

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

**DOMINATION ON NON-CAYLEY VERTEX TRANSITIVE
GRAPHS**

HALİME İLHAN
A THESIS SUBMITTED FOR THE DEGREE OF
MASTER OF SCIENCE
DEPARTMENT OF MATHEMATICS

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PROF. DR. SİBEL ÖZKAN

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T.C.
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FEN BİLİMLERİ ENSTİTÜSÜ

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SUMMARY

A dominating set in a graph G is a vertex subset S such that every vertex not in S has a neighbor in S , and domination number is the minimum cardinality of S . A total dominating set in a graph G is a vertex subset S such that every vertex in V is adjacent to some vertex in S and total domination number is the minimum cardinality of S . Every Cayley graph $\text{Cay}(G, A)$ is vertex-transitive but the converse is not true. A well-known example is the Petersen graph. In this thesis we will focus on non-Cayley vertex transitive graphs. Let n and k be two positive integers, the Kneser graph $K(n, k)$ is the graph whose vertices represent the k -subsets of n -element sets and where two vertices are adjacent if and only if they correspond to disjoint subsets. Johnson graph $J(n, k)$ has same vertex set with the Kneser graph and two vertices are adjacent if and only if their intersection has $k - 1$ elements. For $k = 2$, The Johnson graph $J(n, 2)$ is the complement of the Kneser graph $K(n, 2)$. In this thesis we will study dominating sets and total dominating sets on some specific non-Cayley vertex-transitive graphs such as Odd Kneser graphs and Johnson graphs. Then we will focus on finding the domination numbers or boundaries for the Johnson graph.

Keywords: Domination, Total Domination, Dominating Set, Vertex Transitive Graphs, Kneser graph, Johnson graph.

ÖZET

G çizgesinde bir dominasyon kümesi, S 'de olmayan her köşenin S 'de komşusunun olduğu köşeler alt kümesidir ve S dominasyon sayısı S 'nin sahip olabileceği minimum kardinalitedir. G çizgesinde bir tam dominasyon kümesi V 'deki her köşenin S 'de komşusunun olduğu S köşeler alt kümesidir ve tam dominasyon sayısı S 'nin sahip olabileceği minimum kardinalitedir. Her Cayley çizgesi $Cay(G, A)$ köşe geçişlidir ama tersi doğru değildir. En iyi bilinen örneği Petersen çizgesidir. Bu tezde Cayley olmayan köşe geçişli çizgeler üzerinde çalışılacaktır. n ve k iki pozitif tam sayı olsun. Kneser çizgesi $K(n, k)$, köşeleri n -elemanlı kümenin k - elemanlı alt kümelerinden oluşan ve iki köşenin ancak ve ancak ayrık kümelerle karşılık gelmesi durumunda komşu olduğu çizgedir. Johnson çizgesi $J(n, k)$, Kneser çizgesiyle aynı köşelere sahiptir ve iki köşe ancak ve ancak kesişimleri $k - 1$ elemanlı ise komşudur. $k = 2$ için Johnson çizgesi $J(n, 2)$, Kneser çizgesi $K(n, 2)$ 'nin tümleyenidir. Bu tezde Tek Kneser çizgeleri ve Johnson çizgeleri gibi bazı Cayley olmayan köşe geçişli çizgeler üzerinde dominasyon ve tam dominasyon kümelerini inceleyeceğiz. Ayrıca Johnson çizgesinin dominasyon sayılarını veya iligi sınırları bulmaya odaklanacağız.

Anahtar Kelimeler: Dominasyon, Toplam Dominasyon, Dominasyon kümesi, Köşe Geçişli Çizgeler, Kneser Çizgesi, Johnson Çizgesi.

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LIST of ABBREVIATIONS and ACRONYMS

<u>Abbreviations</u>	<u>Explanations</u>
<u>and Acronyms</u>	
$V(G)$	: Vertex set of a graph G
$E(G)$	: Edge set of a graph G
$(V(G), E(G))$	: Representation of a graph G with the vertex set $V(G)$ and edge set $E(G)$
$d(u, v)$	: Distance between vertices of u and v
$diam(G)$	: Diameter of a graph G
$N(v)$	: Open neighborhood of vertex v
$N[v]$	: Closed neighborhood of a vertex v
$N(S)$	: Open neighborhood of the vertices in vertex set S
$N[S]$	: Closed neighborhood of the vertices in vertex set S
$d_G(v)$	: Degree of a vertex v in a graph G
$\delta(G)$	: Minimum degree of the vertices of a graph G
$\Delta(G)$	: Maximum degree of the vertices of a graph G
$n(G)$	: Order of a graph G
$m(G)$	: Size of a graph G
K_n	: Complete graph with n vertices
$K_{a,b}$	: Complete a-bipartite graph with b vertices in each part
$pn_s(u)$	: Private neighbor of a vertex u
$Cay(G, A)$	: Cayley graph of a group G and a subset A of G
NC	: Set of all non-Cayley numbers
$K(n, k)$	: Kneser graph of a graph G with n -elements and k -subsets
O_{k+1}	: Odd graph $K(2k + 1, k)$
$C(n, t, k)$	: The minimum size of (n, t, k) covering design
$\gamma(G)$	: Domination number of a graph G
$\gamma_t(G)$	: Total domination number of a graph G
$J(n, k)$	: Johnson graph with n -elements and k -subsets

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1. INTRODUCTION

A graph is a structure including a set of vertices and a set of edges where each edge is associated with two vertices. These two vertices do not have to be distinct. An object can be represented by a vertex and a relationship between two objects can be represented by an edge. So, a graph may be used to model problems involving relationships between the pairs of objects. Let us first start with the origins of graph theory. Today, Königsberg is known as Kaliningrad in the Russian Federation and is the capital of the German state of East Prussia. In this city, the Old and New Pregel rivers merge to form the Pregolian River. These rivers divide the city into four regions and the seven bridges built on the river are connected to each other. In Figure 1 [Newman, 1960], the Pregel River forms four pieces of land: island A, B and C on the riverbank, and land D between the two branches. The Königsberg's Seven Bridges Problem can be stated as: Can a city tour start from any point in the city, pass each bridge only once and return to the starting point? Although most of the people of Königsberg thought that there was no such solution, they could not prove it until Euler proved that such a journey was not possible. In order to simplify the solution and simplify the figure, a graph is drawn in which the land pieces are represented by vertices and the bridges are represented by edges (Figure 1.2).

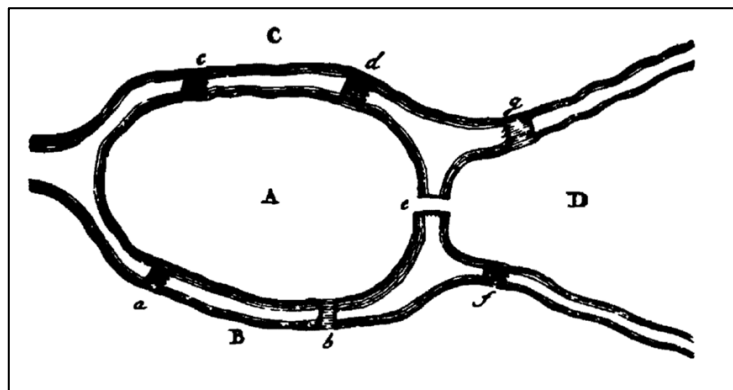


Figure 1.1: Königsberg's seven bridges problem.

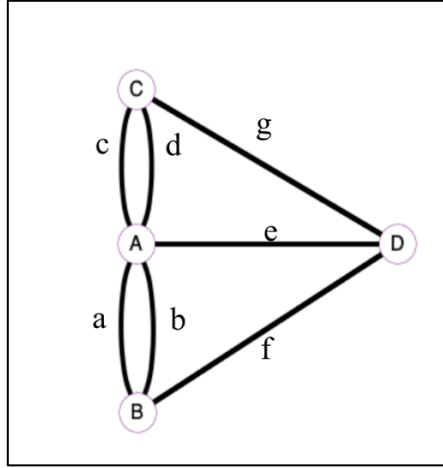


Figure 1.2: Representation of Figure 1.1 as a graph.

Graph theory is an area of mathematics related to mostly pairwise relationships – represented by vertices and connecting edges, and it has applications in many other disciplines such as Chemistry, Social Sciences, Computer Science and Engineering. One of the topics with vast applications in computer science and even in military is the domination concept. There are many domination types emerged in years and there are more than 2000 papers on the concept so far. Also, one of the important graph classes involve vertex-transitive graphs. Due to their relations with group theory they are important class in theoretical studies, and due to their perfect symmetry, they are important in computer science applications. Cayley graphs is a big subclass of vertex-transitive graphs and it is well studied in many aspects, here we will turn our attention to non-Cayley vertex transitive graphs.

In this thesis, we will study domination and total domination number of non-Cayley vertex-transitive graphs. First, we will study domination and total domination number on a well-studied special graph class: Kneser graphs. All Kneser graphs and Johnson graphs are vertex-transitive. Every Cayley graph is vertex transitive but the converse is not true. Odd Kneser graphs are examples of non-Cayley graphs. It is known that Kneser Graph is the complement of Johnson Graph for $k = 2$. Domination results for Kneser graphs when $k = 2$ is well-known, but there is no such study for Johnson graphs when $k = 2$ or bigger. Here we will investigate the domination and total domination number of Johnson graphs when $k = 2$. After giving the basic definitions and notations of graph theory that we frequently use in the thesis, we will briefly talk about the history of domination and give the related theorems and the

related bounds. Then we will give the related results for Kneser graphs and the rest of this thesis will be devoted to proving the results obtained for Johnson graphs.

1.1. Basic Definitions and Notations

In this chapter, we will give the basics of graph theory. We refer reader to [Godsil and Royle, 2001] and [Bondy and Murty, 2008] for more details.

A graph G is a pair of sets (V, E) , where V is a finite set of elements called *vertices* (or *nodes*), and E is a set of distinct vertices called *edges* (or *lines*) where each edge is associated with two vertices (not necessarily distinct). The set V is the *vertex set* and E is the *edge-set* of G , and are often denoted by $V(G)$ and $E(G)$, respectively. A graph is called a *directed graph* (or *digraph*) if all edges are directed from one vertex to another. A graph is called *undirected graph* if all edges are double-sided. A graph is *finite* if both its vertex set and edge set are finite. A graph H is called a *subgraph* of a graph G , if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Each edge is associated with two vertices which are called its *endpoints*. An edge with same endpoints is called a *loop*. A graph is called a *simple graph* if it has no loops and no multiple edges between the same pair of vertices. A *walk* in G is a finite sequence $W = v_0 e_1 v_1 e_2 v_2, \dots, e_k v_k$ whose terms are alternately vertices and edges for $1 \leq i \leq k$. Here W is a (v_0, v_k) -walk. If the vertices $v_0, v_1, v_2, \dots, v_k$ are distinct, W is called a *path*, namely a (v_0, v_k) -path. A graph G is called a *connected graph* if for each pair of vertices u, v of G , there is a (u, v) -path in G . Length of the shortest (u, v) -path in a connected graph G is called *the distance* between vertices u and v , and denoted by $d(u, v)$. The maximum distance between all pairs of vertices of G is called *the diameter* of a graph G and denoted by $diam(G)$. In this thesis, we study only finite, simple, and connected graphs, so the term graph will always be regarded as having these three properties.

If u and v are the endpoints of an edge, then they are called *adjacent*. The *open neighborhood* of a vertex v is the set of vertices adjacent to v and it is denoted by $N(v)$. Namely, $N(v) = \{x \in V \mid vx \in E\}$. The *closed neighborhood* of a vertex v is the set $\{v\} \cup N(v)$ and it is denoted by $N[v]$. Given a set S of vertices, the *open neighborhood* of S is defined to be the union of the neighborhoods of the vertices in S and denoted by $N(S)$. Similarly, the *closed neighborhood* of S is defined to be $S \cup N(S)$ and denoted by $N[S]$. For $S \subseteq V$, a vertex $v \in S$ is called an *isolate* of S if

$N(v) \subseteq V - S$. The number of edges adjacent to a vertex v in a graph G is called the *degree* (or *valency*) of v and it is denoted by $d_G(v)$ or $d(v)$ in short. The *minimum* and *maximum degree* of the vertices of G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. A graph G is called *k-regular* if $d(v) = k$ for all $v \in V$. A vertex with a degree of 0 is called an *isolated vertex*. *Order* of a graph of G is the number of elements in the vertex set of a graph and is denoted by $n(G)$. *Size* of a graph of G is the number of elements in the edge set of a graph and is denoted by $m(G)$. The *complement* $\bar{G}$ of a simple graph G is the simple graph with vertex set V , two vertices being adjacent in $\bar{G}$ if and only if they are not adjacent in G .

A *complete graph* is a simple graph in which each pair of distinct vertices is joined by an edge. The complete graph on n vertices is denoted by K_n . A *bipartite graph* is a graph G one whose vertex set can be partitioned into two subsets X and Y , such that every edge of G has one end in X and the other end in Y . A *complete bipartite graph* is a simple bipartite graph in which each vertex of X is adjacent to each vertex of Y . If $|X| = a$ and $|Y| = b$, such a graph is denoted by $K_{a,b}$.

2. DOMINATION CONCEPT

2.1. Definition and History

Domination is a subject that is widely studied in graph theory and it has connection with other areas, such as computer science and engineering. With the increasing interest in graph theory in recent years, research on this area has also increased. Dominating sets are used in the study of facility location problems such as location of schools, hospitals, fire stations, cargo companies etc. For example, A cargo company would like to open a branch in certain places. Cargo packages delivered to these branches are taken to a meeting center owned by this company and taken out for distribution. A dominating set can be used to create a route where transportation to this gathering center meets certain conditions in terms of time and distance. Dominating sets can also be used to design communication lines and games. For example; we want to ensure that people working in a business use the internet. How many internet providers are needed at minimum so that employees' internet is not interrupted. Dominating and total dominating sets can be used also in military. For instance, let the support group of helicopters which both fight on the field and protect the helicopters in this field. How many helicopters should be used at minimum so that all helicopters can be protected by the support force. In this way, many examples can be given about how dominating and total dominating set is used in mathematical modeling of real-life problems.

Before giving the basic definitions and theorems about domination, let us start briefly with its origins. Mathematical study of domination in graphs begin with Berge who wrote a book on graph theory, in which he mentioned the co-efficient of external stability for the first time [Berge, 1962]. Ore introduced the concepts of dominating set and domination number in his book published in the same year [Ore, 1962]. Approximately 15 years later, Cockayne and Hedetniemi first who used the notation $\gamma(G)$ for domination number of a graph [Cockayne and Hedetniemi, 1977]. After the definition and notation of the domination concept, the concept of domination that have attracted attention and many results have been published.

In general, a *dominating set* in a graph G is a set of vertices S each vertex either in S or is adjacent to a vertex in S . Domination is based on the following chess

problem. Let us talk briefly about this problem. Consider the 8×8 chessboard whose squares can be represented as the vertices of the graph G , as can be seen in Figure 2.1. If the corresponding squares are separated by any number of squares vertically, horizontally or diagonally, there is an edge between the two vertices. Several queens can be placed on squares on chessboard. The question that has originated the domination problem can be stated as: What is the minimum number of queens that can be placed on a chessboard so that each square can be reached by at least one queen? In this problem, we want to find the minimum number of queens that dominate all the squares of the chessboard, namely, the domination number of the graph representing the moves of the queen. One of the solutions is in Figure 2.2.

	✓			✓			✓
		✓		✓		✓	
			✓	✓	✓		
✓	✓	✓	✓	Q	✓	✓	✓
			✓	✓	✓		
		✓		✓		✓	
	✓			✓			✓
✓				✓			

Figure 2.1: Squares that the queen can move (or dominate).

						Q	
		Q					
							Q
			Q				
Q							

Figure 2.2: One solution of minimum dominating set on a chessboard.

Now we can give formal definitions related to these concepts and some examples to understand these definitions better.

Definition 2.1: A dominating set in a graph G is a vertex subset S such that every vertex not in S has a neighbor in S , and the domination number of a graph, written $\gamma(G)$, is the minimum cardinality of S .

Example 2.1:

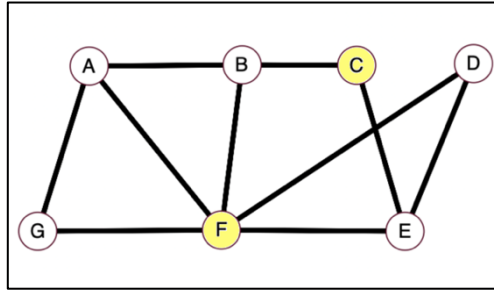


Figure 2.3: A Dominating Set of G .

- $\{C, F\}$ is a minimum dominating set of G and $\gamma(G) = 2$.

In Figure 2.3, $\{B, F\}$ and $\{A, E\}$ are also minimum dominating sets of G . Note that a vertex in a dominating set dominates its open neighborhood and also itself. If we choose a vertex in a graph G , this vertex dominates all its neighbors and itself. So, it can dominate at most $\Delta(G) + 1$ vertices and $\gamma(G) \leq \frac{n}{\Delta(G)+1}$ follows. We will mention from this in Section 2.2.

Definition 2.2: A dominating set S is a minimal dominating set if no proper subset of S is a dominating set.

Note that every minimum dominating set is minimal, but the converse is not true. We give an example of minimum dominating set and minimal dominating set in Figure 2.4 and Figure 2.5, relatively.

Example 2.2:

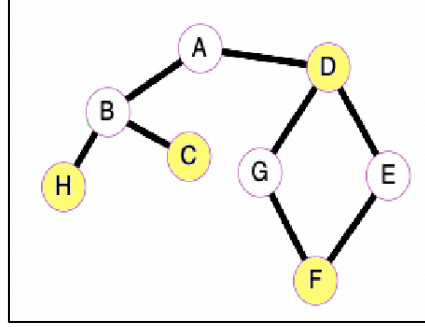


Figure 2.4: A Minimal Dominating Set.

- In Figure 2.4, set $\{C, D, F, H\}$ is a minimal dominating set, but it is not minimum.

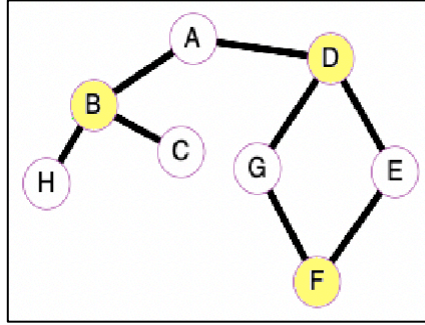


Figure 2.5: A Minimum Dominating Set.

- In Figure 2.5, set $\{B, D, F\}$ is a minimum dominating set and it is also a minimal dominating set.

Definition 2.3: Let $S \subseteq V$ and $u \in S$. Let v be a vertex not in S , then v is a private neighbor of u if u is the only neighbor of v in S . It is denoted by $pn_S(u)$, or $pn(u)$.

By the definition of private neighbor, when S is a dominating set and v is a private neighbor of u , then $S - \{u\}$ is not a dominating set since the vertex v is not dominated by any vertex in S except u . There are some bounds related with the private neighbor. We will give this in Section 2.2.

There are many domination types but among the most studied ones is the total domination. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [Cockayne et al., 1980].

Definition 2.4: A total dominating set in a graph G is a vertex subset S such that every vertex in V is adjacent to some vertex in S and the total domination number of a graph, written $\gamma_t(G)$, is the minimum cardinality of S .

Example 2.3:

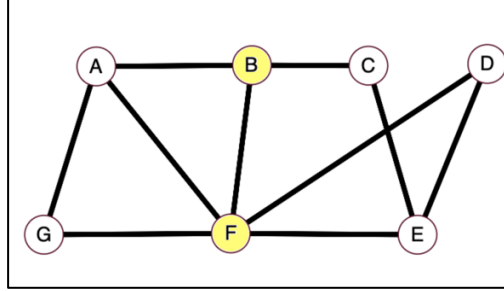


Figure 2.6: A Total dominating set of G .

- $\{B, F\}$ is a minimum total dominating set of G and $\gamma_t(G) = 2$.

In Figure 2.6, $\{E, F\}$ is also a total dominating set of G . This time a vertex can not dominate itself. If we choose a vertex in a graph G , this vertex dominates all its neighbors and. So, it can dominate at most $\Delta(G)$ vertices. So, $\gamma(G) \leq \frac{n}{\Delta(G)}$ follows. We will mention this bound in Section 2.3. It is also clear that every total dominating set is a dominating set, but the converse is not true. We will give a bound that relates domination number to total domination number also in Section 2.3.

2.2. Bounds on Domination

In this section, we will give some bounds on dominating sets in graphs. The first theorems about dominating sets in graphs were given by Ore, in 1962.

Theorem 2.1: [Ore, 1962] A dominating set S is a minimal dominating set if and only if for each vertex $u \in S$, one of the following two conditions holds:

- u is an isolate of S ,
- there exists a vertex $v \in V - S$ for which $N(v) \cap S = \{u\}$.

Proof 2.1: [Haynes et al., 1998] Assume that S is a minimal dominating set of G . Then if we remove a vertex, say u , $S - \{u\}$ cannot dominate G . Namely, some vertex v of $\{V - S\} \cup \{u\}$ is not dominated by any vertex in $S - \{u\}$. If we have $v = u$, then u is an isolate of S . Otherwise, $v \in V - S$. Now, $S - \{u\}$ does not dominate v but S does. It shows that u is the only neighbor of v in S , meaning that $N(v) \cap S = \{u\}$.

Conversely, assume that S is a dominating set for G and one of the above conditions hold for any vertex of S . We will show that S is minimal. Assume on the contrary, that S is not a minimal. Then there exists a vertex in S , say u , such that when we remove u from S , the remaining set $S - \{u\}$ is still a dominating set. Hence, the neighborhood of u and $S - \{u\}$ has a nonempty intersection, but this contradicts with the condition (i). Also, if $S - \{u\}$ is still dominating G , then it means that each v in $V - S$ is adjacent to the vertices in S other than u . And this contradicts with the condition (ii). ■

Theorem 2.2: [Ore, 1962] A dominating set S is minimal if and only if every $u \in S$ that has a neighbor in S has a private neighbor in $V \setminus S$.

Proof 2.2: [Gorodezky, 2007] Assume that S is minimal and a vertex u of S is adjacent to some v of S . If u has no private neighbors in $V \setminus S$ then $S \setminus \{u\}$ is a dominating set because u and each vertex in $V \setminus S - \{u\}$ is adjacent to a vertex in $S \setminus \{u\}$. This contradicts the minimality of S , so u must have a private neighbor.

Conversely, let S be a dominating set and every $u \in S$ that has a neighbor in S has a private neighbor in $V \setminus S$. We will show that S is minimal. Assume that S is not minimal, then there exists a vertex u of S such that $S \setminus \{u\}$ is a dominating set. This implies that u is adjacent to some vertex v of S , so by assumption it has a private neighbor in $V \setminus S$, call it t . Then t is not adjacent to any vertex in $S \setminus \{u\}$, which contradicts our assumption that this is a dominating set. ■

Theorem 2.3: [Ore, 1962] Every connected graph G of order $n \geq 2$ has a dominating set S whose complement $V - S$ is also a dominating set.

Theorem 2.4: [Ore, 1962] If G is a graph with no isolated vertices, then the complement of $V - S$ of every minimal dominating set S is a dominating set.

Proof 2.4: Let S be a minimal dominating set of G . We will show that complement of S is also a dominating set for G . So, assume that complement of S is not a dominating set, then there exists a vertex u of S which is not dominated by the complement of S , namely by $V - S$. This means that vertex u is not adjacent to any vertex in $V - S$. So, since u cannot be an isolated vertex, all the neighbors of u must be in $S - \{u\}$. This u is dominated by a vertex in $S - \{u\}$. And since u is not dominating any vertex in $V - S$, when we remove u from S , $S - \{u\}$ is still a dominating set of G , but this contradicts our assumption on the minimality of S . ■

If we want to dominate the vertices of a graph, then we need at least one vertex and we can at most use all of the vertices of the graph. So $1 \leq \gamma(G) \leq n$ is a trivial bound in graph theory. Ore's theorem gives an upper bound for graphs without isolated vertices in terms of the order of the graph.

Theorem 2.5: [Ore, 1962] If a graph G has no isolated vertices, then $\gamma(G) \leq \frac{n}{2}$.

Proof 2.5 [Gorodezky, 2007]: Let S be a minimum size dominating set in a graph G . So, every vertex u of S is adjacent to some vertex v not in S , otherwise $S \setminus \{u\}$ would be a dominating set but it contradicts with the minimality of S . This shows that $V \setminus S$ is also a dominating set. Namely, the complement of a minimal dominating set is also a dominating set. Hence, $\gamma(G) \leq \frac{n}{2}$. ■

Now, we give a bound in terms of order and size; this theorem was given by Vizing [Vizing, 1965] and it bounds the size m of a graph, given the domination number.

Theorem 2.6: [Vizing, 1965] If a graph G has $\gamma(G) \geq 2$, then $m \leq \left\lfloor \frac{1}{2}(n - \gamma(G))(n - \gamma(G) + 2) \right\rfloor$.

In 1991, for graphs having $\Delta(G) < n - \gamma(G)$, Sanchis has improved the above theorem giving the following result:

Theorem 2.7: [Sanchis, 1991] If a graph G has $\gamma(G) \geq 2$, and $\Delta(G) < n - \gamma(G) - 1$, then $m \leq \left\lfloor \frac{1}{2}(n - \gamma(G))(n - \gamma(G) + 1) \right\rfloor$.

For Theorem 2.8, we refer the upper bound to Berge and the lower bound to Walikar, Acharya, and Sampathkumar.

The following bound involves the order and the maximum degree of the graph:

Theorem 2.8: [Berge, 1962], [Walikar et al., 1979] For any graph G , $\left\lfloor \frac{n}{1 + \Delta(G)} \right\rfloor \leq \gamma(G) \leq n - \Delta(G)$.

Proof 2.8: Let S be a minimum dominating set of G . First, we will show the lower bound. Since every vertex dominates at most itself and $\Delta(G)$ other vertices, we have $\gamma(G) \geq \left\lfloor \frac{n}{1 + \Delta(G)} \right\rfloor$.

Now, we will show the upper bound. Let us construct a dominating set S . If v is a vertex of maximum degree $\Delta(G)$, then v dominates $N[v]$ vertices. Place v in S . Then place the rest of the vertices, namely $V - N[v]$ also in S . Since the vertices in $V - N[v]$ dominate themselves. Hence, S is a dominating set of cardinality $n - \Delta(G)$, so $\gamma(G) \leq n - \Delta(G)$. ■

We also have bounds related to the regular graphs as well:

Theorem 2.9: [Alon and Spencer, 2000] If G is a k -regular graph on n vertices, then $\frac{\gamma(G)}{n} \leq \frac{1 + \log(k+1)}{k+1}$.

Theorem 2.10: [Caro and Roditty, 1990] If G is a k -regular graph on n vertices, then $\frac{\gamma(G)}{n} \leq \frac{k}{3k-1}$.

For the Theorem 2.10, there are also other proofs that can be found in [Liu and Sun, 2004] and also in [Xing et al., 2006].

2.3. Bounds on Total Domination

In this section, we will focus on the bounds related to total domination number in a graph.

Theorem 2.11: [Cockayne et al., 1980] If G is a connected graph of order $n \geq 3$, then $\gamma_t(G) \leq \frac{2n}{3}$.

Favaron et al. made another improvement for $\delta(G) \geq 3$:

Theorem 2.12: [Favaron et al., 2000] If G is a connected graph of order n with $\delta(G) \geq 3$, then $\gamma_t(G) \leq \frac{7n}{13}$.

In one of the proofs of Theorem 2.13, the following Lemma 2.1 is used, which is a result about bipartite graphs proved by Archdeacon et al. For the other proof of Theorem 2.18, see [Tuza, 1990] and [Chvátal and McDiarmid, 1992]

Lemma 2.1: [Archdeacon et al., 2004] Let G be a bipartite graph with partite sets (X, Y) whose vertices in Y are of degree at least 3. Then there exists a set $A \subseteq X$ of size at most $|X \cup Y|/4$ such that every vertex in Y is adjacent to a vertex in A .

Theorem 2.13: [Archdeacon et al., 2004] If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma_t(G) \leq \frac{n}{2}$.

Proof 2.13: Let G be a graph on n vertices $\{v_1, v_2, \dots, v_n\}$. Then construct a bipartite graph H with parts X and Y from G as the following. For each vertex v_i in G , form two vertices x_i, y_i in H . For each edge $v_i v_j$ in G , add two edges $x_i y_j$ and $x_j y_i$ in H . The bipartition is $(X, Y) = (\{x_i\}, \{y_i\})$. A total dominating set A in H corresponds to an $A \subset X$ in G adjacent to every vertex in Y . The result now follows by Lemma 2.1. ■

The following bounds involve order, size and/or maximum degree of the corresponding graph.

Theorem 2.14: [Henning, 2005] If G is a connected graph with $\delta(G) \geq 3$ and of order n and size m , then $\gamma_t(G) \leq n - \frac{m}{3}$.

Theorem 2.15: [Cockayne et al., 1980] If G is a graph of order n with no isolated vertex, then $\gamma_t(G) \leq n - \Delta(G) + 1$.

Proof 2.15: Let v be a vertex with maximum degree. If we remove v and its open neighborhood from V , then remaining set is not dominated by vertex v and let us say this set A , namely $A = V - \{v\} \cup N(v)$. We have two cases.

First, if $A = \emptyset$, then it means that all vertices in V are dominated by vertex v . So, our minimum dominating set includes v and one of its neighbors and total domination number is 2, namely $\gamma_t(G) = 2 = n - \Delta(G) + 1$.

Otherwise, $A \neq \emptyset$. Then it means that vertex v does not dominate all vertices in V . So, let T be a set of isolates of $G[A]$. If T is empty set, then $A \cup \{v\} \cup \{y\}$, where $y \in N(v)$ is a total dominating set of cardinality $n - \Delta(G) + 1$. If T is a non-empty set, then each $t \in T$ is adjacent to some vertex in $N(v)$ and let $M(T)$ to be a set with the smallest cardinality contained in $N(v)$ such that $t \in T$ is adjacent to an element of $M(T)$. It is easy to see that $|M(T)| \leq |T|$. Then $\{v\} \cup M(T) \cup (A - T)$ is a total dominating set and $\gamma_t(G) \leq 1 + |T| + |A - T| = 1 + |A| = n - \Delta(G)$. ■

Theorem 2.16: [Cockayne et al., 1980] If G is a connected graph of order $n \geq 3$ and $\Delta(G) \leq n - 2$, then $\gamma_t(G) \leq n - \Delta(G)$.

Theorem 2.17: [Henning, and Yeo, 2013] If G is a graph of order n with no isolated vertex, then $\gamma_t(G) \geq \frac{n}{\Delta(G)}$.

Proof 2.17: Let S be a total dominating set. Since every vertex is totally dominated by the set S , every vertex belongs to the open neighborhood of at least one vertex in S . Hence, $V(G) = \bigcup_{v \in S} N_G(v)$ implying that;

$$n = \left| \bigcup_{v \in S} N_G(v) \right| \leq \sum_{v \in S} |N_G(v)| \leq |S| \cdot \Delta(G)$$
or, equivalently, $\gamma_t(G) = |S| \geq \frac{n}{\Delta(G)}$. ■

Theorem 2.18: [Bollobas and Cockayne, 1979] For every graph G with no isolated vertex, then $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$.

Proof 2.18: Since a total dominating set gives us a dominating set in the same graph G we naturally have $\gamma(G) \leq \gamma_t(G)$. Let S be a minimum dominating set in a graph G and let S' be a set obtained by selecting an arbitrary neighbor of every vertex in S . Then in S' , there are at most $|S|$ elements, namely $|S'| \leq |S|$. And it is easy to see that the set $S' \cup S$ is a total dominating set in G . Hence, $\gamma_t(G) \leq |S'| + |S| \leq 2|S| = 2\gamma(G)$. ■

3. VERTEX TRANSITIVE GRAPHS

3.1. Basics of Vertex Transitive Graphs

In this section, we give some basic definitions about algebra and group theory to understand vertex-transitive graphs better. First, we start with definition of the binary operation. Let G be a nonempty set and define a function $*$: $G \times G \rightarrow G$, and $(a, b) \rightarrow (a * b)$, such a function is called a *binary operation* on G . Also, a binary operation $*$ on G is called *closed binary operation* on G if all a, b of G , $a * b \in G$. If G is a set with a binary operation, and the following three conditions hold which are, associativity, having identity, and having inverse, then $(G, *)$ is called a *group*. If $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$, then this operation is called *associative*. If there exist an element “ e ” in G such that $a * e = e * a = a$ for all $a \in G$, then e is called *identity*. Also, if G has an identity element, it is unique. If there exist an identity element of G and there is an inverse element in G such that $a \circ a^{-1} = a^{-1} \circ a = e$, then a^{-1} is called the *inverse* of a .

Order of a group G is the number of its elements (cardinality) and it is denoted by $|G|$. Let G be a group and $c \in G$, if there exists an integer $n > 0$ such that $c^n = e$ then we say G has *finite order*. The *order* of an element c in a group G is the smallest integer $n > 0$ such that $c^n = e$ (in additive notation $nc = 0$), and it is denoted by $o(c)$ or $|c|$. If there is no such integer exist, then it is said to have *infinite order* and show it by $o(c) = \infty$. If G is a group and $a \in G$, if $G = \{a^n \mid n \in \mathbb{Z}\} = \langle a \rangle$ for an element a , then G is called a *cyclic group* and a is called the *generator* of G . It can be seen that 1 is a generator of G . The elements of $\mathbb{Z}/n\mathbb{Z}$ are $\{0, 1, 2, \dots, n-1\}$ and the group operation is the addition modulo n . Every finite cyclic group of order n is isomorphic to the group $\mathbb{Z}/n\mathbb{Z}$. Also, every infinite cyclic group is isomorphic to the group $\mathbb{Z}$ under addition.

Example 3.1: Group: Integers $\mathbb{Z}$ under $+$

- *Claim:* $\mathbb{Z} = \langle 1 \rangle$
 - $\langle 1 \rangle = \{\dots -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$

- Also, $\mathbb{Z} = \langle -1 \rangle$
- Thus, $\mathbb{Z}$ is a cyclic group and has precisely two generators; 1 and -1 .

Example 3.2: Group: Real numbers $\mathbb{R}$ under addition.

- *Claim: $\mathbb{R}$ is not a cyclic.*
 - *Proof: Suppose $(\mathbb{R}, +)$ is cyclic.*
 - *Then there exists an element a , $a \in \mathbb{R}$, such that $\mathbb{R} = \langle a \rangle$.*
 - *Note $\mathbb{R} = \langle a \rangle = \{na \mid n \in \mathbb{Z}\}$.*
 - *$\frac{1}{2}a \in \mathbb{R}$ but $\frac{1}{2}a \notin \langle a \rangle$.*
 - *If it were, $\frac{1}{2}a = na$ for some $n \in \mathbb{Z}$.*
 - *$a \neq 0, \frac{1}{2} = n$, contradiction.*

Example 3.3: Find the cyclic subgroups of $\mathbb{Z}_6$.

- The elements of $\mathbb{Z}_6$ are $\{0, 1, 2, 3, 4, 5\}$.
- $\langle 1 \rangle = \{1, 2, 3, 4, 5, 0\}$
- $\langle 2 \rangle = \{2, 4, 0\}$
- $\langle 3 \rangle = \{3, 0\}$
- $\langle 4 \rangle = \{4, 2, 0\}$
- $\langle 5 \rangle = \{5, 4, 3, 2, 1, 0\}$
- $\langle 0 \rangle = \{0\}$
- $\langle 1 \rangle = \langle 5 \rangle = \mathbb{Z}_6$.

A one to one and onto mapping from X to X is called a *permutation of X* . Every possible permutation of a set X is with called *symmetric group*, is denoted by S_n for all integer $n > 0$ and a permutation group is a subgroup of symmetric group. The order of a symmetric group with n elements S_n is $n!$.

Example 3.4: Let us find the elements of symmetric group S_3 .

Order of S_3 is 6.

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, a_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (12), a_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (23),$$

$$a_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (23), \quad a_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123), \quad a_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132).$$

$$S_3 = \{e, a_1, a_2, a_3, a_4, a_5\}$$

$$S_3 = \{e, (12), (13), (23), (123), (132)\}.$$

Let $(G, \odot)$ and $(H, \otimes)$ be two groups and $f: G \rightarrow H$ be a mapping. If for all $a, b \in G$, $f(a \odot b) = f(a) \otimes f(b)$, then f is called a *group homomorphism*. If homomorphism f is one to one and onto then it is called *isomorphism*. If $\phi: G \rightarrow H$ is an isomorphism, then *group G is called isomorphic to a group H* and it is denoted by $G \cong H$. Let $(G, \odot)$ be a group and $f: G \rightarrow G$ be a mapping. If for all $a, b \in G$, $f(a \odot b) = f(a) \odot f(b)$, and f is one to one, onto and a homomorphism, then G is called a *group automorphism*. Actually, a group automorphism is a special kind of isomorphism from a group G onto itself. Set of all possible automorphisms of G is denoted by $\text{Aut}(G)$.

Example 3.5: Let $\beta: \mathbb{Z} \rightarrow \mathbb{Z}_n$ and defined by $\beta(a) = \bar{a}$.

$$\beta(a + b) = \overline{a + b} = \bar{a} + \bar{b} = \beta(a) + \beta(b).$$

So β is a homomorphism.

Example 3.6: Let $\beta: (\mathbb{R}, +) \rightarrow (\mathbb{R}, +)$, and defined by $\beta(a) = a^2$. A and b are non-negative real numbers, $\beta(a + b) = (a + b)^2 = a^2 + 2ab + b^2 \neq a^2 + b^2 = \beta(a) + \beta(b)$. So β is not a homomorphism.

Example 3.7: Let $f: G \rightarrow G$ be a mapping such that $f(x) = axa^{-1}$ where $a \in G$.

We show that f is automorphism, and to show this we will show that these three properties hold;

- *Let us first show that, f is a homomorphism:*
 - $f(xy) = axya^{-1} = (axa^{-1}aya^{-1}) = f(x)f(y)$.
 - *So, f is homomorphism.*
- *Let us show f is one-to-one:*
 - $f(x) = f(y) \Rightarrow axa^{-1} = aya^{-1}$
 - $\Rightarrow x = y$
 - *So, f is 1-1.*

- *Let us show f is onto:*
 - *For any $g \in G$ there exist $x \in G$ such that*
 - *$f(x)=g$*
 - *$\Rightarrow axa^{-1} = g$*
 - *$\Rightarrow x = a^{-1}ga$*
 - *$\Rightarrow f$ is onto.*
 - *So, f is automorphism of G .*

Let G be a group on a nonempty set X . Then G is called a group action on X if there exist a function f such that $f: G \times X \rightarrow X$, usually $f(g, x)$ is written as $g.x$, such that $e.x = x$ for all $x \in X$ and $g.(h.x) = (gh).x$ for all $g, h \in G$ and $x \in X$. “ G acts on X ” is denoted by $G \curvearrowright X$. If for all $x, y \in X$, $\exists g \in G$ such that $g.x = y$ then a group G acting on a set is called *transitive*.

Let $G = ((V(G), E(G)))$ and $H = ((V(H), E(H)))$ be two graphs. A *graph homomorphism* f from graph G to H , $f: V(G) \rightarrow V(H)$, is a mapping where any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H . If a homomorphism of graphs G and H is also a bijection between the vertex sets of G and H then this homomorphism is called an *isomorphism*. Let $G = (V, E)$ be a graph, an *automorphism of a graph* G is a permutation f of the vertex set V such that $\{u, v\}$ is an edge in G if and only if $\{f(u), f(v)\}$ is also an edge in G . Namely, it is a graph isomorphism from G to itself. As an example, the automorphism group of the complete graph on n vertices, K_n , is the symmetric group on n elements S_n , so it consists of all permutations of the vertex set of S_n .

There are many definitions related to vertex-transitivity. The most commonly used definition is: For any two vertices u and v in a graph G , if there is an automorphism, $f: V(G) \rightarrow V(G)$ such that $f(u) = v$ then G is called a *vertex-transitive* graph (or just a transitive graph). Definition of vertex-transitivity related with group action can also be given as: A graph G is called vertex-transitive if its automorphism group acts transitively on $V(G)$. All vertex transitive graphs are regular. Due to their perfect symmetry, there are many graph theory problems like finding Hamiltonian cycles and graph decompositions on those graphs. Actually, Lovász wanted to know that whether each vertex transitive graph that is connected and finite has a Hamiltonian path, namely whether there exists a path that visits all vertices only

once. There are only five connected vertex transitive graphs that do not include Hamilton cycles; The complete graph with 2 vertices, K_2 , the well-known Petersen graph, the Coxeter graph that is a 3 regular graph having 28 vertices and 42 edges, and the two graphs derived from the Petersen and Coxeter graphs by replacing each vertex with a triangle. Except K_2 , remaining graphs are non- Cayley graphs.

Example 3.8:

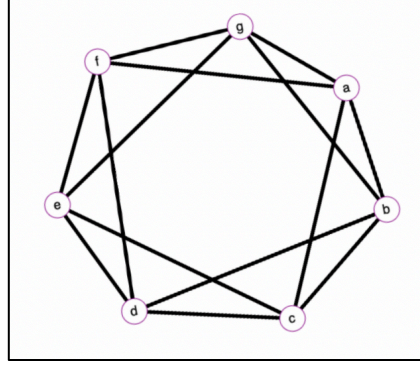


Figure 3.1: A Vertex-Transitive Graph.

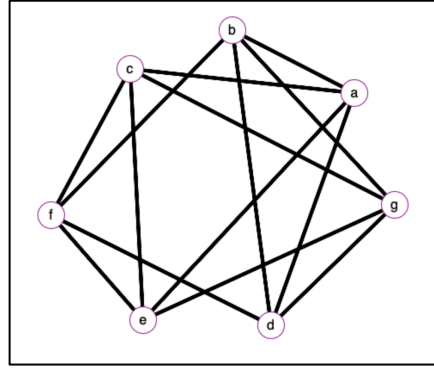


Figure 3.2: Not Vertex-Transitive Graph.

D_n is the set of symmetries of an n –gon, also known as the *dihedral group* on n –element. The group D_n has order $2n$. D_n contains $2n$ actions; n rotations and n reflections. There are only two generators; r = counterclockwise rotation by $2\pi/n$ radians and s = fixing the symmetry axis. There are some presentations of D_n .

$$D_n = \{e, r, r^2, \dots, r^{n-1}, s, rs, r^2s, \dots, r^{n-1}s\} \quad (3.1)$$

$$D_n = \langle r, s : r^n = s^2 = e, rsr = s \rangle.$$

Example 3.9: The Dihedral Group D_4 :

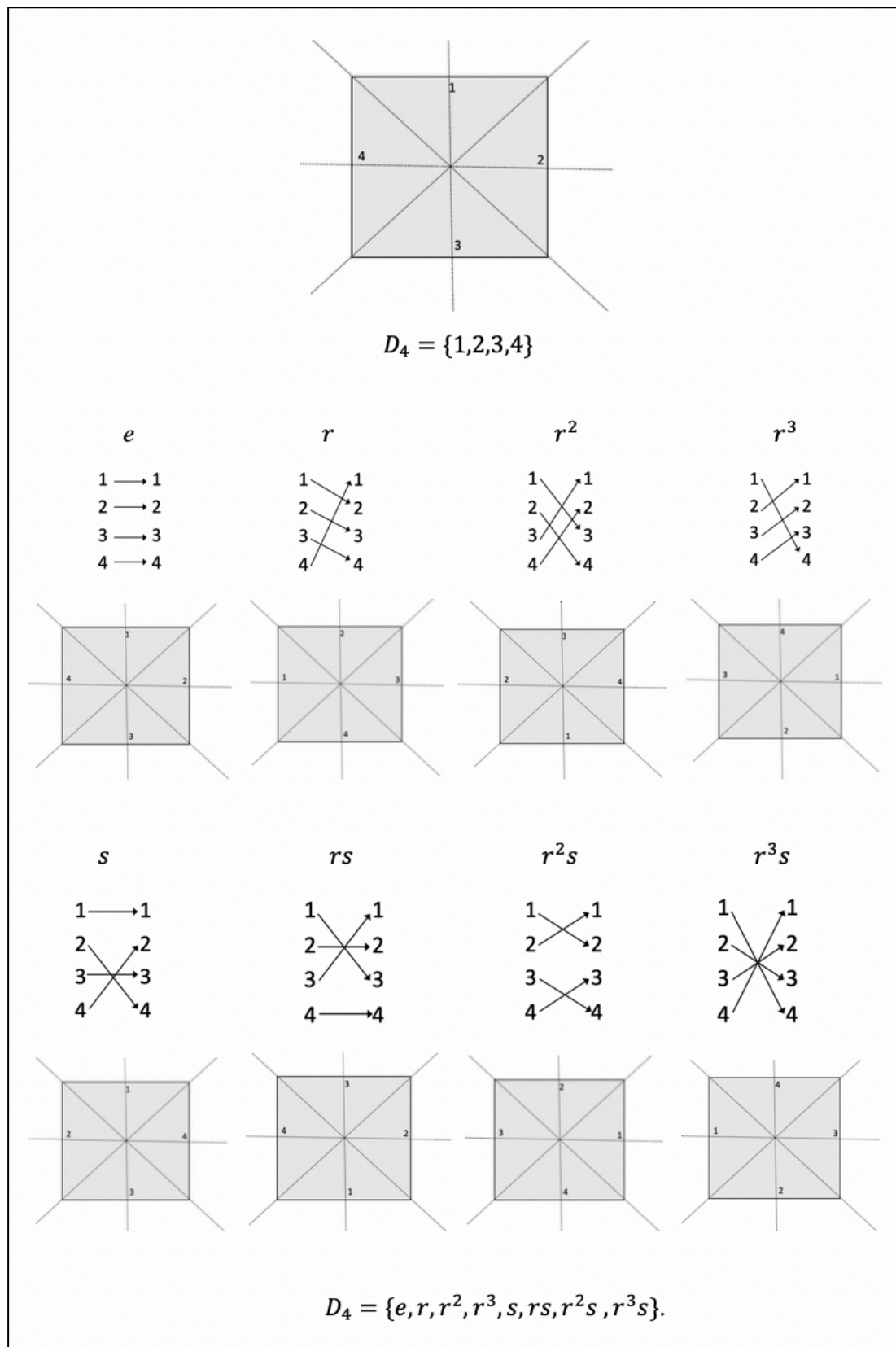


Figure 3.3: Dihedral Group D_4 .

Many people have attempted to find vertex transitive graphs for given degrees. First, H.P Yap has the first attempt to list vertex transitive graphs up to 13 vertices [Yap, 1973], but he did miss three graphs with 12 vertices. Then McKay listed all the vertex-transitive graphs up to 19 vertices [McKay, 1979], he used a different method from Yap and gave two tables about the number of transitive graphs and connected transitive graphs. He showed that Petersen graph which has 10 vertices, is the only non-Cayley vertex transitive graph with number of vertices less than 13. He explained also the methods and calculations used in this study in his doctoral thesis [McKay, 1980]. Vertex transitive graphs from 20 to 23 vertices were found by McKay and Royle [Royle, 1988] and then for 24 vertices were found by Royle and Praeger [Royle and Praeger, 1989] and lastly, vertex transitive graphs at most 26 vertices were found by McKay and G. F. Royle [McKay and Royle, 1990]. Also, the vertex-transitive graphs up to 31 vertices are given in <http://staffhome.ecm.uwa.edu.au/~00013890/trans/> [Web 2, 2020]. The accuracy of values from 27 to 31 is uncertain.

3.2. Cayley Graphs

There are many structures that produce families of vertex-transitive graphs. The most important examples of vertex transitive graphs are Cayley graphs. Cayley graph was introduced by Arthur Cayley in 1878. Cayley graph has many applications in computer science and molecular biology [Bafna and Pevzner, 1996], [Hahn and Sabidussi, 1997]. Cayley graphs have a very symmetric structure and owes their symmetrical features to vertex-transitivity, thanks to this property they are used as models for many interconnection networks which are the whole system created by connecting at least two computers with software and hardware components over wired or wireless communication tools. They are connected by vertices that may have one or more processors. With these connections, data is moved from one processor to another or to main memory and there are edges between systems that are directly connected to each other. This network system provides very fast communication and data transfer. There are many families of Cayley graphs such as complete graphs, circulant graphs, hypercube, butterfly graphs, and Kneser graph (for some values of n and k). Cayley graphs were constructed on groups. In this thesis, we will use finite groups. We will

start by giving some basic definitions and theorems on Cayley graphs. The definitions below are taken from [Beineke et al., 2007].

Definition 3.1: Let G be a finite group and let A be a subset of G . The Cayley diagram (or Cayley digraph) of G with respect to A , denoted by $\text{Cay}(G, A)$, is the digraph with vertex set G and edge set $\{(h, hs) : h \in G, s \in A\}$.

Throughout, e denotes the identity element of the group G and also the corresponding vertex of the digraph $\text{Cay}(G, A)$.

Example 3.10: The Cayley digraph of S_3 with generating set $A = \{(12), (123)\}$.

In Figure 3.2 you can see Cayley digraph S_3 . We will use (12) instead of $(1, 2)$.

Table 3.1: S_3 with generating set $A = \{(12), (123)\}$.

$(e)(12) = (12)$	$(13)(12) = (123)$	$(123)(12) = (13)$
$(e)(123) = (123)$	$(13)(123) = (12)$	$(123)(123) = (132)$
$(12)(12) = e$	$(23)(12) = (132)$	$(132)(12) = (23)$
$(12)(123) = (23)$	$(23)(123) = (13)$	$(132)(123) = e$

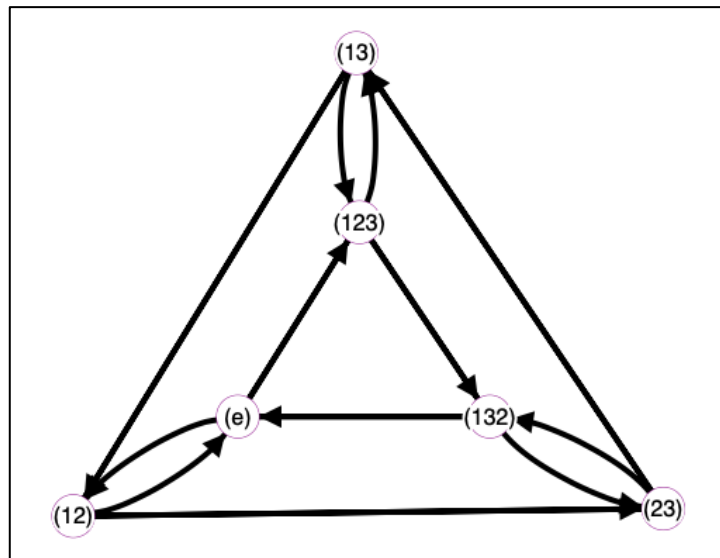


Figure 3.4: $\text{Cay}\{S_3, \{(12), (123)\}\}$.

We can see that in this example subset A is not inverse closed, so all vertices have two incoming and two outgoing edges. Hence, this Cayley graph is a 4-regular graph.

If the identity group element e belongs to A then all vertices have a loop in a Cayley digraph. In this thesis, we will study simple, undirected graphs in which no loops and multiple edges occur. Now, we will give the definition of Cayley graph which does not include loops or multiple edges. So, we remove the identity group element from the connection set A and we want it to be inverse closed.

Definition 3.2: Let G be a finite group with identity 1 . Let A be a subset of G satisfying $1 \notin A$ and $A = A^{-1}$ that is, $s \in A$ if and only if $s^{-1} \in A$. The Cayley graph $\text{Cay}(G, A)$ on G with connection set A is defined as follows:

- *The vertices are the elements of G ;*
- *there is an edge joining g and h if and only if $h = gs$ for some $s \in A$, namely*

$$E(\text{Cay}(G, A)) = \{\{g, h\} \mid g^{-1}h \in A\}$$

The set of all Cayley graphs on G is denoted by $\text{Cay}(G)$. Note that a Cayley graph $\text{Cay}(G)$ is connected if and only if A generates G .

Theorem 3.1: [Marušič, 1983] Every Cayley Graph $\text{Cay}(G, A)$ is vertex transitive.

Proof 3.1: For each vertex g of G , we define a map;

$$\begin{aligned} \Pi_g: G &\rightarrow G \\ x &\mapsto gx, \quad x \in G. \end{aligned} \tag{3.2}$$

- *Let us first show that, Π_g is a permutation of G .*
 - *i.e. Π_g is a bijection (one to one and onto).*
- *Let us show Π_g is one-to-one:*
 - $\Pi_g(y) = \Pi_g(z)$
 - So, $gy = gz$

- $g^{-1}(gy) = g^{-1}(gz), g^{-1} \in G.$
 - $(g^{-1}g)y = (g^{-1}g)z$
 - $y = z$
 - So, Π_g is one-to-one.
- Let us show Π_g is onto:
 - For any $y \in G$ there exist $z \in G$ such that
 - $\Pi_g(y) = z$
 - $gy = z$
 - $y = g^{-1}z.$
 - So, Π_g is onto.
- Let us show Π_g is an automorphism of $\text{Cay}(G, A)$;
 - $\{x, y\} \in E(\text{Cay}(G, A))$ if and only if $x^{-1}y \in A$ by definition of Cayley graph.
 - $\Rightarrow x^{-1}y = x^{-1}g^{-1}gy$, since $g^{-1}g = \text{identity}$
 - $= (gx)^{-1}(gy) \in A$
 - $\Rightarrow \{\Pi_g(x), \Pi_g(y)\} \in E(\text{Cay}(G, A)).$
 - So, Π_g is an automorphism of $\text{Cay}(G, A)$.
- Aut group $\text{Cay}(G, A)$ acts transitively on G for any $x, y \in G$,
 - $\Pi_{yx^{-1}}(x) = (yx^{-1})x = y$
 - Hence, $\text{Cay}(G, A)$ is vertex-transitive. ■

It is well-known that each Cayley graph is vertex-transitive, but the converse is not true. We will give a well-known example in the Section 3.3.

Definition 3.3: A Cayley graph $\text{Cay}(G, A)$ is called circulant graph when G is a cyclic group $\mathbb{Z}_n$.

Figure 3.5 is an example of circulant graph in which G is a cyclic group $\mathbb{Z}_8$ and a generator of G is $\{1, 2\}$.

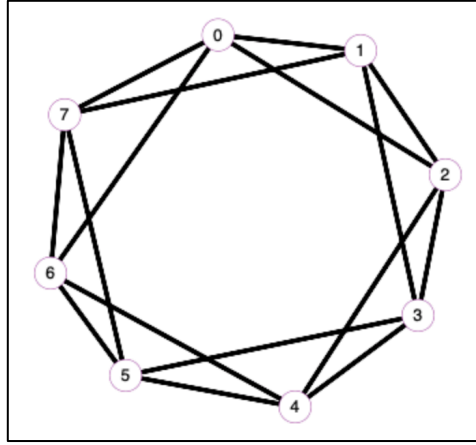


Figure 3.5: $\text{Cay}(\mathbb{Z}_8, \{1, 2\})$.

In Figure 3.5 by the connection set $\{1, 2\}$ what is actually meant is $\{\pm 1, \pm 2\}$ since it needs to be inverse closed. So, $\text{Cay}(\mathbb{Z}_8, \{1, 2\})$ is a 4-regular graph.

“Let G be a transitive permutation group acting on a finite set H . It is easy to show that the following three conditions are equivalent:

- *the only element of G that fixes an element of H is the identity permutation;*
- *$|G| = |H|$*
- *for any $h_1, h_2 \in H$, there is a unique element $g \in G$ satisfying $h_1 g = h_2$.*

A transitive permutation group G that satisfies any one of these conditions is said to be regular.”

Sabidussi characterizes Cayley graphs in terms of their automorphism group.

Theorem 3.2: [Sabidussi, 1958] A graph G is Cayley graph if and only if the automorphism group contains a subgroup which acts regularly on the vertex set.

Example 3.11: $\text{Cay}(S_3, \{(12), (23)\}) \cong C_6$

Table 3.2: S_3 with generating set $A = \{(12), (23)\}$.

$(e)(12) = (12)$	$(13)(12) = (123)$	$(123)(12) = (13)$
$(e)(23) = (23)$	$(13)(23) = (132)$	$(123)(23) = (12)$
$(12)(12) = (e)$	$(23)(12) = (132)$	$(132)(12) = (23)$
$(12)(23) = (123)$	$(23)(23) = (e)$	$(132)(23) = (13)$

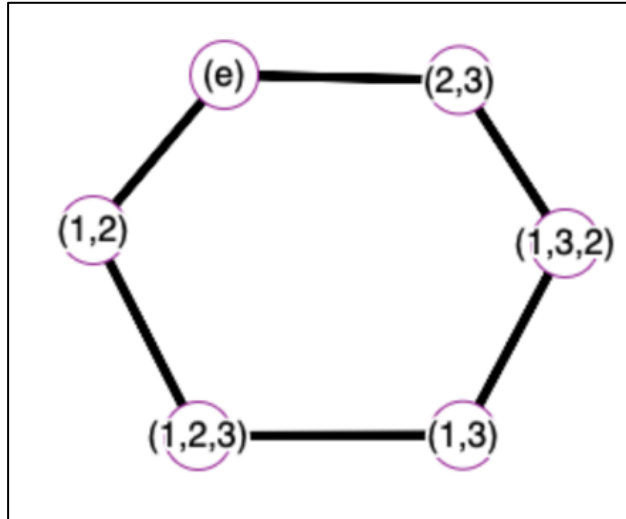


Figure 3.6: $\text{Cay}\{S_3, \{(12), (23)\}\}$.

Definition 3.4: n-dimensional hypercube is a graph whose vertex set is $\{0,1\}^n$, and there is an edge between two vertices if and only if they differ in exactly one bit position.

Example 3.12: Let $A = \{(12), (34), (56)\}$ and G is a permutation group generated by A . We will investigate the $\text{Cay}(G, A)$ which is isomorphic to the Q_3 .

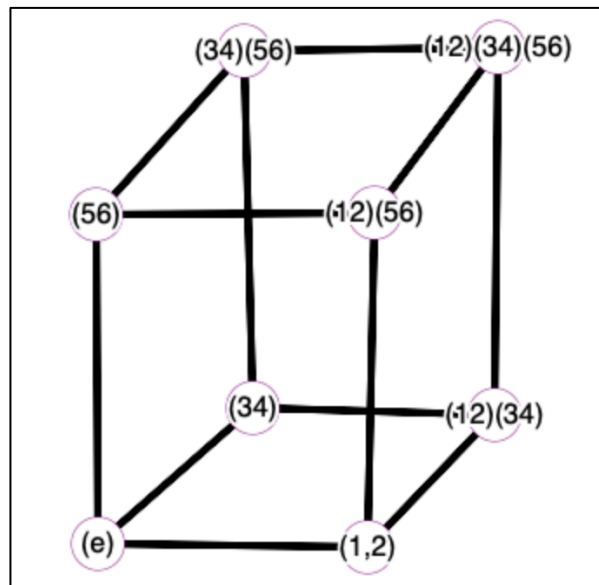


Figure 3.7: $\text{Cay}\{G, \{(12), (34), (56)\}\}$.

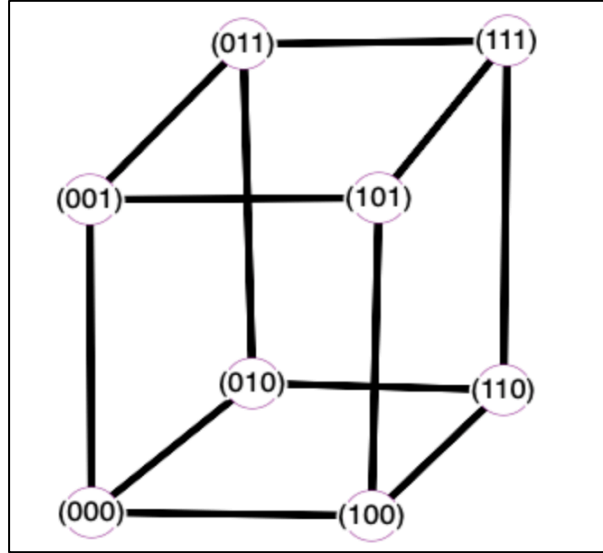


Figure 3.8: 3-dimensional hypercube, Q_3 .

It is known that the $\text{Cay}(G, A)$ of the permutation group generated by the set of n disjoint transpositions $\{(1, 2), (3, 4), \dots, (2n - 1, 2n)\}$ is isomorphic to Q_n . Also, we can say that n -dimensional hypercube is a Cayley graph $\text{Cay}(\mathbb{Z}_2^n, A)$ where A consists of binary n -tuples with exactly one 1 in them.

3.3. Non-Cayley Vertex Transitive Graphs

There are many studies on Cayley graphs, and we know that every Cayley graph is vertex-transitive because we proved in Section 3.2. However, there is not much work on non-Cayley graphs. In this part, we particularly focus on vertex-transitive graphs which are not Cayley graphs. First, we will talk about the basics of non-Cayley graphs and the related studies. The first person to mention the non-Cayley number detection problem is Marušič [Marušič, 1983]. He asked for which positive integer n , there exist a non-Cayley vertex transitive graph. Let G be a vertex-transitive graph and take a union of finitely many copies of G so it is clear that any multiple of a member of non-cayley vertex transitive graph is also in non-Cayley. This union is a Cayley graph if and only if G is a Cayley graph. Every Cayley graph is vertex-transitive but converse is not true. Petersen graph is the smallest example of non-Cayley vertex transitive graphs.

Now, we will give a proof about the well-known result.

Theorem 3.3: The Petersen graph is vertex- transitive but it is not a Cayley graph.

Proof 3.3: [Yap, 1986] The smallest positive integer n such that $G = A \cup A^2 \cup \dots \cup A^n$ where $A^2 = \{xy \mid x, y \in A\}$ and $A^i = A^{i-1}A$ for $i \geq 3$ give us the diameter of a Cayley graph $\text{Cay}(G, A)$. The diameter of the Petersen graph is 2. If, we show that every 3-regular Cayley graph of order 10 have diameter greater than 2 then we are done.

First, there are two groups of order 10: the cyclic group $\mathbb{Z}_{10}$ and the dihedral group D_5 . Here we use the additive group and replace A^{-1} by $-A$.

- *Case 1: Assume $G = \mathbb{Z}_{10} = \{0, 1, \dots, 9\}$.*

Since $-A = A$ and 5 is an involution we have $5 \in A$. Also $|A| = 3$, we have four choices for A ;

$$- A_1 = \{1, 5, 9\}, A_2 = \{2, 5, 8\}, A_3 = \{3, 5, 7\}, A_4 = \{4, 5, 6\}.$$

Now, $|A_i + A_i| = 5$ for each $i = 1, 2, 3, 4$. So, the diameter of G is greater than 2.

- *Case 2: $G = D_5 = \{0, y, 2y, 3y, 4y, x, x + y, x + 2y, x + 3y, x + 4y\}$ where $2x = 0, 5y = 0$ and $y + x = x + 4y$.*

In this case $x, x + y, x + 2y, x + 3y$ and $x + 4y$ are the only elements of order 2 in G . So, we have only three choices for A ;

- $A_1 = \{x + jy, y, 4y\}, \quad j = 0, 1, 2, 3, 4;$
- $A_2 = \{x + jy, 2y, 3y\}, \quad j = 0, 1, 2, 3, 4;$
- $A_3 = \{x + j_1y, x + j_2y, x + j_3y\}, \quad j = 0 \leq j_1 < j_2 < j_3 \leq 4$

It can be verified that $|A_i + A_i| = 5$ for each $i = 1, 2, 3$. Hence, the diameter of G is greater than 2 in this case as well. ■

Definition 3.5: Order of non-Cayley vertex transitive graph is called a non-Cayley number and the set of all non-Cayley number are denoted by NC.

Some previous results of NC can be summarised below.

Theorem 3.4: Let p and q be distinct primes and $p > q$. Then,

- i) [Alspach and Sutcliffe, 1979] $2p \in NC$ if and only if $p \equiv 1 \pmod{4}$*
- ii) [McKay and Praeger, 1994] $2p^2 \in NC$.*
- iii) [McKay and Praeger, 1994] $4p \in NC$ if $p \geq 5$.*
- iv) [McKay, 1979], [McKay and Royle, 1990], [Royle and Praeger, 1989] for $n \leq 24$, n even, $n \in NC$ if and only if one of 10, 16, 18, 20, 24.*

Marušič has constructed a family of non-Cayley vertex-transitive graphs of order p^k , where p is a prime $p \geq 5$ and $k \geq 4$ [Marušič, 1985] but McKay and Praeger have thought that the construction given is invalid and they give an example in [McKay and Praeger, 1994]. Also, in this paper, they gave a table which lists the vertex-transitive graphs and non-Cayley vertex-transitive graphs for order n up to 27. Based on the values in the table, all vertex-transitive graphs for small n values can be said to be a Cayley graph. They thought also that, each vertex transitive graph is Cayley but they could not prove this claim. Later, in 1996, they disproved their own claim and proved that any positive integer n divisible by p^2 for a prime number p , with the exception of $n \in \{12, p^2, p^3\}$, is a non-Cayley number in [McKay and Praeger, 1996]. In [Miller and Praeger, 1992], authors investigate a family of vertex-transitive graphs on $2pq$ vertices which are not Cayley graphs in the case where $p \equiv 1 \pmod{q}$. Also you can find all non-Cayley vertex-transitive graphs up to 31 vertices in <http://staffhome.ecm.uwa.edu.au/~00013890/trans/> [Web 2, 2020].

4. KNESER GRAPHS

In this section, we will study Kneser graphs and related known bounds for domination and total domination. Kneser graph was first investigated by Martin Kneser in 1955 [Kneser, 1955]. He gave a conjecture for chromatic number of Kneser graph, namely, the inequality $\chi(K(n, k)) \leq n - 2k + 2$. About 20 years later, Lovász proved this conjecture [Lovász, 1978]. More information about Kneser graphs can be found on [Godsil and Royle, 2001]. First, we will give definition of the Kneser graph and Odd graph then we will give two proofs related with domination number and total domination number of Kneser graph and transversal.

Definition 4.1: Given positive integers k and n , Kneser graph is the graph with the k -subsets of a fixed n -set as its vertices with two k -subsets adjacent if they are disjoint and it is denoted by $K(n, k)$.

Example 4.1: We will give $K(5, 2)$ as an example of Kneser graph. This is also the well-known Petersen graph.

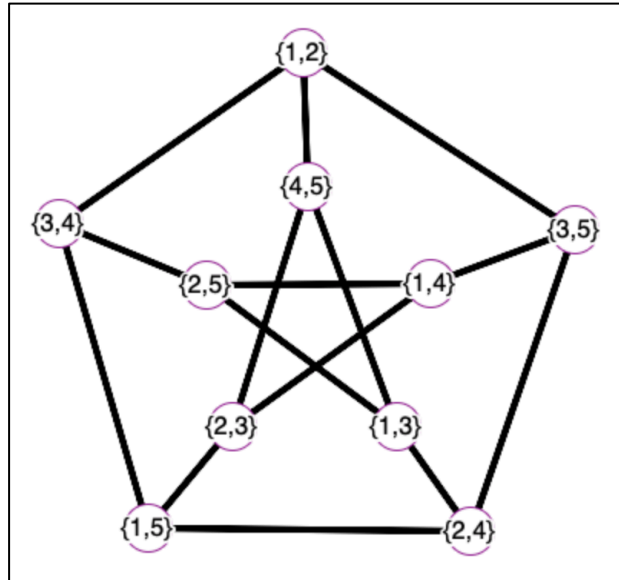


Figure 4.1: Kneser Graph $K(5, 2)$.

The following is a well-known result about Kneser graphs.

Theorem 4.1: All Kneser graphs are vertex transitive.

Proof 4.1: [Heus, 2008] Let Π be a permutation of $\{1, 2, \dots, n\}$. Then Π induces an automorphism of $K(n, k)$ as follows: Let $u = \{x_1, \dots, x_k\}$ and $v = \{y_1, \dots, y_k\}$ be two vertices of $K(n, k)$. Then, for example,

$$(x_1 y_1)(x_2 y_2) \cdots (x_k y_k) \quad (4.1)$$

is a permutation of $\{1, 2, \dots, n\}$. And clearly, this permutation sends u to v and the neighbors of u to the neighbors of v . So (4.1) is an automorphism of $K(n, k)$ and all Kneser graphs are vertex transitive. ■

Definition 4.2: In particular, when $n = 2k + 1$, $K(2k + 1, k)$ is called as the Odd graph, and it is denoted by O_{k+1} .

Odd graphs are important for us since it is proven that all Odd graphs are non-Cayley vertex transitive graphs.

Biggs asked whether there are several values of k for which the Kneser graphs are Cayley graphs in [Biggs, 1978]. As an answer, Godsil proved in [Godsil, 1980] that, except in the following cases, Kneser graphs $K(n, k)$ are not Cayley.

- “ i) $k = 2$, n is a prime power and $n \equiv 3 \pmod{4}$.
- ii) $k = 3$, $n = 8$ or 32 .”

Godsil proved that odd graphs are non-Cayley with using deep group theoretical facts. S. Morteza Mirafzal in [Mirafzal, 2017] showed that;

“If the Kneser graph $K(n, k)$ is of even order where n is an odd integer or both of the integers n, k are even, then $K(n, k)$ is a vertex-transitive non-Cayley graph.”

It is known that Odd graphs are non-Cayley but there are some other non-Cayley graphs for some n and k values in the Kneser graph family. To find these values, studies go on. The well-known Petersen graph is an Odd graph, $O_3 = K(5, 2)$ and it is not a Cayley graph [Godsil, 1980]. We will talk about relation between Kneser graphs, Odd graphs and Johnson graphs in Chapter 5. Now, we will give a definition

of transversal and then we will prove two theorems about dominating and total dominating set of Kneser graph with using the definition of transversal.

Definition 4.3: If S is a collection of subsets of n -set, a transversal of S is a subset of n -set that intersects every subset in S .

The following two observations are mentioned in [Gorodezky, 2007] but no proof is given. Here we give a proof.

Theorem 4.2: Let S be a collection of k -subsets of n -set, then S dominates $K(n, k)$ if and only if every transversal of S of size at most k is contained in some member of S .

Proof 4.2: ($\Rightarrow$) Let S be a set with k subset of n -sets and assume S dominates $K(n, k)$. Let T be a transversal of size at most k . Since V contains all possible k -subsets of the n -set, if no sets in S contains the transversal T , then there is a k -set, say A , in $V - S$ containing the transversal. Since the set of k elements containing this transversal has a common element with each element in S , the vertex A is not adjacent to any element in S . This contradicts with S being a dominating set, then transversal is contained by an element of S .

($\Leftarrow$) Assume that every transversal of S of size at most k is contained in a member of S . Assume also that S is not a dominating set. Then there is a vertex, say A , that is not dominated by any of the elements in S , this means that A is not adjacent to any vertex in S . A is not in S . If it was in S , it would have dominated itself. Since it is not adjacent to any vertex in S , A has at least one common element with each of the vertices in S . So, A contains a transversal, and since A is a k -set this transversal has at most k elements. This contradicts with our assumption that every transversal of S of size at most k being contained in a member of S . So, S must be a dominating set. ■

Theorem 4.3: Let S be a collection of k -subsets of n -set, then S is a total dominating set of $K(n, k)$ if and only if all transversals of S are of size at least $k + 1$.

Proof 4.3: ($\Rightarrow$) Assume that S is a collection of k -subsets of n -set and S dominates $K(n, k)$. Also assume that all transversals of S are of size at most k . Then, there exist

a vertex, say A , that contains this transversal. This vertex A is not adjacent to any of the vertices in S , so A is not dominated by S which contradicts with S being a total dominating set.

($\Leftarrow$) Assume that all transversals of S are of size at least $k + 1$. Also assume that S is not a total dominating set. Then, there exist a vertex A in V that is not adjacent to any vertices in S ; namely A has common elements with each vertex in S . So, A is a transversal of size k which contradicts with our assumption. ■

4.1. Basic Results and Bounds

For the Kneser graph $K(n, k)$, there are 3 choices for the smallest value n can take. First, if $n < 2k$, then Kneser graph only consist of isolated vertices, namely it does not contain any edge. Second, if $n = 2k$, then a dominating set must contain one vertex from each pair of disjoint k -sets. To be specific, $K(2k, k)$ has $\binom{2k}{k}$ vertices and is regular of degree $\binom{k}{k} = 1$. So it is a matching of $\frac{1}{2} \binom{2k}{k}$ edges. So that, for any k , $\gamma(K(2k, k)) = \frac{1}{2} \binom{2k}{k}$.

In this thesis, we take $n > 2k$. Kneser graph has $\binom{n}{k}$ vertices and is clearly a $\binom{n-k}{k}$ -regular graph.

Theorem 4.4: [Marušič, 1985] If $k \geq 3$ and $\frac{3}{4}k(k+1) \leq n \leq k(k+1)$, then

$$\gamma(K(n, k)) = k + 1 + \left\lceil \frac{k(k+1-n)}{\lfloor \frac{k}{2} \rfloor} \right\rceil.$$

Theorem 4.5: [Ivančo and Zelinka, 1993] If $n \geq k(k+1)$, then $\gamma(K(n, k)) = k + 1$.

Proof 4.5: [Gorodezky, 2007] First we prove that $\gamma(K(n, k)) \leq k + 1$. Since $n \geq k(k+1)$ we can choose a set S of $k + 1$ pairwise disjoint k -subsets of n -set. Any k -subset must be disjoint from at least one member of S , since any subset of n -set that intersects them all must contain at least $k + 1$ elements. So, S is a dominating set and $\gamma(K(n, k)) \leq |S| = k + 1$.

Next we prove $K(n, k) \geq k + 1$. Let S be a vertex subset of $K(n, k)$ of size k ; we will show that S cannot be a dominating set. If all k -sets in S are pairwise disjoint then we can create a transversal of S of size k by choosing an arbitrary element from each member of S . This transversal cannot be in S because it cannot be equal to any of the elements of S , which are pairwise disjoint. Thus, there exists a k -subset not in S and not disjoint from any member of S , meaning S is not a dominating set.

If on the other hand some pair of members of S has nontrivial intersection then there exists a set U of size at most $k - 1$ that intersects every member of S . Since there are at least $n - k|S| \geq k$ elements of n -set that are not contained in any member of S , we can use these elements to complete U to a k -subset that is not disjoint from any member of S but cannot itself be in S . Again, this implies that S is not a dominating set. ■

Theorem 4.5 is attributed to Hartman and West in [Web 3, 2020]. Also, this theorem is attributed to Jurkiewicz and Wager by Chris Godsil in [Web 4, 2020]. This theorem is a useful result for $K(n, k)$. Thanks to this, for $n \geq k(k + 1)$, all values of $\gamma(K(n, k))$ can be found easily.

Corollary 4.1: [Ivančo and Zelinka, 1993] If $n \geq 4$, then $\gamma(K(n, 2)) = 3$.

Proof 4.1: For $n \geq 6$, the result follows by Theorem 4.5, so we must verify it only for $n = 4$ and $n = 5$. If $n = 4$ then domination number of $K(4, 2)$ which is a matching on six vertices is 3 and domination number of $K(5, 2)$ which is known as Petersen graph is 3.

Petersen graph is a well-known graph, so we will use it to give examples of domination and total domination number. In Example 4.1 and Example 4.2, you can see dominating and total dominating sets of $K(5, 2)$.

Example 4.2:

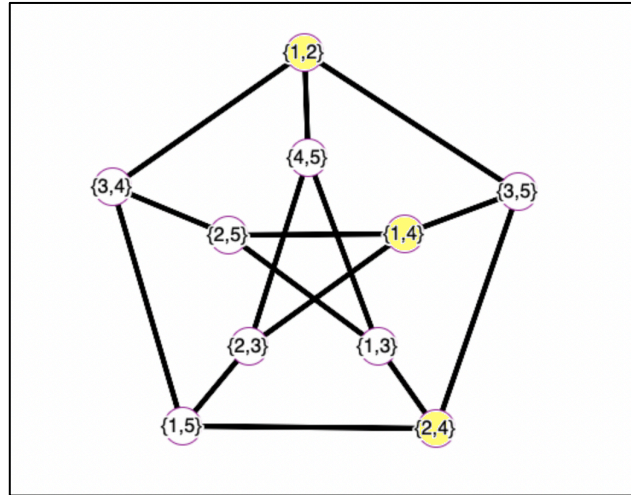


Figure 4.2: A Dominating Set of Petersen Graph.

Petersen Graph is a Kneser Graph $K(5,2)$;

- A minimum dominating set of $K(5,2)$ is $\{\{1,2\}, \{1,4\}, \{2,4\}\}$, and $\gamma(K(5,2)) = 3$.

Example 4.3:

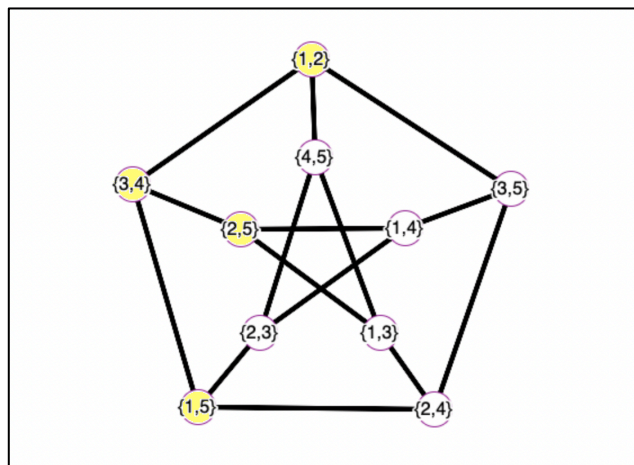


Figure 4.3: A Total Dominating set of Petersen Graph.

- A minimum total dominating set of $K(5,2)$ is $\{\{1,2\}, \{3,4\}, \{1,5\}, \{2,5\}\}$, and $\gamma_t(K(5,2)) = 4$.

Proposition 4.1: [Web 3, 2020] If $n \geq 2k + 1$, then $\gamma(K(n, k)) \geq \gamma(K(n + 1, k))$.

For certain n and k , domination number and total domination number have been studied for Kneser graphs. While doing research on Kneser graphs, we observed that; for certain k values, as the n value increases, the domination number and total domination number are increase to some extent, and then it always gets the same value. As a result of these studies, some bounds are given but a general formula has not been found yet. As a result of the extensive studies related with Kneser graphs, a relationship between the total domination number of the Kneser graph and covering number has been found.

4.2. Covering Design and Total Dominating Set

In this part, we will see how covering designs and total domination number of the Kneser graphs are related. For more information see [Mills and Mullin, 1992].

Definition 4.4: A (n, t, k) covering design is a collection of t element subsets of n -set elements, called blocks, such that any k element subset of n -set is contained in at least one block. The minimum size of (n, t, k) covering design is called covering number and denoted by $C(n, t, k)$ for $n \geq t \geq k$.

Every block of an (n, t, k) covering design covers exactly $\binom{t}{k}$ k -sets and each k -set is covered by at least one block, so we have the trivial lower bound

$$C(n, t, k) \geq \binom{n}{k} / \binom{t}{k}. \quad (4.2)$$

Schönheim [Schönheim, 1964] has observed that $C(n, t, k) \geq L(n, t, k)$, where

$$L(n, t, k) = \left\lceil \frac{n}{t} \left\lceil \frac{n-1}{t-1} \dots \left\lceil \frac{n-k+1}{t-k+1} \right\rceil \dots \right\rceil \right\rceil. \quad (4.3)$$

The best general lower bound on $C(n, t, k)$, due to Schönheim, comes from the following inequality and it is called *Schönheim lower bound*.

$$C(n, t, k) \geq \left\lceil \frac{n}{t} C(n-1, t-1, k-1) \right\rceil. \quad (4.4)$$

Theorem 4.6: [Schönheim, 1964] Suppose that a (n, t, k) design exists, then $C(n+1, t, k) = L(n+1, t, k)$.

The value of $C(n, t, k)$ has been established for certain small values of t and k . It is known that $C(n, 3, 2)$ satisfies the Schönheim bound for all n values, and $C(n, 4, 2)$ meets the bound for $n \neq \{7, 9, 10, 19\}$. $C(n, 5, 2)$ is determined for many values of n , but many are still uncertain. In many cases where n is small enough, $C(n, t, k)$ is known. For small n and k values, the Table 4.1 gives domination number of Kneser graphs and the Table 4.2 gives total domination number of Kneser graphs. You can reach the results for big n and k values from the La Jolla Covering Repository (<https://www.dmgordon.org/cover/>) which is also the source of the information used to create Table 4.1 and Table 4.2. The tables below are taken from [Gorodezky, 2007].

Table 4.1: Some values of $\gamma(K(n, k))$.

n/k	2	3	4	5
4	3	-	-	-
5	3	-	-	-
6	3	10	-	-
7	3	7	-	-
8	3	7	35	-
9	3	7	$22 \leq \dots \leq 27$	-
10	3	6	$14 \leq \dots \leq 20$	126
11	3	5	$10 \leq \dots \leq 17$	66
12	3	4	$10 \leq \dots \leq 12$	$36 \leq \dots \leq 59$
13	3	4	10	$23 \leq \dots \leq 42$
14	3	4	9	$16 \leq \dots \leq 32$
15	3	4	8	$14 \leq \dots \leq 27$
16	3	4	7	$11 \leq \dots \leq 22$
17	3	4	7	$11 \leq \dots \leq 17$
18	3	4	6	$11 \leq \dots \leq 15$
19	3	4	6	$11 \leq \dots \leq 14$
20	3	4	5	$11 \leq \dots \leq 12$
21	3	4	5	$11 \leq \dots \leq 12$

Table 4.2: Some values of $\gamma_t(K(n, k))$.

n/k	2	3	4	5
4	6	-	-	-
5	4	-	-	-
6	3	20	-	-
7	3	12	-	-
8	3	8	70	-
9	3	7	30	-
10	3	6	20	252
11	3	5	17	$96 \leq \dots \leq 100$
12	3	4	12	$55 \leq \dots \leq 59$
13	3	4	10	$33 \leq \dots \leq 42$
14	3	4	9	$28 \leq \dots \leq 32$
15	3	4	8	$24 \leq \dots \leq 27$
16	3	4	7	$18 \leq \dots \leq 22$
17	3	4	7	17
18	3	4	6	15
19	3	4	6	14
20	3	4	5	12
21	3	4	5	12

We know that for $k = 2$ Kneser graph is the complement of Johnson graph. So, we focused on domination number and total domination number for $k = 2$ in Table 4.1 and Table 4.2. We can see that for all n values up to 21, domination number is 3 in Table 4.1. We can also see that except $n = 4, 5$, for all n values up to 21 total domination number is 3. So, we are interested in possible relation between Kneser graphs and Johnson graphs on domination number and total domination number for $k = 2$. And we want to investigate if these domination numbers also constant for any n . We will investigate this relation in Section 5 but now we will study the relation between Kneser graphs and covering designs.

Let S be a total dominating set of $K(n, k)$, then every k -subset of n -elements is disjoint from k -subset in S . Hartman and West, observed that there is a relationship between the total domination number of Kneser graph and Covering number in [Web 3, 2020]. They observed that S is a total dominating set if and only if the complements of the k -subsets in S form a $(n, n - k, k)$ covering design. The following observation is mentioned in [Web 3, 2020]. Here we give another proof.

Theorem 4.7: $\gamma_t(K(n, k)) = C(n, n - k, k)$.

Proof 4.7: Let X be an n -set and assume that as a collection of k -subsets from X , S is a total dominating set of $K(n, k)$. Let D contain the collection of $(n - k)$ -subsets from X that are complements of the vertices (k -subsets) of S . Assume on the contrary D is not a $(n, n - k, k)$ -covering design. Then there is a k -subset A that is not contained in any $(n - k)$ -subset in D . So, A must be k -subset that intersects every element (vertex) of S , meaning that A is a transversal of S . But since it is not disjoint from any vertex of S , A can not be dominated by the elements in S . So S cannot be total dominating set. A contradiction. Therefore, D is a $(n, n - k, k)$ -covering design.

For the converse, let D be a $(n, n - k, k)$ -covering design on X and S contain the k -subsets which are the complements of the blocks of D . Assume on the contrary S is not a total dominating set. Then there must be a vertex, say A , that is not adjacent to any vertex in S . So, A has intersection with each vertex in S ; namely A is a transversal of S . A being a k -subset must be contained in a block (an $(n - k)$ -subset) of D . But then, since S contains the complements of the blocks of D , A cannot have any intersection with the vertices in S . This contradicts with A being a transversal of S . Hence, S is a total dominating set for $K(n, k)$. ■

Example 4.4: Now, we will investigate total domination number of $K(6, 2)$ and covering number of $C(6, 4, 2)$ in Figure 4.4.

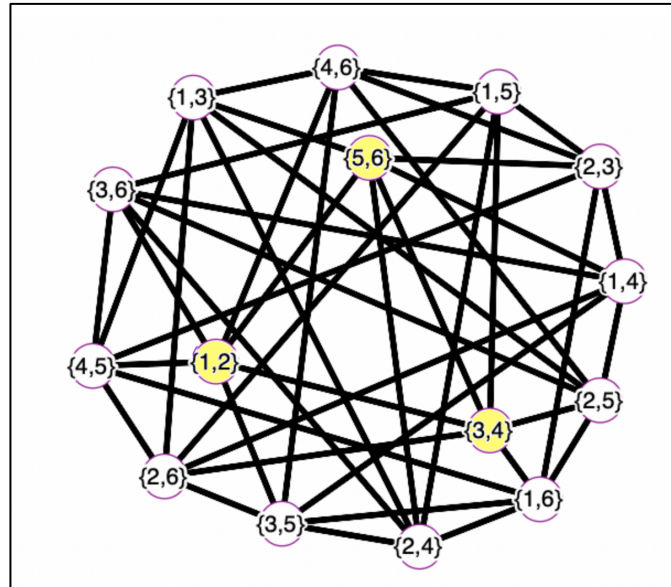


Figure 4.4: $\gamma_t(K(6,2)) = C(6,4,2) = 3$.

$$\gamma_t(K(6,2)) = C(6,4,2).$$

- *Minimum total dominating set of $K(6,2)$ is $\{\{1,2\}, \{3,4\}, \{5,6\}\}$, and $\gamma_t(K(6,2)) = 3$.*
- *$C(6,4,2) = 3$ and its blocks are $[3\ 4\ 5\ 6], [1\ 2\ 5\ 6], [1\ 2\ 3\ 4]$.*

There are some formulas for finding the domination and total domination number for small n and k values, but as the n and k values increase, it becomes difficult to find these values for Kneser graphs. Although there is no general bound for all n and k values, thanks to the relationship between covering design and Kneser graphs, more domination and total domination numbers are known.

5. JOHNSON GRAPH

In this chapter we will study domination and total domination numbers of the Johnson Graph. We have results on $J(n, 2)$ in particular. Before these results, we will first give some definitions and properties about Johnson graphs. These graphs are derived from Johnson schemes and are highly symmetrical and both of Johnson schemes and graphs introduced by Selmer M. Johnson.

Definition 5.1: Let i, j, k be non-negative integers. A graph G is called distance-regular graph if for arbitrary vertices of x, y with $d(x, y) = k$ the number of vertices z such that $d(x, z) = i$ and $d(y, z) = j$ is independent of the choice of x, y .

Definition 5.2: For a graph G , if for given arbitrary vertices u, v, w , and z with $d(u, v) = d(w, z)$, there is an automorphism that carries the first pair onto the second then G is called a distance-transitive graph.

Every distance transitive graph is distance regular, but the converse is not necessarily true [Adelson-Velsky et al., 1969].

Definition 5.3: [Godsil and Royle, 2001] Let n, k , and i be fixed positive integers, with $n \geq k \geq i$; let θ be a fixed set of size n ; and define $J(n, k, i)$ as follows. The vertices of $J(n, k, i)$ are the subsets of θ with size k , where two subsets are adjacent if their intersection has size i .

$J(n, k, i)$ has $\binom{n}{k}$ vertices, and it is a $\binom{k}{i}\binom{n-k}{k-i}$ -regular graph.

Lemma 5.1: [Godsil and Royle, 2001] If $n \geq k \geq i$ then $J(n, k, i) \cong J(n, n - k, n - 2k + i)$.

The graphs $J(n, k, 0)$ are known as the Kneser graphs and the Kneser graph $J(5, 2, 0)$ is well known as the Petersen graph. The graphs $J(n, k, k - 1)$ are known as the Johnson graphs for $n \geq 2k$ and in short Johnson graphs are denoted by $J(n, k)$.

Let us give a formal definition of a Johnson graph $J(n, k)$:

Definition 5.4: For integers $1 \leq k \leq n$, the Johnson graph $J(n, k)$ has vertex set consisting of all k -subsets chosen from n -set and two vertices are adjacent if and only if their intersection has exactly $k - 1$ elements.

- For $k = 1$, Johnson graph $J(n, 1)$ is a complete graph on n vertices.
- For $k = 2$, The Johnson graph $J(n, 2)$ is the complement of the Kneser graph $K(n, 2)$ and it is the line graph of the complete graph on n vertices, and also it is known as the triangular graph $T(n)$ for $n \geq 4$.

Note that, Johnson graphs are $k(n - k)$ - regular graphs.

The Johnson graph $J(n, 2, 1)$, namely $J(n, 2)$ is distance transitive and in general $J(n, k, k - 1)$ is vertex-transitive. The Johnson graph $J(n, k, k - 1)$ is also distance-transitive [Biggs, 1993]. Note that, any distance transitive graph is vertex-transitive but the converse is not true in general.

Odd graphs, O_{k+1} are $J(2k + 1, k, 0)$ and by properties $J(n, k, i)$, it is a regular graph with valency:

$$\binom{k}{0} \binom{k+1}{k} = k + 1. \quad (5.1)$$

The Odd graph O_{k+1} is distance transitive also it is a vertex transitive graph. In Theorem 4.1, we give a proof that all Kneser graphs are vertex transitive.

The Johnson graph $J(n, k)$ is a vertex transitive graph [Godsil and Royle, 2001]. For $k > 2$, Johnson graph is certainly non-Cayley [Konstantinova, 2013]. In general, complement of a Cayley graph may be or may not be Cayley. But our observations show that $J(n, 2)$ is non-Cayley when $K(n, 2)$ is non-Cayley except $n = 3, 4$ and $J(n, 2)$ is Cayley when $K(n, 2)$ is Cayley.

We will first give some examples of dominating sets and total dominating sets of Johnson graphs for some small values of n and k .

Example 5.1: Domination and total domination numbers of the $J(5, 2)$ and $J(6, 2)$ are shown below.

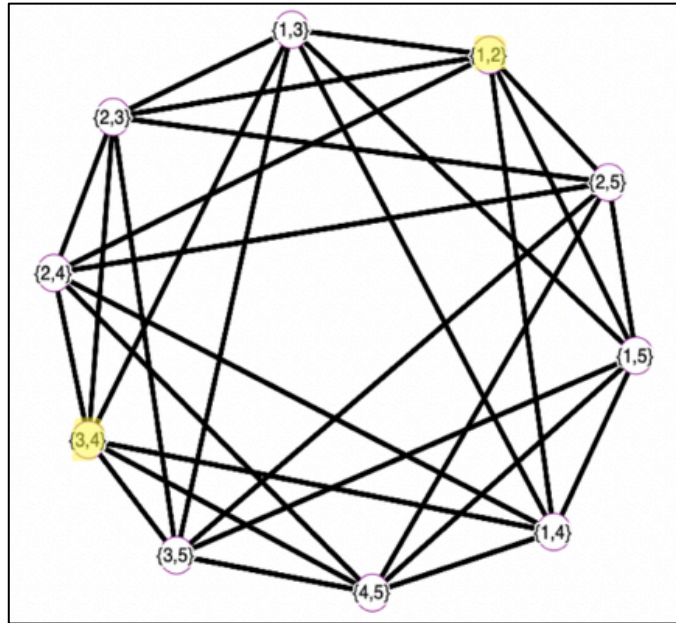


Figure 5.1: A Dominating set of Johnson Graph $J(5, 2)$.

- A minimum dominating set of $J(5, 2)$ is $\{\{1, 2\}, \{3, 4\}\}$, and $\gamma(J(5, 2)) = 2$.

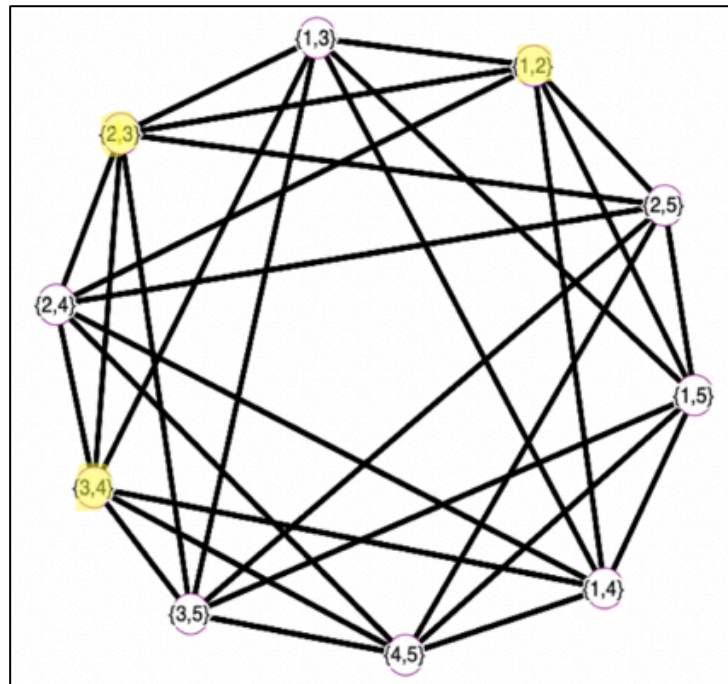


Figure 5.2: A Total Dominating set of Johnson Graph $J(5, 2)$.

- A minimum total dominating set of $J(5,2)$ is $\{\{1,2\}, \{2,3\}, \{3,4\}\}$, and $\gamma_t(J(5,2)) = 3$.

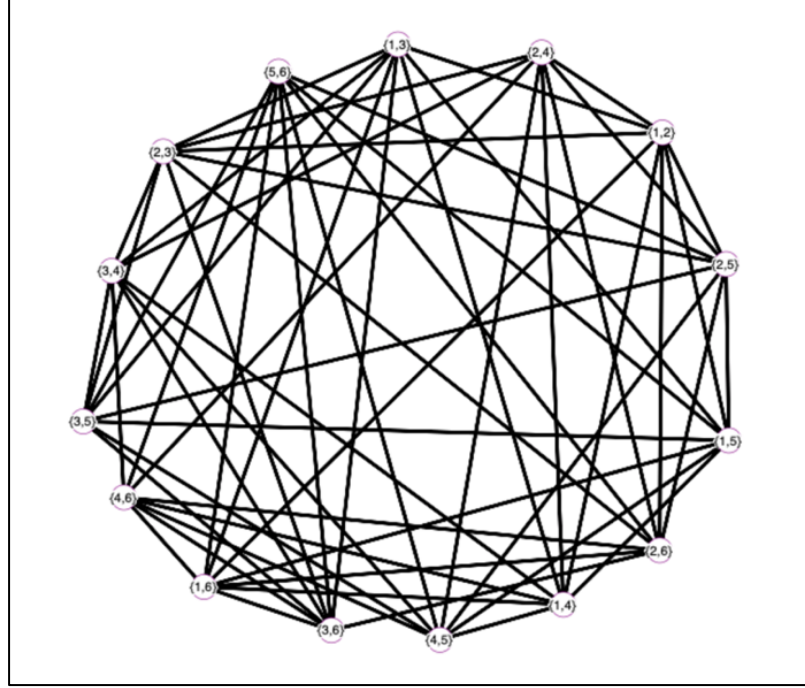


Figure 5.3: Johnson Graph $J(6,2)$.

- A minimum dominating set of $J(6,2)$ is $\{\{1,2\}, \{3,4\}, \{5,6\}\}$, and $\gamma(J(6,2)) = 3$.
- A minimum total dominating set of $J(6,2)$ is $\{\{1,2\}, \{2,3\}, \{4,5\}, \{5,6\}\}$, and $\gamma_t(J(6,2)) = 4$.

According to our observations, as n increases the domination and total domination numbers increase in Johnson graph for $k = 2$ unlike the Kneser graph. During our studies, we reached some results that gives domination number and total domination number for Johnson graphs for all n values and $k = 2$. For some cases, we give only bounds for these numbers.

First, we will give some results on domination number of the Johnson graph $J(n, 2)$. We investigate domination number in two cases: when n is even and when n is odd. To show the equality in Theorem 5.1 and Theorem 5.2, we first show that it is greater than or equal to $\frac{n}{2}$, then we will show that it is less than or equal to $\frac{n}{2}$. Then

we define a dominating set and we will show that this set dominates the remaining vertices.

Theorem 5.1: If $n \equiv 0 \pmod{2}$, then $\gamma(J(n, 2)) = \frac{n}{2}$.

Proof 5.1: Let n be even and $X = \{1, 2, \dots, n\}$. Let V be the vertex set of $J(n, 2)$, that is, 2-element subsets of X . Let S be a dominating set of $J(n, 2)$, and assume that $|S| < \frac{n}{2}$. If $|S| \leq \frac{n}{2} - 1$, then it covers at most $(n - 2)$ elements from X . So, there is one vertex pair (a, b) that can not be dominated by the vertices in S . So, $|S| \geq \frac{n}{2}$. Hence, $\gamma(J(n, 2)) \geq \frac{n}{2}$.

Now, let $S = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \dots, \{n - 1, n\}\}$ be a partition of X into 2-element sets. Let $\{i, j\}$ be a vertex in V .

- *If $\{i, j\} \in S$, then it dominates itself.*
- *If $\{i, j\} \notin S$, $\{i - 1, i\}$ and $\{i, i + 1\}$ (also $\{j - 1, j\}, \{j, j + 1\}$) are among the neighbors of $\{i, j\}$.*
 - *If i is odd, $\{i, i + 1\}$ is in S and dominates $\{i, j\}$.*
 - *If i is even, $\{i - 1, i\}$ is in S and dominates $\{i, j\}$.*

So, any $\{i, j\} \in V$ gets dominated by a vertex in S . So, $\gamma(J(n, 2)) \leq \frac{n}{2}$.

But we have shown that $\gamma(J(n, 2)) \geq \frac{n}{2}$ above. Hence, $\gamma(J(n, 2)) = \frac{n}{2}$. ■

Theorem 5.2: If $n \equiv 1 \pmod{2}$, then $\gamma(J(n, 2)) = \frac{n-1}{2}$.

Proof 5.2: Let n be odd and $X = \{1, 2, \dots, n\}$. Let V be the vertex set of $J(n, 2)$, that is, 2-element subsets of X . Let S be a dominating set of $J(n, 2)$, and assume that $|S| < \frac{n-1}{2}$. If $|S| \leq \frac{n-1}{2} - 1$, then it covers at most $(n - 3)$ elements from X . So, there is three vertex pair that cannot be dominated by the vertices in S . So, $|S| \geq \frac{n-1}{2}$. Hence, $\gamma(J(n, 2)) \geq \frac{n-1}{2}$.

Now, let $S = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \dots, \{n - 2, n - 1\}\}$ be a partition of $X - \{n\}$ into 2-element sets. Let $\{i, j\}$ be a vertex in V and $j > i$.

- If $\{i, j\} \in S$, then it dominates itself.
- If $\{i, j\} \notin S$, since $\{i - 1, i\}$ and $\{i, i + 1\}$, (also $\{j - 1, j\}, \{j, j + 1\}$) are among the neighbors of $\{i, j\}$.
 - If i is odd, $\{i, i + 1\}$ is in S and dominates $\{i, j\}$.
 - If i is even, $\{i - 1, i\}$ is in S and dominates $\{i, j\}$.

So, any $\{i, j\} \in V$ gets dominated by a vertex in S . So, $\gamma(J(n, 2)) \leq \frac{n-1}{2}$.

But we have shown that $\gamma(J(n, 2)) \geq \frac{n-1}{2}$ above. Hence $\gamma(J(n, 2)) = \frac{n-1}{2}$. ■

Now, we will give bounds on total domination number of Johnson graphs $J(n, 2)$.

Theorem 5.3: If $n \equiv 0 \pmod{3}$, then $\gamma_t(J(n, 2)) \leq \frac{2n}{3}$.

Proof 5.3: Let $n = 3s$ for some integer $s \geq 1$ and let $X = \{1, 2, 3, \dots, 3s\}$ and also let $S_t = \{\{1, 2\}, \{2, 3\}, \{4, 5\}, \{5, 6\}, \dots, \{3s - 2, 3s - 1\}, \{3s - 1, 3s\}\}$ be a subset of V . Let $\{i, j\}$ be a vertex in V and assume $j > i$.

If $\{i, j\} \in S_t$, then since $\{i - 1, i\}$ or $\{j, j + 1\} \in S_t$, $\{i, j\}$ is dominated by $\{i - 1, i\}$ or $\{j, j + 1\}$.

If $\{i, j\} \notin S_t$, then;

- If $i = 3k$ and
 - $j = 3k$, then $\{i - 1, i\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$,
 - $j = 3k + 1$, then $\{i - 1, i\}$ and $\{j, j + 1\}$ are in S_t , and dominate $\{i, j\}$,
 - $j = 3k + 2$, then $\{i - 1, i\}$, $\{j, j + 1\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$.
- If $i = 3k + 1$ and
 - $j = 3k$, then $\{i, i + 1\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$,
 - $j = 3k + 1$, then $\{i, i + 1\}$ and $\{j, j + 1\}$ are in S_t , and dominate $\{i, j\}$,
 - $j = 3k + 2$, then $\{i, i + 1\}$, $\{j, j + 1\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$.
- If $i = 3k + 2$ and

- $j = 3k$, then $\{i, i + 1\}$, $\{i - 1, i\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$,
- $j = 3k + 1$, then $\{i, i + 1\}$, $\{i - 1, i\}$ and $\{j, j + 1\}$ are in S_t , and dominate $\{i, j\}$,
- $j = 3k + 2$, then $\{i, i + 1\}$, $\{i - 1, i\}$, $\{j, j + 1\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$.

So, S_t is a total dominating set with $2s = \frac{2n}{3}$ elements, hence we can say that

$$\gamma_t(J(n, 2)) \leq \frac{2n}{3}. \quad \blacksquare$$

Theorem 5.4: If $n \equiv 1 \pmod{3}$ then, $\gamma_t(J(n, 2)) \leq \frac{2n-2}{3}$.

Proof 5.4: Let $n = 3s + 1$ for some integer $s \geq 1$ and let $X = \{1, 2, 3, \dots, 3s + 1\}$ and also let $S_t = \{\{1, 2\}, \{2, 3\}, \{4, 5\}, \{5, 6\}, \dots, \{3s - 2, 3s - 1\}, \{3s - 1, 3s\}\}$ be a subset of V . Let $\{i, j\}$ be a vertex in V and assume $j > i$.

If $\{i, j\} \in S_t$, then since $\{i - 1, i\}$ or $\{j, j + 1\} \in S_t$, $\{i, j\}$ is dominated by $\{i - 1, i\}$ or $\{j, j + 1\}$.

If $\{i, j\} \notin S_t$, then;

- *If $i = 3k$ and*
 - $j = 3k$, then $\{i - 1, i\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$,
 - $j = 3k + 1$, then $\{i - 1, i\}$ and $\{j, j + 1\}$ are in S_t , and dominate $\{i, j\}$,
 - $j = 3k + 2$, then $\{i - 1, i\}$, $\{j, j + 1\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$.
- *If $i = 3k + 1$ and*
 - $j = 3k$, then $\{i, i + 1\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$,
 - $j = 3k + 1$, then $\{i, i + 1\}$ and $\{j, j + 1\}$ are in S_t , and dominate $\{i, j\}$,
 - $j = 3k + 2$, then $\{i, i + 1\}$, $\{j, j + 1\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$.
- *If $i = 3k + 2$ and*

- $j = 3k$, then $\{i, i + 1\}$, $\{i - 1, i\}$ and $\{j - 1, j\}$ is in S_t , and dominate $\{i, j\}$,
- $j = 3k + 1$, then $\{i, i + 1\}$, $\{i - 1, i\}$ and $\{j, j + 1\}$ are in S_t , and dominate $\{i, j\}$,
- $j = 3k + 2$, then $\{i, i + 1\}$, $\{i - 1, i\}$, $\{j, j + 1\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$.
- If $j = 3s + 1$ and
 - $i = 3k$, then $\{i - 1, i\}$ is in S_t , and dominates $\{i, j\}$,
 - $i = 3k + 1$, then $\{i, i + 1\}$ is in S_t , and dominates $\{i, j\}$,
 - $i = 3k + 2$, then $\{i, i + 1\}$ and $\{i - 1, i\}$ are in S_t , and dominate $\{i, j\}$.

So, S_t is a total dominating set with $2s = \frac{2n-2}{3}$ elements, hence we can say that $\gamma_t(J(n, 2)) \leq \frac{2n-2}{3}$. ■

Theorem 5.5: If $n \equiv 2 \pmod{3}$, then $\gamma_t(J(n, 2)) \leq \frac{2n-1}{3}$.

Proof 5.5: Let $n = 3s + 2$ for some integer $s \geq 1$ and let $X = \{1, 2, 3, \dots, 3s + 2\}$ and also let

$$S_t = \{\{1, 2\}, \{2, 3\}, \{4, 5\}, \{5, 6\}, \dots, \{3s - 2, 3s - 1\}, \{3s - 1, 3s\}, \{3s, 3s + 1\}\}$$

be a subset of V . Let $\{i, j\}$ be a vertex in V and assume $j > i$.

If $\{i, j\} \in S_t$, then since $\{i - 1, i\}$ or $\{j, j + 1\} \in S_t$, $\{i, j\}$ is dominated by $\{i - 1, i\}$ or $\{j, j + 1\}$.

If $\{i, j\} \notin S_t$, then;

- If $i = 3k$ and
 - $j = 3k$, then $\{i - 1, i\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$,
 - $j = 3k + 1$, then $\{i - 1, i\}$ and $\{j, j + 1\}$ are in S_t , and dominate $\{i, j\}$,
 - $j = 3k + 2$, then $\{i - 1, i\}$, $\{j, j + 1\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$.
- If $i = 3k + 1$ and
 - $j = 3k$, then $\{i, i + 1\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$,

- $j = 3k + 1$, then $\{i, i + 1\}$ and $\{j, j + 1\}$ are in S_t , and dominate $\{i, j\}$,
- $j = 3k + 2$, then $\{i, i + 1\}$, $\{j, j + 1\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$.
- If $i = 3k + 2$ and
 - $j = 3k$, then $\{i, i + 1\}$, $\{i - 1, i\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$,
 - $j = 3k + 1$, then $\{i, i + 1\}$, $\{i - 1, i\}$ and $\{j, j + 1\}$ are in S_t , and dominate $\{i, j\}$,
 - $j = 3k + 2$, then $\{i, i + 1\}$, $\{i - 1, i\}$, $\{j, j + 1\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$.
- If $i = 3k$ and
 - $j = 3s + 2$, then $\{i - 1, i\}$ is in S_t , and dominates $\{i, j\}$,
 - $j = 3s + 1$, then $\{i - 1, i\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$.
- If $i = 3k + 1$ and
 - $j = 3s + 2$, then $\{i, i + 1\}$ is in S_t , and dominates $\{i, j\}$,
 - $j = 3s + 1$, then $\{i, i + 1\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$.
- If $i = 3k + 2$ and
 - $j = 3s + 2$, then $\{i, i + 1\}$ and $\{i - 1, i\}$ are in S_t , and dominate $\{i, j\}$,
 - $j = 3s + 1$, then $\{i, i + 1\}$, $\{i - 1, i\}$ and $\{j - 1, j\}$ are in S_t , and dominate $\{i, j\}$.
- If $j = 3s + 2$ and
 - $i = 3s$, then $\{i, i + 1\}$ and $\{i - 1, i\}$ are in S_t , and dominate $\{i, j\}$,
 - $i = 3s + 1$, then $\{i - 1, i\}$ is in S_t , and dominates $\{i, j\}$.

So, S_t is a total dominating set with $2s + 1 = \frac{2n-1}{3}$ elements, hence we can

say that $\gamma_t(J(n, 2)) \leq \frac{2n-1}{3}$. ■

6. CONCLUSION

In this thesis we have studied certain non-Cayley vertex-transitive graphs, and their domination and total domination numbers. Graphs included in this thesis were basically Kneser and Johnson graphs. Although there are many results for Kneser graphs, we could not see results on Johnson graphs and domination parameters. It is known and easy to show that $K(n, 2)$ and $J(n, 2)$ are complements of each other. Since the results for $K(n, 2)$ are known we have decided to analyze $J(n, 2)$ first. But then we have seen that their domination parameters are not related. For Kneser graphs the domination numbers are constant but for Johnson graph it increases with n .

To summarize, we can combine the results Theorem 5.1 and Theorem 5.2 to get the following result:

Theorem 6.1: For $n \geq 2$, $\gamma(J(n, 2)) = \left\lfloor \frac{n}{2} \right\rfloor$.

Also, for the total domination on Johnson graphs $J(n, 2)$ we gave upper bounds. Theorem 5.3, 5.4, and 5.5 can be summarized as follows:

Theorem 6.2: For $n \geq 3$, then $\gamma_t(J(n, 2)) \leq \left\lfloor \frac{2n}{3} \right\rfloor$.

We believe that this is actually an equality. In the future, we will try to prove that idea. Studying the cases for $J(n, k)$ when $k > 2$ is also a future plan but the number of edges increases very rapidly as k gets bigger. Finding a relation between designs and the domination numbers of Johnson graphs as in the case of Kneser graphs would also be a good result. We could not find a relation between Johnson graphs and transversals similar to the Kneser graph case yet. We will try to find this relation in the future.

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BIOGRAPHY

Halime İLHAN was born on the 29th of June 1993 in İstanbul, TURKEY. She graduated from Gebze Technical University, Faculty of Science, Mathematics Department in 2016. She worked as a teacher for a while. She has been a MSc student of Mathematics Department in Gebze Technical University Graduate School of Natural and Applied Sciences since 2017.

APPENDICES

Appendix A: Publications Based on The Thesis

İlhan, H., Özkan, S., (2019), “Domination on Non-Cayley Vertex Transitive Graphs”, GTÜ Fen Bilimleri Lisansüstü Araştırmalar Sempozyumu 2019, Gebze, Turkey, 17 – 18 Jun