

T.C.
GEBZE YÜKSEK TEKNOLOJİ ENSTİTÜSÜ
MÜHENDİSLİK VE FEN BİLİMLERİ ENSTİTÜSÜ

HİPERBOLİK-PARABOLİK
DİFERANSİYEL VE FARK
DENKLEMLERİ İÇİN LOKAL OLMAYAN
SINIR DEĞER PROBLEMLERİ

Yıldırım ÖZDEMİR

DOKTORA TEZİ

MATEMATİK

ANABİLİM DALI

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GEBZE

2007

ÖZET

TEZ KONUSU: Hiperbolik-Parabolik Diferansiyel Ve Fark Denklemleri İçin Lokal Olmayan Sınır Değer Problemleri

YAZAR ADI: Yıldırım ÖZDEMİR

Bu çalışmada Hilbert uzayında verilen fark denklemlerinin self-adjoint pozitif tanımlı A operatörlü çok noktalı lokal olmayan sınır değer problemi

$$\left\{ \begin{array}{l} \frac{d^2 u(t)}{dt^2} + Au(t) = f(t), \quad 0 \leq t \leq 1, \\ \frac{du(t)}{dt} + Au(t) = g(t), \quad -1 \leq t \leq 0, \\ u(-1) = \sum_{j=1}^N \alpha_j u(\mu_j) + \sum_{j=1}^L \beta_j u'(\lambda_j) + \varphi, \\ \sum_{j=1}^N |\alpha_j|, \sum_{j=1}^L |\beta_j| \leq 1, 0 < \mu_j, \lambda_j \leq 1 \end{array} \right.$$

ele alınmıştır. Operatör yaklaşımı uygulanarak bu problemin çözümünün kararlılık kestirimleri elde edilmiştir. Bu problemin yaklaşık çözümleri için fark şemalarının kararlılığı gösterilmiştir. Uygulamalarda bu sonuç, hiperbolik-parabolik denklemlerin fark şemalarının çözümü için kararlılık kestirimlerinin elde edilmesini sağlamıştır. Bu fark şemalarının çözümü için yapılan teorik sonuçların doğruluğu, sayısal denemelerle de desteklenmiştir.

Anahtar Kelimeler: Hiperbolik-Parabolik Fark Denklemleri, Fark Şemaları, Kararlılık

SUMMARY

TITLE: Nonlocal Boundary Value Problem for Hyperbolic-Parabolic Differential and Difference Equations

AUTHOR: Yıldırım ÖZDEMİR

In this work the multipoint nonlocal boundary value problem

$$\left\{ \begin{array}{l} \frac{d^2 u(t)}{dt^2} + Au(t) = f(t), \quad 0 \leq t \leq 1, \\ \frac{du(t)}{dt} + Au(t) = g(t), \quad -1 \leq t \leq 0, \\ u(-1) = \sum_{j=1}^N \alpha_j u(\mu_j) + \sum_{j=1}^L \beta_j u'(\lambda_j) + \varphi, \\ \sum_{j=1}^N |\alpha_j|, \sum_{j=1}^L |\beta_j| \leq 1, 0 < \mu_j, \lambda_j \leq 1 \end{array} \right.$$

for the differential equation in a Hilbert space H with the self-adjoint positive definite operator A is considered. Applying the operator approach, the stability estimates for the solution of this problem are obtained. The stability of difference schemes for approximately solving this problem are presented. In applications, this abstract results permit to obtain the stability estimates for the solution of the difference schemes for hyperbolic-parabolic equations. The theoretical statements for the solution of this difference schemes are supported by the results of numerical experiments.

Keywords: Hyperbolic-Parabolic Difference Equation, Difference Schemes, Stability

TEŞEKKÜR

Başlamış olduğum bu yolda, en başından sonuna dek tüm görüşlerini paylaşan,engin bilgi ve tecrübelerini hiçbir zaman esirgemeyen, bilim tutkusunu içinde barındıran,tüm sorularıma sabır ile yanıt veren ve her türlü konuda desteğini eksik etmeyerek yanımda olan saygıdeğer hocam sayın Prof.Dr. Allaberen ASHYRALYEV'e teşekkürü bir borç bilirim.

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1 GİRİŞ

Akışkanlar mekaniğindeki bir çok problemde, ısı akışı, füzyon süreci, matematiksel biyoloji ve diğer pek çok fiziksel alanlarda karşımıza hiperbolik-parabolik tipteki diferansiyel denklemler çıkmaktadır. Bu tip denklemlerle fiziksel olayların modelleri ifade edilebilir. Bu tipteki denklemler için sınır değer probleminin çözüm metotları üzerine kapsamlı olarak bir çok araştırma yapılmıştır. (bkz. [Korzyuk, V. I. ve Lemeshevsky, S. V., 2001], [Vallet, G., 2003], [Glazatov, S. N., 1998], [Karatoprakliev, G. D., 1989], [Karatopraklieva, M. G., 1991], [Gerish, A., Kotschote, M. ve Zacher, R., 2004], [Salahatdinov, M. S., 1974], [Drujaev, T. D., 1979], [Bazarov, D. ve Soltanov, H., 1995], [Vragov, V. N., 1983], [Nakhushev, A. M., 1995], [Ashyralyev, A. ve Yurtsever, A., 2005], [Ashyralyev, A. ve Yurtsever, A., 2004]).

Bu çalışmada, H Hilbert uzayında self-adjoint pozitif tanımlı A operatörlü

$$\begin{cases} \frac{d^2 u(t)}{dt^2} + Au(t) = f(t), & 0 \leq t \leq 1, \\ \frac{du(t)}{dt} + Au(t) = g(t), & -1 \leq t \leq 0, \\ u(-1) = \sum_{j=1}^N \alpha_j u(\mu_j) + \sum_{j=1}^L \beta_j u'(\lambda_j) + \varphi, \\ \sum_{j=1}^N |\alpha_j|, \sum_{j=1}^L |\beta_j| \leq 1, 0 < \mu_j, \lambda_j \leq 1 \end{cases} \quad (1.1)$$

çok noktalı lokal olmayan sınır değer problemi ele alınmıştır. Bilindiği gibi, hiperbolik-parabolik denklemler için çeşitli lokal olmayan sınır değer problemleri (1.1) lokal olmayan sınır değer problemine dönüştürülebilir. (bkz. [Salahatdinov, M.S., 1974], [Dju-raev, T. D., 1978], [Bazarov, D. ve Soltanov, H., 1995], [Krein, S. G., 1966]).

Aşağıdaki şartları sağlayan $u(t)$ fonksiyonu (1.1) problemin çözümüdür:

i. $u(t)$ fonksiyonu $(0, 1]$ aralığında ikinci türevi sürekli olan ve $[-1, 1]$ aralığında türevi sürekli olan bir fonksiyondur. Aralığın sınır noktalarındaki türevler, uygun tek taraflı türevler olarak anlaşılır.

ii. $u(t)$ fonksiyonu, A operatörünün tanım kümesinin elemanıdır ve $Au(t)$ fonksiyonu $[-1, 1]$ aralığında süreklidir.

iii. $u(t)$ fonksiyonu, (1.1) denklemini ve bu denklemin lokal olmayan sınır koşulunu sağlar.

Eğer $f(t)$, $[0, 1]$ aralığında ve $g(t)$, $[-1, 0]$ aralığında türevleri sürekli olan iki fonksiyon ise, [Ashyralyev, A. ve Yurtsever, A., 2001] makalesinde (1.1) probleminin kararlılık

kestirimleri ispatlanmıştır.

Teorem 1.1. Eğer $\varphi \in D(A)$ ve $f(t)$ fonksiyonu, $[0, 1]$ aralığında, $g(t)$ fonksiyonu $[-1, 0]$ aralığında türevleri sürekli ve $\alpha_j = 0$, $j = 2, \dots, N$; $\beta_j = 0$, $j = 1, \dots, L$ ise, bu durumda (1.1) probleminin tek çözümü vardır ve bu çözüm için aşağıdaki

$$\begin{aligned} & \max_{-1 \leq t \leq 1} \|u(t)\|_H \leq M [\|\varphi\|_H \\ & + \max_{-1 \leq t \leq 0} \|g(t)\|_H + \max_{0 \leq t \leq 1} \|A^{-1/2} f(t)\|_H] , \\ & \max_{-1 \leq t \leq 1} \|A^{1/2} u(t)\|_H \leq M [\|A^{1/2} \varphi\|_H \\ & + \|g(0)\|_H + \int_{-1}^0 \|g'(t)\|_H dt + \max_{0 \leq t \leq 1} \|f(t)\|_H] , \\ & \max_{-1 \leq t \leq 0} \left\| \frac{du(t)}{dt} \right\|_H + \max_{0 \leq t \leq 1} \left\| \frac{d^2 u(t)}{dt^2} \right\|_H \\ & + \max_{-1 \leq t \leq 1} \|Au(t)\|_H \leq M [\|A\varphi\|_H + \|A^{1/2} g(0)\|_H \\ & + \|f(0)\|_H + \max_{-1 \leq t \leq 0} \|g'(t)\|_H + \int_0^1 \|f'(t)\|_H dt] \end{aligned}$$

kararlılık kestirimleri sağlanır. Burada, M katsayısı $f(t)$, $g(t)$ ve φ 'den bağımsızdır.

Bu çalışmanın ardından, $\alpha_j = 0$, $j = 2, \dots, N$ ve $\beta_j = 0$, $j = 1, \dots, L$ olduğu durumlarda, (1.1) probleminin sayısal çözümlerini bulmaya yönelik metotlar kapsamlı olarak araştırılmıştır. (bkz. [Ashyralyev, A. ve Yurtsever, A., 2000], [Ashyralyev, A. ve Yurtsever, A., 2001], [Ashyralyev, A. ve Yurtsever, A., 2005], [Ashyralyev, A. ve Yurtsever, A., 2006], [Ashyralyev, A. ve Muradov, I., 1995], [Ashyralyev, A. ve Muradov, I., 1996], [Ashyralyev, A. ve Muradov, I., 1998], [Ashyralyev, A. ve Orazov, M. B., 1999], [Ashyralyev, A. ve Orazov, M. B., 1999], [Ashyralyev, A., 1999]).

Örneğin; [Ashyralyev, A. ve Yurtsever, A., 2000] ve [Ashyralyev, A. ve Yurtsever, A., 2001] makalelerinde, (1.1) probleminin yaklaşık çözümünü bulmak için $A^{1/2}$ operatörü tarafından oluşturulan birinci ve ikinci basamaktan doğruluklu fark şemaları kurulmuştur. Bu fark şemalarının çözümleri için kararlılık kestirimleri elde edilmiştir. Uygulamalarda, hiperbolik-parabolik denklemler için karma tipli sınır değer problemlerinin fark şemalarının çözümlerinin kararlılığı gösterilmiştir. Ancak, bu fark denklemlerinin pratikte kullanılması için, ilk önce $A^{1/2}$ operatörünün oluşturulması gerekmektedir. Fakat, bu işlem bilgisayar için oldukça güçtür. Bu yüzden, iyi teorik sonuçlar

elde edilmesine rağmen, bu problemin sayısal çözümlerini bulmak zordur ve değeri beklenildiği gibi değildir.

Daha sonraları, [Ashyralyev, A. ve Yurtsever, A., 2005] ve [Ashyralyev, A. ve Yurtsever, A., 2006] makalelerinde, (1.1) probleminin yaklaşık çözümünü bulmak için tamsayı kuvvetli A operatörü tarafından oluşturulan birinci ve ikinci basamaktan doğruluklu fark şemaları kurulmuştur. Bu fark şemalarının çözümleri için kararlılık kestirimleri elde edilmiştir. İlgili fark şemalarının doğruluğu sayısal deneylerle de desteklenmiştir.

Son olarak, bazı özel durumlar için benzer sonuçlar, daha erken yıllarda [Ashyralyev, A. ve Muradov, I., 1995], [Ashyralyev, A. ve Muradov, I., 1996], [Ashyralyev, A. ve Muradov, I., 1998], [Ashyralyev, A. ve Orazov, M. B., 1999], [Ashyralyev, A. ve Orazov, M. B., 1999], [Ashyralyev, A., 1999] çalışmalarında karşımıza çıkmaktadır.

Yapılan bu çalışmalardan başka, [Ashyralyev, A. ve Ozdemir, Y., 2005] makalesinde, $\alpha_i = 0, i = 2, \dots, N$ ve $\beta_i = 0, i = 2, \dots, L$ olduğu durumlarda da, $f(t)$ fonksiyonunun $[0, 1]$ aralığında ve $g(t)$ fonksiyonunun $[-1, 0]$ aralığında türevlerinin sürekli olmasından daha güçlü bir koşul ile, (1.1) probleminin çözümünün kararlılık kestirimleri elde edilmiştir.

Teorem 1.2. Eğer $\varphi \in D(A), g(0) \in D(A^{1/2}), g'(0) \in H, f(0) \in D(A^{1/2}), f'(0) \in H$ ve $f(t), [0, 1]$ aralığında, $g(t), [-1, 0]$ aralığında ikinci türevleri sürekli olan iki fonksiyon ve $\alpha_j = 0, j = 2, \dots, N; \beta_j = 0, j = 2, \dots, L$ ise, (1.1) probleminin tek çözümü vardır ve bu çözüm için aşağıdaki

$$\begin{aligned}
& \max_{-1 \leq t \leq 1} \|u(t)\|_H \leq M \left[\|\varphi\|_H + \max_{-1 \leq t \leq 0} \|A^{-1/2} g'(t)\|_H \right. \\
& \left. + \|A^{-1/2} g(0)\|_H + \|A^{-1/2} f(0)\|_H + \max_{0 \leq t \leq 1} \|A^{-1/2} f'(t)\|_H \right], \\
& \max_{-1 \leq t \leq 1} \left\| \frac{du}{dt} \right\|_H + \max_{-1 \leq t \leq 1} \|A^{1/2} u(t)\|_H \leq M \left[\|A^{1/2} \varphi\|_H \right. \\
& \left. + \|g(0)\|_H + \max_{-1 \leq t \leq 0} \|g'(t)\|_H + \|f(0)\|_H + \max_{0 \leq t \leq 1} \|f'(t)\|_H \right], \\
& \max_{-1 \leq t \leq 0} \left\| \frac{du}{dt} \right\|_H + \max_{0 \leq t \leq 1} \left\| \frac{d^2 u}{dt^2} \right\|_H + \max_{-1 \leq t \leq 1} \|Au(t)\|_H \\
& \leq M \left[\|A\varphi\|_H + \|A^{1/2} g(0)\|_H + \|g'(0)\|_H + \max_{-1 \leq t \leq 0} \|g''(t)\|_H \right. \\
& \left. + \|A^{1/2} f(0)\|_H + \|f'(0)\|_H + \max_{0 \leq t \leq 1} \|f''(t)\|_H \right]
\end{aligned}$$

kararlılık kestirimleri sağlanır. Burada, M katsayısı $f(t), g(t)$ ve φ 'den bağımsızdır.

Bu çalışmanın ardından, $\alpha_j = 0, j = 2, \dots, N$ ve $\beta_j = 0, j = 2, \dots, L$ olduğu durumlarda, (1.1) probleminin sayısal çözümlerini bulmaya yönelik metotlar kapsamlı olarak araştırılmıştır. (bkz. [Ozdemir, Y., 2005], [Ashyralyev, A. ve Ozdemir, Y., 2005]). Yani, (1.1) probleminin yaklaşık çözümünü bulmak için birinci ve ikinci basamaktan doğruluklu fark şemaları kurulmuştur. Bu fark şemalarının çözümü için kararlılık kestirimleri elde edilmiştir. Uygulamalarda, hiperbolik-parabolik denklemler için karma tipli sınır değer problemlerinin fark şemalarının çözümlerinin kararlılığı ispatlamıştır. Bu fark şemalarının çözümü için elde edilen teorik sonuçların doğruluğu, sayısal deneylerle de desteklenmiştir.

Bu tezdeki amaç, $f(t), [0, 1]$ aralığında ve $g(t), [-1, 0]$ aralığında ikinci türevleri sürekli olan iki fonksiyon ve $\sum_{j=1}^N |\alpha_j|, \sum_{j=1}^L |\beta_j| \leq 1, 0 < \mu_j, \lambda_j \leq 1$ olduğu zaman, çok noktalı (1.1) probleminin çözümünün kararlılığını araştırmaktır.

Öncelikle, tezin içerdiği konu başlıklarından kısaca bahsedelim. Bu çalışma altı bölümden oluşmaktadır.

Birinci bölüm giriş kısmıdır.

İkinci bölüm iki kısımdan oluşmaktadır. Birinci kısımda; H Hilbert uzayında, hiperbolik-parabolik denklemler için çok noktalı lokal olmayan sınır değer probleminin kararlılığı hakkındaki temel teorem ispat edilmiştir. İkinci kısımda ise; birinci kısımda elde edilen soyut sonuç kullanılarak, uygulamalarda hiperbolik-parabolik denklemler için fark şemalarının çözümünün kararlılık kestirimi elde edilmiştir.

Üçüncü bölüm iki kısımdan oluşmaktadır. İlk kısımda; H Hilbert uzayında self-adjoint pozitif tanımlı A operatörlü, hiperbolik-parabolik denklemler için lokal olmayan sınır değer problemlerinin yaklaşık çözümlerini elde etmek amacıyla, kararlı bi-rinci basamaktan doğruluklu fark şeması kurulmuştur. Ayrıca, bu fark şemalarının çözümleri için kararlılık kestirimleri ispat edilmiştir. İkinci kısımda ise uygulamalar verilmiştir.

Dördüncü bölüm üç kısımdan oluşmaktadır. H Hilbert uzayında self-adjoint pozitif tanımlı A operatörlü, hiperbolik-parabolik denklemler için lokal olmayan sınır değer problemlerinin yaklaşık çözümlerini elde etmek amacıyla, kararlı ikinci

basamaktan doğruluklu fark şemaları kurulmuş, ilgili fark şemalarının çözümleri için kararlılık kestirimleri kanıtlanmış ve uygulamaları verilmiştir.

Beşinci bölüm sayısal sonuçlar kısmıdır. Birinci ve ikinci basamaktan doğruluklu fark şemaları ile çalışılmış ve uygun matlab programları yazılarak ikinci basamaktan doğruluklu fark şemalarının, birinci basamaktan doğruluklu fark şemalarına oranla daha doğru olduğu görülmüştür. Ayrıca bu bölüm, tüm şekiller ve tabloları da içermektedir.

Altıncı bölüm sonuçlardır.

Ayrıca tezin sonunda kaynaklar ve ekler kısımları da verilmiştir.

2 HİPERBOLİK-PARABOLİK DİFERANSİYEL DENKLEM İÇİN ÇOK NOKTALI LOKAL OLMAYAN SINIR DEĞER PROBLEMİ

2.1 Temel Teorem

H Hilbert uzayında, A self-adjoint pozitif tanımlı operatör ve $A \geq \delta I$, $\delta > \delta_0 > 0$ 'dır. Bu çalışma boyunca, güçlü sürekli kosinüs operatör-fonksiyonu $c(t)$, ($t \geq 0$)

$$c(t) = \frac{e^{itA^{1/2}} + e^{-itA^{1/2}}}{2}$$

formülü ile tanımlanır. Bu durumda,

$$s(t)u = \int_0^t c(s)uds$$

sinüs operatör-fonksiyonu $s(t)$ 'nin tanımından

$$s(t) = A^{-1/2} \frac{e^{itA^{1/2}} - e^{-itA^{1/2}}}{2i}$$

formülü ortaya çıkar. Kosinüs operatör-fonksiyonlar teorisi için [Fattorini, H. O., 1985] ve [Piskarev, S. ve Shaw, Y., 1997] kaynaklarına başvurulmuştur.

Öncelikle ileride ihtiyaç duyacağımız yardımcı teoremleri verelim.

Yardımcı Teorem 2.1. Aşağıdaki

$$\|c(t)\|_{H \rightarrow H} \leq 1, \|A^{1/2}s(t)\|_{H \rightarrow H} \leq 1, \quad (2.1)$$

$$\|A^\gamma e^{-tA}\|_{H \rightarrow H} \leq t^{-\gamma} e^{-\delta t}, t > 0, 0 \leq \gamma \leq 1, \delta > 0 \quad (2.2)$$

kestirimler sağlanır.

Yardımcı Teorem 2.2. Öncelikle

$$I - \sum_{j=1}^N \alpha_j [c(\mu_j) - As(\mu_j)] e^{-A} + \sum_{j=1}^L \beta_j [s(\lambda_j) + c(\lambda_j)] Ae^{-A}$$

operatörünün tersi vardır ve

$$T = \left(I - \sum_{j=1}^N \alpha_j [c(\mu_j) - As(\mu_j)] e^{-A} + \sum_{j=1}^L \beta_j [s(\lambda_j) + c(\lambda_j)] Ae^{-A} \right)^{-1}$$

aşağıdaki

$$\|T\|_{H \rightarrow H} \leq M \quad (2.3)$$

kestirim sağlanır. Burada, M katsayısı α_j, β_j, μ_j ve λ_j 'den bağımsızdır.

İspat. Burada, (2.3) kestiriminin ispatı

$$\left\| -\sum_{j=1}^N \alpha_j [c(\mu_j) - As(\mu_j)] e^{-A} + \sum_{j=1}^L \beta_j [s(\lambda_j) + c(\lambda_j)] Ae^{-A} \right\|_{H \rightarrow H} < 1 \quad (2.4)$$

eşitsizliğine dayanmaktadır. $c(\mu)$ ve $s(\mu)$ fonksiyonlarının tanımları ve A operatörünün self-adjoint pozitif tanımlı olması özellikleri kullanılarak

$$\begin{aligned} & \left\| -\sum_{j=1}^N \alpha_j [c(\mu_j) - As(\mu_j)] e^{-A} + \sum_{j=1}^L \beta_j [s(\lambda_j) + c(\lambda_j)] Ae^{-A} \right\|_{H \rightarrow H} \\ & \leq \sup_{\delta \leq \rho < \infty} \left| -\sum_{j=1}^N \alpha_j [\cos(\sqrt{\rho}\mu_j) - \sqrt{\rho} \sin(\sqrt{\rho}\mu_j)] \right. \\ & \quad \left. + \sum_{j=1}^L \beta_j [\sqrt{\rho} \sin(\sqrt{\rho}\lambda_j) + \rho \cos(\sqrt{\rho}\lambda_j)] \right| e^{-\rho} \end{aligned}$$

kestirimi elde edilir. Buradan,

$$\cos(\sqrt{\rho}\mu_j) - \sqrt{\rho} \sin(\sqrt{\rho}\mu_j) = \sqrt{\rho+1} \cos(\sqrt{\rho}\mu_j - \mu_{j0}),$$

$$\sqrt{\rho} \sin(\sqrt{\rho}\lambda_j) + \rho \cos(\sqrt{\rho}\lambda_j) = \sqrt{\rho} \sqrt{\rho+1} \cos(\sqrt{\rho}\lambda_j - \lambda_{j0})$$

olduğu için

$$\begin{aligned} & \left\| -\sum_{j=1}^N \alpha_j [c(\mu_j) - As(\mu_j)] e^{-A} + \sum_{j=1}^L \beta_j [s(\lambda_j) + c(\lambda_j)] Ae^{-A} \right\|_{H \rightarrow H} \\ & \leq \sup_{\delta \leq \rho < \infty} \left[\sum_{j=1}^N |\alpha_j| |\cos(\sqrt{\rho}\mu_j) - \sqrt{\rho} \sin(\sqrt{\rho}\mu_j)| \right. \\ & \quad \left. + \sum_{j=1}^L |\beta_j| |\sqrt{\rho} \sin(\sqrt{\rho}\lambda_j) + \rho \cos(\sqrt{\rho}\lambda_j)| \right] e^{-\rho} \\ & \leq \sup_{\delta \leq \rho < \infty} \left[\sum_{j=1}^N |\alpha_j| \sqrt{\rho+1} + \sum_{j=1}^L |\beta_j| \sqrt{\rho} \sqrt{\rho+1} \right] e^{-\rho} \\ & \leq \sup_{0 < \delta_0 < \delta \leq \rho < \infty} \sqrt{\rho+1} (1 + \sqrt{\rho}) e^{-\rho} \end{aligned}$$

kestirimi alınır. Son olarak, $\sup_{0 < \delta_0 < \delta \leq \rho < \infty} \sqrt{\rho+1} (1 + \sqrt{\rho}) e^{-\rho} < 1$ olması nedeniyle, (2.4) kestirimi elde edilir ve böylelikle Yardımcı Teorem 2.2 ispatlanmış olur.

Şimdi, (1.1) probleminin çözümü için gerekli formüller elde edilecektir. Bilindiği gibi,

$$\begin{cases} u''(t) + Au(t) = f(t), & 0 \leq t \leq 1, \\ u(0) = u_0, \quad u'(0) = u'_0, \end{cases} \quad (2.5)$$

$$\begin{cases} u'(t) + Au(t) = g(t), & -1 \leq t \leq 0, \\ u(-1) = u_{-1} \end{cases} \quad (2.6)$$

başlangıç değer problemlerinin

$$u(t) = c(t)u(0) + s(t)u'(0) + \int_0^t s(t-y)f(y)dy, \quad 0 \leq t \leq 1 \quad (2.7)$$

ve

$$u(t) = e^{-(t+1)A}u_{-1} + \int_{-1}^t e^{-(t-y)A}g(y)dy, \quad -1 \leq t \leq 0 \quad (2.8)$$

formüllerini sağlayan tek çözümleri vardır. Buradaki (2.7), (2.8) formülleri ve (1.1) denklemi kullanılarak,

$$\begin{aligned} u(t) = [c(t) - As(t)] & \left\{ e^{-A}u_{-1} + \int_{-1}^0 e^{yA}g(y)dy \right\} \\ & + s(t)g(0) + \int_0^t s(t-y)f(y)dy \end{aligned} \quad (2.9)$$

formülü elde edilir. Daha sonra,

$$u(-1) = \sum_{j=1}^N \alpha_j u(\mu_j) + \sum_{j=1}^L \beta_j u'(\lambda_j) + \varphi$$

koşulu kullanılarak,

$$\begin{aligned} & \left\{ I - \sum_{j=1}^N \alpha_j [c(\mu_j) - As(\mu_j)] e^{-A} + \sum_{j=1}^L \beta_j [s(\lambda_j) + c(\lambda_j)] Ae^{-A} \right\} u_{-1} \\ & = \sum_{j=1}^N \alpha_j \left\{ [c(\mu_j) - As(\mu_j)] \int_{-1}^0 e^{yA}g(y)dy \right. \\ & \quad \left. + s(\mu_j)g(0) + \int_0^{\mu_j} s(\mu_j - y)f(y)dy \right\} \\ & + \sum_{j=1}^L \beta_j \left\{ [-As(\lambda_j) - Ac(\lambda_j)] \int_{-1}^0 e^{yA}g(y)dy \right. \\ & \quad \left. + c(\lambda_j)g(0) + \int_0^{\lambda_j} c(\lambda_j - y)f(y)dy \right\} + \varphi \end{aligned} \quad (2.10)$$

operatör denklemi sağlanır. Bilindiği gibi,

$$I - \sum_{j=1}^N \alpha_j [c(\mu_j) - As(\mu_j)] e^{-A} + \sum_{j=1}^L \beta_j [s(\lambda_j) + c(\lambda_j)] Ae^{-A}$$

operatörünün

$$T = \left(I - \sum_{j=1}^N \alpha_j [c(\mu_j) - As(\mu_j)] e^{-A} + \sum_{j=1}^L \beta_j [s(\lambda_j) + c(\lambda_j)] A e^{-A} \right)^{-1}$$

tersi var olduğundan, (2.10) operatör denkleminin çözümü için

$$\begin{aligned} u_{-1} = T & \left[\sum_{j=1}^N \alpha_j \{ [c(\mu_j) - As(\mu_j)] \int_{-1}^0 e^{yA} g(y) dy \right. \\ & + s(\mu_j) g(0) + \int_0^{\mu_j} s(\mu_j - y) f(y) dy \} \\ & + \sum_{j=1}^L \beta_j \{ [-As(\lambda_j) - Ac(\lambda_j)] \int_{-1}^0 e^{yA} g(y) dy \\ & \left. + c(\lambda_j) g(0) + \int_0^{\lambda_j} c(\lambda_j - y) f(y) dy \} + \varphi \right] \end{aligned} \quad (2.11)$$

formülü elde edilir. Böylece, (1.1) lokal olmayan sınır değer probleminin çözümü için, (2.8), (2.9) ve (2.11) formülleri belirlenmiş olur.

Teorem 2.1. Eğer $\varphi \in D(A)$, $g(0) \in D(A^{1/2})$, $g'(0) \in H$, $f(0) \in D(A^{1/2})$, $f'(0) \in H$ ve $f(t)$, $[0, 1]$ aralığında, $g(t)$, $[-1, 0]$ aralığında ikinci türevleri sürekli olan iki fonksiyon ise, bu durumda (1.1) probleminin tek çözümü vardır ve bu çözüm için aşağıdaki

$$\begin{aligned} \max_{-1 \leq t \leq 1} \|u(t)\|_H & \leq M \left[\|\varphi\|_H + \max_{-1 \leq t \leq 0} \|A^{-1/2} g'(t)\|_H \right. \\ & \left. + \|A^{-1/2} g(0)\|_H + \|A^{-1/2} f(0)\|_H + \max_{0 \leq t \leq 1} \|A^{-1/2} f'(t)\|_H \right], \end{aligned} \quad (2.12)$$

$$\begin{aligned} \max_{-1 \leq t \leq 1} \left\| \frac{du}{dt} \right\|_H & + \max_{-1 \leq t \leq 1} \|A^{1/2} u(t)\|_H \leq M [\|A^{1/2} \varphi\|_H \\ & + \|g(0)\|_H + \max_{-1 \leq t \leq 0} \|g'(t)\|_H + \|f(0)\|_H + \max_{0 \leq t \leq 1} \|f'(t)\|_H], \end{aligned} \quad (2.13)$$

$$\begin{aligned} \max_{-1 \leq t \leq 0} \left\| \frac{du}{dt} \right\|_H & + \max_{0 \leq t \leq 1} \left\| \frac{d^2 u}{dt^2} \right\|_H + \max_{-1 \leq t \leq 1} \|Au(t)\|_H \\ & \leq M \left[\|A\varphi\|_H + \|A^{1/2} g(0)\|_H + \|g'(0)\|_H + \max_{-1 \leq t \leq 0} \|g''(t)\|_H \right. \\ & \left. + \|A^{1/2} f(0)\|_H + \|f'(0)\|_H + \max_{0 \leq t \leq 1} \|f''(t)\|_H \right] \end{aligned} \quad (2.14)$$

kararlılık kestirimleri sağlanır. Burada, M katsayısı $f(t)$, $g(t)$ ve φ 'den bağımsızdır.

İspat. İlk olarak, (2.12) kestirimi elde edilecektir. Bunun için (2.11) formülü ve kısmi integrasyon kullanılarak,

$$\begin{aligned}
u_{-1} = T & \left[\sum_{j=1}^N \alpha_j \left\{ c(\mu_j) \left[A^{-1} (g(0) - e^{-A}g(-1)) \right. \right. \right. \\
& \left. \left. - \int_{-1}^0 e^{yA} g'(y) dy \right) \right] + As(\mu_j) \left(e^{-A}g(-1) + \int_{-1}^0 e^{yA} g'(y) dy \right) \\
& + A^{-1} \left[f(\mu_j) - c(\mu_j) f(0) - \int_0^{\mu_j} c(\mu_j - y) f'(y) dy \right] \Big\} \\
& + \sum_{j=1}^L \beta_j \left\{ -As(\lambda_j) \left(g(0) - e^{-A}g(-1) - \int_{-1}^0 e^{yA} g'(y) dy \right) \right. \\
& \quad + Ac(\lambda_j) \left(e^{-A}g(-1) + \int_{-1}^0 e^{yA} g'(y) dy \right) \\
& \quad \left. + As(\lambda_j) f(0) + \int_0^{\lambda_j} s(\lambda_j - y) f'(y) dy \right\} + \varphi \Big]
\end{aligned} \tag{2.15}$$

formülü elde edilir. Buradan, (2.3), (2.1) ve (2.2) kestirimleri yardımıyla,

$$\begin{aligned}
\|u_{-1}\|_H & \leq M \left[\|\varphi\|_H + \max_{-1 \leq t \leq 0} \|A^{-1/2} g'(t)\|_H \right. \\
& \quad \left. + \|A^{-1/2} g(0)\|_H + \|A^{-1/2} f(0)\|_H + \max_{0 \leq t \leq 1} \|A^{-1/2} f'(t)\|_H \right]
\end{aligned} \tag{2.16}$$

kestirimi bulunur. Daha sonra, (2.8), (2.9) formülleri ve kısmi integrasyon kullanılarak,

$$\begin{aligned}
u(t) = e^{-(t+1)A} u_{-1} + A^{-1} & \left(g(t) - e^{-A}g(-1) \right. \\
& \left. - \int_{-1}^t e^{yA} g'(y) dy, \quad -1 \leq t \leq 0, \right)
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
u(t) = [c(t) - As(t)] & \left\{ e^{-A} u_{-1} \right. \\
& + A^{-1} \left(g(0) - e^{-A}g(-1) - \int_{-1}^0 e^{yA} g'(y) dy \right) \Big\} + s(t) g(0) \\
& + A^{-1} \left[f(t) - c(t) f(0) - \int_0^t c(t-y) f'(y) dy, \quad 0 \leq t \leq 1 \right]
\end{aligned} \tag{2.18}$$

formülleri elde edilir. Son olarak, (2.1) ve (2.2) kestirimleri kullanılarak,

$$\|u(t)\|_H \leq M \left[\|u_{-1}\|_H + \max_{-1 \leq t \leq 0} \|A^{-1/2} g'(t)\|_H + \|A^{-1/2} g(0)\|_H \right], \quad -1 \leq t \leq 0,$$

$$\|u(t)\|_H \leq M \left[\max_{0 \leq t \leq 1} \|A^{-1/2} f'(t)\|_H + \|A^{-1/2} f(0)\|_H \right]$$

$$+ \|u_{-1}\|_H + \max_{-1 \leq t \leq 0} \|A^{-1/2}g'(t)\|_H + \|A^{-1/2}g(0)\|_H \Big], 0 \leq t \leq 1$$

kestirimleri alınır. Böylece, (2.16) kestirimi ve son iki kestirim yardımıyla (2.12) kestirimi ispatlanmış olur.

İkinci olarak, (2.13) kestirimi elde edilecektir. Bunun için öncelikle, (2.15) formülüne $A^{1/2}$ uygulanarak ve (2.3), (2.1), (2.2) kestirimleri kullanılarak,

$$\begin{aligned} \|A^{1/2}u_{-1}\|_H &\leq M \left[\|A^{1/2}\varphi\|_H + \max_{-1 \leq t \leq 0} \|g'(t)\|_H \right. \\ &\quad \left. + \|g(0)\|_H + \|f(0)\|_H + \max_{0 \leq t \leq 1} \|f'(t)\|_H \right] \end{aligned} \quad (2.19)$$

kestirimi elde edilir. Sonra da (2.17), (2.18) formüllerine $A^{1/2}$ uygulanarak ve (2.1), (2.2) kestirimleri kullanılarak,

$$\|A^{1/2}u(t)\|_H \leq M \left[\|A^{1/2}u_{-1}\|_H + \max_{-1 \leq t \leq 0} \|g'(t)\|_H + \|g(0)\|_H \right], -1 \leq t \leq 0,$$

$$\begin{aligned} \|A^{1/2}u(t)\|_H &\leq M \left[\max_{0 \leq t \leq 1} \|f'(t)\|_H + \|f(0)\|_H \right. \\ &\quad \left. + \|A^{1/2}u_{-1}\|_H + \max_{-1 \leq t \leq 0} \|g'(t)\|_H + \|g(0)\|_H \right], 0 \leq t \leq 1 \end{aligned}$$

kestirimleri alınır. Böylece, (2.19) kestirimi ve son iki kestirim yardımıyla (2.13) kestirimi ispatlanmış olur.

Üçüncü olarak, (2.14) kestirimi elde edilecektir. Bunun için, (2.15) formülü ve kısmi integrasyon kullanılarak,

$$\begin{aligned} u_{-1} &= T \left[\sum_{j=1}^N \alpha_j \left\{ c(\mu_j) \left[A^{-1} (g(0) - e^{-A}g(-1) \right. \right. \right. \\ &\quad \left. \left. - A^{-1} \left[g'(0) - e^{-A}g'(-1) - \int_{-1}^0 e^{yA}g''(y) dy \right] \right) \right. \\ &\quad \left. + As(\mu_j) \left(e^{-A}g(-1) + A^{-1} \left[g'(0) - e^{-A}g'(-1) - \int_{-1}^0 e^{yA}g''(y) dy \right] \right) \right. \\ &\quad \left. + A^{-1} \left[f(\mu_j) - c(\mu_j)f(0) - \left[s(\mu_j)f'(0) - \int_0^{\mu_j} s(\mu_j - y)f''(y) dy \right] \right] \right\} \\ &\quad + \sum_{j=1}^L \beta_j \left\{ -As(\lambda_j) (g(0) - e^{-A}g(-1) - \right. \\ &\quad \left. A^{-1} \left[g'(0) - e^{-A}g'(-1) - \int_{-1}^0 e^{yA}g''(y) dy \right] \right) \end{aligned}$$

$$\begin{aligned}
& +Ac(\lambda_j) \left(e^{-A}g(-1) + A^{-1} \left[g'(0) - e^{-A}g'(-1) - \int_{-1}^0 e^{yA}g''(y) dy \right] \right) \\
& +As(\lambda_i) f(0) - A^{-1} \left[f'(\lambda_i) - c(\lambda_i) f'(0) - \int_0^{\lambda_j} c(\lambda_i - y) f''(y) dy \right] \Big\} + \varphi \Big]
\end{aligned}$$

formülü elde edilir. Buradan, (2.3), (2.1) ve (2.2) kestirimleri yardımıyla,

$$\begin{aligned}
\|Au_{-1}\|_H & \leq M \left[\|A\varphi\|_H + \max_{-1 \leq t \leq 0} \|g''(t)\|_H + \|g'(0)\|_H \right. \\
& \left. + \|A^{1/2}g(0)\|_H + \|A^{1/2}f(0)\|_H + \|f'(0)\|_H + \max_{0 \leq t \leq 1} \|f''(t)\|_H \right] \quad (2.20)
\end{aligned}$$

kestirimi alınır. Daha sonra, (2.17), (2.18) formülleri ve kısmi integrasyon kullanılarak,

$$\begin{aligned}
u(t) & = e^{-(t+1)A}u_{-1} + A^{-1} \left(g(t) - e^{-A}g(-1) \right. \\
& \left. - A^{-1} \left[g'(t) - e^{-A}g'(-1) - \int_{-1}^t e^{yA}g''(y) dy \right] \right), -1 \leq t \leq 0, \\
u(t) & = [c(t) - As(t)] \left\{ e^{-A}u_{-1} + A^{-1} \left(g(0) - e^{-A}g(-1) \right. \right. \\
& \left. \left. - A^{-1} \left[g'(0) - e^{-A}g'(-1) - \int_{-1}^0 e^{yA}g''(y) dy \right] \right) \right\} \\
& + s(t)g(0) + A^{-1}f(t) - c(t)f(0) \\
& - \left[s(t)f'(0) + \int_0^t s(t-y)f''(y) dy \right], 0 \leq t \leq 1
\end{aligned}$$

formülleri elde edilir. Son iki formüle A uygulanarak ve (2.1), (2.2) kestirimleri kullanılarak,

$$\begin{aligned}
\|Au(t)\|_H & \leq M \left[\|Au_{-1}\|_H + \max_{-1 \leq t \leq 0} \|g''(t)\|_H, \right. \\
& \left. + \|A^{1/2}g(0)\|_H + \|g'(0)\|_H \right] - 1 \leq t \leq 0, \\
\|Au(t)\|_H & \leq M \left[\max_{0 \leq t \leq 1} \|f''(t)\|_H + \|A^{1/2}f(0)\|_H + \|f'(0)\|_H \right. \\
& \left. + \|Au_{-1}\|_H + \max_{-1 \leq t \leq 0} \|g''(t)\|_H + \|A^{1/2}g(0)\|_H + \|g'(0)\|_H \right], 0 \leq t \leq 1
\end{aligned}$$

kestirimleri alınır. Böylece, (2.20) ve son iki kestirim yardımıyla (2.14) kestirimi ispatlanmış olur. Dolayısıyla Teorem 2.1'in ispatı tamamlanmıştır.

2.2 Uygulamalar

İlk olarak, bir boyutlu hiperbolik-parabolik denklem için lokal olmayan sınır değer problemi

$$\left\{ \begin{array}{l} v_{tt} - (a(x)v_x)_x + \delta v = f(t, x), 0 < t < 1, 0 < x < 1, \\ v_t - (a(x)v_x)_x + \delta v = g(t, x), -1 < t < 0, 0 < x < 1, \\ v(-1, x) = \sum_{j=1}^N \alpha_j v(\mu_j, x) + \sum_{j=1}^L \beta_j v_t(\lambda_j, x) + \varphi(x), 0 \leq x \leq 1, \\ \sum_{j=1}^N |\alpha_j|, \sum_{j=1}^L |\beta_j| \leq 1, 0 < \mu_j, \lambda_j \leq 1, \\ v(t, 0) = v(t, 1), v_x(t, 0) = v_x(t, 1), -1 \leq t \leq 1, \\ v(0+, x) = v(0-, x), v_t(0+, x) = v_t(0-, x), 0 \leq x \leq 1 \end{array} \right. \quad (2.21)$$

ele alınmıştır. Eğer $a(x) \geq a > 0$ ($x \in (0, 1)$); $\varphi(x)$ ($x \in [0, 1]$); $f(t, x)$ ($t \in [0, 1]$, $x \in [0, 1]$); $g(t, x)$ ($t \in [-1, 0]$, $x \in [0, 1]$) fonksiyonları tanım kümelerinde yeterince pürüzsüz (smooth) ve $\delta = \text{sabit} > 0$ ise, bu durumda (2.21) probleminin çözümü vardır ve tektir. Bunun için, (2.21) problemi, H Hilbert uzayında self-adjoint pozitif tanımlı A operatörlü, (1.1) lokal olmayan sınır değer problemine dönüştürülebilir.

Teorem 2.2. Lokal olmayan sınır değer (2.21) probleminin çözümü için aşağıdaki

$$\begin{aligned} \max_{-1 \leq t \leq 1} \|v_x(t, \cdot)\|_{L_2[0,1]} &\leq M \left[\|f(0, \cdot)\|_{L_2[0,1]} + \max_{0 \leq t \leq 1} \|f_t(t, \cdot)\|_{L_2[0,1]} \right. \\ &\quad \left. + \|g(0, \cdot)\|_{L_2[0,1]} + \max_{-1 \leq t \leq 0} \|g_t(t, \cdot)\|_{L_2[0,1]} + \|\varphi_x\|_{L_2[0,1]} \right], \\ \max_{-1 \leq t \leq 1} \|v_x(t, \cdot)\|_{L_2[0,1]} &\leq M \left[\|f(0, \cdot)\|_{L_2[0,1]} + \max_{0 \leq t \leq 1} \|f_t(t, \cdot)\|_{L_2[0,1]} \right. \\ &\quad \left. + \|g(0, \cdot)\|_{L_2[0,1]} + \max_{-1 \leq t \leq 0} \|g_t(t, \cdot)\|_{L_2[0,1]} + \|\varphi_x\|_{L_2[0,1]} \right], \\ \max_{-1 \leq t \leq 1} \|v_x(t, \cdot)\|_{L_2[0,1]} + \max_{-1 \leq t \leq 0} \|v_t(t, \cdot)\|_{L_2[0,1]} + \max_{0 \leq t \leq 1} \|v_{tt}(t, \cdot)\|_{L_2[0,1]} \\ &\leq M \left[\|\varphi_x\|_{L_2[0,1]} + \|f(0, \cdot)\|_{L_2[0,1]} + \|f_t(0, \cdot)\|_{L_2[0,1]} + \max_{0 \leq t \leq 1} \|f_{tt}(t, \cdot)\|_{L_2[0,1]} \right. \\ &\quad \left. + \|g(0, \cdot)\|_{L_2[0,1]} + \|g_t(0, \cdot)\|_{L_2[0,1]} + \max_{-1 \leq t \leq 0} \|g_{tt}(t, \cdot)\|_{L_2[0,1]} \right] \end{aligned}$$

kararlılık kestirimleri sağlanır. Burada, M katsayısı $f(t, x)$, $g(t, x)$ ve $\varphi(x)$ 'den bağımsızdır.

Teorem 2.2'nin ispatı, soyut Teorem 2.1 ve (2.21) problemini tarafından oluşturulan uzay operatörünün simetri özelliklerine dayanmaktadır.

İkinci olarak, n -boyutlu $\mathbb{R}^n$ Öklid uzayında, $\Omega = (0 < x_k < 1, 1 \leq k \leq n)$ bir açık küme ve S bu kümenin sınırı olsun öyle ki $\bar{\Omega} = \Omega \cup S$ 'dir. $[0, 1] \times \Omega$ kümesinde, çok boyutlu hiperbolik-parabolik denklem için karma sınır değer problemi

$$\left\{ \begin{array}{l} v_{tt} - \sum_{r=1}^n (a_r(x) v_{x_r})_{x_r} = f(t, x), 0 \leq t \leq 1, \\ x = (x_1, \dots, x_n) \in \Omega, \\ v_t - \sum_{r=1}^n (a_r(x) v_{x_r})_{x_r} = g(t, x), -1 \leq t \leq 0, \\ x = (x_1, \dots, x_n) \in \Omega, \\ v(-1, x) = \sum_{j=1}^N \alpha_j v(\mu_j, x) + \sum_{j=1}^L \beta_j v_t(\lambda_j, x) + \varphi(x), x \in \bar{\Omega}, \\ \sum_{j=1}^N |\alpha_j|, \sum_{j=1}^L |\beta_j| \leq 1, 0 < \mu_j, \lambda_j \leq 1, \\ u(t, x) = 0, x \in S, -1 \leq t \leq 1, \\ v(0+, x) = v(0-, x), v_t(0+, x) = v_t(0-, x), x \in \bar{\Omega} \end{array} \right. \quad (2.22)$$

ele alınmıştır. $\bar{\Omega}$ kümesinde tanımlanan tüm integrallenebilen fonksiyonlar, $L_2(\bar{\Omega})$ Hilbert uzayı olarak tanımlanır. Bu uzayda norm

$$\|f\|_{L_2(\bar{\Omega})} = \left\{ \int \cdots \int_{x \in \bar{\Omega}} |f(x)|^2 dx_1 \cdots dx_n \right\}^{1/2}$$

formülü ile alınır. Eğer $a_r(x) \geq a > 0$, $\varphi(x)$ ($x \in \bar{\Omega}$) ve $f(t, x)$ ($t \in (0, 1), x \in \Omega$); $g(t, x)$ ($t \in (-1, 0), x \in \Omega$) fonksiyonları tanım kümelerinde yeterince pürüzsüz ise, bu durumda (2.22) probleminin çözümü vardır ve tektir. Bunun için, (2.22) problemi, H Hilbert uzayında self-adjoint pozitif tanımlı A operatörlü, (1.1) lokal olmayan sınır değer problemine dönüştürülebilir.

Teorem 2.3. Lokal olmayan sınır değer (2.22) probleminin çözümü için aşağıdaki

$$\begin{aligned} \max_{-1 \leq t \leq 1} \sum_{r=1}^n \|v_{x_r}(t, \cdot)\|_{L_2(\bar{\Omega})} &\leq M \left[\|f(0, \cdot)\|_{L_2(\bar{\Omega})} + \max_{0 \leq t \leq 1} \|f_t(t, \cdot)\|_{L_2(\bar{\Omega})} \right. \\ &\quad \left. + \|g(0, \cdot)\|_{L_2(\bar{\Omega})} + \max_{-1 \leq t \leq 0} \|g_t(t, \cdot)\|_{L_2(\bar{\Omega})} + \sum_{r=1}^n \|\varphi_{x_r}\|_{L_2(\bar{\Omega})} \right], \\ \max_{-1 \leq t \leq 1} \sum_{r=1}^n \|v_{x_r}(t, \cdot)\|_{L_2(\bar{\Omega})} &\leq M \left[\|f(0, \cdot)\|_{L_2(\bar{\Omega})} + \max_{0 \leq t \leq 1} \|f_t(t, \cdot)\|_{L_2(\bar{\Omega})} \right. \\ &\quad \left. + \|g(0, \cdot)\|_{L_2(\bar{\Omega})} + \max_{-1 \leq t \leq 0} \|g_t(t, \cdot)\|_{L_2(\bar{\Omega})} + \sum_{r=1}^n \|\varphi_{x_r}\|_{L_2(\bar{\Omega})} \right], \\ \max_{-1 \leq t \leq 1} \sum_{r=1}^n \|v_{x_r}(t)\|_{L_2(\bar{\Omega})} &+ \max_{-1 \leq t \leq 0} \|v_t(t, \cdot)\|_{L_2(\bar{\Omega})} + \max_{0 \leq t \leq 1} \|v_{tt}(t, \cdot)\|_{L_2(\bar{\Omega})} \end{aligned}$$

$$\leq M \left[\sum_{r=1}^n \|\varphi_{x_r}\|_{L_2(\bar{\Omega})} + \|f(0, \cdot)\|_{L_2(\bar{\Omega})} + \|f_t(0, \cdot)\|_{L_2(\bar{\Omega})} + \max_{0 \leq t \leq 1} \|f_{tt}(t, \cdot)\|_{L_2(\bar{\Omega})} \right. \\ \left. + \|g(0, \cdot)\|_{L_2(\bar{\Omega})} + \|g_t(0, \cdot)\|_{L_2(\bar{\Omega})} + \max_{-1 \leq t \leq 0} \|g_{tt}(t, \cdot)\|_{L_2(\bar{\Omega})} \right]$$

kararlılık kestirimleri sağlanır. Burada, M katsayısı $f(t, x)$, $g(t, x)$ ve $\varphi(x)$ 'den bağımsızdır.

Teorem 2.3. ün ispatı, soyut Teorem 2.1, (2.22) problemini tarafından oluşturulan uzay operatörünün simetri özelliklerine ve aşağıdaki $L_2(\Omega)$ uzayında eliptik diferansiyel probleminin çözümü için koersiv kestirimi alınan teoreme dayanmaktadır.

Teorem 2.4. Eliptik diferansiyel probleminin

$$A^x u(x) = \omega(x), x \in \Omega, \quad (2.23)$$

$$u(x) = 0, x \in S$$

çözümü için aşağıdaki

$$\sum_{r=1}^n \|u_{x_r x_r}\|_{L_2(\Omega)} \leq M \|\omega\|_{L_2(\Omega)}.$$

koersiv kestirimi sağlanır [Sobolevskii, P. E., 1975].

3 BİRİNCİ BASAMAKTAN DOĞRULUKLU FARK ŞEMASI

3.1 Fark Şeması

Bu çalışma boyunca, işlemlerde kolaylık sağlaması için $\min \mu_j, \min \lambda_j > 2\tau$ olarak alınacaktır. Bu bölümde, (1.1) sınır değer problemi ile bu probleme karşılık gelen

$$\begin{cases} \tau^{-2} (u_{k+1} - 2u_k + u_{k-1}) + Au_{k+1} = f_k, f_k = f(t_{k+1}), \\ t_{k+1} = (k+1)\tau, 1 \leq k < N, \\ \tau^{-1} (u_k - u_{k-1}) + Au_k = g_k, g_k = g(t_k), t_k = k\tau, -N < k \leq 0, \\ \tau^{-1} (u_1 - u_0) = -Au_0 + g_0, \\ u_{-N} = \sum_{j=1}^N \alpha_j u_{[\mu_j/\tau]} + \sum_{j=1}^L \beta_j \frac{u_{[\lambda_j/\tau]} - u_{[\lambda_j/\tau]-1}}{\tau} + \varphi \end{cases} \quad (3.1)$$

birinci basamaktan doğruluklu fark şeması incelenmiştir.

Bilindiği gibi, H Hilbert uzayında self-adjoint pozitif tanımlı A diferansiyel operatörlü lokal olmayan sınır değer probleminin bir değişkenli diskritizasyon (discretization) fark şemalarını araştırmak demek, H_h Hilbert uzaylarında h 'ye ($0 < h \leq h_0$) göre düzgün self-adjoint pozitif tanımlı A_h fark operatörlü çok değişkenli diskritizasyon fark şemalarını araştırmak demektir. Dolayısıyla bu çalışmada sadece bir değişkenli diskritizasyon fark şemaları incelenmektedir.

Öncelikle ileride ihtiyaç duyacağımız yardımcı teoremleri verelim.

Yardımcı Teorem 3.1. Aşağıdaki

$$\|R(\pm\tau A^{1/2})\|_{H \rightarrow H} \leq 1, \|\tau A^{1/2} R(\pm\tau A^{1/2})\|_{H \rightarrow H} \leq 1, \quad (3.2)$$

$$\|R^k\|_{H \rightarrow H} \leq M(1 + \delta\tau)^{-k}, \|AR^k\|_{H \rightarrow H} \leq M(k\tau)^{-1}, k \geq 1 \quad (3.3)$$

kestirimler sağlanır. Burada, M katsayısı τ 'dan bağımsızdır ve

$$R(\pm\tau A^{1/2}) = (I \pm i\tau A^{1/2})^{-1}, R = R(\tau A) = (I + \tau A)^{-1}$$

dir.

Yardımcı Teorem 3.2. Öncelikle

$$I - \left[\sum_{j=1}^N \alpha_j \left\{ \frac{1}{2} [R^{[\mu_j/\tau]-1} (\tau A^{1/2}) + R^{[\mu_j/\tau]-1} (-\tau A^{1/2})] \right\} \right]$$

$$\begin{aligned}
& + \frac{1}{2i} A^{1/2} [R^{[\mu_j/\tau]-1} (\tau A^{1/2}) + R^{[\mu_j/\tau]-1} (-\tau A^{1/2})] \Big\} \\
& - \frac{1}{2i} A^{1/2} (I + \tau^2 A) [R^{[\mu_j/\tau]} (-\tau A^{1/2}) - R^{[\mu_j/\tau]} (\tau A^{1/2})] \Big\} R^N \\
& - \sum_{j=1}^L \beta_j \tau^{-1} \left(\left\{ \left\{ \frac{1}{2} [R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-1} (-\tau A^{1/2})] \right. \right. \right. \\
& \quad \left. \left. \left. + \frac{1}{2i} A^{1/2} [R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-1} (-\tau A^{1/2})] \right\} \right. \right. \\
& \quad \left. \left. - \frac{1}{2i} A^{1/2} (I + \tau^2 A) [R^{[\lambda_j/\tau]} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]} (\tau A^{1/2})] \right\} \right. \\
& \quad \left. - \left\{ \left\{ \frac{1}{2} [R^{[\lambda_j/\tau]-2} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-2} (-\tau A^{1/2})] \right. \right. \right. \\
& \quad \left. \left. \left. + \frac{1}{2i} A^{1/2} [R^{[\lambda_j/\tau]-2} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-2} (-\tau A^{1/2})] \right\} + \frac{1}{2i} A^{1/2} (I + \tau^2 A) \right. \right. \\
& \quad \left. \left. \times [R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]-1} (\tau A^{1/2})] \right\} \right) R^N]
\end{aligned}$$

operatörünün tersi vardır ve

$$\begin{aligned}
T_\tau &= \left(I - \sum_{j=1}^N \alpha_j \left\{ \left\{ \frac{1}{2} [R^{[\mu_j/\tau]-1} (\tau A^{1/2}) + R^{[\mu_j/\tau]-1} (-\tau A^{1/2})] \right. \right. \right. \\
& \quad \left. \left. \left. + \frac{1}{2i} A^{1/2} [R^{[\mu_j/\tau]-1} (\tau A^{1/2}) + R^{[\mu_j/\tau]-1} (-\tau A^{1/2})] \right\} \right. \right. \\
& \quad \left. \left. - \frac{1}{2i} A^{1/2} (I + \tau^2 A) [R^{[\mu_j/\tau]} (-\tau A^{1/2}) - R^{[\mu_j/\tau]} (\tau A^{1/2})] \right\} R^N \right. \\
& \quad \left. + \sum_{j=1}^L \beta_j \tau^{-1} \left(\left\{ \left\{ \frac{1}{2} [R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-1} (-\tau A^{1/2})] \right. \right. \right. \right. \\
& \quad \left. \left. \left. + \frac{1}{2i} A^{1/2} [R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-1} (-\tau A^{1/2})] \right\} \right. \right. \\
& \quad \left. \left. - \frac{1}{2i} A^{1/2} (I + \tau^2 A) [R^{[\lambda_j/\tau]} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]} (\tau A^{1/2})] \right\} \right. \\
& \quad \left. - \left\{ \left\{ \frac{1}{2} [R^{[\lambda_j/\tau]-2} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-2} (-\tau A^{1/2})] \right. \right. \right. \\
& \quad \left. \left. \left. + \frac{1}{2i} A^{1/2} [R^{[\lambda_j/\tau]-2} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-2} (-\tau A^{1/2})] \right\} + \frac{1}{2i} A^{1/2} (I + \tau^2 A) \right. \right. \\
& \quad \left. \left. \times [R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]-1} (\tau A^{1/2})] \right\} \right) R^N \Big)^{-1}
\end{aligned}$$

aşağıdaki

$$\|T_\tau\|_{H \rightarrow H} \leq M \quad (3.4)$$

kestirimi sağlar. Burada, M katsayısı τ 'dan bağımsızdır.

İspat. İlk olarak,

$$\begin{aligned}
& T_\tau - \left(I - \sum_{j=1}^N \alpha_j ([c([\mu_j/\tau]\tau) - As([\mu_j/\tau]\tau)] \right. \\
& \left. - \sum_{j=1}^L \beta_j [s([\lambda_j/\tau]\tau) + c([\lambda_j/\tau]\tau)] A e^{-A} \right)^{-1} = T_\tau (I - \sum_{j=1}^N \alpha_j ([c([\mu_j/\tau]\tau) \\
& - As([\mu_j/\tau]\tau)] - \sum_{j=1}^L \beta_j [s([\lambda_j/\tau]\tau) + c([\lambda_j/\tau]\tau)] A e^{-A})^{-1} \\
& \times \left(\left[\sum_{j=1}^N \alpha_j \left\{ \left\{ \frac{1}{2} [R^{[\mu_j/\tau]-1} (\tau A^{1/2}) + R^{[\mu_j/\tau]-1} (-\tau A^{1/2})] \right. \right. \right. \right. \\
& \left. \left. \left. + \frac{1}{2i} A^{1/2} [R^{[\mu_j/\tau]-1} (\tau A^{1/2}) + R^{[\mu_j/\tau]-1} (-\tau A^{1/2})] \right\} \right. \right. \\
& \left. \left. - \frac{1}{2i} A^{1/2} (I + \tau^2 A) [R^{[\mu_j/\tau]} (-\tau A^{1/2}) - R^{[\mu_j/\tau]} (\tau A^{1/2})] \right\} \right. \\
& \left. - \sum_{j=1}^L \beta_j \tau^{-1} \left(\left\{ \left\{ \frac{1}{2} [R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-1} (-\tau A^{1/2})] \right. \right. \right. \right. \\
& \left. \left. \left. + \frac{1}{2i} A^{1/2} [R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-1} (-\tau A^{1/2})] \right\} \right. \right. \\
& \left. \left. - \frac{1}{2i} A^{1/2} (I + \tau^2 A) [R^{[\lambda_j/\tau]} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]} (\tau A^{1/2})] \right\} \right. \\
& \left. - \left\{ \left\{ \frac{1}{2} [R^{[\lambda_j/\tau]-2} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-2} (-\tau A^{1/2})] \right. \right. \right. \\
& \left. \left. \left. + \frac{1}{2i} A^{1/2} [R^{[\lambda_j/\tau]-2} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-2} (-\tau A^{1/2})] \right\} \right. \right. \\
& \left. \left. + \frac{1}{2i} A^{1/2} (I + \tau^2 A) [R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]-1} (\tau A^{1/2})] \right\} \right) \right] R^N \\
& - \sum_{j=1}^N \alpha_j ([c([\mu_j/\tau]\tau) - As([\mu_j/\tau]\tau)] \\
& - \sum_{j=1}^L \beta_j [s([\lambda_j/\tau]\tau) + c([\lambda_j/\tau]\tau)] A e^{-A} \Big)
\end{aligned}$$

formülü doğrudur. Buradan,

$$\begin{aligned}
& \left\| \left[\sum_{j=1}^N \alpha_j \left\{ \left\{ \frac{1}{2} [R^{[\mu_j/\tau]-1} (\tau A^{1/2}) + R^{[\mu_j/\tau]-1} (-\tau A^{1/2})] \right. \right. \right. \right. \\
& \left. \left. \left. + \frac{1}{2i} A^{1/2} [R^{[\mu_j/\tau]-1} (\tau A^{1/2}) + R^{[\mu_j/\tau]-1} (-\tau A^{1/2})] \right\} \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2i}A^{1/2}(I+\tau^2A)[R^{[\mu_j/\tau]}(-\tau A^{1/2})-R^{[\mu_j/\tau]}(\tau A^{1/2})]\Big\} \\
& -\sum_{j=1}^L\beta_j\tau^{-1}\left(\left\{\frac{1}{2}[R^{[\lambda_j/\tau]-1}(\tau A^{1/2})+R^{[\lambda_j/\tau]-1}(-\tau A^{1/2})]\right.\right. \\
& \quad \left.\left.+\frac{1}{2i}A^{1/2}[R^{[\lambda_j/\tau]-1}(\tau A^{1/2})+R^{[\lambda_j/\tau]-1}(-\tau A^{1/2})]\right\}\right. \\
& \quad \left.-\frac{1}{2i}A^{1/2}(I+\tau^2A)[R^{[\lambda_j/\tau]}(-\tau A^{1/2})-R^{[\lambda_j/\tau]}(\tau A^{1/2})]\right\} \\
& \quad -\left\{\frac{1}{2}[R^{[\lambda_j/\tau]-2}(\tau A^{1/2})+R^{[\lambda_j/\tau]-2}(-\tau A^{1/2})]\right. \\
& \quad \left.+\frac{1}{2i}A^{1/2}[R^{[\lambda_j/\tau]-2}(\tau A^{1/2})+R^{[\lambda_j/\tau]-2}(-\tau A^{1/2})]\right\} \\
& \quad \left.+\frac{1}{2i}A^{1/2}(I+\tau^2A)[R^{[\lambda_j/\tau]-1}(-\tau A^{1/2})-R^{[\lambda_j/\tau]-1}(\tau A^{1/2})]\right\}\Bigg)R^N \\
& -\sum_{j=1}^N\alpha_j([c([\mu_j/\tau]\tau)-As([\mu_j/\tau]\tau)] \\
& -\sum_{j=1}^L\beta_j[s([\lambda_j/\tau]\tau)+c([\lambda_j/\tau]\tau)]A)e^{-A}\Bigg\|_{H\rightarrow H}\leq M\tau
\end{aligned}$$

kestirimi kullanılarak, (3.4) kestirimi elde edilir. Bu kestirimin ispatı (3.2), (3.3) kestirimlerine ve

$$\left\|\left(I-\sum_{j=1}^N\alpha_j([c(\mu_j)-As(\mu_j)]-\sum_{j=1}^L\beta_j[s(\lambda_j)+c(\lambda_j)]A)e^{-A}\right)^{-1}\right\|_{H\rightarrow H}\leq M \quad (3.5)$$

kestirimine dayanmaktadır. Buradaki (3.5) kestiriminin ispatı ise [Ashyralyev, A. ve Ozdemir, Y., 2005] makalesinde verilmiştir. Böylelikle, Yardımcı Teorem 3.2'nin ispatı tamamlanmış olur.

Teorem 3.1. Eğer $\varphi \in D(A)$, $g_0 \in D(A^{1/2})$ ve $f_1 \in D(A^{1/2})$ ise, bu durumda (3.1) fark şemasının çözümü için aşağıdaki

$$\max_{-N \leq k \leq N} \|u_k\|_H \leq M [\|\varphi\|_H + \|A^{-1/2}f_1\|_H + \|A^{-1/2}g_0\|_H] \quad (3.6)$$

$$+ \max_{2 \leq k \leq N} \|A^{-1/2}(f_k - f_{k-1})\tau^{-1}\|_H + \max_{-N < k \leq 0} \|A^{-1/2}(g_k - g_{k-1})\tau^{-1}\|_H \Big],$$

$$\max_{-N \leq k \leq N} \|A^{1/2}u_k\|_H \leq M [\|A^{1/2}\varphi\|_H + \|f_1\|_H + \|g_0\|_H] \quad (3.7)$$

$$+ \max_{2 \leq k \leq N-1} \|(f_k - f_{k-1})\tau^{-1}\|_H + \max_{-(N-1) \leq k \leq 0} \|(g_k - g_{k-1})\tau^{-1}\|_H \Big],$$

$$\begin{aligned}
& \max_{1 \leq k \leq N-1} \left\| \tau^{-2} (u_{k+1} - 2u_k + u_{k-1}) \right\|_H + \max_{-N \leq k \leq N} \|Au_k\|_H \\
& + \max_{-(N-1) \leq k \leq 0} \left\| \tau^{-1} (u_k - u_{k-1}) \right\|_H \leq M \left[\|A\varphi\|_H + \|A^{1/2}f_1\|_H \right. \\
& + \left\| (f_2 - f_1) \tau^{-1} \right\|_H + \max_{2 \leq k \leq N-2} \left\| (f_{k+1} - 2f_k + f_{k-1}) \tau^{-2} \right\|_H + \|A^{1/2}g_0\|_H \\
& \left. + \left\| (g_0 - g_{-1}) \tau^{-1} \right\|_H + \max_{-(N-1) \leq k \leq -1} \left\| (g_{k+1} - 2g_k + g_{k-1}) \tau^{-2} \right\|_H \right]
\end{aligned} \tag{3.8}$$

kararlılık kestirimleri sağlanır. Burada, M katsayısı $\tau, f_k, 1 \leq k < N, g_k, -N < k \leq 0$ ve φ 'den bağımsızdır.

İspat. Herşeyden önce, (3.1) fark şemasının çözümü için gerekli formüller elde edilecektir. Bilindiği gibi,

$$\begin{cases} \tau^{-2} (u_{k+1} - 2u_k + u_{k-1}) + Au_{k+1} = f_k, f_k = f(t_{k+1}), \\ t_{k+1} = (k+1)\tau, \quad 1 \leq k < N, u_0 = \xi, \tau^{-1} (u_1 - u_0) = \psi, \end{cases} \tag{3.9}$$

$$\begin{cases} \tau^{-1} (u_k - u_{k-1}) + Au_k = g_k, \quad g_k = g(t_k), \\ t_k = k\tau, \quad -N < k \leq 0, \quad u_{-N} \text{ biliniyor} \end{cases} \tag{3.10}$$

başlangıç değer fark problemlerinin tek çözümü vardır ve aşağıdaki formüller

$$u_1 = \xi + \tau\psi,$$

$$\begin{aligned}
u_k &= \left\{ \frac{1}{2} [R^{k-1} (\tau A^{1/2}) + R^{k-1} (-\tau A^{1/2})] \right. \\
&+ \frac{1}{2i} A^{1/2} [R^{k-1} (\tau A^{1/2}) + R^{k-1} (-\tau A^{1/2})] \left. \right\} \xi \\
&+ \frac{1}{2i} A^{-1/2} (I + \tau^2 A) [R^k (-\tau A^{1/2}) - R^k (\tau A^{1/2})] \psi \\
&+ \sum_{s=1}^{k-1} \frac{\tau}{2i} A^{-1/2} [R^{k-s} (-\tau A^{1/2}) - R^{k-s} (\tau A^{1/2})] f_s, \quad 2 \leq k \leq N,
\end{aligned} \tag{3.11}$$

$$u_k = R^{N+k} u_{-N} + \tau \sum_{s=-N+1}^k R^{k-s+1} g_s, \quad -(N-1) \leq k \leq 0 \tag{3.12}$$

sağlanır. Buradaki (3.11), (3.12) denklemleri ve

$$\tau^{-1} (u_1 - u_0) = -Au_0 + g_0, u_0 = \xi, \tau^{-1} (u_1 - u_0) = \psi,$$

formülleri kullanılarak,

$$\xi = u_0 = R^N u_{-N} + \tau \sum_{s=-N+1}^0 R^{-s+1} g_s,$$

$$\psi = \tau^{-1}(u_1 - u_0) = -A \left[R^N u_{-N} + \tau \sum_{s=-N+1}^0 R^{-s+1} g_s \right] + g_0$$

formülleri elde edilir. Bu durumda,

$$u_1 = (I - \tau A) \left[R^N u_{-N} + \tau \sum_{s=-N+1}^0 R^{-s+1} g_s \right] + \tau g_0, \quad (3.13)$$

$$\begin{aligned} u_k = & \left\{ \left\{ \frac{1}{2} [R^{k-1} (\tau A^{1/2}) + R^{k-1} (-\tau A^{1/2})] \right. \right. \\ & + \frac{1}{2i} A^{1/2} [R^{k-1} (\tau A^{1/2}) + R^{k-1} (-\tau A^{1/2})] \Big\} \\ & - \frac{1}{2i} A^{1/2} (I + \tau^2 A) [R^k (-\tau A^{1/2}) - R^k (\tau A^{1/2})] \Big\} \\ & \times \left[R^N u_{-N} + \tau \sum_{s=-N+1}^0 R^{-s+1} g_s \right] \\ & + \frac{1}{2i} A^{-1/2} (I + \tau^2 A) [R^k (-\tau A^{1/2}) - R^k (\tau A^{1/2})] g_0 \\ & + \sum_{s=1}^{k-1} \frac{\tau}{2i} A^{-1/2} [R^{k-s} (-\tau A^{1/2}) - R^{k-s} (\tau A^{1/2})] f_s, \quad 2 \leq k \leq N \end{aligned} \quad (3.14)$$

dir. Buradan,

$$u_{-N} = \sum_{j=1}^N \alpha_j u_{[\mu_j/\tau]} + \sum_{j=1}^L \beta_j \frac{u_{[\lambda_j/\tau]} - u_{[\lambda_j/\tau]-1}}{\tau} + \varphi$$

koşulu kullanılarak,

$$\begin{aligned} u_{-N} = & T_\tau \left(\sum_{j=1}^N \alpha_j \left[\left\{ \frac{1}{2} [R^{[\mu_j/\tau]-1} (\tau A^{1/2}) + R^{[\mu_j/\tau]-1} (-\tau A^{1/2})] \right. \right. \right. \\ & + \frac{1}{2i} A^{1/2} [R^{[\mu_j/\tau]-1} (\tau A^{1/2}) + R^{[\mu_j/\tau]-1} (-\tau A^{1/2})] \Big\} \\ & - \frac{1}{2i} A^{1/2} (I + \tau^2 A) [R^{[\mu_j/\tau]} (-\tau A^{1/2}) - R^{[\mu_j/\tau]} (\tau A^{1/2})] \Big\} \\ & \times \tau \sum_{s=-N+1}^0 R^{-s+1} g_s \\ & + \frac{1}{2i} A^{-1/2} (I + \tau^2 A) [R^{[\mu_j/\tau]} (-\tau A^{1/2}) - R^{[\mu_j/\tau]} (\tau A^{1/2})] g_0 \\ & + \sum_{s=1}^{[\mu_j/\tau]-1} \frac{\tau}{2i} A^{-1/2} [R^{[\mu_j/\tau]-s} (-\tau A^{1/2}) - R^{[\mu_j/\tau]-s} (\tau A^{1/2})] f_s \Big] \\ & + \sum_{j=1}^L \beta_j \tau^{-1} \left(\left\{ \frac{1}{2} [R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-1} (-\tau A^{1/2})] \right. \right. \end{aligned} \quad (3.15)$$

$$\begin{aligned}
& + \frac{1}{2i} A^{1/2} \left[R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) \right] \Big\} \\
& - \frac{1}{2i} A^{1/2} (I + \tau^2 A) \left[R^{[\lambda_j/\tau]} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]} (\tau A^{1/2}) \right] \Big\} \\
& \quad \times \tau \sum_{s=-N+1}^0 R^{-s+1} g_s \\
& + \frac{1}{2i} A^{-1/2} (I + \tau^2 A) \left[R^{[\lambda_j/\tau]} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]} (\tau A^{1/2}) \right] g_0 \\
& + \sum_{s=1}^{[\lambda_j/\tau]-1} \frac{\tau}{2i} A^{-1/2} \left[R^{[\lambda_j/\tau]-s} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]-s} (\tau A^{1/2}) \right] f_s \\
& \quad - \left\{ \left\{ \frac{1}{2} \left[R^{[\lambda_j/\tau]-2} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-2} (-\tau A^{1/2}) \right] \right. \right. \\
& \quad \left. \left. + \frac{1}{2i} A^{1/2} \left[R^{[\lambda_j/\tau]-2} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-2} (-\tau A^{1/2}) \right] \right\} \right. \\
& \quad \left. + \frac{1}{2i} A^{1/2} (I + \tau^2 A) \left[R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) \right] \right\} \\
& \quad \times \tau \sum_{s=-N+1}^0 R^{-s+1} g_s + \varphi \\
& - \frac{1}{2i} A^{-1/2} (I + \tau^2 A) \left[R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) \right] g_0 \\
& - \left. \sum_{s=1}^{[\lambda_j/\tau]-2} \frac{\tau}{2i} A^{-1/2} \left[R^{[\lambda_j/\tau]-1-s} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]-1-s} (\tau A^{1/2}) \right] f_s \right) \Bigg)
\end{aligned}$$

operatör denklemi elde edilir. Böylece, (3.1) fark şemasının çözümü için, (3.12), (3.13), (3.14) ve (3.15) formülleri kurulmuş olur.

Şimdi, (3.6), (3.7) ve (3.8) kestirimlerinin ispatları verilecektir. İlk olarak, $\|u_{-N}\|_H$ için kestirim elde edilecektir. Bunun için, Abel formülü ve (3.15) formülü kullanılarak,

$$\begin{aligned}
u_{-N} = T_\tau & \left(\sum_{j=1}^N \alpha_j \left[\left\{ \left\{ \frac{1}{2} \left[R^{[\mu_j/\tau]-1} (\tau A^{1/2}) + R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) \right] \right. \right. \right. \right. \\
& \quad \left. \left. + \frac{1}{2i} A^{1/2} \left[R^{[\mu_j/\tau]-1} (\tau A^{1/2}) + R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) \right] \right\} \right. \\
& \quad \left. \left. - \frac{1}{2i} A^{1/2} (I + \tau^2 A) \left[R^{[\mu_j/\tau]} (-\tau A^{1/2}) - R^{[\mu_j/\tau]} (\tau A^{1/2}) \right] \right\} \right. \\
& \quad \left. \times A^{-1} \left[-R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} [g_{s-1} - g_s] \right] \right. \\
& \quad \left. + A^{-1} \left[f_{[\mu_j/\tau]-1} - \frac{1}{2} \left[R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) + R^{[\mu_j/\tau]-1} (\tau A^{1/2}) \right] f_1 \right. \right.
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
& + \sum_{s=2}^{[\mu_j/\tau]-1} \frac{1}{2} \left[R^{[\mu_j/\tau]-s} (-\tau A^{1/2}) + R^{[\mu_j/\tau]-s} (\tau A^{1/2}) \right] [f_{s-1} - f_s] \Bigg] \\
& + \sum_{j=1}^L \beta_j \tau^{-1} \left(\left\{ \left\{ \frac{1}{2} [R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-1} (-\tau A^{1/2})] \right. \right. \right. \\
& \quad \left. \left. \left. + \frac{1}{2i} A^{1/2} [R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-1} (-\tau A^{1/2})] \right\} \right. \right. \\
& \quad \left. \left. - \frac{1}{2i} A^{1/2} (I + \tau^2 A) [R^{[\lambda_j/\tau]} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]} (\tau A^{1/2})] \right\} \right. \\
& \quad \left. \times A^{-1} \left[-R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} [g_{s-1} - g_s] \right] \right. \\
& \quad \left. + A^{-1} \left[f_{[\lambda_j/\tau]-1} - \frac{1}{2} [R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) + R^{[\lambda_j/\tau]-1} (\tau A^{1/2})] f_1 \right. \right. \\
& \quad \left. \left. + \sum_{s=2}^{[\lambda_j/\tau]-1} \frac{1}{2} [R^{[\lambda_j/\tau]-s} (-\tau A^{1/2}) + R^{[\lambda_j/\tau]-s} (\tau A^{1/2})] [f_{s-1} - f_s] \right] \right. \\
& \quad \left. - \left\{ \left\{ \frac{1}{2} [R^{[\lambda_j/\tau]-2} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-2} (-\tau A^{1/2})] \right. \right. \right. \\
& \quad \left. \left. \left. + \frac{1}{2i} A^{1/2} [R^{[\lambda_j/\tau]-2} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-2} (-\tau A^{1/2})] \right\} \right. \right. \\
& \quad \left. \left. + \frac{1}{2i} A^{1/2} (I + \tau^2 A) [R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]-1} (\tau A^{1/2})] \right\} \right. \\
& \quad \left. \times A^{-1} \left[-R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} [g_{s-1} - g_s] \right] + \varphi \right. \\
& \quad \left. - A^{-1} \left[f_{[\lambda_j/\tau]-2} - \frac{1}{2} [R^{[\lambda_j/\tau]-2} (-\tau A^{1/2}) + R^{[\lambda_j/\tau]-2} (\tau A^{1/2})] f_1 \right. \right. \\
& \quad \left. \left. + \sum_{s=2}^{[\lambda_j/\tau]-2} \frac{1}{2} [R^{[\lambda_j/\tau]-1-s} (-\tau A^{1/2}) + R^{[\lambda_j/\tau]-1-s} (\tau A^{1/2})] [f_{s-1} - f_s] \right] \right) \Bigg)
\end{aligned}$$

formülü elde edilir. Buradan, (3.16) formülü ve (3.2), (3.3) kestirimleri yardımı ile,

$$\begin{aligned}
\|u_{-N}\|_H & \leq M \left[\|\varphi\|_H + \|A^{-1/2} f_1\|_H + \max_{2 \leq k \leq N-1} \|A^{-1/2} (f_k - f_{k-1}) \tau^{-1}\|_H \right. \\
& \quad \left. + \|A^{-1/2} g_0\|_H + \max_{-(N-1) \leq k \leq 0} \|A^{-1/2} (g_k - g_{k-1}) \tau^{-1}\|_H \right] \quad (3.17)
\end{aligned}$$

kestirimi alınır.

İkinci olarak, $\|u_k\|_H$ için kestirim elde edilecektir. Bunun için önce $-N < k \leq 0$ aralığında inceleyelim. Bu durumda, Abel formülü ve (3.12) formülü kullanılarak,

$$u_k = R^{N+k} u_{-N} \quad (3.18)$$

$$+ A^{-1} \left[g_k - R^{N+k-1} g_{-N+1} + \sum_{s=-N+2}^k R^{k-s+1} [g_{s-1} - g_s] \right]$$

formülü elde edilir. Buradan, (3.18) formülü ve (3.3) kestirimi yardımıyla,

$$\|u_k\|_H \leq \|u_{-N}\|_H + 2 \|A^{-1/2} g_0\|_H + \max_{-N < k \leq 0} \|A^{-1/2} (g_k - g_{k-1}) \tau^{-1}\|_H \quad (3.19)$$

kestirimi alınır. Şimdi ise $1 \leq k \leq N$ aralığında inceleyelim. Bu durumda, Abel formülü ve (3.13), (3.14) formülleri kullanılarak,

$$u_1 = (I - \tau A) [R^N u_{-N} + A^{-1} [g_0 - R^N g_{-N+1} \quad (3.20)$$

$$+ \sum_{s=-N+2}^0 R^{-s+1} [g_{s-1} - g_s]] + \tau g_0,$$

$$\begin{aligned} u_k = & \left\{ \left\{ \frac{1}{2} [R^{k-1} (\tau A^{1/2}) + R^{k-1} (-\tau A^{1/2})] \right. \right. \quad (3.21) \\ & + \frac{1}{2i} A^{1/2} [R^{k-1} (\tau A^{1/2}) + R^{k-1} (-\tau A^{1/2})] \Big\} \\ & - \frac{1}{2i} A^{1/2} (I + \tau^2 A) [R^k (-\tau A^{1/2}) - R^k (\tau A^{1/2})] \Big\} \\ & \times \left[R^N u_{-N} + A^{-1} \left[-R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} [g_{s-1} - g_s] \right] \right] \\ & + A^{-1} \left[f_{k-1} - \frac{1}{2} [R^{k-1} (-\tau A^{1/2}) + R^{k-1} (\tau A^{1/2})] f_1 \right. \\ & \left. + \sum_{s=2}^{k-1} \frac{1}{2} [R^{k-s} (-\tau A^{1/2}) + R^{k-s} (\tau A^{1/2})] [f_{s-1} - f_s] \right], 2 \leq k \leq N \end{aligned}$$

formülleri elde edilir. Buradan, (3.20), (3.21) formülleri ve (3.2), (3.3) kestirimleri yardımıyla,

$$\|u_1\|_H \leq M \left[\|u_{-N}\|_H + \|A^{-1/2} g_0\|_H + \max_{-N < k \leq 0} \|A^{-1/2} (g_k - g_{k-1}) \tau^{-1}\|_H \right], \quad (3.22)$$

$$\|u_k\|_H \leq M \left[\|u_{-N}\|_H + \|A^{-1/2} f_1\|_H + \max_{2 \leq k < N} \|A^{-1/2} (f_k - f_{k-1}) \tau^{-1}\|_H \right]$$

$$+ \|A^{-1/2} g_0\|_H + \max_{-(N-1) \leq k \leq 0} \|A^{-1/2} (g_k - g_{k-1}) \tau^{-1}\|_H], 2 \leq k \leq N \quad (3.23)$$

kestirimleri alınır. Sonuç olarak, (3.17), (3.19), (3.22) ve (3.23) kestirimleri birleştirilerek ve üçgen eşitsizliği kullanılarak, (3.6) kestirimi elde edilir. Burada (3.7) kestiriminin ispatı, (3.16), (3.18) formüllerine ve (3.8), (3.4), (3.2), (3.3) kestirimlerine bağlı kalmak şartıyla, (3.6) kestiriminin ispatında olduğu gibi aynı yolla elde edilir.

Üçüncü olarak, $\|Au_{-N}\|_H$ için kestirim elde edilecektir. Bunun için, Abel formülü ve (3.15) formülü kullanılarak,

$$\begin{aligned}
Au_{-N} = & T_\tau \left(\sum_{j=1}^N \alpha_j \left[\left\{ \left\{ \frac{1}{2} [R^{[\mu_j/\tau]-1} (\tau A^{1/2}) + R^{[\mu_j/\tau]-1} (-\tau A^{1/2})] \right. \right. \right. \\
& + \frac{1}{2i} A^{1/2} [R^{[\mu_j/\tau]-1} (\tau A^{1/2}) + R^{[\mu_j/\tau]-1} (-\tau A^{1/2})] \left. \right\} \\
& - \frac{1}{2i} A^{1/2} (I + \tau^2 A) [R^{[\mu_j/\tau]} (-\tau A^{1/2}) - R^{[\mu_j/\tau]} (\tau A^{1/2})] \left. \right\} \\
& \times [-R^N g_{-N+1} + (\tau A)^{-1} [(g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \\
& + \sum_{s=-N+2}^{-1} R^{-s} (g_{s+1} - 2g_s + g_{s-1})] \left. \right] \\
& + \left[f_{[\mu_j/\tau]-1} - \frac{1}{2} [R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) + R^{[\mu_j/\tau]-1} (\tau A^{1/2})] f_1 \right. \\
& + \left. \sum_{s=2}^{[\mu_j/\tau]-1} \frac{1}{2} [R^{[\mu_j/\tau]-s} (-\tau A^{1/2}) + R^{[\mu_j/\tau]-s} (\tau A^{1/2})] [f_{s-1} - f_s] \right] \\
& + \sum_{j=1}^L \beta_j \tau^{-1} \left(\left\{ \left\{ \frac{1}{2} [R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-1} (-\tau A^{1/2})] \right. \right. \right. \\
& + \frac{1}{2i} A^{1/2} [R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-1} (-\tau A^{1/2})] \left. \right\} \\
& - \frac{1}{2i} A^{1/2} (I + \tau^2 A) [R^{[\lambda_j/\tau]} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]} (\tau A^{1/2})] \left. \right\} \\
& \times [-R^N g_{-N+1} + (\tau A)^{-1} [(g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \\
& + \sum_{s=-N+2}^{-1} R^{-s} (g_{s+1} - 2g_s + g_{s-1})] \left. \right] \\
& + \left[f_{[\lambda_j/\tau]-1} - \frac{1}{2} [R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) + R^{[\lambda_j/\tau]-1} (\tau A^{1/2})] f_1 \right. \\
& + \left. \sum_{s=2}^{[\lambda_j/\tau]-1} \frac{1}{2} [R^{[\lambda_j/\tau]-s} (-\tau A^{1/2}) + R^{[\lambda_j/\tau]-s} (\tau A^{1/2})] [f_{s-1} - f_s] \right] \\
& - \left\{ \left\{ \frac{1}{2} [R^{[\lambda_j/\tau]-2} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-2} (-\tau A^{1/2})] \right. \right. \\
& + \frac{1}{2i} A^{1/2} [R^{[\lambda_j/\tau]-2} (\tau A^{1/2}) + R^{[\lambda_j/\tau]-2} (-\tau A^{1/2})] \left. \right\} \\
& + \frac{1}{2i} A^{1/2} (I + \tau^2 A) [R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]-1} (\tau A^{1/2})] \left. \right\} \\
& \times [-R^N g_{-N+1} + (\tau A)^{-1} [(g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2})
\end{aligned}$$

$$\begin{aligned}
& + \sum_{s=-N+2}^{-1} R^{-s} (g_{s+1} - 2g_s + g_{s-1}) \Bigg] + A\varphi \\
& - \left[f_{[\lambda_j/\tau]-2} - \frac{1}{2} [R^{[\lambda_j/\tau]-2} (-\tau A^{1/2}) + R^{[\lambda_j/\tau]-2} (\tau A^{1/2})] f_1 \right. \\
& \left. + \sum_{s=2}^{[\lambda_j/\tau]-2} \frac{1}{2} [R^{[\lambda_j/\tau]-1-s} (-\tau A^{1/2}) + R^{[\lambda_j/\tau]-1-s} (\tau A^{1/2})] [f_{s-1} - f_s] \right] \Bigg)
\end{aligned}$$

formülü elde edilir. Buradan, (3.16) formülü ve (3.2), (3.3) kestirimleri yardımıyla,

$$\begin{aligned}
\|Au_{-N}\|_H & \leq M [\|A\varphi\|_H + \|A^{1/2}f_1\|_H + \|(f_2 - f_1)\tau^{-1}\|_H \\
& + \max_{2 \leq k \leq N-2} \|(f_{k+1} - 2f_k + f_{k-1})\tau^{-2}\|_H + \|A^{1/2}g_0\|_H \\
& + \|(g_0 - g_{-1})\tau^{-1}\|_H + \max_{-(N-1) \leq k \leq -1} \|(g_{k+1} - 2g_k + g_{k-1})\tau^{-2}\|_H]
\end{aligned} \tag{3.24}$$

kestirimi alınır.

Son olarak, $\|Au_k\|_H$ için kestirim elde edilecektir. Bunun için önce $-N+1 \leq k \leq 0$ aralığında inceleyelim. Bu durumda, Abel formülü ve (3.12) formülü kullanılarak,

$$\begin{aligned}
Au_k & = R^{N+k}Au_{-N} + [g_k - R^{N+k-1}g_{-N+1} + (\tau A)^{-1}[(g_{k-1} - g_k) \\
& - R^{k+N-1}(g_{-N+1} - g_{-N+2}) + \sum_{s=-N+2}^{k-1} R^{k-s}(g_{s+1} - 2g_s + g_{s-1})]
\end{aligned}$$

formülü elde edilir. Buradan, (3.18) formülü ve (3.3) kestirimi yardımıyla,

$$\begin{aligned}
\|Au_k\|_H & \leq M [\|Au_{-N}\|_H + \|\tau^{-1}(g_{-1} - g_0)\|_H + \|A^{1/2}g_0\|_H \\
& + \max_{-(N-2) \leq k \leq -1} \|\tau^{-2}(g_{s+1} - 2g_s + g_{s-1})\|_H]
\end{aligned} \tag{3.25}$$

kestirimi alınır. Şimdi ise $1 \leq k \leq N$ aralığında inceleyelim. Bu durumda, Abel formülü ve (3.13), (3.14) formülleri kullanılarak,

$$\begin{aligned}
Au_1 & = R^N Au_{-N} + [g_0 - R^N g_{-N+1} + (\tau A)^{-1}[(g_{-1} - g_0) \\
& - R^{N-1}(g_{-N+1} - g_{-N+2}) + \sum_{s=-N+2}^{-1} R^{-s}(g_{s+1} - 2g_s + g_{s-1})] \\
& - \tau A (R^N Au_{-N} - R^N g_{-N+1} + (\tau A)^{-1}[(g_{-1} - g_0) \\
& - R^{N-1}(g_{-N+1} - g_{-N+2}) + \sum_{s=-N+2}^{-1} R^{-s}(g_{s+1} - 2g_s + g_{s-1})]
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
Au_k = & \left\{ \left\{ \frac{1}{2} [R^{k-1} (\tau A^{1/2}) + R^{k-1} (-\tau A^{1/2})] \right. \right. \\
& + \frac{1}{2i} A^{1/2} [R^{k-1} (\tau A^{1/2}) + R^{k-1} (-\tau A^{1/2})] \Big\} \\
& - \frac{1}{2i} A^{1/2} (I + \tau^2 A) [R^k (-\tau A^{1/2}) - R^k (\tau A^{1/2})] \Big\} \\
& \times [R^N Au_{-N} - R^N g_{-N+1} + (\tau A)^{-1} [(g_{-1} - g_0) \\
& - R^{N-1} (g_{-N+1} - g_{-N+2}) + \sum_{s=-N+2}^{-1} R^{-s} (g_{s+1} - 2g_s + g_{s-1})] \Big] + f_{k-1} \\
& - \frac{1}{2} [R^{k-1} (-\tau A^{1/2}) + R^{k-1} (\tau A^{1/2})] f_1 + (i\tau A^{1/2})^{-1} \left\{ \frac{1}{2} [-R^{k-2} (-\tau A^{1/2}) \right. \\
& + R^{k-2} (\tau A^{1/2})] [f_1 - f_2] + \sum_{s=2}^{k-2} \frac{1}{2} [-R^{k-s-1} (-\tau A^{1/2}) \\
& + R^{k-s-1} (\tau A^{1/2})] [f_{s+1} - 2f_s + f_{s-1}] \Big\}, 2 \leq k \leq N
\end{aligned} \tag{3.27}$$

formülleri elde edilir. Buradan, son iki formül ve (3.2), (3.3) kestirimleri yardımıyla,

$$\begin{aligned}
\|Au_1\|_H \leq & M [\|Au_{-N}\|_H + \|A^{1/2}g_0\|_H + \|\tau^{-1}(g_{-1} - g_0)\|_H \\
& + \max_{-(N-1) \leq k \leq -1} \|\tau^{-1}[g_{s+1} - 2g_s + g_{s-1}]\|_H] ,
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
\|Au_k\|_H \leq & M [\|Au_{-N}\|_H + \|A^{1/2}f_1\|_H + \|(f_1 - f_2)\tau^{-1}\|_H \\
& + \max_{2 \leq k \leq N-2} \|(f_{k+1} - 2f_k + f_{k-1})\tau^{-2}\|_H + \|A^{1/2}g_0\|_H + \|(g_{-1} - g_0)\tau^{-1}\|_H \\
& + \max_{-(N-2) \leq k \leq -1} \|\tau^{-2}(g_{s+1} - 2g_s + g_{s-1})\|_H] , 2 \leq k \leq N
\end{aligned} \tag{3.29}$$

kestirimi alınır. Sonuç olarak, (3.24), (3.25), (3.28) ve (3.29) kestirimleri birleştirilerek ve üçgen eşitsizliği kullanılarak, (3.8) kestirimi elde edilir. Böylece, Teorem 3.1'in ispatı tamamlanmış olur.

3.2 Uygulamalar

İlk olarak, bir boyutlu hiperbolik-parabolik denklem için, (2.21) lokal olmayan sınır değer problemi ele alınacaktır. Burada (2.21) probleminin diskritizasyonu iki adımda incelenir. Birinci adımda önce,

$$[0, 1]_h = \{x : x_n = nh, 0 \leq n \leq K, Kh = 1\}$$

ağ uzayı tanımlanır. Sonra $[0, 1]_h$ aralığında tanımlanan $\varphi^h(x)$ ağ fonksiyonları, $L_{2h} = L_2([0, 1]_h)$ Hilbert uzayı olarak tanımlanır. Bu uzayda norm

$$\|\varphi^h\|_{L_{2h}} = \left(\sum_{n=1}^{K-1} |\varphi^h(x)|^2 h \right)^{1/2}$$

formülü ile ifade edilir. Daha sonra da, (2.21) problemi tarafından oluşturulan A diferansiyel operatörü yerine

$$A_h^x \varphi^h(x) = \left\{ -(a(x)\varphi_x^-)_{x,n} + \delta \varphi^n \right\}_1^{K-1} \quad (3.30)$$

formülüyle tanımlanan A_h^x fark operatörü alınır. Burada, A_h^x fark operatörü

$$\varphi^0 = \varphi^K, \varphi^1 - \varphi^0 = \varphi^K - \varphi^{K-1}$$

koşulları sağlayan $\varphi^h(x) = \{\varphi^n\}_0^K$ ağ fonksiyonları uzayında tanımlanmıştır. Bilindiği gibi, L_{2h} Hilbert uzayında A_h^x self-adjoint pozitif tanımlı operatördür. O halde, A_h^x fark operatörünün yardımıyla (2.21) lokal olmayan sınır değer problemi

$$\begin{cases} \frac{d^2 v^h(t, x)}{dy^2} + A_h^x v^h(y, x) = f^h(y, x), & 0 \leq y \leq 1, \quad x \in [0, 1]_h, \\ \frac{dv^h(t, x)}{dy} + A_h^x v^h(y, x) = f^h(y, x), & -1 \leq y \leq 0, \quad x \in [0, 1]_h, \\ v^h(-1, x) = \sum_{j=1}^N \alpha_j v^h(\mu_j, x) + \sum_{j=1}^L \beta_j v_t^h(\lambda_j, x) + \varphi^h(x), & x \in [0, 1]_h, \\ v^h(0+, x) = v^h(0-, x), \frac{dv^h(0+, x)}{dy} = \frac{dv^h(0-, x)}{dy}, & x \in [0, 1]_h \end{cases} \quad (3.31)$$

adi diferansiyel denklem sistemine dönüştürülür. İkinci adımda ise, (3.31) problemi için (3.1) fark şeması kullanılarak,

$$\begin{cases} \frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_{k+1}^h = f_k^h(x), & x \in [0, 1]_h, \\ f_{k+1}^h(x) = \{f(y_{k+1}, x_n)\}_1^{K-1}, & y_{k+1} = (k+1)\tau, 1 \leq k \leq N-1, \quad N\tau = 1, \\ \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + A_h^x u_k^h = g_k^h(x), & x \in [0, 1]_h, \\ g_k^h(x) = \{g(y_k, x_n)\}_1^{K-1}, & y_k = k\tau, -N+1 \leq k \leq 0, \\ u_{-N}^h(x) = \sum_{j=1}^N \alpha_j u_{[\mu_j/\tau]}^h(x) + \sum_{j=1}^L \beta_j \frac{u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)}{\tau} + \varphi^h(x), & x \in [0, 1]_h, \\ \frac{u_1^h(x) - u_0^h(x)}{\tau} = -A_h^x u_0^h(x) + g_0^h(x), & g_0^h(x) = g^h(0, x), \quad x \in [0, 1]_h \end{cases} \quad (3.32)$$

fark şeması elde edilir.

Teorem 3.2. Eğer τ ve h yeterince küçük sayılar ise, bu durumda (3.32) fark şemasının çözümü için aşağıdaki

$$\max_{-N \leq k \leq N} \|u_k^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq N} \|\tau^{-1} (u_k^h - u_{k-1}^h)\|_{L_{2h}} + \max_{-N \leq k \leq N} \|(u_k^h)_x\|_{L_{2h}}$$

$$\begin{aligned}
&\leq M_1 \left[\|f_1^h\|_{L_{2h}} + \max_{2 \leq k \leq N-1} \|(f_k^h - f_{k-1}^h) \tau^{-1}\|_{L_{2h}} \right. \\
&\quad \left. + \|g_0^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \|(g_k^h - g_{k-1}^h) \tau^{-1}\|_{L_{2h}} + \left\| \varphi_{\frac{h}{x}}^h \right\|_{L_{2h}} \right], \\
&\quad \max_{1 \leq k \leq N-1} \|\tau^{-2} (u_{k+1}^h - 2u_k^h + u_{k-1}^h)\|_{L_{2h}} \\
&\quad + \max_{-N \leq k \leq N} \left\| \left(u_{\frac{h}{x}}^k \right)_x \right\|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \|\tau^{-1} (u_k^h - u_{k-1}^h)\|_{L_{2h}} \\
&\leq M_1 \left[\|f_{1x}^h\|_{L_{2h}} + \|\tau^{-1} (f_2^h - f_1^h)\|_{L_{2h}} \right. \\
&\quad + \max_{2 \leq k \leq N-1} \|\tau^{-2} (f_{k+1}^h - 2f_k^h + f_{k-1}^h)\|_{L_{2h}} + \|g_{0x}^h\|_{L_{2h}} + \|\tau^{-1} (g_0^h - g_{-1}^h)\|_{L_{2h}} \\
&\quad \left. + \max_{-N+1 \leq k \leq -1} \|\tau^{-2} (g_{k+1}^h - 2g_k^h + g_{k-1}^h)\|_{L_{2h}} + \left\| \left(\varphi_{\frac{h}{x}}^h \right)_x \right\|_{L_{2h}} \right]
\end{aligned}$$

kararlılık kestirimleri sağlanır. Burada, M_1 katsayısı $\tau, h, f_k^h, 1 \leq k < N, g_k^h, -N < k \leq 0$ ve $\varphi^h(x)$ 'den bağımsızdır.

Teorem 3.2'nin ispatı, soyut Teorem 3.1 ve (3.30) formülü ile tanımlanan A_h^x fark operatörünün simetri özelliklerine dayanmaktadır.

İkinci olarak, çok boyutlu hiperbolik-parabolik denklem için (2.22) karma sınır değer problemi ele alınacaktır. Burada da (2.22) probleminin diskritizasyonu iki adımda incelenir. Birinci adımda önce,

$$\tilde{\Omega}_h = \{x = x_m = (h_1 m_1, \dots, h_n m_n), m = (m_1, \dots, m_n),$$

$$0 \leq m_r \leq N_r, h_r N_r = L, r = 1, \dots, n\}, \Omega_h = \tilde{\Omega}_h \cap \Omega, S_h = \tilde{\Omega}_h \cap S$$

ağ uzayı tanımlanır. Sonra $\tilde{\Omega}_h$ kümesinde tanımlanan $\varphi^h(x) = \{\varphi(h_1 m_1, \dots, h_n m_n)\}$ ağ fonksiyonları, $L_{2h} = L_2(\tilde{\Omega}_h)$ Banach uzayı olarak tanımlanır. Bu uzayda norm

$$\|\varphi^h\|_{L_2(\tilde{\Omega}_h)} = \left(\sum_{x \in \tilde{\Omega}_h} |\varphi^h(x)|^2 h_1 \cdots h_n \right)^{1/2}$$

formülüyle ifade edilir. Daha sonra da, (2.22) problemi tarafından oluşturulan A diferansiyel operatörü yerine

$$A_h^x u_x^h = - \sum_{r=1}^n \left(a_r(x) u_{x_r}^h \right)_{x_r, j_r} \quad (3.33)$$

formülüyle tanımlanan A_h^x fark operatörü alınır. Burada, A_h^x fark operatörü $\forall x \in S_h$ değerleri için $u^h(x) = 0$ koşullarını sağlayan $u^h(x)$ ağ fonksiyonları uzayında tanımlanmış-

tır. Bilindiği gibi, $L_2(\tilde{\Omega}_h)$ uzayında A_h^x self-adjoint pozitif tanımlı operatördür. O halde, A_h^x fark operatörünün yardımıyla (2.22) lokal olmayan sınır değer problemi

$$\begin{cases} \frac{d^2 v^h(t, x)}{dy^2} + A_h^x v^h(y, x) = f^h(y, x), & 0 \leq y \leq 1, \quad x \in \tilde{\Omega}_h, \\ \frac{dv^h(y, x)}{dy} + A_h^x v^h(y, x) = f^h(y, x), & -1 \leq y \leq 0, \quad x \in \tilde{\Omega}_h, \\ v^h(-1, x) = \sum_{j=1}^N \alpha_j v^h(\mu_j, x) + \sum_{j=1}^L \beta_j v_t^h(\lambda_j, x) + \varphi^h(x), & x \in \tilde{\Omega}_h, \\ v^h(0+, x) = v^h(0-, x), \quad \frac{dv^h(0+, x)}{dy} = \frac{dv^h(0-, x)}{dy}, & x \in \tilde{\Omega}_h \end{cases} \quad (3.34)$$

adi diferansiyel denklem sistemine dönüştürülür. İkinci adımda ise, (3.34) problemi için (3.1) fark şeması kullanılarak,

$$\begin{cases} \frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_{k+1}^h = f_k^h(x), & x \in \tilde{\Omega}_h, \\ f_{k+1}^h(x) = f^h(y_{k+1}, x_n), & y_{k+1} = (k+1)\tau, \quad 1 \leq k \leq N-1, \quad N\tau = 1, \quad x \in \tilde{\Omega}_h, \\ \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + A_h^x u_k^h = g_k^h(x), & x \in \tilde{\Omega}_h, \\ g_k^h(x) = g^h(y_k, x_n), & y_k = k\tau, \quad -N+1 \leq k \leq -1, \quad x \in \tilde{\Omega}_h, \\ u_{-N}^h(x) = \sum_{j=1}^N \alpha_j u_{[\mu_j/\tau]}^h(x) + \sum_{j=1}^L \beta_j \frac{u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)}{\tau} + \varphi^h(x), & x \in \tilde{\Omega}_h, \\ \frac{u_1^h(x) - u_0^h(x)}{\tau} = -A_h^x u_0^h(x) + g_0^h(x), & g_0^h(x) = g^h(0, x), \quad x \in \tilde{\Omega}_h \end{cases} \quad (3.35)$$

fark şeması elde edilir.

Teorem 3.3. Eğer τ ve $|h|$ yeterince küçük sayılar ise, bu durumda (3.35) fark şemasının çözümü için aşağıdaki

$$\begin{aligned} & \max_{-N \leq k \leq N} \|u_k^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq N} \|\tau^{-1} (u_k^h - u_{k-1}^h)\|_{L_{2h}} + \max_{-N \leq k \leq N} \sum_{r=1}^n \|(u_k^h)_{x_r, j_r}\|_{L_{2h}} \\ & \leq M_1 \left[\|f_1^h\|_{L_{2h}} + \max_{2 \leq k \leq N-1} \|(f_k^h - f_{k-1}^h) \tau^{-1}\|_{L_{2h}} \right. \\ & \quad \left. + \|g_0^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \|(g_k^h - g_{k-1}^h) \tau^{-1}\|_{L_{2h}} + \sum_{r=1}^n \|(\varphi^h)_{\bar{x}_r, j_r}\|_{L_{2h}} \right], \\ & \quad \max_{1 \leq k \leq N-1} \|\tau^{-2} (u_{k+1}^h - 2u_k^h + u_{k-1}^h)\|_{L_{2h}} \\ & \quad + \max_{-N \leq k \leq N} \sum_{r=1}^n \|(u_k^h)_{\bar{x}_r, j_r}\|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \|\tau^{-1} (u_k^h - u_{k-1}^h)\|_{L_{2h}} \\ & \leq M_1 \left[\sum_{r=1}^n \|(f_1^h)_{\bar{x}_r, j_r}\|_{L_{2h}} + \|\tau^{-1} (f_2^h - f_1^h)\|_{L_{2h}} \right. \\ & \quad \left. + \max_{2 \leq k \leq N-1} \|\tau^{-2} (f_{k+1}^h - 2f_k^h + f_{k-1}^h)\|_{L_{2h}} \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{r=1}^n \left\| (g_0^h)_{\bar{x}_r, j_r} \right\|_{L_{2h}} + \left\| \tau^{-1} (g_0^h - g_{-1}^h) \right\|_{L_{2h}} \\
& + \max_{-N+1 \leq k \leq -1} \left\| \tau^{-2} (g_{k+1}^h - 2g_k^h + g_{k-1}^h) \right\|_{L_{2h}} + \sum_{r=1}^n \left\| (\varphi^h)_{\bar{x}_r x_r, j_r} \right\|_{L_{2h}} \Big]
\end{aligned}$$

kararlılık kestirimleri sağlanır. Burada, M_1 katsayısı $\tau, h, f_k^h, 1 \leq k < N, g_k^h, -N < k \leq 0$ ve $\varphi^h(x)$ 'den bağımsızdır.

Teorem 3.3'ün ispatı, soyut Teorem 3.1, (3.33) formülü ile tanımlanan A_h^x fark operatörünün simetri özelliklerine ve aşağıdaki L_{2h} uzayındaki eliptik fark probleminin çözümü için koersiv kestirimi elde edilen teoreme dayanmaktadır.

Teorem 3.4. Eliptik fark probleminin

$$A_h^x u^h(x) = \omega^h(x), x \in \Omega_h, \quad (3.36)$$

$$u^h(x) = 0, x \in S_h$$

çözümü için

$$\sum_{r=1}^n \left\| (u^h)_{\bar{x}_r x_r, j_r} \right\|_{L_{2h}} \leq M \|\omega^h\|_{L_{2h}}$$

koersiv kestirimi sağlanır [Sobolevskii, P. E., 1975].

4 İKİNCİ BASAMAKTAN DOĞRULUKLU FARK ŞEMALARI

4.1 Fark Şeması 1

Bu kısımda, (1.1) lokal olmayan sınır değer probleminin yaklaşık çözümlerini bulmak için

$$\left\{ \begin{array}{l} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k + \frac{\tau^2}{4}A^2u_{k+1} = f_k, f_k = f(t_k), \\ t_k = k\tau, \ 1 \leq k < N, \tau^{-1}(I + \tau^2A)(u_1 - u_0) = Z_1, \\ Z_1 = \frac{\tau}{2}(f(0) - Au_0) + (g(0) - Au_0), \\ \tau^{-1}(u_k - u_{k-1}) + A(I + \frac{\tau}{2}A)u_k = (I + \frac{\tau}{2}A)g_k, \\ g_k = g(t_k - \frac{\tau}{2}), \ t_k = k\tau, \ -(N-1) \leq k \leq 0, \\ u_{-N} = \sum_{j=1}^N \alpha_j \left(u_{[\mu_j/\tau]} + (\mu_j - [\frac{\mu_j}{\tau}]\tau) \frac{u_{[\mu_j/\tau]} - u_{[\mu_j/\tau]-1}}{\tau} \right) \\ + \sum_{j=1}^L \beta_j \left(\frac{u_{[\lambda_j/\tau]} - u_{[\lambda_j/\tau]-1}}{\tau} + \left(\lambda_j - [\frac{\lambda_j}{\tau}]\tau + \frac{\tau}{2} \right) \left(f_{[\frac{\lambda_j}{\tau}]} - Au_{[\frac{\lambda_j}{\tau}]} \right) \right) + \varphi \end{array} \right. \quad (4.1)$$

ikinci basamaktan doğruluklu fark şeması kurulmuştur.

Öncelikle ileride ihtiyaç duyacağımız, yardımcı teoremleri verelim.

Yardımcı Teorem 4.1. Aşağıdaki

$$\|R(\pm\tau A^{1/2})\|_{H \mapsto H} \leq 1, \|\tau A^{1/2}R(\pm\tau A^{1/2})\|_{H \mapsto H} \leq 1, \quad (4.2)$$

$$\|\tau^2 AR(\pm\tau A^{1/2})\|_{H \mapsto H} \leq 1, (k\tau)^\beta \|A^\beta R^k\|_{H \mapsto H} \leq M, k \geq 1, 0 \leq \beta \leq 1 \quad (4.3)$$

kestirimler sağlanırlar. Burada,

$$R(\pm\tau A^{1/2}) = \left(I \pm i\tau A^{1/2} - \frac{\tau^2}{2}A \right)^{-1} \text{ ve } R = \left(I + \tau A + \frac{\tau^2}{2}A^2 \right)^{-1}$$

dir.

Yardımcı Teorem 4.2. Öncelikle,

$$\begin{aligned} I - \sum_{j=1}^N \alpha_j \left(1 + \left(\frac{\mu_j}{\tau} - [\frac{\mu_j}{\tau}] \right) \right) \left\{ \frac{1}{2} \left[R^{[\mu_j/\tau]-1} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \\ \left. \left. + R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] R^N \right\} \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2i}A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) [R^{[\mu_j/\tau]} (-\tau A^{1/2}) - R^{[\mu_j/\tau]} (\tau A^{1/2})] \\
& \quad \times (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) R^N \} \Bigg) \\
& - \sum_{j=1}^N \alpha_j \left(\frac{\mu_j}{\tau} - [\frac{\mu_j}{\tau}] \right) \left\{ \frac{1}{2} \left[R^{[\mu_j/\tau]-2} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \\
& \quad \left. \left. + R^{[\mu_j/\tau]-2} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] R^N \right. \\
& - \frac{1}{2i}A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) [R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) - R^{[\mu_j/\tau]-1} (\tau A^{1/2})] \\
& \quad \times (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) R^N + \sum_{j=1}^L \beta_j \frac{1}{\tau} \left(I - \left(\lambda_j - [\frac{\lambda_j}{\tau}] \tau + \frac{\tau}{2} \right) A \right) \\
& \times \left\{ \frac{1}{2} \left[R^{[\frac{\lambda_j}{\tau}-1} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) + R^{[\frac{\lambda_j}{\tau}-1} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] R^N \right. \\
& \quad \left. - \frac{1}{2i}A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) [R^{[\frac{\lambda_j}{\tau}]} (-\tau A^{1/2}) - R^{[\frac{\lambda_j}{\tau}]} (\tau A^{1/2})] \right. \\
& \quad \left. \times (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) R^N \right\} - \sum_{j=1}^L \beta_j \frac{1}{\tau} \\
& \times \left\{ \frac{1}{2} \left[R^{[\frac{\lambda_j}{\tau}-2} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) + R^{[\frac{\lambda_j}{\tau}-2} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] R^N \right. \\
& \quad \left. - \frac{1}{2i}A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) [R^{[\frac{\lambda_j}{\tau}-1} (-\tau A^{1/2}) - R^{[\frac{\lambda_j}{\tau}-1} (\tau A^{1/2})] \right. \\
& \quad \left. \times (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) R^N \right.
\end{aligned}$$

operatörünün tersi vardır ve

$$\begin{aligned}
T_\tau = & \left(I - \sum_{j=1}^N \alpha_j \left(1 + \left(\frac{\mu_j}{\tau} - [\frac{\mu_j}{\tau}] \right) \right) \left\{ \frac{1}{2} \left[R^{[\mu_j/\tau]-1} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \right. \\
& \quad \left. \left. + R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] R^N \right. \\
& - \frac{1}{2i}A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) [R^{[\mu_j/\tau]} (-\tau A^{1/2}) - R^{[\mu_j/\tau]} (\tau A^{1/2})] \\
& \quad \left. \times (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) R^N \right\} \Bigg) \\
& - \sum_{j=1}^N \alpha_j \left(\frac{\mu_j}{\tau} - [\frac{\mu_j}{\tau}] \right) \left\{ \frac{1}{2} \left[R^{[\mu_j/\tau]-2} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \\
& \quad \left. \left. + R^{[\mu_j/\tau]-2} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] R^N \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2i}A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \left[R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) - R^{[\mu_j/\tau]-1} (\tau A^{1/2}) \right] \\
& \times (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) R^N \} + \sum_{j=1}^L \beta_j \frac{1}{\tau} \left(I - \left(\lambda_j - [\frac{\lambda_j}{\tau}] \tau + \frac{\tau}{2} \right) A \right) \\
& \times \left\{ \frac{1}{2} \left[R^{[\frac{\lambda_j}{\tau}]-1} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) + R^{[\frac{\lambda_j}{\tau}]-1} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] R^N \right. \\
& \quad \left. - \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \left[R^{[\frac{\lambda_j}{\tau}]} (-\tau A^{1/2}) - R^{[\frac{\lambda_j}{\tau}]} (\tau A^{1/2}) \right] \right. \\
& \quad \left. \times (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) R^N \right\} - \sum_{j=1}^L \beta_j \frac{1}{\tau} \\
& \times \left\{ \frac{1}{2} \left[R^{[\frac{\lambda_j}{\tau}]-2} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) + R^{[\frac{\lambda_j}{\tau}]-2} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] R^N \right. \\
& \quad \left. - \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \left[R^{[\frac{\lambda_j}{\tau}]-1} (-\tau A^{1/2}) - R^{[\frac{\lambda_j}{\tau}]-1} (\tau A^{1/2}) \right] \right. \\
& \quad \left. \times (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) R^N \right\}^{-1}
\end{aligned}$$

aşağıdaki

$$\|T_\tau\|_{H \mapsto H} \leq M \quad (4.4)$$

kestirimi sağlar. Burada, M katsayısı τ 'dan bağımsızdır.

İspat. İlk olarak,

$$\begin{aligned}
& T_\tau - \left(I - \sum_{j=1}^N \alpha_j ([c([\mu_j/\tau]\tau) - As([\mu_j/\tau]\tau)] \right. \\
& \quad \left. - \sum_{j=1}^L \beta_j [s([\lambda_j/\tau]\tau) + c([\lambda_j/\tau]\tau)] A \right) e^{-A} \Big)^{-1} \\
& = T_\tau \left(I - \sum_{j=1}^N \alpha_j ([c([\mu_j/\tau]\tau) - As([\mu_j/\tau]\tau)] \right. \\
& \quad \left. - \sum_{j=1}^L \beta_j [s([\lambda_j/\tau]\tau) + c([\lambda_j/\tau]\tau)] A \right) e^{-A} \Big)^{-1} \\
& \times \left[\left[\sum_{j=1}^N \alpha_j \left(1 + \left(\frac{\mu_j}{\tau} - [\frac{\mu_j}{\tau}] \right) \right) \left\{ \frac{1}{2} \left[R^{[\mu_j/\tau]-1} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \right. \right. \\
& \quad \left. \left. \left. + R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] - \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \right. \right. \right. \\
& \quad \left. \left. \times [R^{[\mu_j/\tau]} (-\tau A^{1/2}) - R^{[\mu_j/\tau]} (\tau A^{1/2})] (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) \right\} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^N \alpha_j \left(\frac{\mu_j}{\tau} - \left[\frac{\mu_j}{\tau} \right] \right) \left\{ \frac{1}{2} \left[R^{[\mu_j/\tau]-2} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \\
& \quad \left. \left. + R^{[\mu_j/\tau]-2} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] - \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \right. \\
& \quad \left. \times \left[R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) - R^{[\mu_j/\tau]-1} (\tau A^{1/2}) \right] (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) \right\} \\
& - \sum_{j=1}^L \beta_j \frac{1}{\tau} \left(I - \left(\lambda_j - \left[\frac{\lambda_j}{\tau} \right] \tau - \frac{\tau}{2} \right) A \right) \left\{ \frac{1}{2} \left[R^{[\frac{\lambda_j}{\tau}]-1} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \\
& \quad \left. \left. + R^{[\frac{\lambda_j}{\tau}]-1} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] - \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \right. \\
& \quad \left. \times \left[R^{[\frac{\lambda_j}{\tau}]-1} (-\tau A^{1/2}) - R^{[\frac{\lambda_j}{\tau}]-1} (\tau A^{1/2}) \right] (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) \right\} \\
& \quad + \sum_{j=1}^L \beta_j \frac{1}{\tau} \left\{ \frac{1}{2} \left[R^{[\frac{\lambda_j}{\tau}]-2} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \\
& \quad \left. \left. + R^{[\frac{\lambda_j}{\tau}]-2} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] - \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \right. \\
& \quad \left. \times \left[R^{[\frac{\lambda_j}{\tau}]-1} (-\tau A^{1/2}) - R^{[\frac{\lambda_j}{\tau}]-1} (\tau A^{1/2}) \right] (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) \right\} R^N \\
& \quad - \sum_{j=1}^N \alpha_j \left([c([\mu_j/\tau]\tau) - As([\mu_j/\tau]\tau)] - \right. \\
& \quad \left. \sum_{j=1}^L \beta_j [s([\lambda_j/\tau]\tau) + c([\lambda_j/\tau]\tau)] A \right) e^{-A} \Big]
\end{aligned}$$

formülü doğrudur. O halde,

$$\begin{aligned}
& \left\| \left(\left\{ \sum_{j=1}^N \alpha_j \left(1 + \left(\frac{\mu_j}{\tau} - \left[\frac{\mu_j}{\tau} \right] \right) \right) \frac{1}{2} \left[R^{[\mu_j/\tau]-1} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \right. \right. \\
& \quad \left. \left. + R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] - \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \right. \\
& \quad \left. \times \left[R^{[\mu_j/\tau]} (-\tau A^{1/2}) - R^{[\mu_j/\tau]} (\tau A^{1/2}) \right] (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) \right\} \\
& \quad + \sum_{j=1}^N \alpha_j \left(\frac{\mu_j}{\tau} - \left[\frac{\mu_j}{\tau} \right] \right) \frac{1}{2} \left[R^{[\mu_j/\tau]-2} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \\
& \quad \left. + R^{[\mu_j/\tau]-2} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] - \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \\
& \quad \times \left[R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) - R^{[\mu_j/\tau]-1} (\tau A^{1/2}) \right] \\
& \quad \times \left((I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) + \sum_{j=1}^L \beta_j (1 + \lambda_j) A \right) R^N
\end{aligned}$$

$$- \left(\sum_{j=1}^N \alpha_j [c([\mu_j/\tau]\tau) - As([\mu_j/\tau]\tau)] - \sum_{j=1}^L \beta_j A \right) e^{-A} \Big\|_{H \rightarrow H} \leq M\tau, \quad (4.5)$$

$$\begin{aligned} & \left\| \left[\sum_{j=1}^N \alpha_j (I - \mu_j A) + \sum_{j=1}^L \beta_j \frac{1}{\tau} \left(I - \left(\lambda_j - \left[\frac{\lambda_j}{\tau} \right] \tau - \frac{\tau}{2} \right) A \right) \right. \right. \\ & \quad \times \left\{ \frac{1}{2} \left[R^{[\frac{\lambda_j}{\tau}] - 1} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \\ & \quad \left. \left. + R^{[\frac{\lambda_j}{\tau}] - 1} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] - \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \right. \\ & \quad \left. \times \left[R^{[\frac{\lambda_j}{\tau}]} (-\tau A^{1/2}) - R^{[\frac{\lambda_j}{\tau}]} (\tau A^{1/2}) \right] (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) \right\} \\ & \quad + \sum_{j=1}^L \beta_j \frac{1}{\tau} \left\{ \frac{1}{2} \left[R^{[\frac{\lambda_j}{\tau}] - 2} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) + R^{[\frac{\lambda_j}{\tau}] - 2} (-\tau A^{1/2}) \right. \right. \\ & \quad \left. \left. \times \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] - \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \left[R^{[\frac{\lambda_j}{\tau}] - 1} (-\tau A^{1/2}) \right. \right. \\ & \quad \left. \left. - R^{[\frac{\lambda_j}{\tau}] - 1} (\tau A^{1/2}) \right] (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) \right\} R^N \\ & \left. - \left(\sum_{j=1}^N \alpha_j - \sum_{j=1}^L \beta_j [s([\lambda_j/\tau]\tau) + c([\lambda_j/\tau]\tau)] A \right) e^{-A} \right\|_{H \rightarrow H} \leq M\tau \quad (4.6) \end{aligned}$$

kestirimleri kullanılarak (4.4) kestirimi elde edilir. Buradaki (4.5) ve (4.6) kestirimlerinin ispatları, (4.2), (4.3) kestirimlerine dayanmaktadır. Böylece, Yardımcı Teorem 4.2'nin ispatı tamamlanmış olur.

Teorem 4.1. Eğer $\varphi \in D(A)$, $g_0 \in D(A^{1/2})$ ve $f_0 \in D(A^{1/2})$ ise, bu durumda (4.1) fark şemasının çözümü için aşağıdaki

$$\max_{-N \leq k \leq N} \|u_k\|_H \leq M [\|\varphi\|_H + \|A^{-1/2} f_0\|_H + \|A^{-1/2} g_0\|_H] \quad (4.7)$$

$$+ \max_{1 \leq k \leq N} \left\| \frac{A^{-1/2}}{\tau} (f_k - f_{k-1}) \right\|_H + \max_{-N < k \leq 0} \left\| \frac{A^{-1/2}}{\tau} (g_k - g_{k-1}) \right\|_H \Big],$$

$$\max_{-N \leq k \leq N} \|A^{1/2} u_k\|_H \leq M [\|A^{1/2} \varphi\|_H + \|f_0\|_H + \|g_0\|_H] \quad (4.8)$$

$$+ \max_{1 \leq k \leq N-1} \|(f_k - f_{k-1}) \tau^{-1}\|_H + \max_{-(N-1) \leq k \leq 0} \|(g_k - g_{k-1}) \tau^{-1}\|_H \Big],$$

$$\max_{1 \leq k \leq N-1} \|\tau^{-2} (u_{k+1} - 2u_k + u_{k-1})\|_H + \max_{-N \leq k \leq N} \|Au_k\|_H \quad (4.9)$$

$$\begin{aligned}
& + \max_{-(N-1) \leq k \leq 0} \left\| \tau^{-1} (u_k - u_{k-1}) \right\|_H \leq M \left[\|A\varphi\|_H + \|A^{1/2} f_0\|_H \right. \\
& + \left\| (f_1 - f_0) \tau^{-1} \right\|_H + \max_{1 \leq k \leq N-2} \left\| (f_{k+1} - 2f_k + f_{k-1}) \tau^{-2} \right\|_H + \|A^{1/2} g_0\|_H \\
& \left. + \left\| (g_0 - g_{-1}) \tau^{-1} \right\|_H + \max_{-(N-1) \leq k \leq -1} \left\| (g_{k+1} - 2g_k + g_{k-1}) \tau^{-2} \right\|_H \right]
\end{aligned}$$

kararlılık kestirimleri sağlanır. Burada, M katsayı $\tau, f_k, 0 \leq k < N, g_k, -N < k \leq 0$ ve φ 'den bağımsızdır.

İspat. Herşeyden önce, (4.1) fark şemasının çözümü için formül elde edilecektir. Bilindiği gibi,

$$\begin{cases} \tau^{-2} (u_{k+1} - 2u_k + u_{k-1}) + Au_k + \frac{\tau^2}{4} A^2 u_{k+1} = f_k, f_k = f(t_k), \\ t_k = k\tau, 1 \leq k < N, u_0 = \xi, \tau^{-1}(u_1 - u_0) = \psi, \end{cases} \quad (4.10)$$

$$\begin{cases} \tau^{-1} (u_k - u_{k-1}) + A(I + \frac{\tau}{2} A)u_k = (I + \frac{\tau}{2} A)g_k, \\ g_k = g(t_k - \frac{\tau}{2}), t_k = k\tau, -(N-1) \leq k \leq 0, u_{-N} \text{ biliniyor} \end{cases} \quad (4.11)$$

başlangıç değer fark problemlerinin tek çözümleri vardır ve aşağıdaki formüller

$$u_1 = \xi + \tau\psi, \quad (4.12)$$

$$\begin{aligned}
u_k &= \frac{1}{2} \left[R^{k-1} (\tau A^{1/2}) \left(i - \frac{i\tau A^{1/2}}{2} \right) + R^{k-1} (-\tau A^{1/2}) \left(i + \frac{i\tau A^{1/2}}{2} \right) \right] \xi \\
&+ \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) [R^k (-\tau A^{1/2}) - R^k (\tau A^{1/2})] \psi \\
&+ \sum_{s=1}^{k-1} \frac{\tau}{2i} A^{-1/2} [R^{k-s} (-\tau A^{1/2}) - R^{k-s} (\tau A^{1/2})] f_s, 2 \leq k \leq N,
\end{aligned} \quad (4.13)$$

$$u_k = R^{N+k} u_{-N} + \tau \sum_{s=-N+1}^k \left(I + \frac{\tau}{2} A \right) R^{k-s+1} g_s, -(N-1) \leq k \leq 0 \quad (4.14)$$

sağlanır. Buradaki (4.12), (4.13), (4.14) denklemleri ve

$$\tau^{-1} (u_1 - u_0) = (I + \tau^2 A)^{-1} \left(- \left(\frac{\tau}{2} + 1 \right) A u_0 + \frac{\tau}{2} f(0) + g_0 \right),$$

$$u_0 = \xi, \tau^{-1} (u_1 - u_0) = \psi,$$

formülleri kullanılarak,

$$\xi = u_0 = R^N u_{-N} + \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s,$$

$$\begin{aligned} \psi &= \tau^{-1} (u_1 - u_0) = (I + \tau^2 A)^{-1} \left\{ -A \left(\frac{\tau}{2} + 1 \right) \right. \\ &\quad \times \left[R^N u_{-N} + \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \right] + \frac{\tau}{2} f(0) + g_0 \left. \right\} \end{aligned}$$

formülleri elde edilir. Dolayısıyla,

$$\begin{aligned} u_1 &= \left(I - \tau A \left(\frac{\tau}{2} + I \right) (I + \tau^2 A)^{-1} \right) [R^N u_{-N} \\ &\quad + \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s] + (I + \tau^2 A)^{-1} \left\{ \frac{\tau^2}{2} f(0) + \tau g_0 \right\}, \end{aligned} \quad (4.15)$$

$$\begin{aligned} u_k &= \frac{1}{2} \left[R^{k-1} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) + R^{k-1} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] \\ &\quad \times \left[R^N u_{-N} + \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \right] + \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \\ &\quad \times [R^k (-\tau A^{1/2}) - R^k (\tau A^{1/2})] (I + \tau^2 A)^{-1} \left\{ -A \left(\frac{\tau}{2} + 1 \right) \right. \\ &\quad \times \left[R^N u_{-N} + \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \right] + \frac{\tau}{2} f(0) + g_0 \left. \right\} \\ &\quad + \sum_{s=1}^{k-1} \frac{\tau}{2i} A^{-1/2} [R^{k-s} (-\tau A^{1/2}) - R^{k-s} (\tau A^{1/2})] f_s, \quad 2 \leq k \leq N \end{aligned} \quad (4.16)$$

olur. Buradan,

$$\begin{aligned} u_{-N} &= \sum_{j=1}^N \alpha_j \left(u_{[\mu_j/\tau]} + \left(\mu_j - \left[\frac{\mu_j}{\tau} \right] \tau \right) \frac{u_{[\mu_j/\tau]} - u_{[\mu_j/\tau]-1}}{\tau} \right) \\ &\quad + \sum_{j=1}^L \beta_j \left(\frac{u_{[\lambda_j/\tau]} - u_{[\lambda_j/\tau]-1}}{\tau} + \left(\lambda_j - \left[\frac{\lambda_j}{\tau} \right] \tau - \frac{\tau}{2} \right) (f_{[\lambda_j/\tau]} - A u_{[\lambda_j/\tau]}) \right) + \varphi \end{aligned}$$

koşulu kullanılarak,

$$\begin{aligned} u_{-N} &= T_\tau \left[\sum_{j=1}^N \alpha_j \left(1 + \left(\frac{\mu_j}{\tau} - \left[\frac{\mu_j}{\tau} \right] \right) \right) \left\{ \frac{1}{2} \left[R^{[\mu_j/\tau]-1} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \right. \\ &\quad \left. \left. + R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \right. \right. \\ &\quad \left. \left. + \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) [R^{[\mu_j/\tau]} (-\tau A^{1/2}) - R^{[\mu_j/\tau]} (\tau A^{1/2})] (I + \tau^2 A)^{-1} \right. \right. \\ &\quad \left. \left. \times \left\{ -A \left(\frac{\tau}{2} + 1 \right) \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s + \frac{\tau}{2} f(0) + g_0 \right\} \right. \right. \end{aligned} \quad (4.17)$$

$$\begin{aligned}
& + \sum_{s=1}^{[\mu_j/\tau]-1} \frac{\tau}{2i} A^{-1/2} \left[R^{[\mu_j/\tau]-s} (-\tau A^{1/2}) - R^{[\mu_j/\tau]-s} (\tau A^{1/2}) \right] f_s \Bigg\} \\
& - \sum_{j=1}^N \alpha_j \left(\frac{\mu_j}{\tau} - [\frac{\mu_j}{\tau}] \right) \left\{ \frac{1}{2} \left[R^{[\mu_j/\tau]-2} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \\
& + R^{[\mu_j/\tau]-2} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \Bigg] \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \\
& + \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \left[R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) - R^{[\mu_j/\tau]-1} (\tau A^{1/2}) \right] \\
& \times (I + \tau^2 A)^{-1} \left\{ -A \left(\frac{\tau}{2} + 1 \right) \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s + \frac{\tau}{2} f(0) + g_0 \right\} \\
& + \sum_{s=1}^{[\mu_j/\tau]-2} \frac{\tau}{2i} A^{-1/2} \left[R^{[\mu_j/\tau]-1-s} (-\tau A^{1/2}) - R^{[\mu_j/\tau]-1-s} (\tau A^{1/2}) \right] f_s \Bigg\} \\
& + \sum_{j=1}^L \beta_j \frac{1}{\tau} \left(I - \left(\lambda_j - [\frac{\lambda_j}{\tau}] \tau + \frac{\tau}{2} \right) A \right) \\
& \times \left\{ \frac{1}{2} \left[R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \\
& + R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \Bigg] \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \\
& + \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \left[R^{[\frac{\lambda_j}{\tau}]} (-\tau A^{1/2}) - R^{[\frac{\lambda_j}{\tau}]} (\tau A^{1/2}) \right] \\
& \times (I + \tau^2 A)^{-1} \left\{ -A \left(\frac{\tau}{2} + 1 \right) \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s + \frac{\tau}{2} f(0) + g_0 \right\} \\
& + \sum_{s=1}^{[\lambda_j/\tau]-1} \frac{\tau}{2i} A^{-1/2} \left[R^{[\lambda_j/\tau]-s} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]-s} (\tau A^{1/2}) \right] f_s \Bigg\} \\
& - \sum_{j=1}^L \beta_j \frac{1}{\tau} \left\{ \frac{1}{2} \left[R^{[\lambda_j/\tau]-2} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \\
& + R^{[\lambda_j/\tau]-2} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \Bigg] \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \\
& + \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \left[R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) \right] \\
& \times (I + \tau^2 A)^{-1} \left\{ -A \left(\frac{\tau}{2} + 1 \right) \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s + \frac{\tau}{2} f(0) + g_0 \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{s=1}^{[\lambda_j/\tau]-2} \frac{\tau}{2i} A^{-1/2} \left[R^{[\lambda_j/\tau]-1-s} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]-1-s} (\tau A^{1/2}) \right] f_s \Bigg\} \\
& + \sum_{j=1}^L \beta_j \left(\lambda_j - \left[\frac{\lambda_j}{\tau} \right] \tau + \frac{\tau}{2} \right) f_{[\frac{\lambda_j}{\tau}] + \varphi} \Bigg]
\end{aligned}$$

operatör denklemi elde edilir. Böylece, (4.1) fark şemasının çözümü için (4.14), (4.15), (4.16) ve (4.17) kurulmuş olur.

Şimdi, (4.7), (4.8) ve (4.9) kestirimlerinin ispatları verilecektir. İlk olarak, $\|u_{-N}\|_H$ için kestirim elde edilecektir. Bunun için, Abel formülü ve (4.17) formülü kullanılarak,

$$\begin{aligned}
u_{-N} = T_\tau & \left[\sum_{j=1}^N \alpha_j \left(1 + \left(\frac{\mu_j}{\tau} - \left[\frac{\mu_j}{\tau} \right] \right) \right) \left\{ \frac{1}{2} \left[R^{[\mu_j/\tau]-1} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \right. \\
& \left. \left. + R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] \right. \\
& \left. \times A^{-1} \left(g_0 - R^{-N} g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} (g_{s-1} - g_s) \right) \right. \\
& \left. + \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) [R^{[\mu_j/\tau]} (-\tau A^{1/2}) - R^{[\mu_j/\tau]} (\tau A^{1/2})] (I + \tau^2 A)^{-1} \right. \\
& \left. \times \left\{ - \left(\frac{\tau}{2} + 1 \right) \left(g_0 - R^{-N} g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} (g_{s-1} - g_s) \right) + \frac{\tau}{2} f(0) + g_0 \right\} \right. \\
& \left. + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \left\{ f_{[\mu_j/\tau]-1} - R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) [f_0 - (f_0 - f_1)] \right. \right. \\
& \left. \left. + \sum_{s=2}^{[\mu_j/\tau]-1} R^{[\mu_j/\tau]-s} (-\tau A^{1/2}) [f_{s-1} - f_s] \right\} \right. \\
& \left. + \frac{1}{2} A^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \left\{ f_{[\mu_j/\tau]-1} - R^{[\mu_j/\tau]-1} (\tau A^{1/2}) [f_0 - (f_0 - f_1)] \right. \right. \\
& \left. \left. + \sum_{s=2}^{[\mu_j/\tau]-1} R^{[\mu_j/\tau]-s} (\tau A^{1/2}) [f_{s-1} - f_s] \right\} \right. \\
& \left. - \sum_{j=1}^N \alpha_j \left(\frac{\mu_j}{\tau} - \left[\frac{\mu_j}{\tau} \right] \right) \left\{ \frac{1}{2} \left[R^{[\mu_j/\tau]-2} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \right. \\
& \left. \left. + R^{[\mu_j/\tau]-2} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] \right. \\
& \left. \times A^{-1} \left(g_0 - R^{-N} g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} (g_{s-1} - g_s) \right) \right.
\end{aligned}
\tag{4.18}$$

$$\begin{aligned}
& + \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \left[R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) - R^{[\mu_j/\tau]-1} (\tau A^{1/2}) \right] (I + \tau^2 A)^{-1} \\
& \times \left\{ - \left(\frac{\tau}{2} + 1 \right) \left(g_0 - R^{-N} g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} (g_{s-1} - g_s) \right) + \frac{\tau}{2} f(0) + g_0 \right\} \\
& + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \left\{ f_{[\mu_j/\tau]-2} - R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) [f_0 - (f_0 - f_1)] \right. \\
& \quad \left. + \sum_{s=2}^{[\mu_j/\tau]-2} R^{[\mu_j/\tau]-1-s} (-\tau A^{1/2}) [f_{s-1} - f_s] \right\} \\
& + \frac{1}{2} A^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \left\{ f_{[\mu_j/\tau]-2} - R^{[\mu_j/\tau]-1} (\tau A^{1/2}) [f_0 - (f_0 - f_1)] \right. \\
& \quad \left. + \sum_{s=2}^{[\mu_j/\tau]-2} R^{[\mu_j/\tau]-1-s} (\tau A^{1/2}) [f_{s-1} - f_s] \right\} \\
& + \sum_{j=1}^L \beta_j \frac{1}{\tau} \left(I - \left(\lambda_j - \left[\frac{\lambda_j}{\tau} \right] \tau + \frac{\tau}{2} \right) A \right) \\
& \times \left\{ \frac{1}{2} \left[R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) + R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] \right. \\
& \quad \times A^{-1} \left(g_0 - R^{-N} g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} (g_{s-1} - g_s) \right) \\
& \quad + \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \left[R^{[\lambda_j/\tau]} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]} (\tau A^{1/2}) \right] (I + \tau^2 A)^{-1} \\
& \quad \times \left\{ - \left(\frac{\tau}{2} + 1 \right) \left(g_0 - R^{-N} g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} (g_{s-1} - g_s) \right) + \frac{\tau}{2} f(0) + g_0 \right\} \\
& \quad + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \left\{ f_{[\lambda_j/\tau]-1} - R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) [f_0 - (f_0 - f_1)] \right. \\
& \quad \left. + \sum_{s=2}^{[\lambda_j/\tau]-1} R^{[\lambda_j/\tau]-s} (-\tau A^{1/2}) [f_{s-1} - f_s] \right\} \\
& \quad + \frac{1}{2} A^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \left\{ f_{[\lambda_j/\tau]-1} - R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) [f_0 - (f_0 - f_1)] \right. \\
& \quad \left. + \sum_{s=2}^{[\lambda_j/\tau]-1} R^{[\lambda_j/\tau]-s} (\tau A^{1/2}) [f_{s-1} - f_s] \right\} \\
& \quad - \sum_{j=1}^L \beta_j \frac{1}{\tau} \left\{ \frac{1}{2} \left[R^{[\lambda_j/\tau]-2} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \\
& \quad \left. \left. + R^{[\lambda_j/\tau]-2} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& \times A^{-1} \left(g_0 - R^{-N} g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} (g_{s-1} - g_s) \right) \\
& + \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) [R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]-1} (\tau A^{1/2})] (I + \tau^2 A)^{-1} \\
& \times \left\{ - \left(\frac{\tau}{2} + 1 \right) \left(g_0 - R^{-N} g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} (g_{s-1} - g_s) \right) + \frac{\tau}{2} f(0) + g_0 \right\} \\
& + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \left\{ f_{[\lambda_j/\tau]-2} - R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) [f_0 - (f_0 - f_1)] \right. \\
& \quad \left. + \sum_{s=2}^{[\lambda_j/\tau]-2} R^{[\lambda_j/\tau]-1-s} (-\tau A^{1/2}) [f_{s-1} - f_s] \right\} \\
& + \frac{1}{2} A^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \left\{ f_{[\lambda_j/\tau]-2} - R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) [f_0 - (f_0 - f_1)] \right. \\
& \quad \left. + \sum_{s=2}^{[\lambda_j/\tau]-2} R^{[\lambda_j/\tau]-1-s} (\tau A^{1/2}) [f_{s-1} - f_s] \right\} \\
& \quad \left. + \sum_{j=1}^L \beta_j \left(\lambda_j - \left[\frac{\lambda_j}{\tau} \right] \tau + \frac{\tau}{2} \right) f_{[\lambda_j/\tau]} + \varphi \right]
\end{aligned}$$

formülü elde edilir. Buradan, (4.14), (4.16), (4.18) formülleri ve (4.2), (4.3) kestirimleri yardımıyla,

$$\begin{aligned}
\|u_{-N}\|_H & \leq M \left[\|\varphi\|_H + \|A^{-1/2} f_0\|_H + \max_{1 \leq k \leq N} \|A^{-1/2} (f_k - f_{k-1}) \tau^{-1}\|_H \right. \\
& \quad \left. + \|A^{-1/2} g_0\|_H + \max_{-N \leq k \leq 0} \|A^{-1/2} (g_k - g_{k-1}) \tau^{-1}\|_H \right]
\end{aligned} \tag{4.19}$$

kestirimi alınır.

İkinci olarak, $\|u_k\|_H$ için kestirim elde edilecektir. Bunun için önce $-N < k \leq 0$ aralığında inceleyelim. Bu durumda, Abel formülü ve (4.14) formülü kullanılarak,

$$\begin{aligned}
u_k & = R^{N+k} u_{-N} \\
& + A^{-1} \left[g_k - R^{N+k} g_{-N+1} + \sum_{s=-N+2}^k R^{k-s+1} [g_{s-1} - g_s] \right]
\end{aligned} \tag{4.20}$$

formülü elde edilir. Buradan, (4.20) formülü ve (4.3) kestirimi yardımıyla,

$$\|u_k\|_H \leq \|u_{-N}\|_H + 2 \|A^{-1/2} g_0\|_H + \max_{-N < k \leq 0} \|A^{-1/2} (g_k - g_{k-1}) \tau^{-1}\|_H \tag{4.21}$$

kestirimi bulunur. Şimdi ise $1 \leq k \leq N$ aralığında inceleyelim. Bu durumda, Abel formülü ve (4.15), (4.16) formülleri kullanılarak,

$$u_1 = \left(I - \tau A \left(\frac{\tau}{2} + I \right) (I + \tau^2 A)^{-1} \right) \quad (4.22)$$

$$\begin{aligned} & \times \left[R^N u_{-N} + A^{-1} \left\{ g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} (g_{s-1} - g_s) \right\} \right] \\ & + (I + \tau^2 A)^{-1} \left\{ \frac{\tau^2}{2} f(0) + \tau g_0 \right\} \\ u_k = & \frac{1}{2} \left[R^{k-1} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) + R^{k-1} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] \quad (4.23) \\ & \times \left[R^N u_{-N} + A^{-1} \left\{ g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} (g_{s-1} - g_s) \right\} \right] \\ & + \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) [R^k (-\tau A^{1/2}) - R^k (\tau A^{1/2})] \\ & \times (I + \tau^2 A)^{-1} \left\{ -A \left(\frac{\tau}{2} + 1 \right) \right. \\ & \times \left[R^N u_{-N} + A^{-1} \left\{ g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} (g_{s-1} - g_s) \right\} \right] \\ & \left. + \frac{\tau}{2} f(0) + g_0 \right\} + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \\ & \times \left[f_{k-1} - R^{k-1} (-\tau A^{1/2}) f_1 + \sum_{s=2}^{k-1} R^{k-s} (-\tau A^{1/2}) [f_{s-1} - f_s] \right] \\ & + \frac{1}{2} A^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \\ & \times \left[f_{k-1} - R^{k-1} (\tau A^{1/2}) f_1 + \sum_{s=2}^{k-1} R^{k-s} (\tau A^{1/2}) [f_{s-1} - f_s] \right] \end{aligned}$$

formülleri elde edilir. Buradan, (4.22), (4.23) formülleri ve (4.2), (4.3) kestirimleri yardımıyla,

$$\|u_1\|_H \leq M \left[\|u_{-N}\|_H + \|A^{-1/2} g_0\|_H + \max_{-N < k \leq 0} \|A^{-1/2} (g_k - g_{k-1}) \tau^{-1}\|_H \right], \quad (4.24)$$

$$\|u_k\|_H \leq M \left[\|u_{-N}\|_H + \|A^{-1/2} f_0\|_H + \max_{2 \leq k < N} \|A^{-1/2} (f_k - f_{k-1}) \tau^{-1}\|_H \right.$$

$$\left. + \|A^{-1/2} g_0\|_H + \max_{-(N-1) \leq k \leq 0} \|A^{-1/2} (g_k - g_{k-1}) \tau^{-1}\|_H \right], 1 \leq k \leq N \quad (4.25)$$

kestirimleri alınır. Sonuç olarak, (4.19), (4.21), (4.24), (4.25) kestirimleri birleştirilerek ve üçgen eşitsizliği kullanılarak, (4.7) kestirimi elde edilir. Burada (4.8) kestiriminin ispatı, (4.18), (4.20) formüllerine ve (4.2), (4.3) kestirimlerine bağlı kalmak şartıyla, (4.7) kestiriminin ispatında olduğu gibi aynı yolla elde edilir.

Üçüncü olarak, $\|Au_{-N}\|_H$ için kestirim elde edilecektir. Bunun için, Abel formülü ve (4.17) formülü kullanılarak,

$$\begin{aligned}
u_{-N} = T_\tau & \left[\sum_{j=1}^N \alpha_j \left(1 + \left(\frac{\mu_j}{\tau} - \left\lfloor \frac{\mu_j}{\tau} \right\rfloor \right) \right) \left\{ \frac{1}{2} \left[R^{[\mu_j/\tau]-1} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \right. \\
& \quad \left. \left. + R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] \right. \\
& \quad \times A^{-1} \left(g_0 - R^{-N} g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \\
& \quad \times \left\{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \right. \\
& \quad \left. \left. + \sum_{s=-N+2}^{-1} R^{-s} (g_{s+1} - 2g_s + g_{s-1}) \right\} \right) \\
& \quad + \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) [R^{[\mu_j/\tau]} (-\tau A^{1/2}) - R^{[\mu_j/\tau]} (\tau A^{1/2})] \\
& \quad \times (I + \tau^2 A)^{-1} \left\{ -A \left(\frac{\tau}{2} + 1 \right) A^{-1} \left(\left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \right. \\
& \quad \times \left\{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \right. \\
& \quad \left. \left. + \sum_{s=-N+2}^{-1} R^{-s} (g_{s+1} - 2g_s + g_{s-1}) \right\} \right) \\
& \quad \left. + \frac{\tau}{2} f(0) + g_0 \right\} \\
& \quad + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \left\{ f_{[\mu_j/\tau]-1} - R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) [f_0 + (f_1 - f_0)] \right. \\
& \quad + (-i\tau A^{1/2})^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \left[f_{[\mu_j/\tau]-2} - f_{[\mu_j/\tau]-1} - R^{[\mu_j/\tau]-2} (-\tau A^{1/2}) \right. \\
& \quad \times [-(f_0 - 2f_1 + f_2) - (f_1 - f_0)] \\
& \quad \left. \left. + \sum_{s=2}^{[\mu_j/\tau]} R^{[\mu_j/\tau]-1-s} (-\tau A^{1/2}) [f_{s+1} - 2f_s + f_{s-1}] \right] \right\} \\
& \quad + \frac{1}{2} A^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \left\{ f_{[\mu_j/\tau]-1} - R^{[\mu_j/\tau]-1} (\tau A^{1/2}) [f_0 + (f_1 - f_0)] \right. \\
& \quad + (i\tau A^{1/2})^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \left[f_{[\mu_j/\tau]-2} - f_{[\mu_j/\tau]-1} - R^{[\mu_j/\tau]-2} (\tau A^{1/2}) \right.
\end{aligned}
\tag{4.26}$$

$$\begin{aligned}
& \times [-(f_0 - 2f_1 + f_2) - (f_1 - f_0)] \\
& + \sum_{s=2}^{[\mu_j/\tau]} R^{[\mu_j/\tau]-1-s} (\tau A^{1/2}) [f_{s+1} - 2f_s + f_{s-1}] \Bigg\} \\
& - \sum_{j=1}^N \alpha_j \left(\frac{\mu_j}{\tau} - [\frac{\mu_j}{\tau}] \right) \left\{ \frac{1}{2} \left[R^{[\mu_j/\tau]-2} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \\
& \quad \left. \left. + R^{[\mu_j/\tau]-2} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] \right. \\
& \quad \times A^{-1} \left(g_0 - R^{-N} g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \\
& \quad \times \left\{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \right. \\
& \quad \left. \left. + \sum_{s=-N+2}^{-1} R^{-s} (g_{s+1} - 2g_s + g_{s-1}) \right\} \right) \Bigg] \\
& + \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) [R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) - R^{[\mu_j/\tau]-1} (\tau A^{1/2})] \\
& \times (I + \tau^2 A)^{-1} \left\{ -A \left(\frac{\tau}{2} + 1 \right) A^{-1} \left(\left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \right. \\
& \quad \times \left\{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \right. \\
& \quad \left. \left. + \sum_{s=-N+2}^{-1} R^{-s} (g_{s+1} - 2g_s + g_{s-1}) \right\} \right) \Bigg] \\
& \quad \left. + \frac{\tau}{2} f(0) + g_0 \right\} \\
& + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \left\{ f_{[\mu_j/\tau]-2} - R^{[\mu_j/\tau]-1} (-\tau A^{1/2}) [f_0 + (f_1 - f_0)] \right. \\
& + (-i\tau A^{1/2})^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \left[f_{[\mu_j/\tau]-3} - f_{[\mu_j/\tau]-2} - R^{[\mu_j/\tau]-3} (-\tau A^{1/2}) \right. \\
& \quad \times [-(f_0 - 2f_1 + f_2) - (f_1 - f_0)] \\
& \quad \left. \left. + \sum_{s=2}^{[\mu_j/\tau]-1} R^{[\mu_j/\tau]-2-s} (-\tau A^{1/2}) [f_{s+1} - 2f_s + f_{s-1}] \right] \right\} \\
& + \frac{1}{2} A^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \left\{ f_{[\mu_j/\tau]-2} - R^{[\mu_j/\tau]-1} (\tau A^{1/2}) [f_0 + (f_1 - f_0)] \right. \\
& + (i\tau A^{1/2})^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \left[f_{[\mu_j/\tau]-3} - f_{[\mu_j/\tau]-2} - R^{[\mu_j/\tau]-3} (\tau A^{1/2}) \right. \\
& \quad \times [-(f_0 - 2f_1 + f_2) - (f_1 - f_0)]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{s=2}^{[\mu_j/\tau]-1} R^{[\mu_j/\tau]-2-s} (\tau A^{1/2}) [f_{s+1} - 2f_s + f_{s-1}] \Bigg] \Bigg\} \\
& + \sum_{j=1}^L \beta_j \frac{1}{\tau} \left(I - \left(\lambda_j - \left[\frac{\lambda_j}{\tau} \right] \tau + \frac{\tau}{2} \right) A \right) \\
& \times \left\{ \frac{1}{2} \left[R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \\
& + R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \Bigg] A^{-1} \left(\left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \\
& \times \{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \\
& + \sum_{s=-N+2}^{-1} R^{-s} (g_{s+1} - 2g_s + g_{s-1}) \} \Bigg) \Bigg] \\
& + \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) [R^{[\lambda_j/\tau]} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]} (\tau A^{1/2})] \\
& \times (I + \tau^2 A)^{-1} \left\{ -A \left(\frac{\tau}{2} + 1 \right) A^{-1} \left(\left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \right. \\
& \times \{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \\
& + \sum_{s=-N+2}^{-1} R^{-s} (g_{s+1} - 2g_s + g_{s-1}) \} \Bigg) \Bigg] \\
& \left. + \frac{\tau}{2} f(0) + g_0 \right\} \\
& + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \{ f_{[\lambda_j/\tau]-1} - R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) [f_0 + (f_1 - f_0)] \\
& + (-i\tau A^{1/2})^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} [f_{[\lambda_j/\tau]-2} - f_{[\lambda_j/\tau]-1} - R^{[\lambda_j/\tau]-2} (-\tau A^{1/2}) \\
& \times [-(f_0 - 2f_1 + f_2) - (f_1 - f_0)] \\
& + \sum_{s=2}^{[\lambda_j/\tau]} R^{[\lambda_j/\tau]-1-s} (-\tau A^{1/2}) [f_{s+1} - 2f_s + f_{s-1}] \Bigg] \Bigg\} \\
& + \frac{1}{2} A^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \{ f_{[\lambda_j/\tau]-1} - R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) [f_0 + (f_1 - f_0)] \\
& + (i\tau A^{1/2})^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} [f_{[\lambda_j/\tau]-2} - f_{[\lambda_j/\tau]-1} - R^{[\lambda_j/\tau]-2} (\tau A^{1/2}) \\
& \times [-(f_0 - 2f_1 + f_2) - (f_1 - f_0)] \\
& + \sum_{s=2}^{[\lambda_j/\tau]} R^{[\lambda_j/\tau]-1-s} (\tau A^{1/2}) [f_{s+1} - 2f_s + f_{s-1}] \Bigg] \Bigg\}
\end{aligned}$$

$$\begin{aligned}
& - \sum_{j=1}^L \beta_j \frac{1}{\tau} \left\{ \frac{1}{2} \left[R^{[\lambda_j/\tau]-2} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \right. \\
& + R^{[\lambda_j\tau]-2} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \left. \right] A^{-1} \left(\left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \\
& \times \left\{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \right. \\
& + \left. \left. \sum_{s=-N+2}^{-1} R^{-s} (g_{s+1} - 2g_s + g_{s-1}) \right\} \right) \left. \right] \\
& + \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) [R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) - R^{[\lambda_j/\tau]-1} (\tau A^{1/2})] \\
& \times (I + \tau^2 A)^{-1} \left\{ -A \left(\frac{\tau}{2} + 1 \right) A^{-1} \left(\left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \right. \\
& \times \left\{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \right. \\
& + \left. \left. \sum_{s=-N+2}^{-1} R^{-s} (g_{s+1} - 2g_s + g_{s-1}) \right\} \right) \left. \right] \\
& + \frac{\tau}{2} f(0) + g_0 \Big\} \\
& + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \left\{ f_{[\lambda_j/\tau]-2} - R^{[\lambda_j/\tau]-1} (-\tau A^{1/2}) [f_0 + (f_1 - f_0)] \right. \\
& + (-i\tau A^{1/2})^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} [f_{[\lambda_j/\tau]-3} - f_{[\lambda_j/\tau]-2} - R^{[\lambda_j/\tau]-3} (-\tau A^{1/2}) \\
& \times [-(f_0 - 2f_1 + f_2) - (f_1 - f_0)] \\
& + \left. \left. \sum_{s=2}^{[\lambda_j/\tau]-1} R^{[\lambda_j/\tau]-2-s} (-\tau A^{1/2}) [f_{s+1} - 2f_s + f_{s-1}] \right] \right\} \\
& + \frac{1}{2} A^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \left\{ f_{[\lambda_j/\tau]-2} - R^{[\lambda_j/\tau]-1} (\tau A^{1/2}) [f_0 + (f_1 - f_0)] \right. \\
& + (i\tau A^{1/2})^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} [f_{[\lambda_j/\tau]-3} - f_{[\lambda_j/\tau]-2} - R^{[\lambda_j/\tau]-3} (\tau A^{1/2}) \\
& \times [-(f_0 - 2f_1 + f_2) - (f_1 - f_0)] \\
& + \left. \left. \sum_{s=2}^{[\lambda_j/\tau]-1} R^{[\lambda_j/\tau]-2-s} (\tau A^{1/2}) [f_{s+1} - 2f_s + f_{s-1}] \right] \right\} \\
& + \sum_{j=1}^L \beta_j \left(\lambda_j - \left[\frac{\lambda_j}{\tau} \right] \tau + \frac{\tau}{2} \right) f_{[\lambda_j/\tau]} + \varphi \Big]
\end{aligned}$$

formülü elde edilir. Buradan, (4.26) formülü ve (4.2), (4.3) kestirimleri yardımıyla,

$$\|Au_{-N}\|_H \leq M [\|A\varphi\|_H + \|A^{1/2}f_0\|_H + \|(f_1 - f_0)\tau^{-1}\|_H]$$

$$\begin{aligned}
& + \max_{2 \leq k \leq N-2} \left\| (f_{k+1} - 2f_k + f_{k-1}) \tau^{-2} \right\|_H + \left\| A^{1/2} g_0 \right\|_H \\
& + \left\| (g_0 - g_{-1}) \tau^{-1} \right\|_H + \max_{-(N-1) \leq k \leq -1} \left\| (g_{k+1} - 2g_k + g_{k-1}) \tau^{-2} \right\|_H \Big] \quad (4.27)
\end{aligned}$$

kestirimi alınır.

Son olarak, $\|Au_k\|_H$ için kestirim elde edilecektir. Bunun için önce $-N+1 \leq k \leq 0$ aralığında inceleyelim. Bu durumda, Abel formülü ve (4.14) formülü kullanılarak,

$$\begin{aligned}
Au_k &= R^{N+k} Au_{-N} + \left[g_k - R^{N+k} g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \\
&\quad \times \left\{ (g_{k-1} - g_k) - R^{k+N-1} (g_{-N+1} - g_{-N+2}) \right. \\
&\quad \left. \left. + \sum_{s=-N+2}^{k-1} R^{k-s} [g_{s+1} - 2g_s + g_{s-1}] \right\} \right] \quad (4.28)
\end{aligned}$$

formülü elde edilir. Buradan, (4.28) formülü ve (4.3) kestirimi yardımıyla,

$$\begin{aligned}
\|Au_k\|_H &\leq M \left[\|Au_{-N}\|_H + \left\| \tau^{-1} (g_{-1} - g_0) \right\|_H + \left\| A^{1/2} g_0 \right\|_H \right. \\
&\quad \left. + \max_{-(N-2) \leq k \leq -1} \left\| \tau^{-2} (g_{k+1} - 2g_k + g_{k-1}) \right\|_H \right] \quad (4.29)
\end{aligned}$$

kestirimi alınır. Şimdi ise $1 \leq k \leq N$ aralığında inceleyelim. Bu durumda, Abel formülü ve (4.15), (4.16) formülleri kullanılarak,

$$\begin{aligned}
Au_1 &= \left\{ I - \tau A \left(\frac{\tau}{2} + I \right) \left((I + \tau^2 A)^{-1} \right) \right\} \\
&\times \left[R^N Au_{-N} + \left(g_0 - R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \right. \\
&\quad \times \left\{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \right. \\
&\quad \left. \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \right\} \right) \Big] \\
&\quad + A (I + \tau^2 A)^{-1} \left(\frac{\tau^2}{2} f(0) + \tau g_0 \right) \quad (4.30) \\
Au_k &= \frac{1}{2} \left[R^{k-1} (\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) + R^{k-1} (-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] \\
&\quad \times \left[R^N Au_{-N} + \left(g_0 - R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \right. \\
& \quad \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \right\} \Bigg] \\
& \quad + \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \\
& \times [R^k (-\tau A^{1/2}) - R^k (\tau A^{1/2})] (I + \tau^2 A)^{-1} \left\{ -A \left(\frac{\tau}{2} + 1 \right) \right. \\
& \times \left[R^N A u_{-N} + \left(g_0 - R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \right. \\
& \quad \times \left\{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \right. \\
& \quad \left. \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \right\} \right] \Bigg] \\
& \quad + \frac{\tau}{2} f(0) + g_0 \Big\} \\
& + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \{ f_{k-1} - R^{k-1} (-\tau A^{1/2}) f_1 \\
& + (-i\tau A^{1/2})^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} [f_{k-2} - f_{k-1} - R^{k-2} (-\tau A^{1/2}) \\
& \quad \times [-(f_0 - 2f_1 + f_2) - (f_1 - f_0)] \\
& \quad + \sum_{s=2}^{k-1} R^{k-1-s} (-\tau A^{1/2}) [f_{s+1} - 2f_s + f_{s-1}] \Big\} \\
& + \frac{1}{2} A^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \{ f_{k-1} - R^{k-1} (\tau A^{1/2}) f_1 \\
& + (i\tau A^{1/2})^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} [f_{k-2} - f_{k-1} - R^{k-2} (\tau A^{1/2}) \\
& \quad \times [-(f_0 - 2f_1 + f_2) - (f_1 - f_0)] \\
& \quad + \sum_{s=2}^{k-1} R^{k-1-s} (\tau A^{1/2}) [f_{s+1} - 2f_s + f_{s-1}] \Big\}, 2 \leq k \leq N \tag{4.31}
\end{aligned}$$

formülleri elde edilir. Buradan, (4.30), (4.31) formülleri ve (4.2), (4.3) kestirimleri kullanılarak ise

$$\begin{aligned}
\|Au_1\|_H & \leq M [\|Au_{-N}\|_H + \|A^{1/2}g_0\|_H + \|\tau^{-1}(g_{-1} - g_0)\|_H \\
& \quad + \max_{-(N-1) \leq k \leq -1} \|\tau^{-1}[g_{s+1} - 2g_s + g_{s-1}]\|_H] , \tag{4.32} \\
\|Au_k\|_H & \leq M [\|Au_{-N}\|_H + \|A^{1/2}f_0\|_H + \|(f_1 - f_0)\tau^{-1}\|_H
\end{aligned}$$

$$\begin{aligned}
& + \max_{2 \leq k \leq N-2} \left\| (f_{k+1} - 2f_k + f_{k-1}) \tau^{-2} \right\|_H + \|A^{1/2} g_0\|_H + \|(g_{-1} - g_0) \tau^{-1}\|_H \\
& + \max_{-(N-2) \leq k \leq -1} \left\| \tau^{-2} (g_{s+1} - 2g_s + g_{s-1}) \right\|_H \Big], 2 \leq k \leq N \quad (4.33)
\end{aligned}$$

kestirimleri alınır. Sonuç olarak, (4.27), (4.29), (4.32), (4.33) kestirimleri birleştirilerek ve üçgen eşitsizliği kullanılarak, (4.9) kestirimi elde edilir. Böylece, Teorem 4.1'in ispatı tamamlanmış olur.

Şimdi, Teorem 4.1'in uygulamaları verilecektir. İlk olarak, bir boyutlu hiperbolik-parabolik denklem için, (2.21) lokal olmayan sınır değer problemi ele alınmıştır. (2.21) probleminin diskritizasyonu iki adımda incelenir. Birinci adımda, A_h^x fark operatörünün yardımıyla, (2.21) sınır değer problemi, (3.31) adi diferansiyel denklem sistemine dönüşür. İkinci adımda ise, (3.31) problemi için (4.1) fark şeması kullanılarak,

$$\left\{ \begin{aligned} & \frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h(x) + \frac{\tau^2}{4} (A_h^x)^2 u_{k+1}^h(x) = f_k^h(x), \\ & f_k^h(x) = \{f(y_k, x_n)\}_1^{K-1}, x \in [0, 1]_h, \\ & y_k = k\tau, 1 \leq k \leq N-1, N\tau = 1, \tau^{-1}(I + \tau^2 A_h^x)(u_1^h(x) - u_0^h(x)) = Z_1^h(x), \\ & Z_1^h(x) = \frac{\tau}{2}(f_0^h(x) - A_h^x u_0^h(x)) + (g_0^h(x) - A_h^x u_0^h(x)), \\ & f_0^h(x) = \{f(0, x_n)\}_1^{K-1}, g_0^h(x) = \{g(0, x_n)\}_1^{K-1}, \\ & \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + A_h^x(I + \frac{\tau}{2} A_h^x) u_k^h(x) = (I + \frac{\tau}{2} A_h^x) g_k^h(x), x \in [0, 1]_h, \\ & g_k^h(x) = \{g(y_k - \frac{\tau}{2}, x_n)\}_1^{K-1}, y_k = k\tau, -(N-1) \leq k \leq 0, \\ & u_{-N}^h(x) = \sum_{j=1}^N \alpha_j \left(u_{[\mu_j/\tau]}^h(x) + (\mu_j - [\frac{\mu_j}{\tau}]\tau) \frac{u_{[\mu_j/\tau]}^h(x) - u_{[\mu_j/\tau]-1}^h(x)}{\tau} \right) \\ & + \sum_{j=1}^L \beta_j \left(\frac{u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)}{\tau} + \left(\lambda_j - [\frac{\lambda_j}{\tau}]\tau - \frac{\tau}{2} \right) \right. \\ & \left. \times \left(f_{[\frac{\lambda_j}{\tau}]}^h(x) - A_h^x u_{[\frac{\lambda_j}{\tau}]}^h(x) \right) \right) + \varphi^h(x), x \in [0, 1]_h \end{aligned} \right. \quad (4.34)$$

fark şeması elde edilir.

Teorem 4.2. Eğer τ ve h yeterince küçük sayılar ise, bu durumda (4.34) fark şemasının çözümü için aşağıdaki

$$\begin{aligned}
& \max_{-N \leq k \leq N} \|u_k^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq N} \|\tau^{-1}(u_k^h - u_{k-1}^h)\|_{L_{2h}} + \max_{-N \leq k \leq N} \|(u_k^h)_x\|_{L_{2h}} \\
& \leq M_1 \left[\|f_0^h\|_{L_{2h}} + \max_{2 \leq k \leq N-1} \|(f_k^h - f_{k-1}^h) \tau^{-1}\|_{L_{2h}} \right. \\
& \left. + \|g_0^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \|(g_k^h - g_{k-1}^h) \tau^{-1}\|_{L_{2h}} + \left\| \varphi_x^h \right\|_{L_{2h}} \right], \\
& \max_{1 \leq k \leq N-1} \|\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\|_{L_{2h}}
\end{aligned}$$

$$\begin{aligned}
& + \max_{-N \leq k \leq N} \left\| \left(u_{\frac{x}{h}}^k \right)_x \right\|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \left\| \tau^{-1} (u_k^h - u_{k-1}^h) \right\|_{L_{2h}} \\
& \leq M_1 \left[\left\| f_{0x}^h \right\|_{L_{2h}} + \left\| \tau^{-1} (f_1^h - f_0^h) \right\|_{L_{2h}} \right. \\
& + \max_{2 \leq k \leq N-1} \left\| \tau^{-2} (f_{k+1}^h - 2f_k^h + f_{k-1}^h) \right\|_{L_{2h}} + \left\| g_{0x}^h \right\|_{L_{2h}} + \left\| \tau^{-1} (g_0^h - g_{-1}^h) \right\|_{L_{2h}} \\
& \left. + \max_{-N+1 \leq k \leq -1} \left\| \tau^{-2} (g_{k+1}^h - 2g_k^h + g_{k-1}^h) \right\|_{L_{2h}} + \left\| \left(\varphi_{\frac{x}{h}}^h \right)_x \right\|_{L_{2h}} \right]
\end{aligned}$$

kararlılık kestirimleri sağlanır. Burada, M_1 katsayısı $\tau, h, f_k^h, 1 \leq k < N, g_k^h, -N < k \leq 0$ ve $\varphi^h(x)$ 'den bağımsızdır.

Teorem 4.2'nin ispatı, soyut Teorem 4.1 ve (3.30) formülü ile tanımlanan A_h^x fark operatörünün simetri özelliklerine dayanmaktadır.

İkinci olarak, çok boyutlu hiperbolik-parabolik denklem için karma sınır değer problemi (2.22) ele alınmıştır. (2.22) probleminin diskritizasyonu iki adımda incelenebilir. Birinci adımda, A_h^x fark operatörünün yardımıyla, (2.22) sınır değer problemi, (3.34) adi diferansiyel denklem sistemine dönüştürülür. İkinci adımda ise, (3.34) problemi için (4.1) fark şeması kullanılarak,

$$\left\{ \begin{array}{l} \frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h(x) + \frac{\tau^2}{4} (A_h^x)^2 u_{k+1}^h(x) = f_k^h(x), \\ f_k^h(x) = f^h(y_k, x), x \in \tilde{\Omega}_h, \\ y_k = k\tau, 1 \leq k < N, \tau^{-1} (I + \tau^2 A_h^x) (u_1^h(x) - u_0^h(x)) = Z_1^h(x), x \in \tilde{\Omega}_h, \\ Z_1^h(x) = \frac{\tau}{2} (f_0^h(x) - A_h^x u_0^h) + (g_0^h(x) - A_h^x u_0^h), \\ f_0^h(x) = \{f(0, x_n)\}_1^{K-1}, g_0^h(x) = \{g(0, x_n)\}_1^{K-1} \\ \tau^{-1} (u_k^h(x) - u_{k-1}^h(x)) + A_h^x (I + \frac{\tau}{2} A_h^x) u_k^h(x) = (I + \frac{\tau}{2} A_h^x) g_k^h(x), x \in \tilde{\Omega}_h, \\ g_k^h(x) = g^h(y_k - \frac{\tau}{2}, x), y_k = k\tau, -(N-1) \leq k \leq 0, x \in \tilde{\Omega}_h, \\ u_{-N}^h(x) = \sum_{j=1}^N \alpha_j \left(u_{[\mu_j/\tau]}^h(x) + (\mu_j - [\frac{\mu_j}{\tau}]\tau) \frac{u_{[\mu_j/\tau]}^h(x) - u_{[\mu_j/\tau]-1}^h(x)}{\tau} \right) \\ + \sum_{j=1}^L \beta_j \left(\frac{u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)}{\tau} + \left(\lambda_j - [\frac{\lambda_j}{\tau}]\tau - \frac{\tau}{2} \right) \right. \\ \left. \times \left(f_{[\frac{\lambda_j}{\tau}]}^h(x) - A_h^x u_{[\frac{\lambda_j}{\tau}]}^h(x) \right) \right) + \varphi^h, x \in \tilde{\Omega}_h \end{array} \right. \quad (4.35)$$

fark şeması elde edilir.

Teorem 4.3. Eğer τ ve $|h|$ yeterince küçük sayılar ise, bu durumda (4.35) fark şemasının çözümü için aşağıdaki

$$\max_{-N \leq k \leq N} \|u_k^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq N} \left\| \tau^{-1} (u_k^h - u_{k-1}^h) \right\|_{L_{2h}} + \max_{-N \leq k \leq N} \sum_{r=1}^n \left\| (u_k^h)_{x_r, j_r} \right\|_{L_{2h}}$$

$$\begin{aligned}
&\leq M_1 \left[\|f_0^h\|_{L_{2h}} + \max_{2 \leq k \leq N-1} \|(f_k^h - f_{k-1}^h) \tau^{-1}\|_{L_{2h}} \right. \\
&\quad + \|g_0^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \|(g_k^h - g_{k-1}^h) \tau^{-1}\|_{L_{2h}} + \sum_{r=1}^n \left\| (\varphi^h)_{\bar{x}_r, j_r} \right\|_{L_{2h}} \left. \right] \\
&\quad + \max_{1 \leq k \leq N-1} \|\tau^{-2} (u_{k+1}^h - 2u_k^h + u_{k-1}^h)\|_{L_{2h}} \\
&\quad + \max_{-N \leq k \leq N} \sum_{r=1}^n \left\| (u_k^h)_{\bar{x}_r, j_r} \right\|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \|\tau^{-1} (u_k^h - u_{k-1}^h)\|_{L_{2h}} \\
&\leq M_1 \left[\sum_{r=1}^n \left\| (f_0^h)_{\bar{x}_r, j_r} \right\|_{L_{2h}} + \|\tau^{-1} (f_1^h - f_0^h)\|_{L_{2h}} \right. \\
&\quad + \max_{2 \leq k \leq N-1} \|\tau^{-2} (f_{k+1}^h - 2f_k^h + f_{k-1}^h)\|_{L_{2h}} \\
&\quad + \sum_{r=1}^n \left\| (g_0^h)_{\bar{x}_r, j_r} \right\|_{L_{2h}} + \|\tau^{-1} (g_0^h - g_{-1}^h)\|_{L_{2h}} \\
&\quad + \max_{-N+1 \leq k \leq -1} \|\tau^{-2} (g_{k+1}^h - 2g_k^h + g_{k-1}^h)\|_{L_{2h}} + \sum_{r=1}^n \left\| (\varphi^h)_{\bar{x}_r, j_r} \right\|_{L_{2h}} \left. \right]
\end{aligned}$$

kararlılık kestirimleri sağlanır. Burada, M_1 katsayısı $\tau, h, f_k^h, 1 \leq k < N, g_k^h, -N < k \leq 0$ ve $\varphi^h(x)$ 'den bağımsızdır.

Teorem 4.3'ün ispatı, soyut Teorem 4.1, (3.33) formülü ile tanımlanan A_h^x fark operatörünün simetri özelliklerine ve Teorem 3.4'e dayanmaktadır.

4.2 Fark Şeması 2

Bu kısımda, (1.1) lokal olmayan sınır değer probleminin yaklaşık çözümlerini bulmak için

$$\left\{ \begin{array}{l} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + \frac{1}{2}Au_k + \frac{1}{4}A(u_{k+1} + u_{k-1}) = f_k, f_k = f(t_k), \\ t_k = k\tau, \ 1 \leq k < N, \\ \left(I + \frac{\tau^2 A}{4}\right) \left(\left(I + \frac{\tau^2 A}{4}\right) \tau^{-1} (u_1 - u_0) - \frac{\tau}{2} (g_0 - Au_0) \right) = f_0 - Au_0, \\ \tau^{-1}(u_k - u_{k-1}) + A(I + \frac{\tau}{2}A)u_k = (I + \frac{\tau}{2}A)g_k, \\ g_k = g(t_k - \frac{\tau}{2}), \ t_k = k\tau, \ -(N-1) \leq k \leq 0, \\ u_{-N} = \sum_{j=1}^N \alpha_j \left(u_{[\mu_j/\tau]} + (\mu_j - [\frac{\mu_j}{\tau}]\tau) \frac{u_{[\mu_j/\tau]} - u_{[\mu_j/\tau]-1}}{\tau} \right) \\ + \sum_{j=1}^L \beta_j \left(\frac{u_{[\lambda_j/\tau]} - u_{[\lambda_j/\tau]-1}}{\tau} + \left(\lambda_j - [\frac{\lambda_j}{\tau}]\tau + \frac{\tau}{2} \right) (f_{[\lambda_j/\tau]} - Au_{[\lambda_j/\tau]}) \right) + \varphi \end{array} \right. \quad (4.36)$$

ikinci basamaktan doğruluklu fark şeması kurulmuştur.

Öncelikle ileride ihtiyaç duyacağımız yardımcı teoremleri verelim.

Yardımcı Teorem 4.3. Aşağıdaki

$$\|D\|_{H \mapsto H} \leq 1, \quad \|\tilde{D}\|_{H \mapsto H} \leq 1, \quad \left\| \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right\|_{H \mapsto H} \leq 1 \quad (4.37)$$

$$(k\tau)^\beta \|A^\beta R^k\|_{H \mapsto H} \leq M, \quad k \geq 1, \quad 0 \leq \beta \leq 1 \quad (4.38)$$

kestirimler sağlanırlar. Burada,

$$D = \left(I - \frac{i\tau A^{1/2}}{2} \right) \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1},$$

$$\tilde{D} = \left(I + \frac{i\tau A^{1/2}}{2} \right) \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1}$$

ve

$$R = \left(I + \tau A + \frac{\tau^2}{2} A^2 \right)^{-1}$$

dir.

Yardımcı Teorem 4.4. Öncelikle,

$$\begin{aligned} & I - \sum_{j=1}^N \alpha_j \left\{ \frac{1}{2} \left\{ D^{[\mu_j/\tau]} + \tilde{D}^{[\mu_j/\tau]} \right\} R^N \right. \\ & \quad + \frac{A^{-1/2}}{2i} \left(I + \frac{\tau^2 A}{4} \right)^{-1} \left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} (-A) R^N \\ & \quad \left. + \frac{\tau A^{1/2}}{4i} \left[\left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} R^N \right] + \left(\mu_j - \left[\frac{\mu_j}{\tau} \right] \tau \right) \right. \\ & \quad \times \left[\frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\mu_j/\tau]-1} - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\mu_j/\tau]-1} \right\} \right. \\ & \quad \left. + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\mu_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\mu_j/\tau]-1} \right\} \right. \\ & \quad \left. \times \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} (-A) R^N + \frac{\tau A}{2} R^N \right] \right] \Bigg\} \\ & \quad - \sum_{j=1}^L \beta_j \left\{ \frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\lambda_j/\tau]-1} \right. \right. \\ & \quad \left. \left. - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\lambda_j/\tau]-1} \right\} R^N \right. \\ & \quad \left. + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\lambda_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\lambda_j/\tau]-1} \right\} \right\} \end{aligned}$$

$$\begin{aligned}
& \times \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} (-A) R^N + \frac{\tau A}{2} R^N \right] + \left(\lambda_j - \left[\frac{\lambda_j}{\tau} \right] \tau + \frac{\tau}{2} \right) \\
& \quad \times \left\{ \frac{1}{2} \left\{ D^{[\lambda_j/\tau]} + \tilde{D}^{[\lambda_j/\tau]} \right\} (-A) R^N \right. \\
& \quad \left. + \frac{A^{-1/2}}{2i} \left(I + \frac{\tau^2 A}{4} \right)^{-1} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} (-A) R^N \right. \\
& \quad \left. + \frac{\tau A^{1/2}}{4i} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} R^N \right\} \Big\}
\end{aligned}$$

operatörünün tersi vardır ve

$$\begin{aligned}
T_\tau &= \left(I - \sum_{j=1}^N \alpha_j \left\{ \frac{1}{2} \left\{ D^{[\mu_j/\tau]} + \tilde{D}^{[\mu_j/\tau]} \right\} R^N \right. \right. \\
& \quad \left. \left. + \frac{A^{-1/2}}{2i} \left(I + \frac{\tau^2 A}{4} \right)^{-1} \left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} (-A) R^N \right. \right. \\
& \quad \left. \left. + \frac{\tau A^{1/2}}{4i} \left[\left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} R^N \right] + \left(\mu_j - \left[\frac{\mu_j}{\tau} \right] \tau \right) \right. \right. \\
& \quad \times \left[\frac{i A^{1/2}}{2} \left\{ \left(I - \frac{i \tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\mu_j/\tau]-1} - \left(I + \frac{i \tau A^{1/2}}{2} \right)^{-1} D^{[\mu_j/\tau]-1} \right\} \right. \\
& \quad \left. \left. + \frac{1}{2} \left\{ \left(I + \frac{i \tau A^{1/2}}{2} \right) D^{[\mu_j/\tau]-1} + \left(I - \frac{i \tau A^{1/2}}{2} \right) \tilde{D}^{[\mu_j/\tau]-1} \right\} \right. \right. \\
& \quad \left. \left. \times \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} (-A) R^N + \frac{\tau A}{2} R^N \right] \right] \right\} \\
& \quad - \sum_{j=1}^L \beta_j \left\{ \frac{i A^{1/2}}{2} \left\{ \left(I - \frac{i \tau A^{1/2}}{2} \right)^{-1} D^{[\lambda_j/\tau]-1} \right. \right. \\
& \quad \left. \left. - \left(I + \frac{i \tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\lambda_j/\tau]-1} \right\} R^N \right. \\
& \quad \left. \left. + \frac{1}{2} \left\{ \left(I + \frac{i \tau A^{1/2}}{2} \right) D^{[\lambda_j/\tau]-1} + \left(I - \frac{i \tau A^{1/2}}{2} \right) \tilde{D}^{[\lambda_j/\tau]-1} \right\} \right. \right. \\
& \quad \times \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} (-A) R^N + \frac{\tau A}{2} R^N \right] + \left(\lambda_j - \left[\frac{\lambda_j}{\tau} \right] \tau + \frac{\tau}{2} \right) \\
& \quad \times \left\{ \frac{1}{2} \left\{ D^{[\lambda_j/\tau]} + \tilde{D}^{[\lambda_j/\tau]} \right\} (-A) R^N \right. \\
& \quad \left. \left. + \frac{A^{-1/2}}{2i} \left(I + \frac{\tau^2 A}{4} \right)^{-1} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} (-A) R^N \right. \right. \\
& \quad \left. \left. + \frac{\tau A^{1/2}}{4i} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} R^N \right\} \right\} \Big)^{-1}
\end{aligned}$$

aşağıdaki kesitirimi

$$\|T_\tau\|_{H \rightarrow H} \leq M \quad (4.39)$$

sağlar. Burada, M katsayısı τ 'dan bağımsızdır.

İspat. İlk olarak,

$$\begin{aligned}
& T_\tau - \left(I - \sum_{j=1}^N \alpha_j ([c([\mu_j/\tau]\tau) - As([\mu_j/\tau]\tau)] \right. \\
& \quad \left. - \sum_{j=1}^L \beta_j [s([\lambda_j/\tau]\tau) + c([\lambda_j/\tau]\tau)] A \right) e^{-A} \Big)^{-1} \\
& = T_\tau \left(I - \sum_{j=1}^N \alpha_j ([c([\mu_j/\tau]\tau) - As([\mu_j/\tau]\tau)] \right. \\
& \quad \left. - \sum_{j=1}^L \beta_j [s([\lambda_j/\tau]\tau) + c([\lambda_j/\tau]\tau)] A \right) e^{-A} \Big)^{-1} \\
& \quad \times \left[\left[\sum_{j=1}^N \alpha_j \left\{ \frac{1}{2} \left\{ D^{[\mu_j/\tau]} + \tilde{D}^{[\mu_j/\tau]} \right\} R^N \right. \right. \right. \\
& \quad + \frac{A^{-1/2}}{2i} \left(I + \frac{\tau^2 A}{4} \right)^{-1} \left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} (-A) R^N \\
& \quad + \frac{\tau A^{1/2}}{4i} \left[\left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} R^N \right] + \left(\mu_j - \left[\frac{\mu_j}{\tau} \right] \tau \right) \\
& \quad \times \left[\frac{i A^{1/2}}{2} \left\{ \left(I - \frac{i \tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\mu_j/\tau]-1} - \left(I + \frac{i \tau A^{1/2}}{2} \right)^{-1} D^{[\mu_j/\tau]-1} \right\} \right. \\
& \quad + \frac{1}{2} \left\{ \left(I + \frac{i \tau A^{1/2}}{2} \right) D^{[\mu_j/\tau]-1} + \left(I - \frac{i \tau A^{1/2}}{2} \right) \tilde{D}^{[\mu_j/\tau]-1} \right\} \\
& \quad \left. \left. \left. \times \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} (-A) R^N + \frac{\tau A}{2} R^N \right] \right] \right\} \right. \\
& \quad - \sum_{j=1}^L \beta_j \left\{ \frac{i A^{1/2}}{2} \left\{ \left(I - \frac{i \tau A^{1/2}}{2} \right)^{-1} D^{[\lambda_j/\tau]-1} \right. \right. \\
& \quad \left. \left. - \left(I + \frac{i \tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\lambda_j/\tau]-1} \right\} R^N \right. \\
& \quad + \frac{1}{2} \left\{ \left(I + \frac{i \tau A^{1/2}}{2} \right) D^{[\lambda_j/\tau]-1} + \left(I - \frac{i \tau A^{1/2}}{2} \right) \tilde{D}^{[\lambda_j/\tau]-1} \right\} \\
& \quad \times \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} (-A) R^N + \frac{\tau A}{2} R^N \right] + \left(\lambda_j - \left[\frac{\lambda_j}{\tau} \right] \tau + \frac{\tau}{2} \right) \\
& \quad \left. \left. \left. \times \left\{ \frac{1}{2} \left\{ D^{[\lambda_j/\tau]} + \tilde{D}^{[\lambda_j/\tau]} \right\} (-A) R^N \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{A^{-1/2}}{2i} \left(I + \frac{\tau^2 A}{4} \right)^{-1} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} (-A) R^N \\
& + \frac{\tau A^{1/2}}{4i} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} R^N \Bigg\} \\
& - \sum_{j=1}^N \alpha_j \left([c([\mu_j/\tau]\tau) - As([\mu_j/\tau]\tau)] \right. \\
& \left. - \sum_{j=1}^L \beta_j [s([\lambda_j/\tau]\tau) + c([\lambda_j/\tau]\tau)] A \right) e^{-A} \Bigg]
\end{aligned}$$

formülü doğrudur. Buradan,

$$\begin{aligned}
& \left\| \left(\sum_{j=1}^N \alpha_j \left([c([\mu_j/\tau]\tau) - As([\mu_j/\tau]\tau)] - \right. \right. \right. \\
& \left. \left. \sum_{j=1}^L \beta_j [s([\lambda_j/\tau]\tau) + c([\lambda_j/\tau]\tau)] A \right) \right. \\
& \left. - \sum_{j=1}^N \alpha_j \left\{ \frac{1}{2} \left\{ D^{[\mu_j/\tau]} + \tilde{D}^{[\mu_j/\tau]} \right\} \right. \right. \\
& \left. + \frac{A^{-1/2}}{2i} \left(I + \frac{\tau^2 A}{4} \right)^{-1} \left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} (-A) \right. \\
& \left. + \frac{\tau A^{1/2}}{4i} \left[\left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} \right] + \left(\mu_j - \left[\frac{\mu_j}{\tau} \right] \tau \right) \\
& \times \left[\frac{i A^{1/2}}{2} \left\{ \left(I - \frac{i \tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\mu_j/\tau]-1} - \left(I + \frac{i \tau A^{1/2}}{2} \right)^{-1} D^{[\mu_j/\tau]-1} \right\} \right. \\
& \left. + \frac{1}{2} \left\{ \left(I + \frac{i \tau A^{1/2}}{2} \right) D^{[\mu_j/\tau]-1} + \left(I - \frac{i \tau A^{1/2}}{2} \right) \tilde{D}^{[\mu_j/\tau]-1} \right\} \right. \\
& \left. \times \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} (-A) + \frac{\tau A}{2} \right] \right] \Bigg\} \\
& - \sum_{j=1}^L \beta_j \left\{ \frac{i A^{1/2}}{2} \left\{ \left(I - \frac{i \tau A^{1/2}}{2} \right)^{-1} D^{[\lambda_j/\tau]-1} \right. \right. \\
& \left. \left. - \left(I + \frac{i \tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\lambda_j/\tau]-1} \right\} \right. \\
& \left. + \frac{1}{2} \left\{ \left(I + \frac{i \tau A^{1/2}}{2} \right) D^{[\lambda_j/\tau]-1} + \left(I - \frac{i \tau A^{1/2}}{2} \right) \tilde{D}^{[\lambda_j/\tau]-1} \right\} \right. \\
& \times \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} (-A) + \frac{\tau A}{2} \right] + \left(\lambda_j - \left[\frac{\lambda_j}{\tau} \right] \tau + \frac{\tau}{2} \right) \\
& \times \left\{ \frac{1}{2} \left\{ D^{[\lambda_j/\tau]} + \tilde{D}^{[\lambda_j/\tau]} \right\} (-A) \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{A^{-1/2}}{2i} \left(I + \frac{\tau^2 A}{4} \right)^{-1} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} (-A) \\
& + \frac{\tau A^{1/2}}{4i} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} \Bigg\} e^{-A} \Bigg\|_H \leq M\tau, \tag{4.40} \\
& \left\| \sum_{j=1}^N \alpha_j \left\{ \frac{1}{2} \left\{ D^{[\mu_j/\tau]} + \tilde{D}^{[\mu_j/\tau]} \right\} \right. \right. \\
& + \frac{A^{-1/2}}{2i} \left(I + \frac{\tau^2 A}{4} \right)^{-1} \left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} (-A) \\
& + \frac{\tau A^{1/2}}{4i} \left[\left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} \right] + \left(\mu_j - \left\lfloor \frac{\mu_j}{\tau} \right\rfloor \tau \right) \\
& \times \left[\frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\mu_j/\tau]-1} - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\mu_j/\tau]-1} \right\} \right. \\
& + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\mu_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\mu_j/\tau]-1} \right\} \\
& \times \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} (-A) + \frac{\tau A}{2} \Bigg] \Bigg\} \\
& - \sum_{j=1}^L \beta_j \left\{ \frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\lambda_j/\tau]-1} \right. \right. \\
& \quad \left. \left. - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\lambda_j/\tau]-1} \right\} \right. \\
& + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\lambda_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\lambda_j/\tau]-1} \right\} \\
& \times \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} (-A) + \frac{\tau A}{2} \right] + \left(\lambda_j - \left\lfloor \frac{\lambda_j}{\tau} \right\rfloor \tau + \frac{\tau}{2} \right) \\
& \times \left\{ \frac{1}{2} \left\{ D^{[\lambda_j/\tau]} + \tilde{D}^{[\lambda_j/\tau]} \right\} (-A) \right. \\
& + \frac{A^{-1/2}}{2i} \left(I + \frac{\tau^2 A}{4} \right)^{-1} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} (-A) \\
& \left. + \frac{\tau A^{1/2}}{4i} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} \right\} \Bigg\} (e^{-A} - R^N) \Bigg\|_H \leq M\tau \tag{4.41}
\end{aligned}$$

kestirimleri kullanılarak, (4.39) kestirimi elde edilir. Buradaki (4.40) ve (4.41) kestirimlerinin ispatları, (4.37) ve (4.38) kestirimlerine dayanmaktadır. Böylece, Yardımcı Teorem 4.4'ün ispatı tamamlanmış olur.

Teorem 4.4. Eğer $\varphi \in D(A)$, $g_0 \in D(A^{1/2})$ ve $f_0 \in D(A^{1/2})$ ise, bu durumda, (4.36) fark şemasının çözümü için aşağıdaki

$$\begin{aligned} \max_{-N \leq k \leq N} \|u_k\|_H &\leq M \left[\|\varphi\|_H + \|A^{-1/2} f_0\|_H + \|A^{-1/2} g_0\|_H \right. \\ &\left. + \max_{1 \leq k \leq N} \|A^{-1/2} (f_k - f_{k-1}) \tau^{-1}\|_H + \max_{-N < k \leq 0} \|A^{-1/2} (g_k - g_{k-1}) \tau^{-1}\|_H \right], \end{aligned} \quad (4.42)$$

$$\begin{aligned} \max_{-N \leq k \leq N} \|A^{1/2} u_k\|_H &\leq M \left[\|A^{1/2} \varphi\|_H + \|f_0\|_H + \|g_0\|_H \right. \\ &\left. + \max_{1 \leq k \leq N-1} \|(f_k - f_{k-1}) \tau^{-1}\|_H + \max_{-(N-1) \leq k \leq 0} \|(g_k - g_{k-1}) \tau^{-1}\|_H \right], \end{aligned} \quad (4.43)$$

$$\begin{aligned} \max_{1 \leq k \leq N-1} \|\tau^{-2} (u_{k+1} - 2u_k + u_{k-1})\|_H &+ \max_{-N+1 \leq k \leq 0} \|Au_k\|_H \\ &+ \max_{1 \leq k \leq N} \left\| A \left(\frac{u_k + u_{k-1}}{2} \right) \right\|_H \\ &+ \max_{-(N-1) \leq k \leq 0} \|\tau^{-1} (u_k - u_{k-1})\|_H \leq M \left[\|A\varphi\|_H + \|A^{1/2} f_0\|_H \right. \\ &+ \|(f_1 - f_0) \tau^{-1}\|_H + \max_{1 \leq k \leq N-2} \|(f_{k+1} - 2f_k + f_{k-1}) \tau^{-2}\|_H + \|A^{1/2} g_0\|_H \\ &\left. + \|(g_0 - g_{-1}) \tau^{-1}\|_H + \max_{-(N-1) \leq k \leq -1} \|(g_{k+1} - 2g_k + g_{k-1}) \tau^{-2}\|_H \right] \end{aligned} \quad (4.44)$$

kararlılık kestirimleri sağlanır. Burada, M katsayısı τ , $f_k, 0 \leq k < N$, $g_k, -N < k \leq 0$ ve φ 'den bağımsızdır.

İspat. Öncelikle, (4.36) fark şemasının çözümü için formül elde edilecektir. Bilindiği gibi,

$$\begin{cases} \tau^{-2} (u_{k+1} - 2u_k + u_{k-1}) + \frac{1}{2} Au_k + \frac{1}{4} A (u_{k+1} + u_{k-1}) = f_k, f_k = f(t_k), \\ t_k = k\tau, 1 \leq k < N, N\tau = 1, \\ \left(I + \frac{\tau^2 A}{4} \right) \tau^{-1} (u_1 - u_0) - \frac{\tau}{2} (g_0 - Au_0) = \omega, u_0 = \mu, \end{cases} \quad (4.45)$$

$$\begin{cases} \tau^{-1} (u_k - u_{k-1}) + A \left(I + \frac{\tau}{2} A \right) u_k = \left(I + \frac{\tau}{2} A \right) g_k, \\ g_k = g \left(t_k - \frac{\tau}{2} \right), t_k = k\tau, -(N-1) \leq k \leq 0, u_{-N} \text{ biliniyor} \end{cases} \quad (4.46)$$

başlangıç değer fark problemlerinin tek çözümü vardır ve aşağıdaki

$$u_1 = \left(I + \frac{\tau^2 A}{4} \right)^{-1} \left[\left(I - \frac{\tau^2 A}{4} \right) \mu - \tau \omega + \frac{\tau^2}{2} g_0 \right], \quad (4.47)$$

$$u_k = \frac{1}{2} \left(D^k + \tilde{D}^k \right) \mu + \frac{A^{-1/2}}{2i} \left(\tilde{D}^k - D^k \right) \left(\omega + \frac{\tau}{2} f_0 + \frac{\tau}{2} A \mu \right) \quad (4.48)$$

$$- \sum_{s=1}^{k-1} \frac{\tau}{2i} A^{-1/2} \left[D^{k-s} - \tilde{D}^{k-s} \right] f_s, \quad 2 \leq k \leq N,$$

$$u_k = R^{N+k} u_{-N} + \tau \sum_{s=-N+1}^k \left(I + \frac{\tau}{2} A \right) R^{k-s+1} g_s, \quad -(N-1) \leq k \leq 0 \quad (4.49)$$

formüller sağlanır. Buradaki (4.47), (4.48), (4.49) denklemleri ve

$$\left(I + \frac{\tau^2 A}{4} \right) \tau^{-1} (u_1 - u_0) - \frac{\tau}{2} (g_0 - A u_0) = \left(I + \frac{\tau^2 A}{4} \right)^{-1} (f_0 - A u_0), \quad u_0 = \mu$$

formülleri kullanılarak,

$$\mu = u_0 = R^N u_{-N} + \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s,$$

$$\omega = \left(I + \frac{\tau^2 A}{4} \right)^{-1} \left(f_0 - A \left\{ R^N u_{-N} + \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \right\} \right)$$

formülleri elde edilir. Dolayısıyla,

$$u_1 = \left(I + \frac{\tau^2 A}{4} \right)^{-1} \left[\left(I - \frac{\tau^2 A}{4} \right) \right.$$

$$\times \left\{ R^N u_{-N} + \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \right\} + \tau \left(I + \frac{\tau^2 A}{4} \right)^{-1}$$

$$\times \left(f_0 - A \left\{ R^N u_{-N} + \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \right\} \right) + \frac{\tau^2}{2} g_0 \Big], \quad (4.50)$$

$$u_k = \frac{1}{2} \left(D^k + \tilde{D}^k \right) \left\{ R^N u_{-N} + \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \right\}$$

$$+ \frac{A^{-1/2}}{2i} \left(\tilde{D}^k - D^k \right) \left(\left(I + \frac{\tau^2 A}{4} \right)^{-1} \right.$$

$$\times \left(f_0 - A \left\{ R^N u_{-N} + \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \right\} \right)$$

$$+ \frac{\tau}{2} f_0 + \frac{\tau}{2} A \left\{ R^N u_{-N} + \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \right\}$$

$$- \sum_{s=1}^{k-1} \frac{\tau}{2i} A^{-1/2} \left[D^{k-s} - \tilde{D}^{k-s} \right] f_s, \quad 2 \leq k \leq N \quad (4.51)$$

$$\begin{aligned}
& - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\lambda_j/\tau]-1} \Big\} \\
& \times \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \\
& + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\lambda_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\lambda_j/\tau]-1} \right\} \\
& \times \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \left(f_0 - A\tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \right) \right. \\
& \left. + \frac{\tau^2 A}{2} \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \right] + \tau \left(I + \frac{\tau^2 A}{4} \right)^{-1} f_{[\lambda_j/\tau]-1} \\
& + \sum_{s=1}^{[\lambda_j/\tau]-2} \tau \left[\left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\lambda_j/\tau]-1-s} + \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\lambda_j/\tau]-1-s} \right] f_s \\
& \quad + \left(\lambda_j - \left\lfloor \frac{\lambda_j}{\tau} \right\rfloor \tau + \frac{\tau}{2} \right) \\
& \times \left(f_{[\lambda_j/\tau]} - A \left\{ \frac{1}{2} \left[D^{[\lambda_j/\tau]} + \tilde{D}^{[\lambda_j/\tau]} \right] \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} \right) R^{-s+1} g_s \right. \right. \\
& \quad \left. \left. + \frac{A^{-1/2}}{2i} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} \right. \right. \\
& \quad \times \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \left(f_0 - A\tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \right) \right. \\
& \quad \left. \left. + \frac{\tau}{2} f_0 + \frac{\tau^2 A}{2} \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \right] \right. \\
& \quad \left. \left. - \sum_{s=1}^{[\lambda_j/\tau]-1} \frac{\tau}{2i} A^{-1/2} \left[D^{[\lambda_j/\tau]-s} + \tilde{D}^{[\lambda_j/\tau]-s} \right] f_s \right\} \right\} + \varphi \Big]
\end{aligned}$$

operatör denklemi elde edilir. Böylece, (4.36) fark şemasının çözümü için, (4.49), (4.50), (4.51) ve (4.52) formülleri kurulmuş olur.

Şimdi, (4.7), (4.8) ve (4.9) kestirimlerinin ispatları verilecektir. İlk olarak, $\|u_{-N}\|_H$ için kestirim elde edilecektir. Bunun için, Abel formülü ve (4.52) formülü kullanılarak,

$$\begin{aligned}
u_{-N} = & T_\tau \left[\sum_{j=1}^N \alpha_j \left\{ \frac{1}{2} A^{-1} \left\{ D^{[\mu_j/\tau]} + \tilde{D}^{[\mu_j/\tau]} \right\} \right. \right. \\
& \times \left[g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} \{g_{s-1} - g_s\} \right] \\
& \left. \left. + \frac{A^{-1/2}}{2i} \left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} \right] \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \right. \right.
\end{aligned} \tag{4.53}$$

$$\begin{aligned}
& \times \left(f_0 - \left[g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} \{g_{s-1} - g_s\} \right] \right) \\
& + \frac{\tau}{2} f_0 + \frac{\tau}{2} \left[g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} \{g_{s-1} - g_s\} \right] \\
& \quad + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right) \\
& \times \left[f_{[\mu_j/\tau]} - D^{[\mu_j/\tau]-1} f_1 + \sum_{s=2}^{[\mu_j/\tau]-1} D^{[\mu_j/\tau]-s} \{f_{s-1} - f_s\} \right] \\
& \quad - \frac{1}{2} A^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right) \\
& \times \left[f_{[\mu_j/\tau]} - \tilde{D}^{[\mu_j/\tau]-1} f_1 + \sum_{s=2}^{[\mu_j/\tau]-1} \tilde{D}^{[\mu_j/\tau]-s} \{f_{s-1} - f_s\} \right] \\
& \quad + \left(\mu_j - \left[\frac{\mu_j}{\tau} \right] \tau \right) \\
& \times \left[\frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\mu_j/\tau]-1} - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\mu_j/\tau]-1} \right\} \right. \\
& \quad \times A^{-1} \left[g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} \{g_{s-1} - g_s\} \right] \\
& \quad + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\mu_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\mu_j/\tau]-1} \right\} \\
& \times \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \left(f_0 - \left[g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} \{g_{s-1} - g_s\} \right] \right) \right. \\
& \quad + \frac{\tau}{2} \left[g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} \{g_{s-1} - g_s\} \right] \\
& \quad \left. + \tau \left(I + \frac{\tau^2 A}{4} \right)^{-1} f_{[\mu_j/\tau]-1} \right. \\
& + iA^{-1/2} \left\{ \left(\tilde{D} - D \right) f_{[\mu_j/\tau]-2} + \left(D^{[\mu_j/\tau]-1} - \tilde{D}^{[\mu_j/\tau]-1} \right) f_1 \right. \\
& \quad \left. + \sum_{s=2}^{[\mu_j/\tau]-2} \left[\tilde{D}^{[\mu_j/\tau]-s} - D^{[\mu_j/\tau]-s} \right] \{f_{s-1} - f_s\} \right\} \\
& + \sum_{j=1}^L \beta_j \left\{ \frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\lambda_j/\tau]-1} \right. \right. \\
& \quad \left. \left. - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\lambda_j/\tau]-1} \right\} \right\}
\end{aligned}$$

$$\begin{aligned}
& \times A^{-1} \left[g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} \{g_{s-1} - g_s\} \right] \\
& + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\lambda_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\lambda_j/\tau]-1} \right\} \\
& \times \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \left(f_0 - \left[g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} \{g_{s-1} - g_s\} \right] \right) \right. \\
& \quad \left. + \frac{\tau}{2} \left[g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} \{g_{s-1} - g_s\} \right] \right] \\
& \quad + \tau \left(I + \frac{\tau^2 A}{4} \right)^{-1} f_{[\lambda_j/\tau]-1} \\
& \quad + iA^{-1/2} \left\{ \left(\tilde{D} - D \right) f_{[\lambda_j/\tau]-2} + \left(D^{[\lambda_j/\tau]-1} - \tilde{D}^{[\lambda_j/\tau]-1} \right) f_1 \right. \\
& \quad \left. + \sum_{s=2}^{[\lambda_j/\tau]-2} \left[\tilde{D}^{[\lambda_j/\tau]-s} - D^{[\lambda_j/\tau]-s} \right] \{f_{s-1} - f_s\} \right\} + \left(\lambda_j - \left\lfloor \frac{\lambda_j}{\tau} \right\rfloor \tau + \frac{\tau}{2} \right) \\
& \quad \times \left(f_{[\lambda_j/\tau]} - A \left\{ \frac{1}{2} A^{-1} \left\{ D^{[\lambda_j/\tau]} + \tilde{D}^{[\lambda_j/\tau]} \right\} \right. \right. \\
& \quad \times \left[g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} \{g_{s-1} - g_s\} \right] \\
& \quad \left. \left. + \frac{A^{-1/2}}{2i} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \right. \right. \right. \\
& \quad \times \left(f_0 - \left[g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} \{g_{s-1} - g_s\} \right] \right) \\
& \quad \left. \left. \left. + \frac{\tau}{2} f_0 + \frac{\tau}{2} \left[g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} \{g_{s-1} - g_s\} \right] \right] \right] \right. \\
& \quad \left. + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right) \left[f_{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]-1} f_1 + \sum_{s=2}^{[\lambda_j/\tau]-1} D^{[\lambda_j/\tau]-s} \{f_{s-1} - f_s\} \right] \right. \\
& \quad \left. - \frac{1}{2} A^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right) \left[f_{[\lambda_j/\tau]} - \tilde{D}^{[\lambda_j/\tau]-1} f_1 \right. \right. \\
& \quad \left. \left. + \sum_{s=2}^{[\lambda_j/\tau]-1} \tilde{D}^{[\lambda_j/\tau]-s} \{f_{s-1} - f_s\} \right] \right\} + \varphi
\end{aligned}$$

formülü elde edilir. Buradan, (4.49), (4.51), (4.53) formülleri ve (4.37), (4.38) kestirimleri yardımıyla,

$$\|u_{-N}\|_H \leq M \left[\|\varphi\|_H + \|A^{-1/2} f_0\|_H + \max_{1 \leq k \leq N} \|A^{-1/2} (f_k - f_{k-1}) \tau^{-1}\|_H \right]$$

$$+ \left\| A^{-1/2} g_0 \right\|_H + \max_{-N \leq k \leq 0} \left\| A^{-1/2} (g_k - g_{k-1}) \tau^{-1} \right\|_H \right] \quad (4.54)$$

kestirimi alınır.

İkinci olarak, $\|u_k\|_H$ için kestirim elde edilecektir. Bunun için önce $-N < k \leq 0$ aralığında inceleyelim. Bu durumda, Abel formülü ve (4.49) formülü kullanılarak,

$$u_k = R^{N+k} u_{-N} + A^{-1} \left[g_k - R^{N+k} g_{-N+1} + \sum_{s=-N+2}^k R^{k-s+1} [g_{s-1} - g_s] \right] \quad (4.55)$$

formülü elde edilir. Buradan, (4.55) formülünün ve (4.38) kestiriminin yardımıyla,

$$\|u_k\|_H \leq \|u_{-N}\|_H + 2 \left\| A^{-1/2} g_0 \right\|_H + \max_{-N < k \leq 0} \left\| A^{-1/2} (g_k - g_{k-1}) \tau^{-1} \right\|_H \quad (4.56)$$

kestirimi bulunur. Şimdi ise, $1 \leq k \leq N$ aralığında inceleyelim. Bu durumda, Abel formülü ve (4.50), (4.51) formülleri kullanılarak,

$$\begin{aligned} u_1 &= \left(I - \tau A \left(\frac{\tau}{2} + I \right) (I + \tau^2 A)^{-1} \right) \\ &\times \left[R^N u_{-N} + A^{-1} \left\{ g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} (g_{s-1} - g_s) \right\} \right] \\ &+ (I + \tau^2 A)^{-1} \left\{ \frac{\tau^2}{2} f(0) + \tau g_0 \right\}, \quad (4.57) \\ u_k &= \frac{1}{2} (D^k + \tilde{D}^k) \left\{ R^N u_{-N} \right. \\ &+ A^{-1} \left[g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} \{g_{s-1} - g_s\} \right] \left. \right\} \\ &+ \frac{1}{2i} A^{-1/2} (\tilde{D}^k - D^k) \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} (f_0 - A \{R^N u_{-N} + \right. \\ &+ A^{-1} \left[g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} \{g_{s-1} - g_s\} \right] \left. \right\} + \frac{\tau}{2} f_0 \\ &+ \frac{\tau}{2} A \left\{ R^N u_{-N} + A^{-1} \left[g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^0 R^{-s+1} \{g_{s-1} - g_s\} \right] \right\} \left. \right] \\ &+ \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right) \left[f_{k-1} - D^{k-1} f_1 + \sum_{s=2}^{k-1} D^{k-s} \{f_{s-1} - f_s\} \right] \end{aligned}$$

$$+\frac{1}{2}A^{-1}\left(I+\frac{i\tau A^{1/2}}{2}\right)\left[f_{k-1}-\tilde{D}^{k-1}f_1+\sum_{s=2}^{k-1}D^{k-s}\{f_{s-1}-f_s\}\right] \quad (4.58)$$

formülleri elde edilir. Buradan, (4.57), (4.58) formülleri ve (4.37), (4.38) kestirimlerinin yardımıyla,

$$\|u_1\|_H \leq M \left[\|u_{-N}\|_H + \|A^{-1/2}g_0\|_H + \max_{-N < k \leq 0} \|A^{-1/2}(g_k - g_{k-1})\tau^{-1}\|_H \right], \quad (4.59)$$

$$\begin{aligned} \|u_k\|_H &\leq M \left[\|u_{-N}\|_H + \|A^{-1/2}f_0\|_H + \max_{2 \leq k < N} \|A^{-1/2}(f_k - f_{k-1})\tau^{-1}\|_H \right. \\ &\quad \left. + \|A^{-1/2}g_0\|_H + \max_{-(N-1) \leq k \leq 0} \|A^{-1/2}(g_k - g_{k-1})\tau^{-1}\|_H \right], 1 \leq k \leq N \end{aligned} \quad (4.60)$$

kestirimleri alınır. Sonuç olarak, (4.54), (4.57), (4.59), (4.60) kestirimleri birleştirilerek ve üçgen eşitsizliği kullanılarak, (4.42) kestirimi elde edilir. Burada (4.43) kestiriminin ispatı, (4.53), (4.55) formüllerine ve (4.37), (4.38) kestirimlerine bağlı kalmak şartıyla, (4.42) kestiriminin ispatında olduğu gibi aynı yolla elde edilir.

Üçüncü olarak, $\left\| A \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} u_{-N} \right\|_H$ için kestirim elde edilecektir. Bunun için, Abel formülü ve (4.52) formülü kullanılarak,

$$\begin{aligned} Au_{-N} &= T_\tau \left[\sum_{j=1}^N \alpha_j \left\{ \frac{1}{2} \left\{ D^{[\mu_j/\tau]} + \tilde{D}^{[\mu_j/\tau]} \right\} \right. \right. \\ &\quad \times \left[g_0 - R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \\ &\quad \times \left\{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \right. \\ &\quad \left. \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \right\} \right] \\ &\quad + \frac{1}{2i} \left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \right. \\ &\quad \times \left(A^{1/2} f_0 - \left[A^{1/2} g_0 - A^{1/2} R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \right. \\ &\quad \times \left\{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \right. \\ &\quad \left. \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \right\} \right] \left. \right] \\ &\quad \left. + \frac{\tau}{2} A^{1/2} f_0 + \frac{\tau}{2} \left[A^{1/2} g_0 - A^{1/2} R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \right. \end{aligned} \quad (4.61)$$

$$\begin{aligned}
& \times \left\{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \right. \\
& \quad \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \right\} \Bigg] \\
& + f_k - \frac{1}{2} \left[A^{1/2} \left(I - \frac{i\tau A^{1/2}}{2} \right) D^{k-1} + A^{1/2} \left(I + \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{k-1} \right] A^{1/2} f_1 \\
& + \frac{1}{2} (i\tau A^{1/2})^{-1} \left[f_{[\mu_j/\tau]-2} - f_{[\mu_j/\tau]-1} + D^{[\mu_j/\tau]-2} (f_1 - f_0) \right. \\
& \quad \left. + \sum_{s=1}^{[\mu_j/\tau]-2} D^{[\mu_j/\tau]-1-s} \{f_{s+1} - 2f_s + f_{s-1}\} \right] \\
& + \frac{1}{2} (i\tau A^{1/2})^{-1} \left[f_{[\mu_j/\tau]-2} - f_{[\mu_j/\tau]-1} + \tilde{D}^{[\mu_j/\tau]-2} (f_1 - f_0) \right. \\
& \quad \left. + \sum_{s=1}^{[\mu_j/\tau]-2} \tilde{D}^{[\mu_j/\tau]-1-s} \{f_{s+1} - 2f_s + f_{s-1}\} \right] + \left(\mu_j - \left[\frac{\mu_j}{\tau} \right] \tau \right) \\
& \times \left[\frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\mu_j/\tau]-1} - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\mu_j/\tau]-1} \right\} \right. \\
& \quad \times \left[g_0 - R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \\
& \quad \times \left\{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \right. \\
& \quad \quad \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \right\} \Bigg] \\
& + \frac{1}{2} \left\{ A^{1/2} \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\mu_j/\tau]-1} + A^{1/2} \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\mu_j/\tau]-1} \right\} \\
& \times \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \left(A^{1/2} f_0 - \left[A^{1/2} g_0 - A^{1/2} R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \right. \right. \\
& \quad \times \left\{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \right. \\
& \quad \quad \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \right\} \Bigg] \Bigg) \\
& + \frac{\tau}{2} \left[A^{1/2} g_0 - A^{1/2} R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \\
& \quad \times \left\{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \right. \\
& \quad \quad \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \right\} \Bigg] + \tau \left(I + \frac{\tau^2 A}{4} \right)^{-1} f_{[\mu_j/\tau]-1} \\
& + iA^{-1/2} \left\{ \left(\tilde{D} - D \right) f_{[\mu_j/\tau]-2} + \left(D^{[\mu_j/\tau]-1} - \tilde{D}^{[\mu_j/\tau]-1} \right) f_1 \right.
\end{aligned}$$

$$\begin{aligned}
& - \left(i\tau A^{1/2} \right)^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right) \\
& \times \left\{ \left[\tilde{D} \left(f_{[\mu_j/\tau]-3} - f_{[\mu_j/\tau]-2} \right) + \tilde{D}^{[\mu_j/\tau]-2} (f_1 - f_0) \right. \right. \\
& \quad \left. \left. + \sum_{s=1}^{[\mu_j/\tau]-3} \tilde{D}^{[\mu_j/\tau]-1-s} \{f_{s+1} - 2f_s + f_{s-1}\} \right] \right. \\
& + \left(I - \frac{i\tau A^{1/2}}{2} \right) \left[D \left(f_{[\mu_j/\tau]-3} - f_{[\mu_j/\tau]-2} \right) + D^{[\mu_j/\tau]-2} (f_1 - f_0) \right. \\
& \quad \left. \left. + \sum_{s=1}^{[\mu_j/\tau]-3} D^{[\mu_j/\tau]-1-s} \{f_{s+1} - 2f_s + f_{s-1}\} \right] \right\} \\
& + \sum_{j=1}^L \beta_j \left\{ \frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\lambda_j/\tau]-1} \right. \right. \\
& \quad \left. \left. - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\lambda_j/\tau]-1} \right\} \right. \\
& \times \left[A^{1/2} g_0 - A^{1/2} R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \\
& \quad \times \{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \\
& \quad \left. \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \right\} \right] \\
& + \frac{1}{2} A^{1/2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\lambda_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\lambda_j/\tau]-1} \right\} \\
& \times \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \left(A^{1/2} f_0 - \left[A^{1/2} g_0 - A^{1/2} R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \right. \right. \\
& \quad \times \{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \\
& \quad \left. \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \right\} \right] \right] \\
& + \frac{\tau}{2} \left[A^{1/2} g_0 - A^{1/2} R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \\
& \quad \times \{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \\
& \quad \left. \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \right\} \right] + \tau \left(I + \frac{\tau^2 A}{4} \right)^{-1} f_{[\lambda_j/\tau]-1} \\
& + \left\{ \left(\tilde{D} - D \right) f_{[\lambda_j/\tau]-2} + \left(D^{[\lambda_j/\tau]-1} - \tilde{D}^{[\lambda_j/\tau]-1} \right) f_1 \right. \\
& \left. - \tau^{-1} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) \left[\tilde{D} \left(f_{[\lambda_j/\tau]-3} - f_{[\lambda_j/\tau]-2} \right) + \tilde{D}^{[\lambda_j/\tau]-2} (f_1 - f_0) \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \sum_{s=1}^{[\lambda_j/\tau]-3} \tilde{D}^{[\lambda_j/\tau]-1-s} \{f_{s+1} - 2f_s + f_{s-1}\} \Big] \\
& + \left(I - \frac{i\tau A^{1/2}}{2} \right) \left[D \left(f_{[\mu_j/\tau]-3} - f_{[\mu_j/\tau]-2} \right) + D^{[\lambda_j/\tau]-2} (f_1 - f_0) \right. \\
& + \left. \sum_{s=1}^{[\lambda_j/\tau]-3} D^{[\lambda_j/\tau]-1-s} \{f_{s+1} - 2f_s + f_{s-1}\} \right] \Bigg\} + \left(\lambda_j - \left\lfloor \frac{\lambda_j}{\tau} \right\rfloor \tau + \frac{\tau}{2} \right) \\
& \times \left(f_{[\lambda_j/\tau]} - \left\{ \frac{1}{2} A^{1/2} \left\{ D^{[\lambda_j/\tau]} + \tilde{D}^{[\lambda_j/\tau]} \right\} \right. \right. \\
& \times \left[A^{1/2} g_0 - A^{1/2} R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \\
& \times \left\{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \right. \\
& \left. \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \right\} \right] \\
& + \frac{1}{2i} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \right. \\
& \times \left(A^{1/2} f_0 - \left[A^{1/2} g_0 - A^{1/2} R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \right. \\
& \times \left\{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \right. \\
& \left. \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \right\} \right] \Bigg) \\
& + \frac{\tau}{2} A^{1/2} f_0 + \frac{\tau}{2} \left[A^{1/2} g_0 - A^{1/2} R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \\
& \times \left\{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \right. \\
& \left. \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \right\} \right] \Bigg] \\
& + \frac{1}{2} \left(I - \frac{i\tau A^{1/2}}{2} \right) [f_{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]-1} f_1] \\
& + \frac{1}{2} (i\tau A^{1/2})^{-1} [f_{[\lambda_j/\tau]-2} - f_{[\lambda_j/\tau]-1} + D^{[\lambda_j/\tau]-2} (f_1 - f_0) \\
& + \sum_{s=1}^{[\lambda_j/\tau]-2} D^{[\lambda_j/\tau]-1-s} \{f_{s+1} - 2f_s + f_{s-1}\} \Big] \\
& + \frac{1}{2} \left(I + \frac{i\tau A^{1/2}}{2} \right) [f_{[\lambda_j/\tau]} - \tilde{D}^{[\lambda_j/\tau]-1} f_1] \\
& + \frac{1}{2} (i\tau A^{1/2})^{-1} [f_{[\lambda_j/\tau]-2} - f_{[\lambda_j/\tau]-1} + \tilde{D}^{[\lambda_j/\tau]-2} (f_1 - f_0) \Bigg) \Bigg\}
\end{aligned}$$

$$+ \sum_{s=1}^{[\lambda_j/\tau]-2} \tilde{D}^{[\lambda_j/\tau]-1-s} \{f_{s+1} - 2f_s + f_{s-1}\} \Big] \Big) \Big) \Big\} + A\varphi \Big]$$

formülü elde edilir. Buradan, (4.53) formülünün ve (4.37), (4.38) kestirimlerinin yardımıyla,

$$\begin{aligned} \left\| A \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} u_{-N} \right\|_H &\leq M \left[\|A\varphi\|_H + \|A^{1/2}f_0\|_H + \|(f_1 - f_0)\tau^{-1}\|_H \right. \\ &\quad + \max_{2 \leq k \leq N-2} \|(f_{k+1} - 2f_k + f_{k-1})\tau^{-2}\|_H + \|A^{1/2}g_0\|_H \\ &\quad \left. + \|(g_0 - g_{-1})\tau^{-1}\|_H + \max_{-(N-1) \leq k \leq -1} \|(g_{k+1} - 2g_k + g_{k-1})\tau^{-2}\|_H \right] \end{aligned} \quad (4.62)$$

kestirimi alınır.

Son olarak, $\|Au_k\|_H$ için kestirim elde edilecektir. Bunun için önce $-N+1 \leq k \leq 0$ aralığında inceleyelim. Bu durumda, Abel formülü ve (4.49) formülü kullanılarak,

$$\begin{aligned} Au_k &= R^{N+k} \left(I + \frac{i\tau A^{1/2}}{2} \right) \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} Au_{-N} \\ &\quad + \left[g_k - R^{N+k}g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \\ &\quad \times \left\{ (g_{k-1} - g_k) - R^{k+N-1} (g_{-N+1} - g_{-N+2}) \right. \\ &\quad \left. \left. + \sum_{s=-N+2}^{k-1} R^{k-s} [g_{s+1} - 2g_s + g_{s-1}] \right\} \right] \end{aligned} \quad (4.63)$$

formülü elde edilir. Buradan, (4.63) formülünün ve (4.38) kestiriminin yardımıyla,

$$\begin{aligned} \|Au_k\|_H &\leq M \left[\left\| A \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} u_{-N} \right\|_H \right. \\ &\quad + \|\tau^{-1}(g_{-1} - g_0)\|_H + \|A^{1/2}g_0\|_H \\ &\quad \left. + \max_{-(N-2) \leq k \leq -1} \|\tau^{-2}(g_{k+1} - 2g_k + g_{k-1})\|_H \right] \end{aligned} \quad (4.64)$$

kestirimi alınır. Şimdi ise $1 \leq k \leq N$ aralığında inceleyelim. Bu durumda, Abel formülü ve (4.50), (4.51) formülleri kullanılarak,

$$\begin{aligned} \frac{Au_1 + Au_0}{2} &= \frac{1}{2} \left(R^N u_{-N} + \tau \sum_{s=-N+1}^0 \left(I + \frac{\tau}{2} A \right) R^{-s+1} g_s \right) \\ &\quad + \frac{1}{2} \left(I - \tau A \left(\frac{\tau}{2} + I \right) (I + \tau^2 A)^{-1} \right) \end{aligned}$$

$$\begin{aligned}
& \times \left[R^N A \left(I + \frac{i\tau A^{1/2}}{2} \right) \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} u_{-N} \right. \\
& \quad + \left\{ g_0 - R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \\
& \quad \times \{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \\
& \quad \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \} \} \right] \\
& \quad + A (I + \tau^2 A)^{-1} \left\{ \frac{\tau^2}{2} f_0 + \tau g_0 \right\}, \tag{4.65} \\
u_k = & \frac{1}{2} (D^k + \tilde{D}^k) \left\{ R^N \left(I + \frac{i\tau A^{1/2}}{2} \right) \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} u_{-N} \right. \\
& + A^{-1} \left[g_0 - R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \\
& \times \{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \\
& \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \} \right] \} \\
& + \frac{1}{2i} A^{-1/2} (\tilde{D}^k - D^k) \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} (f_0 - \{ R^N A u_{-N} + \right. \\
& \quad + \left[g_0 - R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \\
& \quad \times \{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \\
& \quad \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \} \} \right] \right] + \frac{\tau}{2} f_0 \\
& + \frac{\tau}{2} \left\{ R^N \left(I + \frac{i\tau A^{1/2}}{2} \right) \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} A u_{-N} \right. \\
& + \left[A^{1/2} g_0 - A^{1/2} R^N g_{-N+1} + \left(\tau A + \frac{\tau^2 A^2}{2} \right)^{-1} \right. \\
& \times \{ (g_{-1} - g_0) - R^{N-1} (g_{-N+1} - g_{-N+2}) \\
& \left. + \sum_{s=-N+2}^{-1} R^{-s} [g_{s+1} - 2g_s + g_{s-1}] \} \right] \} \\
& + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right) \left[f_{k-1} - D^{k-1} f_1 + (i\tau A^{1/2})^{-1} \left(1 - \frac{i\tau A^{1/2}}{2} \right) \right. \\
& \times \left[D (f_{k-2} - f_{k-1}) + D^{k-1} (f_1 - f_0) + \sum_{s=1}^{k-2} D^{k-s} \{ f_{s+1} - 2f_s + f_{s-1} \} \right] \left. \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} A^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right) \left[f_{k-1} - \tilde{D}^{k-1} f_1 - (i\tau A^{1/2})^{-1} \left(1 + \frac{i\tau A^{1/2}}{2} \right) \right. \\
& \times \left. \left[\tilde{D}(f_{k-2} - f_{k-1}) + \tilde{D}^{k-1}(f_1 - f_0) + \sum_{s=1}^{k-2} \tilde{D}^{k-s} \{f_{s+1} - 2f_s + f_{s-1}\} \right] \right] \quad (4.66)
\end{aligned}$$

formülleri elde edilir. Buradan, (4.65), (4.66) formüllerinin ve (4.37), (4.38) kestirimlerinin yardımıyla,

$$\begin{aligned}
& \left\| A \left(\frac{u_1 + u_0}{2} \right) \right\|_H \leq M \left[\left\| A \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} u_{-N} \right\|_H \right. \\
& \quad + \left\| A^{1/2} g_0 \right\|_H + \left\| \tau^{-1} (g_{-1} - g_0) \right\|_H \\
& \quad \left. + \max_{-(N-1) \leq k \leq -1} \left\| \tau^{-2} [g_{s+1} - 2g_s + g_{s-1}] \right\|_H \right], \quad (4.67) \\
& \left\| A \left(\frac{u_k + u_{k-1}}{2} \right) \right\|_H \leq M \left[\left\| A \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} u_{-N} \right\|_H \right. \\
& \quad + \left\| A^{1/2} f_0 \right\|_H + \left\| (f_1 - f_0) \tau^{-1} \right\|_H \\
& \quad + \max_{2 \leq k \leq N-2} \left\| (f_{k+1} - 2f_k + f_{k-1}) \tau^{-2} \right\|_H + \left\| A^{1/2} g_0 \right\|_H + \left\| (g_{-1} - g_0) \tau^{-1} \right\|_H \\
& \quad \left. + \max_{-(N-2) \leq k \leq -1} \left\| \tau^{-2} (g_{s+1} - 2g_s + g_{s-1}) \right\|_H \right], \quad 2 \leq k \leq N \quad (4.68)
\end{aligned}$$

kestirimleri alınır. Sonuç olarak, (4.62), (4.64), (4.67), (4.68) kestirimleri birleştirilerek ve üçgen eşitsizliği kullanılarak, (4.44) kestirimi elde edilir. Böylece, Teorem 4.4'ün ispatı tamamlanmış olur.

Şimdi, Teorem 4.4'ün uygulamaları verilecektir. İlk olarak, bir boyutlu hiperbolik-parabolik denklem için, (2.21) lokal olmayan sınır değer problemi ele alınmıştır. Burada (2.21) probleminin diskritizasyonu iki adımda incelenir. Birinci adımda, A_h^x fark operatörünün yardımıyla, (2.21) sınır değer problemi, (3.31) adi diferansiyel denklem sistemine dönüştürülür. İkinci adımda ise, (3.31) problemi için (4.36) fark şeması

kullanılarak

$$\left\{ \begin{array}{l} \frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + \frac{1}{2}A_h^x u_k^h(x) + \frac{1}{4}A_h^x (u_{k+1}^h(x) + u_{k-1}^h(x)) = f_k^h(x), \\ f_k^h(x) = \{f(y_k, x_n)\}_1^{K-1}, x \in [0, 1]_h, y_k = k\tau, 1 \leq k \leq N-1, N\tau = 1, \\ \left(I + \frac{\tau^2}{4}A_h^x\right) \left[\left(I + \frac{\tau^2}{4}A_h^x\right) \tau^{-1} (u_1^h(x) - u_0^h(x)) \right. \\ \left. - \frac{\tau}{2} (g_0^h(x) - A_h^x u_0^h(x))\right] = Z_1^h(x), \\ Z_1^h(x) = f_0^h(x) - A_h^x u_0^h(x), \\ f_0^h(x) = \{f(0, x_n)\}_1^{K-1}, g_0^h(x) = \{g(0, x_n)\}_1^{K-1}, \\ \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + A_h^x \left(I + \frac{\tau}{2}A_h^x\right) u_k^h(x) = \left(I + \frac{\tau}{2}A_h^x\right) g_k^h(x), x \in [0, 1]_h, \\ g_k^h(x) = \{g(y_k - \frac{\tau}{2}, x_n)\}_1^{K-1}, y_k = k\tau, -(N-1) \leq k \leq 0, \\ u_N^h(x) = \sum_{j=1}^N \alpha_j \left(u_{[\mu_j/\tau]}^h(x) + (\mu_j - [\frac{\mu_j}{\tau}]\tau) \frac{u_{[\mu_j/\tau]}^h(x) - u_{[\mu_j/\tau]-1}^h(x)}{\tau} \right) \\ + \sum_{j=1}^L \beta_j \left[\frac{u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)}{\tau} + \left(\lambda_j - [\frac{\lambda_j}{\tau}]\tau + \frac{\tau}{2} \right) \right. \\ \left. \times \left(f_{[\frac{\lambda_j}{\tau}]}^h(x) - A_h^x u_{[\frac{\lambda_j}{\tau}]}^h(x) \right) \right] + \varphi^h(x), x \in [0, 1]_h \end{array} \right. \quad (4.69)$$

fark şeması elde edilir.

Teorem 4.5. Eğer τ ve h yeterince küçük sayılar ise, bu durumda (4.69) fark şemasının çözümü için aşağıdaki

$$\begin{aligned} & \max_{-N \leq k \leq N} \|u_k^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq N} \|\tau^{-1} (u_k^h - u_{k-1}^h)\|_{L_{2h}} + \max_{-N \leq k \leq N} \|(u_k^h)_x\|_{L_{2h}} \\ & \leq M_1 \left[\|f_0^h\|_{L_{2h}} + \max_{2 \leq k \leq N-1} \|(f_k^h - f_{k-1}^h) \tau^{-1}\|_{L_{2h}} \right. \\ & \quad \left. + \|g_0^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \|(g_k^h - g_{k-1}^h) \tau^{-1}\|_{L_{2h}} + \left\| \varphi_x^h \right\|_{L_{2h}} \right], \\ & \quad \max_{1 \leq k \leq N-1} \|\tau^{-2} (u_{k+1}^h - 2u_k^h + u_{k-1}^h)\|_{L_{2h}} \\ & \quad + \max_{-N+1 \leq k \leq 0} \left\| \left(\frac{u_x^k}{x} \right)_x \right\|_{L_{2h}} + \max_{1 \leq k \leq N} \left\| \frac{\left(\frac{u_x^k}{x} \right)_x + \left(\frac{u_x^{k-1}}{x} \right)_x}{2} \right\|_{L_{2h}} \\ & \quad + \max_{-N+1 \leq k \leq 0} \|\tau^{-1} (u_k^h - u_{k-1}^h)\|_{L_{2h}} \\ & \leq M_1 \left[\|f_{0x}^h\|_{L_{2h}} + \|\tau^{-1} (f_1^h - f_0^h)\|_{L_{2h}} \right. \\ & \quad + \max_{2 \leq k \leq N-1} \|\tau^{-2} (f_{k+1}^h - 2f_k^h + f_{k-1}^h)\|_{L_{2h}} + \|g_{0x}^h\|_{L_{2h}} + \|\tau^{-1} (g_0^h - g_{-1}^h)\|_{L_{2h}} \\ & \quad \left. + \max_{-N+1 \leq k \leq -1} \|\tau^{-2} (g_{k+1}^h - 2g_k^h + g_{k-1}^h)\|_{L_{2h}} + \left\| \left(\varphi_x^h \right)_x \right\|_{L_{2h}} \right] \end{aligned}$$

kararlılık kestirimleri sağlanır. Burada, M_1 katsayısı $\tau, h, f_k^h, 1 \leq k < N, g_k^h,$

$-N < k \leq 0$ ve $\varphi^h(x)$ 'den bağımsızdır.

Teorem 4.5'in ispatı, soyut Teorem 4.4 ve (3.30) formülü ile tanımlanan A_h^x fark operatörünün simetri özelliklerine dayanmaktadır.

İkinci olarak, çok boyutlu hiperbolik-parabolik denklem için, (2.22) karma sınır değer problemi ele alınmıştır. Burada da (2.22) probleminin diskritizasyonu iki adımda incelenir. Birinci adımda, A_h^x operatörünün yardımıyla, (2.22) sınır değer problemi, (3.34) adi diferansiyel denklem sistemine dönüştürülür. İkinci adımda ise, (3.34) problemi için (4.36) fark şeması kullanılarak

$$\left\{ \begin{array}{l} \frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + \frac{1}{2}A_h^x u_k^h(x) + \frac{1}{4}A_h^x (u_{k+1}^h(x) + u_{k-1}^h(x)) \\ = f_k^h(x), \quad f_k^h(x) = f^h(y_k, x), \quad x \in \tilde{\Omega}_h, y_k = k\tau, \quad 1 \leq k < N, \\ \left(I + \frac{\tau^2}{4}A_h^x \right) \left[\left(I + \frac{\tau^2}{4}A_h^x \right) \tau^{-1} (u_1^h(x) - u_0^h(x)) \right. \\ \left. - \frac{\tau}{2} (g_0^h(x) - A_h^x u_0^h(x)) \right] = Z_1^h(x), \quad x \in \tilde{\Omega}_h, \\ Z_1^h(x) = f_0^h(x) - A_h^x u_0^h(x), \quad x \in \tilde{\Omega}_h, \\ f_0^h(x) = \{f(0, x_n)\}_1^{K-1}, g_0^h(x) = \{g(0, x_n)\}_1^{K-1} \\ \tau^{-1} (u_k^h(x) - u_{k-1}^h(x)) + A_h^x \left(I + \frac{\tau}{2}A_h^x \right) u_k^h(x) \\ = \left(I + \frac{\tau}{2}A_h^x \right) g_k^h(x), \quad x \in \tilde{\Omega}_h, \\ g_k^h(x) = g^h(y_k - \frac{\tau}{2}, x), \quad y_k = k\tau, \quad -(N-1) \leq k \leq 0, x \in \tilde{\Omega}_h, \\ u_{-N}^h(x) = \sum_{j=1}^N \alpha_j \left(u_{[\mu_j/\tau]}^h(x) + (\mu_j - [\frac{\mu_j}{\tau}]\tau) \frac{u_{[\mu_j/\tau]}^h(x) - u_{[\mu_j/\tau]-1}^h(x)}{\tau} \right) \\ + \sum_{j=1}^L \beta_j \left[\frac{u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)}{\tau} + \left(\lambda_j - [\frac{\lambda_j}{\tau}]\tau - \frac{\tau}{2} \right) \right. \\ \left. \times \left(f_{[\frac{\lambda_j}{\tau}]}^h(x) - A_h^x u_{[\frac{\lambda_j}{\tau}]}^h(x) \right) \right] + \varphi^h, \quad x \in \tilde{\Omega}_h \end{array} \right. \quad (4.70)$$

fark şeması elde edilir.

Teorem 4.6. Eğer τ ve $|h|$ yeterince küçük sayılar ise, bu durumda (4.70) fark şemasının çözümü için aşağıdaki

$$\begin{aligned} & \max_{-N \leq k \leq N} \|u_k^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq N} \|\tau^{-1} (u_k^h - u_{k-1}^h)\|_{L_{2h}} \\ & \quad + \max_{-N \leq k \leq N} \sum_{r=1}^n \|(u_k^h)_{x_r, j_r}\|_{L_{2h}} \\ & \leq M_1 \left[\|f_0^h\|_{L_{2h}} + \max_{2 \leq k \leq N-1} \|(f_k^h - f_{k-1}^h) \tau^{-1}\|_{L_{2h}} \right. \\ & \quad \left. + \|g_0^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \|(g_k^h - g_{k-1}^h) \tau^{-1}\|_{L_{2h}} + \sum_{r=1}^n \|(\varphi^h)_{x_r, j_r}^-\|_{L_{2h}} \right], \\ & \max_{1 \leq k \leq N-1} \|\tau^{-2} (u_{k+1}^h - 2u_k^h + u_{k-1}^h)\|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \sum_{r=1}^n \|(u_k^h)_{x_r, j_r}^-\|_{L_{2h}} \end{aligned}$$

$$\begin{aligned}
& + \max_{1 \leq k \leq N} \sum_{r=1}^n \left\| \frac{(u_k^h)_{\bar{x}_r, x_r, j_r} + (u_{k-1}^h)_{\bar{x}_r, x_r, j_r}}{2} \right\|_{L_{2h}} \\
& + \max_{-N+1 \leq k \leq 0} \left\| \tau^{-1} (u_k^h - u_{k-1}^h) \right\|_{L_{2h}} \\
& \leq M_1 \left[\sum_{r=1}^n \left\| (f_0^h)_{\bar{x}_r, j_r} \right\|_{L_{2h}} + \left\| \tau^{-1} (f_1^h - f_0^h) \right\|_{L_{2h}} \right. \\
& + \max_{2 \leq k \leq N-1} \left\| \tau^{-2} (f_{k+1}^h - 2f_k^h + f_{k-1}^h) \right\|_{L_{2h}} \\
& + \sum_{r=1}^n \left\| (g_0^h)_{\bar{x}_r, j_r} \right\|_{L_{2h}} + \left\| \tau^{-1} (g_0^h - g_{-1}^h) \right\|_{L_{2h}} \\
& \left. + \max_{-N+1 \leq k \leq -1} \left\| \tau^{-2} (g_{k+1}^h - 2g_k^h + g_{k-1}^h) \right\|_{L_{2h}} + \sum_{r=1}^n \left\| (\varphi^h)_{\bar{x}_r, x_r, j_r} \right\|_{L_{2h}} \right]
\end{aligned}$$

kararlılık kestirimleri sağlanır. Burada, M_1 katsayısı $\tau, h, f_k^h, 1 \leq k < N, g_k^h, -N < k \leq 0$ ve $\varphi^h(x)$ 'den bağımsızdır.

Teorem 4.6'nın ispatı, soyut Teorem 4.4, (3.33) formülü ile tanımlanan A_h^x fark operatörünün simetri özelliklerine ve Teorem 3.4'e dayanmaktadır.

4.3 Fark Şeması 3

Bu kısımda, (1.1) lokal olmayan sınır değer probleminin yaklaşık çözümlerini bulmak için, r -iyileştirilmiş Crank-Nicholson tarafından oluşturulan

$$\left\{ \begin{array}{l} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + \frac{1}{2}Au_k + \frac{1}{4}A(u_{k+1} + u_{k-1}) = f_k, f_k = f(t_k), \\ t_k = k\tau, \ 1 \leq k < N, \\ \left(I + \frac{\tau^2 A}{4}\right) \left(\left(I + \frac{\tau^2 A}{4}\right) \tau^{-1} (u_1 - u_0) - \frac{\tau}{2} (g_0 - Au_0) \right) = f_0 - Au_0, \\ \tau^{-1}(u_k - u_{k-1}) + Au_k = g_k, \quad -N+1 \leq k \leq -N+r, \\ \tau^{-1}(u_k - u_{k-1}) + \frac{A}{2}(u_k + u_{k-1}) = g_k, \quad -N+r+1 \leq k \leq 0, \\ g_k = g(t_k - \frac{\tau}{2}), \quad t_k = k\tau, \quad -N+1 \leq k \leq 0, \\ u_{-N} = \sum_{j=1}^N \alpha_j \left(u_{[\mu_j/\tau]} + (\mu_j - [\frac{\mu_j}{\tau}]\tau) \frac{u_{[\mu_j/\tau]} - u_{[\mu_j/\tau]-1}}{\tau} \right) \\ + \sum_{j=1}^L \beta_j \left(\frac{u_{[\lambda_j/\tau]} - u_{[\lambda_j/\tau]-1}}{\tau} + \left(\lambda_j - [\frac{\lambda_j}{\tau}]\tau + \frac{\tau}{2} \right) (f_{[\lambda_j/\tau]} - Au_{[\lambda_j/\tau]}) \right) + \varphi \end{array} \right. \quad (4.71)$$

ikinci basamaktan doğruluklu fark şeması kurulmuştur.

Öncelikle, ileride ihtiyaç duyacağımız yardımcı teoremleri verelim.

Yardımcı Teorem 4.5. Aşağıdaki

$$\|D\|_{H \mapsto H} \leq 1, \quad \|\tilde{D}\|_{H \mapsto H} \leq 1, \quad \left\| \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right\|_{H \mapsto H} \leq 1, \quad (4.72)$$

$$\|R^k\|_{H \rightarrow H} \leq M(1 + \delta\tau)^{-k}, \|AR^k\|_{H \rightarrow H} \leq M(k\tau)^{-1}, k \geq 1, \quad (4.73)$$

$$\|AB^{k-r}R^r\|_{H \rightarrow H} \leq M(k\tau)^{-1}, r \geq 2, \|B^k\|_{H \rightarrow H} \leq 1, \|C^k\|_{H \rightarrow H} \leq 1, k \geq 1 \quad (4.74)$$

kestirimler sağlanırlar. Burada,

$$\begin{aligned} D &= \left(I - \frac{i\tau A^{1/2}}{2}\right) \left(I + \frac{i\tau A^{1/2}}{2}\right)^{-1}, \\ \tilde{D} &= \left(I + \frac{i\tau A^{1/2}}{2}\right) \left(I - \frac{i\tau A^{1/2}}{2}\right)^{-1}, \\ R &= (I + \tau A)^{-1}, B = \left(I - \frac{\tau A}{2}\right) \left(I + \frac{\tau A}{2}\right)^{-1} \text{ ve } C = \left(I + \frac{\tau A}{2}\right)^{-1} \end{aligned}$$

dir.

Yardımcı Teorem 4.6. Öncelikle,

$$\begin{aligned} &I - \sum_{j=1}^N \alpha_j \left\{ \frac{1}{2} \left\{ D^{[\mu_j/\tau]} + \tilde{D}^{[\mu_j/\tau]} \right\} B^{N-r} R^r \right. \\ &+ \frac{1}{2i} \left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] B^{N-r} A^{1/2} R^r \\ &\quad \left. + \left(\mu_j - \left\lfloor \frac{\mu_j}{\tau} \right\rfloor \tau \right) \right. \\ &\quad \times \left[\frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\mu_j/\tau]-1} \right. \right. \\ &\quad \left. \left. - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\mu_j/\tau]-1} \right\} B^{N-r} R^r \right. \\ &+ \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\mu_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\mu_j/\tau]-1} \right\} \\ &\quad \left. \times \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] \right] AB^{N-r} R^r \Big\} \\ &+ \sum_{j=1}^L \beta_j \left\{ \frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\lambda_j/\tau]-1} \right. \right. \\ &\quad \left. \left. - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\lambda_j/\tau]-1} \right\} B^{N-r} R^r \right. \\ &\quad \left. + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\lambda_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\lambda_j/\tau]-1} \right\} \right. \end{aligned}$$

$$\begin{aligned}
& \times \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] AB^{N-r} R^r + \left(\lambda_j - \left[\frac{\lambda_j}{\tau} \right] \tau + \frac{\tau}{2} \right) \\
& \times \left\{ \frac{1}{2} \left\{ D^{[\lambda_j/\tau]} + \tilde{D}^{[\lambda_j/\tau]} \right\} (-A) B^{N-r} R^r \right. \\
& \left. + \frac{1}{2i} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] B^{N-r} A^{1/2} R^r \right\}
\end{aligned}$$

operatörünün tersi vardır ve

$$\begin{aligned}
T_\tau &= \left(I - \sum_{j=1}^N \alpha_j \left\{ \frac{1}{2} \left\{ D^{[\mu_j/\tau]} + \tilde{D}^{[\mu_j/\tau]} \right\} B^{N-r} R^r \right. \right. \\
&+ \frac{1}{2i} \left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] B^{N-r} A^{1/2} R^r \\
&\quad \left. \left. + \left(\mu_j - \left[\frac{\mu_j}{\tau} \right] \tau \right) \right. \right. \\
&\quad \times \left[\frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\mu_j/\tau]-1} \right. \right. \\
&\quad \left. \left. - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\mu_j/\tau]-1} \right\} B^{N-r} R^r \right. \\
&+ \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\mu_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\mu_j/\tau]-1} \right\} \\
&\quad \times \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] \left. \right] AB^{N-r} R^r \left. \right\} \\
&+ \sum_{j=1}^L \beta_j \left\{ \frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\lambda_j/\tau]-1} \right. \right. \\
&\quad \left. \left. - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\lambda_j/\tau]-1} \right\} B^{N-r} R^r \right. \\
&+ \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\lambda_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\lambda_j/\tau]-1} \right\} \\
&\quad \times \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] AB^{N-r} R^r + \left(\lambda_j - \left[\frac{\lambda_j}{\tau} \right] \tau + \frac{\tau}{2} \right) \\
&\quad \times \left\{ \frac{1}{2} \left\{ D^{[\lambda_j/\tau]} + \tilde{D}^{[\lambda_j/\tau]} \right\} (-A) B^{N-r} R^r \right. \\
&\quad \left. \left. + \frac{1}{2i} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] B^{N-r} A^{1/2} R^r \right\} \right\} \Big)^{-1}
\end{aligned}$$

aşağıdaki

$$\|T_\tau\|_{H \mapsto H} \leq M \quad (4.75)$$

kestirimi sağlar. Burada, M katsayısı τ 'dan bağımsızdır.

İspat. İlk olarak,

$$\begin{aligned}
& T_\tau - \left(I - \sum_{j=1}^N \alpha_j ([c([\mu_j/\tau]\tau) - As([\mu_j/\tau]\tau)] \right. \\
& \quad \left. - \sum_{j=1}^L \beta_j [s([\lambda_j/\tau]\tau) + c([\lambda_j/\tau]\tau)] A \right) e^{-A} \Big)^{-1} \\
& = T_\tau \left(I - \sum_{j=1}^N \alpha_j ([c([\mu_j/\tau]\tau) - As([\mu_j/\tau]\tau)] \right. \\
& \quad \left. - \sum_{j=1}^L \beta_j [s([\lambda_j/\tau]\tau) + c([\lambda_j/\tau]\tau)] A \right) e^{-A} \Big)^{-1} \\
& \quad \times \left[\sum_{j=1}^N \alpha_j \left\{ \frac{1}{2} \left\{ D[\mu_j/\tau] + \tilde{D}[\mu_j/\tau] \right\} B^{N-r} R^r \right. \right. \\
& \quad \left. \left. + \frac{1}{2i} \left\{ \tilde{D}[\mu_j/\tau] - D[\mu_j/\tau] \right\} \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] B^{N-r} A^{1/2} R^r \right. \right. \\
& \quad \left. \left. + \left(\mu_j - \left\lfloor \frac{\mu_j}{\tau} \right\rfloor \tau \right) \right. \right. \\
& \quad \left. \left. \times \left[\frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}[\mu_j/\tau]^{-1} \right. \right. \right. \right. \\
& \quad \left. \left. \left. - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D[\mu_j/\tau]^{-1} \right\} B^{N-r} R^r \right. \right. \\
& \quad \left. \left. + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D[\mu_j/\tau]^{-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}[\mu_j/\tau]^{-1} \right\} \right. \right. \\
& \quad \left. \left. \times \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] \right] AB^{N-r} R^r \right\} \\
& \quad + \sum_{j=1}^L \beta_j \left\{ \frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} D[\lambda_j/\tau]^{-1} \right. \right. \\
& \quad \left. \left. - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}[\lambda_j/\tau]^{-1} \right\} B^{N-r} R^r \right. \\
& \quad \left. + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D[\lambda_j/\tau]^{-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}[\lambda_j/\tau]^{-1} \right\} \right. \\
& \quad \left. \times \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] AB^{N-r} R^r + \left(\lambda_j - \left\lfloor \frac{\lambda_j}{\tau} \right\rfloor \tau + \frac{\tau}{2} \right) \right. \\
& \quad \left. \times \left\{ \frac{1}{2} \left\{ D[\lambda_j/\tau] + \tilde{D}[\lambda_j/\tau] \right\} (-A) B^{N-r} R^r \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2i} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] B^{N-r} A^{1/2} R^r \Bigg\} \\
& - \sum_{j=1}^N \alpha_j \left([c([\mu_j/\tau]\tau) - As([\mu_j/\tau]\tau)] \right. \\
& \left. - \sum_{j=1}^L \beta_j [s([\lambda_j/\tau]\tau) + c([\lambda_j/\tau]\tau)] A \right) e^{-A} \Bigg]
\end{aligned}$$

formülü doğrudur. Buradan,

$$\begin{aligned}
& \left\| \left(\sum_{j=1}^N \alpha_j \left([c([\mu_j/\tau]\tau) - As([\mu_j/\tau]\tau)] \right. \right. \right. \\
& \left. \left. - \sum_{j=1}^L \beta_j [s([\lambda_j/\tau]\tau) + c([\lambda_j/\tau]\tau)] A \right) \right. \\
& \left. - \sum_{j=1}^N \alpha_j \left\{ \frac{1}{2} \left\{ D^{[\mu_j/\tau]} + \tilde{D}^{[\mu_j/\tau]} \right\} B^{N-r} R^r \right. \right. \\
& \left. \left. + \frac{1}{2i} \left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] B^{N-r} A^{1/2} R^r \right. \right. \\
& \left. \left. + \left(\mu_j - \left[\frac{\mu_j}{\tau} \right] \tau \right) \right. \right. \\
& \left. \left. \times \left[\frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\mu_j/\tau]-1} \right. \right. \right. \\
& \left. \left. \left. - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\mu_j/\tau]-1} \right\} B^{N-r} R^r \right. \right. \\
& \left. \left. + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\mu_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\mu_j/\tau]-1} \right\} \right. \right. \\
& \left. \left. \times \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] \right] AB^{N-r} R^r \right\} \\
& + \sum_{j=1}^L \beta_j \left\{ \frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\lambda_j/\tau]-1} \right. \right. \\
& \left. \left. - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\lambda_j/\tau]-1} \right\} B^{N-r} R^r \right. \\
& \left. + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\lambda_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\lambda_j/\tau]-1} \right\} \right. \\
& \left. \times \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] AB^{N-r} R^r + \left(\lambda_j - \left[\frac{\lambda_j}{\tau} \right] \tau + \frac{\tau}{2} \right) \right. \\
& \left. \times \left\{ \frac{1}{2} \left\{ D^{[\lambda_j/\tau]} + \tilde{D}^{[\lambda_j/\tau]} \right\} (-A) B^{N-r} R^r + \frac{1}{2i} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \left\| \left\{ \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] B^{N-r} A^{1/2} R^r \right\} \right\| e^{-A} \Big\|_H \leq M\tau, \quad (4.76) \\
& \left\| \sum_{j=1}^N \alpha_j \left\{ \frac{1}{2} \left\{ D^{[\mu_j/\tau]} + \tilde{D}^{[\mu_j/\tau]} \right\} B^{N-r} R^r \right. \right. \\
& + \frac{1}{2i} \left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] B^{N-r} A^{1/2} R^r \\
& \quad + \left(\mu_j - \left[\frac{\mu_j}{\tau} \right] \tau \right) \\
& \quad \times \left[\frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\mu_j/\tau]-1} \right. \right. \\
& \quad \left. \left. - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\mu_j/\tau]-1} \right\} B^{N-r} R^r \right. \\
& + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\mu_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\mu_j/\tau]-1} \right\} \\
& \quad \times \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] \Big\| AB^{N-r} R^r \Big\| \\
& + \sum_{j=1}^L \beta_j \left\{ \frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\lambda_j/\tau]-1} \right. \right. \\
& \quad \left. \left. - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\lambda_j/\tau]-1} \right\} B^{N-r} R^r \right. \\
& + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\lambda_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\lambda_j/\tau]-1} \right\} \\
& \quad \times \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] AB^{N-r} R^r + \left(\lambda_j - \left[\frac{\lambda_j}{\tau} \right] \tau + \frac{\tau}{2} \right) \\
& \times \left\{ \frac{1}{2} \left\{ D^{[\lambda_j/\tau]} + \tilde{D}^{[\lambda_j/\tau]} \right\} (-A) B^{N-r} R^r + \frac{1}{2i} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} \right. \\
& \quad \left. \left[\frac{\tau}{2} - \left(I + \frac{\tau^2 A}{4} \right)^{-1} \right] B^{N-r} A^{1/2} R^r \right\} \Big\| (e^{-A} - B^{N-r} R^r) \Big\|_H \leq M\tau \quad (4.77)
\end{aligned}$$

kestirimleri kullanılarak, (4.75) kestirimi elde edilir. Buradaki (4.76) ve (4.77) kestirimlerinin ispatları, (4.72), (4.73) ve (4.74) kestirimlerine dayanmaktadır. O halde, Yardımcı Teorem 4.6'nın ispatı tamamlanmış olur.

Teorem 4.7. Eğer $\varphi \in D(A)$, $g_0 \in D(A^{1/2})$ ve $f_0 \in D(A^{1/2})$ ise, bu durumda (4.71) fark şemasının çözümü için aşağıdaki

$$\max_{-N \leq k \leq N} \|u_k\|_H \leq M [\|\varphi\|_H + \|A^{-1/2} f_0\|_H + \|A^{-1/2} g_0\|_H] \quad (4.78)$$

$$+ \max_{1 \leq k < N} \|A^{-1/2} (f_k - f_{k-1}) \tau^{-1}\|_H + \max_{-N < k \leq 0} \|A^{-1/2} (g_k - g_{k-1}) \tau^{-1}\|_H \Big],$$

$$\max_{-N \leq k \leq N} \|A^{1/2} u_k\|_H \leq M [\|A^{1/2} \varphi\|_H + \|f_0\|_H + \|g_0\|_H \quad (4.79)$$

$$+ \max_{1 \leq k \leq N-1} \|(f_k - f_{k-1}) \tau^{-1}\|_H + \max_{-(N-1) \leq k \leq 0} \|(g_k - g_{k-1}) \tau^{-1}\|_H \Big],$$

$$\max_{1 \leq k \leq N-1} \|\tau^{-2} (u_{k+1} - 2u_k + u_{k-1})\|_H + \max_{-N+1 \leq k \leq 0} \|Au_k\|_H \quad (4.80)$$

$$+ \max_{1 \leq k \leq N} \left\| A \left(\frac{u_k + u_{k-1}}{2} \right) \right\|_H$$

$$+ \max_{-(N-1) \leq k \leq 0} \|\tau^{-1} (u_k - u_{k-1})\|_H \leq M [\|A\varphi\|_H + \|A^{1/2} f_0\|_H$$

$$+ \|(f_1 - f_0) \tau^{-1}\|_H + \max_{1 \leq k \leq N-2} \|(f_{k+1} - 2f_k + f_{k-1}) \tau^{-2}\|_H + \|A^{1/2} g_0\|_H$$

$$+ \|(g_0 - g_{-1}) \tau^{-1}\|_H + \max_{-(N-1) \leq k \leq -1} \|(g_{k+1} - 2g_k + g_{k-1}) \tau^{-2}\|_H \Big]$$

kararlılık kestirimleri sağlanır. Burada, M katsayısı τ , $f_k, 0 \leq k < N$, $g_k, -N < k \leq 0$ ve φ 'den bağımsızdır.

İspat. Öncelikle, (4.71) fark şemasının çözümü için gerekli formüller elde edilecektir. Bilindiği gibi,

$$\begin{cases} \tau^{-2} (u_{k+1} - 2u_k + u_{k-1}) + \frac{1}{2} Au_k + \frac{1}{4} A (u_{k+1} + u_{k-1}) = f_k, f_k = f(t_k), \\ t_k = k\tau, 1 \leq k < N, N\tau = 1, \\ \left(I + \frac{\tau^2 A}{4} \right) \tau^{-1} (u_1 - u_0) - \frac{\tau}{2} (g_0 - Au_0) = \omega, u_0 = \mu, \end{cases} \quad (4.81)$$

$$\begin{cases} \tau^{-1} (u_k - u_{k-1}) + Au_k = g_k, & -N+1 \leq k \leq -N+r, \\ \tau^{-1} (u_k - u_{k-1}) + \frac{A}{2} (u_k + u_{k-1}) = g_k, & -N+r+1 \leq k \leq 0, \\ g_k = g\left(t_k - \frac{\tau}{2}\right), t_k = k\tau, & -(N-1) \leq k \leq 0, u_{-N} \text{ biliniyor} \end{cases} \quad (4.82)$$

başlangıç değer fark problemlerinin tek çözümü vardır ve aşağıdaki

$$u_1 = \left(I + \frac{\tau^2 A}{4} \right)^{-1} \left[\left(I - \frac{\tau^2 A}{4} \right) \mu - \tau \omega + \frac{\tau^2}{2} g_0 \right], \quad (4.83)$$

$$u_k = \frac{1}{2} (D^k + \tilde{D}^k) \mu + \frac{1}{2i} A^{-1/2} (\tilde{D}^k - D^k) \left(\omega + \frac{\tau}{2} f_0 + \frac{\tau}{2} A \mu \right) \quad (4.84)$$

$$- \sum_{s=1}^{k-1} \frac{\tau}{2i} A^{-1/2} [D^{k-s} - \tilde{D}^{k-s}] f_s, \quad 2 \leq k \leq N,$$

$$u_k = R^{N+k}u_{-N} + \tau \sum_{s=-N+1}^k R^{k-s+1}g_s, \quad -N+1 \leq k \leq -N+r, \quad (4.85)$$

$$u_k = B^{N-r+k}u_{-N+r} + \tau \sum_{s=-N+r+1}^k B^{k-s}Cg_s \quad (4.86)$$

formüller sağlanır. Buradaki (4.85) denkleminde k yerine $-N+r$ yazılarak,

$$u_{-N+r} = R^r u_{-N} + \tau \sum_{s=-N+1}^{-N+r} R^{-N+r-s+1}g_s \quad (4.87)$$

elde edilir. Böylece, (4.86) formülü kullanılarak,

$$\begin{aligned} u_k = & B^{N-r+k}R^r u_{-N} + \tau \sum_{s=-N+1}^{-N+r} B^{N-r+k}R^{-N+r-s+1}g_s \\ & + \tau \sum_{s=-N+r+1}^k B^{k-s}Cg_s, \quad -N+r+1 \leq k \leq 0 \end{aligned} \quad (4.88)$$

formülü elde edilir. Buradan, (4.83), (4.84), (4.88) denklemleri ve

$$\left(I + \frac{\tau^2 A}{4}\right) \tau^{-1}(u_1 - u_0) - \frac{\tau}{2}(g_0 - Au_0) = \left(I + \frac{\tau^2 A}{4}\right)^{-1} (f_0 - Au_0), \quad u_0 = \mu$$

formülleri kullanılarak,

$$\begin{aligned} \mu = u_0 = & B^{N-r}R^r u_{-N} + \tau \sum_{s=-N+1}^{-N+r} B^{N-r}R^{-N+r-s+1}g_s + \tau \sum_{s=-N+r+1}^0 B^{-s}Cg_s, \\ \omega = & \left(I + \frac{\tau^2 A}{4}\right)^{-1} \left(f_0 - A \left\{ B^{N-r}R^r u_{-N} \right. \right. \\ & \left. \left. + \tau \sum_{s=-N+1}^{-N+r} B^{N-r}R^{-N+r-s+1}g_s + \tau \sum_{s=-N+r+1}^0 B^{-s}Cg_s \right\} \right) \end{aligned}$$

formülleri elde edilir. Dolayısıyla,

$$\begin{aligned} u_1 = & \left(I + \frac{\tau^2 A}{4}\right)^{-1} \left[\left(I - \frac{\tau^2 A}{4}\right) \left\{ B^{N-r}R^r u_{-N} \right. \right. \\ & + \tau \sum_{s=-N+1}^{-N+r} B^{N-r}R^{-N+r-s+1}g_s + \tau \sum_{s=-N+r+1}^0 B^{-s}Cg_s \left. \right\} - \tau \left(I + \frac{\tau^2 A}{4}\right)^{-1} \\ & \times \left(f_0 - A \left\{ B^{N-r}R^r u_{-N} + \tau \sum_{s=-N+1}^{-N+r} B^{N-r}R^{-N+r-s+1}g_s \right. \right. \\ & \left. \left. + \tau \sum_{s=-N+r+1}^0 B^{-s}Cg_s \right\} \right) + \frac{\tau^2}{2}g_0 \left. \right], \end{aligned} \quad (4.89)$$

$$\begin{aligned}
u_k = & \frac{1}{2} \left(D^k + \tilde{D}^k \right) \left\{ B^{N-r} R^r u_{-N} + \tau \sum_{s=-N+1}^{-N+r} B^{N-r} R^{-N+r-s+1} g_s \right. \\
& \left. + \tau \sum_{s=-N+r+1}^0 B^{-s} C g_s \right\} + \frac{1}{2i} A^{-1/2} \left(\tilde{D}^k - D^k \right) \left(\left(I + \frac{\tau^2 A}{4} \right)^{-1} \right. \\
& \left. \times f_0 - A \left\{ B^{N-r} R^r u_{-N} \right. \right. \\
& \left. \left. + \tau \sum_{s=-N+1}^{-N+r} B^{N-r} R^{-N+r-s+1} g_s + \tau \sum_{s=-N+r+1}^0 B^{-s} C g_s \right\} \right) + \frac{\tau}{2} f_0 \\
& + \frac{\tau}{2} A \left\{ B^{N-r} R^r u_{-N} + \tau \sum_{s=-N+1}^{-N+r} B^{N-r} R^{-N+r-s+1} g_s + \tau \sum_{s=-N+r+1}^0 B^{-s} C g_s \right\} \\
& - \sum_{s=1}^{k-1} \frac{\tau}{2i} A^{-1/2} \left[D^{k-s} - \tilde{D}^{k-s} \right] f_s, \quad 2 \leq k \leq N
\end{aligned} \tag{4.90}$$

dir. Buradan,

$$\begin{aligned}
u_{-N} = & \sum_{j=1}^N \alpha_j \left(u_{[\mu_j/\tau]} + \left(\mu_j - \left[\frac{\mu_j}{\tau} \right] \tau \right) \frac{u_{[\mu_j/\tau]} - u_{[\mu_j/\tau]-1}}{\tau} \right) \\
& + \sum_{j=1}^L \beta_j \left(\frac{u_{[\lambda_j/\tau]} - u_{[\lambda_j/\tau]-1}}{\tau} + \left(\lambda_j - \left[\frac{\lambda_j}{\tau} \right] \tau + \frac{\tau}{2} \right) (f_{[\lambda_j/\tau]} - A u_{[\lambda_j/\tau]}) \right) + \varphi
\end{aligned}$$

koşulu kullanılarak,

$$\begin{aligned}
u_{-N} = & T_\tau \left[\sum_{j=1}^N \alpha_j \left\{ \frac{\tau}{2} \left[D^{[\mu_j/\tau]} + \tilde{D}^{[\mu_j/\tau]} \right] \right. \right. \\
& \times \left\{ \sum_{s=-N+1}^{-N+r} B^{N-r} R^{-N+r-s+1} g_s + \sum_{s=-N+r+1}^0 B^{-s} C g_s \right\} \\
& \left. + \frac{A^{-1/2}}{2i} \left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \right. \right. \\
& \times \left(f_0 - A \tau \left\{ \sum_{s=-N+1}^{-N+r} B^{N-r} R^{-N+r-s+1} g_s + \sum_{s=-N+r+1}^0 B^{-s} C g_s \right\} \right) \\
& \left. \left. + \frac{\tau}{2} f_0 + \frac{\tau^2 A}{2} \left\{ \sum_{s=-N+1}^{-N+r} B^{N-r} R^{-N+r-s+1} g_s + \sum_{s=-N+r+1}^0 B^{-s} C g_s \right\} \right] \right. \\
& \left. - \sum_{s=1}^{[\mu_j/\tau]-1} \frac{\tau}{2i} A^{-1/2} \left[D^{[\mu_j/\tau]-s} - \tilde{D}^{[\mu_j/\tau]-s} \right] f_s + \left(\mu_j - \left[\frac{\mu_j}{\tau} \right] \tau \right) \right. \\
& \times \left[\frac{i A^{1/2}}{2} \left\{ \left(I - \frac{i \tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\mu_j/\tau]-1} - \left(I + \frac{i \tau A^{1/2}}{2} \right)^{-1} D^{[\mu_j/\tau]-1} \right\} \right.
\end{aligned} \tag{4.91}$$

$$\begin{aligned}
& \times \tau \left\{ \sum_{s=-N+1}^{-N+r} B^{N-r} R^{-N+r-s+1} g_s + \sum_{s=-N+r+1}^0 B^{-s} C g_s \right\} \\
& + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\mu_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\mu_j/\tau]-1} \right\} \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \right. \\
& \times \left(f_0 - A\tau \left\{ \sum_{s=-N+1}^{-N+r} B^{N-r} R^{-N+r-s+1} g_s + \sum_{s=-N+r+1}^0 B^{-s} C g_s \right\} \right) \\
& \left. + \frac{\tau^2 A}{2} \left\{ \sum_{s=-N+1}^{-N+r} B^{N-r} R^{-N+r-s+1} g_s + \sum_{s=-N+r+1}^0 B^{-s} C g_s \right\} \right] \\
& + \tau \left(I + \frac{\tau^2 A}{4} \right)^{-1} f_{[\mu_j/\tau]-1} \\
& + \sum_{s=1}^{[\mu_j/\tau]-2} \tau \left[\left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\mu_j/\tau]-1-s} \right. \\
& \left. + \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\mu_j/\tau]-1-s} \right] f_s \Bigg\} \\
& + \sum_{j=1}^L \beta_j \left\{ \frac{iA^{1/2}}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\lambda_j/\tau]-1} \right. \right. \\
& \left. \left. - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\lambda_j/\tau]-1} \right\} \right. \\
& \times \tau \left\{ \sum_{s=-N+1}^{-N+r} B^{N-r} R^{-N+r-s+1} g_s + \sum_{s=-N+r+1}^0 B^{-s} C g_s \right\} \\
& + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\lambda_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\lambda_j/\tau]-1} \right\} \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \right. \\
& \times \left(f_0 - A\tau \left\{ \sum_{s=-N+1}^{-N+r} B^{N-r} R^{-N+r-s+1} g_s + \sum_{s=-N+r+1}^0 B^{-s} C g_s \right\} \right) \\
& \left. + \frac{\tau^2 A}{2} \left\{ \sum_{s=-N+1}^{-N+r} B^{N-r} R^{-N+r-s+1} g_s + \sum_{s=-N+r+1}^0 B^{-s} C g_s \right\} \right] \\
& + \tau \left(I + \frac{\tau^2 A}{4} \right)^{-1} f_{[\lambda_j/\tau]-1} \\
& + \sum_{s=1}^{[\lambda_j/\tau]-2} \tau \left[\left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\lambda_j/\tau]-1-s} \right. \\
& \left. + \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\lambda_j/\tau]-1-s} \right] f_s \\
& + \left(\lambda_j - \left\lfloor \frac{\lambda_j}{\tau} \right\rfloor \tau + \frac{\tau}{2} \right) \left(f_{[\lambda_j/\tau]} - A \left\{ \frac{\tau}{2} \left[D^{[\lambda_j/\tau]} + \tilde{D}^{[\lambda_j/\tau]} \right] \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ \sum_{s=-N+1}^{-N+r} B^{N-r} R^{-N+r-s+1} g_s + \sum_{s=-N+r+1}^0 B^{-s} C g_s \right\} \\
& + \frac{A^{-1/2}}{2i} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \right. \\
& \times \left(f_0 - A\tau \left\{ \sum_{s=-N+1}^{-N+r} B^{N-r} R^{-N+r-s+1} g_s + \sum_{s=-N+r+1}^0 B^{-s} C g_s \right\} \right) \\
& + \frac{\tau}{2} f_0 + \frac{\tau^2 A}{2} \left\{ \sum_{s=-N+1}^{-N+r} B^{N-r} R^{-N+r-s+1} g_s + \sum_{s=-N+r+1}^0 B^{-s} C g_s \right\} \\
& \left. - \sum_{s=1}^{[\lambda_j/\tau]-1} \frac{\tau}{2i} A^{-1/2} \left[D^{[\lambda_j/\tau]-s} + \tilde{D}^{[\lambda_j/\tau]-s} \right] f_s \right\} \Bigg] + \varphi
\end{aligned}$$

operatör denklemi elde edilir. Böylece, (4.71) fark şemasının çözümü için, (4.88), (4.89), (4.90) ve (4.91) formülleri kurulmuş olur.

Şimdi, (4.78), (4.79) ve (4.80) kestirimlerinin ispatları verilecektir. İlk olarak, $\|u_{-N}\|_H$ için kestirim elde edilecektir. Bunun için, Abel formülü ve (4.91) formülü kullanılarak,

$$\begin{aligned}
u_{-N} &= T_\tau \left[\sum_{j=1}^N \alpha_j \left\{ \frac{1}{2} A^{-1} \left[D^{[\mu_j/\tau]} + \tilde{D}^{[\mu_j/\tau]} \right] \right. \right. \\
& \times \left\{ B^{N-r} \left[g_{-N+r} - R^r g_{-N+1} + \sum_{s=-N+2}^{-N+r} R^{-N+r-s+1} \{g_{s-1} - g_s\} \right] \right. \\
& + \left. \left[g_0 - B^{N-r} g_{-N+r+1} + \sum_{s=-N+r+2}^0 B^{-s+1} \{g_{s-1} - g_s\} \right] \right\} \\
& + \frac{1}{2i} A^{-1/2} \left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \right. \\
& \times \left(f_0 - \left\{ B^{N-r} \left[g_{-N+r} - R^r g_{-N+1} + \sum_{s=-N+2}^{-N+r} R^{-N+r-s+1} \{g_{s-1} - g_s\} \right] \right. \right. \\
& + \left. \left[g_0 - B^{N-r} g_{-N+r+1} + \sum_{s=-N+r+2}^0 B^{-s+1} \{g_{s-1} - g_s\} \right] \right\} \Bigg] \\
& + \frac{\tau}{2} f_0 + \frac{\tau}{2} \left\{ B^{N-r} \left[g_{-N+r} - R^r g_{-N+1} + \sum_{s=-N+2}^{-N+r} R^{-N+r-s+1} \{g_{s-1} - g_s\} \right] \right. \\
& + \left. \left[g_0 - B^{N-r} g_{-N+r+1} + \sum_{s=-N+r+2}^0 B^{-s+1} \{g_{s-1} - g_s\} \right] \right\} \Bigg] \\
& + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right)
\end{aligned} \tag{4.92}$$

$$\begin{aligned}
& \times \left[f_{[\mu_j/\tau]-1} - D^{[\mu_j/\tau]-1} f_1 + \sum_{s=2}^{[\mu_j/\tau]-1} D^{[\mu_j/\tau]-s} \{f_{s-1} - f_s\} \right] \\
& \quad + \frac{1}{2} A^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right) \\
& \times \left[f_{[\mu_j/\tau]-1} - \tilde{D}^{[\mu_j/\tau]-1} f_1 + \sum_{s=2}^{[\mu_j/\tau]-1} \tilde{D}^{[\mu_j/\tau]-s} \{f_{s-1} - f_s\} \right] \\
& \quad + \left(\mu_j - \left[\frac{\mu_j}{\tau} \right] \tau \right) \\
& \times \left[\frac{i}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\mu_j/\tau]-1} - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\mu_j/\tau]-1} \right\} \right. \\
& \times A^{-1/2} \left\{ B^{N-r} \left[g_{-N+r} - R^r g_{-N+1} + \sum_{s=-N+2}^{-N+r} R^{-N+r-s+1} \{g_{s-1} - g_s\} \right] \right. \\
& \quad \left. \left. + \left[g_0 - B^{N-r} g_{-N+r+1} + \sum_{s=-N+r+2}^0 B^{-s+1} \{g_{s-1} - g_s\} \right] \right\} \right. \\
& + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\mu_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\mu_j/\tau]-1} \right\} \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \right. \\
& \times \left(f_0 - \left\{ B^{N-r} \left[g_{-N+r} - R^r g_{-N+1} + \sum_{s=-N+2}^{-N+r} R^{-N+r-s+1} \{g_{s-1} - g_s\} \right] \right. \right. \\
& \quad \left. \left. + \left[g_0 - B^{N-r} g_{-N+r+1} + \sum_{s=-N+r+2}^0 B^{-s+1} \{g_{s-1} - g_s\} \right] \right\} \right) \right] \\
& + \frac{\tau A}{2} \left\{ B^{N-r} \left[g_{-N+r} - R^r g_{-N+1} + \sum_{s=-N+2}^{-N+r} R^{-N+r-s+1} \{g_{s-1} - g_s\} \right] \right. \\
& \quad \left. + \left[g_0 - B^{N-r} g_{-N+r+1} + \sum_{s=-N+r+2}^0 B^{-s+1} \{g_{s-1} - g_s\} \right] \right\} \\
& \quad + \tau \left(I + \frac{\tau^2 A}{4} \right)^{-1} f_{[\mu_j/\tau]-1} \\
& + \frac{A^{-1/2}}{i} \left[\left(D - \tilde{D} \right) f_{[\mu_j/\tau]-1} + \left(\tilde{D}^{[\mu_j/\tau]-1} - D^{[\mu_j/\tau]-1} \right) f_1 \right. \\
& \quad \left. + \sum_{s=2}^{[\mu_j/\tau]-2} \left[D^{[\mu_j/\tau]-s} - \tilde{D}^{[\mu_j/\tau]-s} \right] \{f_{s-1} - f_s\} \right] \\
& + \sum_{j=1}^L \beta_j \left\{ \frac{i}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\lambda_j/\tau]-1} - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\lambda_j/\tau]-1} \right\} \right.
\end{aligned}$$

$$\begin{aligned}
& \times A^{-1/2} \left\{ B^{N-r} \left[g_{-N+r} - R^r g_{-N+1} + \sum_{s=-N+2}^{-N+r} R^{-N+r-s+1} \{g_{s-1} - g_s\} \right] \right. \\
& \quad \left. + \left[g_0 - B^{N-r} g_{-N+r+1} + \sum_{s=-N+r+2}^0 B^{-s+1} \{g_{s-1} - g_s\} \right] \right\} \\
& + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\lambda_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\lambda_j/\tau]-1} \right\} \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \right. \\
& \quad \times \left(f_0 - \left\{ B^{N-r} \left[g_{-N+r} - R^r g_{-N+1} + \sum_{s=-N+2}^{-N+r} R^{-N+r-s+1} \{g_{s-1} - g_s\} \right] \right. \right. \\
& \quad \left. \left. + \left[g_0 - B^{N-r} g_{-N+r+1} + \sum_{s=-N+r+2}^0 B^{-s+1} \{g_{s-1} - g_s\} \right] \right\} \right) \\
& \quad + \frac{\tau}{2} \left\{ B^{N-r} \left[g_{-N+r} - R^r g_{-N+1} + \sum_{s=-N+2}^{-N+r} R^{-N+r-s+1} \{g_{s-1} - g_s\} \right] \right. \\
& \quad \left. + \left[g_0 - B^{N-r} g_{-N+r+1} + \sum_{s=-N+r+2}^0 B^{-s+1} \{g_{s-1} - g_s\} \right] \right\} \\
& \quad \left. + \tau \left(I + \frac{\tau^2 A}{4} \right)^{-1} f_{[\lambda_j/\tau]-1} \right. \\
& \quad + \frac{A^{-1/2}}{i} \left[\left(D - \tilde{D} \right) f_{[\lambda_j/\tau]-1} + \left(\tilde{D}^{[\lambda_j/\tau]-1} - D^{[\lambda_j/\tau]-1} \right) f_1 \right. \\
& \quad \left. + \sum_{s=2}^{[\lambda_j/\tau]-2} \left[D^{[\lambda_j/\tau]-s} - \tilde{D}^{[\lambda_j/\tau]-s} \right] \{f_{s-1} - f_s\} \right] \\
& \quad + \left(\lambda_j - \left\lfloor \frac{\lambda_j}{\tau} \right\rfloor \tau + \frac{\tau}{2} \right) \left(f_{[\lambda_j/\tau]} - \left\{ \frac{1}{2} \left[D^{[\lambda_j/\tau]} + \tilde{D}^{[\lambda_j/\tau]} \right] \right. \right. \\
& \quad \times \left\{ B^{N-r} \left[g_{-N+r} - R^r g_{-N+1} + \sum_{s=-N+2}^{-N+r} R^{-N+r-s+1} \{g_{s-1} - g_s\} \right] \right. \\
& \quad \left. \left. + \left[g_0 - B^{N-r} g_{-N+r+1} + \sum_{s=-N+r+2}^0 B^{-s+1} \{g_{s-1} - g_s\} \right] \right\} \right. \\
& \quad \left. + \frac{A^{-1/2}}{2i} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \right. \right. \\
& \quad \times \left(f_0 - \left\{ B^{N-r} \left[g_{-N+r} - R^r g_{-N+1} + \sum_{s=-N+2}^{-N+r} R^{-N+r-s+1} \{g_{s-1} - g_s\} \right] \right. \right. \\
& \quad \left. \left. + \left[g_0 - B^{N-r} g_{-N+r+1} + \sum_{s=-N+r+2}^0 B^{-s+1} \{g_{s-1} - g_s\} \right] \right\} \right) \\
& \quad \left. + \frac{\tau}{2} f_0 + \frac{\tau}{2} \left\{ B^{N-r} \left[g_{-N+r} - R^r g_{-N+1} + \sum_{s=-N+2}^{-N+r} R^{-N+r-s+1} \{g_{s-1} - g_s\} \right] \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \left[g_0 - B^{N-r} g_{-N+r+1} + \sum_{s=-N+r+2}^0 B^{-s+1} \{g_{s-1} - g_s\} \right] \Bigg] \\
& + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right) \\
& \times \left[f_{[\lambda_j/\tau]-1} - D^{[\lambda_j/\tau]-1} f_1 + \sum_{s=2}^{[\lambda_j/\tau]-1} D^{[\lambda_j/\tau]-s} \{f_{s-1} - f_s\} \right] \\
& + \frac{1}{2} A^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right) \left[f_{[\lambda_j/\tau]-1} - \tilde{D}^{[\lambda_j/\tau]-1} f_1 + \right. \\
& \left. \sum_{s=2}^{[\lambda_j/\tau]-1} \tilde{D}^{[\lambda_j/\tau]-s} \{f_{s-1} - f_s\} \right] \Bigg] \Bigg) \Bigg] + \varphi
\end{aligned}$$

formülü elde edilir. Buradan, (4.88), (4.90), (4.92) formülleri ve (4.72), (4.73), (4.74) kesitirimleri yardımıyla,

$$\begin{aligned}
\|u_{-N}\|_H & \leq M \left[\|\varphi\|_H + \|A^{-1/2} f_0\|_H + \max_{1 \leq k \leq N} \|A^{-1/2} (f_k - f_{k-1}) \tau^{-1}\|_H \right. \\
& \left. + \|A^{-1/2} g_0\|_H + \max_{-N \leq k \leq 0} \|A^{-1/2} (g_k - g_{k-1}) \tau^{-1}\|_H \right] \quad (4.93)
\end{aligned}$$

kestirimi alınır.

İkinci olarak, $\|u_k\|_H$ için kestirim elde edilecektir. Öncelikle $-N < k < -N + r$ aralığında inceleyelim. Bu durumda, Abel formülü ve (4.85) formülü kullanılarak,

$$u_k = R^{N+k} u_{-N} + A^{-1} \left[g_k - R^{N+k} g_{-N+1} + \sum_{s=-N+2}^k R^{k-s+1} \{g_{s-1} + g_s\} \right] \quad (4.94)$$

formülü elde edilir. Sonra da $-N + r + 1 \leq k \leq 0$ aralığında inceleyelim. Bu durumda, Abel formülü ve (4.88) formülü kullanılarak,

$$\begin{aligned}
u_k & = B^{N-r+k} R^r u_{-N} \\
& + A^{-1} B^{N-r+k} \left[g_{-N+r} - R^r g_{-N+1} + \sum_{s=-N+2}^{-N+r} R^{-N+r-s+1} \{g_{s-1} - g_s\} \right] \\
& + A^{-1} \left[g_k - B^{N-r+k} g_{-N+r+1} + \sum_{s=-N+r+2}^k B^{k-s+1} \{g_{s-1} - g_s\} \right] \quad (4.95)
\end{aligned}$$

formülü elde edilir. Buradan, (4.94), (4.95) formüllerinin ve (4.73), (4.74) kesitirimlerinin yardımıyla,

$$\|u_k\|_H \leq \|u_{-N}\|_H + 2 \|A^{-1/2} g_0\|_H + \max_{-N < k \leq 0} \|A^{-1/2} (g_k - g_{k-1}) \tau^{-1}\|_H \quad (4.96)$$

kesitirimi alınır. Şimdi ise $1 \leq k \leq N$ aralığında inceleyelim. Bu durumda, Abel formülü ve (4.89), (4.90) formülleri kullanılarak,

$$\begin{aligned}
u_1 = & \left(I + \frac{\tau^2 A}{4} \right)^{-1} \left[\left(I - \frac{\tau^2 A}{4} \right) A^{-1} \{ B^{N-r} R^r u_{-N} \right. \\
& + B^{N-r} \left[g_{-N+r} - R^r g_{-N+1} + \sum_{s=-N+2}^{-N+r} R^{-N+r-s+1} \{ g_{s-1} - g_s \} \right] \\
& + \left. \left[g_0 - B^{N-r} g_{-N+r+1} + \sum_{s=-N+r+2}^0 B^{-s+1} \{ g_{s-1} - g_s \} \right] \right\} \\
& - \tau \left(I + \frac{\tau^2 A}{4} \right)^{-1} \left(f_0 - \{ B^{N-r} A R^r u_{-N} \right. \\
& + B^{N-r} \left[g_{-N+r} - R^r g_{-N+1} + \sum_{s=-N+2}^{-N+r} R^{-N+r-s+1} \{ g_{s-1} - g_s \} \right] \\
& + \left. \left[g_0 - B^{N-r} g_{-N+r+1} + \sum_{s=-N+r+2}^0 B^{-s+1} \{ g_{s-1} - g_s \} \right] \right\} \Bigg) + \frac{\tau^2}{2} g_0 \Bigg], \quad (4.97)
\end{aligned}$$

$$\begin{aligned}
u_k = & \frac{1}{2} \left(D^k + \tilde{D}^k \right) \{ B^{N-r} R^r u_{-N} \\
& + A^{-1} B^{N-r} \left[g_{-N+r} - R^r g_{-N+1} + \sum_{s=-N+2}^{-N+r} R^{-N+r-s+1} \{ g_{s-1} - g_s \} \right] \\
& + A^{-1} \left[g_0 - B^{N-r} g_{-N+r+1} + \sum_{s=-N+r+2}^0 B^{-s+1} \{ g_{s-1} - g_s \} \right] \Bigg\} \\
& + \frac{1}{2i} A^{-1/2} \left(\tilde{D}^k - D^k \right) \left(\left(I + \frac{\tau^2 A}{4} \right)^{-1} \left(f_0 - \{ A B^{N-r} R^r u_{-N} \right. \right. \\
& + B^{N-r} \left[g_{-N+r} - R^r g_{-N+1} + \sum_{s=-N+2}^{-N+r} R^{-N+r-s+1} \{ g_{s-1} - g_s \} \right] \\
& + \left. \left[g_0 - B^{N-r} g_{-N+r+1} + \sum_{s=-N+r+2}^0 B^{-s+1} \{ g_{s-1} - g_s \} \right] \right\} \Bigg) + \frac{\tau}{2} f_0 \\
& + \frac{\tau}{2} \{ A B^{N-r} R^r u_{-N} \\
& + B^{N-r} \left[g_{-N+r} - R^r g_{-N+1} + \sum_{s=-N+2}^{-N+r} R^{-N+r-s+1} \{ g_{s-1} - g_s \} \right] \\
& + \left. \left[g_0 - B^{N-r} g_{-N+r+1} + \sum_{s=-N+r+2}^0 B^{-s+1} \{ g_{s-1} - g_s \} \right] \right\} \\
& + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right) \left[f_{k-1} - D^{k-1} f_1 + \sum_{s=2}^{k-1} D^{k-s} \{ f_{s-1} - f_s \} \right]
\end{aligned}$$

$$+\frac{1}{2}A^{-1}\left(I+\frac{i\tau A^{1/2}}{2}\right)\left[f_{k-1}-\tilde{D}^{k-1}f_1+\sum_{s=2}^{k-1}\tilde{D}^{k-s}\{f_{s-1}-f_s\}\right] \quad (4.98)$$

formülleri elde edilir. Buradan, (4.97), (4.98) formüllerinin ve (4.72), (4.73), (4.74) kestirimlerinin yardımıyla,

$$\|u_1\|_H \leq M \left[\|u_{-N}\|_H + \|A^{-1/2}g_0\|_H + \max_{-N < k \leq 0} \|A^{-1/2}(g_k - g_{k-1})\tau^{-1}\|_H \right], \quad (4.99)$$

$$\begin{aligned} \|u_k\|_H &\leq M \left[\|u_{-N}\|_H + \|A^{-1/2}f_0\|_H + \max_{2 \leq k < N} \|A^{-1/2}(f_k - f_{k-1})\tau^{-1}\|_H \right. \\ &\quad \left. + \|A^{-1/2}g_0\|_H + \max_{-(N-1) \leq k \leq 0} \|A^{-1/2}(g_k - g_{k-1})\tau^{-1}\|_H \right], 1 \leq k \leq N \end{aligned} \quad (4.100)$$

kestirimleri alınır. Sonuç olarak, (4.93), (4.97), (4.99), (4.100) kestirimleri birleştirilerek ve üçgen eşitsizliği kullanılarak, (4.78) kestirimi elde edilir. Burada (4.79) kestiriminin ispatı, (4.92), (4.95) formüllerine ve (4.72), (4.73), (4.74) kestirimlerine bağlı kalmak şartıyla, (4.78) kestiriminin ispatında olduğu gibi aynı yolla bulunabilir.

Üçüncü olarak, $\left\| A \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} u_{-N} \right\|_H$ için kestirim elde edilecektir. Bunun için, Abel formülü ve (4.91) formülü kullanılarak,

$$\begin{aligned} u_{-N} &= T_\tau \left[\sum_{j=1}^N \alpha_j \left\{ \frac{1}{2} A^{-1} \left[D^{[\mu_j/\tau]} + \tilde{D}^{[\mu_j/\tau]} \right] \{ B^{N-r} [g_{-N+r} - R^r g_{-N+1} \right. \right. \\ &\quad \left. \left. + (\tau A)^{-1} \{ (g_{-N+r-1} - g_{-N+r}) - R^{r-1} (g_{-N+1} - g_{-N+2}) \right. \right. \\ &\quad \left. \left. + \sum_{s=-N+3}^{-N+r} R^{-N+r-s+1} \{ g_{s-2} - 2g_{s-1} + g_s \} \} \right\} \right] \\ &\quad + \left[g_0 - B^{N-r} g_{-N+r+1} + (\tau A)^{-1} \left(1 - \frac{\tau A}{2} \right) \right. \\ &\quad \times \left\{ (g_{-1} - g_0) - B^{N-r-1} (g_{-N+r+1} - g_{-N+r+2}) \right. \\ &\quad \left. \left. + \sum_{s=-N+r+3}^0 B^{-s+1} \{ g_{s-2} - 2g_{s-1} + g_s \} \} \right] \right\} \\ &\quad + \frac{1}{2i} A^{-1/2} \left\{ \tilde{D}^{[\mu_j/\tau]} - D^{[\mu_j/\tau]} \right\} \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \right. \\ &\quad \times (f_0 - \{ B^{N-r} [g_{-N+r} - R^r g_{-N+1} + \\ &\quad (\tau A)^{-1} \{ (g_{-N+r-1} - g_{-N+r}) - R^{r-1} (g_{-N+1} - g_{-N+2}) \\ &\quad \left. \left. + \sum_{s=-N+3}^{-N+r} R^{-N+r-s+1} \{ g_{s-2} - 2g_{s-1} + g_s \} \} \right) \right] \end{aligned} \quad (4.101)$$

$$\begin{aligned}
& + \left[g_0 - B^{N-r} g_{-N+r+1} + (\tau A)^{-1} \left(1 - \frac{\tau A}{2} \right) \right. \\
& \times \left\{ (g_{-1} - g_0) - B^{N-r-1} (g_{-N+r+1} - g_{-N+r+2}) \right. \\
& + \left. \sum_{s=-N+r+3}^0 B^{-s+1} \{g_{s-2} - 2g_{s-1} + g_s\} \right\} \left. \right] \\
& + \frac{\tau}{2} f_0 + \frac{\tau}{2} \left\{ B^{N-r} [g_{-N+r} - R^r g_{-N+1} \right. \\
& + (\tau A)^{-1} \left\{ (g_{-N+r-1} - g_{-N+r}) - R^{r-1} (g_{-N+1} - g_{-N+2}) \right. \\
& + \left. \sum_{s=-N+3}^{-N+r} R^{-N+r-s+1} \{g_{s-2} - 2g_{s-1} + g_s\} \right\} \left. \right] \\
& + \left[g_0 - B^{N-r} g_{-N+r+1} + (\tau A)^{-1} \left(1 - \frac{\tau A}{2} \right) \right. \\
& \times \left\{ (g_{-1} - g_0) - B^{N-r-1} (g_{-N+r+1} - g_{-N+r+2}) \right. \\
& + \left. \sum_{s=-N+r+3}^0 B^{-s+1} \{g_{s-2} - 2g_{s-1} + g_s\} \right\} \left. \right] \\
& + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right) \\
& \times \left[f_{[\mu_j/\tau]-1} - D^{[\mu_j/\tau]-1} f_1 + (i\tau A^{1/2})^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \\
& \times \left(\left(f_{[\mu_j/\tau]-2} - f_{[\mu_j/\tau]-1} \right) - D^{[\mu_j/\tau]-2} (f_1 - f_2) \right. \\
& + \left. \sum_{s=2}^{[\mu_j/\tau]-2} D^{[\mu_j/\tau]-1-s} \{f_{s+1} - 2f_s + f_{s-1}\} \right) \left. \right] \\
& + \frac{1}{2} A^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right) \\
& \times \left[f_{[\mu_j/\tau]-1} - \tilde{D}^{[\mu_j/\tau]-1} f_1 - (i\tau A^{1/2})^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right) \right. \\
& \times \left(\left(f_{[\mu_j/\tau]-2} - f_{[\mu_j/\tau]-1} \right) - \tilde{D}^{[\mu_j/\tau]-2} (f_1 - f_2) \right. \\
& + \left. \sum_{s=2}^{[\mu_j/\tau]-2} \tilde{D}^{[\mu_j/\tau]-1-s} \{f_{s+1} - 2f_s + f_{s-1}\} \right) \left. \right] + \left(\mu_j - \left[\frac{\mu_j}{\tau} \right] \tau \right) \\
& \times \left[\frac{i}{2} \left\{ \left(I - \frac{i\tau A^{1/2}}{2} \right)^{-1} \tilde{D}^{[\mu_j/\tau]-1} - \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} D^{[\mu_j/\tau]-1} \right\} \right. \\
& \times A^{-1/2} \left\{ B^{N-r} [g_{-N+r} - R^r g_{-N+1} \right. \\
& + (\tau A)^{-1} \left\{ (g_{-N+r-1} - g_{-N+r}) - R^{r-1} (g_{-N+1} - g_{-N+2}) \right.
\end{aligned}$$

$$\begin{aligned}
& + \sum_{s=-N+3}^{-N+r} R^{-N+r-s+1} \{g_{s-2} - 2g_{s-1} + g_s\} \Bigg] \\
& + \left[g_0 - B^{N-r} g_{-N+r+1} + (\tau A)^{-1} \left(1 - \frac{\tau A}{2} \right) \right. \\
& \times \left\{ (g_{-1} - g_0) - B^{N-r-1} (g_{-N+r+1} - g_{-N+r+2}) \right. \\
& \left. + \sum_{s=-N+r+3}^0 B^{-s+1} \{g_{s-2} - 2g_{s-1} + g_s\} \Bigg] \Bigg\} \\
& + \frac{1}{2} \left\{ \left(I + \frac{i\tau A^{1/2}}{2} \right) D^{[\mu_j/\tau]-1} + \left(I - \frac{i\tau A^{1/2}}{2} \right) \tilde{D}^{[\mu_j/\tau]-1} \right\} \\
& \times \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} (f_0 - \{B^{N-r} [g_{-N+r} - R^r g_{-N+1} \right. \\
& + (\tau A)^{-1} \{ (g_{-N+r-1} - g_{-N+r}) - R^{r-1} (g_{-N+1} - g_{-N+2}) \\
& \left. + \sum_{s=-N+3}^{-N+r} R^{-N+r-s+1} \{g_{s-2} - 2g_{s-1} + g_s\} \Bigg] \right. \\
& \left. + \left[g_0 - B^{N-r} g_{-N+r+1} + (\tau A)^{-1} \left(1 - \frac{\tau A}{2} \right) \right. \right. \\
& \times \left\{ (g_{-1} - g_0) - B^{N-r-1} (g_{-N+r+1} - g_{-N+r+2}) \right. \\
& \left. + \sum_{s=-N+r+3}^0 B^{-s+1} \{g_{s-2} - 2g_{s-1} + g_s\} \Bigg] \Bigg] \Bigg\} \\
& + \frac{\tau A}{2} \{B^{N-r} [g_{-N+r} - R^r g_{-N+1} \\
& + (\tau A)^{-1} \{ (g_{-N+r-1} - g_{-N+r}) - R^{r-1} (g_{-N+1} - g_{-N+2}) \\
& + \sum_{s=-N+3}^{-N+r} R^{-N+r-s+1} \{g_{s-2} - 2g_{s-1} + g_s\} \Bigg] \\
& + \left[g_0 - B^{N-r} g_{-N+r+1} + (\tau A)^{-1} \left(1 - \frac{\tau A}{2} \right) \right. \\
& \times \left\{ (g_{-1} - g_0) - B^{N-r-1} (g_{-N+r+1} - g_{-N+r+2}) \right. \\
& \left. + \sum_{s=-N+r+3}^0 B^{-s+1} \{g_{s-2} - 2g_{s-1} + g_s\} \Bigg] \Bigg] \Bigg\} + \tau \left(I + \frac{\tau^2 A}{4} \right)^{-1} f_{[\mu_j/\tau]-1} \\
& + \frac{A^{-1/2}}{i} \left[(D - \tilde{D}) f_{[\mu_j/\tau]-1} + (\tilde{D}^{[\mu_j/\tau]-1} - D^{[\mu_j/\tau]-1}) f_1 \right. \\
& \quad \left. + (i\tau A^{1/2})^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \\
& \quad \times \left(D (f_{[\mu_j/\tau]-3} - f_{[\mu_j/\tau]-2}) - D^{[\mu_j/\tau]-2} (f_1 - f_2) \right)
\end{aligned}$$

$$\begin{aligned}
& + \left[g_0 - B^{N-r} g_{-N+r+1} + (\tau A)^{-1} \left(1 - \frac{\tau A}{2} \right) \right. \\
& \quad \times \left\{ (g_{-1} - g_0) - B^{N-r-1} (g_{-N+r+1} - g_{-N+r+2}) \right. \\
& \quad \left. + \sum_{s=-N+r+3}^0 B^{-s+1} \{g_{s-2} - 2g_{s-1} + g_s\} \right\} \left. \right] + \tau \left(I + \frac{\tau^2 A}{4} \right)^{-1} f_{[\lambda_j/\tau]-1} \\
& \quad + \frac{A^{-1/2}}{i} \left[\left(D - \tilde{D} \right) f_{[\lambda_j/\tau]-1} + \left(\tilde{D}^{[\lambda_j/\tau]-1} - D^{[\lambda_j/\tau]-1} \right) f_1 \right. \\
& \quad \quad \left. + \left(i\tau A^{1/2} \right)^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \\
& \quad \quad \times \left(D (f_{[\lambda_j/\tau]-3} - f_{[\lambda_j/\tau]-2}) - D^{[\lambda_j/\tau]-2} (f_1 - f_2) \right. \\
& \quad \quad \left. + \sum_{s=2}^{[\lambda_j/\tau]-3} D^{[\lambda_j/\tau]-1-s} \{f_{s+1} - 2f_s + f_{s-1}\} \right) + \left(i\tau A^{1/2} \right)^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right) \\
& \quad \quad \times \left(\tilde{D} (f_{[\lambda_j/\tau]-3} - f_{[\lambda_j/\tau]-2}) - \tilde{D}^{[\lambda_j/\tau]-2} (f_1 - f_2) \right. \\
& \quad \quad \left. + \sum_{s=2}^{[\lambda_j/\tau]-3} \tilde{D}^{[\lambda_j/\tau]-1-s} \{f_{s+1} - 2f_s + f_{s-1}\} \right) \left. \right] \\
& \quad + \left(\lambda_j - \left\lfloor \frac{\lambda_j}{\tau} \right\rfloor \tau + \frac{\tau}{2} \right) \left(f_{[\lambda_j/\tau]} - \left\{ \frac{1}{2} [D^{[\lambda_j/\tau]} + \tilde{D}^{[\lambda_j/\tau]}] \right. \right. \\
& \quad \quad \times \left\{ B^{N-r} [g_{-N+r} - R^r g_{-N+1}] \right. \\
& \quad \quad \left. + (\tau A)^{-1} \left\{ (g_{-N+r-1} - g_{-N+r}) - R^{r-1} (g_{-N+1} - g_{-N+2}) \right. \right. \\
& \quad \quad \left. \left. + \sum_{s=-N+3}^{-N+r} R^{-N+r-s+1} \{g_{s-2} - 2g_{s-1} + g_s\} \right\} \right. \\
& \quad \quad \left. + \left[g_0 - B^{N-r} g_{-N+r+1} + (\tau A)^{-1} \left(1 - \frac{\tau A}{2} \right) \right. \right. \\
& \quad \quad \times \left\{ (g_{-1} - g_0) - B^{N-r-1} (g_{-N+r+1} - g_{-N+r+2}) \right. \\
& \quad \quad \left. \left. + \sum_{s=-N+r+3}^0 B^{-s+1} \{g_{s-2} - 2g_{s-1} + g_s\} \right\} \right. \\
& \quad \quad \left. + \frac{A^{-1/2}}{2i} \left\{ \tilde{D}^{[\lambda_j/\tau]} - D^{[\lambda_j/\tau]} \right\} \left[\left(I + \frac{\tau^2 A}{4} \right)^{-1} \right. \right. \\
& \quad \quad \times \left(f_0 - \left\{ B^{N-r} [g_{-N+r} - R^r g_{-N+1}] \right. \right. \\
& \quad \quad \left. + (\tau A)^{-1} \left\{ (g_{-N+r-1} - g_{-N+r}) - R^{r-1} (g_{-N+1} - g_{-N+2}) \right. \right. \\
& \quad \quad \left. \left. + \sum_{s=-N+3}^{-N+r} R^{-N+r-s+1} \{g_{s-2} - 2g_{s-1} + g_s\} \right\} \right. \\
& \quad \quad \left. \left. + \left[g_0 - B^{N-r} g_{-N+r+1} + (\tau A)^{-1} \left(1 - \frac{\tau A}{2} \right) \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \left\{ (g_{-1} - g_0) - B^{N-r-1} (g_{-N+r+1} - g_{-N+r+2}) \right. \\
& + \left. \sum_{s=-N+r+3}^0 B^{-s+1} \{g_{s-2} - 2g_{s-1} + g_s\} \right\} \Bigg] \\
& + \frac{\tau}{2} f_0 + \frac{\tau}{2} \{B^{N-r} [g_{-N+r} - R^r g_{-N+1} \\
& + (\tau A)^{-1} \{ (g_{-N+r-1} - g_{-N+r}) - R^{r-1} (g_{-N+1} - g_{-N+2}) \\
& + \sum_{s=-N+3}^{-N+r} R^{-N+r-s+1} \{g_{s-2} - 2g_{s-1} + g_s\} \} \Bigg] \\
& + \left[g_0 - B^{N-r} g_{-N+r+1} + (\tau A)^{-1} \left(1 - \frac{\tau A}{2} \right) \right. \\
& \times \{ (g_{-1} - g_0) - B^{N-r-1} (g_{-N+r+1} - g_{-N+r+2}) \\
& + \sum_{s=-N+r+3}^0 B^{-s+1} \{g_{s-2} - 2g_{s-1} + g_s\} \} \Bigg] \Bigg] \\
& + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right) [f_{[\lambda_j/\tau]-1} - D^{[\lambda_j/\tau]-1} f_1 \\
& + (i\tau A^{1/2})^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right) \\
& \times ((f_{[\lambda_j/\tau]-2} - f_{[\lambda_j/\tau]-1}) - D^{[\lambda_j/\tau]-2} (f_1 - f_2) \\
& + \sum_{s=2}^{[\lambda_j/\tau]-2} D^{[\lambda_j/\tau]-1-s} \{f_{s+1} - 2f_s + f_{s-1}\}) \Bigg] \\
& + \frac{1}{2} A^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right) [f_{[\lambda_j/\tau]-1} - \tilde{D}^{[\lambda_j/\tau]-1} f_1 \\
& - (i\tau A^{1/2})^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right) \\
& \times ((f_{[\lambda_j/\tau]-2} - f_{[\lambda_j/\tau]-1}) - \tilde{D}^{[\lambda_j/\tau]-2} (f_1 - f_2) \\
& + \sum_{s=2}^{[\lambda_j/\tau]-2} \tilde{D}^{[\lambda_j/\tau]-1-s} \{f_{s+1} - 2f_s + f_{s-1}\}) \Bigg] \Bigg] \Bigg\} + \varphi \Bigg]
\end{aligned}$$

formülü elde edilir. Buradan, (4.101) formülü ve (4.72), (4.73), (4.74) kestirimlerinin yardımıyla,

$$\begin{aligned}
& \left\| A \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} u_{-N} \right\|_H \leq M [\|A\varphi\|_H + \|A^{1/2} f_0\|_H + \|(f_1 - f_0) \tau^{-1}\|_H \\
& + \max_{2 \leq k \leq N-2} \|(f_{k+1} - 2f_k + f_{k-1}) \tau^{-2}\|_H + \|A^{1/2} g_0\|_H \\
& + \|(g_0 - g_{-1}) \tau^{-1}\|_H + \max_{-(N-1) \leq k \leq -1} \|(g_{k+1} - 2g_k + g_{k-1}) \tau^{-2}\|_H] \quad (4.102)
\end{aligned}$$

kestirimi alınır.

Son olarak, $\|Au_k\|_H$ için kestirim elde edilecektir. Öncelikle $-N+1 < k < -N+r$ aralığında inceleyelim. Bu durumda, Abel formülü ve (4.85) formülü kullanılarak,

$$\begin{aligned} u_k &= R^{N+k}u_{-N} + A^{-1} \left[g_k - R^{N+k}g_{-N+1} \right. \\ &\quad \left. + (\tau A)^{-1} \left((g_{k-1} - g_k) - R^{N+k-1}(g_{-N+1} - g_{-N+2}) \right. \right. \\ &\quad \left. \left. + \sum_{s=-N+2}^{k-1} R^{k-s} \{g_{s+1} - 2g_s + g_{s-1}\} \right) \right] \end{aligned} \quad (4.103)$$

formülü elde edilir. Sonra da $-N+r+1 \leq k \leq 0$ aralığında inceleyelim. Bu durumda, Abel formülü ve (4.88) formülü kullanılarak,

$$\begin{aligned} u_k &= B^{N-r+k}R^r u_{-N} + A^{-1}B^{N-r+k} \left[g_{-N+r} - R^r g_{-N+1} \right. \\ &\quad \left. + (\tau A)^{-1} \left((g_{-N+r-1} - g_{-N+r}) - R^{r-1}(g_{-N+1} - g_{-N+2}) \right. \right. \\ &\quad \left. \left. + \sum_{s=-N+2}^{-N+r-1} R^{-N+r-s} \{g_{s+1} - 2g_s + g_{s-1}\} \right) \right] \\ &\quad + A^{-1} \left[g_k - B^{N-r+k}g_{-N+r+1} \right. \\ &\quad \left. + (\tau A)^{-1} \left(I - \frac{\tau A}{2} \right) \left((g_{k-1} - g_k) - B^{N-r+k-1}(g_{-N+r+1} - g_{-N+r+2}) \right. \right. \\ &\quad \left. \left. + \sum_{s=-N+r+2}^{k-1} B^{k-s} \{g_{s+1} - 2g_s + g_{s-1}\} \right) \right] \end{aligned} \quad (4.104)$$

formülü elde edilir. Buradan, (4.103), (4.104) formülleri ve (4.73), (4.74) kestirimlerinin yardımıyla,

$$\begin{aligned} \|Au_k\|_H &\leq M \left[\left\| A \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} u_{-N} \right\|_H \right. \\ &\quad \left. + \left\| \tau^{-1}(g_{-1} - g_0) \right\|_H + \left\| A^{1/2}g_0 \right\|_H \right. \\ &\quad \left. + \max_{-(N-2) \leq k \leq -1} \left\| \tau^{-2}(g_{k+1} - 2g_k + g_{k-1}) \right\|_H \right] \end{aligned} \quad (4.105)$$

kestirimi alınır. Şimdi ise $1 \leq k \leq N$ aralığında inceleyelim. Bu durumda, Abel formülü ve (4.89), (4.90) formülleri kullanılarak,

$$\begin{aligned} u_1 &= \left(I + \frac{\tau^2 A}{4} \right)^{-1} \left[\left(I - \frac{\tau^2 A}{4} \right) \right. \\ &\quad \left. \times A^{-1} \{ B^{N-r}R^r u_{-N} + B^{N-r}g_{-N+r} - R^r g_{-N+1} \} \right] \end{aligned}$$

$$\begin{aligned}
& + (\tau A)^{-1} \left((g_{-N+r-1} - g_{-N+r}) - R^{r-1} (g_{-N+1} - g_{-N+2}) \right. \\
& + \left. \sum_{s=-N+2}^{-N+r-1} R^{-N+r-s} \{g_{s+1} - 2g_s + g_{s-1}\} \right) \Bigg] \Bigg\} + [g_0 - B^{N-r} g_{-N+r+1} \\
& + (\tau A)^{-1} \left(I - \frac{\tau A}{2} \right) \left((g_{-1} - g_0) - B^{N-r-1} (g_{-N+r+1} - g_{-N+r+2}) \right. \\
& + \left. \sum_{s=-N+r+2}^{-1} B^{-s} \{g_{s+1} - 2g_s + g_{s-1}\} \right) \Bigg] \Bigg\} \\
& - \tau \left(I + \frac{\tau^2 A}{4} \right)^{-1} \left(f_0 - \{B^{N-r} A R^r u_{-N} + B^{N-r} [g_{-N+r} - R^r g_{-N+1} \right. \\
& + (\tau A)^{-1} \left((g_{-N+r-1} - g_{-N+r}) - R^{r-1} (g_{-N+1} - g_{-N+2}) \right. \\
& + \left. \sum_{s=-N+2}^{-N+r-1} R^{-N+r-s} \{g_{s+1} - 2g_s + g_{s-1}\} \right) \Bigg] + [g_0 - B^{N-r} g_{-N+r+1} \\
& + (\tau A)^{-1} \left(I - \frac{\tau A}{2} \right) \left((g_{-1} - g_0) - B^{N-r-1} (g_{-N+r+1} - g_{-N+r+2}) \right. \\
& + \left. \sum_{s=-N+r+2}^{-1} B^{-s} \{g_{s+1} - 2g_s + g_{s-1}\} \right) \Bigg] \Bigg\} + \frac{\tau^2}{2} g_0 \Bigg], \tag{4.106} \\
& u_k = \frac{1}{2} \left(D^k + \tilde{D}^k \right) \left\{ B^{N-r} R^r u_{-N} + A^{-1} B^{N-r} [g_{-N+r} - R^r g_{-N+1} \right. \\
& + (\tau A)^{-1} \left((g_{-N+r-1} - g_{-N+r}) - R^{r-1} (g_{-N+1} - g_{-N+2}) \right. \\
& + \left. \sum_{s=-N+2}^{-N+r-1} R^{-N+r-s} \{g_{s+1} - 2g_s + g_{s-1}\} \right) \Bigg] + A^{-1} [g_0 - B^{N-r} g_{-N+r+1} \\
& + (\tau A)^{-1} \left(I - \frac{\tau A}{2} \right) \left((g_{-1} - g_0) - B^{N-r-1} (g_{-N+r+1} - g_{-N+r+2}) \right. \\
& + \left. \sum_{s=-N+r+2}^{-1} B^{-s} \{g_{s+1} - 2g_s + g_{s-1}\} \right) \Bigg] \Bigg\} \\
& + \frac{1}{2i} A^{-1/2} \left(\tilde{D}^k - D^k \right) \left(\left(I + \frac{\tau^2 A}{4} \right)^{-1} \right. \\
& \times \left(f_0 - \{A B^{N-r} R^r u_{-N} + B^{N-r} [g_{-N+r} - R^r g_{-N+1} \right. \\
& + (\tau A)^{-1} \left((g_{-N+r-1} - g_{-N+r}) - R^{r-1} (g_{-N+1} - g_{-N+2}) \right. \\
& + \left. \sum_{s=-N+2}^{-N+r-1} R^{-N+r-s} \{g_{s+1} - 2g_s + g_{s-1}\} \right) \Bigg] + [g_0 - B^{N-r} g_{-N+r+1} \\
& + (\tau A)^{-1} \left(I - \frac{\tau A}{2} \right) \left((g_{-1} - g_0) - B^{N-r-1} (g_{-N+r+1} - g_{-N+r+2}) \right. \\
& + \left. \sum_{s=-N+r+2}^{-1} B^{-s} \{g_{s+1} - 2g_s + g_{s-1}\} \right) \Bigg] \Bigg\} + \frac{\tau}{2} f_0
\end{aligned}$$

$$\begin{aligned}
& + \frac{\tau}{2} \left\{ AB^{N-r} R^r u_{-N} + B^{N-r} [g_{-N+r} - R^r g_{-N+1} \right. \\
& + (\tau A)^{-1} ((g_{-N+r-1} - g_{-N+r}) - R^{r-1} (g_{-N+1} - g_{-N+2})) \\
& + \left. \sum_{s=-N+2}^{-N+r-1} R^{-N+r-s} \{g_{s+1} - 2g_s + g_{s-1}\} \right\} + [g_0 - B^{N-r} g_{-N+r+1} \\
& (\tau A)^{-1} \left(I - \frac{\tau A}{2} \right) ((g_{-1} - g_0) - B^{N-r-1} (g_{-N+r+1} - g_{-N+r+2})) \\
& + \sum_{s=-N+r+2}^{-1} B^{-s} \{g_{s+1} - 2g_s + g_{s-1}\} \Bigg] \Bigg\} \\
& + \frac{1}{2} A^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right) \left[f_{k-1} - D^{k-1} f_1 + (i\tau A^{1/2})^{-1} \left(I - \frac{i\tau A^{1/2}}{2} \right) \right. \\
& \times \left((f_{k-2} - f_{k-1}) - D^{k-2} (f_1 - f_2) + \sum_{s=2}^{k-2} D^{k-1-s} \{f_{s+1} - 2f_s + f_{s-1}\} \right) \Bigg] \\
& + \frac{1}{2} A^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right) \left[f_{k-1} - \tilde{D}^{k-1} f_1 - (i\tau A^{1/2})^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right) \right. \\
& \times \left. \left((f_{k-2} - f_{k-1}) - \tilde{D}^{k-2} (f_1 - f_2) + \sum_{s=2}^{k-2} \tilde{D}^{k-1-s} \{f_{s+1} - 2f_s + f_{s-1}\} \right) \right] \quad (4.107)
\end{aligned}$$

formülleri elde edilir. Buradan, (4.106), (4.107) formülleri ve (4.72), (4.73), (4.74) kestirimleri yardımıyla,

$$\begin{aligned}
& \left\| A \left(\frac{u_1 + u_0}{2} \right) \right\|_H \leq M \left[\left\| A \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} u_{-N} \right\|_H \right. \\
& + \left\| A^{1/2} g_0 \right\|_H + \left\| \tau^{-1} (g_{-1} - g_0) \right\|_H \\
& + \left. \max_{-(N-1) \leq k \leq -1} \left\| \tau^{-2} [g_{s+1} - 2g_s + g_{s-1}] \right\|_H \right], \quad (4.108) \\
& \left\| A \left(\frac{u_k + u_{k-1}}{2} \right) \right\|_H \leq M \left[\left\| A \left(I + \frac{i\tau A^{1/2}}{2} \right)^{-1} u_{-N} \right\|_H \right. \\
& + \left\| A^{1/2} f_0 \right\|_H + \left\| (f_1 - f_0) \tau^{-1} \right\|_H \\
& + \max_{2 \leq k \leq N-2} \left\| (f_{k+1} - 2f_k + f_{k-1}) \tau^{-2} \right\|_H + \left\| A^{1/2} g_0 \right\|_H + \left\| (g_{-1} - g_0) \tau^{-1} \right\|_H \\
& + \left. \max_{-(N-2) \leq k \leq -1} \left\| \tau^{-2} (g_{s+1} - 2g_s + g_{s-1}) \right\|_H \right], \quad 2 \leq k \leq N \quad (4.109)
\end{aligned}$$

kestirimleri alırız. Sonuç olarak, (4.102), (4.105), (4.108), (4.109) kestirimleri birleştirilerek ve üçgen eşitsizliği kullanılarak, (4.80) kestirimi elde edilir. Böylece, Teorem 4.7'nin ispatı tamamlanmış olur.

Şimdi, Teorem 4.7'nin uygulamaları verilecektir. İlk olarak, (2.21) lokal olmayan sınır değer problemi alınmıştır. Burada (2.21) probleminin diskritizasyonu iki adımda incelenir. Birinci adımda, A_h^x fark operatörünün yardımıyla, (2.21) problemi, (3.31) adi diferansiyel denklem sistemine dönüştürülür. İkinci adımda ise (3.31) problemi için (4.71) fark şeması kullanılarak

$$\left\{ \begin{array}{l} \frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + \frac{1}{2}A_h^x u_k^h(x) + \frac{1}{4}A_h^x (u_{k+1}^h(x) + u_{k-1}^h(x)) = f_k^h(x), \\ f_k^h(x) = \{f(y_k, x_n)\}_1^{K-1}, x \in [0, 1]_h, y_k = k\tau, 1 \leq k \leq N-1, N\tau = 1, \\ \left(I + \frac{\tau^2}{4}A_h^x \right) \left[\left(I + \frac{\tau^2}{4}A_h^x \right) \tau^{-1} (u_1^h(x) - u_0^h(x)) \right. \\ \left. - \frac{\tau}{2} (g_0^h(x) - A_h^x u_0^h(x)) \right] = Z_1^h(x), Z_1^h(x) = f_0^h(x) - A_h^x u_0^h(x), \\ f_0^h(x) = \{f(0, x_n)\}_1^{K-1}, g_0^h(x) = \{g(0, x_n)\}_1^{K-1}, \\ \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + A_h^x u_k^h(x) = g_k^h(x), x \in [0, 1]_h, \\ \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + \frac{1}{2}A_h^x (u_k^h(x) + u_{k-1}^h(x)) = g_k^h(x), x \in [0, 1]_h, \\ g_k^h(x) = \{g(y_k - \frac{\tau}{2}, x_n)\}_1^{K-1}, y_k = k\tau, -(N-1) \leq k \leq 0, \\ u_{-N}^h(x) = \sum_{j=1}^N \alpha_j \left(u_{[\mu_j/\tau]}^h(x) + (\mu_j - [\frac{\mu_j}{\tau}]\tau) \frac{u_{[\mu_j/\tau]}^h(x) - u_{[\mu_j/\tau]-1}^h(x)}{\tau} \right) \\ + \sum_{j=1}^L \beta_j \left[\frac{u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)}{\tau} + \left(\lambda_j - [\frac{\lambda_j}{\tau}]\tau + \frac{\tau}{2} \right) \right. \\ \left. \times \left(f_{[\frac{\lambda_j}{\tau}]}^h(x) - A_h^x u_{[\frac{\lambda_j}{\tau}]}^h(x) \right) \right] + \varphi^h(x), x \in [0, 1]_h \end{array} \right. \quad (4.110)$$

fark şeması elde edilir.

Teorem 4.8. Eğer τ ve h yeterince küçük sayılar ise, bu durumda (4.110) fark şemasının çözümü için aşağıdaki

$$\begin{aligned} & \max_{-N \leq k \leq N} \|u_k^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq N} \|\tau^{-1} (u_k^h - u_{k-1}^h)\|_{L_{2h}} + \max_{-N \leq k \leq N} \|(u_k^h)_x\|_{L_{2h}} \\ & \leq M_1 \left[\|f_0^h\|_{L_{2h}} + \max_{2 \leq k \leq N-1} \|(f_k^h - f_{k-1}^h) \tau^{-1}\|_{L_{2h}} \right. \\ & \quad \left. + \|g_0^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \|(g_k^h - g_{k-1}^h) \tau^{-1}\|_{L_{2h}} + \left\| \varphi_x^h \right\|_{L_{2h}} \right], \\ & \quad \max_{1 \leq k \leq N-1} \|\tau^{-2} (u_{k+1}^h - 2u_k^h + u_{k-1}^h)\|_{L_{2h}} \\ & \quad + \max_{-N+1 \leq k \leq 0} \left\| \left(u_{\frac{-}{x}}^k \right)_x \right\|_{L_{2h}} + \max_{1 \leq k \leq N} \left\| \frac{\left(u_{\frac{-}{x}}^k \right)_x + \left(u_{\frac{-}{x}}^{k-1} \right)_x}{2} \right\|_{L_{2h}} \\ & \quad + \max_{-N+1 \leq k \leq 0} \|\tau^{-1} (u_k^h - u_{k-1}^h)\|_{L_{2h}} \\ & \leq M_1 \left[\|f_{0x}^h\|_{L_{2h}} + \|\tau^{-1} (f_1^h - f_0^h)\|_{L_{2h}} \right] \end{aligned}$$

$$\begin{aligned}
& + \max_{2 \leq k \leq N-1} \left\| \tau^{-2} (f_{k+1}^h - 2f_k^h + f_{k-1}^h) \right\|_{L_{2h}} + \|g_{0x}^h\|_{L_{2h}} + \left\| \tau^{-1} (g_0^h - g_{-1}^h) \right\|_{L_{2h}} \\
& + \max_{-N+1 \leq k \leq -1} \left\| \tau^{-2} (g_{k+1}^h - 2g_k^h + g_{k-1}^h) \right\|_{L_{2h}} + \left\| \left(\varphi_{\frac{h}{x}}^h \right)_x \right\|_{L_{2h}} \Bigg]
\end{aligned}$$

kararlılık kestirimleri sağlanır. Burada, M_1 katsayısı $\tau, h, f_k^h, 1 \leq k < N, g_k^h, -N < k \leq 0$ ve $\varphi^h(x)$ 'den bağımsızdır.

Teorem 4.8'in ispatı, soyut Teorem 4.7 ve (3.30) formülü ile tanımlanan A_h^x fark operatörünün simetri özelliklerine dayanmaktadır.

İkinci olarak, (2.22) karma sınır değer problemi ele alınmıştır. (2.22) probleminin diskritizasyonu iki adımda incelenir. Birinci adımda, A_h^x fark opertörünün yardımıyla, (2.22) problemi (3.34) adi diferansiyel denklem sistemine dönüştürülür. İkinci adımda ise (3.34) problemi için (4.71) fark şeması kullanılarak

$$\left\{ \begin{aligned}
& \frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + \frac{1}{2} A_h^x u_k^h(x) + \frac{1}{4} A_h^x (u_{k+1}^h(x) + u_{k-1}^h(x)) \\
& = f_k^h(x), \quad f_k^h(x) = f^h(y_k, x), \quad x \in \tilde{\Omega}_h, \quad y_k = k\tau, \quad 1 \leq k < N, \\
& \left(I + \frac{\tau^2}{4} A_h^x \right) \left[\left(I + \frac{\tau^2}{4} A_h^x \right) \tau^{-1} (u_1^h(x) - u_0^h(x)) \right. \\
& \quad \left. - \frac{\tau}{2} (g_0^h(x) - A_h^x u_0^h(x)) \right] = Z_1^h(x), \quad x \in \tilde{\Omega}_h, \\
& Z_1^h(x) = f_0^h(x) - A_h^x u_0^h(x), \quad x \in \tilde{\Omega}_h, \\
& f_0^h(x) = \{f(0, x_n)\}_1^{K-1}, \quad g_0^h(x) = \{g(0, x_n)\}_1^{K-1} \\
& \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + A_h^x u_k^h(x) = g_k^h(x), \quad x \in \tilde{\Omega}_h, \\
& \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + \frac{1}{2} A_h^x (u_k^h(x) + u_{k-1}^h(x)) = g_k^h(x), \quad x \in \tilde{\Omega}_h, \\
& g_k^h(x) = g^h(y_k - \frac{\tau}{2}, x), \quad y_k = k\tau, \quad -(N-1) \leq k \leq 0, \quad x \in \tilde{\Omega}_h, \\
& u_{-N}^h(x) = \sum_{j=1}^N \alpha_j \left(u_{[\mu_j/\tau]}^h(x) + (\mu_j - [\frac{\mu_j}{\tau}]\tau) \frac{u_{[\mu_j/\tau]}^h(x) - u_{[\mu_j/\tau]-1}^h(x)}{\tau} \right) \\
& + \sum_{j=1}^L \beta_j \left[\frac{u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)}{\tau} + \left(\lambda_j - [\frac{\lambda_j}{\tau}]\tau + \frac{\tau}{2} \right) \right. \\
& \quad \left. \times \left(f_{[\frac{\lambda_j}{\tau}]}^h(x) - A_h^x u_{[\frac{\lambda_j}{\tau}]}^h(x) \right) \right] + \varphi^h, \quad x \in \tilde{\Omega}_h
\end{aligned} \right. \quad (4.111)$$

fark şeması elde edilir.

Teorem 4.9. Eğer τ ve $|h|$ yeterince küçük sayılar ise, bu durumda (4.111) fark şemasının çözümü için aşağıdaki

$$\begin{aligned}
& \max_{-N \leq k \leq N} \|u_k^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq N} \left\| \tau^{-1} (u_k^h - u_{k-1}^h) \right\|_{L_{2h}} \\
& + \max_{-N \leq k \leq N} \sum_{r=1}^n \left\| (u_k^h)_{x_r, j_r} \right\|_{L_{2h}} \\
& \leq M_1 \left[\|f_0^h\|_{L_{2h}} + \max_{2 \leq k \leq N-1} \left\| (f_k^h - f_{k-1}^h) \tau^{-1} \right\|_{L_{2h}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \|g_0^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \|(g_k^h - g_{k-1}^h) \tau^{-1}\|_{L_{2h}} + \sum_{r=1}^n \left\| (\varphi^h)_{\bar{x}_r, j_r} \right\|_{L_{2h}} \Bigg], \\
& \max_{1 \leq k \leq N-1} \|\tau^{-2} (u_{k+1}^h - 2u_k^h + u_{k-1}^h)\|_{L_{2h}} \\
& + \max_{-N+1 \leq k \leq 0} \sum_{r=1}^n \left\| (u_k^h)_{\bar{x}_r, j_r} \right\|_{L_{2h}} \\
& + \max_{1 \leq k \leq N} \sum_{r=1}^n \left\| \frac{(u_k^h)_{\bar{x}_r, j_r} + (u_{k-1}^h)_{\bar{x}_r, j_r}}{2} \right\|_{L_{2h}} \\
& + \max_{-N+1 \leq k \leq 0} \|\tau^{-1} (u_k^h - u_{k-1}^h)\|_{L_{2h}} \\
& \leq M_1 \left[\sum_{r=1}^n \left\| (f_0^h)_{\bar{x}_r, j_r} \right\|_{L_{2h}} + \|\tau^{-1} (f_1^h - f_0^h)\|_{L_{2h}} \right. \\
& + \max_{2 \leq k \leq N-1} \|\tau^{-2} (f_{k+1}^h - 2f_k^h + f_{k-1}^h)\|_{L_{2h}} \\
& + \sum_{r=1}^n \left\| (g_0^h)_{\bar{x}_r, j_r} \right\|_{L_{2h}} + \|\tau^{-1} (g_0^h - g_{-1}^h)\|_{L_{2h}} \\
& \left. + \max_{-N+1 \leq k \leq -1} \|\tau^{-2} (g_{k+1}^h - 2g_k^h + g_{k-1}^h)\|_{L_{2h}} + \sum_{r=1}^n \left\| (\varphi^h)_{\bar{x}_r, j_r} \right\|_{L_{2h}} \right]
\end{aligned}$$

kararlılık kestirimleri sağlanır. Burada, M_1 katsayısı $\tau, h, f_k^h, 1 \leq k < N, g_k^h, -N < k \leq 0$ ve $\varphi^h(x)$ 'den bağımsızdır.

Teorem 4.9'un ispatı, soyut Teorem 4.7, (3.33) formülü ile tanımlanan A_h^x fark operatörünün simetri özelliklerine ve Teorem 3.4'e dayanmaktadır.

5 SAYISAL SONUÇLAR

Yukarıdaki kararlılık eşitsizliklerindeki sabit sayılar için keskin bir kestirim alınmaktadır.

Bu yüzden, hiperbolik-parabolik denklem için lokal olmayan sınır değer probleminin

$$\left\{ \begin{array}{l} \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), \quad 0 < t < 1, \quad 0 < x < 1, \\ \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = g(t,x), \quad -1 < t < 0, \quad 0 < x < 1, \\ u(0+,x) = u(0-,x), u_t(0+,x) = u_t(0-,x), \quad 0 \leq x \leq 1, \\ u(-1,x) = u(1,x) + u_t(1,x) + \varphi(x), \quad 0 \leq x \leq 1, \\ u(t,0) = u(t,1) = 0, \quad -1 \leq t \leq 1, \\ f(t,x) = e^{(1-t^2)} (-2 + 4t^2 + \pi^2) \sin \pi x, \\ g(t,x) = e^{(1-t^2)} (-2t + \pi^2) \sin \pi x, \\ \varphi(x) = 2 \sin \pi x \end{array} \right. \quad (5.1)$$

sayısal çözümleri verilecektir. Bu problemin tam çözümü

$$u(t,x) = e^{(1-t^2)} \sin \pi x$$

şeklindedir.

Buradaki (5.1) lokal olmayan sınır değer probleminin yaklaşık çözümleri için, birinci ve i-kinici basamaktan doğruluklu fark şemaları kullanılacaktır. İkinci ve dördüncü mertebeden, katsayıları matris olan, n 'ye göre fark denklemleri elde edilecektir. Bu fark denklemlerini çözmek için, iyileştirilmiş Gauss eliminasyon yöntemi kullanılacaktır. Sayısal denemelerin sonucu olarak ikinci basamaktan doğruluklu fark şemalarının birinci basamaktan doğruluklu fark şemalarına oranla daha doğru olduğu gösterilecektir.

5.1 Birinci Basamaktan Doğruluklu Fark Şeması

Burada, birinci basamaktan doğruluklu fark şeması (3.1) uygulanarak, (5.1) probleminin yaklaşık çözümleri için

$$\left\{ \begin{array}{l} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} = f(t_{k+1}, x_n), \quad t_{k+1} = (k+1)\tau, \\ 1 \leq k \leq N-1, \quad N\tau = 1, \quad x_n = nh, \quad 1 \leq n \leq M-1, \quad Mh = \pi, \\ \frac{u_n^k - u_n^{k-1}}{\tau} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} = g(t_k, x_n), \quad t_k = k\tau, \\ -N+1 \leq k \leq 0, \quad x_n = nh, \quad 1 \leq n \leq M-1, \\ u_n^1 - u_n^0 = u_n^0 - u_n^{-1}, \quad 1 \leq n \leq M-1, \\ u_n^{-N} = u_n^N + \frac{u_n^N - u_n^{N-1}}{\tau} + 2 \sin \pi x_n, \quad x_n = nh, \quad 1 \leq n \leq M-1, \\ u_0^k = u_M^k = 0, \quad 0 \leq k \leq N \end{array} \right. \quad (5.2)$$

birinci basamaktan doğruluklu fark şeması kurulur. Böylece, $(2N+1) \times (2N+1)$ boyutlu doğrusal denklem sistemi elde edilmiş olur. Bu doğrusal denklem sistemi düzenlenerek matris formunda yazılırsa,

$$\left\{ \begin{array}{l} A U_{n+1} + B U_n + C U_{n-1} = D \varphi_n, \quad 1 \leq n \leq M-1, \\ U_0 = \tilde{0}, \quad U_M = \tilde{0} \end{array} \right.$$

elde edilir. Burada,

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & a & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdot & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdot & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(2N+1) \times (2N+1)},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\tau} & -\frac{1}{\tau} - 1 \\ b & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & b & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & e & f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & e & f & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & d & e & f \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \end{bmatrix}_{(2N+1) \times (2N+1)},$$

ve

$$C = A, D = \begin{bmatrix} 1 & 0 & \cdot & 0 \\ 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 1 \end{bmatrix}_{(2N+1) \times (2N+1)},$$

$$U_s = \begin{bmatrix} U_s^{-N} \\ \dots \\ U_s^0 \\ U_s^1 \\ \dots \\ U_s^N \end{bmatrix}_{(2N+1) \times (1)}, s = n \pm 1, n$$

dir. Ayrıca,

$$a = -\frac{1}{h^2}, b = \frac{1}{\tau}, c = \frac{1}{\tau} + \frac{2}{h^2}, d = \frac{1}{\tau^2}, e = -\frac{2}{\tau^2}, f = \frac{1}{\tau^2} + \frac{2}{h^2},$$

$$\varphi_n^k = \begin{cases} 2 \sin(\pi x_n), \\ g(t_k, x_n), & -N+1 \leq k \leq 0, \\ f(t_{k+1}, x_n), & 1 \leq k \leq N-1, \\ 0, & k = N, \end{cases}, \varphi_n = \begin{bmatrix} \varphi_n^{-N} \\ \dots \\ \varphi_n^0 \\ \varphi_n^1 \\ \dots \\ \varphi_n^N \end{bmatrix}_{(2N+1) \times (1)}$$

dir. Dolayısıyla, ikinci mertebeden katsayıları matris olan, n 'ye göre fark denklemleri elde edilmiş olur. Bu fark denklemlerini çözmek için, iyileştirilmiş Gauss eliminasyon

yöntemini kullanalım. Yani, matris denkleminin

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}, \quad n = M-1, \dots, 2, 1, 0 \quad (5.3)$$

formunda bir çözümünü araştıralım. Bu son denklemde α_j 'ler ($j = 1, \dots, M-1$), $(2N+1) \times (2N+1)$ boyutlu kare matrisler ve β_j 'ler ($j = 1, \dots, M-1$), $(2N+1) \times 1$ boyutlu sütun matrislerdir. $U_0 = \tilde{0}$ ve $U_M = \tilde{0}$ olduğu için $\alpha_1, (2N+1) \times (2N+1)$ ve $\beta_1, (2N+1) \times 1$ boyutlu sıfır matrislerdir. Buradan, (5.3) denkleminde n yerine $s-1$ ve s yazılarak ve ayrıca

$$A U_{n+1} + B U_n + C U_{n-1} = D\varphi_n$$

denklemini kullanarak,

$$[A + B\alpha_{n+1} + C\alpha_n\alpha_{n+1}]U_{n+1} + [B\beta_{n+1} + C\alpha_n\beta_{n+1} + C\beta_n] = D\varphi_n \quad (5.4)$$

denklemini elde edilir. Eğer

$$\begin{cases} A + B\alpha_{n+1} + C\alpha_n\alpha_{n+1} = 0, \\ B\beta_{n+1} + C\alpha_n\beta_{n+1} + C\beta_n = D\varphi_n, 1 \leq n \leq M-1 \end{cases}$$

olarak seçilirse (5.4) denklemini sağlanır. Böylece,

$$\begin{cases} \alpha_{n+1} = -(B + C\alpha_n)^{-1} A, \\ \beta_{n+1} = (B + C\alpha_n)^{-1} (D\varphi_n - C\beta_n), 1 \leq n \leq M-1 \end{cases}$$

formülleri elde edilmiş olur. Farklı N ve M değerleri verildiğinde, (5.1) probleminin sayısal çözümlerini yukarıdaki yöntem ile bulan bir matlab programı Ek 1 kısmında verilmiştir.

5.2 İkinci Basamaktan Doğruluklu Fark Şemaları

İlk olarak, ikinci basamaktan doğruluklu fark şeması (4.1) ve

$$\begin{aligned} \frac{2u(0) - 5u(h) + 4u(2h) - u(3h)}{h^2} - u''(0) &= O(h^2), \\ \frac{2u(1) - 5u(1-h) + 4u(1-2h) - u(1-3h)}{h^2} - u''(1) &= O(h^2), \\ \frac{u(x_{n+2}) - 4u(x_{n+1}) + 6u(x_n) - 4u(x_{n-1}) + u(x_{n-2}))}{h^4} - u^{(4)}(x_n) &= O(h^2) \end{aligned}$$

uygulanarak, (5.1) probleminin yaklaşık çözümleri için

$$\left\{ \begin{array}{l} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} + \tau^2 \left(\frac{u_{n+2}^{k+1}-4u_{n+1}^{k+1}+6u_n^{k+1}-4u_{n-1}^{k+1}+u_{n-2}^{k+1}}{4h^4} \right) \\ = f(t_k, x_n), \quad 1 \leq k \leq N-1, 2 \leq n \leq M-2, \\ \frac{u_n^k-u_n^{k-1}}{\tau} - \frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} + \tau \left(\frac{u_{n+2}^k-4u_{n+1}^k+6u_n^k-4u_{n-1}^k+u_{n-2}^k}{2h^4} \right) \\ = g(t_k - \frac{\tau}{2}, x_n), \quad -N+1 \leq k \leq 0, 2 \leq n \leq M-2, \\ \frac{u_n^1-u_n^0}{\tau} - \tau \left(\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1-u_{n+1}^0+2u_n^0-u_{n-1}^0}{h^2} \right) = \frac{\tau}{2} (f(0, x_n) + \frac{1}{h^2} (u_{n+1}^0 - 2u_n^0 + u_{n-1}^0)) \\ + (g(0, x_n) + \frac{1}{h^2} (u_{n+1}^0 - 2u_n^0 + u_{n-1}^0)), \quad x_n = nh, \quad 1 \leq n \leq M-1, \\ u_n^{-N} = u_n^N + \frac{3u_n^N-4u_n^{N-1}+u_n^{N-2}}{2\tau} + \varphi(x_n), \quad 0 \leq n \leq M, \\ u_0^k = u_M^k = 0, \quad 0 \leq k \leq N, \\ u_1^k = \frac{4}{5}u_2^k - \frac{1}{5}u_3^k, \quad u_{M-1}^k = \frac{4}{5}u_{M-2}^k - \frac{1}{5}u_{M-3}^k, \quad 0 \leq k \leq N. \end{array} \right. \quad (5.5)$$

fark şeması kurulur. Böylece, yine $(2N+1) \times (2N+1)$ boyutlu doğrusal denklem sistemi elde edilmiş olur. Bu doğrusal denklem sistemi düzenlenerek matris formunda yazılırsa

$$\left\{ \begin{array}{l} A U_{n+2} + B U_{n+1} + C U_n + D U_{n-1} + E U_{n-2} = R \varphi_n, \quad 2 \leq n \leq M-2, \\ U_0 = 0, \quad U_M = 0, \\ U_1 = \frac{4}{5}U_2 - \frac{1}{5}U_3, \\ U_{M-1} = \frac{4}{5}U_{M-2} - \frac{1}{5}U_{M-3} \end{array} \right.$$

elde edilir. Burada

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \end{bmatrix}_{(2N+1) \times (2N+1)},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & w & v & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & w & v & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & w & v \\ 0 & 0 & 0 & \cdot & m & n & \cdot & 0 & 0 \end{bmatrix}_{(2N+1) \times (2N+1)},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & 0 & -\frac{1}{2\tau} & \frac{2}{\tau} & -\frac{3}{2\tau} - 1 \\ 0 & a & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & b & c & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & b & c & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & b & c \\ 0 & 0 & \cdot & p & q & p & \cdot & 0 & 0 \end{bmatrix}_{(2N+1) \times (2N+1)},$$

ve

$$D = B, \quad E = A,$$

$$R = \begin{bmatrix} 1 & 0 & \cdot & 0 \\ 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 1 \end{bmatrix}_{(2N+1) \times (2N+1)},$$

$$U_s = \begin{bmatrix} U_s^{-N} \\ \dots \\ U_s^0 \\ U_s^1 \\ \dots \\ U_s^N \end{bmatrix}_{(2N+1) \times (1)}, \quad s = n \pm 2, n \pm 1, n$$

dir. Ayrıca,

$$x = \frac{\tau}{2h^4}, \quad y = \frac{\tau^2}{4h^4}, \quad z = -\frac{1}{h^2} - \frac{2\tau}{h^4}, \quad w = -\frac{1}{h^2},$$

$$v = -\frac{\tau^2}{h^4}, \quad m = \frac{\tau}{h^2} - \frac{\tau}{2h^2} - \frac{1}{h^2}$$

$$\begin{aligned}
n &= \frac{1}{\tau} + \frac{2\tau}{h^2}, \quad a = \frac{3\tau}{h^4} + \frac{1}{\tau} + \frac{2}{h^2}, \quad b = -\frac{2}{\tau^2} + \frac{2}{h^2}, \\
c &= \frac{1}{\tau^2} + \frac{3\tau^2}{2h^4}, \quad p = \frac{1}{\tau} + \frac{2\tau}{h^2}, \quad q = -\frac{1}{\tau} - \frac{2}{h^2} + \frac{\tau}{h^2} + \frac{2}{h^2}, \\
\varphi_n^k &= \begin{cases} 2 \sin(\pi x_n), \\ g(t_k - \frac{\tau}{2}, x_n) - \frac{\tau}{2h^2} [g(t_k - \frac{\tau}{2}, x_{n+1}) + g(t_k - \frac{\tau}{2}, x_n) + g(t_k - \frac{\tau}{2}, x_{n-1})], \\ -N + 1 \leq k \leq 0, \\ f(t_k, x_n), \quad 1 \leq k \leq N - 1, \\ 0, \quad k = N, \end{cases} \\
\varphi_n &= \begin{bmatrix} \varphi_n^{-N} \\ \vdots \\ \varphi_n^0 \\ \varphi_n^1 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(2N+1) \times (1)}
\end{aligned}$$

dir. Dolayısıyla, dördüncü mertebeden katsayıları matris olan, n 'ye göre fark denklemleri elde edilmiş olur. Bu fark denklemlerini çözmek için, iyileştirilmiş Gauss eliminasyon yöntemini kullanalım. Yani, matris denkleminin

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}U_{n+2} + \gamma_{n+1}, \quad n = M - 2, \dots, 2, 1, 0$$

formunda bir çözümünü araştıralım. Bu son denklemde α_j, β_j 'ler ($j = 1, \dots, M - 1$), $(2N + 1) \times (2N + 1)$ boyutlu kare matrisler ve γ_j 'ler ($j = 1, \dots, M - 1$), $(2N + 1) \times 1$ boyutlu sütun matrislerdir. Böylece, $\alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}$ için $n = 2, \dots, M - 1$ olmak üzere

$$\begin{cases} \alpha_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(B + D\beta_n + E\alpha_{n-1}\beta_n), \\ \beta_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(A), \\ \gamma_{n+1} = (C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(R\varphi_n - D\gamma_n - E\alpha_{n-1}\gamma_n - E\gamma_{n-1}) \end{cases}$$

formülleri elde edilir. Burada,

$$\begin{cases} \alpha_1 \text{ ve } \beta_1, (2N + 1) \times (2N + 1) \text{ boyutlu sıfır matrisler,} \\ I, (2N + 1) \times (2N + 1) \text{ boyutlu birim matris,} \\ \alpha_2 = \frac{4}{5}I \text{ ve } \beta_2 = -\frac{1}{5}I, \\ \gamma_1 \text{ ve } \gamma_2, (2N + 1) \times 1 \text{ boyutlu sıfır matrislerdir.} \end{cases}$$

Ayrıca,

$$\begin{cases} U_M = \tilde{0}, \\ U_{M-1} = [(\beta_{M-2} + 5I) - (4I - \alpha_{M-2})\alpha_{M-1}]^{-1}[(4I - \alpha_{M-2})\gamma_{M-1} - \gamma_{M-2}], \\ U_{M-2} = [(4I - \alpha_{M-2})]^{-1}[(\beta_{M-2} + 5I)U_{M-1} + \gamma_{M-2}] \end{cases}$$

dir. Farklı N ve M değerleri verildiğinde, (5.1) probleminin sayısal çözümlerini, yukarıdaki yöntem ile bulan bir matlab programı Ek 2 kısmında verilmiştir.

İkinci olarak, ikinci basamaktan doğruluklu fark şeması (4.36), (5.1) probleminin yaklaşık çözümleri için

$$\left\{ \begin{array}{l} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{2h^2} - \frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}+u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{4h^2} \\ = f(t_k, x_n), \quad x_n = nh, \quad t_k = k\tau, \quad 1 \leq k \leq N-1, \quad 1 \leq n \leq M-1, \\ \frac{u_n^k-u_n^{k-1}}{\tau} - \frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} + \tau \left(\frac{u_{n+2}^k-4u_{n+1}^k+6u_n^k-4u_{n-1}^k+u_{n-2}^k}{2h^4} \right) \\ = g(t_k - \frac{\tau}{2}, x_n) - \frac{\tau}{2h^2} [g(t_k - \frac{\tau}{2}, x_{n+1}) \\ - 2g(t_k - \frac{\tau}{2}, x_n) + g(t_k - \frac{\tau}{2}, x_{n-1})], \\ x_n = nh, \quad t_k = k\tau, \quad -N+1 \leq k \leq 0, \quad 1 \leq n \leq M-1, \\ \frac{u_n^1-u_n^0}{\tau} - \frac{\tau}{h^2} (u_{n+1}^1 - 2u_n^1 + u_{n-1}^1 - u_{n+1}^0 + 2u_n^0 - u_{n-1}^0) \\ = \frac{\tau}{2} (f(0, x_n) + \frac{1}{h^2} (u_{n+1}^0 - 2u_n^0 + u_{n-1}^0)) \\ + (g(0, x_n) + \frac{1}{h^2} (u_{n+1}^0 - 2u_n^0 + u_{n-1}^0)), \quad x_n = nh, \quad 2 \leq n \leq M-2, \\ u_n^{-N} = u_n^N + \frac{3u_n^N-4u_n^{N-1}+u_n^{N-2}}{2\tau} + \varphi(x_n), \quad x_n = nh, \quad 1 \leq n \leq M-1, \\ u_0^k = u_M^k = 0, \quad 0 \leq k \leq N, \\ u_1^k = \frac{4}{5}u_2^k - \frac{1}{5}u_3^k, \quad u_{M-1}^k = \frac{4}{5}u_{M-2}^k - \frac{1}{5}u_{M-3}^k, \quad 0 \leq k \leq N \end{array} \right. \quad (5.6)$$

ikinci basamaktan doğruluklu fark şeması kurulur. Böylece, yine $(2N+1) \times (2N+1)$ doğrusal denklem sistemi elde edilmiş olur. Bu doğrusal denklem sistemi düzenlenerek matris formunda yazılırsa,

$$\left\{ \begin{array}{l} A U_{n+2} + B U_{n+1} + C U_n + D U_{n-1} + E U_{n-2} = R \varphi_n, \quad 2 \leq n \leq M-2, \\ U_0 = 0, \quad U_M = 0, \\ U_1 = \frac{4}{5}U_2 - \frac{1}{5}U_3, \\ U_{M-1} = \frac{4}{5}U_{M-2} - \frac{1}{5}U_{M-3} \end{array} \right.$$

matris denklemi elde edilir. Burada,

$$\begin{aligned}
 A &= \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \end{bmatrix}_{(2N+1) \times (2N+1)}, \\
 B &= \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & e & f & e & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & e & f & e & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & e & f & e \\ 0 & 0 & 0 & \cdot & m & n & \cdot & 0 & 0 \end{bmatrix}_{(2N+1) \times (2N+1)}, \\
 C &= \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & 0 & -\frac{1}{2\tau} & \frac{2}{\tau} & -\frac{3}{2\tau} - 1 \\ z & w & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & z & w & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & z & w & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & c & d & c & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & c & d & c & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & c & d & c \\ 0 & 0 & \cdot & p & q & p & \cdot & 0 & 0 \end{bmatrix}_{(2N+1) \times (2N+1)}, \\
 D = B, \quad E = A, \quad R &= \begin{bmatrix} 1 & 0 & \cdot & 0 \\ 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 1 \end{bmatrix}_{(2N+1) \times (2N+1)},
 \end{aligned}$$

$$U_s = \begin{bmatrix} U_s^{-N} \\ \dots \\ U_s^0 \\ U_s^1 \\ \dots \\ U_s^N \end{bmatrix}_{(2N+1) \times (1)}, \quad s = n \pm 2, n \pm 1, n$$

dir. Ayrıca,

$$\begin{aligned} x &= \frac{\tau}{2h^4}, \quad y = -\frac{2\tau}{h^4} - \frac{1}{2h^2}, \quad z = -\frac{1}{\tau}, \\ w &= \frac{1}{\tau} + \frac{2}{h^2} + \frac{3\tau}{h^4}, \quad e = -\frac{1}{4h^2}, \quad f = -\frac{1}{2h^2} \\ m &= \frac{\tau}{h^2} - \frac{\tau}{2h^2} - \frac{1}{h^2}, \quad n = -\frac{\tau}{h^2}, \\ c &= \frac{1}{\tau^2} + \frac{1}{2h^2}, \quad d = -\frac{2}{\tau^2} + \frac{1}{2h^2} \\ p &= \frac{1}{\tau} + \frac{2\tau}{h^2}, \quad q = -\frac{1}{\tau} - \frac{2\tau}{h^2} + \frac{\tau}{h^2} + \frac{2}{h^2}, \end{aligned}$$

$$\varphi_n^k = \begin{cases} 2 \sin(\pi x_n), \\ g(t_k - \frac{\tau}{2}, x_n) - \frac{\tau}{2h^2} [g(t_k - \frac{\tau}{2}, x_{n+1}) + g(t_k - \frac{\tau}{2}, x_n) + g(t_k - \frac{\tau}{2}, x_{n-1})], \\ -N + 1 \leq k \leq 0, \\ f(t_k, x_n), 1 \leq k \leq N - 1, \\ 0, \quad k = N \end{cases}$$

ve

$$\varphi_n = \begin{bmatrix} \varphi_n^{-N} \\ \dots \\ \varphi_n^0 \\ \varphi_n^1 \\ \dots \\ \varphi_n^N \end{bmatrix}_{(2N+1) \times 1}$$

dir. Dolayısıyla, dördüncü mertebeden katsayıları matris olan, n 'ye göre fark denklemleri elde edilmiş olur. Bu fark denklemlerini çözmek için, iyileştirilmiş Gauss eliminasyon yöntemini kullanalım. Yani, matris denkleminin

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}U_{n+2} + \gamma_{n+1}, \quad n = M - 2, \dots, 2, 1, 0$$

formunda bir çözümünü araştıralım. Bu son denklemde α_j, β_j 'ler ($j = 1, \dots, M-1$), $(2N+1) \times (2N+1)$ boyutlu kare matrisler ve γ_j 'ler ($j = 1, \dots, M-1$), $(2N+1) \times 1$ boyutlu sütun matrislerdir. Böylece, $\alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}$ için $n = 2, \dots, M-1$ olmak üzere

$$\begin{cases} \alpha_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(B + D\beta_n + E\alpha_{n-1}\beta_n), \\ \beta_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(A), \\ \gamma_{n+1} = (C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(R\varphi_n - D\gamma_n - E\alpha_{n-1}\gamma_n - E\gamma_{n-1}) \end{cases}$$

formülleri elde edilir. Burada,

$$\begin{cases} \alpha_1 \text{ ve } \beta_1, (2N+1) \times (2N+1) \text{ boyutlu sıfır matrisler,} \\ I, (2N+1) \times (2N+1) \text{ boyutlu birim matris,} \\ \alpha_2 = \frac{4}{5}I \text{ ve } \beta_2 = -\frac{1}{5}I, \\ \gamma_1 \text{ ve } \gamma_2, (2N+1) \times 1 \text{ boyutlu sıfır matrislerdir.} \end{cases}$$

Ayrıca,

$$\begin{cases} U_M = \tilde{0}, \\ U_{M-1} = [(\beta_{M-2} + 5I) - (4I - \alpha_{M-2})\alpha_{M-1}]^{-1}[(4I - \alpha_{M-2})\gamma_{M-1} - \gamma_{M-2}], \\ U_{M-2} = [(4I - \alpha_{M-2})]^{-1}[(\beta_{M-2} + 5I)U_{M-1} + \gamma_{M-2}] \end{cases}$$

dir. Farklı N ve M değerleri verildiğinde, (5.1) probleminin sayısal çözümlerini, yukarıdaki yöntem ile bulan bir matlab programı Ek 3 kısmında verilmiştir.

Üçüncü olarak, r -iyileştirilmiş Crank-Nicholson fark şeması tarafından oluşturulan ikinci basamakta doğruluklu fark şeması (4.71) uygulanarak, (5.1) probleminin yaklaşık çözümleri için

$$\begin{cases} \frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{2h^2} - \frac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1} + u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{4h^2} \\ = f(t_k, x_n), \quad x_n = nh, \quad t_k = k\tau, \quad 1 \leq k \leq N-1, \quad 1 \leq n \leq M-1, \\ \frac{u_n^k - u_n^{k-1}}{\tau} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} = g(t_k - \frac{\tau}{2}, x_n), \\ x_n = nh, \quad t_k = k\tau, \quad -N+1 \leq k \leq -N+r, \quad 1 \leq n \leq M-1, \\ \frac{u_n^k - u_n^{k-1}}{\tau} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k + u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{2h^2} = g(t_k - \frac{\tau}{2}, x_n), \\ x_n = nh, \quad t_k = k\tau, \quad -N+r+1 \leq k \leq 0, \quad 1 \leq n \leq M-1, \\ \frac{u_n^1 - u_n^0}{\tau} - \frac{\tau}{h^2} (u_{n+1}^1 - 2u_n^1 + u_{n-1}^1 - u_{n+1}^0 + 2u_n^0 - u_{n-1}^0) \\ = \frac{\tau}{2} (f(0, x_n) + \frac{1}{h^2} (u_{n+1}^0 - 2u_n^0 + u_{n-1}^0)) \\ + (g(0, x_n) + \frac{1}{h^2} (u_{n+1}^0 - 2u_n^0 + u_{n-1}^0)), \quad x_n = nh, \quad 2 \leq n \leq M-2, \\ u_n^{-N} = u_n^N + \frac{3u_n^N - 4u_n^{N-1} + u_n^{N-2}}{2\tau} + \varphi(x_n), \quad x_n = nh, \quad 1 \leq n \leq M-1, \\ u_0^k = u_M^k = 0, \quad 0 \leq k \leq N, \\ u_1^k = \frac{4}{5}u_2^k - \frac{1}{5}u_3^k, \quad u_{M-1}^k = \frac{4}{5}u_{M-2}^k - \frac{1}{5}u_{M-3}^k, \quad 0 \leq k \leq N \end{cases} \quad (5.7)$$

fark şeması kurulur. Böylece, yine $(2N + 1) \times (2N + 1)$ doğrusal denklem sistemi elde edilmiş olur. Bu doğrusal denklem sistemi düzenlenerek

$$\begin{cases} A U_{n+2} + B U_{n+1} + C U_n + D U_{n-1} + E U_{n-2} = R \varphi_n, 2 \leq n \leq M - 2, \\ U_0 = 0, \quad U_M = 0, \\ U_1 = \frac{4}{5} U_2 - \frac{1}{5} U_3 \\ U_{M-1} = \frac{4}{5} U_{M-2} - \frac{1}{5} U_{M-3}, \end{cases}$$

matris denklemini elde edilir. Burada,

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(2N+1) \times (2N+1)},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & e & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & b & b & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & b & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & a & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b & a & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & b & a & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(2N+1) \times (2N+1)},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -\frac{1}{2\tau} & \frac{2}{\tau} & -\frac{3}{2\tau} - 1 \\ f & g & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & f & g & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & h & k & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h & k & 0 & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & d & c & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & d & c & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & p & q & p & \cdot & \cdot & 0 & 0 & 0 \end{bmatrix}_{(2N+1) \times (2N+1)}$$

$$D = B, \quad E = A, \quad R = \begin{bmatrix} 1 & 0 & \cdot & 0 \\ 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 1 \end{bmatrix}_{(2N+1) \times (2N+1)},$$

$$U_s = \begin{bmatrix} U_s^{-N} \\ \dots \\ U_s^0 \\ U_s^1 \\ \dots \\ U_s^N \end{bmatrix}_{(2N+1) \times (1)}, \quad s = n \pm 2, n \pm 1, n$$

dir. Ayrica,

$$a = -\frac{1}{4h^2}, b = -\frac{1}{2h^2}, c = \frac{1}{\tau^2} + \frac{1}{2h^2}, d = -\frac{2}{\tau^2} + \frac{1}{h^2},$$

$$e = -\frac{1}{h^2}, f = -\frac{1}{\tau}, g = \frac{1}{\tau} + \frac{2}{h^2}, h = -\frac{1}{\tau} + \frac{1}{h^2}, k = \frac{1}{\tau} + \frac{1}{h^2},$$

$$p = \frac{1}{\tau} + \frac{2\tau}{h^2}, q = -\frac{1}{\tau} - \frac{2\tau}{h^2} + \frac{\tau}{h^2} + \frac{2}{h^2},$$

$$\varphi_n^k = \begin{cases} 2 \sin(\pi x_n), \\ g(t_k - \frac{\tau}{2}, x_n), -N + 1 \leq k \leq 0, \\ f(t_k, x_n), 1 \leq k \leq N - 1, \\ 0, \quad k = N, \end{cases}$$

ve

$$\varphi_n = \begin{bmatrix} \varphi_n^{-N} \\ \dots \\ \varphi_n^0 \\ \varphi_n^1 \\ \dots \\ \varphi_n^N \end{bmatrix}_{(2N+1) \times 1}$$

dir. Dolayısıyla, dördüncü mertebeden katsayıları matris olan, n 'ye göre fark denklemleri elde edilmiş olur. Bu fark denklemlerini çözmek için, iyileştirilmiş Gauss eliminasyon yöntemini kullanalım. Yani, matris denkleminin

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}U_{n+2} + \gamma_{n+1}, n = M-2, \dots, 2, 1, 0$$

formunda bir çözümünü araştıralım. Bu son denklemde α_j, β_j 'ler ($j = 1, \dots, M-1$), $(2N+1) \times (2N+1)$ boyutlu kare matrisler ve γ_j 'ler ($j = 1, \dots, M-1$), $(2N+1) \times 1$ boyutlu sütun matrislerdir. Böylece, $\alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}$ için $n = 2, \dots, M-1$ olmak üzere

$$\begin{cases} \alpha_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(B + D\beta_n + E\alpha_{n-1}\beta_n), \\ \beta_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(A), \\ \gamma_{n+1} = (C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(R\varphi_n - D\gamma_n - E\alpha_{n-1}\gamma_n - E\gamma_{n-1}) \end{cases}$$

formülleri elde edilir. Burada,

$$\begin{cases} \alpha_1 \text{ ve } \beta_1, (2N+1) \times (2N+1) \text{ boyutlu sıfır matrisler,} \\ I, (2N+1) \times (2N+1) \text{ boyutlu birim matris,} \\ \alpha_2 = \frac{4}{5}I \text{ ve } \beta_2 = -\frac{1}{5}I \\ \gamma_1 \text{ ve } \gamma_2, (2N+1) \times 1 \text{ boyutlu sıfır matrislerdir.} \end{cases}$$

Ayrıca,

$$\begin{cases} U_M = \tilde{0}, \\ U_{M-1} = [(\beta_{M-2} + 5I) - (4I - \alpha_{M-2})\alpha_{M-1}]^{-1}[(4I - \alpha_{M-2})\gamma_{M-1} - \gamma_{M-2}], \\ U_{M-2} = [(4I - \alpha_{M-2})]^{-1}[(\beta_{M-2} + 5I)U_{M-1} + \gamma_{M-2}] \end{cases}$$

dir. Farklı N ve M değerleri verildiğinde, (5.1) probleminin sayısal çözümlerini, yukarıdaki yöntem ile bulan bir matlab programı Ek 4 kısmında verilmiştir.

5.3 Hata Analizi

Hiperbolik-parabolik denklem için (5.1) lokal olmayan sınır değer problemi ele alınmıştır. Bu problemin yaklaşık çözümleri için birinci ve ikinci basamaktan doğruluklu fark

şemaları kullanılmıştır. (5.1) probleminin tam ve yaklaşık çözümleri aşağıdaki şekillerde verilmiştir.

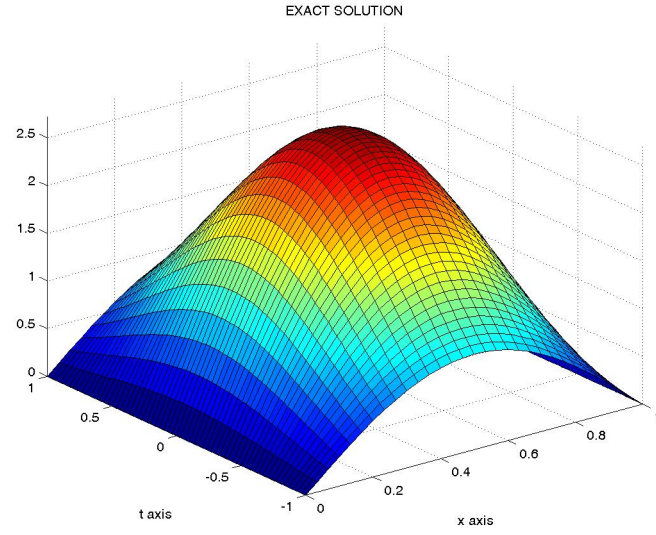


Figure 1: Şekil 5.1. Tam çözüm

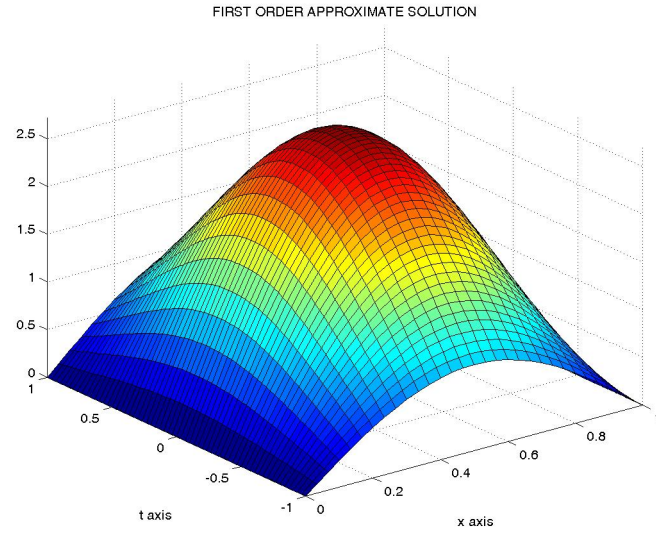


Figure 2: Şekil 5.2. Birinci basamaktan doğruluklu fark şeması

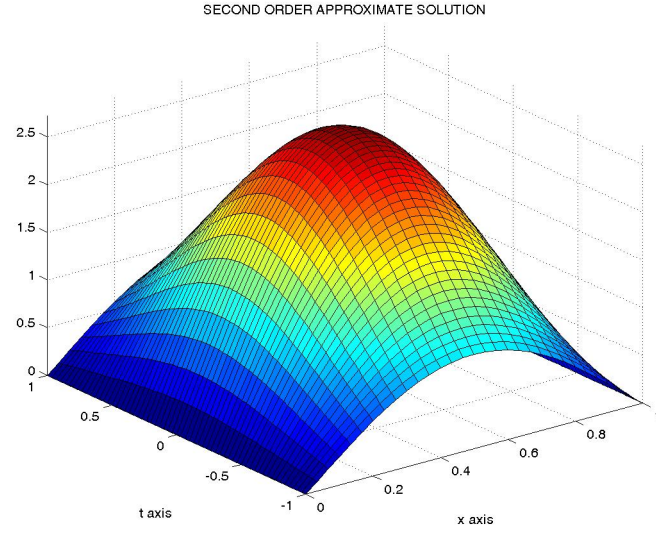


Figure 3: Şekil 5.3. İkinci basamaktan doğrulukluk fark şemaları (4.1)

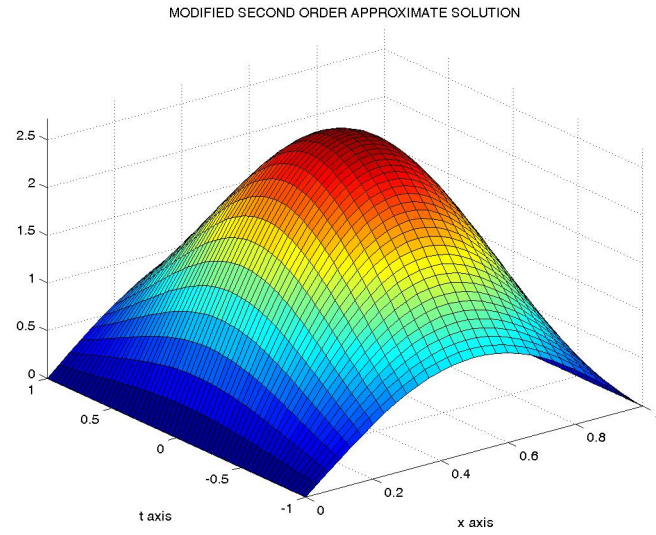


Figure 4: Şekil 5.4. İkinci basamaktan doğruluklu fark şemaları (4.36)

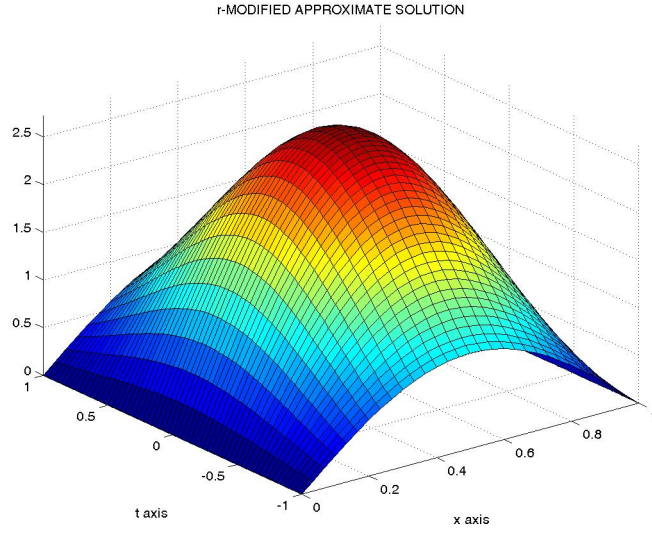


Figure 5: Şekil 5.5. İkinci basamaktan doğruluklu fark şemaları (4.71)

Sonuçların karşılaştırılması için hatalar

$$E_M^N = \max_{1 \leq k \leq N-1} \left(\sum_{n=1}^{M-1} |u(t_k, x_n) - u_n^k|^2 h \right)^{\frac{1}{2}}$$

formülüyle hesaplanır. Burada $u(t_k, x_n)$, problemin gerçek çözümünü ve u_n^k de, (t_k, x_n) noktasında sayısal çözümü temsil etmektedir. Sonuçlar aşağıdaki tablolarda gösterilmiştir:

Tablo 5.1. $N = M = 30$ değerleri için farklı fark şemalarının hatalarının karşılaştırılması.

Fark Şemaları	E_M^N
Birinci basamaktan doğruluklu fark şeması (3.1)	0.0370
İkinci basamaktan doğruluklu fark şeması (4.1)	0.0069
İkinci basamaktan doğruluklu fark şeması (4.36)	0.0059
İkinci basamaktan doğruluklu fark şeması (4.71) ($r = 3$)	0.0056
İkinci basamaktan doğruluklu fark şeması (4.71) ($r = 2$)	0.0033
İkinci basamaktan doğruluklu fark şeması (4.71) ($r = 1$)	0.0012

Tablo 5.2. $N = 30$ ve $M = 100$ değerleri için farklı fark şemalarının hatalarının

karşılaştırılması.

Fark Şemaları	E_M^N
Birinci basamaktan doğruluklu fark şeması (3.1)	0.0362
İkinci basamaktan doğruluklu fark şeması (4.1)	0.0039
İkinci basamaktan doğruluklu fark şeması (4.36)	0.0036
İkinci basamaktan doğruluklu fark şeması (4.71) ($r = 3$)	0.0053
İkinci basamaktan doğruluklu fark şeması (4.71) ($r = 2$)	0.0028
İkinci basamaktan doğruluklu fark şeması (4.71) ($r = 1$)	0.0005

Tablo 5.3. $N = M = 100$ değerleri için farklı fark şemalarının hatalarının karşılaştırılması.

Fark Şemaları	E_M^N
Birinci basamaktan doğruluklu fark şeması (3.1)	0.0125
İkinci basamaktan doğruluklu fark şeması (4.1)	0.0004
İkinci basamaktan doğruluklu fark şeması (4.36)	0.0003
İkinci basamaktan doğruluklu fark şeması (4.71) ($r = 3$)	0.0006
İkinci basamaktan doğruluklu fark şeması (4.71) ($r = 2$)	0.0003
İkinci basamaktan doğruluklu fark şeması (4.71) ($r = 1$)	0.0001

Tablolardan da açık bir şekilde görüldüğü gibi, ikinci basamaktan doğruluklu fark şemaları yöntemi, birinci basamaktan doğruluklu fark şemalarına kıyasla gerçek çözümlere daha yakın sonuçlar vermektedir.

Son olarak, $r = 0$ ve $N = M = 30$ olduğu zaman, ikinci basamaktan doğruluklu fark şeması (4.71) için hata 0.6924'tür. Bilindiği gibi $r = 0$ olduğunda tam Crank-Nicholson fark şeması elde edilir. Ne yazık ki bu fark şeması çalıştığımız problem için kararlı değildir. Bu yüzden, r -iyileştirilmiş Crank-Nicholson fark şeması tarafından oluşturulan ikinci basamaktan doğruluklu fark şeması (4.71) kurulmuştur ve bu fark şeması da birinci basamaktan doğruluklu fark şemasından daha doğru sonuçlar vermektedir.

6 SONUÇLAR

Bu çalışmanın esas amacı hiperbolik-parabolik denklemler için çok noktalı lokal olmayan sınır değer problemlerinin kararlılık kestirimlerini elde etmektir. Yapılan bu çalışmanın sonucunda aşağıdaki orijinal sonuçlar elde edilmiştir:

- H Hilbert uzayında self-adjoint pozitif tanımlı A operatörlü, soyut hiperbolik-parabolik denklem için çok noktalı lokal olmayan sınır değer probleminin

$$\begin{cases} \frac{d^2 u(t)}{dt^2} + Au(t) = f(t) (0 \leq t \leq 1), \\ \frac{du(t)}{dt} + Au(t) = g(t) (-1 \leq t \leq 0), \\ u(-1) = \sum_{j=1}^N \alpha_j u(\mu_j) + \sum_{j=1}^L \beta_j u'(\lambda_j) + \varphi, \\ \sum_{j=1}^N |\alpha_j|, \sum_{j=1}^L |\beta_j| \leq 1, 0 < \mu_j, \lambda_j \leq 1 \end{cases} \quad (6.1)$$

çözümünün kararlılık kestirimleri elde edilmiştir.

- Hiperbolik-parabolik denklemler için iki çeşit lokal olmayan sınır değer problemlerinin çözümlerinin kararlılık kestirimleri ispatlanmıştır.
- Soyut (6.1) lokal olmayan sınır değer probleminin yaklaşık çözümlerini bulmak için, tamsayı kuvvetli olan bir A operatörü tarafından oluşturulan, birinci ve ikinci basamaktan doğruluklu fark şemaları

$$\begin{cases} \tau^{-2} (u_{k+1} - 2u_k + u_{k-1}) + Au_{k+1} = f_k, f_k = f(t_{k+1}), \\ t_{k+1} = (k+1)\tau, 1 \leq k < N, \\ \tau^{-1} (u_k - u_{k-1}) + Au_k = g_k, g_k = g(t_k), t_k = k\tau, -N < k \leq 0, \\ \tau^{-1} (u_1 - u_0) = -Au_0 + g_0, \\ u_{-N} = \sum_{j=1}^N \alpha_j u_{[\mu_j/\tau]} + \sum_{j=1}^L \beta_j \frac{u_{[\lambda_j/\tau]} - u_{[\lambda_j/\tau]-1}}{\tau} + \varphi, \end{cases} \quad (6.2)$$

$$\begin{cases} \tau^{-2} (u_{k+1} - 2u_k + u_{k-1}) + Au_k + \frac{\tau^2}{4} A^2 u_{k+1} = f_k, f_k = f(t_k), \\ t_k = k\tau, 1 \leq k < N, \tau^{-1} (I + \tau^2 A) (u_1 - u_0) = Z_1, \\ Z_1 = \frac{\tau}{2} (f(0) - Au_0) + (g(0) - Au_0), \\ \tau^{-1} (u_k - u_{k-1}) + A(I + \frac{\tau}{2} A) u_k = (I + \frac{\tau}{2} A) g_k, \\ g_k = g(t_k - \frac{\tau}{2}), t_k = k\tau, -(N-1) \leq k \leq 0, \\ u_{-N} = \sum_{j=1}^N \alpha_j \left(u_{[\mu_j/\tau]} + (\mu_j - [\frac{\mu_j}{\tau}]\tau) \frac{u_{[\mu_j/\tau]} - u_{[\mu_j/\tau]-1}}{\tau} \right) \\ + \sum_{j=1}^L \beta_j \left(\frac{u_{[\lambda_j/\tau]} - u_{[\lambda_j/\tau]-1}}{\tau} + \left(\lambda_j - [\frac{\lambda_j}{\tau}]\tau + \frac{\tau}{2} \right) \left(f_{[\frac{\lambda_j}{\tau}]} - Au_{[\frac{\lambda_j}{\tau}]} \right) \right) + \varphi, \end{cases} \quad (6.3)$$

$$\left\{ \begin{array}{l} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + \frac{1}{2}Au_k + \frac{1}{4}A(u_{k+1} + u_{k-1}) = f_k, f_k = f(t_k), \\ t_k = k\tau, \ 1 \leq k < N, \\ \left(I + \frac{\tau^2 A}{4}\right) \left(\left(I + \frac{\tau^2 A}{4}\right) \tau^{-1}(u_1 - u_0) - \frac{\tau}{2}(g_0 - Au_0) \right) = f_0 - Au_0, \\ \tau^{-1}(u_k - u_{k-1}) + A(I + \frac{\tau}{2}A)u_k = (I + \frac{\tau}{2}A)g_k, \\ g_k = g(t_k - \frac{\tau}{2}), \ t_k = k\tau, \ -(N-1) \leq k \leq 0, \\ u_{-N} = \sum_{j=1}^N \alpha_j \left(u_{[\mu_j/\tau]} + \left(\mu_j - [\frac{\mu_j}{\tau}] \right) \tau \frac{u_{[\mu_j/\tau]} - u_{[\mu_j/\tau]-1}}{\tau} \right) \\ + \sum_{j=1}^L \beta_j \left(\frac{u_{[\lambda_j/\tau]} - u_{[\lambda_j/\tau]-1}}{\tau} + \left(\lambda_j - [\frac{\lambda_j}{\tau}] \right) \tau + \frac{\tau}{2} \right) (f_{[\lambda_j/\tau]} - Au_{[\lambda_j/\tau]}) + \varphi, \end{array} \right. \quad (6.4)$$

$$\left\{ \begin{array}{l} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + \frac{1}{2}Au_k + \frac{1}{4}A(u_{k+1} + u_{k-1}) = f_k, f_k = f(t_k), \\ t_k = k\tau, \ 1 \leq k < N, \\ \left(I + \frac{\tau^2 A}{4}\right) \left(\left(I + \frac{\tau^2 A}{4}\right) \tau^{-1}(u_1 - u_0) - \frac{\tau}{2}(g_0 - Au_0) \right) = f_0 - Au_0, \\ \tau^{-1}(u_k - u_{k-1}) + Au_k = g_k, \ -N+1 \leq k \leq -N+r, \\ \tau^{-1}(u_k - u_{k-1}) + \frac{A}{2}(u_k + u_{k-1}) = g_k, \ -N+r+1 \leq k \leq 0, \\ g_k = g(t_k - \frac{\tau}{2}), \ t_k = k\tau, \ -N+1 \leq k \leq 0, \\ u_{-N} = \sum_{j=1}^N \alpha_j \left(u_{[\mu_j/\tau]} + \left(\mu_j - [\frac{\mu_j}{\tau}] \right) \tau \frac{u_{[\mu_j/\tau]} - u_{[\mu_j/\tau]-1}}{\tau} \right) \\ + \sum_{j=1}^L \beta_j \left(\frac{u_{[\lambda_j/\tau]} - u_{[\lambda_j/\tau]-1}}{\tau} + \left(\lambda_j - [\frac{\lambda_j}{\tau}] \right) \tau + \frac{\tau}{2} \right) (f_{[\lambda_j/\tau]} - Au_{[\lambda_j/\tau]}) + \varphi \end{array} \right. \quad (6.5)$$

kurulmuştur.

- (6.2)-(6.5) fark şemalarının çözümlerinin kararlılık kestirimleri elde edilmiştir.
- Hiperbolik-parabolik denklemlerin iki çeşit lokal olmayan sınır değer problemleri için fark şemalarının çözümlerinin kararlılık kestirimleri ispatlanmıştır.
- Bu fark şemalarının çözümleri için sayısal sonuçlar alınmıştır.

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EKLER

Ek 1. Birinci Basamaktan Doğruluklu Fark Şeması için Matlab Programı

```

function [table,es,p]=firstorder(N,M)

close; close;

if nargin<1; N=30 ; M=30 ;end;

tau=1/N; h=1/M;

A=zeros(2*N+1,2*N+1);

for i=2:N+1;

A(i,i)=-1/(h^2); end;

for i=N+2:2*N;

A(i,i+1)=-1/(h^2); end;

B=zeros(2*N+1,2*N+1);

B(1,1)=1; B(1,2*N)=1/tau;

B(1,2*N+1)=-1/tau-1;

for i=1:N; B(i+1,i)=-1/tau; end;

for i=2:N+1;

B(i,i)=1/tau+2/(h^2); end;

for i=N+2:2*N;

B(i,i)=-2/(tau^2); end;

for i=N+2:2*N-1;

B(i,i+1)=1/(tau^2)+2/(h^2); end;

for i=N+2:2*N;

B(i,i+1)=1/(tau^2)+2/(h^2); end;

for i=N+1:2*N-1;

B(i+1,i)=1/(tau^2); end;

B(2*N+1,N)=1;

B(2*N+1,N+1)=-2;

```

```

B(2*N+1,N+2)=1;

for i=1:2*N+1;

C(i,i)=1; end ;

C(2*N+1,2*N+1)=0;

alpha(2*N+1,2*N+1,1:1)= 0 ;

betha(2*N+1,1:1) = 0 ;

'fi(j) = fi(k,j) hesaplanıyor ' ;

for j=1:M; x=j*h;

fii(1,j:j)=2*sin(pi*x); %%% alfa(xn)

for k=2:N+1;

x=j*h;

t=(-N+k-1)*tau ;

fii( k, j:j ) = g(t,x); %(pi^2)*sin(pi*x); %%g(t,x,aaa); end;

for k=N+2:2*N+1;

t=(-N+k-1)*tau+tau;

x=j*h;

fii( k, j:j ) = f(t,x) ; %(pi^2)*sin(pi*x); %%f(t,x,aaa); end; end;

'alpha(N+1,N+1,j) ve betha(N+1,j) ler hesaplanacak' ;

for j=1:M-1;

alpha( :, :, j+1:j+1 ) = - inv(B+A*alpha(:, :, j:j))*A ;

betha( :, j+1:j+1 ) = inv(B+A*alpha(:, :, j:j ))*(C*( fii(:, j:j ))-(A* betha(:,j:j) ));

end;

U( 2*N+1,1, M:M ) = 0;

for z = M-1:-1:1 ;

U(:,z, z) = alpha(:,z,z+1)* U(:,z,z+1) + betha(:,z,z+1); end;

for z = 1:M ;

p(:,z+1:z+1)=U(:,z,z); end;

```

```

'EXACT SOLUTION OF THIS PDE' ;

for j=1:M+1;

for k=1:2*N+1;

t=(-N+k-1)*tau;

x=(j-1)*h; %exact solution on grid points,

es(k,j) = exact(t,x); end; end;

'ERROR ANALYSIS';

maxes=max(max(es));

maxapp=max(max(p));

maxerror=max(max(abs(es-p)));

relativeerror=max(max((abs(es-p)))/max(max(abs(p))));

cevap=[maxes,maxapp,maxerror,relativeerror];

table=[es;p]; table(1:2:end,:)=es; table(2:2:end,:)=p;

%%%%%%%%%%%%GRAPH OF THE SOLUTION %%%%%%%%%%%%%%

q=min(min(table)); w=max(max(table));

figure;

[xler,tler]=meshgrid(0:h:1,-1:tau:1);

surf(xler,tler,es); XLabel('x axis');YLabel('t axis');

title('EXACT SOLUTION');

set(gca,'ZLim',[q w]);

rotate3d; figure;

surf(xler,tler,p); XLabel('x axis');YLabel('t axis');

title('FIRST ORDER APPROXIMATE SOLUTION');

set(gca,'ZLim',[q w]);

function estx=exact(t,x)

estx= exp((1 - t)*(1 + t))*sin(pi*x);

function ftx=f(t,x)

```

```

ftx=(-2 +4*t^2+ pi^2)*exp((1 - t)*(1 + t))*sin(pi*x);

function gtx=g(t,x)

gtx=(-2*t + pi^2)*exp((1 - t)*(1 + t))*sin(pi*x);

```

Ek 2. İkinci Basamaktan Doğruluklu Fark Şemaları (4.1) için Matlab Programı

```

function [table,es,p]=secondorder1(N,M)

if nargin <1; N=30 ; M=30 ;end;

tau= 1/N ; h= 1 /M ;

A=zeros(2*N+1,2*N+1);

for i=2:N+1;

A(i,i)=tau/(2*(h^4)); end; %diagonal (parabolic part)

for i=N+2:2*N;

A(i,i+1)=(tau^2)/(4*(h^4)); end; %diagonalın yukarısı (hyperbolic part)

B=zeros(2*N+1,2*N+1);

for i=2:N+1;

B(i,i)=-1/(h^2)-2*tau/(h^4); end; %diagonal (parabolic part)

for i=N+2:2*N;

B(i,i)=-1/(h^2); end; %diagonal (hyperbolic part)

for i=N+2:2*N;

B(i,i+1)=- (tau^2)/(h^4) ; end; %diagonalın yukarısı (hyperbolic part)

B(2*N+1,N+1)= tau/(h^2)-tau /(2*(h^2))-1/(h^2);

B(2*N+1,N+2)=-tau/(h^2);

C=zeros(2*N+1,2*N+1);

C(1,1)=1;

C(1,2*N-1)=-1/(2*tau);

C(1,2*N)=2/(tau);

C(1,2*N+1)=-3/(2*tau)-1;

```

```

for i=2:N+1;

C(i,i-1)=-1/tau; end; %diagonalın aşağısı (parabolic part)

for i=2:N+1;

C(i,i)=2/(h^2)+1/tau+3*tau/(h^4); end; %diagonal (parabolic part)

for i=N+2:2*N;

C(i,i)=2/(h^2)-2/(tau^2); end; %diagonal (hyperbolic part)

for i=N+2:2*N;

C(i,i-1)=1/(tau^2); end; %diagonalın aşağısı (hyperbolic part)

for i=N+2:2*N;

C(i,i+1)=3*(tau^2)/(2*h^4)+1/(tau^2) ; end; %diagonalın yukarısı (hyperbolic
part)

C(2*N+1,N+1)=-1/tau-2*tau/(h^2)+tau/(h^2)+2/(h^2);

C(2*N+1,N+2)=1/tau+2*tau/(h^2);

E=A ; D=B ;

for i=1:2*N+1; R(i,i)= 1 ; end;

alpha(1:2*N+1,1:2*N+1,1:1) = 0*eye(2*N+1) ;

betha(1:2*N+1,1:2*N+1,1:1) = 0*eye(2*N+1) ;

gamma(2*N+1,1:1)= 0 ;

alpha(1:2*N+1,1:2*N+1,2:2) = (4/5)*eye(2*N+1) ; %%%%%%%%%

betha(1:2*N+1,1:2*N+1,2:2) = (-1/5)*eye(2*N+1);

gamma(2*N+1,2:2)= 0 ;

'fi(j) = fi(k,j) hesaplanıyor' ;

for j=2:M-1;

x=j*h;

fi(1,j:j)=2*sin(pi*x); %%% alfa(xn)

for k=2:N+1;

x=j*h;

```

```

t=(-N+k-1)*tau ;

t=t-tau/2;

fii( k, j:j ) = g(t,x)-(tau/(2*h^2))*(g(t,x+h)-2*g(t,x)+g(t,x-h)); end;

for k=N+2:2*N+1;

t=(-N+k-1)*tau ;

x=j*h;

fii( k, j:j ) = f(t,x) ; end; %(pi^2)*sin(pi*x); %%f(t,x,aaa);

fii(2*N+1,j:j)=(tau/2)*f(0,x)+g(0,x); end;

fii;

'alpha(N+1,N+1,j) ve betha(N+1,j) ler hesaplanacak' ;

for n = 2:M-2 ;

bebek = C + D*alpha(:, , n:n ) + E*betha(:, ,n-1 : n-1)...

+ E*alpha(:, ,n-1:n-1)*alpha(:, , n:n) ;

betha(:, ,n+1:n+1 ) = - inv( bebek )*(A) ;

alpha(:, ,n+1:n+1) = - inv(bebek )*(B +D*betha(:, ,n:n) ...

+ E * alpha(:, ,n-1:n-1)* betha(:, ,n) ) ;

gamma(:, ,n+1:n+1) = inv( bebek )*...

(R*fii(:, ,n:n) - D * gamma(:, ,n:n)...

-E * alpha(:, ,n-1:n-1)* gamma(:, ,n:n) - E*gamma(:, , n-1 : n-1) ) ; end;

'U deęerleri oluřturulup tutuluyor ve matrisine atılıyor.';

U(1:2*N+1,1:2*N+1)=nan;

U( 1:2*N+1, M:M ) = 0 ;

U( :, M-1:M-1 ) = inv( (betha(:, ,M-2:M-2) + 5*eye(2*N+1)) ...

- (4*eye(2*N+1)-alpha(:, ,M-2:M-2) )*alpha(:, ,M-1:M-1))...

*((4*eye(2*N+1)-alpha(:, ,M-2:M-2))*gamma(:, , M-1:M-1)- gamma(:, , M-2:M-2)

);

```

```

U(:, M-2:M-2) = inv(4*eye(2*N+1)-alpha(:, :, M-2:M-2))*( betha(:, :, M-2:M-2)
+...
5* eye(2*N+1) )*U(:, M-1:M-1)+gamma(:, M-2:M-2) );
'INITIAL VALUEs of U is OBTAINED HERE' ;
for z = M-3:-1:1 ;
U(:, z:z) =alpha(:, :, z+1:z+1)*U(:, z+1:z+1)+ ...
betha(:, :, z+1:z+1)*U(:, z+2:z+2)+gamma(:, z+1:z+1); end;
for z = 1 : M ;
p(:, z+1:z+1)=U(:, z:z); end;
'EXACT SOLUTION OF THIS PDE' ;
for j=1:M+1;
for k=1:2*N+1;
t=(-N+k-1)*tau;
x=(j-1)*h;
es(k,j) = exact(t,x); end; end;
'ERROR ANALYSIS' ;
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)));
relativeerror=max(max((abs(es-p)))/max(max(abs(p)) ) );
cevap = [maxes,maxapp,maxerror,relativeerror]
table=[es;p];table(1:2:end,:)=es; table(2:2:end,:)=p;
%%%%%%%%%%%%GRAPH OF THE SOLUTION %%%%%%%%%%%%%%
q=min(min(table)); w=max(max(table));
figure;
[xler,tler]=meshgrid(0:h:1,-1:tau:1);
surf(xler,tler,es); XLabel('x axis');YLabel('t axis');

```

```

title('EXACT SOLUTION'); set(gca,'ZLim',[q w]);

rotate3d; figure;

surf(xler,tlr,p); xlabel('x axis');ylabel('t axis');

title('SECOND ORDER APPROXIMATE SOLUTION'); rotate3d ;

set(gca,'ZLim',[q w]);

function estx=exact(t,x)

estx= exp((1 - t)*(1 + t))*sin(pi*x);

function ftx=f(t,x)

ftx=(-2 +4*t^2+ pi^2)*exp((1 - t)*(1 + t))*sin(pi*x);

function gtx=g(t,x)

gtx=(-2*t + pi^2)*exp((1 - t)*(1 + t))*sin(pi*x);

```

Ek 3. İkinci Basamaktan Doğruluklu Fark Şemaları (4.36) için Matlab Programı

```

function [table,es,p]=secondorder2(N,M)

close;close;

close;close;

if nargin<1;N= 30 ; M= 30 ;end

tau= 1/N ; h= 1 /M ;

A=zeros(2*N+1,2*N+1);

for i=2:N+1;

A(i,i)=tau/(2*(h^4)); %diagonal (parabolic part)

end;

B=zeros(2*N+1,2*N+1);

for i=2:N+1;

B(i,i)=-1/(h^2)-2*tau/(h^4); %diagonal (parabolic part)

end;

for i=N+2:2*N;

```

```

B(i,i)=-1/(2*h^2); %diagonal (hyperbolic part)

end;

for i=N+2:2*N;

B(i,i-1)=-1/(4*h^2); %diagonalın aşağısı (hyperbolic part)

end;

for i=N+2:2*N;

B(i,i+1)=-1/(4*h^2) ; %diagonalın yukarısı (hyperbolic part)

end;

B(2*N+1,N+1)= tau/(h^2)-tau /(2*(h^2))-1/(h^2);

B(2*N+1,N+2)=-tau/(h^2);

C=zeros(2*N+1,2*N+1);

C(1,1)=1;

C(1,2*N-1)=-1/(2*tau);

C(1,2*N)=4/(2*tau);

C(1,2*N+1)=-3/(2*tau)-1;

for i=2:N+1;

C(i,i-1)=-1/tau; %diagonalın aşağısı (parabolic part)

end;

for i=2:N+1;

C(i,i)=2/(h^2)+1/tau+3*tau/(h^4); %diagonal (parabolic part)

end;

for i=N+2:2*N;

C(i,i)=1/(h^2)-2/(tau^2); %diagonal (hyperbolic part)

end;

for i=N+2:2*N;

C(i,i-1)=1/(2*h^2)+1/(tau^2); %diagonalın aşağısı (hyperbolic part)

end;

```

```

for i=N+2:2*N;

C(i,i+1)=1/(2*h^2)+1/(tau^2) ; %diagonalın yukarısı (hyperbolic part)

end;

C(2*N+1,N+1)=-1/tau-2*tau/(h^2)+tau/(h^2)+2/(h^2);

C(2*N+1,N+2)=1/tau+2*tau/(h^2);

E=A; D=B;

for i=1:2*N+1; R(i,i)= 1 ; end;

alpha(1:2*N+1,1:2*N+1,1:1) = 0*eye(2*N+1) ;

betha(1:2*N+1,1:2*N+1,1:1) = 0*eye(2*N+1) ;

gamma(2*N+1,1:1)= 0 ;

alpha(1:2*N+1,1:2*N+1,2:2) = (4/5)*eye(2*N+1) ; %%%%%%%%%

betha(1:2*N+1,1:2*N+1,2:2) = (-1/5)*eye(2*N+1);

gamma(2*N+1,2:2)= 0 ;

'fi(j) = fi(k,j) hesaplanıyor' ;

for j=2:M-1;

x=j*h;

fii(1,j:j)=2*sin(pi*x); %%% alfa(xn)

for k=2:N+1;

x=j*h;

t=(-N+k-1)*tau ;

t=t-tau/2;

fii( k, j:j ) = g(t,x)-(tau/(2*h^2))*(g(t,x+h)-2*g(t,x)+g(t,x-h));

end;

for k=N+2:2*N+1;

t=(-N+k-1)*tau ;

x=j*h;

fii( k, j:j ) = f(t,x) ; %(pi^2)*sin(pi*x); %%f(t,x,aaa);

```

```

end;

fii(2*N+1,j:j)=(tau/2)*f(0,x)+g(0,x);

end;

fii;

'alpha(N+1,N+1,j) ve betha(N+1,j) ler hesaplanacak' ;

for n = 2:M-2 ;

bebek = C + D*alpha(:, , n:n ) + E*betha(:, ,n-1 : n-1)...

+ E*alpha(:, ,n-1:n-1)*alpha(:, , n:n) ;

betha(:, ,n+1:n+1 ) = - inv( bebek )*(A) ;

alpha(:, ,n+1:n+1) = - inv(bebek )*(B +D*betha(:, ,n:n) ...

+ E * alpha(:, ,n-1:n-1)* betha(:, ,n) ) ;

gamma(:,n+1:n+1) = inv( bebek )*...

(R*fii(:,n:n) - D * gamma(:,n:n)...

-E * alpha(:, ,n-1:n-1)* gamma(:,n:n) - E*gamma(:, n-1 : n-1) ) ;

end;

U(1:2*N+1,1:2*N+1)=nan;

U( 1:2*N+1, M:M ) = 0 ;

U( :, M-1:M-1 ) = inv( (betha(:, ,M-2:M-2) + 5*eye(2*N+1)) ...

- (4*eye(2*N+1)-alpha(:, ,M-2:M-2) )*alpha(:, ,M-1:M-1))...

*((4*eye(2*N+1)-alpha(:, ,M-2:M-2))*gamma(:, , M-1:M-1)- gamma(:, , M-2:M-2)

);

U(:, , M-2:M-2 ) = inv(4*eye(2*N+1)-alpha(:, ,M-2:M-2))*( (betha(:, ,M-2:M-2)

+...

5* eye(2*N+1) ))*U(:,M-1:M-1)+gamma(:,M-2:M-2) );

'INITIAL VALUEs of U is OBTAINED HERE' ;

for z = M-3:-1:1 ;

U(:,z:z )=alpha(:, ,z+1:z+1)*U(:,z+1:z+1)+ ...

```

```

betha(:,z+1:z+1)*U(:,z+2:z+2)+gamma(:,z+1:z+1);

end;

for z = 1 : M ;

p(:,z+1:z+1)=U(:,z:z);

end;

'EXACT SOLUTION OF THIS PDE' ;

for j=1:M+1;

for k=1:2*N+1;

t=(-N+k-1)*tau;

x=(j-1)*h;

es(k,j) = exact(t,x);

end;

end;

'ERROR ANALYSIS' ;

maxes=max(max(es)) ;

maxapp=max(max(p)) ;

maxerror=max(max(abs(es-p)));

relativeerror=max(max((abs(es-p)))/max(max(abs(p)) ));

cevap = [maxes,maxapp,maxerror,relativeerror]

table=[es;p];table(1:2:end,:)=es; table(2:2:end,:)=p;

%%%%%%%%%%%%GRAPH OF THE SOLUTION %%%%%%%%%%%%%%

q=min(min(table)); w=max(max(table));

figure;

[xler,tler]=meshgrid(0:h:1,-1:tau:1);

surf(xler,tler,es); XLabel('x axis');YLabel('t axis');

title('EXACT SOLUTION');

set(gca,'ZLim',[q w]);

```

```

rotate3d;

figure;

surf(xler, tler, p); XLabel('x axis'); YLabel('t axis');

title('MODIFIED SECOND ORDER APPROXIMATE SOLUTION');

set(gca, 'ZLim', [q w]);

function estx=exact(t,x)

estx= exp((1 - t)*(1 + t))*sin(pi*x);

function ftx=f(t,x)

ftx=(-2 + 4*t^2 + pi^2)*exp((1 - t)*(1 + t))*sin(pi*x);

function gtx=g(t,x)

gtx=(-2*t + pi^2)*exp((1 - t)*(1 + t))*sin(pi*x);

```

Ek 4. İkinci Basamaktan Doğruluklu Fark Şemaları (4.71) için Matlab Programı

```

function [table, es, p]=secondorder3(N, M, r)

close; close;

close; close;

if nargin<1; N= 30; M= 30; end;

tau=1/N;

h=1/M;

A=zeros(2*N+1, 2*N+1);

B=zeros(2*N+1, 2*N+1);

for i=2:r;

B(i,i)=-1/(h^2); %diagonal

end;

for i=r+1:N+1;

B(i,i)=-1/(2*h^2); %diagonal

end;

```

```

for i=r+1:N+1;

B(i,i-1)=-1/(2*h^2); %diagonal altı

end;

for i=N+2:2*N;

B(i,i)=-1/(2*h^2); %% 2. dia

end;

for i=N+2:2*N;

B(i,i-1)=-1/(4*h^2); % 2. dia altılıyor

end;

for i=N+2:2*N;

B(i,i+1)=-1/(4*h^2) ; % 2. diagonalın üstü

end;

B(2*N+1,N+1)= tau/(h^2)-tau /(2*(h^2))-1/(h^2);

B(2*N+1,N+2)=-tau/(h^2);

C=zeros(2*N+1,2*N+1);

C(1,1)=1;

C(1,2*N-1)=-1/(2*tau);

C(1,2*N)=4/(2*tau);

C(1,2*N+1)=-3/(2*tau)-1;

for i=2:r;

C(i,i-1)=-1/tau; %diagonal altı (parabolic part)

end;

for i=r+1:N+1;

C(i,i-1)=-1/tau+1/(h^2); %diagonal altı (parabolic part)

end;

for i=2:r;

C(i,i)=2/(h^2)+1/tau; %diagonal (parabolic part)

```

```

end;

for i=r+1:N+1;

C(i,i)=1/(h^2)+1/tau; %diagonal (parabolic part)

end;

for i=N+2:2*N;

C(i,i)=1/(h^2)-2/(tau^2); %% diagonal (hyperbolic part)

end;

for i=N+2:2*N;

C(i,i-1)=1/(2*h^2)+1/(tau^2); % diagonal altı (hyperbolic part)

end;

for i=N+2:2*N;

C(i,i+1)=1/(2*h^2)+1/(tau^2) ; % diagonal üstü (hyperbolic part)

end;

C(2*N+1,N+1)=-1/tau-2*tau/(h^2)+tau/(h^2)+2/(h^2);

C(2*N+1,N+2)=1/tau+2*tau/(h^2);

E=A ; D=B ;

for i=1:2*N+1; R(i,i)= 1 ; end;

alpha(1:2*N+1,1:2*N+1,1:1) = 0*eye(2*N+1) ;

betha(1:2*N+1,1:2*N+1,1:1) = 0*eye(2*N+1) ;

gamma(2*N+1,1:1)= 0 ;

alpha(1:2*N+1,1:2*N+1,2:2) = (4/5)*eye(2*N+1) ; %%%%%%%%%

betha(1:2*N+1,1:2*N+1,2:2) = (-1/5)*eye(2*N+1);

gamma(2*N+1,2:2)= 0 ;

'fi(j) = fi(k,j) hesaplanıyor' ;

for j=2:M-1;

x=j*h;

fii(1,j:j)=2*sin(pi*x); %%% alfa(xn)

```

```

for k=2:N+1;

x=j*h;

t=(-N+k-1)*tau ;

t=t-tau/2;

fii(k,j:j) = g(t,x);

end;

for k=N+2:2*N+1;

t=(-N+k-1)*tau ;

x=j*h;

fii(k,j:j) = f(t,x) ; %(pi^2)*sin(pi*x); %%f(t,x,aaa);

end;

fii(2*N+1,j:j)=(tau/2)*f(0,x)+g(0,x);

end;

fii;

'alpha(N+1,N+1,j) ve betha(N+1,j) ler hesaplanacak' ;

for n = 2:M-2 ;

bebek = C + D*alpha(:, , n:n ) + E*betha(:, :,n-1 : n-1)...

+ E*alpha(:, :,n-1:n-1)*alpha(:, , n:n) ;

betha(:, :,n+1:n+1 ) = - inv( bebek )*(A) ;

alpha(:, :,n+1:n+1) = - inv(bebek )*(B +D*betha(:, :,n:n) ...

+ E * alpha(:, :,n-1:n-1)* betha(:, :,n) ) ;

gamma(:,n+1:n+1) = inv( bebek )*...

(R*fii(:,n:n) - D * gamma(:,n:n)...

-E * alpha(:, :,n-1:n-1)* gamma(:,n:n) - E*gamma(:, n-1 : n-1) ) ;

end;

U(1:2*N+1,1:2*N+1)=nan;

U( 1:2*N+1, M:M ) = 0 ;

```

```

U( :, M-1:M-1 ) = inv( (betha(:, :, M-2:M-2) + 5*eye(2*N+1)) ...
- (4*eye(2*N+1)-alpha(:, :, M-2:M-2) ) * alpha(:, :, M-1:M-1)) ...
*((4*eye(2*N+1)-alpha(:, :, M-2:M-2)) * gamma(:, M-1:M-1) - gamma(:, M-2:M-2)
);

U( :, M-2:M-2 ) = inv(4*eye(2*N+1)-alpha(:, :, M-2:M-2)) * ( (betha(:, :, M-2:M-2)
+ ...
5* eye(2*N+1) ) * U(:, M-1:M-1) + gamma(:, M-2:M-2) );

'INITIAL VALUEs OF U Is OBTAINED HERE' ;

for z = M-3:-1:1 ;

U(:, z:z ) = alpha(:, :, z+1:z+1) * U(:, z+1:z+1) + ...
betha(:, :, z+1:z+1) * U(:, z+2:z+2) + gamma(:, z+1:z+1);

end;

for z = 1 : M ;

p(:, z+1:z+1) = U(:, z:z);

end;

'EXACT SOLUTION OF THIS PDE' ;

for j=1:M+1;

for k=1:2*N+1;

t=(-N+k-1)*tau;

x=(j-1)*h;

es(k,j) = exact(t,x);

end;

end;

'ERROR ANALYSIS' ;

maxes = max(max(es)) ;

maxapp = max(max(p)) ;

maxerror = max(max(abs(es-p)));

```

```

relativeerror=max(max((abs(es-p)))/max(max(abs(p)) ));

cevap = [maxes,maxapp,maxerror,relativeerror]

table=[es;p];table(1:2:end,:)=es; table(2:2:end,:)=p;

%%%%%%%%%%%%GRAPH OF THE SOLUTION %%%%%%%%%%%%%%

q=min(min(table)); w=max(max(table));

figure;

[xler,tlr]=meshgrid(0:h:1,-1:tau:1);

surf(xler,tlr,es); XLabel('x axis');YLabel('t axis');

title('EXACT SOLUTION');

set(gca,'ZLim',[q w]);

rotate3d;

figure;

surf(xler,tlr,p); XLabel('x axis');YLabel('t axis');

title('r-MODIFIED APPROXIMATE SOLUTION');

set(gca,'ZLim',[q w]);

function estx=exact(t,x)

estx= exp((1 - t)*(1 + t))*sin(pi*x);

function ftx=f(t,x)

ftx=(-2 +4*t^2+ pi^2)*exp((1 - t)*(1 + t))*sin(pi*x);

function gtx=g(t,x)

gtx=(-2*t + pi^2)*exp((1 - t)*(1 + t))*sin(pi*x);

```

ÖZGEÇMİŞ

1980 yılında İstanbul'un Bakırköy ilçesinde dünyaya geldi. İlk, orta ve lise öğrenimini bu şehirde tamamladı. 1997 yılında Fatih Üniversitesi Fen-Edebiyat Fakültesi Matematik Bölümünde lisans eğitimine başladı. 2002 yılında mezun oldu. Aynı sene Fatih Üniversitesi Fen ve Mühendislik Bilimleri Enstitüsü Matematik Anabilim dalında yüksek lisans eğitimine başladı. 2005 yılında mezun oldu ve Gebze Yüksek Teknoloji Enstitüsü, Mühendislik ve Fen Bilimleri Enstitüsü Matematik Anabilim dalında doktora eğitimine başladı. Bu süre içerisinde 2005-2006 yılları arasında Fatih Üniversitesi Matematik Bölümünde yarı zamanlı öğretim görevlisi olarak çalıştı. 2006 yılından itibaren aynı kurumda tam zamanlı öğretim görevlisi olarak çalışmaya başladı.

Aynı zamanda yazarın çeşitli akademik çalışmaları vardır. Bunlar;

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şeklindedir.