\partial H|_\alpha}{\partial q^j}. \quad (3.85)$$

These two equations give us that $L|_\alpha$ is the (inverse) Legendre transformation of the Hamiltonian function $H|_\alpha$ by the relation

$$L|_\alpha(q, \dot{q}) = \dot{q}^i p_i - H|_\alpha(q, p). \quad (3.86)$$

One way to obtain the equation of motion for the Lagrangian function is to use the Euler-Lagrange equation (3.69) as

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L_\alpha}{\partial \dot{q}^i}\right) - \frac{\partial L_\alpha}{\partial q^i} &= \frac{d}{dt}\left(\frac{\partial e^{-\sigma_\alpha} L|_\alpha}{\partial \dot{q}^i}\right) - \frac{\partial e^{-\sigma_\alpha} L|_\alpha}{\partial q^i} \\ &= e^{-\sigma_\alpha} \left(-\varphi_j \dot{q}^j \left(\frac{\partial L|_\alpha}{\partial \dot{q}^i}\right) + \frac{d}{dt}\left(\frac{\partial L|_\alpha}{\partial \dot{q}^i}\right) + \varphi_i L|_\alpha - \frac{\partial L|_\alpha}{\partial q^i} \right). \end{aligned} \quad (3.87)$$

So we have the local realization of the locally conformal Euler-Lagrange (abbreviated as LCEL) equations as in (3.82). Another way is arrive at this system of equations is to substitute the relations (3.83), (3.84) and (3.85) into the LCS Hamilton's equation (3.29). $\square$

Variational Analysis of LCEL Dynamics. We elaborate the variational foundations of LCEL dynamics. First, introduce the local action integral

$$S_\alpha(q) = \int_a^b L_\alpha(q, \dot{q}) dt \quad (3.88)$$

referring to the local Lagrangian function L_α . Then substitute the local-global relation (3.71) into the integral. This turns the action as a functional defined on the global function L . Accordingly, we compute the variation as

$$\begin{aligned}
\delta \int_a^b L_\alpha(q, \dot{q}) dt &= \delta \int_a^b e^{-\sigma_\alpha(q)} L|_\alpha(q, \dot{q}) dt \\
&= \int_a^b \left(-e^{-\sigma_\alpha(q)} \frac{\partial \sigma_\alpha}{\partial q^i} L|_\alpha \delta q^i + e^{-\sigma_\alpha(q)} \frac{\partial L|_\alpha}{\partial q^i} \delta q^i + e^{-\sigma_\alpha(q)} \frac{\partial L|_\alpha}{\partial \dot{q}^i} \delta \dot{q}^i \right) dt \\
&= \int_a^b \left(-e^{-\sigma_\alpha(q)} \frac{\partial \sigma_\alpha}{\partial q^i} L|_\alpha \delta q^i + e^{-\sigma_\alpha(q)} \frac{\partial L|_\alpha}{\partial q^i} \delta q^i \right) dt + e^{-\sigma_\alpha(q)} \frac{\partial L|_\alpha}{\partial \dot{q}^i} \delta q^i \Big|_a^b \\
&\quad - \int_a^b \left(\left(-e^{-\sigma_\alpha(q)} \frac{\partial L|_\alpha}{\partial \dot{q}^i} \frac{\partial \sigma_\alpha}{\partial q^j} \dot{q}^j + e^{-\sigma_\alpha(q)} \frac{d}{dt} \frac{\partial L|_\alpha}{\partial \dot{q}^i} \right) \delta q^i \right) dt \\
&= \int_a^b \left(e^{-\sigma_\alpha(q)} \left(-\frac{\partial \sigma_\alpha}{\partial q^i} L|_\alpha + \frac{\partial L|_\alpha}{\partial q^i} - \frac{d}{dt} \frac{\partial L|_\alpha}{\partial \dot{q}^i} + \frac{\partial L|_\alpha}{\partial \dot{q}^i} \frac{\partial \sigma_\alpha}{\partial q^j} \dot{q}^j \right) \delta q^i \right) dt.
\end{aligned} \tag{3.89}$$

Then the fundamental theorem of calculus of variations gives that the variation of the action integral is zero if

$$\frac{d}{dt} \frac{\partial L|_\alpha}{\partial \dot{q}^i} - \frac{\partial L|_\alpha}{\partial q^i} = \frac{\partial L|_\alpha}{\partial \dot{q}^i} \frac{\partial \sigma_\alpha}{\partial q^j} \dot{q}^j - \frac{\partial \sigma_\alpha}{\partial q^i} L|_\alpha \tag{3.90}$$

which is precisely the locally conformal Euler-Lagrange equations in (3.82). See that when the conformal factor is zero, we obtain the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0. \tag{3.91}$$

4. DISCRETE DYNAMICS ON LOCALLY CONFORMAL FRAMEWORK

4.1. Discrete Lagrangian and Discrete Hamiltonian Dynamics

In continuous dynamics, Lagrangian function is defined on the tangent bundle TQ . In this case, solving the Euler-Lagrange equations (2.46) involves determining a smooth curve $q = q(t)$ in Q . This curve, along with its velocity $(q(t), \dot{q}(t))$, is substituted into the Lagrangian function to achieve an extremum of the action integral. We cite [64–67] for more details on the discrete formulations presented in this subsection.

To discretize the Lagrangian dynamics we replace the role of the smooth curve with a finite sequence of points. For example, let $a, b \in \mathbb{R}$ and $a < b$ then define the step $h = (b - a)/N$, where N is the number of divisions, that is the number of elements in the finite sequence. We determine such a sequence by

$$[\mathbf{q}] = (\mathbf{q}_a = \mathbf{q}_0, \dots, \mathbf{q}_b = \mathbf{q}_N). \quad (4.1)$$

Here, the bold notation stands for $\mathbf{q} = (q^i)$. In the discrete framework, the Lagrangian function is substituted by a discrete Lagrangian function $L^d : Q \times Q \rightarrow \mathbb{R}$ which is an approximation to the exact discrete Lagrangian

$$L^{d-ex}(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) = \int_{t_\kappa}^{t_{\kappa+1}} L(\mathbf{q}(t), \dot{\mathbf{q}}(t)) dt, \quad (4.2)$$

where $\mathbf{q} : [t_\kappa, t_{\kappa+1}] \rightarrow Q$ is the solution of the continuous Euler-Lagrange equation with boundary conditions $\mathbf{q}(t_\kappa) = \mathbf{q}_\kappa$ and $\mathbf{q}(t_{\kappa+1}) = \mathbf{q}_{\kappa+1}$. In the following proposition we state discrete Euler-Lagrange equations, see, for example, [64].

Proposition 4.1: The discrete Euler-Lagrange (dEL) equation, arising from a discrete Lagrangian function L^d , is

$$D_2 L^d(\mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa) + D_1 L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) = 0, \quad (4.3)$$

where D_1 denotes the partial derivative with respect to the first argument whereas D_2 is the partial derivative with respect to the second argument.

Proof. Let us write action sum for discrete Lagrangian L^d as the following finite sum:

$$S^d([\mathbf{q}]) = \sum_{\kappa=0}^{N-1} L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}). \quad (4.4)$$

We take the variation of this action and endow the discrete variational principle $\delta S^d(\mathbf{q}) = 0$. This gives us the equality:

$$\delta S^d(\mathbf{q}) = \sum_{\kappa=0}^{N-1} (D_1 L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) \delta \mathbf{q}_\kappa + D_2 L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) \delta \mathbf{q}_{\kappa+1}) = 0. \quad (4.5)$$

Considering that at the boundary the variations are identically zero that is $\delta \mathbf{q}_0 = \delta \mathbf{q}_N = 0$, we have the following expression:

$$\sum_{\kappa=1}^{N-1} D_1 L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) \delta \mathbf{q}_\kappa + \sum_{\kappa=0}^{N-2} D_2 L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) \delta \mathbf{q}_{\kappa+1} = 0. \quad (4.6)$$

In the second term, we change κ with $\kappa - 1$ in order to take the whole expression into the parenthesis of $\delta \mathbf{q}_\kappa$. This reads

$$\sum_{\kappa=1}^{N-1} (D_1 L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) + D_2 L^d(\mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa)) \delta \mathbf{q}_\kappa = 0. \quad (4.7)$$

For each point in the sequence, we get the discrete Euler-Lagrange equation (4.3). $\square$

The Discrete Legendre Transformation. As in the case of continuous dynamics, we employ the Legendre transformation to pass from the discrete Lagrangian dynamics to the discrete Hamiltonian dynamics. In the discrete case, we have the right and left Legendre transformations

$$\mathbb{F}^\pm L^d : Q \times Q \longrightarrow T^*Q \quad (4.8)$$

given respectively by

$$\begin{aligned} \mathbb{F}^+ L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) &= (\mathbf{q}_{\kappa+1}, D_2 L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1})), \\ \mathbb{F}^- L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) &= (\mathbf{q}_\kappa, -D_1 L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1})). \end{aligned} \quad (4.9)$$

We determine the momenta as the second entry of the image space of the discrete Legendre transformations. Accordingly, we have the right and the left momenta as

$$\begin{aligned} \mathbf{p}_{\kappa, \kappa+1}^+ &= D_2 L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}), \\ \mathbf{p}_{\kappa, \kappa+1}^- &= -D_1 L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}), \end{aligned} \quad (4.10)$$

respectively. See that from the discrete Euler-Lagrange equation (4.3) we can define momenta depending on a single point as

$$\mathbf{p}_\kappa := \mathbf{p}_{\kappa-1,\kappa}^+ = \mathbf{p}_{\kappa,\kappa+1}^- \quad (4.11)$$

Accordingly, we can write

$$\begin{aligned} \mathbf{p}_\kappa &= -D_1 L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}), \\ \mathbf{p}_{\kappa+1} &= D_2 L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}). \end{aligned} \quad (4.12)$$

In the continuous case, the Legendre transformation is non-degenerate if the Lagrangian function is regular, meaning that the rank of the Hessian of the Lagrangian function is at its maximum. To analyze this phenomenon in the discrete case, we introduce discrete (right and left) Poincaré-Cartan one-forms:

$$\begin{aligned} \theta^+ &= D_2 L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) d\mathbf{q}_{\kappa+1}, \\ \theta^- &= -D_1 L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) d\mathbf{q}_\kappa. \end{aligned} \quad (4.13)$$

The exterior derivatives of these one-forms coincide and determine the discrete Poincaré-Cartan two-form as

$$\omega^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) = -d\theta^\pm = -D_1 D_2 L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) d\mathbf{q}_\kappa \wedge d\mathbf{q}_{\kappa+1}. \quad (4.14)$$

The discrete Legendre transformation (4.8) is a local diffeomorphism if the discrete Poincaré-Cartan two-form is non-degenerate (hence symplectic). In this case, the discrete Lagrangian function is called regular. We determine the discrete Hamiltonian dynamics under this assumption.

For a regular discrete Lagrangian function L^d , referring to the Darboux coordinates, we define the right discrete Hamiltonian function as the difference between the coupling function and the discrete Lagrangian function as

$$H^{d+}(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}) = \mathbf{p}_{\kappa+1} \cdot \mathbf{q}_{\kappa+1} - L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}). \quad (4.15)$$

On the other hand, we define the left discrete Hamiltonian function as the difference between the *minus* of the coupling function and the discrete Lagrangian function as

$$H^{d-}(\mathbf{q}_{\kappa+1}, \mathbf{p}_\kappa) = -\mathbf{p}_\kappa \cdot \mathbf{q}_\kappa - L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}). \quad (4.16)$$

The following proposition presents the discrete Hamiltonian dynamics for both the left and right Hamiltonian functions.

Proposition 4.2: The right discrete Hamilton's equations generated by the right Hamiltonian function (4.15) are

$$\begin{cases} \mathbf{q}_{\kappa+1} = D_2 H^{d+}(\mathbf{q}_{\kappa}, \mathbf{p}_{\kappa+1}), \\ \mathbf{p}_{\kappa} = D_1 H^{d+}(\mathbf{q}_{\kappa}, \mathbf{p}_{\kappa+1}), \end{cases} \quad (4.17)$$

whereas the left discrete Hamilton's equations generated by the left Hamiltonian function (4.16) are

$$\begin{cases} \mathbf{q}_{\kappa} = -D_2 H^{d-}(\mathbf{q}_{\kappa+1}, \mathbf{p}_{\kappa}), \\ \mathbf{p}_{\kappa+1} = -D_1 H^{d-}(\mathbf{q}_{\kappa+1}, \mathbf{p}_{\kappa}), \end{cases} \quad (4.18)$$

where D_a denotes the partial derivative with respect to the a -th entry.

Proof. We consider the right Hamiltonian function (4.15), then take the partial derivatives of it with respect to its first entry $\mathbf{q}_{\kappa}$ and its second entry $\mathbf{p}_{\kappa+1}$. This gives us

$$\begin{aligned} D_1 H^{d+}(\mathbf{q}_{\kappa}, \mathbf{p}_{\kappa+1}) &= -D_1 L^d(\mathbf{q}_{\kappa}, \mathbf{q}_{\kappa+1}) \\ D_2 H^{d+}(\mathbf{q}_{\kappa}, \mathbf{p}_{\kappa+1}) &= \mathbf{q}_{\kappa+1}. \end{aligned} \quad (4.19)$$

Substituting the momentum definition from (4.12), we obtain the right discrete Hamilton's equations (4.17).

On the other hand, in the left case, we compute the partial derivatives of the left Hamiltonian function (4.16) with respect to its first entry $\mathbf{q}_{\kappa+1}$ and its second entry $\mathbf{p}_{\kappa}$, yielding:

$$\begin{aligned} D_1 H^{d-}(\mathbf{q}_{\kappa+1}, \mathbf{p}_{\kappa}) &= -D_2 L^d(\mathbf{q}_{\kappa}, \mathbf{q}_{\kappa+1}) \\ D_2 H^{d-}(\mathbf{q}_{\kappa+1}, \mathbf{p}_{\kappa}) &= -\mathbf{q}_{\kappa}. \end{aligned} \quad (4.20)$$

Substituting (4.12), we derive the left discrete Hamilton's equations (4.18). $\square$

4.2. Locally Conformal Discrete Lagrangian Dynamics

In this subsection, we generalize the discrete dynamics discussions presented in the previous subsection to the locally conformal framework. In this novel construction, we start with elaborating the discretization of the locally conformally Euler-Lagrange (LCEL) dynamics given in Subsection 3.5.

For a manifold Q , we assume an open covering as in (3.17), consisting of local charts $\{V_\alpha\}$. For each local chart V_α , we define the discrete local Lagrangian function L_α^d on the product vector space $V_\alpha \times V_\alpha$ as follows

$$L_\alpha^d : V_\alpha \times V_\alpha \longrightarrow \mathbb{R}. \quad (4.21)$$

Referring to Proposition 4.1 and the dynamical equation (4.3), we express the local discrete dynamics generated by this local discrete Lagrangian as

$$D_2 L_\alpha^d(\mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa) + D_1 L_\alpha^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) = 0. \quad (4.22)$$

Recalling from the construction in Subsection 3.5, we assume locally conformal character determined by the family of scalar functions $\{\sigma_\alpha\}$ so that the local discrete Lagrangian functions are satisfying

$$e^{\sigma_\alpha} L_\alpha^d = e^{\sigma_\beta} L_\beta^d \quad (4.23)$$

on the intersection $V_\alpha \cap V_\beta$. Following (3.71) in Subsection 3.5, we define a global discrete Lagrangian function L^d on the whole manifold Q whose local realization is

$$L^d|_\alpha = e^{\sigma_\alpha} L_\alpha^d. \quad (4.24)$$

The following proposition establishes discrete dynamics in terms of the global discrete Lagrangian function L^d , presenting the discretization of the locally conformal Euler-Lagrange equations (3.82).

Proposition 4.3: The discrete dynamics generated by the global discrete Lagrangian function L^d defined in (4.24) is given by:

$$\begin{aligned} e^{\sigma_\alpha(\mathbf{q}_\kappa) - \sigma_\alpha(\mathbf{q}_{\kappa-1})} D_2 L^d|_\alpha(\mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa) \\ = D\sigma_\alpha(\mathbf{q}_\kappa) \cdot L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) - D_1 L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}), \end{aligned} \quad (4.25)$$

where D_a denotes the partial derivative with respect to the a -th entry whereas $D\sigma_\alpha$ is the partial derivative of the function σ_α .

Proof. We begin with the discrete action

$$S^d([\mathbf{q}]) = \sum_{\kappa=0}^{N-1} L_\alpha^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) \quad (4.26)$$

where L_α^d represents the local discrete Lagrangian function. To write the action sum for the global discrete Lagrangian function, we substitute (4.24) into the discrete action (4.26) yields

$$S^d([\mathbf{q}]) = \sum_{\kappa=0}^{N-1} e^{-\sigma_\alpha(\mathbf{q}_\kappa)} L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}). \quad (4.27)$$

See that the action sum takes different values for different sequences. The variation of the action is computed to be

$$\begin{aligned} \delta S^d([\mathbf{q}]) &= \sum_{\kappa=0}^{N-1} (-e^{-\sigma_\alpha(\mathbf{q}_\kappa)} D\sigma_\alpha(\mathbf{q}_\kappa) L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) \delta \mathbf{q}_\kappa \\ &\quad + e^{-\sigma_\alpha(\mathbf{q}_\kappa)} D_1 L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) \delta \mathbf{q}_\kappa \\ &\quad + e^{-\sigma_\alpha(\mathbf{q}_\kappa)} D_2 L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) \delta \mathbf{q}_{\kappa+1}). \end{aligned} \quad (4.28)$$

In order to extremize the action sum, we set $\delta S^d([\mathbf{q}]) = 0$ while fixing the boundary conditions: $\delta \mathbf{q}_0 = 0$ and $\delta \mathbf{q}_N = 0$. Accordingly, we obtain

$$\begin{aligned} &\sum_{\kappa=1}^{N-1} e^{-\sigma_\alpha(\mathbf{q}_\kappa)} (-D\sigma_\alpha(\mathbf{q}_\kappa) L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) + D_1 L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1})) \delta \mathbf{q}_\kappa \\ &\quad + \sum_{\kappa=0}^{N-2} e^{-\sigma_\alpha(\mathbf{q}_\kappa)} D_2 L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) \delta \mathbf{q}_{\kappa+1} = 0. \end{aligned} \quad (4.29)$$

By relabeling κ with $\kappa - 1$ in the last term we write

$$\begin{aligned} &\sum_{\kappa=1}^{N-1} \left(e^{-\sigma_\alpha(\mathbf{q}_\kappa)} (-D\sigma_\alpha(\mathbf{q}_\kappa) L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) + D_1 L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1})) \right. \\ &\quad \left. + e^{-\sigma_\alpha(\mathbf{q}_{\kappa-1})} D_2 L^d|_\alpha(\mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa) \right) \delta \mathbf{q}_\kappa = 0. \end{aligned} \quad (4.30)$$

Thus for arbitrary $\delta \mathbf{q}_\kappa$, the equality holds if (4.25) is satisfied. $\square$

We call (4.25) as discrete locally conformally Euler-Lagrange (dLCEL) equations. It is noteworthy that if there is no locally conformal character within the setting, meaning $L|_\alpha = L_\alpha$, then Proposition 4.3 aligns with Proposition 4.1, and the discrete locally conformally Euler-Lagrange equation (4.25) reduces to the discrete Euler-Lagrange equation (4.3). The following diagram summarizes this discussion.

$$\begin{array}{ccc} LCEL & \xrightarrow{\varphi=0} & EL \\ (3.82) & & (2.46) \\ \downarrow \text{discretization} & & \downarrow \text{discretization} \\ dLCEL & \xrightarrow{\varphi=0} & dEL \\ (4.25) & & (4.3) \end{array} \quad (4.31)$$

The Discrete Legendre Transformation. For a local chart V_α we introduce the local right and left Legendre Transformations as

$$\mathbb{F}^\pm L_\alpha^d : V_\alpha \times V_\alpha \longrightarrow T^*V_\alpha \quad (4.32)$$

given explicitly by

$$\begin{aligned} \mathbb{F}^+ L_\alpha^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) &= (\mathbf{q}_{\kappa+1}, r_{\kappa, \kappa+1}^+ = D_2 L_\alpha^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1})), \\ \mathbb{F}^- L_\alpha^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) &= (\mathbf{q}_\kappa, r_{\kappa, \kappa+1}^- = -D_1 L_\alpha^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1})), \end{aligned} \quad (4.33)$$

respectively. Here, $r_{\kappa, \kappa+1}^+$ represents the right local momenta, while $r_{\kappa, \kappa+1}^-$ denotes the left local momenta. To express the discrete Legendre transformations in terms of the global discrete Lagrangian function L^d , we utilize the identification (4.24) and substitute it into the definitions of local momenta. Consequently, we obtain

$$\begin{aligned} \mathbf{r}_{\kappa, \kappa+1}^+ &= e^{-\sigma_\alpha(\mathbf{q}_\kappa)} D_2 L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}), \\ \mathbf{r}_{\kappa, \kappa+1}^- &= e^{-\sigma_\alpha(\mathbf{q}_\kappa)} \left(D\sigma_\alpha(\mathbf{q}_\kappa) \cdot L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) - D_1 L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) \right). \end{aligned} \quad (4.34)$$

Under dLCEL equation (4.25), and after a parameter shift, the right and the left momenta turn out to be the same and this results in the following definition

$$\mathbf{r}_\kappa := \mathbf{r}_{\kappa-1, \kappa}^+ = \mathbf{r}_{\kappa, \kappa+1}^-. \quad (4.35)$$

Recall that in the continuous case, there exist two sets of momenta in a relation given by $r_i = e^{-\sigma_\alpha} p_i$. This is depicted in (3.20). Motivated by this, and following the order in (4.34), we define two new sets of local momenta

$$\begin{aligned} \mathbf{p}_{\kappa, \kappa+1}^+ &= e^{\sigma_\alpha(\mathbf{q}_\kappa)} r_{\kappa, \kappa+1}^+ = D_2 L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}), \\ \mathbf{p}_{\kappa, \kappa+1}^- &= e^{\sigma_\alpha(\mathbf{q}_\kappa)} r_{\kappa, \kappa+1}^- = D\sigma_\alpha(\mathbf{q}_\kappa) \cdot L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) - D_1 L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}). \end{aligned} \quad (4.36)$$

A naive shift of parameters is not enough to define a momenta $\mathbf{p}$ depending on a single point. More precisely one has that

$$\mathbf{p}_\kappa := e^{\sigma_\alpha(\mathbf{q}_\kappa) - \sigma_\alpha(\mathbf{q}_{\kappa-1})} \mathbf{p}_{\kappa-1, \kappa}^+ = \mathbf{p}_{\kappa, \kappa+1}^-. \quad (4.37)$$

It is immediate to see that $\mathbf{r}$ -momenta in (4.35) and $\mathbf{p}$ -momenta in (4.37) are related as

$$\mathbf{r}_\kappa = e^{-\sigma_\alpha(\mathbf{q}_\kappa)} \mathbf{p}_\kappa \quad (4.38)$$

as we expected.

Poincaré-Cartan Forms. Referring to (4.13), we pull back the local canonical one-form Θ_α in (3.20), defined on the cotangent bundle T^*V_α , to the product space $V_\alpha \times V_\alpha$ via the right and left Legendre transformations (4.33). Then, in a respective manner, we define the right and left local Poincaré-Cartan one-forms for the local Lagrangian function L_α as follows

$$\begin{aligned}\theta_\alpha^+ &= \mathbb{F}^+ L_\alpha^d(\Theta_\alpha) = D_2 L_\alpha^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) d\mathbf{q}_{\kappa+1}, \\ \theta_\alpha^- &= \mathbb{F}^- L_\alpha^d(\Theta_\alpha) = -D_1 L_\alpha^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) d\mathbf{q}_\kappa.\end{aligned}\tag{4.39}$$

Exterior derivatives of these one-forms coincide and determine local Poincaré-Cartan two-form

$$\omega_\alpha^\pm = -d\theta_\alpha^\pm\tag{4.40}$$

This is a symplectic two-form if the local Lagrangian function is regular.

Let us use the discrete Legendre transformations to elucidate the locally conformal character of Proposition 4.3 and the dLCEL equation presented in (4.25). We do this for the right Poincaré-Cartan one-form the left case is more or less following the same lines of calculation. Referring to (4.24), we write the right Poincaré-Cartan one-form in terms of the global discrete Lagrangian function as

$$\begin{aligned}\theta_\alpha^+ &= D_2(e^{-\sigma_\alpha(\mathbf{q}_\kappa)} L|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1})) d\mathbf{q}_{\kappa+1} \\ &= e^{-\sigma_\alpha(\mathbf{q}_\kappa)} D_2 L|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) d\mathbf{q}_{\kappa+1}.\end{aligned}\tag{4.41}$$

This one-form fails to be global as in the continuous case. To have a global one we introduce a one-form θ^{d+} whose restriction to V_α is

$$\theta^{d+}|_\alpha = e^{\sigma_\alpha} \theta_\alpha^+ = D_2 L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) d\mathbf{q}_{\kappa+1}.\tag{4.42}$$

Taking the negative exterior derivative of this one-form we obtain a global two-form

$$\begin{aligned}\omega^{d+}|_\alpha &= -d\theta^{d+}|_\alpha = -e^{\sigma_\alpha(\mathbf{q}_\kappa)} d\theta_\alpha^+ - e^{\sigma_\alpha(\mathbf{q}_\kappa)} d\sigma_\alpha(\mathbf{q}_\kappa) \wedge \theta_\alpha^+ \\ &= e^{\sigma_\alpha(\mathbf{q}_\kappa)} (\omega_\alpha^+ - d\sigma_\alpha(\mathbf{q}_\kappa) \wedge \theta_\alpha^+).\end{aligned}\tag{4.43}$$

We introduce discrete locally conformally Poincaré-Cartan two-form as

$$\omega_\varphi^{d+}|_\alpha = e^{\sigma_\alpha(\mathbf{q}_\kappa)} \omega_\alpha^+,\tag{4.44}$$

where $\omega_\varphi^{d+}|_\alpha$ denotes the restriction of ω_φ^{d+} to local chart TV_α . Referring to (4.43), it is a direct observation to see that

$$\omega_\varphi^{d+}|_\alpha = \omega^{d+}|_\alpha + d\sigma_\alpha(\mathbf{q}_\kappa) \wedge \theta^+|_\alpha.\tag{4.45}$$

is satisfying the LCS two-form construction in (3.10).

Proposition 4.4: The discrete locally conformally Poincaré-Cartan two-form ω_φ^{d+} defined in (4.45) is a LCS two-form with the Lee one-form is $d\sigma_\alpha$.

Proof. Let us now perform the following calculation to assure once more that the discrete locally conformally Poincaré-Cartan two-form ω_φ^{d+} is indeed an LCS two-form:

$$\begin{aligned}
\omega_\varphi^{d+}|_\alpha &= -d(D_2 L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1})d\mathbf{q}_{\kappa+1}) \\
&\quad + d\sigma_\alpha(\mathbf{q}_\kappa) \wedge D_2 L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1})d\mathbf{q}_{\kappa+1} \\
&= -\frac{\partial^2 L^d|_\alpha}{\partial \mathbf{q}_\kappa^i \partial \mathbf{q}_{\kappa+1}^j} d\mathbf{q}_\kappa^i \wedge d\mathbf{q}_{\kappa+1}^j - \frac{\partial^2 L^d|_\alpha}{\partial \mathbf{q}_{\kappa+1}^i \partial \mathbf{q}_{\kappa+1}^j} d\mathbf{q}_{\kappa+1}^i \wedge d\mathbf{q}_{\kappa+1}^j \\
&\quad + \frac{\partial \sigma_\alpha}{\partial \mathbf{q}_\kappa^i} \frac{\partial L^d|_\alpha}{\partial \mathbf{q}_{\kappa+1}^j} d\mathbf{q}_\kappa^i \wedge d\mathbf{q}_{\kappa+1}^j.
\end{aligned} \tag{4.46}$$

Then we take the exterior derivative of this explicit expression as

$$\begin{aligned}
d\omega_\varphi^{d+}|_\alpha &= \frac{\partial \sigma_\alpha}{\partial \mathbf{q}_\kappa^i} \frac{\partial^2 L^d|_\alpha}{\partial \mathbf{q}_\kappa^l \partial \mathbf{q}_{\kappa+1}^j} d\mathbf{q}_\kappa^l \wedge d\mathbf{q}_\kappa^i \wedge d\mathbf{q}_{\kappa+1}^j \\
&= \frac{\partial \sigma_\alpha}{\partial \mathbf{q}_\kappa^i} d\mathbf{q}_\kappa^i \wedge \left(-\frac{\partial^2 L^d|_\alpha}{\partial \mathbf{q}_\kappa^l \partial \mathbf{q}_{\kappa+1}^j} d\mathbf{q}_\kappa^l \wedge d\mathbf{q}_{\kappa+1}^j \right) \\
&= d\sigma_\alpha(\mathbf{q}_\kappa) \wedge \omega^+|_\alpha.
\end{aligned} \tag{4.47}$$

reading that that the discrete locally conformally Poincaré-Cartan two-form satisfies the LCS two-form condition in (3.8). $\square$

4.3. Locally Conformal Discrete Hamiltonian Dynamics

We have defined the discrete locally conformal Euler-Lagrange equations (4.25) in Proposition 4.3. We have also determined the right and left Legendre transformations $\mathbb{F}^\pm L_\alpha^d$ in (4.33). In this subsection, our goal is to derive discrete versions of the right and the left discrete Hamiltonian dynamics exhibited in Proposition 4.2.

Let us start with the local picture first by concentrating on the plus momenta. Accordingly, we pass to the Hamiltonian picture applying the right discrete Legendre transformation to the discrete locally conformally Euler-Lagrange equations (4.25). At first, we introduce the Hamiltonian function for $\mathbf{r}$ -momenta as

$$H_\alpha^{d+}(\mathbf{q}_\kappa, \mathbf{r}_{\kappa+1}) = \mathbf{r}_{\kappa+1} \cdot \mathbf{q}_{\kappa+1} - L_\alpha^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}). \tag{4.48}$$

A direct computation reads the local right discrete Hamiltonian dynamics in terms of the $\mathbf{r}$ -momenta as

$$\begin{cases} \mathbf{q}_{\kappa+1} = D_2 H_\alpha^{d+}(\mathbf{q}_\kappa, \mathbf{r}_{\kappa+1}), \\ \mathbf{r}_\kappa = D_1 H_\alpha^{d+}(\mathbf{q}_\kappa, \mathbf{r}_{\kappa+1}). \end{cases}$$

We assume the locally conformal character for the local discrete Hamiltonian functions. Motivated by the continuous case depicted in (3.31), we take

$$e^{\sigma_\alpha} H_\alpha^{d+} = e^{\sigma_\beta} H_\beta^{d+} \quad (4.49)$$

on the intersection of the tangent bundles $TV_\alpha \cap TV_\beta$. The equality determines an obstruction to glue up local functions $\{H_\alpha^d\}$ to a real-valued function on the cotangent bundle T^*Q . However, the same identity determines a global discrete Hamiltonian function H^d whose local characterization is given in terms of the $\mathbf{p}$ -momenta as

$$H^{d+}|_\alpha(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}) := e^{\sigma_\alpha(\mathbf{q}_\kappa)} H_\alpha^{d+}(\mathbf{q}_\kappa, \mathbf{r}_{\kappa+1}). \quad (4.50)$$

Here the $\mathbf{r}$ -momenta and the $\mathbf{p}$ -momenta are related as shown in (4.38).

Proposition 4.5: The right discrete locally conformal Hamiltonian dynamics generated by the right discrete Hamiltonian function H^{d+} given in (4.50) is

$$\begin{cases} \mathbf{q}_{\kappa+1} = e^{\sigma_\alpha(\mathbf{q}_{\kappa+1}) - \sigma_\alpha(\mathbf{q}_\kappa)} D_2 H^{d+}|_\alpha(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}), \\ \mathbf{p}_\kappa = D_1 H^{d+}|_\alpha(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}) - D\sigma_\alpha(\mathbf{q}_\kappa) \cdot H^{d+}|_\alpha(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}). \end{cases} \quad (4.51)$$

Proof. Referring to the global discrete Lagrangian function L^d in (4.24), the equation (4.48) can be written as

$$\begin{aligned} e^{-\sigma_\alpha(\mathbf{q}_\kappa)} H^{d+}|_\alpha(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}) \\ = e^{-\sigma_\alpha(\mathbf{q}_{\kappa+1})} \mathbf{p}_{\kappa+1} \cdot \mathbf{q}_{\kappa+1} - e^{-\sigma_\alpha(\mathbf{q}_\kappa)} L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}). \end{aligned} \quad (4.52)$$

Taking the partial derivative of this equation with respect to its first entry $\mathbf{q}_\kappa$, we arrive at

$$\begin{aligned} -e^{-\sigma_\alpha(\mathbf{q}_\kappa)} D\sigma_\alpha(\mathbf{q}_\kappa) H^{d+}|_\alpha(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}) + e^{-\sigma_\alpha(\mathbf{q}_\kappa)} D_1 H^{d+}|_\alpha(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}) \\ = e^{-\sigma_\alpha(\mathbf{q}_\kappa)} (D\sigma_\alpha(\mathbf{q}_\kappa) L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) - D_1 L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1})) \end{aligned} \quad (4.53)$$

then use the definition $\mathbf{p}_\kappa$ in (4.37) we get the second line in (4.51). In a similar manner, to obtain the first line in (4.51), we differentiate (4.52) with respect to the second entry $\mathbf{p}_{\kappa+1}$, and a direct calculation yields the result. $\square$

Notice that if one assumes that the conformal functions $\sigma_\alpha(\mathbf{q}_\kappa)$ are constant functions, then the right discrete locally conformally symplectic Hamiltonian dynamics in (4.51) reduce to the discrete Hamilton's equation (4.17).

Equivalently, with the left Legendre transformation, we can obtain the local left discrete Hamiltonian

$$H_\alpha^{d-}(\mathbf{q}_{\kappa+1}, \mathbf{r}_\kappa) = -\mathbf{r}_\kappa \mathbf{q}_\kappa - L_\alpha^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) \quad (4.54)$$

and the local left discrete Hamilton's equations

$$\begin{cases} \mathbf{q}_\kappa = -D_2 H_\alpha^{d-}(\mathbf{q}_{\kappa+1}, \mathbf{p}_\kappa), \\ \mathbf{r}_{\kappa+1} = -D_1 H_\alpha^{d-}(\mathbf{q}_{\kappa+1}, \mathbf{p}_\kappa). \end{cases}$$

We express the relation (3.31) for the left local discrete Hamiltonian function to define the left global discrete Hamiltonian function H^{d-} with the following local expression:

$$H^{d-}|_\alpha(\mathbf{q}_{\kappa+1}, \mathbf{p}_\kappa) := e^{\sigma_\alpha(\mathbf{q}_{\kappa+1})} H_\alpha^{d-}(\mathbf{q}_{\kappa+1}, \mathbf{r}_\kappa). \quad (4.55)$$

This leads us to write (4.54) in terms of the global discrete Hamiltonian function as

$$e^{-\sigma_\alpha(\mathbf{q}_{\kappa+1})} H^{d-}|_\alpha(\mathbf{q}_{\kappa+1}, \mathbf{p}_\kappa) = -e^{-\sigma_\alpha(\mathbf{q}_\kappa)} \mathbf{p}_\kappa \cdot \mathbf{q}_\kappa - e^{-\sigma_\alpha(\mathbf{q}_\kappa)} L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}). \quad (4.56)$$

By performing similar calculations done for the case of the right Hamiltonian, we obtain the left discrete LCS Hamilton's equation stated in the following proposition.

Proposition 4.6: The left discrete LCS Hamilton's equation generated by the left discrete Hamiltonian function H^{d-} in (4.55) is

$$\begin{cases} \mathbf{q}_\kappa = -e^{\sigma_\alpha(\mathbf{q}_\kappa) - \sigma_\alpha(\mathbf{q}_{\kappa+1})} D_2 H^{d-}|_\alpha(\mathbf{q}_{\kappa+1}, \mathbf{p}_\kappa), \\ \mathbf{p}_{\kappa+1} = e^{\sigma_\alpha(\mathbf{q}_\kappa) - \sigma_\alpha(\mathbf{q}_{\kappa+1})} (D\sigma_\alpha(\mathbf{q}_{\kappa+1}) \cdot H^{d-}|_\alpha(\mathbf{q}_{\kappa+1}, \mathbf{p}_\kappa) \\ \quad - D_1 H^{d-}|_\alpha(\mathbf{q}_{\kappa+1}, \mathbf{p}_\kappa)). \end{cases} \quad (4.57)$$

The following diagram gives the hierarchy of discrete and continuous Hamiltonian

dynamics discussed in this work:

$$\begin{array}{ccc}
\begin{array}{c} LCS\ HE \\ (3.29) \end{array} & \xrightarrow{\varphi=0} & \begin{array}{c} HE \\ (2.27) \end{array} \\
\downarrow \text{discretization} & & \downarrow \text{discretization} \\
\begin{array}{c} dLCS\ HE \\ \text{right : (4.51)} \\ \text{left : (4.57)} \end{array} & \xrightarrow{\varphi=0} & \begin{array}{c} dHE \\ \text{right : (4.17)} \\ \text{left : (4.18)} \end{array}
\end{array} \tag{4.58}$$

where d refers to term discrete whereas HE stands for an abbreviation of Hamilton's equation.

4.4. Second-Order Discrete Locally Conformal Lagrangian Dynamics

In this subsection, we present the second-order discrete Euler–Lagrange equations and the discrete locally conformal Euler–Lagrange equations. For more information on higher-order discrete Lagrangian dynamics, see [68]. We follow a similar set of steps to those used in the first-order case.

Second-Order Discrete Euler-Lagrange Equations. We consider 3 times Cartesian product of manifold Q that is $Q^3 = Q \times Q \times Q$. We determine two end points $\mathbf{q}_0 = 0$ and $\mathbf{q}_N = 0$ in Q and a sequence $[\mathbf{q}]$ given in (4.1) connecting these two points. Consider the discrete Lagrangian function

$$L^d := Q \times Q \times Q \longrightarrow \mathbb{R}, \quad L^d = L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}) \tag{4.59}$$

where κ is from 0 to $N - 2$. We write action sum for L^d as

$$S^d([\mathbf{q}]) = \sum_{\kappa=0}^{N-2} L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}). \tag{4.60}$$

We compute the variation of the action sum

$$\delta S^d([\mathbf{q}]) = 0 \tag{4.61}$$

under fixed boundary conditions, which impose $\delta \mathbf{q}_0 = 0$ and $\delta \mathbf{q}_N = 0$. This yields the following equation for the coefficients of $\delta \mathbf{q}_1$:

$$D_1 L^d(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) + D_2 L^d(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2) = 0. \tag{4.62}$$

For $2 \leq \kappa \leq N - 2$, the equations for the coefficients of $\delta \mathbf{q}_\kappa$ are obtained as follows:

$$\begin{aligned} D_1 L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}) + D_2 L^d(\mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) \\ + D_3 L^d(\mathbf{q}_{\kappa-2}, \mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa) = 0. \end{aligned} \quad (4.63)$$

Finally, for $\delta \mathbf{q}_{N-1}$, we obtain:

$$D_2 L^d(\mathbf{q}_{N-2}, \mathbf{q}_{N-1}, \mathbf{q}_N) + D_3 L^d(\mathbf{q}_{N-3}, \mathbf{q}_{N-2}, \mathbf{q}_{N-1}) = 0. \quad (4.64)$$

The equations (4.62), (4.63), and (4.64) are known as the second-order discrete Euler-Lagrange equations. The following proposition formalizes this.

Proposition 4.7: The discrete dynamics generated by the global discrete Lagrangian function L^d defined in (4.59), is given by:

$$\begin{cases} D_1 L^d(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) + D_2 L^d(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2) = 0, \\ D_1 L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}) + D_2 L^d(\mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) \\ \quad + D_3 L^d(\mathbf{q}_{\kappa-2}, \mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa) = 0, \quad 2 \leq \kappa \leq N - 2, \\ D_2 L^d(\mathbf{q}_{N-2}, \mathbf{q}_{N-1}, \mathbf{q}_N) + D_3 L^d(\mathbf{q}_{N-3}, \mathbf{q}_{N-2}, \mathbf{q}_{N-1}) = 0 \end{cases} \quad (4.65)$$

where D_a denotes the partial derivative with respect to the a -th entry.

Second-Order Discrete Locally Conformal Euler-Lagrange Equations.

To derive discrete locally conformal Lagrangian dynamics, we begin by modifying the classical discrete Lagrangian framework with the introduction of a conformal factor. First, this is applied to the second-order discrete locally conformal Euler-Lagrange equations. Utilizing the equality (4.68), we express the local discrete Lagrangian in terms of a global discrete Lagrangian function. By mimicking the processes of discrete Lagrangian dynamics for second-order, we derive the second-order discrete locally conformal Euler-Lagrange equations. The following process illustrates this.

We assume a local chart V_α in an open covering of manifold Q , then take the Cartesian product space $V_\alpha^3 = V_\alpha \times V_\alpha \times V_\alpha$ of V_α . Recall a sequence (4.1) of points $[\mathbf{q}]$ joining two end points $\mathbf{q}_0$ and $\mathbf{q}_N$. On such sequences we determine a local Lagrangian function as

$$L_\alpha^d : V_\alpha \times V_\alpha \times V_\alpha \longrightarrow \mathbb{R}, \quad L_\alpha^d = L_\alpha^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}) \quad (4.66)$$

for κ runs from 0 to $N - 2$. Then we write the discrete local action as

$$S^d([\mathbf{q}]) = \sum_{\kappa=0}^{N-2} L_{\alpha}^d(\mathbf{q}_{\kappa}, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}). \quad (4.67)$$

We define a global discrete Lagrangian function L^d on the local chart as follows

$$L^d|_{\alpha}(\mathbf{q}_{\kappa}, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}) = e^{\sigma_{\alpha}(\mathbf{q}_{\kappa})} L_{\alpha}^d(\mathbf{q}_{\kappa}, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}) \quad (4.68)$$

and calculate the variation of the action

$$S^d([\mathbf{q}]) = \sum_{\kappa=0}^{N-2} e^{-\sigma_{\alpha}(\mathbf{q}_{\kappa})} L^d|_{\alpha}(\mathbf{q}_{\kappa}, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}). \quad (4.69)$$

$\delta S^d([\mathbf{q}]) = 0$ while fixing the boundary conditions $\delta \mathbf{q}_0 = 0$ and $\delta \mathbf{q}_N = 0$. As seen in the action sum (4.69), we have a conformal factor. When we calculate the variation of (4.69), we obtain equations similar to those in (4.65), with the difference arising from the conformal factor. Now, we express these equations explicitly. From the coefficients $\delta \mathbf{q}_1$ we have

$$\begin{aligned} & e^{-\sigma_{\alpha}(\mathbf{q}_1)} \left(-D\sigma_{\alpha}(\mathbf{q}_1) L^d|_{\alpha}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) + D_1 L^d|_{\alpha}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \right) \\ & + e^{-\sigma_{\alpha}(\mathbf{q}_0)} D_2 L^d|_{\alpha}(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2) = 0. \end{aligned} \quad (4.70)$$

From the coefficients $\delta \mathbf{q}_{\kappa}$ where $2 \leq \kappa \leq N - 2$, we obtain

$$\begin{aligned} & e^{-\sigma_{\alpha}(\mathbf{q}_{\kappa})} \left(-D\sigma_{\alpha}(\mathbf{q}_{\kappa}) L^d|_{\alpha}(\mathbf{q}_{\kappa}, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}) + D_1 L^d|_{\alpha}(\mathbf{q}_{\kappa}, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}) \right) \\ & + e^{-\sigma_{\alpha}(\mathbf{q}_{\kappa-1})} D_2 L^d|_{\alpha}(\mathbf{q}_{\kappa-1}, \mathbf{q}_{\kappa}, \mathbf{q}_{\kappa+1}) + e^{-\sigma_{\alpha}(\mathbf{q}_{\kappa-2})} D_3 L^d|_{\alpha}(\mathbf{q}_{\kappa-2}, \mathbf{q}_{\kappa-1}, \mathbf{q}_{\kappa}) = 0. \end{aligned} \quad (4.71)$$

Lastly, we have an equation from the coefficients $\delta \mathbf{q}_{N-1}$

$$\begin{aligned} & e^{-\sigma_{\alpha}(\mathbf{q}_{N-2})} D_2 L^d|_{\alpha}(\mathbf{q}_{N-2}, \mathbf{q}_{N-1}, \mathbf{q}_N) \\ & + e^{-\sigma_{\alpha}(\mathbf{q}_{N-3})} D_3 L^d|_{\alpha}(\mathbf{q}_{N-3}, \mathbf{q}_{N-2}, \mathbf{q}_{N-1}) = 0. \end{aligned} \quad (4.72)$$

We called (4.70), (4.71) and (4.72) second-order discrete locally conformal Euler-Lagrange equations, which we present in the following proposition.

Proposition 4.8: The discrete dynamics generated by the global discrete Lagrangian function L^d defined in (4.68) is given by:

$$\left\{ \begin{aligned} & e^{-\sigma_{\alpha}(\mathbf{q}_1)} \left(-D\sigma_{\alpha}(\mathbf{q}_1) L^d|_{\alpha}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) + D_1 L^d|_{\alpha}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \right) \\ & + e^{-\sigma_{\alpha}(\mathbf{q}_0)} D_2 L^d|_{\alpha}(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2) = 0, \\ & e^{-\sigma_{\alpha}(\mathbf{q}_{\kappa})} \left(-D\sigma_{\alpha}(\mathbf{q}_{\kappa}) L^d|_{\alpha}(\mathbf{q}_{\kappa}, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}) + D_1 L^d|_{\alpha}(\mathbf{q}_{\kappa}, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}) \right) \\ & + e^{-\sigma_{\alpha}(\mathbf{q}_{\kappa-1})} D_2 L^d|_{\alpha}(\mathbf{q}_{\kappa-1}, \mathbf{q}_{\kappa}, \mathbf{q}_{\kappa+1}) + e^{-\sigma_{\alpha}(\mathbf{q}_{\kappa-2})} D_3 L^d|_{\alpha}(\mathbf{q}_{\kappa-2}, \mathbf{q}_{\kappa-1}, \mathbf{q}_{\kappa}) = 0, \\ & e^{-\sigma_{\alpha}(\mathbf{q}_{N-2})} D_2 L^d|_{\alpha}(\mathbf{q}_{N-2}, \mathbf{q}_{N-1}, \mathbf{q}_N) \\ & + e^{-\sigma_{\alpha}(\mathbf{q}_{N-3})} D_3 L^d|_{\alpha}(\mathbf{q}_{N-3}, \mathbf{q}_{N-2}, \mathbf{q}_{N-1}) = 0, \end{aligned} \right. \quad (4.73)$$

for $2 \leq \kappa \leq N - 2$.

4.5. Third-Order Discrete Locally Conformal Lagrangian Dynamics

In this subsection, we construct the discrete Euler–Lagrange equations and the discrete locally conformal Euler–Lagrange equations associated with the third order.

Third-Order Discrete Euler-Lagrange Equations. We can calculate similarly third-order discrete Euler Lagrange equations. Let us take the discrete Lagrangian function as

$$L^d = Q \times Q \times Q \times Q \longrightarrow \mathbb{R}, \quad L^d = L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}, \mathbf{q}_{\kappa+3}) \quad (4.74)$$

where κ is from 0 to $N - 3$. We take the variation of action sum

$$S^d([\mathbf{q}]) = \sum_{\kappa=0}^{N-3} L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}, \mathbf{q}_{\kappa+3}) \quad (4.75)$$

$\delta S^d([\mathbf{q}]) = 0$ while fixing the boundary conditions: $\delta \mathbf{q}_0 = 0$ and $\delta \mathbf{q}_N = 0$. Then we get the set equations in (4.76). Notice that the first two equations arise from the coefficients $\delta \mathbf{q}_1$ and $\delta \mathbf{q}_2$. The third set of equations arises from the coefficients $\delta \mathbf{q}_\kappa$ for $3 \leq \kappa \leq N - 3$. The final two equations come from the coefficients $\delta \mathbf{q}_{N-2}$ and $\delta \mathbf{q}_{N-1}$. We give the third-order discrete Euler-Lagrange equations in the following proposition.

Proposition 4.9: The discrete dynamics generated by the global discrete Lagrangian function L^d defined in (4.74), is given by:

$$\left\{ \begin{array}{l} D_1 L^d(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) + D_2 L^d(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = 0, \\ D_1 L^d(\mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5) + D_2 L^d(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) + D_3 L^d(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = 0, \\ D_1 L^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}, \mathbf{q}_{\kappa+3}) + D_2 L^d(\mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}) \\ \quad + D_3 L^d(\mathbf{q}_{\kappa-2}, \mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) + D_4 L^d(\mathbf{q}_{\kappa-3}, \mathbf{q}_{\kappa-2}, \mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa) = 0, \\ D_2 L^d(\mathbf{q}_{N-3}, \mathbf{q}_{N-2}, \mathbf{q}_{N-1}, \mathbf{q}_N) + D_3 L^d(\mathbf{q}_{N-4}, \mathbf{q}_{N-3}, \mathbf{q}_{N-2}, \mathbf{q}_{N-1}) \\ \quad + D_4 L^d(\mathbf{q}_{N-5}, \mathbf{q}_{N-4}, \mathbf{q}_{N-3}, \mathbf{q}_{N-2}) = 0, \\ D_3 L^d(\mathbf{q}_{N-3}, \mathbf{q}_{N-2}, \mathbf{q}_{N-1}, \mathbf{q}_N) + D_4 L^d(\mathbf{q}_{N-4}, \mathbf{q}_{N-3}, \mathbf{q}_{N-2}, \mathbf{q}_{N-1}) = 0. \end{array} \right. \quad (4.76)$$

for $3 \leq \kappa \leq N - 3$.

The equations in (4.76) are called third-order discrete Euler-Lagrange equations.

Third-Order Discrete Locally Conformal Euler-Lagrange Equations.

Next, we derive the discrete locally conformal Euler-Lagrange equations for the third-order. Here, the local discrete Lagrangian function is expressed in terms of a global Lagrangian function using (4.79). By performing calculations similar to those in third-order discrete Euler-Lagrange equations, we obtain third-order discrete locally conformal Euler-Lagrange equations. The calculations are as follows:

Consider the Cartesian product $V_\alpha \times V_\alpha \times V_\alpha \times V_\alpha$ of the local chart V_α . To obtain third-order discrete locally conformal Euler-Lagrange equations consider the local Lagrangian function

$$L_\alpha^d : V_\alpha \times V_\alpha \times V_\alpha \times V_\alpha \longrightarrow \mathbb{R}, \quad L_\alpha^d = L_\alpha^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}, \mathbf{q}_{\kappa+3}). \quad (4.77)$$

For this Lagrangian discrete action can be written as

$$S^d([\mathbf{q}]) = \sum_{\kappa=0}^{N-3} L_\alpha^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}, \mathbf{q}_{\kappa+3}). \quad (4.78)$$

We define a global discrete Lagrangian function L^d on the local chart as follows

$$L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}, \mathbf{q}_{\kappa+3}) = e^{\sigma_\alpha(\mathbf{q}_\kappa)} L_\alpha^d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}, \mathbf{q}_{\kappa+3}) \quad (4.79)$$

and calculate the variation of the action

$$S^d([\mathbf{q}]) = \sum_{\kappa=0}^{N-3} e^{-\sigma_\alpha(\mathbf{q}_\kappa)} L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}, \mathbf{q}_{\kappa+3}) \quad (4.80)$$

$\delta S^d([\mathbf{q}]) = 0$ when $\delta \mathbf{q}_0 = 0$ and $\delta \mathbf{q}_N = 0$. After some calculations, we obtain similar equations third-order discrete Euler Lagrange equations in (4.76) with the difference of the terms arising from conformal factor. We have first two equations obtained from the coefficients $\delta \mathbf{q}_1$ and $\delta \mathbf{q}_2$. A set of equations arises from the coefficients $\delta \mathbf{q}_\kappa$ for $3 \leq \kappa \leq N-3$. The final two equations come from the coefficients $\delta \mathbf{q}_{N-2}$ and $\delta \mathbf{q}_{N-1}$. We now present these equations explicitly. The following equations from coefficients $\delta \mathbf{q}_1$ and $\delta \mathbf{q}_2$, respectively.

$$\begin{cases} e^{-\sigma_\alpha(\mathbf{q}_1)} \left(-D\sigma_\alpha(\mathbf{q}_1) L^d|_\alpha(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) + D_1 L^d|_\alpha(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \right) \\ + e^{-\sigma_\alpha(\mathbf{q}_0)} D_2 L^d|_\alpha(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = 0, \\ + e^{-\sigma_\alpha(\mathbf{q}_2)} \left(-D\sigma_\alpha(\mathbf{q}_2) L^d|_\alpha(\mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5) + D_1 L^d|_\alpha(\mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5) \right) \\ + e^{-\sigma_\alpha(\mathbf{q}_1)} D_2 L^d|_\alpha(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) + e^{-\sigma_\alpha(\mathbf{q}_0)} D_3 L^d|_\alpha(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = 0. \end{cases} \quad (4.81)$$

From coefficients $\delta \mathbf{q}_\kappa$ when $3 \leq \kappa \leq N-3$, we have the equations as follows

$$\begin{aligned}
& e^{-\sigma_\alpha(\mathbf{q}_\kappa)} \left(-D\sigma_\alpha(\mathbf{q}_\kappa) L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}, \mathbf{q}_{\kappa+3}) + D_1 L^d|_\alpha(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}, \mathbf{q}_{\kappa+3}) \right) \\
& + e^{-\sigma_\alpha(\mathbf{q}_{\kappa-1})} D_2 L^d|_\alpha(\mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}) \\
& + e^{-\sigma_\alpha(\mathbf{q}_{\kappa-2})} D_3 L^d|_\alpha(\mathbf{q}_{\kappa-2}, \mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}) \\
& + e^{-\sigma_\alpha(\mathbf{q}_{\kappa-3})} D_4 L^d|_\alpha(\mathbf{q}_{\kappa-3}, \mathbf{q}_{\kappa-2}, \mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa) = 0.
\end{aligned} \tag{4.82}$$

Lastly, we get the equations from the coefficients $\delta \mathbf{q}_{N-2}$ and $\delta \mathbf{q}_{N-1}$ as follows

$$\begin{cases}
e^{-\sigma_\alpha(\mathbf{q}_{N-3})} D_2 L^d|_\alpha(\mathbf{q}_{N-3}, \mathbf{q}_{N-2}, \mathbf{q}_{N-1}, \mathbf{q}_N) \\
+ e^{-\sigma_\alpha(\mathbf{q}_{N-4})} D_3 L^d|_\alpha(\mathbf{q}_{N-4}, \mathbf{q}_{N-3}, \mathbf{q}_{N-2}, \mathbf{q}_{N-1}) \\
+ e^{-\sigma_\alpha(\mathbf{q}_{N-5})} D_4 L^d|_\alpha(\mathbf{q}_{N-5}, \mathbf{q}_{N-4}, \mathbf{q}_{N-3}, \mathbf{q}_{N-2}) = 0 \\
e^{-\sigma_\alpha(\mathbf{q}_{N-3})} D_3 L^d|_\alpha(\mathbf{q}_{N-3}, \mathbf{q}_{N-2}, \mathbf{q}_{N-1}, \mathbf{q}_N) \\
+ e^{-\sigma_\alpha(\mathbf{q}_{N-4})} D_4 L^d|_\alpha(\mathbf{q}_{N-4}, \mathbf{q}_{N-3}, \mathbf{q}_{N-2}, \mathbf{q}_{N-1}) = 0.
\end{cases} \tag{4.83}$$

The equations (4.81), (4.82) and (4.83) are called the third-order discrete locally conformal Euler-Lagrange equations.

Proposition 4.10: The third-order discrete locally conformal Euler-Lagrange equations generated by the Lagrangian function defined in (4.79) are given by the following sets of equations (4.81), (4.82) and (4.83).

5. LOCALLY CONFORMAL KINETIC DYNAMICS

In this section, we describe a geometric derivation of the locally conformal kinetic dynamics that governs the collective behaviour of particles under LCS Hamiltonian flow. These dynamics is expressed in terms of both momentum and density variables.

5.1. Kinetic Lift of LCS Hamiltonian Dynamics in Momentum Variables

Consider a locally conformally symplectic manifold (M, Ω, φ) . We denote the set of Hamiltonian vector fields on LCS as

$$\mathfrak{X}_{\text{LCS-ham}}(M) = \{\xi_H \in \mathfrak{X}(M) : \iota_{\xi_H} \Omega = d_\varphi H\}. \quad (5.1)$$

Note that the space $\mathfrak{X}_{\text{LCS-ham}}(M)$ is a Lie subalgebra of the space $\mathfrak{X}(M)$ of all vector fields by the isomorphism

$$[\xi_F, \xi_H]_{JL} = -\xi_{\{F, H\}^{(LCS)}}, \quad (5.2)$$

where the bracket on the right-hand side is the LCS Jacobi bracket in (3.51). In light of (5.2), and assuming the Lie algebra bracket on $\mathfrak{X}_{\text{LCS-ham}}(M)$ being minus of the Jacobi-Lie bracket of vector fields, we determine a one-to-one Lie algebra homomorphism from the space of smooth functions to the space of LCS Hamiltonian vector fields as

$$\Phi : \mathcal{F}(M) \longrightarrow \mathfrak{X}_{\text{LCS-ham}}(M), \quad H \mapsto \xi_H := \Omega^\sharp(dH) - HZ_\varphi, \quad (5.3)$$

where Z_φ is the Lee vector field defined in (3.49).

The dual space of $\mathfrak{X}_{\text{LCS-ham}}(M)$ is consisting of one-form densities in $\Gamma^1(M) \otimes \text{Den}(M)$, along with the L_2 -pairing

$$\langle \bullet, \bullet \rangle_{L_2} : \Gamma^1(M) \otimes \text{Den}(M) \times \mathfrak{X}_{\text{LCS-ham}} \longrightarrow \mathbb{R}, \quad (\Pi \otimes dv, \xi_H) \mapsto \int_M \langle \Pi, \xi_H \rangle dv \quad (5.4)$$

where dv is the LCS volume-form in (3.3). The following proposition concretely determines the dual space as well as the dual map of the Lie algebra isomorphism (5.3).

Proposition 5.1: The dual space of LCS Hamiltonian vector fields $\mathfrak{X}_{\text{LCS-ham}}(M)$ is

$$\mathfrak{X}_{\text{LCS-ham}}^*(M) = \left\{ \Pi \otimes dv \in \Gamma^1(M) \otimes \text{Den}(M) : \text{div} \Omega^\sharp(\Pi) + \langle \varphi, \Omega^\sharp(\Pi) \rangle \neq 0 \right\} \cup \{0\} \quad (5.5)$$

and the dual operation of the Lie algebra mapping Φ is

$$\begin{aligned} \Phi^* : \mathfrak{X}_{\text{LCS-ham}}^*(M) &\longrightarrow \Gamma^1(M) \otimes \text{Den}(M), \\ \Pi \otimes dv &\mapsto f dv = \left(\text{div} \Omega^\sharp(\Pi) + \langle \varphi, \Omega^\sharp(\Pi) \rangle \right) dv. \end{aligned} \quad (5.6)$$

Proof. Let us perform the following calculation to arrive at the precise definition of the dual space:

$$\begin{aligned} \langle \Pi \otimes dv, \xi_H \rangle_{L_2} &= \int_M \langle \Pi, \xi_H \rangle dv = \int_M \langle \Pi, \Omega^\sharp(dH) - H Z_\varphi \rangle dv \\ &= \int_M \langle \Pi, \Omega^\sharp(dH) \rangle dv - \int_M H \langle \Pi, Z_\varphi \rangle dv \\ &= - \int_M \langle \Omega^\sharp(\Pi), dH \rangle dv - \int_M H \langle \Pi, \Omega^\sharp(\varphi) \rangle dv \\ &= - \int_M \iota_{\Omega^\sharp(\Pi)} dH dv + \int_M H \langle \Omega^\sharp(\Pi), \varphi \rangle dv \\ &= - \int_M \iota_{\Omega^\sharp(\Pi)} (dH \wedge dv) - \int_M dH \wedge \iota_{\Omega^\sharp(\Pi)} (dv) \\ &\quad + \int_M H \langle \varphi, \Omega^\sharp(\Pi) \rangle dv \\ &= - \int_M dH \wedge \iota_{\Omega^\sharp(\Pi)} (dv) + \int_M H \langle \varphi, \Omega^\sharp(\Pi) \rangle dv \\ &= - \int_M d(H \iota_{\Omega^\sharp(\Pi)} dv) + \int_M H d\iota_{\Omega^\sharp(\Pi)} dv + \int_M H \langle \varphi, \Omega^\sharp(\Pi) \rangle dv \\ &= \int_M H \mathcal{L}_{\Omega^\sharp(\Pi)} dv + \int_M H \langle \varphi, \Omega^\sharp(\Pi) \rangle dv \\ &= \int_M H \text{div} \Omega^\sharp(\Pi) dv + \int_M H \langle \varphi, \Omega^\sharp(\Pi) \rangle dv \\ &= \int_M H \left(\text{div} \Omega^\sharp(\Pi) + \langle \varphi, \Omega^\sharp(\Pi) \rangle \right) dv \\ &= \int_M H f dv, \end{aligned} \quad (5.7)$$

where div stands for the divergence with respect to the LCS volume-form dv . If we assume the non-degeneracy of the L_2 -pairing we arrive at the definition of the

dual space of LCS Hamiltonian vector fields in (5.5). We record here the density function in terms of the momenta as

$$f = \operatorname{div} \Omega^\sharp(\Pi) + \langle \varphi, \Omega^\sharp(\Pi) \rangle. \quad (5.8)$$

The calculation (5.7) also leads us to determine the dual operation of the Lie algebra mapping (5.3). So we arrive at the dual map Φ^* given in (5.6). Since Φ^* is dual to a Lie algebra mapping, it is a momentum and a Jacobi mapping. $\square$

According to the identity (5.2), the space $\mathfrak{X}_{\text{LCS-ham}}(M)$ of LCS Hamiltonian vector fields, in (5.1), on an LCS manifold (M, Ω, φ) is a Lie algebra. The Lie algebra structure on $\mathfrak{X}_{\text{LCS-ham}}(M)$, on the other hand, is given by the minus Jacobi-Lie bracket of vector fields. This leads us to define the adjoint action of $\mathfrak{X}_{\text{LCS-ham}}(M)$ on itself as

$$ad_{\xi_H} \xi_F = [\xi_H, \xi_F]_{\mathfrak{X}} = -[\xi_H, \xi_F]_{JL} = -\mathcal{L}_{\xi_H} \xi_F, \quad (5.9)$$

where $\mathcal{L}_{\xi_H}$ denotes the Lie derivative along the vector field ξ_H .

Proposition 5.2: The coadjoint action of the Lie algebra $\mathfrak{X}_{\text{LCS-ham}}(M)$ of LCS Hamiltonian vector fields on its dual space $\mathfrak{X}_{\text{LCS-ham}}^*(M)$ is

$$ad_{\xi_H}^* \Pi = \left(\mathcal{L}_{\xi_H} + n \langle \varphi, \xi_H \rangle \right) \Pi, \quad (5.10)$$

where the dimension of M is assumed to be $2n$.

Proof. The coadjoint action of the Lie algebra onto its dual is obtained along the lines of

$$\begin{aligned} \langle ad_{\xi_H}^* (\Pi \otimes dv), \xi_F \rangle &= \langle \Pi \otimes dv, ad_{\xi_H} \xi_F \rangle = - \int_M \langle \Pi, \mathcal{L}_{\xi_H} \xi_F \rangle dv \\ &= \int_M \langle \mathcal{L}_{\xi_H} \Pi + \operatorname{div}(\xi_H) \Pi, \xi_F \rangle dv \end{aligned} \quad (5.11)$$

where $\operatorname{div}(\xi_H)$ stands for the divergence of the vector field ξ_H with respect to the LCS volume-form dv . Then, fixing the LCS volume dv , we compute the coadjoint action of the Hamiltonian vector fields $\mathfrak{X}_{\text{LCS-ham}}(M)$ onto the dual space $\mathfrak{X}_{\text{LCS-ham}}^*(M)$ as in (5.10) where we have employed the explicit expression of the divergence in (3.62). $\square$

Lie-Poisson Dynamics in Momentum Variables. The dual space $\mathfrak{X}_{\text{LCS-ham}}^*(M)$ of the Lie algebra $\mathfrak{X}_{\text{LCS-ham}}(M)$ admits Lie-Poisson bracket. For two functionals

$\mathfrak{F}$ and $\mathfrak{H}$ on the dual space, the Lie-Poisson bracket is given by

$$\{\mathfrak{F}, \mathfrak{H}\}_{\mathfrak{X}_{\text{LCS-ham}}^*(M)}(\Pi) = \left\langle \Pi, \left[\frac{\delta \mathfrak{F}}{\delta \Pi}, \frac{\delta \mathfrak{H}}{\delta \Pi} \right] \right\rangle, \quad (5.12)$$

where the pairing on the right-hand side is the natural pairing between the dual space $\mathfrak{X}_{\text{LCS-ham}}^*(M)$ and the Lie algebra $\mathfrak{X}_{\text{LCS-ham}}(M)$. Note that $\delta \mathfrak{F}/\delta \Pi$ and $\delta \mathfrak{H}/\delta \Pi$ stand for the Fréchet derivatives. Assuming the reflexivity, we shall view both $\delta \mathfrak{F}/\delta \Pi$ and $\delta \mathfrak{H}/\delta \Pi$ in $\mathfrak{X}_{\text{LCS-ham}}(M)$, justifying the existence of the Lie bracket on the right-hand side of (5.12). We cite [3, 6, 7] for the Lie-Poisson formalism in a general context. Accordingly, given a Hamiltonian functional $\mathfrak{H}$, the Lie-Poisson dynamics is governed by the Lie-Poisson equations which can be expressed in terms of the coadjoint action as

$$\dot{\Pi} = -ad_{\delta \mathfrak{H}/\delta \Pi}^* \Pi. \quad (5.13)$$

Let $\mathfrak{H}$ be a linear Hamiltonian, on the dual space such that

$$\mathfrak{H}(\Pi) = \int_M \langle \Pi, \xi_H \rangle dv. \quad (5.14)$$

Then, the Fréchet derivative $\delta \mathfrak{H}/\delta \Pi$ of $\mathfrak{H}$ with respect to the momenta Π is the LCS Hamiltonian vector field ξ_H . So, referring to the coadjoint action (5.10), the Lie-Poisson equations on $\mathfrak{X}_{\text{LCS-ham}}^*(M)$ is computed to be

$$\dot{\Pi} = -\mathcal{L}_{\xi_H} \Pi - n \langle \varphi, \xi_H \rangle \Pi. \quad (5.15)$$

We call this dynamics as locally conformal momentum-Vlasov equations.

Symplectic Case. A particular instance of the previous assertion is the case where the conformal factor $\varphi = 0$. This reads global symplectic geometry in its very classical sense. In this time, we have a symplectic manifold (M, Ω) and the space of Hamiltonian vector fields given by

$$\mathfrak{X}_{\text{S-ham}}(M) = \{X_H \in \mathfrak{X}(M) : \iota_{X_H} \Omega = dH\}. \quad (5.16)$$

The identity (5.2) still holds and written in form

$$[X_F, X_H]_{JL} = -X_{\{F, H\}^{(S)}}, \quad (5.17)$$

where the bracket on the right-hand side is the canonical Poisson bracket on the symplectic manifold (M, Ω) . In the present case, we write the following Lie

algebra homomorphism

$$\widehat{\Phi} : \mathcal{F}(M) \longrightarrow \mathfrak{X}_{\text{S-ham}}(M), \quad H \mapsto X_H := \Omega^\sharp(dH). \quad (5.18)$$

By choosing φ is identically zero, the calculation (5.7) becomes

$$\langle \Pi \otimes d\mu, X_H \rangle_{L_2} = \int_M H \operatorname{div} \Omega^\sharp(\Pi) d\mu. \quad (5.19)$$

Based on this result, we can derive the following corollary [11, 12].

Corollary 5.3: The dual space of the Hamiltonian vector fields $\mathfrak{X}_{\text{S-ham}}(M)$ is

$$\mathfrak{X}_{\text{S-ham}}^*(M) = \left\{ \Pi \otimes d\mu \in \Gamma^1(M) \otimes \operatorname{Den}(M) : \operatorname{div} \Omega^\sharp(\Pi) \neq 0 \right\} \cup \{0\} \quad (5.20)$$

and the dual operation of the Lie algebra homomorphism $\widehat{\Phi}$ is

$$\widehat{\Phi}^* : \mathfrak{X}_{\text{S-ham}}^*(M) \longrightarrow \operatorname{Den}(M), \quad \Pi \otimes d\mu \mapsto f d\mu = (\operatorname{div} \Omega^\sharp(\Pi)) d\mu. \quad (5.21)$$

In this pure symplectic case, the coadjoint flow turns out to be

$$ad_{X_H}^* \Pi = \mathcal{L}_{X_H} \Pi \quad (5.22)$$

while the Lie-Poisson dynamics becomes the momentum-Vlasov equations

$$\dot{\Pi} = -\mathcal{L}_{X_H} \Pi. \quad (5.23)$$

In the following subsection, we will provide a more concrete explanation of how the intermediate level of equations, namely the locally conformal momentum-Vlasov equation (5.15) and the momentum-Vlasov equation (5.22), are related to the Vlasov dynamics.

5.2. Kinetic Lift of LCS Hamiltonian Dynamics in Densities

We consider an LCS manifold (M, Ω, φ) . The space of smooth functions, $\mathcal{F}(M)$, possesses a Lie algebra structure attributed to the LCS Jacobi bracket (3.51). Accordingly we write the adjoint action of $\mathcal{F}(M)$ on itself as

$$ad_H F = \{H, F\}^{LCS}. \quad (5.24)$$

The dual space $\mathcal{F}^*(M)$ of $\mathcal{F}(M)$ is the space of densities $\text{Den}(M)$ consisting of top forms along with L_2 -pairing

$$\langle \bullet, \bullet \rangle_{L_2} : \text{Den}(M) \times \mathcal{F}(M) \longrightarrow \mathbb{R}, \quad (f \, dv, H) \mapsto \int_M H f \, dv. \quad (5.25)$$

We fix the LCS volume form dv defined in (3.3). This permits us to identify the dual space with the space of smooth functions. In order to determine the coadjoint action of $\mathcal{F}(M)$ onto its dual space $\mathcal{F}^*(M)$, we need the following proposition.

Proposition 5.4: For a $2n$ -dimensional LCS manifold (M, Ω, φ) equipped with the LCS Jacobi bracket (3.51), the following identity holds:

$$\int_M f \{H, F\}^{(LCS)} dv = \int_M \{f, H\}^{(LCS)} F dv + \int (n+2) \langle \varphi, \xi_H \rangle f F dv, \quad (5.26)$$

where ξ_H is the LCS Hamiltonian vector field.

Proof. We start with the left-hand side of (5.26) by employing the explicit expression of the LCS Jacobi bracket (3.51) as

$$\int_M f \{H, F\}^{(LCS)} dv = \int_M f \Lambda(dH, dF) dv + \int_M f H Z_\varphi(F) dv - \int_M f F Z_\varphi(H) dv. \quad (5.27)$$

To obtain the right-hand side of (5.26), we need to single out the function F in (5.27). We do this for each term one by one. This is immediate for the third integral in the last line of (5.27). For the first integral, we perform the following calculation

$$\begin{aligned} \int_M f \Lambda(dH, dF) dv &= - \int_M f \langle dF, \Lambda^\sharp(dH) \rangle dv = - \int_M f \iota_{\Lambda^\sharp(dH)}(dF) dv \\ &= - \int_M f dF \wedge \iota_{\Lambda^\sharp(dH)} dv \\ &= \int_M F df \wedge \iota_{\Lambda^\sharp(dH)} dv + \int_M F f d\iota_{\Lambda^\sharp(dH)} dv \\ &= \int_M F \left(df \wedge \iota_{\Lambda^\sharp(dH)} + f \text{div}(\Lambda^\sharp(dH)) \right) dv. \end{aligned} \quad (5.28)$$

For the second integral term in (5.27) we have the following

$$\begin{aligned} \int_M f H Z_\varphi(F) dv &= \int_M f H \iota_{Z_\varphi} dF dv = \int_M f H dF \wedge \iota_{Z_\varphi} dv \\ &= - \int_M F H df \wedge \iota_{Z_\varphi} dv - \int_M F f dH \wedge \iota_{Z_\varphi} dv - \int_M F f H d\iota_{Z_\varphi} dv \\ &= - \int_M F \left(H Z_\varphi(f) + f Z_\varphi(H) + f H \text{div}(Z_\varphi) \right) dv. \end{aligned} \quad (5.29)$$

We collect all the terms in (5.28) and (5.29) with the third integral in (5.27). This gives

$$\begin{aligned}
\int_M f \Lambda(dH, dF) dv &= \int_M \left(\{f, H\}^{(LCS)} + f \operatorname{div}(\Lambda^\sharp(dH)) - 3f Z_\varphi(H) \right. \\
&\quad \left. - f H \operatorname{div}(Z_\varphi) \right) F dv \\
&= \int_M \left(\{f, H\}^{(LCS)} + f \operatorname{div}(\xi_H) - 2f Z_\varphi(H) \right) F dv \\
&= \int_M \left(\{f, H\}^{(LCS)} + (n+2) \langle \varphi, \xi_H \rangle f \right) F dv
\end{aligned} \tag{5.30}$$

where, in the second line, we have substituted the divergence of the Hamiltonian vector field referring to the decomposition (3.59) whereas, in the fourth line, we have used the equality

$$\langle \varphi, \xi_H \rangle = -Z_\varphi(H). \tag{5.31}$$

□

Now, we are ready to compute the coadjoint action of the Lie algebra $\mathcal{F}(M)$ on the dual space.

Proposition 5.5: Given a $2n$ -dimensional LCS manifold (M, Ω, φ) equipped with the LCS Jacobi bracket (3.51) as the Lie algebra bracket on $\mathcal{F}(M)$, the coadjoint action on its dual space $\mathcal{F}^*(M)$ is given by

$$ad_H^* f = \{f, H\}^{(LCS)} + (n+2) \langle \varphi, \xi_H \rangle f, \tag{5.32}$$

where ξ_H is the LCS Hamiltonian vector field.

Proof. We examine the coadjoint action referring to the L_2 -pairing in (5.25) as follows:

$$\begin{aligned}
\langle ad_H^* f, F \rangle_{L^2} &= \int_M (ad_H^* f) F dv = \int_M f (ad_H F) dv \\
&= \int_M f \{H, F\}^{(LCS)} dv \\
&= \int_M \left(\{f, H\}^{(LCS)} + (n+2) \langle \varphi, \xi_H \rangle f \right) F dv,
\end{aligned} \tag{5.33}$$

where we have used Proposition 5.4 in the third line. A direct observation reveals the assertion in (5.32). □

Lie-Poisson Dynamics in Densities. Once more we recall the Lie algebra $\mathcal{F}(M)$ of smooth functions equipped with the LCS Jacobi bracket (3.51). As the

dual space of this Lie algebra, $\mathcal{F}^*(M)$ admits a Lie-Poisson bracket. For two functionals $\mathfrak{F}$ and $\mathfrak{H}$ defined on the dual space $\mathcal{F}^*(M)$ the Lie-Poisson bracket is

$$\{\mathfrak{F}, \mathfrak{H}\}_{\mathcal{F}^*(M)}(f) = \int_M f \left\{ \frac{\delta \mathfrak{F}}{\delta f}, \frac{\delta \mathfrak{H}}{\delta f} \right\}^{(LCS)} dv \quad (5.34)$$

where $\delta \mathfrak{F}/\delta f$ and $\delta \mathfrak{H}/\delta f$ stand for the Fréchet derivatives, and assuming the reflexivity, we consider them as elements of the Lie algebra. Given a Hamiltonian functional $\mathfrak{H}$, the Lie-Poisson dynamics is governed by the Lie-Poisson equations which can be expressed in terms of the coadjoint action as

$$\frac{\partial f}{\partial t} = -ad_{\delta \mathfrak{H}/\delta f}^* f. \quad (5.35)$$

We take a linear Hamiltonian functional on the dual space as

$$\mathfrak{H}(f) = \int_M H f dv. \quad (5.36)$$

By employing the coadjoint action (5.32) into (5.35) we arrive at the Lie-Poisson dynamics

$$\frac{\partial f}{\partial t} + \{f, H\}^{(LCS)} = -(n+2)\langle \varphi, \xi_H \rangle f. \quad (5.37)$$

We call this system as locally conformal Vlasov equation. In the local coordinates (q^i, p_i) , the locally conformal Vlasov equation (5.37) is computed to be

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q^i} = -\varphi_i \frac{\partial f}{\partial p_i} H - A_{ij} \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial p_j} - (n+1)\varphi_i \frac{\partial H}{\partial p_i} f. \quad (5.38)$$

In terms of the Darboux coordinates (q^i, p_i) , we take the Hamiltonian function

$$H(q, p) = \frac{1}{2m} \delta^{ij} p_i p_j + e\phi(q) \quad (5.39)$$

as the total energy of the single particle. Here, m is the mass and e is the electrical charge while ϕ is the electric potential. Under this choice, the locally conformal Vlasov equation (5.38) takes the following form

$$\frac{\partial f}{\partial t} + p_i \frac{\partial f}{\partial q^i} - e \frac{\partial \phi}{\partial q^i} \frac{\partial f}{\partial p_i} = -\varphi_i \frac{\partial f}{\partial p_i} H - A_{ij} \delta^{jk} \frac{\partial f}{\partial p_i} \frac{p_k}{m} - \frac{(n+1)}{m} \delta^{ij} \varphi_i p_j f. \quad (5.40)$$

The left-hand side describes the reversible motion, as we shall illustrate in the following paragraph, while the right-hand side consists of dissipative terms. If φ vanishes identically then the LCS Jacobi bracket on the left-hand side of (5.37)

reduces to the canonical Poisson bracket and the term on the right-hand side drops and we arrive at the Vlasov dynamics in its very classical form

$$\frac{\partial f}{\partial t} + \{f, H\}^{(S)} = 0. \quad (5.41)$$

In the coordinates (q^i, p_i) , the Vlasov equations (5.41) turn out to be

$$\frac{\partial f}{\partial t} + p_i \frac{\partial f}{\partial q^i} - e \frac{\partial \phi}{\partial q^i} \frac{\partial f}{\partial p_i} = 0 \quad (5.42)$$

governing the collisionless non-relativistic plasma motion. This system gives the reversible motion by protecting the energy and the symplectic volume.

From Momentum Formulation to Density Formulation. One of the manifestations of the calculation (5.7) is the equality of the Hamiltonian functionals (5.14) and (5.36). Since the dual mapping Φ^* in (5.6) relating the momentum variable Π and the density f is a Jacobi mapping, the locally conformal momentum-Vlasov equation (5.15) and the locally conformal Vlasov equation (5.37) are equivalent.

Proposition 5.6: The locally conformal momentum-Vlasov equation (5.15) and the locally conformal Vlasov equation (5.37) are related with the Jacobi mapping Φ^* in (5.6).

Proof. We start with the definition (5.8) of the density function and take the time derivative of this identity:

$$\begin{aligned} \frac{\partial f}{\partial t} &= \text{div} \Omega^\#(\dot{\Pi}) + \langle \varphi, \Omega^\#(\dot{\Pi}) \rangle \\ &= -\text{div} \Omega^\#(\mathcal{L}_{\xi_H} \Pi + n \langle \varphi, \xi_H \rangle \Pi) - \langle \varphi, \Omega^\#(\mathcal{L}_{\xi_H} \Pi + n \langle \varphi, \xi_H \rangle \Pi) \rangle \\ &= -\text{div} \Omega^\#(\mathcal{L}_{\xi_H} \Pi) - n \langle \varphi, \xi_H \rangle \text{div} \Omega^\#(\Pi) - n \Omega^\#(\Pi) \text{grad}(\langle \varphi, \xi_H \rangle) - \langle \varphi, \Omega^\#(\mathcal{L}_{\xi_H} \Pi) \rangle \\ &\quad - n \langle \varphi, \xi_H \rangle \langle \varphi, \Omega^\#(\Pi) \rangle. \end{aligned} \quad (5.43)$$

Let us examine the first and the fourth terms as follows

$$\begin{aligned}
& - \int_M (\operatorname{div} \Omega^\sharp(\mathcal{L}_{\xi_H} \Pi) + \langle \varphi, \Omega^\sharp(\mathcal{L}_{\xi_H} \Pi) \rangle) K dv = - \int_M \langle \mathcal{L}_{\xi_H} \Pi, \xi_K \rangle dv \\
& = \int_M \langle \Pi, \mathcal{L}_{\xi_H} \xi_K \rangle dv + \int_M \langle \Pi, \xi_K \rangle \operatorname{div}(\xi_H) dv \\
& = - \int_M \langle \Pi, \xi_{\{H, K\}^{(LCS)}} \rangle dv + \int_M \langle \operatorname{div}(\xi_H) \Pi, \xi_K \rangle dv \\
& = - \int_M f \{H, K\}^{(LCS)} dv + \int_M \operatorname{div}(\operatorname{div}(\xi_H) \Omega^\sharp(\Pi)) K dv \\
& \quad + \int_M \langle \varphi, \Omega^\sharp(\operatorname{div}(\xi_H) \Pi) \rangle K dv \\
& = - \int_M \{f, H\}^{(LCS)} K dv - \int_M (n+2) \langle \varphi, \xi_H \rangle f K dv \\
& \quad + \int_M \operatorname{div}(\xi_H) \operatorname{div}(\Omega^\sharp(\Pi)) K dv + \int_M \Omega^\sharp(\Pi) \operatorname{grad}(\operatorname{div}(\xi_H)) K dv \\
& \quad + \int_M \operatorname{div}(\xi_H) \langle \varphi, \Omega^\sharp(\Pi) \rangle K dv \\
& = - \int_M \{f, H\}^{(LCS)} K dv - \int_M (n+2) \langle \varphi, \xi_H \rangle f K dv + \int_M n \langle \varphi, \xi_H \rangle f K dv \\
& \quad + \int_M \Omega^\sharp(\Pi) \operatorname{grad}(\operatorname{div}(\xi_H)) K dv \\
& = - \int_M \{f, H\}^{(LCS)} K dv - 2 \int_M \langle \varphi, \xi_H \rangle f K dv + \int_M \Omega^\sharp(\Pi) \operatorname{grad}(\operatorname{div}(\xi_H)) K dv.
\end{aligned} \tag{5.44}$$

So that we arrive at

$$\begin{aligned}
& - (\operatorname{div} \Omega^\sharp(\mathcal{L}_{\xi_H} \Pi) + \langle \varphi, \Omega^\sharp(\mathcal{L}_{\xi_H} \Pi) \rangle) \\
& = - \{f, H\}^{(LCS)} - 2 \langle \varphi, \xi_H \rangle f + \Omega^\sharp(\Pi) \operatorname{grad}(n \langle \varphi, \xi_H \rangle).
\end{aligned} \tag{5.45}$$

Collecting all terms in (5.43) and using the definition of density function f in (5.8) we have

$$\begin{aligned}
\frac{\partial f}{\partial t} & = - \{f, H\}^{(LCS)} - 2 \langle \varphi, \xi_H \rangle f - n \langle \varphi, \xi_H \rangle f \\
& = - \{f, H\}^{(LCS)} - (n+2) \langle \varphi, \xi_H \rangle f.
\end{aligned} \tag{5.46}$$

This completes the proof. $\square$

Symplectic Case. To establish the symplectic case, we set $\varphi = 0$. For the symplectic structure, (5.26) reveals

$$\int_M \{F, H\}^{(S)} K d\mu = \int_M F \{H, K\}^{(S)} d\mu \tag{5.47}$$

for all smooth functions F , H , and K defined on M with symplectic volume $d\mu$. Assuming the adjoint action on the space of smooth functions $\mathcal{F}(M)$ is the

canonical Poisson bracket $\{\bullet, \bullet\}^{(S)}$, the coadjoint action of the Lie algebra $\mathcal{F}(M)$ on its dual space $\text{Den}(M)$ can be found, with reference to the calculation (5.47), as

$$\langle ad_H^*(fd\mu), K \rangle = \langle fd\mu, ad_H(K) \rangle = \int_M \{H, K\}^{(S)} fd\mu = \int_M \{f, H\}^{(S)} K d\mu. \quad (5.48)$$

After we fix the symplectic volume as the top form and then identify $\text{Den}(M)$ with the space of smooth functions $\mathcal{F}(M)$, it is immediate to conclude that

$$ad_H^* f = \{f, H\}^{(S)}. \quad (5.49)$$

Let us examine the coadjoint flow in the light of the dual mapping (5.21)

$$\begin{aligned} \int_M \langle ad_{X_H}^* \Pi \otimes d\mu, X_K \rangle &= \int_M \langle \Pi, ad_{X_H} X_K \rangle d\mu = - \int_M \langle \Pi, [X_H, X_K] \rangle d\mu \\ &= \int_M \langle \Pi, X_{\{H, K\}^{(S)}} \rangle d\mu = \int_M \text{div} \Omega^\#(\Pi) \{H, K\}^{(S)} d\mu \\ &= \int_M \{ \text{div} \Omega^\#(\Pi), H \}^{(S)} K d\mu, \end{aligned} \quad (5.50)$$

where we employed the identity (5.17) in the second line and we used (5.47) in the last line. On the other hand, one has that

$$\int_M \langle ad_{X_H}^* \Pi, X_K \rangle d\mu = \int_M \text{div} \Omega^\#(ad_{X_H}^* \Pi) K d\mu. \quad (5.51)$$

Accordingly, we have that

$$\text{div}(\Omega^\#(\mathcal{L}_{X_H} \Pi)) = \{ \text{div} \Omega^\#(\Pi), H \}^{(S)}. \quad (5.52)$$

This equality is a particular instance of (5.45) exhibiting the relationship between the locally conformal Vlasov equation (5.37) and the Vlasov equation (5.41) as

$$\begin{aligned} \frac{\partial f}{\partial t} &= \text{div} \Omega^\#(\dot{\Pi}) = -\text{div}(\Omega^\#(\mathcal{L}_{X_H} \Pi)) \\ &= \{H, \text{div} \Omega^\#(\Pi)\}^{(S)} = \{H, f\}^{(S)}. \end{aligned} \quad (5.53)$$

5.3. A Geometric Pathway to Locally Conformal Kinetic Motion

In this section, we introduce divergence and collective lifts. After providing these definitions, we determine the holonomic and vertical parts of these lifts.

Then, we provide a proposition that reveals the relationship between the vertical representative of the collective lift of the LCS Hamiltonian vector field and the locally conformal momentum-Vlasov dynamics.

Divergence Lift. Consider a volume manifold $(M, d\mu)$ and determine vector fields whose divergence is in form

$$\operatorname{div}(X) = \langle \psi, X \rangle \quad (5.54)$$

for a fixed closed one-form ψ . We refer to these types of vector fields as vector fields with linear divergence. The computation

$$\begin{aligned} \operatorname{div}([X, Y]) &= X(\operatorname{div}(Y)) - Y(\operatorname{div}(X)) = X(\langle \psi, Y \rangle) - Y(\langle \psi, X \rangle) \\ &= \mathcal{L}_X \circ \iota_Y \psi - \mathcal{L}_Y \circ \iota_X \psi = \mathcal{L}_X \circ \iota_Y \psi - \iota_Y \circ \mathcal{L}_X \psi \\ &= \iota_{[X, Y]} \psi = \langle \psi, [X, Y] \rangle, \end{aligned} \quad (5.55)$$

shows that the space

$$\mathfrak{X}^\psi(M) = \{X \in \mathfrak{X}(M) : \operatorname{div}(X) = \langle \psi, X \rangle, d\psi = 0\} \subset \mathfrak{X}(M) \quad (5.56)$$

of vector fields with linear divergence (5.54) is a Lie subalgebra of the space of vector fields on M .

One can lift such vector fields to the cotangent bundle. To see this, we first recall the symplectic dilation vector field $\mathcal{S}$ on the cotangent bundle T^*M defined as

$$\mathcal{S} = \Omega_M^\sharp(\Theta_M) \quad (5.57)$$

where $\Omega_M^\sharp$ is the musical mapping induced from the canonical symplectic two-form on T^*M [69]. The symplectic dilation vector field $\mathcal{S}$ is the unique field satisfying

$$\mathcal{L}_{\mathcal{S}}\Theta_M = -\Theta_M. \quad (5.58)$$

Note that the divergence of the symplectic dilation vector field with respect to the symplectic volume-form on the cotangent bundle is computed to be

$$\operatorname{div}(\mathcal{S}) = -\dim(M). \quad (5.59)$$

We define the lift from the space $\mathfrak{X}^\psi(M)$ of vector fields with linear divergence to the space of vector fields on T^*M as

$$\mathfrak{s}: \mathfrak{X}^\psi(M) \longrightarrow \mathfrak{X}(T^*M), \quad X \mapsto X^\mathfrak{s} = \langle \psi, X \rangle \mathcal{S}. \quad (5.60)$$

It is noteworthy that since the divergence of the symplectic dilation vector field is $\dim(M)$, the divergence of the vector field $X^\mathfrak{s}$ is computed to be

$$\operatorname{div}(X^\mathfrak{s}) = -\dim(M)\langle\psi, X\rangle. \quad (5.61)$$

If the mapping (5.60) is applied to the space of all vector fields on M then the kernel is the space of divergence-free vector fields. The image space of the mapping (5.60) is one-dimensional. So the Jacobi-Lie bracket of two vector fields in the image space

$$\mathfrak{s} = \left\{ \langle\psi, X\rangle \mathcal{S} \in \mathfrak{X}(T^*M) : X \in \mathfrak{X}^\psi(M) \right\} \quad (5.62)$$

is identically zero. This turns $\mathfrak{s}$ into a Lie subalgebra of $\mathfrak{X}(T^*M)$. In Darboux coordinates (x^a, y_a) , the symplectic dilation vector field $\mathcal{S}$ and the divergence lift of an arbitrary vector field $X = X^a \partial/\partial x^a$ in $\mathfrak{X}^\psi(M)$ are

$$\mathcal{S} = -y_a \frac{\partial}{\partial y_a}, \quad X^\mathfrak{s} = -X^b \psi_b y_a \frac{\partial}{\partial y_a}, \quad (5.63)$$

respectively. Here, we take the closed one-form $\psi = \psi_a dx^a$.

Being a Hamiltonian vector field, a complete cotangent lift X^{c*} is divergence-free. So, the space $\mathfrak{X}^{c*}(T^*M)$ of complete cotangent lifts and the space $\mathfrak{s}$ of vector fields with linear divergence have trivial intersections. So that we can define the direct sum $\mathfrak{X}^{c*}(T^*M) \oplus \mathfrak{s}$ of these spaces. Let us examine the Lie algebra structure in this space. To get this, we compute the left action of the space $\mathfrak{s}$ on $\mathfrak{X}^{c*}(T^*M)$

$$\mathfrak{X}^{c*}(T^*M) \times \mathfrak{s} \longrightarrow \mathfrak{s}, \quad [X^{c*}, Y^\mathfrak{s}] \mapsto X \langle\psi, Y\rangle \mathcal{S}. \quad (5.64)$$

This determines a semi-direct product Lie algebra $\mathfrak{X}^{c*}(T^*M) \ltimes \mathfrak{s}$ with the semi-direct product Lie bracket

$$\begin{aligned} [X^{c*} + X^\mathfrak{s}, Y^{c*} + Y^\mathfrak{s}] &= [X^{c*}, Y^{c*}] + [X^\mathfrak{s}, Y^{c*}] + [X^{c*}, Y^\mathfrak{s}] \\ &= [X^{c*}, Y^{c*}] - Y \langle\psi, X\rangle \mathcal{S} + X \langle\psi, Y\rangle \mathcal{S}. \end{aligned} \quad (5.65)$$

Note that $\mathfrak{X}^{c*}(T^*M)$ is a Lie subalgebra of the space of vector fields on the cotangent bundle T^*M .

Collective lift. Now we collect the complete cotangent lift and the divergence lift (5.60) to define a mapping from the space of vector fields with linear divergence

into the semi-direct product space $\mathfrak{X}^{c*}(T^*M) \ltimes \mathfrak{s}$ of vector fields. We define the collective lift, denoted by Ξ , by adding the complete cotangent lift (2.91) and the divergence lift (5.60):

$$\Xi : \mathfrak{X}^\psi(M) \longrightarrow \mathfrak{X}^{c*}(T^*M) \ltimes \mathfrak{s}, \quad X \mapsto X^{c*} + X^\mathfrak{s}, \quad (5.66)$$

which is an injective map. Since the complete cotangent lift is Hamiltonian hence divergence-free, the divergence of the collective lift is equal to the divergence of $X^\mathfrak{s}$.

$$\operatorname{div}(\Xi(X)) = -\dim(M)\langle\psi, X\rangle. \quad (5.67)$$

The image of a vector field $X = X^a\partial/\partial x^a$ under the collective lift Ξ mapping is computed to be

$$\Xi(X) = X^a \frac{\partial}{\partial x^a} - \left(\frac{\partial X^b}{\partial x^b} y_a + \frac{\partial X^b}{\partial x^a} y_b \right) \frac{\partial}{\partial y_a}. \quad (5.68)$$

Proposition 5.7: The collective lift operator Ξ defined in (5.66) is a Lie algebra homomorphism that is

$$[\Xi(X), \Xi(Y)] = \Xi[X, Y], \quad (5.69)$$

for arbitrary vector fields X and Y in $\mathfrak{X}^\psi(M)$.

Proof. A direct calculation proves the following equality, which is valid for vector fields with linear divergence:

$$[X^{c*}, Y^\mathfrak{s}] + [X^\mathfrak{s}, Y^{c*}] = [X, Y]^\mathfrak{s}. \quad (5.70)$$

We remark that the crucial role for the validation of this identity is the closure of the differential one-form ψ . We use the bilinearity of the Jacobi-Lie bracket to compute

$$\begin{aligned} [\Xi(X), \Xi(Y)] &= [X^{c*} + X^\mathfrak{s}, Y^{c*} + Y^\mathfrak{s}] \\ &= [X^{c*}, Y^{c*}] + [X^{c*}, Y^\mathfrak{s}] + [X^\mathfrak{s}, Y^{c*}] + [X^\mathfrak{s}, Y^\mathfrak{s}] \\ &= [X, Y]^{c*} + [X, Y]^\mathfrak{s} = \Xi[X, Y], \end{aligned} \quad (5.71)$$

where we have employed the Lie algebra homomorphism property of the complete cotangent lift stated in (2.89) and the identity (5.70). $\square$

Now consider holonomic part of a projectable vector field in (2.100). For a given vector field $X = X^a \partial / \partial x^a$, the collective lift $\Xi(X)$ in (5.68) is a projectable vector field. So that we can compute its holonomic part as

$$\mathfrak{H}\Xi(X) = X^a(x) \frac{\partial}{\partial x^a} + X^a(x) \frac{\partial y_b}{\partial x^a} \frac{\partial}{\partial y_b}. \quad (5.72)$$

Notice that the term $\text{div}(X)\mathcal{S}$ in $\Xi(X)$ does not give any contribution to the holonomic part since it is a pure vertical vector. For vector fields with linear divergence, since both the mapping $X \mapsto \Xi(X)$ (Proposition 5.7) and the holonomic part operation $\Xi(X) \mapsto \mathfrak{H}\Xi(X)$ (Proposition 2.2) are Lie algebra homomorphisms, their composition is a Lie algebra homomorphism.

Corollary 5.8: The map (5.73)

$$\mathfrak{H}\Xi : \mathfrak{X}^\psi(M) \longrightarrow \mathfrak{X}^{\text{gen}}(T^*M), \quad X \mapsto \mathfrak{H}\Xi(X) \quad (5.73)$$

is a Lie algebra homomorphism that is

$$\mathfrak{H}\Xi[X, Y] = [\mathfrak{H}\Xi(X), \mathfrak{H}\Xi(Y)]_{\text{pro}}. \quad (5.74)$$

Recall from (2.107) that the vertical representative operation is not a Lie algebra homomorphism. However, the following proposition demonstrates that it becomes a Lie algebra homomorphism when its domain is restricted to the image space of the collective lift mapping Ξ .

Proposition 5.9: The map to the vertical representative $\mathfrak{V}\Xi(X)$ of collective lift $\Xi(X)$:

$$\mathfrak{V}\Xi : \mathfrak{X}^\psi(M) \longrightarrow \mathfrak{X}^{\text{gen}}(T^*M), \quad X \mapsto \mathfrak{V}\Xi(X), \quad (5.75)$$

is a Lie algebra homomorphism

$$\mathfrak{V}\Xi[X, Y] = [\mathfrak{V}\Xi(X), \mathfrak{V}\Xi(Y)]_{\text{pro}}. \quad (5.76)$$

Proof. We first recall that vertical representatives of complete cotangent lifted vector fields is a Lie subalgebra by satisfying

$$[\mathfrak{V}X^{c*}, \mathfrak{V}Y^{c*}]_{\text{pro}} = \mathfrak{V}[X, Y]^{c*}, \quad (5.77)$$

for arbitrary vector fields X and Y in $\mathfrak{X}(M)$, see [13, 14]. On the other hand, a direct calculation proves the following observation:

$$[\mathfrak{V}X^{c*}, Y^s]_{\text{pro}} = [X^{c*}, Y^s]. \quad (5.78)$$

So, referring to the identity (5.70) we deduce

$$[\mathfrak{V}X^{c*}, Y^{\mathfrak{s}}]_{\text{pro}} + [X^{\mathfrak{s}}, \mathfrak{V}Y^{c*}]_{\text{pro}} = [X, Y]^{\mathfrak{s}} \quad (5.79)$$

where X^{c*} is the complete cotangent lift whereas $X^{\mathfrak{s}}$ is the divergence lift. Next, we compute

$$\begin{aligned} [\mathfrak{V}\Xi(X), \mathfrak{V}\Xi(Y)]_{\text{pro}} &= [\mathfrak{V}X^{c*} + \mathfrak{V}X^{\mathfrak{s}}, \mathfrak{V}Y^{c*} + \mathfrak{V}Y^{\mathfrak{s}}]_{\text{pro}} \\ &= [\mathfrak{V}X^{c*}, \mathfrak{V}Y^{c*}]_{\text{pro}} + [\mathfrak{V}X^{c*}, \mathfrak{V}Y^{\mathfrak{s}}]_{\text{pro}} + [\mathfrak{V}X^{\mathfrak{s}}, \mathfrak{V}Y^{c*}]_{\text{pro}} \\ &\quad + [\mathfrak{V}X^{\mathfrak{s}}, \mathfrak{V}Y^{\mathfrak{s}}]_{\text{pro}} \\ &= [\mathfrak{V}X^{c*}, \mathfrak{V}Y^{c*}]_{\text{pro}} + [\mathfrak{V}X^{c*}, Y^{\mathfrak{s}}]_{\text{pro}} + [X^{\mathfrak{s}}, \mathfrak{V}Y^{c*}]_{\text{pro}} \\ &\quad + [X^{\mathfrak{s}}, Y^{\mathfrak{s}}]_{\text{pro}} \\ &= \mathfrak{V}[X, Y]^{c*} + [X, Y]^{\mathfrak{s}} = \mathfrak{V}[X, Y]^{c*} + \mathfrak{V}[X, Y]^{\mathfrak{s}} \\ &= \mathfrak{V}\Xi[X, Y], \end{aligned} \quad (5.80)$$

where we used the observation that the vertical representative of any divergence-lifted vector field remains unchanged, as it is inherently a vertical vector field, that is, $\mathfrak{V}X^{\mathfrak{s}} = X^{\mathfrak{s}}$. In the fourth line of the computation, we used the equalities (5.77) and (5.79). $\square$

In coordinates, if X has linear divergence, then its collective lift is in (5.68) and the vertical representative of the latter is

$$\mathfrak{V}\Xi(X) = -\left(\frac{\partial X^b}{\partial x^b}y_a + \frac{\partial X^b}{\partial x^a}y_b + X^b\frac{\partial y_a}{\partial x^b}\right)\frac{\partial}{\partial y_a}. \quad (5.81)$$

Geometric Pathway. Let Π be a one-form on T^*Q . The vertical lift of Π gives a vertical vector field on the iterated cotangent bundle T^*T^*Q :

$$\Pi^v : T^*T^*Q \longrightarrow TT^*T^*Q. \quad u \mapsto \mathcal{S} \circ \Pi \circ \pi_{T^*Q}(u). \quad (5.82)$$

Following proposition provides a geometric pathway to Lie-Poisson dynamics of LCS particles, as obtained in Subsection 5.1. More precisely, it states that locally conformal momentum-Vlasov equations (5.15) is the vertical representative of collective lift of LCS Hamiltonian vector field ξ_H generating particle motion.

Proposition 5.10: The locally conformal momentum-Vlasov dynamics (5.15) on LCS manifold $(T^*Q, \Omega_\varphi, \varphi)$ is the vertical representative $\mathfrak{V}\Xi(\xi_H)$ of the collective lift of the LCS Hamiltonian vector field ξ_H

$$\dot{\Pi}^v = -\mathfrak{V}\Xi(\xi_H). \quad (5.83)$$

Proof. The following identity holds for an arbitrary LCS Hamiltonian vector field ξ_H

$$\mathfrak{V}\Xi(\xi_H) = \left(\mathcal{L}_{\xi_H}(\Pi) + \text{div}(\xi_H)\Pi \right)^v, \quad (5.84)$$

where the left-hand side is the vertical representative of the collective lift whereas the right-hand side is the vertical lift operation. Therefore, a direct comparison of the locally conformal momentum-Vlasov equations (5.15) with the right-hand side of (5.84) yields the proof. $\square$

We provided generalizations of Vlasov equation both at density and momentum levels to locally conformal framework. This approach is addressing globalization problem of Vlasov plasma dynamics. The following diagram summarizes novel formulations, namely locally conformal momentum-Vlasov (Subsection 5.1) and locally conformal Vlasov equations (Subsection 5.2), located on the left side of the diagram, as well as their relationships.

$$\begin{array}{ccc}
\begin{array}{c} \text{LCS } m - \text{Vlasov} \\ \text{Eq. (5.15)} \end{array} & \xrightarrow{\varphi=0} & \begin{array}{c} m - \text{Vlasov} \\ \text{Eq. (5.23)} \end{array} \\
\downarrow \Phi^* \text{ in (5.6)} & & \downarrow \widehat{\Phi}^* \text{ in (5.21)} \\
\begin{array}{c} \text{LCS Vlasov} \\ \text{Eq. (5.37)} \end{array} & \xrightarrow{\varphi=0} & \begin{array}{c} \text{Vlasov} \\ \text{Eq. (5.41)} \end{array}
\end{array}$$

Section 5.3 introduced a geometric pathway to LCS momentum-Vlasov dynamics starting from particle motion. The principal outcome is Proposition 5.10. This geometric framework operates independently of a bracket or Hamiltonian functional.

6. CONFORMAL KINETIC DYNAMICS

6.1. Conformal Hamiltonian Dynamics

Let M be a manifold, which is called a symplectic manifold if it admits a closed non-degenerate two form Ω , [70, 71]. Accordingly, we shall denote a symplectic manifold by a pair (M, Ω) .

A Hamiltonian vector field on M is defined through the covariance equation (2.19). Recall from (2.20) that the Lie derivative vanishes identically and the divergence is zero. Relaxing this condition, we define a conformal vector field X_H^c as [72–74]

$$\mathcal{L}_{X_H^c} \Omega = c\Omega \quad (6.1)$$

for a fixed real number c called conformal parameter. In the sequel, we shall work on several conformal vector fields making use of subindexes (such as c_H) to be more precise about the parameter. Let us note also that a Hamiltonian vector field is a conformal vector field with the conformal factor being zero, while the divergence of a conformal vector field is non-zero. More precisely (for a symplectic manifold of dimension $2n$) it is computed to be

$$\mathcal{L}_{X_H^c} d\mu = cnd\mu, \quad \text{div}(X_H^c) = nc, \quad (6.2)$$

where $d\mu$ is the symplectic volume defined in (2.6). Integration of the defining identity (6.1) yields that the flow of a conformal vector field preserves the symplectic two-form up to the conformal factor c and the volume up to the conformal factor cn that are

$$\varphi_t^*(\Omega) = \exp(ct)\Omega, \quad \varphi_t^*(d\mu) = \exp(nct)d\mu, \quad (6.3)$$

respectively, where $\exp$ stands for the exponential function.

For an exact symplectic manifold where $\Omega = -d\Theta$, we define a conformal vector field as

$$\iota_{X_H^c} \Omega = dH - c\Theta. \quad (6.4)$$

For the exact symplectic manifolds, as announced in [74], the formulation (6.4) is generic for conformal vector fields. That is the conformal vector field X_H^c defined

in (6.1) admits a function H realizing the covariance equation (6.4). Further, again by citing [74], if the first cohomology of the manifold is vanishing then X_H^c is written as a combination of the symplectic Hamiltonian vector field X_H and the Liouville vector field Z as follows. Following [22] we define Z as the image of the Liouville (canonical) one-form Θ under the musical isomorphism $\Omega^\sharp$ (recall from (2.3) and (2.5)) induced from the symplectic form Ω , namely,

$$Z := \Omega^\sharp(\Theta), \quad \Theta = \Omega^\flat(Z). \quad (6.5)$$

It is worth noting that the Liouville vector field is not a Hamiltonian vector field but a conformal vector field with the conformal factor -1 , and with the divergence $-n$, as

$$\mathcal{L}_Z \Omega = -\Omega, \quad \mathcal{L}_Z d\mu = -n d\mu, \quad \operatorname{div}(Z) = -n. \quad (6.6)$$

In terms of the Liouville vector field, defined in (6.5), we can express a conformal vector field as the linear combination

$$X_H^c = X_H - cZ, \quad (6.7)$$

where X_H is the Hamiltonian vector field for the Hamiltonian function. Accordingly, we compute the change of the Hamiltonian function along the conformal vector field, and its flow ϕ_t , as

$$\mathcal{L}_{X_H^c}(H) = X_H(H) - cZ(H) = -cZ(H), \quad \phi_t^*(H) = H - c\psi_t^*(H), \quad (6.8)$$

respectively, where ψ_t denotes the integral curve of the Liouville vector field Z .

We consider the Darboux coordinates (q^i, p_i) on M assuming that it is locally isomorphic to the momentum phase space $M = T^*Q$ of a configuration manifold Q . The symplectic Hamiltonian vector field X_H and the Liouville vector field Z are computed to be

$$X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i}, \quad Z = -p_i \frac{\partial}{\partial p_i}, \quad (6.9)$$

respectively. So the conformal vector field X_H^c becomes

$$X_H^c = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} - cp_i \right) \frac{\partial}{\partial p_i}. \quad (6.10)$$

Then, the dynamics governed by a conformal vector field X_H^c is given by

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} + cp_i. \quad (6.11)$$

Let us note that in case the conformal factor c is trivial, then (6.11) reduces to the classical Hamilton's equations as expected.

6.2. Lie Algebra of Conformal Vector Fields

Consider a symplectic manifold (M, Ω) . Let us first note that the Jacobi-Lie bracket of two conformal vector fields is a local Hamiltonian vector field. Indeed,

$$\mathcal{L}_{[X_H^c, X_K^c]} \Omega = \mathcal{L}_{X_H^c} \mathcal{L}_{X_K^c} \Omega - \mathcal{L}_{X_K^c} \mathcal{L}_{X_H^c} \Omega = (c_H c_K - c_K c_H) \Omega = 0. \quad (6.12)$$

As such, we can argue that the space of conformal vector fields is a Lie algebra

$$\mathfrak{X}_{\text{ham}}^c(M) := \{X_H^c \in \mathfrak{X}(M) : \mathcal{L}_{X_H^c} \Omega = c_H \Omega\} \quad (6.13)$$

and contains the space of Hamiltonian vector fields $\mathfrak{X}_{\text{ham}}(M)$ as an ideal.

Let, now, $\mathfrak{z}$ denotes the space of vector fields spanned by scalar multiples (that is $\mathbb{R}$) of the Liouville vector field Z . Evidently, this space is one-dimensional over $\mathbb{R}$ and may be obtained as

$$\mathbb{R} \longleftrightarrow \mathfrak{z}, \quad c \leftrightarrow cZ, \quad (6.14)$$

and thus acquires the structure of a trivial Lie algebra. More precisely, being the linear span (with $\mathbb{R}$) of Z , $\mathfrak{z}$ is a trivial Lie subalgebra of the one-dimensional projective module over the ring of functions generated by the vector field Z . Moreover, $\mathfrak{z}$ acts on the space of Hamiltonian vector fields $\mathfrak{X}_{\text{ham}}(M)$ from the left as

$$\mathfrak{z} \times \mathfrak{X}_{\text{ham}}(M) \longrightarrow \mathfrak{X}_{\text{ham}}(M), \quad (Z, X_H) \mapsto [Z, X_H] = X_{Z(H)+H}. \quad (6.15)$$

The realization (6.7) motivates us to have the space $\mathfrak{X}_{\text{ham}}^c(M)$ of conformal vector fields as the Cartesian product of the space $\mathfrak{X}_{\text{ham}}(M)$ of Hamiltonian vector fields and $\mathfrak{z} \simeq \mathbb{R}$. Accordingly, we can recast the space $\mathfrak{X}_{\text{ham}}^c(M)$ of conformal vector fields as a central extension of the space $\mathfrak{X}_{\text{ham}}(M)$ of Hamiltonian vector fields as

$$\mathfrak{X}_{\text{ham}}(M) \times \mathfrak{z} \longleftrightarrow \mathfrak{X}_{\text{ham}}^c(M), \quad (X_H, c) \mapsto X_H^c = X_H - cZ. \quad (6.16)$$

Now we are ready to determine the Lie algebra structure on $\mathfrak{X}_{\text{ham}}^c(M)$. To this

end, given two conformal fields X_H^c and X_F^c , we compute their Lie bracket

$$\begin{aligned}
[X_H^c, X_F^c]_{\mathfrak{X}} &= -[X_H^c, X_F^c] = -[X_H - c_H Z, X_F - c_F Z] \\
&= -[X_H, X_F] + c_H[Z, X_F] + c_F[X_H, Z] \\
&= X_{\{H, F\}^{(S)}} + c_H X_{Z(F)+F} - c_F X_{Z(H)+H} \\
&= X_{\{H, F\}^{(S)} + c_H(Z(F)+F) - c_F(Z(H)+H)},
\end{aligned} \tag{6.17}$$

where we have employed the action (6.15) on the fourth equality.

Let us conclude the present subsection with another characterization of conformal vector fields that will be useful in the sequel.

To this end, let us recall that we have identified the space $\mathfrak{X}_{\text{ham}}(M)$ of Hamiltonian vector fields with the space $\mathcal{F}(M)$ of smooth functions (modulo constants). We now employ this to the identification in (6.16) to obtain

$$\Phi^c : \mathcal{F}(M) \times \mathfrak{z} \longrightarrow \mathfrak{X}_{\text{ham}}^c(M), \quad (H, c) \mapsto X_H^c = X_H - cZ. \tag{6.18}$$

In view of (6.17), it is thus possible to endow $\mathcal{F}(M) \times \mathfrak{z}$ with a Lie algebra structure so that the mapping Φ^c is a Lie algebra homomorphism. Accordingly, we define the bracket

$$[(H, c_H), (F, c_F)] := \left(\{H, F\}^{(S)} + c_H(Z(F) + F) - c_F(Z(H) + H), 0 \right) \tag{6.19}$$

which happens to be a Lie algebra bracket, satisfying the Jacobi identity.

6.3. Conformal Kinetic Dynamics in Momentum Formulation

In order to characterize the dual space $\mathfrak{X}_{\text{ham}}^{c*}(M)$ of the space $\mathfrak{X}_{\text{ham}}^c(M)$ of conformal vector fields we shall now consider the Lie algebra $\mathcal{F}(M) \times \mathfrak{z}$.

For the function space $\mathcal{F}(M)$, the dual space is the space of densities $\text{Den}(M)$. Fixing the symplectic volume $d\mu$, the L_2 pairing allows us to identify the dual space $\mathcal{F}^*(M)$ with $\mathcal{F}(M)$ itself. Further, the identification $\mathfrak{z} \simeq \mathbb{R}$ implies the isomorphism $\mathfrak{z}^* \simeq \mathbb{R}$ on the level of dual spaces. Accordingly, we may consider the dual space $\mathcal{F}^*(M) \times \mathfrak{z}^*$ as $\mathcal{F}(M) \times \mathfrak{z}$ itself. More precisely, given (H, c_H) in $\mathcal{F}(M) \times \mathfrak{z}$, and a dual element (f, c^*) in $\mathcal{F}^*(M) \times \mathfrak{z}^*$, we shall consider the pairing

given by

$$\langle (f, c^*), (H, c_H) \rangle = c^* c_H + \int_M f H d\mu. \quad (6.20)$$

Accordingly, the dual space $\mathfrak{X}_{\text{ham}}^{c*}(M)$ is determined by the pairing

$$\begin{aligned} \langle \Pi \otimes d\mu, X_H^c \rangle_{L_2} &= \int_M \langle \Pi, X_H - cZ \rangle d\mu = \int_M \langle \Pi, \Omega^\sharp(dH) - c\Omega^\sharp(\Theta) \rangle d\mu \\ &= \int_M \langle \Pi, \Omega^\sharp(dH) \rangle d\mu - \int_M \langle \Pi, c\Omega^\sharp(\Theta) \rangle d\mu \\ &= - \int_M \langle \Omega^\sharp(\Pi), dH \rangle d\mu - c \int_M \langle \Pi, \Omega^\sharp(\Theta) \rangle d\mu \\ &= - \int_M \iota_{\Omega^\sharp(\Pi)} dH d\mu + c \int_M \langle \Theta, \Omega^\sharp(\Pi) \rangle d\mu \\ &= - \int_M \iota_{\Omega^\sharp(\Pi)} (dH \wedge d\mu) - \int_M dH \wedge \iota_{\Omega^\sharp(\Pi)} (d\mu) + c \int_M \langle \Theta, \Omega^\sharp(\Pi) \rangle d\mu \\ &= - \int_M dH \wedge \iota_{\Omega^\sharp(\Pi)} (d\mu) + c \int_M \langle \Theta, \Omega^\sharp(\Pi) \rangle d\mu \\ &= - \int_M d(H \iota_{\Omega^\sharp(\Pi)} d\mu) + \int_M H d\iota_{\Omega^\sharp(\Pi)} d\mu + c \int_M \langle \Theta, \Omega^\sharp(\Pi) \rangle d\mu \\ &= \int_M H \mathcal{L}_{\Omega^\sharp(\Pi)} d\mu + c \int_M \langle \Theta, \Omega^\sharp(\Pi) \rangle d\mu \\ &= \int_M H \operatorname{div} \Omega^\sharp(\Pi) d\mu + c \int_M \langle \Theta, \Omega^\sharp(\Pi) \rangle d\mu. \end{aligned}$$

We thus arrive at the dual mapping of the Lie algebra homomorphism Φ^c in (6.18), that is,

$$\begin{aligned} \Phi^{c*} : \mathfrak{X}_{\text{ham}}^{c*}(M) &\longrightarrow \operatorname{Den}(M) \times \mathbb{R}, \\ (\Pi \otimes d\mu) &\mapsto \left(\operatorname{div} \Omega^\sharp(\Pi) d\mu, \int_M \langle \Theta, \Omega^\sharp(\Pi) \rangle d\mu \right). \end{aligned} \quad (6.21)$$

In order to ensure the non-degeneracy of the pairing, therefore, we present the dual space as

$$\begin{aligned} \mathfrak{X}_{\text{ham}}^{c*}(M) &= \left\{ \Pi \otimes d\mu \in \Gamma^1(M) \otimes \operatorname{Den}(M) : \operatorname{div} \Omega^\sharp(\Pi) \neq 0, \right. \\ &\quad \left. \int_M \langle \Theta, \Omega^\sharp(\Pi) \rangle d\mu \neq 0 \right\} \cup \{0\}. \end{aligned} \quad (6.22)$$

This determines the following identification for the two tuple $(fd\mu, c^*)$ with the one-form sections as

$$f(z) = \operatorname{div} \Omega^\sharp(\Pi), \quad c^* = \int_M \langle \Theta, \Omega^\sharp(\Pi) \rangle d\mu. \quad (6.23)$$

Hence, following (2.75) we compute the coadjoint action of the Lie algebra $\mathfrak{X}_{\text{ham}}^c(M)$ on its dual space $\mathfrak{X}_{\text{ham}}^{c*}(M)$ as

$$ad_{X_H^c}^* (\Pi \otimes d\mu) = \left(\mathcal{L}_{X_H^c} \Pi + \operatorname{div}(X_H^c) \Pi \right) \otimes d\mu = \left(\mathcal{L}_{X_H^c} \Pi + nc\Pi \right) \otimes d\mu, \quad (6.24)$$

where we have substituted the divergence of X_H^c from (6.2). Accordingly, in view of (2.81) the Lie-Poisson equation, or equivalently the coadjoint flow, is computed to be

$$\dot{\Pi} = -\mathcal{L}_{X_H^c} \Pi - cn\Pi. \quad (6.25)$$

The equality (6.25) is the conformal kinetic equation in terms of momenta.

Momentum-Vlasov Equation. Let us particularly consider the case where the conformal factor is zero. In this case, one arrives at the pure Hamiltonian flow, and the Lie algebra associated with this particular case is the space $\mathfrak{X}_{\text{ham}}(M)$ of Hamiltonian vector fields. Moreover, the domain of the Lie algebra homomorphism Φ^c in (6.18) turns out to be the function space $\mathcal{F}(M)$, without the extension, that is

$$\Phi : \mathcal{F}(M) \longrightarrow \mathfrak{X}_{\text{ham}}(M), \quad H \mapsto X_H. \quad (6.26)$$

The dual of this linear mapping can be computed similarly to (6.21). As a result, one can see that, only the first term in the dual operation $(\Phi^c)^*$ in (6.21) remains. Namely,

$$\Phi^* : \mathfrak{X}_{\text{ham}}^*(M) \longrightarrow \text{Den}(M), \quad \Pi \mapsto f d\mu := \text{div} \Omega^\sharp(\Pi) d\mu. \quad (6.27)$$

This calculation allows us to define the dual space $\mathfrak{X}_{\text{ham}}^*(M)$ of the space of Hamiltonian vector fields as

$$\mathfrak{X}_{\text{ham}}^*(M) := \{\Pi \otimes d\mu \in \Gamma^1(M) \otimes \text{Den}(M) : \text{div} \Omega^\sharp(\Pi) \neq 0\} \cup \{0\}. \quad (6.28)$$

Having determined the dual space properly, we can then determine the Lie-Poisson equations on $\mathfrak{X}_{\text{ham}}^*(M)$. It follows from (2.20) that a Hamiltonian vector field is divergence-free. As such, in view of the calculation (2.82), the Lie-Poisson equation is given by

$$\dot{\Pi} = -ad_{X_H}^* \Pi = -\mathcal{L}_{X_H} \Pi, \quad (6.29)$$

which is called the momentum-Vlasov equation in the literature [11, 12, 15]. We remark that this system is precisely equal to conformal kinetic dynamics (6.25) with c being zero.

6.4. Conformal Kinetic Dynamics in Density Formulation

As was shown above, the dual space of the Lie algebra $\mathcal{F}(M) \times \mathfrak{z}$ is the product $\text{Den}(M) \times \mathfrak{z}$ of the space of densities with real numbers. Accordingly, in view of the bracket formula (6.19), the coadjoint action of $\mathcal{F}(M) \times \mathfrak{z}$ on its dual $\text{Den}(M) \times \mathfrak{z}$ is given by

$$\begin{aligned}
& \langle ad_{(H, c_H)}^*(fd\mu, c^*), (F, c_F) \rangle \\
&= \langle (fd\mu, c^*), ad_{(H, c_H)}(F, c_F) \rangle \\
&= \left\langle (fd\mu, c^*), \left(\{H, F\}^{(S)} + c_H(Z(F) + F) - c_F(Z(H) + H), 0 \right) \right\rangle \\
&= \int_M (f\{H, F\}^{(S)} + fc_H(Z(F) + F) - fc_F(Z(H) + H))d\mu.
\end{aligned} \tag{6.30}$$

Let us analyze the terms on the right-hand side of the last line one by one. The first term reads

$$\int_M f\{H, F\}^{(S)}d\mu = \int_M \{f, H\}^{(S)}Fd\mu, \tag{6.31}$$

while the third term can be written as a multiple of c_F . The second term, on the other hand, may be examined through

$$\begin{aligned}
\int_M fc_H(Z(F) + F)d\mu &= \int_M fc_HZ(F)d\mu + \int_M fFc_Hd\mu \\
&= \int_M fc_H(\iota_Z dF)d\mu + \int_M fFc_Hd\mu \\
&= \int_M fc_H\iota_Z(dF \wedge d\mu) + \int_M fc_HdF \wedge \iota_Z d\mu + \int_M fFc_Hd\mu \\
&= \int_M fc_HdF \wedge \iota_Z d\mu + \int_M fFc_Hd\mu \\
&= - \int_M Fc_Hdf \wedge \iota_Z d\mu - \int_M fc_HFd\iota_Z d\mu + \int_M fFc_Hd\mu \\
&= - \int_M Fc_H(\iota_Z df)d\mu - \int_M fc_HF\text{div}(Z)d\mu + \int_M fFc_Hd\mu \\
&= - \int_M (c_HZ(f) + \text{div}(Z)fc_H - fc_H)Fd\mu.
\end{aligned} \tag{6.32}$$

Notice that, in the third line, we have used the distribution of the interior product to the wedge product of differential forms:

$$(\iota_Z dF)d\mu = \iota_Z(dF \wedge d\mu) + dF \wedge \iota_Z d\mu. \tag{6.33}$$

See, $dF \wedge d\mu$ is identically zero since its degree is more than the dimension of the manifold. This observation reads the fourth line in (6.32). Now, in case M is $2n$ dimensional, we recall from (6.6) that the divergence of the Liouville vector field Z is $\text{div}(Z) = -n$. Then,

$$ad_{(H, c_H)}^*(f, c^*) = \left(\{f, H\}^{(S)} - c_H Z(f) + c_H(n+1)f, - \int_M f(Z(H) + H) d\mu \right). \quad (6.34)$$

We refer to subsection 6.5 for the pure geometric link between the conformal kinetic equations (6.25) in momentum formulation and the conformal kinetic equation (6.35) in terms of the density.

The calculation (6.34) gives us the Lie-Poisson dynamics generated by a conformal vector field X_H^c as a coupled PDE system

$$\begin{aligned} \frac{\partial f}{\partial t} &= \{H, f\}^{(S)} + c_H Z(f) - c_H(n+1)f, \\ \frac{\partial c^*}{\partial t} &= \int_M f(Z(H) + H) d\mu, \end{aligned} \quad (6.35)$$

which we call the conformal kinetic equations.

Consider the Darboux coordinates (q^i, p_i) on $M = T^*Q$, and let the Hamiltonian function be the total energy $H = p^2/2m + e\phi(q)$ of a single particle. Then the conformal kinetic equations in density formulation (6.35) take the particular form

$$\begin{aligned} \frac{\partial f}{\partial t} + \frac{1}{m} \delta^{ij} p_i \frac{\partial f}{\partial q^j} - e \frac{\partial \phi}{\partial q^i} \frac{\partial f}{\partial p_i} + c_H(n+1)f - c_H p_i \frac{\partial f}{\partial p_i} &= 0, \\ \frac{\partial c^*}{\partial t} - \int_M f(H - p_i \frac{\partial H}{\partial p_i}) d\mu &= 0. \end{aligned} \quad (6.36)$$

If the conformal factor is trivial, then the equation in (6.35) becomes

$$\frac{\partial f}{\partial t} = \{H, f\}^{(S)}, \quad (6.37)$$

and, in local coordinates, we are left with the first equation in (6.36) which turns out to be the Vlasov equation

$$\frac{\partial f}{\partial t} + \frac{1}{m} \delta^{ij} p_i \frac{\partial f}{\partial q^j} - e \frac{\partial \phi}{\partial q^i} \frac{\partial f}{\partial p_i} = 0. \quad (6.38)$$

An Algebraic Route to Conformal Kinetic Equations. Let us now note that the algebraic structure of $\mathcal{F}(M) \times \mathfrak{z}$ formulated in (6.19) fits the abstract formalism presented in Appendix A, more precisely the bracket (A.4). Indeed, given the left action

$$\triangleright : \mathfrak{z} \times \mathcal{F}(M) \longrightarrow \mathcal{F}(M), \quad (c, H) \mapsto c(Z(H) + H) \quad (6.39)$$

of $\mathfrak{z}$ on $\mathcal{F}(M)$, and the adjoint action

$$ad_H F = \{H, F\}^{(S)} \quad (6.40)$$

of $\mathcal{F}(M)$ on itself, the bracket (6.19) may be written as

$$[(H, c_H), (F, c_F)] = (ad_H F + c_H \triangleright F - c_F \triangleright H, 0). \quad (6.41)$$

A direct computation, then, yields the coadjoint action as

$$ad_{(H, c_H)}^*(f, c^*) = (ad_H^* f + f \triangleleft^* c_H, -\mathfrak{b}_H^* f), \quad (6.42)$$

where

$$\begin{aligned} ad_H^* f &= \{f, H\}^{(S)}, & f \triangleleft^* c_H &= (n+1)c_H f - c_H Z(f), \\ \mathfrak{b}_H^* f &= \int_M f(Z(H) + H) d\mu. \end{aligned} \quad (6.43)$$

Let us note also that the coadjoint action (6.42) fits exactly the one in (6.34).

6.5. From Momentum to Density Formulations

We link now the conformal kinetic equations (6.25) in momentum formulation and the conformal kinetic equation (6.35) in terms of the density. To this end, we start with the density function f given in (6.23) and compute its time derivative in view of the conformal kinetic equations (6.25) in momentum formulation. This reads

$$\begin{aligned} \frac{\partial f}{\partial t} &= \operatorname{div} \Omega^\sharp(\dot{\Pi}) = -(\operatorname{div} \Omega^\sharp(\mathcal{L}_{X_H^c} \Pi + c_H n \Pi)) \\ &= -\operatorname{div} \Omega^\sharp(\mathcal{L}_{X_H^c} \Pi) - (c_H n) \operatorname{div} \Omega^\sharp(\Pi), \end{aligned} \quad (6.44)$$

where, keeping (6.23) in mind, the second term is equal to $-c_H n f$ whereas the first one needs a more detailed observation. To write the first term as a function

of f we pair it with an arbitrary function using the L_2 -pairing and compute

$$\begin{aligned}
\int_M \operatorname{div} \Omega^\sharp(\mathcal{L}_{X_H^c} \Pi) K d\mu &= \int_M \langle \mathcal{L}_{X_H^c} \Pi, X_K \rangle d\mu = \int_M \langle \mathcal{L}_{X_H - c_H Z} \Pi, X_K \rangle d\mu \\
&= \int_M \langle \mathcal{L}_{X_H} \Pi, X_K \rangle d\mu - c_H \int_M \langle \mathcal{L}_Z \Pi, X_K \rangle d\mu \\
&= - \int_M \langle \Pi, \mathcal{L}_{X_H} X_K \rangle d\mu + c_H \int_M \langle \Pi, \mathcal{L}_Z X_K \rangle d\mu + c_H \int_M \langle \Pi, X_K \rangle \operatorname{div}(Z) d\mu \\
&= \int_M \langle \Pi, X_{\{H, K\}^{(S)}} \rangle d\mu + c_H \int_M \langle \Pi, X_{Z(K)+K} \rangle d\mu + c_H \int_M \langle \Pi, X_K \rangle \operatorname{div}(Z) d\mu \\
&= \int_M \operatorname{div} \Omega^\sharp(\Pi) \{H, K\}^{(S)} d\mu + c_H \int_M \langle \Pi, X_{Z(K)+K} \rangle d\mu - c_H n \int_M \langle \Pi, X_K \rangle d\mu \\
&= \int_M \{ \operatorname{div} \Omega^\sharp(\Pi), H \}^{(S)} K d\mu + c_H \int_M \langle \Pi, X_{Z(K)} \rangle d\mu - c_H (n-1) \int_M \langle \Pi, X_K \rangle d\mu \\
&= \int_M \{ \operatorname{div} \Omega^\sharp(\Pi), H \}^{(S)} K d\mu + c_H \int_M \operatorname{div} \Omega^\sharp(\Pi) Z(K) d\mu \\
&\quad - c_H (n-1) \int_M \operatorname{div} \Omega^\sharp(\Pi) K d\mu \\
&= \int_M \{ \operatorname{div} \Omega^\sharp(\Pi), H \}^{(S)} K d\mu - c_H \int_M Z(\operatorname{div} \Omega^\sharp(\Pi)) K d\mu \\
&\quad - c_H \int_M (\operatorname{div} \Omega^\sharp(\Pi)) K \operatorname{div}(Z) d\mu - c_H (n-1) \int_M \operatorname{div} \Omega^\sharp(\Pi) K d\mu \\
&= \int_M \{f, H\}^{(S)} K d\mu - c_H \int_M Z(f) K d\mu + c_H \int_M f K n d\mu - c_H (n-1) \int_M f K d\mu \\
&= \int_M (\{f, H\}^{(S)} - c_H Z(f) + c_H f) K d\mu.
\end{aligned} \tag{6.45}$$

As a result, we have

$$\operatorname{div} \Omega^\sharp(\mathcal{L}_{X_H^c} \Pi) = \{f, H\}^{(S)} - c_H Z(f) + c_H f, \tag{6.46}$$

via which we obtain

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \operatorname{div} \Omega^\sharp(\dot{\Pi}) = -\operatorname{div} \Omega^\sharp(\mathcal{L}_{X_H^c} \Pi) - (c_H n) \operatorname{div} \Omega^\sharp(\Pi) \\
&= \{H, f\}^{(S)} + c_H Z(f) - c_H (n+1) f.
\end{aligned} \tag{6.47}$$

This is exactly the same as the evolution of the density variable given in the first line of (6.35). Let us perform a similar analysis for the real variable as well. We thus compute the time derivative of the scalar variable established in (6.23) to

arrive at

$$\begin{aligned}
\frac{\partial c^*}{\partial t} &= - \int_M \langle \dot{\Pi}, Z \rangle d\mu = \int_M \langle \mathcal{L}_{X_H^c} \Pi, Z \rangle d\mu + \int_M \langle cn\Pi, Z \rangle d\mu \\
&= \int_M \langle \mathcal{L}_{X_H - cZ} \Pi, Z \rangle d\mu + \int_M \langle cn\Pi, Z \rangle d\mu \\
&= \int_M \langle \mathcal{L}_{X_H} \Pi, Z \rangle d\mu - c \int_M \langle \mathcal{L}_Z \Pi, Z \rangle d\mu + \int_M \langle cn\Pi, Z \rangle d\mu \\
&= - \int_M \langle \Pi, \mathcal{L}_{X_H} Z \rangle d\mu + c \int_M \langle \Pi, \mathcal{L}_Z Z \rangle d\mu \\
&\quad + c \int_M \langle \Pi, Z \rangle \operatorname{div}(Z) d\mu + \int_M \langle cn\Pi, Z \rangle d\mu \\
&= \int_M \langle \Pi, X_{Z(H)+H} \rangle d\mu \\
&= \int_M \operatorname{div} \Omega^\sharp(\Pi)(Z(H) + H) d\mu \\
&= \int_M f(Z(H) + H) d\mu,
\end{aligned} \tag{6.48}$$

which coincides with the evolution of the real variable given in the second line of (6.35).

6.6. Kinetic Conformal Dynamics as Vertical Representative

Now we refer to the definitions in Section (2.8) and Section (5.3). Based on these sections, we establish the following Lie algebra homomorphisms, which describe the relationship between particle motion and the motion of the continuum:

$$\begin{array}{ccccc}
\text{Particle} & \xrightarrow{\Xi \text{ in (5.66)}} & \text{Lifted} & \xrightarrow{\mathfrak{V} \text{ in (5.81)}} & \text{Kinetic motion} \\
\text{motion, } X & & \text{motion, } \Xi(X) & & \text{of fibers, } \mathfrak{V}\Xi(X).
\end{array} \tag{6.49}$$

We do note that this geometric path provides the geometrization of the conformal kinetic dynamics in (6.25). For the autonomous cases, one reduces the mapping Ξ to the cotangent bundle and arrives at the momentum Vlasov dynamics in (6.29). The conformal kinetic dynamics, formulated in terms of the vertical representative of the collective lift of the conformal Hamiltonian vector field, can be expressed in the following proposition.

Proposition 6.1: The conformal momentum-Vlasov dynamics (6.25) is the vertical representative $\mathfrak{V}\Xi(X_H^c)$ of the collective lift of the conformal Hamiltonian vector field X_H^c

$$\dot{\Pi}^v = -\mathfrak{V}\Xi(X_H^c). \tag{6.50}$$

7. CONTACT KINETIC DYNAMICS

7.1. Contact Manifolds

Let $\bar{M}$ be an odd, say $(2n + 1)$, dimensional manifold. A contact structure on $\bar{M}$ is a maximally non-integrable smooth distribution of codimension one, and it is locally given by the kernel of a one-form η such that

$$d\eta^n \wedge \eta \neq 0. \quad (7.1)$$

Such a one-form η is called a (local) contact form [19, 22]. In the present manuscript, we shall consider the existence of a global contact one-form. A contact one-form for a given contact structure is not unique. Indeed, if η is a contact one-form for a fixed contact structure, then $\lambda\eta$ also defines the same contact structure for any non-zero real-valued function λ defined on $\bar{M}$. In short, we call a $(2n + 1)$ -dimensional manifold $\bar{M}$ as a contact manifold if it is equipped with a contact one-form η satisfying $d\eta^n \wedge \eta \neq 0$, and we shall denote a contact manifold by $(\bar{M}, \eta)$.

Given a contact one-form η , the vector field $\mathcal{R}$ satisfying

$$\iota_{\mathcal{R}}\eta = 1, \quad \iota_{\mathcal{R}}d\eta = 0 \quad (7.2)$$

is unique, and it is called the Reeb vector field. There is, on the other hand, a musical isomorphism $\flat$ from the space of sections of the tangent bundle $T\bar{M}$ to the sections of the cotangent bundle $T^*\bar{M}$ defined by

$$\flat : \mathfrak{X}(\bar{M}) \longrightarrow \Gamma^1(\bar{M}), \quad Y \mapsto \iota_Y d\eta + \eta(Y)\eta. \quad (7.3)$$

It is worth noting that the image of the Reeb vector field $\mathcal{R}$ under the musical mapping is the contact one-form η . We shall denote the inverse of (7.3) by $\sharp$. Referring to this, we define a bivector field Λ on M as

$$\Lambda(\alpha, \beta) = d\eta(\sharp\alpha, \sharp\beta). \quad (7.4)$$

Referring to the bivector field Λ we introduce the following musical mapping

$$\Lambda^\sharp : \Gamma^1(\bar{M}) \longrightarrow \mathfrak{X}(\bar{M}), \quad \alpha \mapsto \Lambda(\bullet, \alpha) = \sharp\alpha - \alpha(\mathcal{R})\mathcal{R}. \quad (7.5)$$

The kernel of the mapping $\Lambda^\sharp$ is spanned by the contact one-form η so it fails to be an isomorphism. A contact manifold $(\bar{M}, \eta)$ admits a Jacobi manifold structure [37, 75], see [76] for a more recent exposition. This realization permits us to define a Jacobi (contact) bracket

$$\{\bar{F}, \bar{H}\}^{(C)} = \Lambda(d\bar{F}, d\bar{H}) + \bar{F}\mathcal{R}(\bar{H}) - \bar{H}\mathcal{R}(\bar{F}). \quad (7.6)$$

See that the bracket satisfies the Jacobi identity but the Leibniz's identity is violated due to the last two terms (the Reeb terms) on the right-hand side.

Contact Hamiltonian Motion. For a Hamiltonian function $\bar{H}$ on a contact manifold $(\bar{M}, \eta)$, there is a corresponding contact Hamiltonian vector field $\xi_{\bar{H}}$ given by

$$\iota_{\xi_{\bar{H}}} \eta = -\bar{H}, \quad \iota_{\xi_{\bar{H}}} d\eta = d\bar{H} - \mathcal{R}(\bar{H})\eta, \quad (7.7)$$

where $\mathcal{R}$ is the Reeb vector field, and $\bar{H}$ is called the contact Hamiltonian function [77–79]. We also have

$$\flat(\xi_{\bar{H}}) = d\bar{H} - (\mathcal{R}(\bar{H}) + \bar{H})\eta. \quad (7.8)$$

Dissipation. Let us denote a contact Hamiltonian system as a three-tuple $(\bar{M}, \eta, \bar{H})$, where $(\bar{M}, \eta)$ is a contact manifold and $\bar{H}$ is a smooth real function on $\bar{M}$. A direct computation determines a conformal factor for a given contact vector field via

$$\mathcal{L}_{\xi_{\bar{H}}} \eta = d\iota_{\xi_{\bar{H}}} \eta + \iota_{\xi_{\bar{H}}} d\eta = -\mathcal{R}(\bar{H})\eta. \quad (7.9)$$

According to (7.9), the flow of a contact Hamiltonian system preserves the contact structure, but it preserves neither the contact one-form nor the Hamiltonian function. Instead, we obtain

$$\mathcal{L}_{\xi_{\bar{H}}} \bar{H} = -\mathcal{R}(\bar{H})\bar{H}. \quad (7.10)$$

Being a non-vanishing top-form we may consider $d\eta^n \wedge \eta$ as a volume form on $\bar{M}$. The Hamiltonian motion does not preserve the volume form since

$$\mathcal{L}_{\xi_{\bar{H}}} (d\eta^n \wedge \eta) = -(n+1)\mathcal{R}(\bar{H})d\eta^n \wedge \eta. \quad (7.11)$$

Assuming the dimension of $\bar{M}$ to be $2n+1$, we compute the divergence of a contact vector field as

$$\text{div}(\xi_{\bar{H}}) = -(n+1)\mathcal{R}(\bar{H}). \quad (7.12)$$

However, it is immediate to see that, for a nowhere vanishing Hamiltonian function $\bar{H}$, the quantity $\bar{H}^{-(n+1)}(d\eta)^n \wedge \eta$ is preserved along the motion (see [80]).

A direct computation proves that the contact bracket and the contact vector field are related as

$$\begin{aligned} \{\bar{F}, \bar{H}\}^{(C)} &= -\iota_{[\xi_{\bar{H}}, \xi_{\bar{F}}]}\eta = -\mathcal{L}_{\xi_{\bar{H}}}\iota_{\xi_{\bar{F}}}\eta + \iota_{\xi_{\bar{F}}}\mathcal{L}_{\xi_{\bar{H}}}\eta \\ &= -\mathcal{L}_{\xi_{\bar{H}}}(-\bar{F}) + \iota_{\xi_{\bar{F}}}(-\mathcal{R}(\bar{H})\eta) \\ &= \xi_{\bar{H}}(\bar{F}) + \bar{F}\mathcal{R}(\bar{H}). \end{aligned} \quad (7.13)$$

This observation is important. We remark that the flow generated by the contact vector field and the flow generated by the contact bracket are not the same.

Darboux Coordinates and Contactization of a Symplectic Manifold. We start with an exact symplectic manifold $(M, \Omega = -d\Theta)$ and consider the principal line bundle

$$(\bar{M}, \eta) \xrightarrow{pr} (M, \Omega = -d\Theta). \quad (7.14)$$

For a local coordinate system z on the circle (which we shall consider being $\mathbb{R}$), and the Darboux coordinates (q^i, p_i) on the symplectic manifold, the contact manifold admits the Darboux coordinates (q^i, p_i, z) on $\bar{M}$. In this realization, the contact one-form and the associated Reeb field are

$$\eta = dz - \bar{\Theta} = dz - p_i dq^i, \quad \mathcal{R} = \frac{\partial}{\partial z}, \quad (7.15)$$

where $\bar{\Theta}$ is the pullback of the potential one-form on M . This suggests the coordinates (q^i, p_i, z) on the contact manifold $\bar{M}$. In this case, the volume form $\bar{d}\mu$ on the contact manifold is computed to be

$$\bar{d}\mu = dz \wedge d\mu \quad (7.16)$$

where $d\mu$ is the symplectic volume in (2.6). The bivector Λ in (7.4) is computed to be

$$\Lambda = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + p_i \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial p_i}. \quad (7.17)$$

In terms of the Darboux coordinates, we compute the musical mapping $\Lambda^\sharp$ in (7.5) as

$$\Lambda^\sharp : \alpha_i dq^i + \alpha^i dp_i + u dz \mapsto \alpha^i \frac{\partial}{\partial q^i} - (\alpha_i + p_i u) \frac{\partial}{\partial p_i} + \alpha^i p_i \frac{\partial}{\partial z}. \quad (7.18)$$

Then, in this local picture, the contact bracket (7.6) is

$$\{\bar{F}, \bar{H}\}^{(C)} = \frac{\partial \bar{F}}{\partial q^i} \frac{\partial \bar{H}}{\partial p_i} - \frac{\partial \bar{F}}{\partial p_i} \frac{\partial \bar{H}}{\partial q^i} + \left(\bar{F} - p_i \frac{\partial \bar{F}}{\partial p_i} \right) \frac{\partial \bar{H}}{\partial z} - \left(\bar{H} - p_i \frac{\partial \bar{H}}{\partial p_i} \right) \frac{\partial \bar{F}}{\partial z}. \quad (7.19)$$

A symplectic manifold M locally looks like a cotangent manifold. Accordingly, without loss of generalization, one may substitute the symplectic manifold M with the cotangent bundle T^*Q . In this case, the contact manifold $\bar{M}$ locally turns out to be the extended cotangent bundle $T^*Q \times \mathbb{R}$.

7.2. Dynamics on Contact Manifolds

This subsection introduces two different dynamical vector fields that can be determined on a contact manifold $(\bar{M}, \eta)$. To have these realizations, for a given Hamiltonian function $\bar{H}$, we first recall the contact Hamiltonian vector field definition in (7.8) and write it as

$$\xi_{\bar{H}} = \sharp(d\bar{H}) - \mathcal{R}(\bar{H})\mathcal{R} - \bar{H}\mathcal{R}. \quad (7.20)$$

As we depict in the sequel, the space of such vector fields determines a Lie algebra as a manifestation of the Jacobi manifold structure of the contact manifold. By using only the first and third terms on the right-hand side we define a strict contact Hamiltonian vector field as

$$Y_{\bar{H}} = \sharp(d\bar{H}) - \bar{H}\mathcal{R}. \quad (7.21)$$

Let us now depict all the algebraic properties of these dynamics in detail.

Contact Diffeomorphisms and Contact Hamiltonian Vector Fields. For a contact manifold $(\bar{M}, \eta)$, a contact diffeomorphism (contactomorphism) is the one that preserves the contact structure. We denote the group of contact diffeomorphisms by [81]

$$\text{Diff}_{\text{con}}(\bar{M}) = \left\{ \varphi \in \text{Diff}(\bar{M}) : \varphi^* \eta = \gamma \eta, \quad \gamma \in \mathcal{F}(\bar{M}) \right\}. \quad (7.22)$$

Here, $\text{Diff}(\bar{M})$ is standing for the group of all diffeomorphism on $\bar{M}$. Notice that the existence of γ in the definition manifests the conformal definition of the contact structure. A vector field on the contact manifold $(\bar{M}, \eta)$ is a contact vector field (called also infinitesimal conformal contact diffeomorphism) if it generates

a one-parameter group of contact diffeomorphisms. Accordingly, the space of contact vector fields is given by

$$\mathfrak{X}_{\text{con}}(\bar{M}) = \left\{ X \in \mathfrak{X}(\bar{M}) : \mathcal{L}_X \eta = -\lambda \eta, \quad \lambda \in \mathcal{F}(\bar{M}) \right\}. \quad (7.23)$$

Sometimes a contact vector field is denoted by a two-tuple (X, λ) to exhibit the conformal factor λ . A direct observation reads from (7.9) that a contact Hamiltonian vector field $\xi_{\bar{H}}$ is a contact vector field with conformal parameter $\lambda = \mathcal{R}(\bar{H})$ so it belongs to $\mathfrak{X}_{\text{con-ham}}(\bar{M})$, for more details, see [77, 78, 82, 83]. In this work, our interest is the space of contact Hamiltonian vector fields

$$\mathfrak{X}_{\text{con-ham}}(\bar{M}) = \left\{ \xi_{\bar{H}} \in \mathfrak{X}(\bar{M}) : \iota_{\xi_{\bar{H}}} \eta = -\bar{H}, \quad \iota_{\xi_{\bar{H}}} d\eta = d\bar{H} - \mathcal{R}(\bar{H})\eta \right\}. \quad (7.24)$$

This space is a Lie subalgebra of space of all vector fields as a manifestation of the identity

$$[\xi_{\bar{F}}, \xi_{\bar{H}}] = -\xi_{\{\bar{F}, \bar{H}\}^{(C)}}. \quad (7.25)$$

So, one may establish the following isomorphism from the space of real smooth functions on $\bar{M}$ to the space of contact Hamiltonian vector fields

$$\Psi : (\mathcal{F}(\bar{M}), \{\bullet, \bullet\}^{(C)}) \longrightarrow (\mathfrak{X}_{\text{con-ham}}(\bar{M}), -[\bullet, \bullet]), \quad \bar{H} \mapsto \xi_{\bar{H}}. \quad (7.26)$$

Referring to the Darboux coordinates (q^i, p_i, z) , for a function $\bar{H} = \bar{H}(q^i, p_i, z)$, the contact Hamiltonian vector field determined in (7.7) becomes

$$\xi_{\bar{H}} = \frac{\partial \bar{H}}{\partial p_i} \frac{\partial}{\partial q^i} - \left(\frac{\partial \bar{H}}{\partial q^i} + \frac{\partial \bar{H}}{\partial z} p_i \right) \frac{\partial}{\partial p_i} + \left(p_i \frac{\partial \bar{H}}{\partial p_i} - \bar{H} \right) \frac{\partial}{\partial z}. \quad (7.27)$$

Thus, we obtain the contact Hamilton's equations for $\bar{H}$ as

$$\frac{dq^i}{dt} = \frac{\partial \bar{H}}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial \bar{H}}{\partial q^i} - p_i \frac{\partial \bar{H}}{\partial z}, \quad \frac{dz}{dt} = p_i \frac{\partial \bar{H}}{\partial p_i} - \bar{H}. \quad (7.28)$$

In particular, the Reeb vector field becomes $\mathcal{R} = \partial/\partial z$. The divergence of a contact Hamiltonian vector field (7.12) is then

$$\text{div}(\xi_{\bar{H}}) = -(n+1)\mathcal{R}(\bar{H}) = -(n+1)\frac{\partial \bar{H}}{\partial z}. \quad (7.29)$$

Contact dynamics finds many applications in various fields of physics especially in thermodynamics see, for example, [84–88].

Quantomorphisms and Strict Contact Hamiltonian Vector Fields. Let us consider a contact manifold $(\bar{M}, \eta)$. By asking the conformal factor γ to be the

unity for a contact diffeomorphism in (7.22), one arrives at the conservation of the contact form $\varphi^*\eta = \eta$. We call such a mapping as a strict contact diffeomorphism (or a quantomorphism). For a contact manifold $(\bar{M}, \eta)$, we denote the group of all strict contact diffeomorphisms as

$$\text{Diff}_{\text{st-con}}(\bar{M}) = \left\{ \varphi \in \text{Diff}(\bar{M}) : \varphi^*\eta = \eta \right\} \subset \text{Diff}_{\text{con}}(\bar{M}). \quad (7.30)$$

The Lie algebra of this group consists of so-called strict contact vector fields (or, infinitesimal quantomorphisms, or infinitesimal strict contact diffeomorphism)

$$\mathfrak{X}_{\text{st-con}}(\bar{M}) = \left\{ X \in \mathfrak{X}(\bar{M}) : \mathcal{L}_X \eta = 0 \right\} \subset \mathfrak{X}_{\text{con}}(\bar{M}). \quad (7.31)$$

Notice that for a given Hamiltonian function $\bar{H}$, the contact Hamiltonian vector field ξ_H defined in (7.7) is a strict contact vector field if and only if $d\bar{H}(\mathcal{R}) = 0$. This reads the following space of strict contact Hamiltonian vector fields

$$\mathfrak{X}_{\text{st-con-ham}}(\bar{M}) = \left\{ Y_{\bar{H}} \in \mathfrak{X}(\bar{M}) : \iota_{Y_{\bar{H}}} \eta = -\bar{H}, \iota_{Y_{\bar{H}}} d\eta = d\bar{H} \right\} \subset \mathfrak{X}_{\text{con-ham}}(\bar{M}). \quad (7.32)$$

Referring to the local realization in (7.15) given in terms of the Darboux coordinates (q^i, p_i, z) , it is possible to see that to generate a strict contact Hamiltonian vector field, a function $\bar{H}$ must be independent of the fiber variable z . For two functions, those that are not dependent on the fiber variable z , the contact bracket $\{\bullet, \bullet\}^{(C)}$ in (7.19) locally turns out to be equal to the canonical Poisson bracket on the symplectic manifold M . Accordingly, a direct calculation reads that

$$[Y_{\bar{H}}, Y_{\bar{F}}] = -Y_{\{\bar{H}, \bar{F}\}^{(C)}}. \quad (7.33)$$

Note that, one has the following identities in terms of the musical mapping $\flat$ in (7.3) and its inverse $\sharp$ as

$$\flat(Y_{\bar{H}}) = d\bar{H} - \bar{H}\eta, \quad Y_{\bar{H}} = \sharp(d\bar{H}) - \bar{H}\mathcal{R}. \quad (7.34)$$

Referring to the Darboux coordinates (q^i, p_i, z) , for a Hamiltonian function $\bar{H} = \bar{H}(q^i, p_i)$ independent of the fiber variable z , the strict contact Hamiltonian vector field is

$$Y_{\bar{H}} = \frac{\partial \bar{H}}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial \bar{H}}{\partial q^i} \frac{\partial}{\partial p_i} + \left(p_i \frac{\partial \bar{H}}{\partial p_i} - \bar{H} \right) \frac{\partial}{\partial z}. \quad (7.35)$$

Thus, we obtain strict contact Hamilton's equations as

$$\frac{dq^i}{dt} = \frac{\partial \bar{H}}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial \bar{H}}{\partial q^i}, \quad \frac{dz}{dt} = p_i \frac{\partial \bar{H}}{\partial p_i} - \bar{H}. \quad (7.36)$$

See that this flow is divergence-free.

7.3. Contact Kinetic Dynamics in Momentum Formulation

Let $\bar{M}$ be the extended cotangent bundle with the contact one-form η . We shall now determine the kinetic motion of contact particles. To this end, we shall lift the particle motion.

The Dual Space of Contact Hamiltonian Vector Fields. Let us now determine the dual space $\mathfrak{X}_{\text{con-ham}}^*(\bar{M})$ of the space contact vector fields $\mathfrak{X}_{\text{con-ham}}(\bar{M})$ given in (7.24). We first note that $\mathfrak{X}_{\text{con-ham}}^*(\bar{M})$ is a subspace of the space $\Gamma^1(\bar{M}) \otimes \text{Den}(\bar{M})$ of one-form densities. To be more precise, we compute the L_2 -pairing (simply multiply-and-integrate) of an arbitrary contact vector field $\xi_{\bar{H}}$ with a one-form density $\bar{\Pi} \otimes \bar{d}\mu$. Making use of the identities of the Cartan calculus, we obtain

$$\begin{aligned} \langle \bar{\Pi} \otimes \bar{d}\mu, \xi_{\bar{H}} \rangle_{L_2} &= \int \langle \bar{\Pi}, \xi_{\bar{H}} \rangle \bar{d}\mu = \int \langle \bar{\Pi}, \sharp(d\bar{H}) - (\mathcal{R}(\bar{H}) + \bar{H})\mathcal{R} \rangle \bar{d}\mu \\ &= - \int \langle \sharp\bar{\Pi}, d\bar{H} \rangle \bar{d}\mu + \int (\mathcal{R}(\bar{H}) + \bar{H}) \langle \sharp\bar{\Pi}, \eta \rangle \bar{d}\mu \\ &= - \int (\iota_{\sharp\bar{\Pi}} d\bar{H}) \bar{d}\mu + \int \mathcal{R}(\bar{H}) \langle \sharp\bar{\Pi}, \eta \rangle \bar{d}\mu + \int \bar{H} \langle \sharp\bar{\Pi}, \eta \rangle \bar{d}\mu \\ &= - \int d\bar{H} \iota_{\sharp\bar{\Pi}} \bar{d}\mu + \int \iota_{\mathcal{R}} d\bar{H} \langle \sharp\bar{\Pi}, \eta \rangle \bar{d}\mu + \int \bar{H} \langle \sharp\bar{\Pi}, \eta \rangle \bar{d}\mu \\ &= \int \bar{H} d\iota_{\sharp\bar{\Pi}} \bar{d}\mu + \int \langle \sharp\bar{\Pi}, \eta \rangle d\bar{H} \wedge \iota_{\mathcal{R}} \bar{d}\mu + \int \bar{H} \langle \sharp\bar{\Pi}, \eta \rangle \bar{d}\mu \\ &= \int \bar{H} d\iota_{\sharp\bar{\Pi}} \bar{d}\mu - \int \bar{H} d\langle \sharp\bar{\Pi}, \eta \rangle \wedge \iota_{\mathcal{R}} \bar{d}\mu \\ &\quad - \int \bar{H} \langle \sharp\bar{\Pi}, \eta \rangle d\iota_{\mathcal{R}} \bar{d}\mu + \int \bar{H} \langle \sharp\bar{\Pi}, \eta \rangle \bar{d}\mu \\ &= \int \bar{H} \left(\text{div}(\sharp\bar{\Pi}) \bar{d}\mu - \iota_{\mathcal{R}} d\langle \sharp\bar{\Pi}, \eta \rangle - \langle \sharp\bar{\Pi}, \eta \rangle \text{div}(\mathcal{R}) + \langle \sharp\bar{\Pi}, \eta \rangle \right) \bar{d}\mu \\ &= \int \bar{H} \left(\text{div}(\sharp\bar{\Pi}) - \mathcal{L}_{\mathcal{R}} \langle \sharp\bar{\Pi}, \eta \rangle + \langle \sharp\bar{\Pi}, \eta \rangle \right) \bar{d}\mu, \end{aligned} \quad (7.37)$$

where div stands for the divergence with respect to the contact volume $\bar{d}\mu$ in (7.16). Accordingly, once the volume form is fixed, the non-degeneracy of the

pairing motivates us to define the dual space as

$$\mathfrak{X}_{\text{con-ham}}^*(\bar{M}) = \left\{ \bar{\Pi} \in \Gamma^1(\bar{M}) : \text{div}(\sharp\bar{\Pi}) - \mathcal{L}_{\mathcal{R}}\langle\sharp\bar{\Pi}, \eta\rangle + \langle\sharp\bar{\Pi}, \eta\rangle \neq 0 \right\} \cup \{0\}. \quad (7.38)$$

We next recall the Lie algebra isomorphism $\bar{H} \mapsto \xi_{\bar{H}}$ of (7.26). Identifying the dual $\mathcal{F}(\bar{M})$ with the space of densities on the contact manifold, and fixing the contact volume form, (7.37) determines the dual of (7.26) as

$$\Psi^* : \mathfrak{X}_{\text{con-ham}}^*(\bar{M}) \longrightarrow \mathcal{F}^*(\bar{M}), \quad \bar{\Pi} \mapsto \bar{f} = \text{div}(\sharp\bar{\Pi}) - \mathcal{L}_{\mathcal{R}}\langle\sharp\bar{\Pi}, \eta\rangle + \langle\sharp\bar{\Pi}, \eta\rangle. \quad (7.39)$$

In terms of the Darboux coordinates (q^i, p_i, z) , we can compute the density function $\bar{f}$ from (7.39) as follows. Consider a one-form section

$$\bar{\Pi} = \bar{\Pi}_i dq^i + \bar{\Pi}^i dp_i + \bar{\Pi}_z dz. \quad (7.40)$$

Let us present the three terms on the right-hand side of (7.39) one by one. A direct calculation reads the contact divergence of $\bar{\Pi}$ as

$$\text{div}(\sharp\bar{\Pi}) = \frac{\partial \bar{\Pi}^i}{\partial q^i} - \frac{\partial \bar{\Pi}_i}{\partial p_i} - n\bar{\Pi}_z - p_i \frac{\partial \bar{\Pi}_z}{\partial p_i} + \frac{\partial \bar{\Pi}_z}{\partial z} + p_i \frac{\partial \bar{\Pi}^i}{\partial z}. \quad (7.41)$$

Then the second Lie derivative term is computed to be

$$\mathcal{L}_{\mathcal{R}}\langle\sharp\bar{\Pi}, \eta\rangle = \mathcal{L}_{\mathcal{R}}\bar{\Pi}_z = \iota_R d\bar{\Pi}_z = \frac{\partial \bar{\Pi}_z}{\partial z}. \quad (7.42)$$

The third term is simply $\langle\sharp\bar{\Pi}, \eta\rangle = \bar{\Pi}_z$. Adding all of these terms we arrive at the definition of the density function

$$\bar{f} = \frac{\partial \bar{\Pi}^i}{\partial q^i} - \frac{\partial \bar{\Pi}_i}{\partial p_i} - p_i \left(\frac{\partial \bar{\Pi}_z}{\partial p_i} - \frac{\partial \bar{\Pi}^i}{\partial z} \right) - (n-1)\bar{\Pi}_z. \quad (7.43)$$

Coadjoint Flow on $\mathfrak{X}_{\text{con-ham}}^*(\bar{M})$. Let, as above, $\mathfrak{X}_{\text{con-ham}}(\bar{M})$ be the Lie algebra of contact vector fields with the opposite Jacobi-Lie bracket. That is,

$$ad_{\xi_{\bar{H}}} \xi_{\bar{F}} = -[\xi_{\bar{H}}, \xi_{\bar{F}}], \quad (7.44)$$

which we consider to be the left adjoint action of $\mathfrak{X}_{\text{con-ham}}(\bar{M})$ on itself. Now dualizing the adjoint action, we arrive at the coadjoint action of $\mathfrak{X}_{\text{con-ham}}(\bar{M})$ on its dual $\mathfrak{X}_{\text{con-ham}}^*(\bar{M})$ as

$$\begin{aligned} ad^* : \mathfrak{X}_{\text{con-ham}}(\bar{M}) \times \mathfrak{X}_{\text{con-ham}}^*(\bar{M}) &\longrightarrow \mathfrak{X}_{\text{con-ham}}^*(\bar{M}), \\ \langle ad_{\xi_{\bar{H}}}^* \bar{\Pi}, \xi_{\bar{F}} \rangle &= \langle \bar{\Pi}, ad_{\xi_{\bar{H}}} \xi_{\bar{F}} \rangle. \end{aligned} \quad (7.45)$$

More explicitly, given an arbitrary field $\xi_{\bar{F}}$ we have

$$\begin{aligned}
\langle ad_{\xi_{\bar{H}}}^* \bar{\Pi}, \xi_{\bar{F}} \rangle &= \langle \bar{\Pi}, ad_{\xi_{\bar{H}}} \xi_{\bar{F}} \rangle = - \int \langle \bar{\Pi}, [\xi_{\bar{H}}, \xi_{\bar{F}}] \rangle d\bar{\mu} \\
&= - \int \langle \bar{\Pi}, \mathcal{L}_{\xi_{\bar{H}}} \xi_{\bar{F}} \rangle d\bar{\mu} = \int \langle \mathcal{L}_{\xi_{\bar{H}}} \bar{\Pi} + \text{div}(\xi_{\bar{H}}) \bar{\Pi}, \xi_{\bar{F}} \rangle d\bar{\mu} \\
&= \int \langle \mathcal{L}_{\xi_{\bar{H}}} \bar{\Pi} - (n+1) \mathcal{R}(\bar{H}) \bar{\Pi}, \xi_{\bar{F}} \rangle d\bar{\mu},
\end{aligned} \tag{7.46}$$

where we used (7.12) for the divergence of the contact vector field $\xi_{\bar{H}}$. As a result, the coadjoint action may be presented as

$$ad_{\xi_{\bar{H}}}^* \bar{\Pi} = \mathcal{L}_{\xi_{\bar{H}}} \bar{\Pi} - (n+1) \mathcal{R}(\bar{H}) \bar{\Pi}. \tag{7.47}$$

Being the dual space of a Lie algebra, $\mathfrak{X}_{\text{con-ham}}^*(\bar{M})$ admits a Poisson bracket called the Lie-Poisson bracket [6–8]. More precisely, given two functionals A and B on $\mathfrak{X}_{\text{con-ham}}^*(\bar{M})$ the Lie-Poisson bracket on $\mathfrak{X}_{\text{con-ham}}^*(\bar{M})$ is defined to be

$$\{A, B\}^{\mathfrak{X}_{\text{con-ham}}^*}(\bar{\Pi}) = \int \left\langle \bar{\Pi}, ad_{\delta A / \delta \bar{\Pi}} \frac{\delta B}{\delta \bar{\Pi}} \right\rangle d\bar{\mu} = - \int \left\langle \bar{\Pi}, \left[\frac{\delta A}{\delta \bar{\Pi}}, \frac{\delta B}{\delta \bar{\Pi}} \right] \right\rangle d\bar{\mu} \tag{7.48}$$

where $\delta A / \delta \bar{\Pi}$ stands for the Fréchet derivative of the functional A . Given a Hamiltonian functional $\mathcal{H}$, the Lie-Poisson dynamics is governed by the Lie-Poisson equations computed in terms of the coadjoint action, that is,

$$\dot{\bar{\Pi}} = \{\bar{\Pi}, \mathcal{H}\}^{\mathfrak{X}_{\text{con-ham}}^*} = -ad_{\delta \mathcal{H} / \delta \bar{\Pi}}^* \bar{\Pi}. \tag{7.49}$$

In particular, for the Hamiltonian functional defined by means of the contact vector field $\xi_{\bar{H}}$ as

$$\mathcal{H}(\bar{\Pi}) = \int \langle \bar{\Pi}, \xi_{\bar{H}} \rangle d\bar{\mu}, \tag{7.50}$$

the Fréchet derivative $\delta \mathcal{H} / \delta \bar{\Pi}$ of $\mathcal{H}$ with respect to the momenta becomes the vector field $\xi_{\bar{H}}$. In this case, the Lie-Poisson equation (7.49) takes the form of

$$\dot{\bar{\Pi}} = -\mathcal{L}_{\xi_{\bar{H}}} \bar{\Pi} + (n+1) \mathcal{R}(\bar{H}) \bar{\Pi}. \tag{7.51}$$

Notice that this system governs the dynamics of one-form sections, which reside in the dual space of contact Hamiltonian vector fields, defined on the contact manifold. We refer to (7.51) as the contact momentum-Vlasov equation. It represents an intermediate system that paves the way for what we call the kinetic equations of contact particles. Before examining the kinetic equations in terms of density functions, we will first focus on the strict contact case. Later, we will

arrive at the kinetic equation of contact particles and relate it to the contact momentum-Vlasov equation (7.51).

The Dual Space of Strict Contact Hamiltonian Vector Fields. Now we consider the algebra (7.32) of strict contact Hamiltonian vector fields $\mathfrak{X}_{\text{st-con-ham}}(\bar{M})$. Similar to the calculation (7.37) done above, we compute the precise dual of this vector space by means of L_2 -pairing. Accordingly, we have

$$\langle \bar{\Sigma} \otimes \bar{d\mu}, Y_{\bar{H}} \rangle_{L_2} = \int \langle \bar{\Sigma}, Y_{\bar{H}} \rangle \bar{d\mu} = \int \bar{H} \left(\text{div}(\sharp \bar{\Sigma}) + \langle \sharp \bar{\Sigma}, \eta \rangle \right) \bar{d\mu}. \quad (7.52)$$

Once again, we fix the volume form. Then the non-degeneracy of the pairing (7.52) leads us to define the dual space as

$$\mathfrak{X}_{\text{st-con-ham}}^*(\bar{M}) = \left\{ \bar{\Sigma} \in \Gamma^1(\bar{M}) : \text{div}(\sharp \bar{\Sigma}) + \langle \sharp \bar{\Sigma}, \eta \rangle \neq 0 \right\} \cup \{0\}. \quad (7.53)$$

For the contact manifold $\bar{M}$, consider the contactization bundle $\tau : \bar{M} \mapsto M$ over the symplectic base manifold M . A real-valued function H on the base manifold can be pulled back to the contact manifold by means of the projection τ . This gives a real-valued function τ^*H which satisfies $\mathcal{R}(\tau^*H) = 0$. So that τ^*H generates a strict contact Hamiltonian vector field Y_{τ^*H} . As a matter of fact, this picture is generic for all functions on the contact manifold that do not depend on the fiber variable. So we arrive at the following isomorphism

$$\Gamma : (\mathcal{F}(M)), \{\bullet, \bullet\} \longrightarrow \left(\mathfrak{X}_{\text{st-con-ham}}(\bar{M}), -[\bullet, \bullet] \right), \quad H \mapsto Y_{\tau^*H}. \quad (7.54)$$

The computation (7.52) provides the dual of this as

$$\Gamma^* : \mathfrak{X}_{\text{st-con-ham}}^*(\bar{M}) \longrightarrow \text{Den}(M), \quad \bar{\Sigma} \mapsto \int_{S^1} \left(\text{div}(\sharp \bar{\Sigma}) + \langle \sharp \bar{\Sigma}, \eta \rangle \right) dz \otimes d\mu, \quad (7.55)$$

where $\text{Den}(M)$ is the space of densities on the symplectic manifold (M, Ω) . Accordingly, we have that the density function

$$f(q, p) = \int_{S^1} \left(\text{div}(\sharp \bar{\Sigma}) + \langle \sharp \bar{\Sigma}, \eta \rangle \right) dz \quad (7.56)$$

defined on the base manifold (that is the symplectic manifold) M . Referring to the Darboux coordinates (q^i, p_i) on M , and the induced Darboux coordinates (q^i, p_i, z) on $\bar{M}$ we compute the density function as

$$f(q, p) = \int_{S^1} \left(\frac{\partial \bar{\Sigma}^i}{\partial q^i} - \frac{\partial \bar{\Sigma}_i}{\partial p_i} - p_i \left(\frac{\partial \bar{\Sigma}_z}{\partial p_i} - \frac{\partial \bar{\Sigma}^i}{\partial z} \right) + \frac{\partial \bar{\Sigma}_z}{\partial z} - (n-1)\bar{\Sigma}_z \right) dz. \quad (7.57)$$

Note that this distribution function is not in the form of a divergence of a vector field, and thus is not normalized to zero.

Coadjoint Flow on $\mathfrak{X}_{\text{st-con-ham}}^*(\bar{M})$. Let, as above, $\mathfrak{X}_{\text{st-con-ham}}(\bar{M})$ be the Lie algebra of contact vector fields with the opposite Jacobi-Lie bracket. That is,

$$ad_{Y_{\bar{H}}} Y_{\bar{F}} = -[Y_{\bar{H}}, Y_{\bar{F}}], \quad (7.58)$$

which we consider to be the left adjoint action of $\mathfrak{X}_{\text{st-con-ham}}(\bar{M})$ on itself. Now dualizing the adjoint action, we arrive at the coadjoint action of $\mathfrak{X}_{\text{st-con-ham}}(\bar{M})$ on its dual $\mathfrak{X}_{\text{st-con-ham}}^*(\bar{M})$ as

$$\begin{aligned} ad^* : \mathfrak{X}_{\text{st-con-ham}}(\bar{M}) \times \mathfrak{X}_{\text{st-con-ham}}^*(\bar{M}) &\longrightarrow \mathfrak{X}_{\text{st-con-ham}}^*(\bar{M}), \\ \langle ad_{\xi_{\bar{H}}}^* \bar{\Sigma}, \xi_{\bar{F}} \rangle &= \langle \bar{\Sigma}, ad_{\xi_{\bar{H}}} \xi_{\bar{F}} \rangle. \end{aligned} \quad (7.59)$$

More explicitly, given an arbitrary field $Y_{\bar{F}}$ we have

$$\langle ad_{Y_{\bar{H}}}^* \bar{\Sigma}, Y_{\bar{F}} \rangle = \langle \bar{\Sigma}, ad_{\xi_{\bar{H}}} \xi_{\bar{F}} \rangle = - \int \langle \bar{\Sigma}, [Y_{\bar{H}}, Y_{\bar{F}}] \rangle d\bar{\mu} = \int \langle \mathcal{L}_{Y_{\bar{H}}} \bar{\Sigma}, Y_{\bar{F}} \rangle d\bar{\mu}. \quad (7.60)$$

So we arrive at the coadjoint action as

$$ad_{Y_{\bar{H}}}^* \bar{\Sigma} = \mathcal{L}_{Y_{\bar{H}}} \bar{\Sigma}. \quad (7.61)$$

The dual space $\mathfrak{X}_{\text{st-con-ham}}^*(\bar{M})$ has the Lie-Poisson bracket

$$\{A, B\}_{\mathfrak{X}_{\text{st-con-ham}}^*}(\bar{\Sigma}) = - \int \left\langle \bar{\Sigma}, \left[\frac{\delta A}{\delta \bar{\Sigma}}, \frac{\delta B}{\delta \bar{\Sigma}} \right] \right\rangle d\bar{\mu} \quad (7.62)$$

and for a given Hamiltonian functional $\mathcal{H}$, the Lie-Poisson dynamics is

$$\dot{\bar{\Sigma}} = \{\bar{\Sigma}, \mathcal{H}\}_{\mathfrak{X}_{\text{st-con-ham}}^*} = -ad_{\delta \mathcal{H} / \delta \bar{\Sigma}}^* \bar{\Sigma} = -\mathcal{L}_{Y_{\bar{H}}} \bar{\Sigma}, \quad (7.63)$$

where we chose $\mathcal{H}(\bar{\Sigma}) = \int \langle \bar{\Sigma}, Y_{\bar{H}} \rangle d\bar{\mu}$. This system governs the dynamics of one-form sections, which are elements of the dual space $\mathfrak{X}_{\text{st-con-ham}}^*(\bar{M})$. We refer to (7.63) as the strict contact momentum-Vlasov equation. Notice that if the contact Hamiltonian vector field in the contact momentum-Vlasov equation (7.51) does not depend on the action variable z , then (7.51) reduces to the strict contact momentum-Vlasov equation (7.63). In the following subsection, we will compute the kinetic equations for both contact and strict contact cases in terms of the density variable.

7.4. Contact Kinetic Dynamics in Density Formulation

Given a Hamiltonian function $\bar{H}$ on the contact manifold $\bar{M}$, one may define two particle motions on the manifold. One is due to the contact bracket given by $\dot{a} = \{a, \bar{H}\}$, and the other is due to the contact vector field $\xi_{\bar{H}}$ given by $\dot{a} = \xi_{\bar{H}}(a)$. In the symplectic framework, these two definitions coincide but not for the contact geometry. So we treat these two situations one by one. Let us start with the kinetic lift of the contact bracket motion.

Kinetic Lift of Contact Bracket Dynamics. In view of the contact bracket (7.19) of smooth functions, let now

$$ad_{\bar{H}}\bar{K} = \{\bar{H}, \bar{K}\}^{(C)} \quad (7.64)$$

be the adjoint action of $\mathcal{F}(\bar{M})$ on itself. As was noted above, we shall make use of the identification $\mathcal{F}^*(\bar{M}) \simeq \mathcal{F}(\bar{M})$ with the dual space. This way, the coadjoint action $\mathcal{F}(\bar{M})$ on $\mathcal{F}^*(\bar{M})$ is computed from

$$\begin{aligned} \int \{\bar{F}, \bar{H}\}^{(C)} \bar{K} \bar{d}\mu &= \int (\xi_{\bar{H}}(\bar{F}) + \bar{F}\mathcal{R}(\bar{H})) \bar{K} \bar{d}\mu \\ &= \int \bar{K} (\iota_{\xi_{\bar{H}}} d\bar{F}) \bar{d}\mu + \int \bar{K} \bar{F} \mathcal{R}(\bar{H}) \bar{d}\mu \\ &= \int \bar{K} d\bar{F} \wedge \iota_{\xi_{\bar{H}}} \bar{d}\mu + \int \bar{K} \bar{F} \mathcal{R}(\bar{H}) \bar{d}\mu \\ &= - \int \bar{F} d\bar{K} \wedge \iota_{\xi_{\bar{H}}} \bar{d}\mu - \int \bar{F} \bar{K} d\iota_{\xi_{\bar{H}}} \bar{d}\mu + \int \bar{K} \bar{F} \mathcal{R}(\bar{H}) \bar{d}\mu \\ &= - \int \bar{F} (\iota_{\xi_{\bar{H}}} d\bar{K}) \bar{d}\mu - \int \bar{F} \bar{K} \operatorname{div}(\xi_{\bar{H}}) \bar{d}\mu + \int \bar{K} \bar{F} \mathcal{R}(\bar{H}) \bar{d}\mu \\ &= - \int \bar{F} \xi_{\bar{H}}(\bar{K}) \bar{d}\mu + \int \bar{F} \bar{K} (n+1) \mathcal{R}(\bar{H}) \bar{d}\mu + \int \bar{K} \bar{F} \mathcal{R}(\bar{H}) \bar{d}\mu \\ &= - \int \bar{F} \xi_{\bar{H}}(\bar{K}) \bar{d}\mu + \int \bar{F} \bar{K} (n+1) \mathcal{R}(\bar{H}) \bar{d}\mu + \int \bar{K} \bar{F} \mathcal{R}(\bar{H}) \bar{d}\mu \\ &= - \int \bar{F} (\{\bar{K}, \bar{H}\}^{(C)} - \bar{K} \mathcal{R}(\bar{H})) \bar{d}\mu + (n+2) \int \bar{F} \bar{K} \mathcal{R}(\bar{H}) \bar{d}\mu \\ &= \int \bar{F} \{\bar{H}, \bar{K}\}^{(C)} \bar{d}\mu + (n+3) \int \bar{F} \bar{K} \mathcal{R}(\bar{H}) \bar{d}\mu, \end{aligned} \quad (7.65)$$

that is,

$$\int \{\bar{F}, \bar{H}\}^{(C)} \bar{K} \bar{d}\mu = \int \bar{F} \{\bar{H}, \bar{K}\}^{(C)} \bar{d}\mu + (n+3) \int \bar{F} \bar{K} \mathcal{R}(\bar{H}) \bar{d}\mu \quad (7.66)$$

for all smooth functions $\bar{F}$, $\bar{H}$, and $\bar{K}$ defined on the contact manifold $\bar{M}$. Accordingly, the coadjoint action appears as

$$ad_{\bar{H}}^* \bar{f} = -\{\bar{H}, \bar{f}\}^{(C)} - (n+3)\bar{f}\mathcal{R}(\bar{H}). \quad (7.67)$$

As discussed in the previous subsection, the dynamics on the density level is determined through the coadjoint action. In particular, for the Hamiltonian functional

$$\mathcal{H}(\bar{f}) = \int \bar{H} \bar{f} d\mu \quad (7.68)$$

on $\mathcal{F}^* \simeq \mathcal{F}$, where $\bar{H}$ is the Hamiltonian function defined on the extended cotangent bundle, the Fréchet derivative $\delta\mathcal{H}/\delta\bar{f}$ becomes $\bar{H}$. In this case, the coadjoint flow may be computed to be

$$\dot{\bar{f}} = -ad_{\delta\mathcal{H}/\delta\bar{f}}^* \bar{f} = -ad_{\bar{H}}^* \bar{f}. \quad (7.69)$$

Substituting the action in (7.67) into the coadjoint dynamics, we compute the kinetic equation of contact particles as

$$\dot{\bar{f}} - \{\bar{H}, \bar{f}\}^{(C)} = (n+3)\bar{f}\mathcal{R}(\bar{H}). \quad (7.70)$$

Keeping in mind that the Lie-Poisson dynamics (7.51) in momentum variables and the Lie-Poisson dynamics (7.70) are related with the Poisson mapping $\bar{\Pi} \mapsto \bar{f}$ given in (7.39), the kinetic equation is computed in Darboux coordinates as

$$\frac{\partial \bar{f}}{\partial t} = -\frac{\partial \bar{H}}{\partial p_i} \frac{\partial \bar{f}}{\partial q^i} + \frac{\partial \bar{H}}{\partial q^i} \frac{\partial \bar{f}}{\partial p_i} + p_i \left(\frac{\partial \bar{f}}{\partial p_i} \frac{\partial \bar{H}}{\partial z} - \frac{\partial \bar{H}}{\partial p_i} \frac{\partial \bar{f}}{\partial z} \right) + (n+2)\bar{f} \frac{\partial \bar{H}}{\partial z} + \frac{\partial \bar{f}}{\partial z} \bar{H}. \quad (7.71)$$

Kinetic Lift of Contact V-Field Dynamics. Given a contact vector field $\xi_{\bar{H}}$, let us consider the linear mapping

$$\Psi_{\xi_{\bar{H}}} : \mathcal{F}(M) \longrightarrow \mathcal{F}(M), \quad (\xi_{\bar{H}}, \bar{K}) \mapsto \xi_{\bar{H}}(\bar{K}) \quad (7.72)$$

that takes a function to its directional derivative along $\xi_{\bar{H}}$. A similar calculation to the one presented in (7.65) hence yields

$$\int \xi_{\bar{H}}(\bar{F}) \bar{K} d\bar{\mu} = \int \bar{F} \{\bar{H}, \bar{K}\}^{(C)} d\bar{\mu} + (n+2) \int \bar{F} \bar{K} \mathcal{R}(\bar{H}) d\bar{\mu}. \quad (7.73)$$

Accordingly, the dual of (7.72) is given by

$$\Psi_{\xi_{\bar{H}}}^* : \mathcal{F}^*(M) \longrightarrow \mathcal{F}^*(M), \quad \bar{f} \mapsto \Psi_{\xi_{\bar{H}}}^*(\bar{f}) = \{\bar{H}, \bar{f}\}^{(C)} + (n+2)\bar{f}\mathcal{R}(\bar{H}). \quad (7.74)$$

We then define the dynamics generated by the dual action as

$$\dot{\bar{f}} = \Psi_{\xi_{\bar{H}}}^*(\bar{f}) = \{\bar{H}, \bar{f}\}^{(C)} + (n+2)\bar{f}\mathcal{R}(\bar{H}). \quad (7.75)$$

In terms of the Darboux coordinates (q^i, p_i, z) , the kinetic dynamics turns out to be

$$\frac{\partial \bar{f}}{\partial t} = -\frac{\partial \bar{H}}{\partial p_i} \frac{\partial \bar{f}}{\partial q^i} + \frac{\partial \bar{H}}{\partial q^i} \frac{\partial \bar{f}}{\partial p_i} + p_i \left(\frac{\partial \bar{f}}{\partial p_i} \frac{\partial \bar{H}}{\partial z} - \frac{\partial \bar{H}}{\partial p_i} \frac{\partial \bar{f}}{\partial z} \right) + (n+1)\bar{f} \frac{\partial \bar{H}}{\partial z} + \frac{\partial \bar{f}}{\partial z} \bar{H}. \quad (7.76)$$

A Direct Calculation to Kinetic Dynamics. Instead of geometric constructions, we may use a simplified method of derivation. Evolution of an observable function $a = a(q^i, p_i, z)$ along the vector field reads

$$\frac{da}{dt} = \xi_{\bar{H}}(a) = \frac{\partial \bar{H}}{\partial p_i} \frac{\partial a}{\partial q^i} - \left(\frac{\partial \bar{H}}{\partial q^i} + p_i \frac{\partial \bar{H}}{\partial z} \right) \frac{\partial a}{\partial p_i} + \left(-\bar{H} + p_i \frac{\partial \bar{H}}{\partial p_i} \right) \frac{\partial a}{\partial z}. \quad (7.77)$$

In order to construct the kinetic theory, we need to introduce the distribution function $\bar{f} = \bar{f}(q^i, p_i, z)$, which makes it possible to define the averaged functional

$$\bar{A}(\bar{f}) = \int a(q^i, p_i, z) \bar{f}(q^i, p_i, z) \overline{d\mu}. \quad (7.78)$$

Evolution of this functional is on the one hand given by

$$\frac{d\bar{A}}{dt} = \int \frac{da}{dt} \bar{f}(q^i, p_i, z) \overline{d\mu}, \quad (7.79)$$

while on the other hand, it can also be seen as an evolution of the distribution function itself,

$$\frac{d\bar{A}}{dt} = \int a \frac{\partial \bar{f}(q^i, p_i, z)}{\partial t} \overline{d\mu}. \quad (7.80)$$

Rewriting the former expression in the form of the latter (integrating by parts while dropping the boundary terms), we obtain (7.76).

Dynamics of Densities for Strict Contact Dynamics. Let us recall the Poisson mapping in (7.55). This turns the contact kinetic dynamics in (7.70) and in (7.75) to the Vlasov equation

$$\dot{\bar{f}} + \{\bar{H}, \bar{f}\}^{(S)} = 0 \quad (7.81)$$

where the bracket is the canonical bracket. The dynamics on the Lie algebra dual of quantomorphisms can be thus seen as the standard dynamics of the distribution function on the phase space of particles.

7.5. A Hierarchy from Contact to Conformal Dynamics

We have so far provided the generalizations of the Vlasov dynamics for conformal and contact settings on a pure geometrical setting. To sum up and relate the dynamical equations we obtained, we shall present in the present section the hierarchy of the relevant Lie algebras (by means of Lie algebra homomorphisms) of both the function spaces and the vector fields. We shall then dualize the Lie algebra homomorphisms to arrive at the momentum and Poisson mappings between different levels of descriptions, namely the reversible Hamiltonian dynamics, the conformal Hamiltonian dynamics, and the contact Hamiltonian dynamics. For the level of particle dynamics the relationship between the conformal and the contact Hamiltonian dynamics discussed in, for example, [89, 90].

Lie Algebra Hierarchy. In order to begin with the contact geometry let us first consider the extended cotangent bundle $T^*M \times \mathbb{R}$ (as a contact manifold), along with a contact Hamiltonian function

$$\bar{H}(q^i, p_i, z) = H(q^i, p_i) - cz \quad (7.82)$$

on it. Then, the contact Hamiltonian dynamics (7.28) takes the particular form

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} + cp_i, \quad \frac{dz}{dt} = p_i \frac{\partial H}{\partial p_i} - H(q^i, p_i) + cz. \quad (7.83)$$

The first two equations of (7.83) can be projected to the cotangent bundle T^*M , which gives a reduction to the conformal Hamiltonian dynamics (6.11). In order to conduct further analysis, we consider two functions

$$\bar{F}(q^i, p_i, z) = F(q^i, p_i) - c_F z, \quad \bar{H}(q^i, p_i, z) = H(q^i, p_i) - c_H z, \quad (7.84)$$

and compute their contact bracket (7.19) as

$$\begin{aligned} \{\bar{F}, \bar{H}\}^{(C)} &= \{F - c_F z, H - c_H z\}^{(C)} \\ &= \{F, H\}^{(C)} - c_H \{F, z\}^{(C)} - c_F \{z, H\}^{(C)} + c_F c_H \{z, z\}^{(C)} \\ &= \{F, H\}^{(S)} - c_H \{F, z\}^{(C)} - c_F \{z, H\}^{(C)} \\ &= \{F, H\}^{(S)} - c_H (F + Z(F)) + c_F (H + Z(H)), \end{aligned} \quad (7.85)$$

where the contact bracket reduces to (the pullback of) the canonical Poisson bracket in the third line. A direct comparison of (7.85) with (6.19) reveals that they look very similar. Accordingly, the choice of the Hamiltonian function (7.82) motivates us to determine the Lie algebra homomorphism (more precisely, an embedding)

$$\Xi : \mathcal{F}(T^*M) \times \mathbb{R} \longrightarrow \mathcal{F}(T^*M \times \mathbb{R}), \quad (H, c_H) \mapsto \bar{H}(q^i, p_i, z) = H(q^i, p_i) - c_H z, \quad (7.86)$$

endowing $\mathcal{F}(T^*M) \times \mathbb{R}$ with the Lie algebra bracket in (6.19), and $\mathcal{F}(T^*M \times \mathbb{R})$ with the contact bracket in (7.19).

It is possible to carry this Lie algebra homomorphism to the level of vector fields. To this end, we employ the isomorphisms (7.26) and (6.18) on the domain and the range of (7.86) in a respected manner to arrive at the Lie algebra homomorphism

$$\Upsilon : \mathfrak{X}_{\text{ham}}(T^*M) \times \mathbb{R} \longrightarrow \mathfrak{X}_{\text{con-ham}}(T^*M \times \mathbb{R}), \quad X_H^c \mapsto \xi_{\bar{H}} \quad (7.87)$$

where $\bar{H}$ is the contact Hamiltonian function in (7.82). Finally, in view of the canonical inclusions of the Lie algebras $\mathfrak{X}_{\text{ham}}(T^*M)$ and $\mathcal{F}(T^*M)$ into their extensions $\mathfrak{X}_{\text{ham}}(T^*M) \times \mathbb{R}$ and $\mathcal{F}(T^*M) \times \mathbb{R}$, we present the following commutative diagram.

$$\begin{array}{ccccc} \mathfrak{X}_{\text{ham}}(T^*M) & \hookrightarrow & \mathfrak{X}_{\text{ham}}(T^*M) \times \mathbb{R} & \xrightarrow{\Upsilon \text{ in (7.87)}} & \mathfrak{X}_{\text{con-ham}}(T^*M \times \mathbb{R}) \\ \uparrow \Phi \text{ in (6.26)} & & \uparrow \Phi^c \text{ in (6.18)} & & \uparrow \Psi \text{ in (7.26)} \\ \mathcal{F}(T^*M) & \hookrightarrow & \mathcal{F}(T^*M) \times \mathbb{R} & \xrightarrow{\Xi \text{ in (7.86)}} & \mathcal{F}(T^*M \times \mathbb{R}) \end{array} \quad (7.88)$$

Poisson Maps Hierarchy. Let us next dualize the Lie algebra homomorphisms that appear in the above diagram. To this end, we consider a density function $\bar{f} = \bar{f}(q^i, p_i, z)$ in the dual space $\mathcal{F}^*(T^*M \times \mathbb{R})$ and examine the mapping Ξ in (7.86). We thus obtain the dual mapping

$$\begin{aligned} \Xi^* : \mathcal{F}^*(T^*M \times \mathbb{R}) &\longrightarrow \mathcal{F}^*(T^*M) \times \mathbb{R}^*, \\ \bar{f}(q^i, p_i, z) &\mapsto \left(\int_{\mathbb{R}} \bar{f}(q^i, p_i, z) dz, \int_{T^*M \times \mathbb{R}} z \bar{f}(q^i, p_i, z) d\bar{\mu} \right). \end{aligned} \quad (7.89)$$

Let us remark that the first term on the range is indeed in $\mathcal{F}^*(T^*M)$, while the

second one is a real number in $\mathbb{R}^* \simeq \mathbb{R}$. More precisely,

$$f(q^i, p_i) := \int_{\mathbb{R}} \bar{f}(q^i, p_i, z) dz, \quad c^* := \int_{T^*M \times \mathbb{R}} z \bar{f}(q^i, p_i, z) d\bar{\mu}. \quad (7.90)$$

Let us note also that (7.89) being a dual of a Lie algebra homomorphism, the moments (7.90) constitute a Poisson map. Therefore, we can argue that the moments in (7.90) map the coadjoint flow (7.70) on the contact level to the coadjoint flow (6.35) on the conformal Hamiltonian geometry. In terms of one-forms, given $\bar{\Pi} = \bar{\Pi}_i dq^i + \bar{\Pi}^i dp_i + \bar{\Pi}_z dz$ we have the projection

$$\Pi_i(q^i, p_i) = \int_{\mathbb{R}} \bar{\Pi}_i(q^i, p_i, z) dz, \quad \Pi^i(q^i, p_i) = \int_{\mathbb{R}} \bar{\Pi}^i(q^i, p_i, z) dz, \quad \bar{\Pi}_z = 0. \quad (7.91)$$

These maps take the kinetic dynamics in (7.51) to the kinetic dynamics in (6.25). All these dynamics and projections may now be summarized through the following commutative diagram.

$$\begin{array}{ccccc}
 & & \textit{Conformal} & & \textit{Contact} \\
 M - \textit{Vlasov} & \xleftarrow{c=0} & M - \textit{Vlasov} & \xleftarrow{(7.91)} & M - \textit{Vlasov} \\
 (6.29) & & (6.25) & & (7.51) \\
 \downarrow \Phi^* \text{ in (6.27)} & & \downarrow \Phi^{c*} \text{ in (6.21)} & & \downarrow \Psi^* \text{ in (7.39)} \\
 \textit{Vlasov} & \xleftarrow{c=0} & \textit{Vlasov} & \xleftarrow{(7.90)} & \textit{Vlasov} \\
 (6.38) & & (6.35) & & (7.76)
 \end{array} \quad (7.92)$$

8. CONCLUSIONS

In this thesis, we introduce locally conformally symplectic, conformal, and contact manifolds. Additionally, we explore discrete dynamics on locally conformally symplectic manifolds and present the kinetic theory of locally conformal dynamics, as well as conformal and contact dynamics.

In the second chapter, we provide the fundamental definitions of symplectic and Poisson manifolds, along with Hamiltonian and Lagrangian dynamics. Furthermore, we introduce Lie algebras and Lie-Poisson dynamics. We also present Jacobi manifolds, as well as the complete lift and jet decompositions of a vector field.

In the third chapter, we introduce locally conformally symplectic manifolds and locally conformal Hamiltonian dynamics. We provide a locally conformally symplectic manifold as an example of a Jacobi manifold. Furthermore, we derive the locally conformal Euler-Lagrange (LCEL) equations.

We also analyze locally conformal systems from the perspective of discrete dynamics in the fourth chapter. We examine discrete dynamics in the context of locally conformal Lagrangian and Hamiltonian systems; see [63]. In this setting, we present discrete locally conformal Euler-Lagrange and Hamiltonian equations. Subsequently, we extend this approach to second- and third-order cases, obtaining Lagrangian dynamics for higher orders.

In the fifth chapter, we develop the kinetic theory of locally conformal dynamics. We derive the locally conformal momentum-Vlasov equation and the locally conformal Vlasov equation, see details in [91].

In the sixth chapter, we introduce conformal Hamiltonian dynamics and the Lie algebra of conformal vector fields. Additionally, we present the formulation of conformal kinetic dynamics.

We explore contact manifolds and contact dynamics in the seventh chapter. Furthermore, we derive the Vlasov dynamics in the contact setting, extending the kinetic theory to this geometric framework. For more information on conformal and contact settings, see [92].

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BIOGRAPHY

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PUBLICATIONS WITHIN the SCOPE of THESIS STUDY

Here are the publications done during the PhD Thesis:

[1] Esen O., Gezici A., Gümral H., (2024), “Variational aspect and kinetic theory of locally conformal dynamics”, J. Phys. A, 57(36), Paper No. 365201, 37.

[2] Esen O., Gezici A., Gümral H., (2024), “Discrete Dynamics on Locally Conformal Framework”, Proceedings of Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, 50(1), 133-151.

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APPENDICES

APPENDIX A: DOUBLE CROSS SUM LIE ALGEBRAS

The present section of the appendix contains the construction of a Lie algebra out of two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, admitting mutual actions

$$\triangleright : \mathfrak{h} \otimes \mathfrak{g} \rightarrow \mathfrak{g}, \quad \eta \otimes \xi \mapsto \eta \triangleright \xi, \quad \triangleleft : \mathfrak{h} \otimes \mathfrak{g} \rightarrow \mathfrak{h}, \quad \eta \otimes \xi = \eta \triangleleft \xi, \quad (\text{A.1})$$

which are assumed to satisfy

$$\begin{aligned} \eta \triangleright [\xi_1, \xi_2] &= [\eta \triangleright \xi_1, \xi_2] + [\xi_1, \eta \triangleright \xi_2] + (\eta \triangleleft \xi_1) \triangleright \xi_2 - (\eta \triangleleft \xi_2) \triangleright \xi_1, \\ [\eta_1, \eta_2] \triangleleft \xi &= [\eta_1, \eta_2 \triangleleft \xi] + [\eta_1 \triangleleft \xi, \eta_2] + \eta_1 \triangleleft (\eta_2 \triangleright \xi) - \eta_2 \triangleleft (\eta_1 \triangleright \xi). \end{aligned} \quad (\text{A.2})$$

Such a pair $(\mathfrak{g}, \mathfrak{h})$ of Lie algebras is called a *matched pair* of Lie algebras, see for instance [93], see also [94]. Then the vector space direct sum $\mathfrak{g} \oplus \mathfrak{h}$ happens to be a Lie algebra along with the bracket given by $\mathfrak{g} \oplus \mathfrak{h}$ as

$$[(\xi_1, \eta_1), (\xi_2, \eta_2)] = ([\xi_1, \xi_2] + \eta_1 \triangleright \xi_2 - \eta_2 \triangleright \xi_1, [\eta_1, \eta_2] + \eta_1 \triangleleft \xi_2 - \eta_2 \triangleleft \xi_1) \quad (\text{A.3})$$

for any $(\xi_1, \eta_1), (\xi_2, \eta_2) \in \mathfrak{g} \oplus \mathfrak{h}$. The Lie algebra $\mathfrak{g} \bowtie \mathfrak{h} := \mathfrak{g} \oplus \mathfrak{h}$ is called the *double cross sum* of the pair $(\mathfrak{g}, \mathfrak{h})$.

Double cross sum construction extends the semi-direct sum construction. Indeed, choosing for instance the right action in (A.3) to be trivial, that is $\eta \triangleleft \xi = 0$ for all ξ in $\mathfrak{g}$ and η in $\mathfrak{h}$, and letting the Lie algebra $\mathfrak{h}$ to be trivial, that is $[\eta_1, \eta_2] = 0$ for all η_1 and η_2 in $\mathfrak{h}$, we see at once that the bracket (A.3) reduces to

$$[(\xi_1, \eta_1), (\xi_2, \eta_2)] = ([\xi_1, \xi_2] + \eta_1 \triangleright \xi_2 - \eta_2 \triangleright \xi_1, 0). \quad (\text{A.4})$$

Physically the double cross sum construction corresponds to the collective motion of two dynamical systems. The Lagrangian (Euler-Poincaré) dynamics on double cross sum Lie algebras were studied in [95–97], and it is followed by the analysis of the Hamiltonian dynamics on the dual spaces of double cross sum Lie algebras, [98]. The case of discrete dynamics, on the other hand, has been treated in [99]. The present work, along these lines, concerns the coadjoint flow on the dual space of a double cross sum Lie algebra.

In order to formulate the coadjoint action of a double cross sum Lie algebra on its dual space, we shall now fix a number of notations.

Fixing the algebra element $\xi \in \mathfrak{g}$ in the left action we define the linear operation

$$\mathfrak{b}_\xi : \mathfrak{h} \rightarrow \mathfrak{g}, \quad \eta \mapsto \eta \triangleright \xi. \quad (\text{A.5})$$

Then, for an arbitrary $\zeta \in \mathfrak{g}^*$, the dual of the mapping $\mathfrak{b}_\xi$ is obtained to be

$$\mathfrak{b}_\xi^* : \mathfrak{g}^* \rightarrow \mathfrak{h}^*, \quad \langle \mathfrak{b}_\xi^* \zeta, \eta \rangle = \langle \zeta, \mathfrak{b}_\xi \eta \rangle. \quad (\text{A.6})$$

Next, we dualize the left action of $\mathfrak{h}$ on $\mathfrak{g}$ into a dual right action of $\mathfrak{h}$ on $\mathfrak{g}^*$, which may be given as

$$\triangleleft^* \eta : \mathfrak{g}^* \rightarrow \mathfrak{g}^*, \quad \zeta \rightarrow \zeta \triangleleft^* \eta, \quad \langle \zeta \triangleleft^* \eta, \xi \rangle = \langle \zeta, \eta \triangleright \xi \rangle. \quad (\text{A.7})$$

The linear algebraic dual of the adjoint action gives the coadjoint action. The dual of (A.4) is then given by

$$\begin{aligned} \langle ad_{(\xi_1, \eta_1)}^*(\zeta, \nu), (\xi_2, \eta_2) \rangle &= \langle (\zeta, \nu), ad_{(\xi_1, \eta_1)}(\xi_2, \eta_2) \rangle \\ &= \langle (\zeta, \nu), ([\xi_1, \xi_2] + \eta_1 \triangleright \xi_2 - \eta_2 \triangleright \xi_1, 0) \rangle \\ &= \langle \zeta, [\xi_1, \xi_2] + \eta_1 \triangleright \xi_2 - \eta_2 \triangleright \xi_1 \rangle \\ &= \langle ad_{\xi_1}^* \zeta, \xi_2 \rangle + \langle \zeta \triangleleft^* \eta_1, \xi_2 \rangle - \langle \mathfrak{b}_{\xi_1}^* \zeta, \eta_2 \rangle. \end{aligned} \quad (\text{A.8})$$

To sum up, for given any $(\xi, \eta) \in \mathfrak{g} \oplus \mathfrak{h}$ and any $(\zeta, \nu) \in \mathfrak{g}^* \oplus \mathfrak{h}^*$, the coadjoint action may be formulated as

$$ad_{(\xi, \eta)}^*(\zeta, \nu) = (ad_{\xi}^* \zeta + \zeta \triangleleft^* \eta, -\mathfrak{b}_{\xi}^* \zeta). \quad (\text{A.9})$$

Finally, given a Hamiltonian functional $H = H(\zeta, \nu)$ on the direct sum $\mathfrak{g}^* \oplus \mathfrak{h}^*$, the Lie-Poisson equations (1.8) turn out to be

$$\begin{aligned} \dot{\zeta} &= -ad_{\frac{\delta H}{\delta \zeta}}^* \zeta - \zeta \triangleleft^* \frac{\delta H}{\delta \nu}, \\ \dot{\nu} &= \mathfrak{b}_{\frac{\delta H}{\delta \zeta}}^* \zeta, \end{aligned} \quad (\text{A.10})$$

which is the abstract evolution equation for ζ and ν .