

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

**COMPARISON OF CONFORMING DISCRETIZATION
SCHEMES FOR COMBINED FIELD INTEGRAL EQUATION**

AHMET FURKAN YILMAZ
A THESIS SUBMITTED FOR THE DEGREE OF
MASTER OF SCIENCE
DEPARTMENT OF ELECTRONICS ENGINEERING

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SUMMARY

In this thesis, discretization schemes of magnetic, electric, combined field integral equations (MFIE, EFIE, and CFIE) are investigated. MFIE is known for its robustness and well-conditioned behavior, and EFIE is known for its better accuracy compared to MFIE. For MFIE, classical (C-MFIE) and mixed (M-MFIE) discretization schemes are implemented. It is shown that using M-MFIE, accuracy of the solution is greatly improved, compared to C-MFIE. For EFIE, three different discretization schemes, i.e. classic (C-EFIE), Calderon preconditioned (CMP-EFIE), and Yukawa-Calderon (YUK-EFIE) preconditioned discretization schemes, are implemented. It is shown that using CMP-EFIE and YUK-EFIE greatly improves the conditioning of MoM matrices. However both MFIE and EFIE contain the interior resonance problem.

CFIE, which is obtained by the weighted sum of EFIE and MFIE using a combination factor, produces relatively well-conditioned matrices and remedies the interior resonance problem when the necessary conditions are met. In this thesis combination strategies for CFIE is analyzed and investigated. Specifically, classical (C-CFIE), mixed (M-CFIE), range space projected (RSP-CFIE), and Yukawa-Calderon preconditioned (YUK-CFIE) CFIEs are implemented and their performances are investigated. It is shown that even if C-CFIE remedies the interior resonance problem, its accuracy is low and depends on the combination factor. M-CFIE improves the accuracy but still supports the interior resonance problem. RSP-CFIE remedies both accuracy and the interior resonance problem. YUK-CFIE on the other hand produces well-conditioned and resonance-free matrices.

Finally, all the schemes mentioned above are compared and advantages and disadvantages are discussed.

Key Words: Calderon Preconditioning, Combined Field Integral Equation, Electric Field Integral Equation, Magnetic Field Integral Equation, Range Space Projection.

ÖZET

Bu tezde manyetik, elektrik ve birleşik alan integral denklemleri (MAİD, EAİD ve BAİD) için ayırıklaştırma yöntemleri analiz edilmiştir. MAİD kolay uygulanabilirliği ve iyi-koşullu matrisler üretmesiyle, EAİD ise MAİD'e göre doğruluğu yüksek çözümler üretmesiyle bilinmektedir. MAİD için, klasik (K-MAİD) ve karma ayırıklaştırma (KA-MAİD) yöntemleri uygulanmıştır ve KA-BAİD'in, K-BAİD'e göre doğruluğunu arttırdığı gösterilmiştir. EAİD için, üç farklı ayırıklaştırma yöntemi uygulanmıştır: klasik (K-EAİD), Calderon ön-koşullu (CÖK-EAİD) ve Yukawa-Calderon (YUK-EAİD) ön-koşullu ayırıklaştırma. CÖK-EAİD ve YUK-EAİD'in iyi-koşullu MoM matrisleri ürettiği gösterilmiştir. Ancak gerek EAİD, gerekse MAİD iç-rezonans problemini barındırmaktadır.

EAİD ve MAİD'in, birleşim katsayısı kullanılarak ağırlıklı toplamından elde edilen BAİD, göreceli olarak iyi-koşullu matrisler üretmekte ve gerekli koşullar sağlandığında iç-rezonans problemini gidermektedir. Bu tez kapsamında BAİD için birleşim stratejileri incelenmiştir. Klasik (K-BAİD), karma (KA-BAİD), hedef uzayı izdüşümü alınarak ayırıklaştırılmış (H-BAİD) ve Yukawa-Calderon ön-koşullanmış (YUK-BAİD) BAİD'ler uygulanmış ve performansları incelenmiştir. K-BAİD'in iç-rezonans problemini gidermesine rağmen çözümün doğruluğunun düşük ve birleşim katsayısına bağlı olduğu gösterilmiştir. KA-BAİD'in çözümün doğruluğunu attırdığı ancak uyumsuz hedef uzayları sebebiyle iç-rezonans problemini gideremediği, H-BAİD'in hem doğruluğu yükselttiği hem de iç-rezonans problemini giderdiği, YUK-BAİD'in hem iyi-koşullu hem de iç-rezonans problemi içermeyen çözümler ürettiği gösterilmiştir.

Son olarak bahsedilen tüm ayırıklaştırma yöntemleri karşılaştırılmış, avantaj ve dezavantajları tartışılmıştır.

Anahtar Kelimeler: Birleşik Alan İntegral Denklemi, Calderon Ön-Koşullama, Elektrik Alan İntegral Denklemi, Hedef Uzayı İzdüşümü, Manyetik Alan İntegral Denklemi.

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LIST of ABBREVIATIONS and ACRONYMS

<u>Abbreviations and Acronyms</u>	<u>Explanations</u>
$\mathcal{L}\{\}, \mathcal{O}\{\}$	: Linear and time invariant integro-differential operator
$f(x)$	: Unknown function
$g(x)$	: Known function
$b_n(x)$	: n^{th} basis function
I_n	: Unknown coefficients
$w_m(x), \mathbf{t}_m(\mathbf{r})$	: m^{th} testing functions
$\mathbf{Z}$	: MoM matrix
$\mathbf{I}$	: Unknown current vector
$\mathbf{V}$	: Incident field vector
$\mathbf{E}^{\text{inc}}(\mathbf{r})$	: Incident electric field
$\mathbf{H}^{\text{inc}}(\mathbf{r})$	: Incident magnetic field
S	: Surface of the scatterer
ε	: Permittivity of the free space
μ	: Permeability of the free space
η	: Characteristic impedance
k	: Wave number
ω	: Radial frequency
$\mathbf{J}(\mathbf{r})$	: Current density induced on the scatterer's surface
$\mathbf{E}^{\text{sca}}(\mathbf{r})$	: Scattered electric field
$\mathbf{r}$	: Field/observation point
$\mathbf{r}'$	: Source point
$\hat{\mathbf{n}}(\mathbf{r})$	: Unit normal vector of S
σ	: Conductivity
α	: Combination factor
χ	: Tolerance of iterative solver
$\mathbf{f}(\mathbf{r})$	: Rao-Wilton-Glisson functions
$\mathbf{g}(\mathbf{r})$	: Buffa-Christiansen-Chen-Wilton functions
$\mathcal{K}\{\}$	: Magnetic field integral equation operator
$\mathcal{T}\{\}$	: Electric field integral equation operator
$\mathcal{T}^{\text{loc}}\{\}$	: Localized electric field integral equation operator

$\mathbf{G}_{\hat{n} \times f, f}$	:	Gram matrix linking curl-conf. RWG and div-conf. RWG spaces
$\mathbf{G}_{\hat{n} \times g, f}$	:	Gram matrix linking curl-conf. BCCW and div-conf. RWG spaces
$\mathbf{G}_{f, f}$	:	Gram matrix of div-conf. RWG functions
$\mathbf{G}_{g, g}$	:	Gram matrix of BCCW functions
ABC	:	Absorbing boundary condition
BCCW	:	Buffa-Christiansen-Chen-Wilton functions
C-CFIE	:	Classical discretization of CFIE
C-EFIE	:	Classical discretization of EFIE
CEM	:	Computational electromagnetic
CFIE	:	Combined field integral equation
C-MFIE	:	Classical discretization of MFIE
CMP-EFIE	:	Calderon multiplicative preconditioned EFIE
EFIE	:	Electric field integral equation
EM	:	Electromagnetic
EMC	:	Electromagnetic compatibility
EMI	:	Electromagnetic immunity
FDTD	:	Finite difference time domain
FEM	:	Finite element method
GTD	:	Geometrical theory of diffraction
M-CFIE	:	Mixed discretized CFIE
MFIE	:	Magnetic field integral equation
M-MFIE	:	Mixed discretized MFIE
MoM	:	Method of moments
MOT	:	Marching on-in time
MPI	:	Message passing interface
PEC	:	Perfect electric conductor
PML	:	Perfectly matched layer
PO	:	Physical optic
PTD	:	Physical theory of diffraction
RADAR	:	Radio detection and ranging
RCS	:	Radar cross section
RSP-CFIE	:	Range space projected CFIE
RWG	:	Rao-Wilton-Glisson

SBR	:	Shooting and bouncing ray
SIE	:	Surface integral equation
TFQMR	:	Transpose-free quasi-minimal residual
VIE	:	Volume integral equation
YUKC-CFIE	:	Yukawa-Calderon preconditioned classical CFIE
YUK-EFIE	:	Yukawa-Calderon preconditioned EFIE
YUKM-CFIE	:	Yukawa-Calderon Preconditioned mixed CFIE

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1. INTRODUCTION

Effects of electromagnetic (EM) fields have to be considered while designing electronic devices within various industries such as automotive, aerospace, naval, communication industries, and consumer products. A plenty of applications in the fields of antenna design [1], visibility of the aerial vehicles and naval vessels [2], EM compatibility and immunity (EMC/EMI) [3], investigation of the lightning phenomena on the airborne vehicles or ground structures [4], [5], and interaction with organic tissues involve scattering or radiation of EM waves. One of the most common EM problem is the radio detection and ranging (radar). Especially, after the Second World War, use of radar systems on the warfare increased greatly [2]. Radar systems are not only used for military purposes but also has found its use into civil life, e.g. speed radars, surveillance radar [6]. Visibility of the objects on radar is defined as radar cross section (RCS). RCS of an object can be determined by measurement, which is costly and time consuming.

Simulating the EM problems is a tedious but extremely crucial task. As use of the EM frequency spectrum increases, the design process becomes more complex [7]. Accurate simulation of EM problems reduces measurement and production costs, and usually consumes less time. Simulation of EM problems requires solving Maxwell's equations with the appropriate formulation and boundary conditions. Analytical solutions to EM problems are limited and only available for a few number of objects, such as a sphere [8]. Since analytical solutions of EM problems involving complex arbitrary structures are not possible, simulating EM problems numerically using computers has great importance. Before development of digital computers, the analysis of EM problems using computers were very limited. Once the computation power started to increase tremendously, researchers, engineers, and scientists immediately began using them to simulate EM problems that could not be solved analytically.

Parallel to the development of the fast computers, many different computational EM (CEM) methods, which are applicable to wide range of EM problems, are also developed. These methods can be classified in two groups:

- i) Full-wave methods,
- ii) Asymptotic techniques.

Full-wave methods are generally used at the low-frequency region and asymptotic techniques are generally used at the high-frequency region. Some of the most important and well known full-wave methods are, Finite Difference Time Domain (FDTD) [9], Finite Element Method (FEM) [10], Method of Moments (MoM) [11], [12], and Marching on-in-Time (MOT) [13]. Some of the most important asymptotic techniques are: Geometrical Theory of Diffraction (GTD) [14], Physical Optics (PO) [15], Physical Theory of Diffraction (PTD) [16], and Shooting and Bouncing Ray (SBR) method [17].

Method of Moments (MoM) is a full-wave method that transforms linear integro-differential equations into matrix system of linear equations and is a popular choice for analyzing time-harmonic and transient EM interactions with scatterers residing in an unbounded background [11], [12]. Perfect electrically conducting (PEC) and piecewise homogeneous objects can be analyzed by MoM solution of surface integral equations (SIEs) [18] and inhomogeneous objects can be analyzed by MoM solution of volume integral equations (VIEs) [19]. MoM solution of integral equations has some advantages over the differential based methods like FDTD and FEM: (i) in FDTD and FEM, equations are solved for the electric and magnetic fields for the entire computational domain, whereas in MoM, instead of entire computational domain, only the surface and volume of the objects for SIE and VIE, respectively, are discretized. (ii) If the scatterer resides in an unbounded medium as in analysis of RCS or antenna radiation, FDTD and FEM require truncation of the computational domain with absorbing boundary conditions (ABC) or perfectly matched layer (PML) [20], [21]. However integral equation formulations naturally satisfy the radiation condition, i.e. the boundary condition at the infinity, and don't require any artificial truncation, which makes the MoM based integral equation solvers suitable for RCS or antenna design/analysis problems.

PEC objects residing in an unbounded medium can be analyzed using the MoM solution of different SIE formulations, e.g. current-charge integral equation [22], however frequency-domain electric field integral equation (EFIE) and frequency-domain magnetic field integral equation (MFIE) are widely used [23]. Both EFIE and MFIE have their advantages and disadvantages:

- i) EFIE can be used in the analysis of both open and closed scatterers, on the other hand, MFIE is only applicable to the closed scatterers, preferably smooth surfaces.
- ii) EFIE is a first kind Fredholm integral equation while MFIE is a second kind. First kind Fredholm integral equations tend to produce a matrix with a high condition number¹, while second kind Fredholm integral equations tend to produce well-conditioned matrix with a low condition number.
- iii) It is well known that the MoM solutions of the EFIE and MFIE is corrupted for closed scatterers when they are solved for the frequencies that coincides with the resonance modes of the scatterer, phenomena named interior resonance problem [24]. It is also known that this interior resonance problem can be remedied with the use of a linear combination of the EFIE and MFIE, i.e. combined field integral equation (CFIE) [25]. CFIE is also only limited to closed scatterers, due to the limitation of the MFIE.

The MoM solution of the EFIE and MFIE, requires the expansion of the unknown current density with a set of basis functions that belongs to divergence conforming function space due to the equation of continuity [26]. In other words, the domain of the integral operator should be discretized by divergence conforming basis functions. Among the different variety of basis functions, Rao-Wilton-Glisson (RWG) functions [27], which are almost became the norm for the discretization hitherto for evaluating scattering from arbitrary objects, are widely used by many researchers due to its robustness and ease to integrate. The expanded current density is then substituted into EFIE and MFIE. To obtain a matrix system, resulting equations are tested with suitable set of testing functions, which discretizes the range of the integral operator. There are different strategies to choose the testing functions. Historically testing functions are chosen same as the basis functions, which is called Galerkin testing [12]. The aim on Galerkin testing is to obtain a well-conditioned matrix, i.e. the matrix has a low condition number, in other words, to obtain a solvable matrix. For this purpose, EFIE is tested with curl conforming (or rotated) RWG functions and MFIE is tested with divergence conforming RWG functions. These discretization schemes are called

¹ The condition number of a matrix is the ratio of the largest and smallest singular values of the matrix.

classical EFIE (C-EFIE) and classical MFIE (C-MFIE). However it is experienced that the MoM solution of C-EFIE provides more accurate results than MoM solution of C-MFIE, even MoM matrix for C-MFIE is well-conditioned compared to the MoM matrix for C-EFIE. It is also shown that the low-frequency behavior of the charge density is retained in MoM solution of the C-EFIE [28] but this behavior does not supported by the C-MFIE [28]–[30].

The problem of the MFIE is investigated by many researchers [26], [31], [32]. Recently, in [31] it is shown that testing with divergence conforming RWG functions does not satisfy the conforming-testing criterion and it is suggested that testing MFIE with curl conforming (rotated) Buffa-Christiansen-Chen-Wilton (BCCW) functions [33], [34] successfully retains the low-frequency behavior of the charge density [30], satisfies the conforming testing criterion, yields a well-conditioned MoM matrix, and provides accurate solutions compared to C-MFIE. This scheme is named mixed discretization of the MFIE (M-MFIE) because the unknown current density is expanded using RWG functions, but the resulting equation is tested with BCCW functions.

As a result, the choice of testing function to discretize the integral equations' range space has two important testing criteria:

- i) Well-tested criterion: the resulting MoM matrix should have low condition number (well-conditioned) enough to be solved.
- ii) Conforming-tested criterion: The testing function should be selected in the dual of the range space of the integral operator [35], [36].

The conforming-tested criterion is satisfied for C-EFIE since the range space of the EFIE operator is divergence conforming function space and the testing is performed by curl conforming RWG functions. However the range space of the MFIE operator belongs to divergence conforming function space, and in C-MFIE the testing is performed by divergence conforming RWG functions. This explains the inaccuracy problem of C-MFIE [37]–[39]. Testing with curl conforming RWG functions does not provide a well-conditioned MoM matrix for MFIE, contrary the resulting MoM matrix is singular [26], which means the well-tested criterion is not satisfied. On the other hand, M-MFIE satisfies the both criteria and provides accurate results.

The well-tested criterion depends on the kind of the equation and the choice of the testing functions. Since EFIE is a first kind Fredholm integral equation, even it is conforming- and well-tested, the resulting MoM matrix will not be as well-conditioned as M-MFIE or C-MFIE, because MFIE is a second kind Fredholm integral equation. Consequently solution of C-EFIE with an iterative solver converges very slowly. This problem of EFIE can be eliminated with the use of Calderon identities [35] and EFIE can be transformed into second kind integral equation, having advantage of the self-regularization properties of the EFIE operator [40]–[43]. This scheme is named Calderon multiplicative preconditioned EFIE (CMP-EFIE), where the EFIE operator is applied to the EFIE itself as a preconditioner. The application of CMP-EFIE requires the mapping of the discretized range space of EFIE to the domain of the EFIE. This mapping is performed by inverting the Gram matrix. However because of the usage of divergence conforming and curl conforming RWG functions as basis and testing functions on the discretization of C-EFIE, the resulting Gram matrix is singular, therefore construction of MoM matrices for CMP-EFIE with only RWG functions is not feasible [43]. Instead, keeping C-EFIE same and discretizing the second EFIE operator using BCCW functions, results in a well-conditioned Gram matrix that maps the curl conforming RWG function space to divergence conforming BCCW function space. Similar to CMP-EFIE, another approach is using Yukawa-Calderon preconditioned EFIE (YUK-EFIE) [44], where the EFIE operator used as preconditioner is obtained for imaginary wave number. The resulting MoM matrices for CMP-EFIE and YUK-EFIE have a very low condition number and converges rapidly using an iterative solver similar to MFIE.

Even if the conditioning of EFIE and the accuracy problem of MFIE are improved, interior resonance problem, explained above, corrupts the solution of them at the resonance frequencies of the scatterer.

1.1. Purpose and Content of the Thesis

In this thesis, combined field integral equation (CFIE) formulations and combination strategies are investigated to obtain interior resonance free MoM solution for the analysis of a closed PEC object. CFIE, as mentioned above, is a linear combination of EFIE and MFIE with a combination factor, which determines the weight of the contribution from the EFIE and MFIE and is selected between 0 and 1.

Combining C-EFIE and C-MFIE to obtain the classical discretization of the CFIE (C-CFIE) remedies the interior resonance problem [25]. However the accuracy of the solution depends on the combination factor because non-conforming-tested C-MFIE limits the accuracy of the solution. If the combination is weighted to C-EFIE to improve the accuracy than ill-conditioned nature of the MoM matrix for C-EFIE affects the solution. Hence combination factor has a key role in the overall accuracy of the solution vs. the iterative solver's convergence.

It is suggested to combine C-EFIE and M-MFIE, since both of them provide accurate solutions. This scheme is named as mixed discretization of the CFIE (M-CFIE) [45], since EFIE is tested with RWG and MFIE is tested with BCCW functions. This scheme ensures the conforming-tested criteria of both EFIE and MFIE, hence the accuracy is improved greatly. However it is shown that, this combination does not remedy the interior resonance problem because the range spaces of the both equations are not in the same function space, this non-conforming combination still supports the interior resonance problem as showed in numerical examples section.

A conforming and interior resonance free combination of C-EFIE and M-MFIE can be achieved by projecting the range spaces onto each other [46], [47], called range space projected CFIE (RSP-CFIE), i.e. projecting BCCW function space used to discretize the range space of M-MFIE onto the RWG function space used to discretize the range of C-EFIE, vice versa. This scheme ensures the conforming-tested criterion for both C-EFIE and M-MFIE thus the accuracy is improved greatly and does not depend on the combination factor. In addition range space projection ensures the conformity of the range spaces, therefore interior resonance problem is remedied.

In RSP-CFIE, conditioning of the C-EFIE can affect the overall conditioning of the MoM matrix of RSP-CFIE, therefore combining CMP-EFIE and MFIE can be considered. However, it is emphasized in [44] that CMP-EFIE shares the same interior resonance frequencies with MFIE, therefore combination of CMP-EFIE and MFIE does not remove the interior resonance problem. In this case, combination YUK-EFIE and MFIE is suggested for interior resonance free solutions. There are two possible ways of combination. First is to combine YUK-EFIE with C-MFIE as in [48] obtaining YUKC-CFIE. Second is to combine YUK-EFIE with M-MFIE as in [44]. In the work of Andriulli et al. [44] it is shown that YUKM-CFIE enhances the accuracy of the YUKC-CFIE even further.

In this thesis, performances of the CFIE formulations mentioned above are investigated in terms of accuracy and conditioning around the interior resonance frequencies, and compared with the EFIE and MFIE.

1.2. Thesis Organization

In the light of information gained during this work, this thesis is organized as follows: Section 2 gives the formulation in general form. Specifically, in Section 2.1 MoM formulation is given briefly, and in Section 2.2 preliminary definitions are defined for the problem of scattering from a PEC object. In Section 2.3, formulations of MFIE for a PEC object is given and under subsections 2.3.1 and 2.3.2 discretization schemes for MFIE are given. In Section 2.4, EFIE for a PEC object is given and under the subsections 2.4.1, 2.4.2, and 2.4.3 discretization schemes for EFIE are given. In section 2.5 general form of the CFIE is given. Different combination strategies for CFIE are presented in sections 2.5.1, 2.5.2, 2.5.3, and, 2.5.4. Section 3, presents the numerical examples for different scatterers. In Section 4, conclusive remarks are discussed. In Appendix A publications, which are outcome of this thesis are listed and in Appendix B, basis functions are explained.

2. FORMULATION

In this section, MoM formulation will be revisited for the sake of completion. Then MFIE operator is given and selection of testing functions, namely discretization schemes of C-MFIE and M-MFIE is presented. Then EFIE operator is given and selection of testing functions, namely discretization schemes of C-EFIE, CMP-EFIE, and YUK-EFIE is presented. After the discretization schemes of EFIE and MFIE, combination strategies of CFIE, C-CFIE, M-CFIE, RSP-CFIE, YUKC-CFIE, and YUKM-CFIE is given.

2.1. Method of Moments

Consider an inhomogeneous, linear, time-invariant equation [12]

$$\mathcal{L}\{f(x)\} = g(x) \quad (2.1)$$

where $\mathcal{L}\{\cdot\}$ is the integro-differential operator, $f(x)$ is the unknown function to be solved, and $g(x)$ is a known function. Sum of weighted basis functions $b_n(x)$ are used to approximate $f(x)$

$$f(x) = \sum_{n=1}^N I_n b_n(x) \quad (2.2)$$

where I_n is the unknown coefficients to be determined and N is the number of basis functions. Since $\mathcal{L}\{\cdot\}$ is a linear operator, $\mathcal{L}\{\cdot\}$ can be distributed as the sum of basis functions,

$$\sum_{n=1}^N I_n \mathcal{L}\{b_n(x)\} = g(x). \quad (2.3)$$

Testing (2.3) with testing functions $w_m(x)$, $m = 1, 2, \dots, N$,

$$\sum_{n=1}^N I_n \langle w_m(x), \mathcal{L}\{b_n(x)\} \rangle = \langle w_m(x), g(x) \rangle \quad (2.4)$$

where testing operation can be defined as follows,

$$\langle a(x), b(x) \rangle = \int a(x)b(x)dx, \quad (2.5)$$

results in a matrix equation in the following form:

$$\mathbf{V} = \mathbf{Z}\mathbf{I} \quad (2.6)$$

Here elements of unknown coefficient vector $\{\mathbf{I}\}_n = I_n$, known function vector $\{\mathbf{V}\}_m = \langle w_m(x), g(x) \rangle$, and the MoM matrix $\mathbf{Z}$ are,

$$\{\mathbf{Z}\}_{m,n} = \langle w_m(x), \mathcal{L}\{b_n(x)\} \rangle. \quad (2.7)$$

As a result once the matrix equation in (2.6) is solved, an approximate solution to $f(x)$ can be obtained.

2.2. Preliminary Definitions

In this subsection scattering problem is defined and preliminary definitions are presented. A scattering problem is depicted in Figure 2.1, where a time-harmonic incident wave with the electric and magnetic fields of $\mathbf{E}^{\text{inc}}(\mathbf{r})$ and $\mathbf{H}^{\text{inc}}(\mathbf{r})$, respectively, excites the closed PEC scatterer residing in a unbounded homogeneous background medium with the electric permittivity ε and the magnetic permeability μ . As a result of the excitation, the surface current density $\mathbf{J}(\mathbf{r})$ is induced on the scatterer's surface S . In return, $\mathbf{J}(\mathbf{r})$ radiates and generates the scattered electric and magnetic fields, $\mathbf{E}^{\text{sca}}(\mathbf{r})$ and $\mathbf{H}^{\text{sca}}(\mathbf{r})$, respectively. In this thesis, formulation is based on the time dependence of $e^{j\omega t}$, where ω and t denote the angular frequency and time, respectively, and j is the imaginary unit.

The total electric and magnetic fields, i.e. the sum of incident and scattered fields, should satisfy the boundary conditions and the scattering problem can be represented by an integral equation [49], as mentioned in Section 1. In this thesis, EFIE and MFIE are considered and the MoM is used to solve them. In MoM solution of EFIE and MFIE, S is discretized with triangular patches and the domain of the integral equation is expanded using suitable basis functions. In this case, the domain

of the EFIE and MFIE is the surface current density $\mathbf{J}(\mathbf{r})$, which is expanded with divergence conforming Rao-Wilton-Glisson (RWG) functions [27]:

$$\mathbf{J}(\mathbf{r}) = \sum_{n=1}^N I_n \mathbf{f}_n(\mathbf{r}) \quad (2.8)$$

where, $\mathbf{f}_n(\mathbf{r})$ denote the RWG functions defined on a pair of triangles (see Appendix B), I_n are the unknown coefficients. Assuming that $\mathcal{O}\{\cdot\}$ represents the integral equation operator, the expansion in (2.8) is substituted in the integral equation operator, then the resulting equation is tested with testing functions $\mathbf{t}_m(\mathbf{r})$.

$$\langle \mathbf{t}_m(\mathbf{r}), \mathcal{O}\{\mathbf{J}(\mathbf{r})\} \rangle = \sum_{n=1}^N I_n \langle \mathbf{t}_m(\mathbf{r}), \mathcal{O}\{\mathbf{f}_n(\mathbf{r})\} \rangle \quad (2.9)$$

As explained in previous subsection, the summation in (2.9) can be written in a matrix form $\mathbf{Z}_{t,f}^{\mathcal{O}} \mathbf{I}$, where the superscript denotes the equation type and the subscript denotes the testing and basis functions. In (2.9), $\langle \mathbf{a}(\mathbf{r}), \mathbf{b}(\mathbf{r}) \rangle = \int \mathbf{a}(\mathbf{r}) \cdot \mathbf{b}(\mathbf{r}) d\mathbf{r}$.

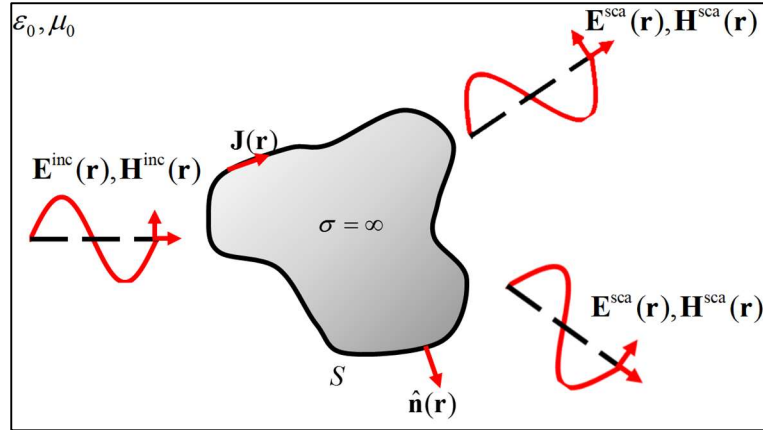


Figure 2.1: Definition of the scattering problem.

The testing functions $\mathbf{t}_m(\mathbf{r})$ can be chosen as curl or divergence conforming RWG ($\mathbf{f}_m(\mathbf{r})$) or Buffa-Christiansen-Chen-Wilton (BCCW) ($\mathbf{g}_m(\mathbf{r})$) functions [33], [34], depending on the well- and conforming-tested criteria explained in Section 1. As is mentioned before, the RWG functions are defined on a pair of triangle that share an edge. These triangles are same as the triangles used to discretize the scatterer's surface S . However, BCCW functions are defined on sub-triangles on the initial mesh (see

Appendix B) and they cannot be obtained in a straight-forward manner. A useful and elegant method to obtain the MoM matrix elements for BCCW functions is presented in [43]. Specifically, the every triangle in the initial mesh that discretizes S is subdivided to 6 sub-triangles, i.e. called barycentrically refined mesh, as shown in Appendix B. Then RWG functions are defined on pair of the barycentrically refined triangles, called barycentrically refined RWG (bRWG) functions and denoted by $\mathbf{f}_n^b(\mathbf{r})$. It is possible to obtain RWG or BCCW functions with the linear combination of bRWG functions. $\mathbf{P}$ and $\mathbf{R}$ operators are defined to map the bRWG function space to BCCW and RWG function spaces, respectively. The matrix elements for BCCW functions are first obtained for bRWG functions then mapped to the BCCW and RWG function spaces.

2.3. Magnetic Field Integral Equation

Considering the scattering problem defined in Figure 2.1, for a smooth surface, magnetic field integral equation (MFIE) is given as [23], [50],

$$\hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{H}^{\text{inc}}(\mathbf{r}) = \frac{\mathbf{J}(\mathbf{r})}{2} - \mathcal{K}\{\mathbf{J}(\mathbf{r})\}, \quad (2.10)$$

where $\mathcal{K}\{\cdot\}$ is the MFIE operator,

$$\mathcal{K}\{\mathbf{J}(\mathbf{r})\} = \hat{\mathbf{n}}(\mathbf{r}) \times \nabla \times \int_S \mathbf{J}(\mathbf{r}') \frac{e^{-jkR}}{4\pi R} d\mathbf{r}', \quad (2.11)$$

$R = |\mathbf{r} - \mathbf{r}'|$ is the distance between the source point $\mathbf{r}'$ and observation point $\mathbf{r}$, $k = \omega\sqrt{\epsilon\mu}$ is the wave number, and $\hat{\mathbf{n}}(\mathbf{r})$ denotes the unit normal vector pointing outwards S .

2.3.1. Classical Discretization of MFIE

In classical discretization of MFIE (C-MFIE), the current density is expanded with divergence conforming RWG functions as in (2.8) and MFIE in (2.10) is tested with divergence conforming RWG functions ($\mathbf{t}_m(\mathbf{r}) = \mathbf{f}_m(\mathbf{r})$). The MoM matrix for C-MFIE can be obtained as

$$\mathbf{Z}_{\mathbf{f},\mathbf{f}}^{\mathbf{H}}\mathbf{I} = \mathbf{V}_{\mathbf{f}}^{\mathbf{H}} \quad (2.12)$$

where the elements of the unknown coefficients $\{\mathbf{I}\}_n = I_n$, tested right hand side vector

$$\{\mathbf{V}_{\mathbf{f}}^{\mathbf{H}}\}_m = \langle \mathbf{f}_m(\mathbf{r}), \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{H}^{\text{inc}}(\mathbf{r}) \rangle, \quad (2.13)$$

and the MoM matrix $\mathbf{Z}_{\mathbf{f},\mathbf{f}}^{\mathbf{H}}$

$$\{\mathbf{Z}_{\mathbf{f},\mathbf{f}}^{\mathbf{H}}\}_{m,n} = \frac{1}{2} \langle \mathbf{f}_m(\mathbf{r}), \mathbf{f}_n(\mathbf{r}) \rangle - \langle \mathbf{f}_m(\mathbf{r}), \mathcal{K} \{ \mathbf{f}_n(\mathbf{r}) \} \rangle. \quad (2.14)$$

Note that in C-MFIE the resulting MoM matrix $\mathbf{Z}_{\mathbf{f},\mathbf{f}}^{\mathbf{H}}$ is well-conditioned, since C-MFIE satisfies the well-tested criterion but the testing function does not belong to the dual space of the range space of MFIE, therefore conforming-tested criterion is not satisfied, hence the accuracy is still significantly lower compared to C-EFIE and M-MFIE [38].

2.3.2. Mixed Discretization of MFIE

In mixed discretization of MFIE (M-MFIE), the current density is expanded with divergence conforming RWG functions as in (2.8) and MFIE in (2.10) is tested with curl conforming BCCW functions ($\mathbf{t}_m(\mathbf{r}) = \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{g}_m(\mathbf{r})$). As explained in the Section 1, curl conforming BCCW functions are suggested to use as testing functions to satisfy the well- and conforming-tested criteria. In this selection, curl conforming BCCW functions belong to the dual space of range of the operator MFIE operator. This scheme is named as mixed discretization of MFIE (M-MFIE) due to the difference in the selection of basis and testing functions. The MoM matrix for M-MFIE can be obtained as

$$\mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{f}}^{\mathbf{H}} \mathbf{I} = \mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{g}}^{\mathbf{H}} \quad (2.15)$$

where the elements of the unknown coefficients $\{\mathbf{I}\}_n = I_n$, tested right hand side vector

$$\{\mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{g}}^{\mathbf{H}}\}_m = \langle \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{g}_m(\mathbf{r}), \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{H}^{\text{inc}}(\mathbf{r}) \rangle, \quad (2.16)$$

and the MoM matrix $\mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{f}}^{\text{H}}$

$$\left\{ \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{f}}^{\text{H}} \right\}_{m,n} = \frac{1}{2} \left\langle \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{g}_m(\mathbf{r}), \mathbf{f}_n(\mathbf{r}) \right\rangle - \left\langle \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{g}_m(\mathbf{r}), \mathcal{K} \{ \mathbf{f}_n(\mathbf{r}) \} \right\rangle. \quad (2.17)$$

As mentioned above in M-MFIE the resulting MoM matrix $\mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{f}}^{\text{H}}$ is well-conditioned and testing functions belong to dual of the range space of MFIE. As a result, M-MFIE satisfies the well- and conforming-tested criteria and provides more accurate results than C-MFIE.

Since M-MFIE involves the use of BCCW functions, the MoM matrix $\mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{f}}^{\text{H}}$ and the right hand side vector $\mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{g}}^{\text{H}}$ in (2.15) are obtained using bRWG functions:

$$\mathbf{P} \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}^{\text{b}}, \mathbf{f}^{\text{b}}}^{\text{H}} \mathbf{R}^{\text{T}} \mathbf{I} = \mathbf{P} \mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{f}^{\text{b}}}^{\text{H}} \quad (2.18)$$

where $\hat{\mathbf{n}} \times \mathbf{f}^{\text{b}}$ and $\mathbf{f}^{\text{b}}$ denote the curl and divergence conforming bRWG functions, $\mathbf{P}$ and $\mathbf{R}$ are defined in Section 2.2, and “T” denotes the transpose operation.

2.4. Electric Field Integral Equation

Similar to MFIE, considering the scattering problem defined in the Figure 2.1, electric field integral equation (EFIE) is given as

$$\hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{E}^{\text{inc}}(\mathbf{r}) = \mathcal{T} \{ \mathbf{J}(\mathbf{r}) \} \quad (2.19)$$

where $\mathcal{T} \{ \cdot \}$ is the EFIE operator

$$\begin{aligned} \mathcal{T} \{ \mathbf{J}(\mathbf{r}) \} = & \frac{j\omega\mu}{4\pi} \hat{\mathbf{n}}(\mathbf{r}) \times \int_S \mathbf{J}(\mathbf{r}') \frac{e^{-jkR}}{R} d\mathbf{r}' \\ & - \frac{1}{4\pi j\omega\epsilon} \hat{\mathbf{n}}(\mathbf{r}) \times \nabla \int_S [\nabla' \cdot \mathbf{J}(\mathbf{r}')] \frac{e^{-jkR}}{R} d\mathbf{r}' \end{aligned} \quad (2.20)$$

2.4.1. Classical Discretization of EFIE

In classical discretization of EFIE (C-EFIE), the current density is expanded with divergence conforming RWG functions as in (2.8) and EFIE in (2.19) is tested with

curl conforming RWG functions ($\mathbf{t}_m(\mathbf{r}) = \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{f}_m(\mathbf{r})$). The MoM matrix for C-EFIE can be obtained as

$$\mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{f}}^{\mathbf{E}} \mathbf{I} = \mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{f}}^{\mathbf{E}}, \quad (2.21)$$

where the elements of the unknown coefficients $\{\mathbf{I}\}_n = I_n$, tested right hand side vector

$$\{\mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{f}}^{\mathbf{E}}\}_m = \langle \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{f}_m(\mathbf{r}), \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{E}^{\text{inc}}(\mathbf{r}) \rangle, \quad (2.22)$$

and the MoM matrix $\mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{f}}^{\mathbf{E}}$ are,

$$\{\mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{f}}^{\mathbf{E}}\}_{m,n} = \langle \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{f}_m(\mathbf{r}), \mathcal{T} \{\mathbf{f}_n(\mathbf{r})\} \rangle. \quad (2.23)$$

Note that in C-EFIE the conforming-tested criterion is satisfied and solution of C-EFIE provides accurate results. However, the resulting MoM matrix $\mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{f}}^{\mathbf{E}}$ is not well-conditioned and its iterative solution usually converges very slowly. Therefore improving the conditioning of $\mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{f}}^{\mathbf{E}}$ is an ongoing research [40]–[42]. One of the most effective technique is Calderon multiplicative preconditioned EFIE (CMP-EFIE) [43], which is explained next.

2.4.2. Calderon Multiplicative Preconditioned EFIE

The conditioning of MoM matrix for EFIE can be improved with the help of Calderon identities [35] and $\mathcal{T}\{\cdot\}$ operator can be transformed into a second kind Fredholm integral equation [42], as follows

$$\begin{aligned} \mathcal{T}^2 \{\mathbf{J}(\mathbf{r})\} &= \mathcal{T} \left\{ \mathcal{T} \{\mathbf{J}(\mathbf{r})\} \right\} \\ &= -\frac{\mathbf{J}(\mathbf{r})}{4} + \mathcal{K}^2 \{\mathbf{J}(\mathbf{r})\} \\ &= \left(\mathcal{K} \{\mathbf{J}(\mathbf{r})\} - \frac{\mathbf{J}(\mathbf{r})}{2} \right) \left(\mathcal{K} \{\mathbf{J}(\mathbf{r})\} + \frac{\mathbf{J}(\mathbf{r})}{2} \right). \end{aligned} \quad (2.24)$$

Equation (2.24) implies that, if $\mathcal{T}\{\cdot\}$ operator applied to the EFIE in (2.19),

$$\mathcal{T} \{\hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{E}^{\text{inc}}(\mathbf{r})\} = \mathcal{T}^2 \{\mathbf{J}(\mathbf{r})\} \quad (2.25)$$

then $\mathcal{T}\{\cdot\}$ preconditions itself and the resulting MoM matrix is well-conditioned [43].

The discretization of $\mathcal{T}^2\{\cdot\}$ results in two MoM matrix for EFIE and a Gram matrix that's linking the discretized range space of the first MoM matrix to the domain of the second MoM matrix of EFIE ($\mathcal{T}^2 \approx \mathbf{Z}^E \mathbf{G}^{-1} \mathbf{Z}^E$). However, direct discretization of the Calderon multiplicative preconditioned EFIE (CMP-EFIE) using RWG functions, as in C-EFIE, is not feasible [43], since in this case the Gram matrix is singular. Instead, one of EFIE operators is discretized with BCCW functions, the other one is discretized with RWG functions, and resulting Gram matrix linking divergence conforming BCCW and curl conforming RWG functions, $\mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{g}}$, is well conditioned. Hence discretization of (2.25) is obtained as follows,

$$\mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{g}}^E \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{g}}^{-1} \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{f}}^E \mathbf{I} = \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{g}}^E \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{g}}^{-1} \mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{f}}^E \quad (2.26)$$

where the elements of the unknown coefficients $\{\mathbf{I}\}_n = I_n$, tested right hand side vector $\mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{f}}^E$ and the MoM matrix $\mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{f}}^E$ are given in (2.22) and (2.23), the MoM matrix $\mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{g}}^E$ are,

$$\left\{ \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{g}}^E \right\}_{m,n} = \left\langle \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{g}_m(\mathbf{r}), \mathcal{T} \left\{ \mathbf{g}_n(\mathbf{r}) \right\} \right\rangle, \quad (2.27)$$

and the Gram matrix $\mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{g}}$ is

$$\left\{ \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{g}} \right\}_{m,n} = \left\langle \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{f}_m(\mathbf{r}), \mathbf{g}_n(\mathbf{r}) \right\rangle. \quad (2.28)$$

Note that CMP-EFIE satisfies well- and conforming-tested criteria, however its implementation requires two EFIE matrices. In [43], it is shown that the CMP-EFIE can be obtained using a single MoM matrix for the bRWG functions, which is suitable for iterative solvers:

$$\left(\mathbf{P} \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}^b, \mathbf{f}^b}^E \mathbf{P}^T \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{g}}^{-1} \mathbf{R} \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}^b, \mathbf{f}^b}^E \mathbf{R}^T \right) \mathbf{I} = \mathbf{P} \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}^b, \mathbf{f}^b}^E \mathbf{P}^T \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{g}}^{-1} \mathbf{R} \mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{f}^b}^E \quad (2.29)$$

and the Gram matrix can be obtained by

$$\mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{g}} = \mathbf{R} \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f}^b, \mathbf{f}^b} \mathbf{P}^T. \quad (2.30)$$

where $\mathbf{P}$ and $\mathbf{R}$ are defined in Section 2.2, and “T” denotes the transpose operation.

2.4.3. Yukawa-Calderon Preconditioned EFIE

Another suitable way to precondition $\mathcal{T}\{\cdot\}$ operator is using the localized $\mathcal{T}^{\text{loc}}\{\cdot\}$ operator [44], [48]:

$$\begin{aligned} \mathcal{T}^{\text{loc}}\{\mathbf{J}(\mathbf{r})\} = & -\frac{\omega\mu}{4\pi} \int_S \mathbf{J}(\mathbf{r}') \frac{e^{-kR}}{R} d\mathbf{r}' \\ & + \frac{1}{4\pi\omega\epsilon} \nabla \int_S [\nabla' \cdot \mathbf{J}(\mathbf{r}')] \frac{e^{-kR}}{R} d\mathbf{r}'. \end{aligned} \quad (2.31)$$

$\mathcal{T}^{\text{loc}}\{\cdot\}$ can be obtained by replacing the wave number in (2.20) with an imaginary wave number $k \rightarrow -jk$ ($\omega \rightarrow -j\omega$). It can be seen in (2.31) that, excluding $\mathbf{J}(\mathbf{r})$, $\mathcal{T}^{\text{loc}}\{\cdot\}$ is purely real. Yukawa-Calderon preconditioned EFIE (YUK-EFIE) can be obtained similar to CMP-EFIE, but discretization of YUK-EFIE requires two different matrices, since $\mathcal{T}^{\text{loc}}\{\cdot\}$ produces a completely different matrix than $\mathcal{T}^2\{\cdot\}$ and $\mathcal{T}\{\cdot\}$. The discretization of YUK-EFIE is obtained as

$$\mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{g}}^{\text{loc}} \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{g}}^{-1} \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{f}}^{\text{E}} \mathbf{I} = \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{g}}^{\text{loc}} \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{g}}^{-1} \mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{f}}^{\text{E}} \quad (2.32)$$

where the elements of the unknown coefficients $\{\mathbf{I}\}_n = I_n$, tested right hand side vector $\mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{f}}^{\text{E}}$, the MoM matrix $\mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{f}}^{\text{E}}$, and the Gram matrix $\mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{g}}$ are given in (2.22), (2.23), (2.28), respectively, and the MoM matrix $\mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{g}}^{\text{loc}}$ are,

$$\left\{ \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{g}}^{\text{loc}} \right\}_{m,n} = \left\langle \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{g}_m(\mathbf{r}), \mathcal{T}^{\text{loc}}\{\mathbf{g}_n(\mathbf{r})\} \right\rangle. \quad (2.33)$$

Note that YUK-EFIE also satisfies well- and conforming-tested criteria, however in Section 3, it is shown that CMP-EFIE provides more accurate results than YUK-EFIE and also YUK-EFIE requires two MoM matrices which increases the memory consumption. YUK-EFIE is more suitable when it is combined with MFIE as shown in Section 2.5.4. Similar to CMP-EFIE, YUK-EFIE can be discretized with the MoM matrices for the bRWG functions:

$$\left(\mathbf{P} \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}^b, \mathbf{f}^b}^{\text{loc}} \mathbf{P}^T \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{g}}^{-1} \mathbf{R} \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}^b, \mathbf{f}^b}^{\text{E}} \mathbf{R}^T \right) \mathbf{I} = \mathbf{P} \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}^b, \mathbf{f}^b}^{\text{loc}} \mathbf{P}^T \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{g}}^{-1} \mathbf{R} \mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{f}^b}^{\text{E}} \quad (2.34)$$

where $\mathbf{P}$ and $\mathbf{R}$ are defined in Appendix B and C, respectively, and “T” denotes the transpose operation.

2.5. Combined Field Integral Equation

Linear combination of EFIE and MFIE with a combination factor α , called combined field integral equation (CFIE), is proven to eliminate the interior resonance problem [24]. α decides the weight of the contribution from EFIE and MFIE. In the following subsections, different combination strategies using the EFIE and MFIE formulations given in Section 2.3 and 2.4 are presented.

2.5.1. Classical Discretization of CFIE

Classical discretization of CFIE (C-CFIE) is obtained by combining C-EFIE and C-MFIE as follows:

$$\left[\alpha \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{f}}^{\text{E}} + \eta(1 - \alpha) \mathbf{Z}_{\mathbf{f}, \mathbf{f}}^{\text{H}} \right] \mathbf{I} = \alpha \mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{f}}^{\text{E}} + \eta(1 - \alpha) \mathbf{V}_{\mathbf{f}}^{\text{H}} \quad (2.35)$$

where $\eta = \sqrt{\mu / \varepsilon}$ is the intrinsic impedance of the medium and used to unify the units of MFIE and EFIE. The classical discretization approach eliminates the effect of the interior resonances in the solution [25]. But accuracy of MFIE affects the accuracy of the solution of C-CFIE, since the selection of testing function doesn't meet the conforming-tested criterion for MFIE. Therefore selection of combination factor has a key role in accuracy of the solution versus the condition number of the resulting MoM matrix. In other words, accuracy of the solution of C-CFIE depends on the combination factor.

2.5.2. Mixed Discretization of CFIE

It is reasonable to combine C-EFIE and M-MFIE to improve the accuracy [51], since both C-EFIE and M-MFIE satisfy the conforming-tested criterion. This

combination is named mixed discretization of the CFIE (M-CFIE), because different testing functions used to test EFIE and MFIE. M-CFIE can be obtained as follows:

$$\left[\alpha \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{f}}^{\mathbf{E}} + \eta(1 - \alpha) \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{f}}^{\mathbf{H}} \right] \mathbf{I} = \alpha \mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{f}}^{\mathbf{E}} + \eta(1 - \alpha) \mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{g}}^{\mathbf{H}}. \quad (2.36)$$

This scheme, indeed, improves the accuracy greatly. However due to the selection of different testing functions, ranges of the discretized EFIE and MFIE belong to the different function spaces. Even though the accuracy is improved, M-CFIE still supports the interior resonances [46] and effect of the interior resonances can be seen in the solution as shown in Section 3.

2.5.3. Range Space Projected CFIE

As explained in the previous sections, C-CFIE remedies the interior resonance problem however does not produce accurate results. On the other hand M-CFIE improves the accuracy greatly but cannot totally eliminates the interior resonance problem, due to the difference in the testing functions. A conforming and interior resonance free combination of CFIE can be achieved by projecting range spaces onto each other [46], [47], [52], which is called range space projected CFIE (RSP-CFIE). There are two ways to project the range spaces:

i) The range space of M-MFIE can be projected onto the range space of C-EFIE, which is the RWG function space as,

$$\sum_{n=1}^N \left\{ \frac{\mathbf{f}_n(\mathbf{r})}{2} - \mathcal{K} \{ \mathbf{f}_n(\mathbf{r}) \} \right\} \{ \mathbf{I} \}_n \approx \sum_{m=1}^N \mathbf{f}_m(\mathbf{r}) \{ \mathbf{b} \}_m. \quad (2.37)$$

Testing the projected M-MFIE operator in (2.37) with curl conforming BCCW functions yields,

$$\mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{f}} \mathbf{b} = \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{f}}^{\mathbf{H}} \mathbf{I} \quad (2.38)$$

where the elements of the Gram matrix $\mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{f}}$ is given as

$$\{ \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{f}} \}_{m,n} = \langle \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{g}_m(\mathbf{r}), \mathbf{f}_n(\mathbf{r}) \rangle. \quad (2.39)$$

and $\mathbf{b}$ holds the coefficients for the range space projection. (2.38) can be rearranged as

$$\mathbf{b} = \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{f}}^{-1} \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{f}}^{\mathbf{H}} \mathbf{I}. \quad (2.40)$$

After similar projection is applied to the incident magnetic field, range space projected M-MFIE and C-EFIE can be combined to obtain range space projected CFIE (RSP-CFIE) as

$$\begin{aligned} -\alpha \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}}^{\mathbf{E}} \mathbf{I} + \eta(1-\alpha) \mathbf{G}_{\mathbf{f}, \mathbf{f}} \mathbf{b} &= -\alpha \mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{f}}^{\mathbf{E}} + \eta(1-\alpha) \mathbf{G}_{\mathbf{f}, \mathbf{f}} \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{f}}^{-1} \mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{g}}^{\mathbf{H}} \\ -\alpha \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{f}}^{\mathbf{E}} \mathbf{I} + \eta(1-\alpha) \mathbf{G}_{\mathbf{f}, \mathbf{f}} \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{f}}^{-1} \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{f}}^{\mathbf{H}} \mathbf{I} &= \\ \left[-\alpha \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{f}}^{\mathbf{E}} + \eta(1-\alpha) \mathbf{G}_{\mathbf{f}, \mathbf{f}} \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{f}}^{-1} \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{f}}^{\mathbf{H}} \right] \mathbf{I} &= \end{aligned} \quad (2.41)$$

where Gram matrix $\mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{g}, \mathbf{f}}$ appears because of the solution of $\mathbf{b}$ and Gram matrix $\mathbf{G}_{\mathbf{f}, \mathbf{f}}$ appears because of the testing of the projection expansion in (2.37). The elements of $\mathbf{G}_{\mathbf{f}, \mathbf{f}}$ is given as

$$\{\mathbf{G}_{\mathbf{f}, \mathbf{f}}\}_{m, n} = \langle \mathbf{f}_m(\mathbf{r}), \mathbf{f}_n(\mathbf{r}) \rangle. \quad (2.42)$$

ii) Alternatively, the range space of the C-EFIE can be projected to the range space of M-MFIE. Similarly as the first step, the range space of M-EFIE is projected onto the BCCW function space,

$$\sum_{n=1}^N \{ \mathcal{T} \{ \mathbf{f}_n(\mathbf{r}) \} \} \{ \mathbf{I} \}_n \approx \sum_{m=1}^N \mathbf{g}_m(\mathbf{r}) \{ \mathbf{c} \}_m. \quad (2.43)$$

Then testing the projected C-EFIE in (2.43) with curl conforming RWG functions yields,

$$\mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{g}} \mathbf{c} = \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{f}}^{\mathbf{E}} \mathbf{I} \quad (2.44)$$

where $\mathbf{c}$ holds the coefficients for the range space projection. (2.44) and can be rearranged as,

$$\mathbf{c} = \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{g}}^{-1} \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f}, \mathbf{f}}^{\mathbf{E}} \mathbf{I}. \quad (2.45)$$

After similar projection is applied to the incident electric field, range space projected C-EFIE and M-MFIE can be combined to obtain RSP-CFIE as,

$$\begin{aligned}
& -\alpha \mathbf{G}_{\mathbf{g},\mathbf{g}} \mathbf{c} + \eta(1-\alpha) \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g},\mathbf{f}}^H \mathbf{I} = -\alpha \mathbf{G}_{\mathbf{g},\mathbf{g}} \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f},\mathbf{g}}^{-1} \mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{f}}^E + \eta(1-\alpha) \mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{g}}^H \\
& -\alpha \mathbf{G}_{\mathbf{g},\mathbf{g}} \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f},\mathbf{g}}^{-1} \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f},\mathbf{f}}^E \mathbf{I} + \eta(1-\alpha) \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g},\mathbf{f}}^H \mathbf{I} = \\
& \left[-\alpha \mathbf{G}_{\mathbf{g},\mathbf{g}} \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f},\mathbf{g}}^{-1} \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f},\mathbf{f}}^E + \eta(1-\alpha) \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g},\mathbf{f}}^H \right] \mathbf{I} =
\end{aligned} \tag{2.46}$$

where Gram matrix $\mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f},\mathbf{g}}$ is defined in (2.28) and appears because of the solution of $\mathbf{c}$, the Gram matrix $\mathbf{G}_{\mathbf{g},\mathbf{g}}$ appears because of testing of the projection expansion in (2.43), and the elements of $\mathbf{G}_{\mathbf{g},\mathbf{g}}$ is given as

$$\{\mathbf{G}_{\mathbf{g},\mathbf{g}}\}_{m,n} = \langle \mathbf{g}_m(\mathbf{r}), \mathbf{g}_n(\mathbf{r}) \rangle. \tag{2.47}$$

This scheme ensures the conforming-tested criterion for both EFIE and MFIE, thus the accuracy improved greatly. In addition, range space projection ensures conformity of the range spaces of C-EFIE and M-MFIE, therefore the interior resonance problem is remedied as shown in Section 3.

2.5.4. Yukawa-Calderon Preconditioned CFIE

Another approach to obtain CFIE can be combining CMP-EFIE and MFIE to benefit well-conditioned behavior of both equations, but it is noted in [53] that second kind behavior of CMP-EFIE shares same interior resonance frequencies with MFIE, resulting combination might not eliminates the interior resonance problem. Therefore YUK-EFIE that doesn't share the same interior resonance frequencies with MFIE is a suitable choice for the combination.

There are two ways to obtain the Yukawa-Calderon preconditioned CFIE (YUK-CFIE). First is to combine YUK-EFIE and C-MFIE (YUKC-CFIE) as in [48], which yields

$$\alpha \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g},\mathbf{g}}^{\text{loc}} \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f},\mathbf{g}}^{-1} \mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{f}}^E + (1-\alpha) \eta \mathbf{V}_{\mathbf{f}}^H = \left(\alpha \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g},\mathbf{g}}^{\text{loc}} \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f},\mathbf{g}}^{-1} \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f},\mathbf{f}}^E + (1-\alpha) \eta \mathbf{Z}_{\mathbf{f},\mathbf{f}}^H \right) \mathbf{I}. \tag{2.48}$$

In [48], it is emphasized that YUKC-CFIE remedies the interior resonance problem as well as the misbehavior when using an iterative solver.

Second approach is to combine YUK-EFIE and M-MFIE (YUKM-CFIE) as in [44],

$$\alpha \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g},\mathbf{g}}^{\text{loc}} \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f},\mathbf{g}}^{-1} \mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{f}}^E + (1-\alpha) \eta \mathbf{V}_{\hat{\mathbf{n}} \times \mathbf{g}}^H = \left(\alpha \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g},\mathbf{g}}^{\text{loc}} \mathbf{G}_{\hat{\mathbf{n}} \times \mathbf{f},\mathbf{g}}^{-1} \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{f},\mathbf{f}}^E + (1-\alpha) \eta \mathbf{Z}_{\hat{\mathbf{n}} \times \mathbf{g},\mathbf{f}}^H \right) \mathbf{I} \tag{2.49}$$

It is emphasized that YUKM-CFIE improves the accuracy of the YUKC-CFIE even further.

3. NUMERICAL RESULTS

In this section advantages and disadvantages of the presented schemes are demonstrated via their application to the analysis of scattering from different scatterers. Solution time, accuracy, iteration number, and consistency of the solutions obtained from the different schemes are analyzed thoroughly.

In all examples, a $\hat{\mathbf{p}}$ polarized plane wave propagating in $\hat{\mathbf{k}}$ direction with the electric field component $\mathbf{E}^{\text{inc}}(\mathbf{r}) = \hat{\mathbf{p}} e^{-jk\hat{\mathbf{k}}\cdot\mathbf{r}}$ illuminates the scatterers. Unless specified, an $\hat{\mathbf{x}}$ polarized plane wave propagating in $\hat{\mathbf{z}}$ direction is used. Transpose-free quasi-minimal residual (TFQMR) iterative solver [54] is used to solve the resulting matrix equation and iteration is stopped at the i th iteration when $\|\mathbf{I}_i - \mathbf{I}_{i-1}\| < \chi$ condition is satisfied. In all examples iterative solver's tolerance is set to $\chi = 10^{-10}$. The surface integrals in the MoM matrix elements are calculated using 7-point Gaussian quadrature rule [55], and singularity extraction is employed if it is necessary [56]. In all CFIE schemes, combination factor is selected as $\alpha = 0.5$, unless specified. Error in scattered electric far field is calculated w.r.t a reference solution for $\theta_i = 0:179$ and $\phi = 0$ as,

$$\text{Error in far field} = \frac{1}{180} \sqrt{\sum_{i=0}^{179} \frac{|\mathbf{E}_{MoM}^{\text{scat, far}}(\theta_i, \phi = 0) - \mathbf{E}_{ref}^{\text{scat, far}}(\theta_i, \phi = 0)|^2}{|\mathbf{E}_{ref}^{\text{scat, far}}(\theta_i, \phi = 0)|^2}}. \quad (3.1)$$

Error in current is calculated in a similar manner w.r.t a reference solution:

$$\text{Error in current} = \frac{1}{N} \sqrt{\sum_{n=1}^N \frac{|\{\mathbf{I}_{MoM}\}_n - \{\mathbf{I}_{ref}\}_n|^2}{|\{\mathbf{I}_{ref}\}_n|^2}}. \quad (3.2)$$

3.1. Unit Sphere

In this section, comparison of all the schemes mentioned earlier is presented. To avoid cluttering and ease of read, first EFIE and MFIE schemes is discussed in terms of their accuracy and iterative solvers' behavior. Then CFIE schemes is discussed. In this example, a PEC unit sphere discretized using 2430 unknowns as shown in Figure

3.1 is analyzed. Frequency is swept from 100 MHz to 300 MHz, and obtained results are then compared w.r.t analytical Mie series solution [8].

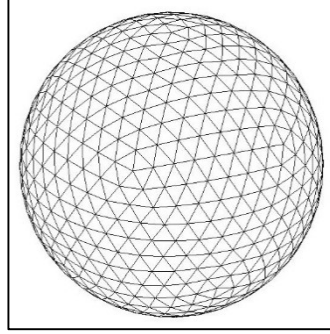


Figure 3.1: Unit sphere discretized using 2430 unknowns.

In Figure 3.2, error in the solution of C-EFIE, C-MFIE, M-MFIE, CMP-EFIE, and YUK-EFIE relative to analytical Mie series solution are shown in logarithmic scale. Dashed vertical lines indicate the interior resonance frequencies of the unit sphere, around 131 MHz, 186 MHz, 214 MHz, 240 MHz, and 270 MHz. In Figure 3.2, it is seen that C-MFIE and M-MFIE schemes are affected by the interior resonances and cannot produce accurate solutions, while MoM solution of C-EFIE, CMP-EFIE and YUK-EFIE schemes seems immune to the interior resonances in terms of accuracy. However this effect is observed if the resonances excited by the incident field. With this example we can conclude that excitation plays a key role in the accuracy of the EFIE schemes regarding the resonance frequencies.

In Figure 3.3, iteration numbers are presented. EFIE schemes without preconditioning shows worse convergence performance compared to MFIE schemes. With the Calderon preconditioning, CMP-EFIE scheme shows better accuracy and iterative convergence. Presence of interior resonance corrupts the conditioning of the MoM matrix in all schemes mentioned. Even if the EFIE schemes seem to be immune to interior resonances in the context of accuracy, iterative solver convergence is still corrupted around the resonance frequencies.

In Figure 3.4, error relative to analytical Mie series solution of discretization schemes of C-CFIE, M-CFIE, RSP-CFIE, YUKC-CFIE, and YUKM-CFIE are shown in logarithmic scale. In Figure 3.4, it is seen that none of the schemes are affected by the interior resonances except M-CFIE and YUKC-CFIE, which is caused because of the non-conforming combination as expected. It should be noted that, error peak occurs

around 290 MHz is caused by the inaccuracy in the modelling of current density. The accuracy could be enhanced by simply increasing the number of basis functions.

In Figure 3.5, iteration numbers of CFIE discretization schemes are presented. CFIE schemes are immune to the interior resonance frequencies as expected, except YUKC-CFIE. It is claimed in [48] that YUKC-CFIE is immune to the resonance frequencies, but since MFIE's range resides in RWG function space and YUK-EFIE's range resides in BCCW function space, the combination of them are non-conforming, thus resulting combination is susceptible to the interior resonance. Also it should be noted that no significant difference observed between both mappings of RSP-CFIE, i.e. mapping from EFIE space to MFIE space or MFIE space to EFIE space.

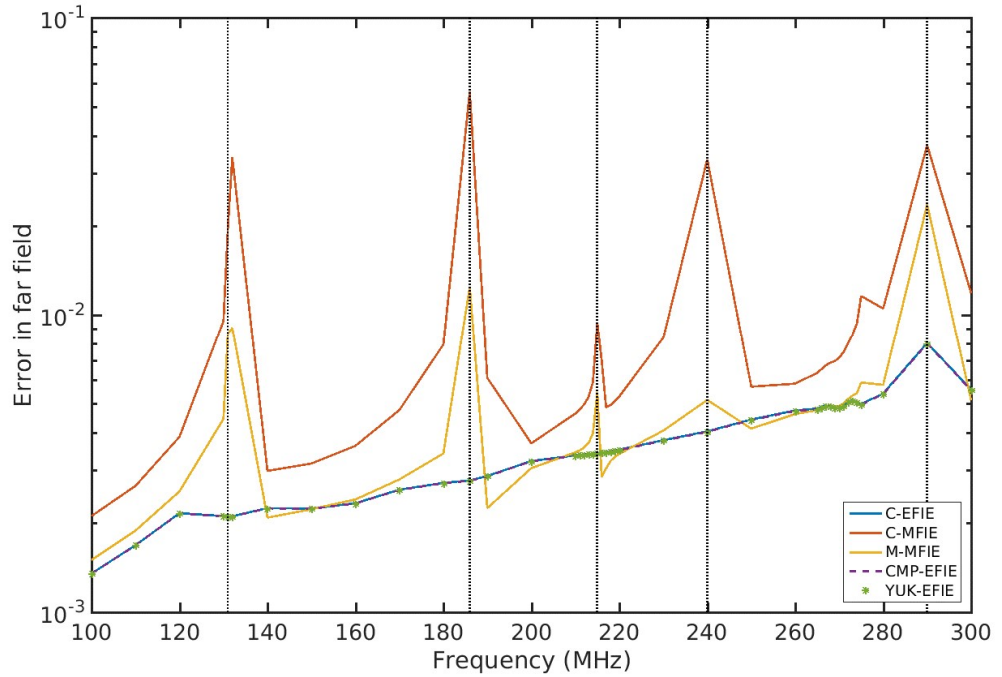


Figure 3.2: Comparison of error levels in scattered electric far field obtained by EFIE and MFIE schemes for the unit sphere.

Sheer iteration number shows how much steps needed to converge to a solution, however it doesn't inform about the iteration characteristics. In Figure 3.6, comparison of iteration characteristics for all the discretization schemes at 200 MHz are presented. As expected, C-EFIE shows the slowest convergence rate to desired solution level while CMP-EFIE is the fastest. YUK-EFIE and CMP-EFIE show similar convergence characteristics, however CMP-EFIE is still more advantageous and even performs better than the C-MFIE, which is a second kind Fredholm integral equation.

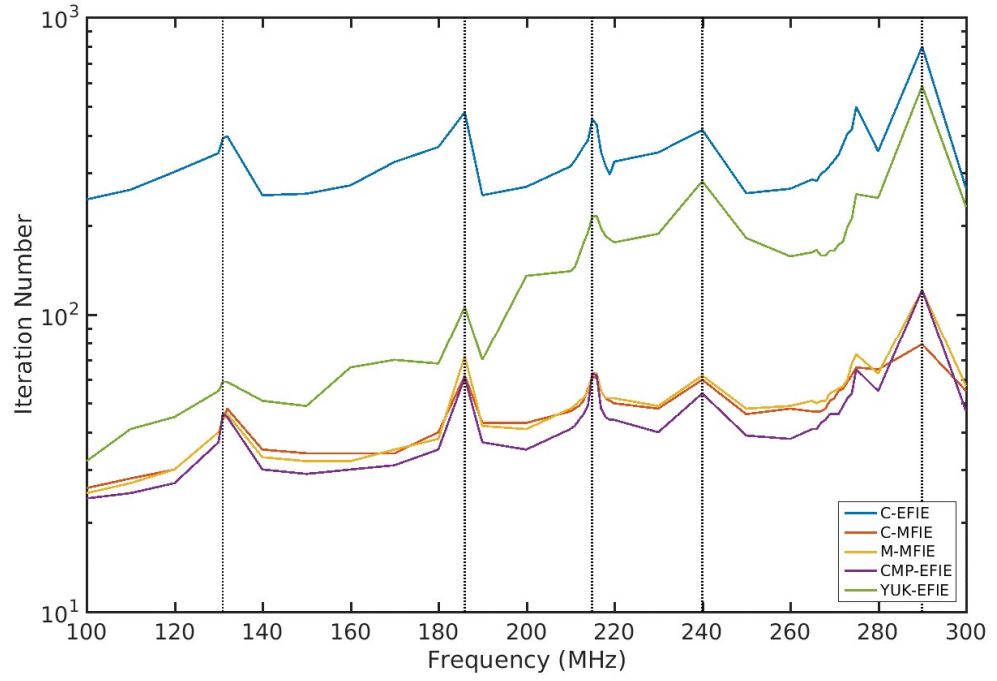


Figure 3.3: Comparison of iteration numbers of EFIE and MFIE schemes for the unit sphere.

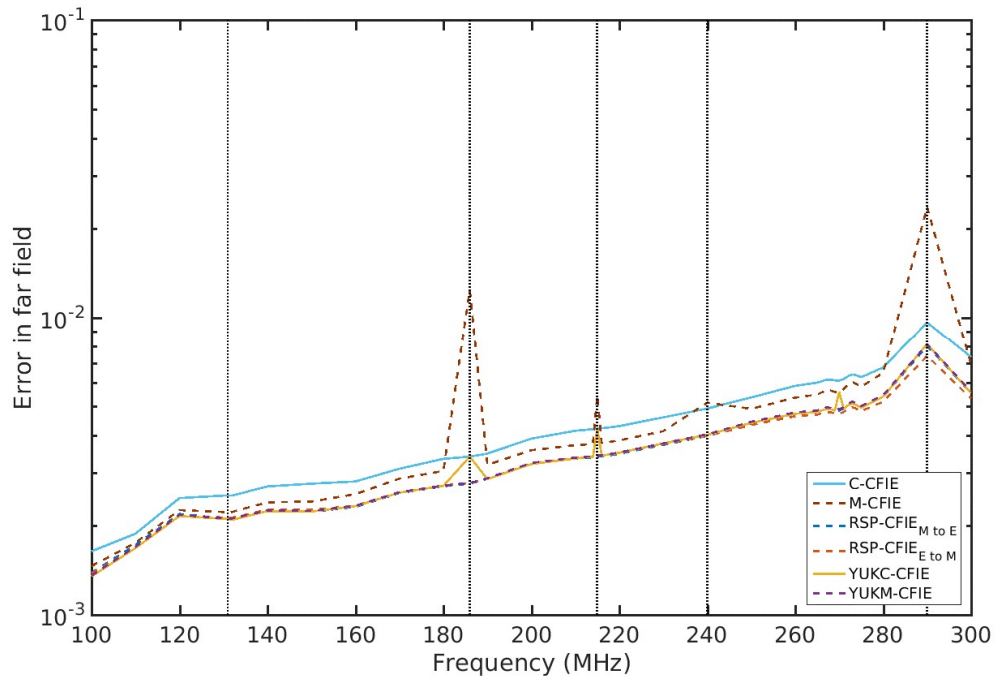


Figure 3.4: Comparison of error levels in scattered electric far field obtained by CFIE schemes for the unit sphere.

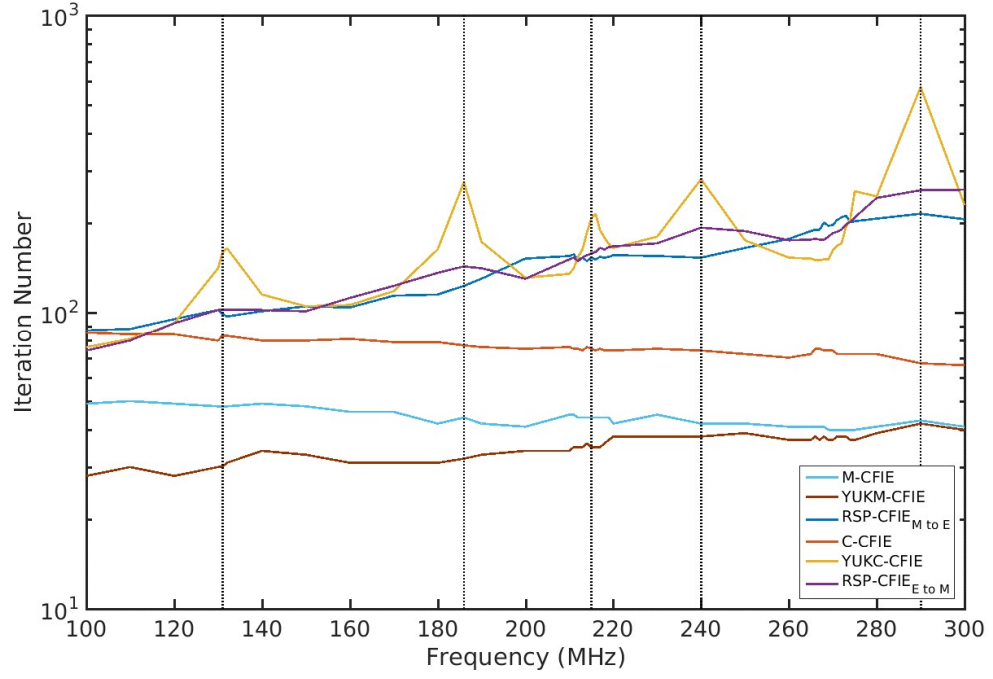


Figure 3.5: Comparison of iteration numbers of CFIE schemes for the unit sphere.

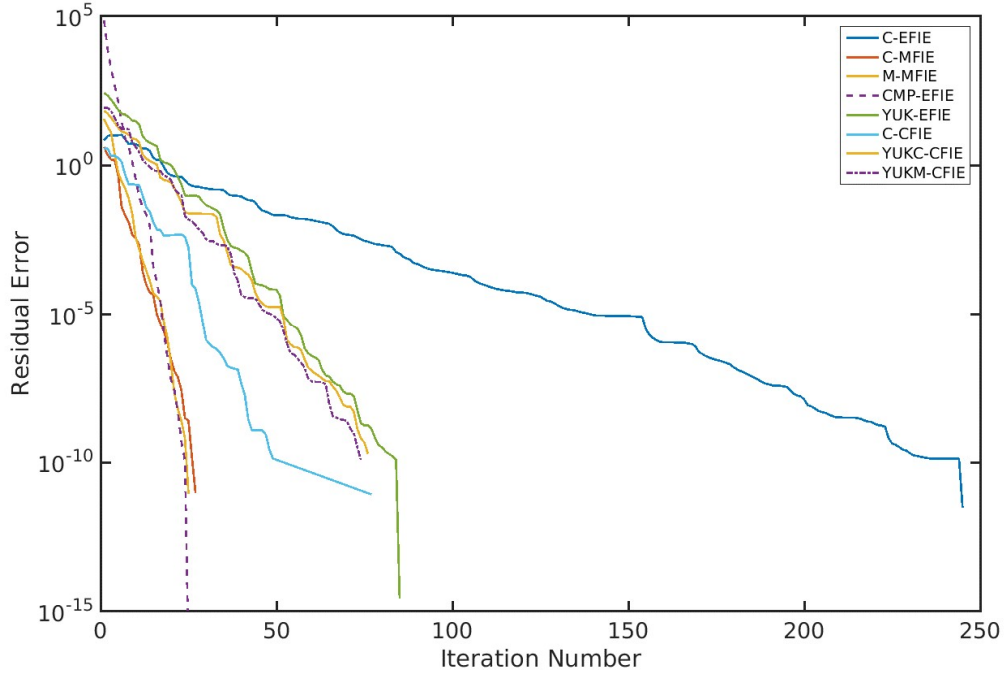


Figure 3.6: Iteration number characteristics of schemes for the unit sphere.

In the next example, significance of the contribution of EFIE and MFIE parts to CFIE is investigated for the same problem. To make it more challenging, resonance frequency of 130 MHz for the sphere is selected and the combination factor (α) is

swept from 0.1 to 0.9. In Figure 3.7, this example clearly states, accuracy of the RSP-CFIE and YUKM-CFIE is almost non-dependent to the combination factor while C-CFIE and M-CFIE get affected by the selection of the combination factor.

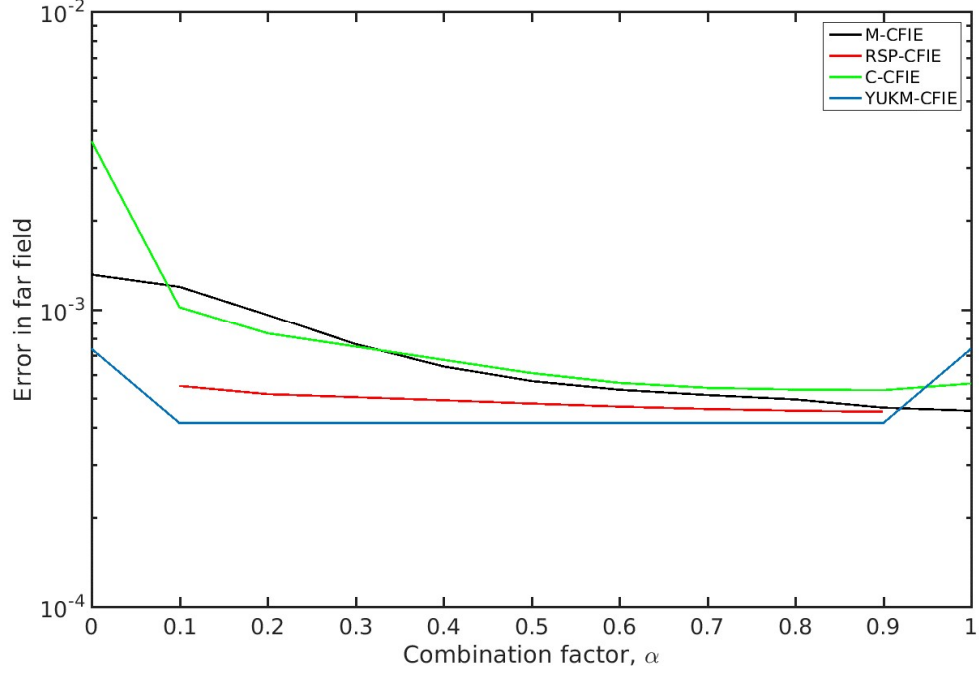


Figure 3.7: Comparison of error levels in scattered electric far field obtained by CFIE schemes for the unit sphere for the combination factor sweep.

3.2. Cuboid

In this section, scattering from a PEC cuboid, which has dimensions of 1 by 0.5 by 0.25 m³, discretized using 1236 unknowns as shown in Figure 3.8, is analyzed. With the prior information gained from the previous numerical examples, solution of M-CFIE and YUKC-CFIE are shown to be inconsistent and susceptible to the interior resonance problem. Therefore only CMP-EFIE, C-CFIE, RSP-CFIE, and YUKM-CFIE are used for the analysis of the cuboid. Frequency is swept from 30 MHz to 1 GHz, with a step of 30 MHz, obtained results are then compared to the reference solution of C-EFIE. The resonance frequencies of the cuboid [8] with dimensions of $a \times b \times c$ is calculated as follows for the resonant modes of m, n, l :

$$f_{m,n,l} = \sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2 + \left(\frac{l}{2c}\right)^2}. \quad (3.3)$$

The interior resonance frequencies are calculated at 423.97, 540.46, 618.04, 686.91, and 749.48 MHz.

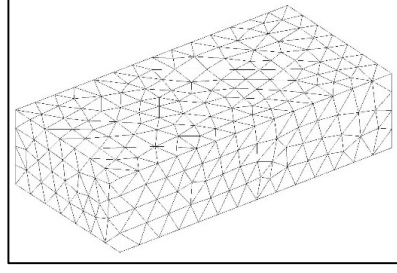


Figure 3.8: Cuboid discretized using 1236 unknowns.

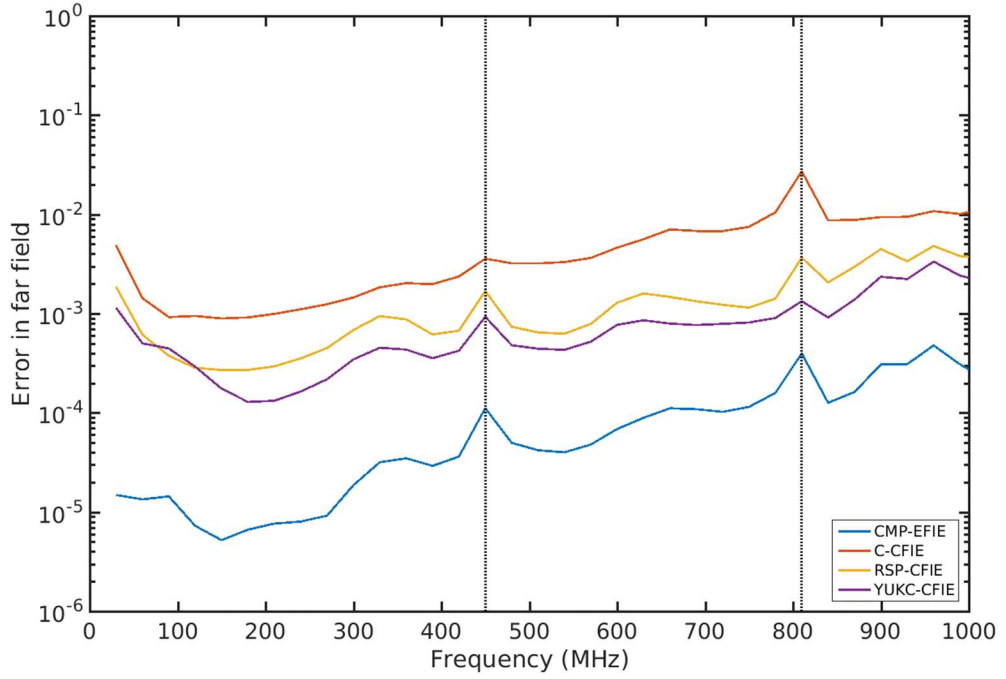


Figure 3.9: Comparison of error levels in scattered electric far field w.r.t. C-EFIE.

In Figure 3.9, effects of the interior resonances can be seen as peaks. However, other resonance frequencies' effects to the error level cannot be observed in the far fields. Their effects can be seen in Figure 3.10 in the currents. From these figures, it can be concluded that, contribution from sharp edges/wedges lowers the accuracy of the MFIE schemes.

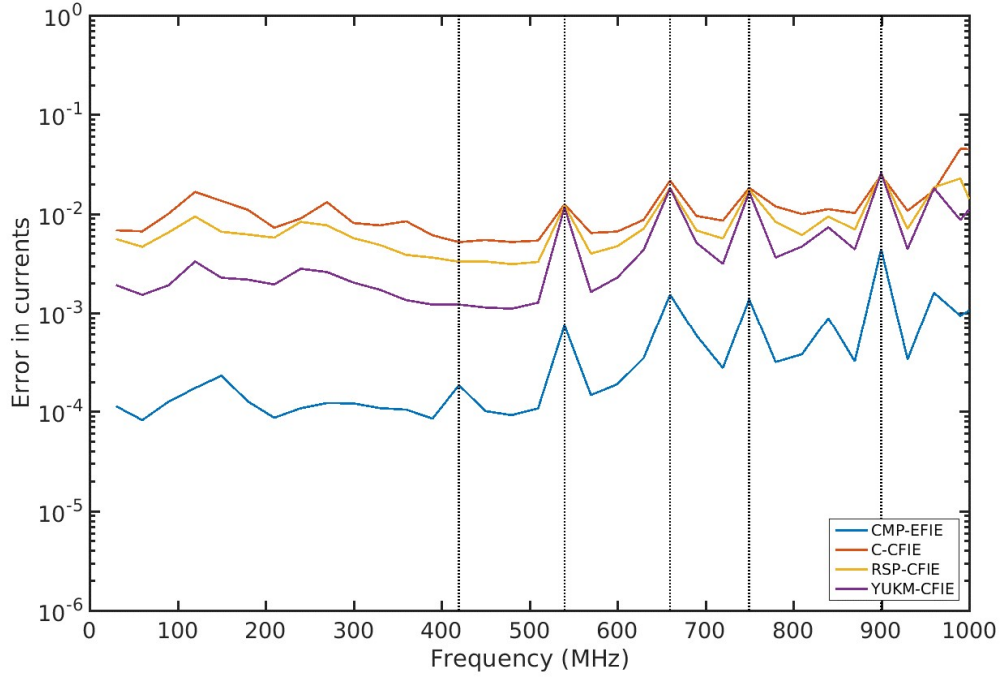


Figure 3.10: Comparison of error level in currents induced on the surface for cuboid w.r.t. C-EFIE.

3.3. NASA Almond

In this section, scattering from an almond shaped object, specifically designed for the measurements of RCS is investigated [57]. Almond shaped test object, i.e. NASA almond, is invented for use in measuring the performance characteristics of microwave anechoic chambers [58] and became a benchmark target for RCS tests.

The almond, has a height of 0.0575 m, width of 1.15 m, and length of 3 m and is discretized using 3312 unknowns as shown in Figure 3.11. After the information gained from the previous examples, only solutions selected from the consistent schemes of C-CFIE, CMP-EFIE, RSP-CFIE, and YUKM-CFIE are applied and compared with the reference solution of C-EFIE.

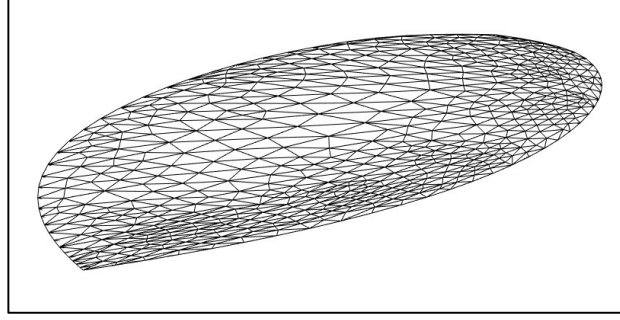


Figure 3.11: NASA almond discretized with 3312 unknowns.

In Figure 3.12 and Figure 3.13, RCS of the NASA almond for the frequencies of 150 MHz and 250 MHz are given, respectively. Results obtained by the CMP-EFIE, RSP-CFIE, and YUKM-CFIE agrees well, however C-CFIE differs. This fact is more apparent for the results at the 250 MHz as seen in Figure 3.13. It is clear that C-MFIE contribution in the C-CFIE lowers the accuracy because of the sharp wedges. Since YUKM-CFIE and RSP-CFIE satisfy the conforming-tested and conforming combination criteria, they produce similar results. CMP-EFIE and C-EFIE, in the essence should reach to same accuracy, however conditioning of C-EFIE fails to converge to the desired level at a reasonable time. In Figure 3.14, iteration numbers for the almond scatterer at 150 MHz is given.

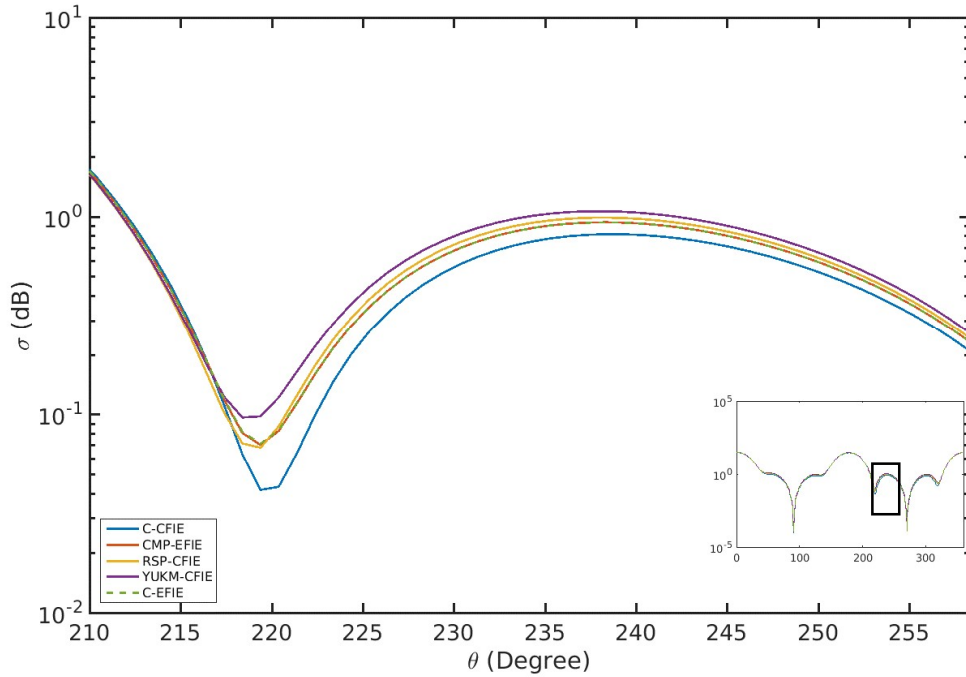


Figure 3.12: RCS of NASA almond at 150 MHz.

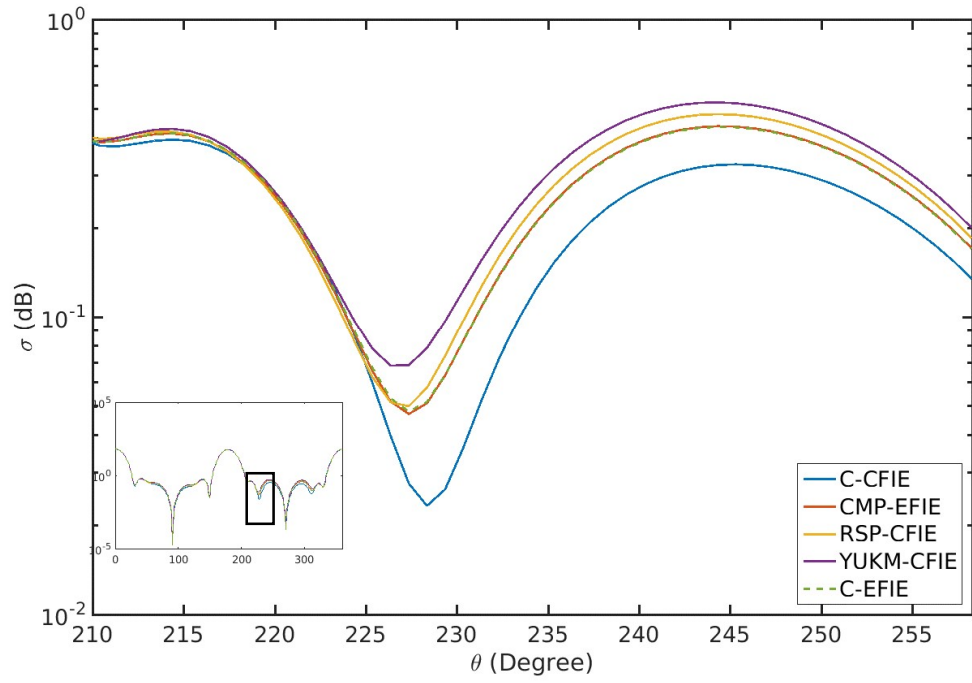


Figure 3.13: RCS of NASA almond at 250 MHz.

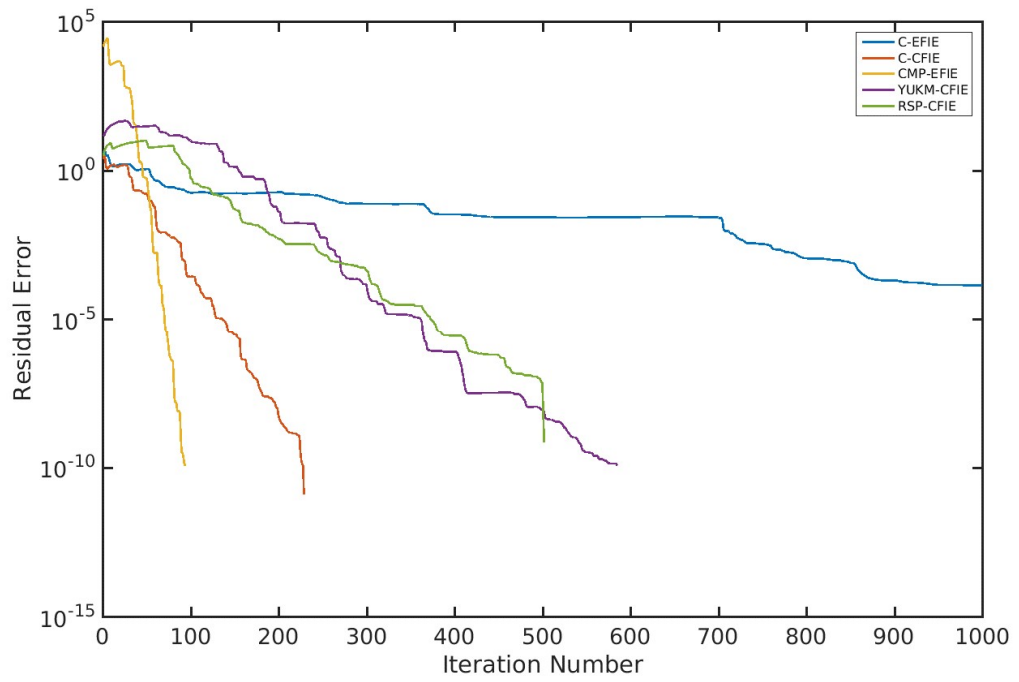


Figure 3.14: Iteration number for the almond at 150 MHz.

3.4. Missile Model

This and next section aims to demonstrate the applicability of the mentioned schemes to real life problems. For this aim, a simplified missile model which has a length of 3.14 meters and maximum width and height of 1.5 meters and is discretized using 3885 unknowns is analyzed as shown in Figure 3.15 is analyzed. An $\hat{x}$ polarized plane wave propagating in the $-\hat{z}$ direction at 100 MHz is used to illuminate the missile.

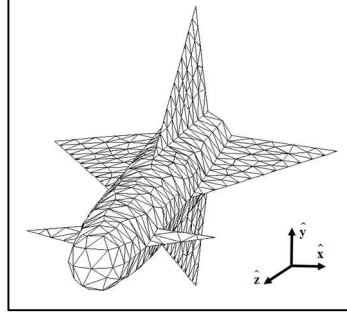


Figure 3.15: Missile model discretized with 3885 unknowns.

In Figure 3.16, RCS of the missile model is given. Solutions obtained by YUKM-CFIE, RSP-CFIE, and C-CFIE schemes with the combination factor selection of $\alpha = 0.1$ and $\alpha = 0.5$ are compared with CMP-EFIE and C-EFIE. It is shown that, contribution of the non-conforming-tested MFIE contribution lowers the accuracy in C-CFIE, while RSP-CFIE and YUKM-CFIE almost do not depend on the selection of the combination factor.

3.5. Plane Model

In this example, a simplified plane model which has a length of 95 meters and wingspan of 77 meters is discretized using 3945 unknowns as shown in Figure 3.17 is analyzed. An $\hat{x}$ polarized plane wave of 100 MHz propagating in the $-\hat{z}$ direction is used to illuminate the plane.

In this example, it is seen in Figure 3.18 while RSP-CFIE, YUKM-CFIE, and CMP-EFIE are in almost perfect agreement, contribution from C-MFIE to C-CFIE lowers the accuracy immensely and fails to reach an accurate solution in a reasonable time.

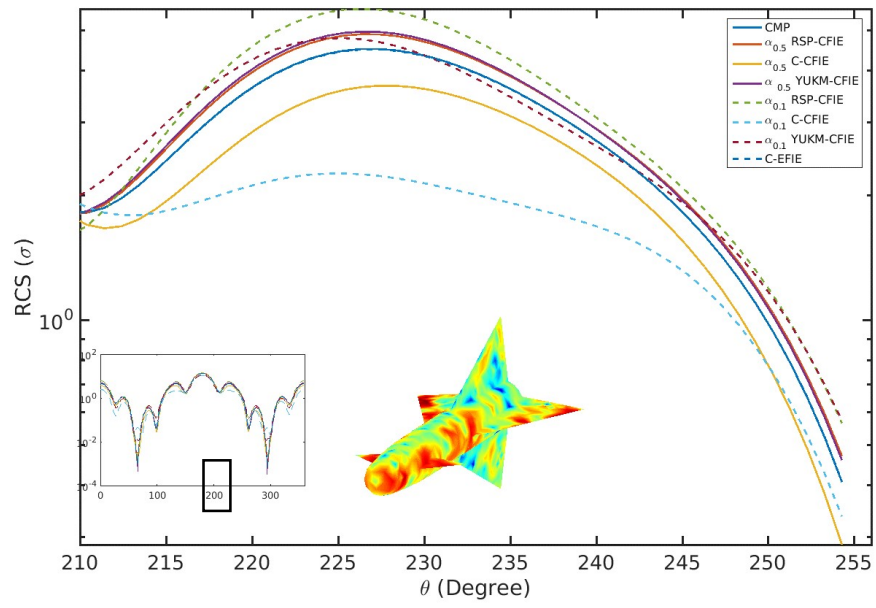


Figure 3.16: RCS of the missile model at 300 MHz.

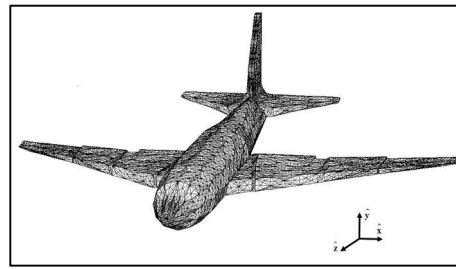


Figure 3.17: Simplified plane model discretized with 3945 unknowns.

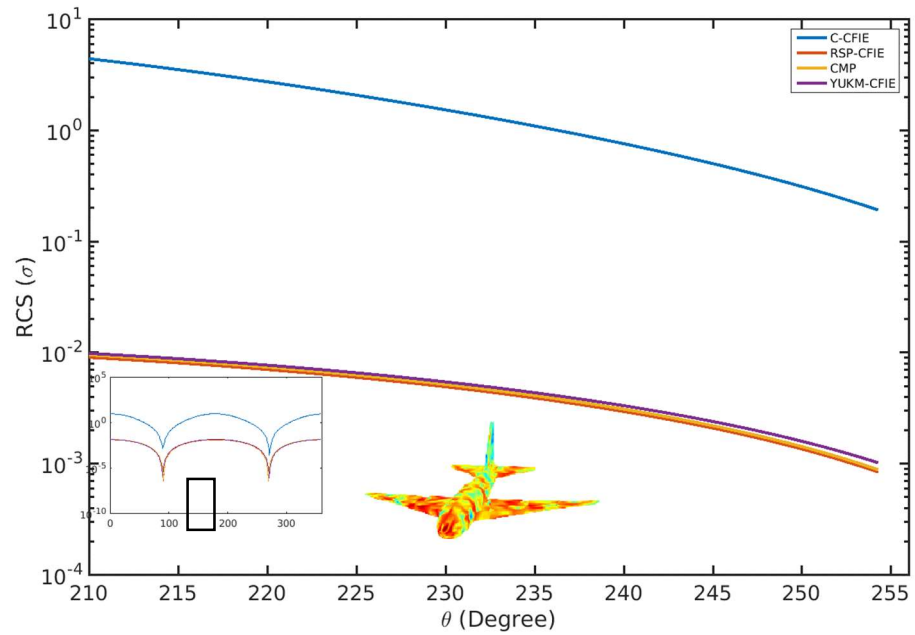


Figure 3.18: RCS of the plane model at 100 MHz.

4. CONCLUSIONS

In this thesis, discretization schemes of MFIE, EFIE, and combination strategies of CFIE are investigated thoroughly. It is shown that classically discretized MFIE produces well-conditioned matrices, however its accuracy is low. By selecting the testing function in dual of the MFIE operators range space mixed discretization is achieved, as a result conforming-tested criterion is satisfied, and accuracy of the MFIE is improved. On the other hand C-EFIE produces more accurate solutions compared to C-MFIE, because conforming-tested criterion is satisfied, however resulting matrix for EFIE is not as well-conditioned as matrices produced by MFIE schemes. Condition number of the EFIE matrix is improved by using Calderon or Yukawa-Calderon preconditioners. Despite of improvement in the accuracy and conditioning, both EFIE and MFIE suffer from the interior resonance problem.

It is shown that interior resonance affects the conditioning of EFIE schemes and affects both accuracy of the solution and conditioning of MFIE schemes. It is possible to remedy the interior resonance problem using CFIE, which is weighted sum of EFIE and MFIE. C-CFIE eliminates the interior resonance problem, and produces a relatively well-conditioned matrix but accuracy of the solution obtained by C-CFIE is lower compared to EFIE solution, because contribution from MFIE lowers the accuracy of the solution. To overcome the accuracy problem, M-CFIE is proposed in the literature. Even if the accuracy of the solution for M-CFIE is improved, it is seen that interior resonance problem still corrupts the solution due to the difference in the range spaces of C-EFIE and M-MFIE. Therefore, range space projection is applied to ensure the conformity of the range spaces. It is shown that, projecting range spaces onto each other before combination, i.e. RSP-CFIE, eliminates the interior resonance problem and improves the accuracy as expected. To further improve the conditioning of CFIE, combination of MFIE schemes with CMP-EFIE can be considered. However this scheme is not able to eliminate the interior resonance problem, because both CMP-EFIE and MFIE share the same resonance frequencies. Therefore, YUK-CFIE is suggested to improve the both the condition number and accuracy. However it is shown that, combination of C-MFIE and YUK-EFIE fails to eliminate the interior resonance problem, due to the difference in the range spaces. Combination of M-MFIE and YUK-EFIE improves the accuracy and the conditioning as expected, since

Yukawa-Calderon preconditioning technique requires mapping from RWG function space to BCCW function space that ensures conformity of the range spaces.

As a conclusion, results similar to the literature are observed, however some remarks has to be emphasized about the interior resonance problem. According to the numerical examples, interior resonance problem affects the conditioning of the MoM matrix produced by EFIE, but not the overall accuracy of the solution obtained. Solution obtained by EFIE only gets affected if the interior resonance is excited by the incident field. On the other hand, it affects both conditioning of the MFIE matrix produced and the accuracy of the solution obtained by MFIE. It can be concluded that, for the problems around the scatterer's interior resonance region, YUKM-CFIE and RSP-CFIE are good choices for accurate and fast solutions. However these techniques are more computationally expensive w.r.t classical schemes due to the barycentric refinement of the mesh.

As future works, excitation criterion of interior resonance problem for the EFIE schemes needs further investigation to come to a decisive conclusion. Acceleration methods, such as fast multiple methods (FMM), will be implemented to solve large scale problems [59]. Development of basis functions which satisfy well- and conforming-tested criteria that do not require barycentric refinement will be investigated.

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BIOGRAPHY

Ahmet Furkan Yilmaz was born in Malatya, 1991. He graduated from Kirimli Fazilet Olcay Anatolian High School in Istanbul and graduated from Gebze Technical University, Department of Electronics Engineering, in 2016. He is a research assistant at Gebze Technical University, Department of Electronics Engineering, Electromagnetic Fields and Microwave Techniques division. His research interests include numerical evaluation of electromagnetic scattering problems.

APPENDICES

Appendix A: Publications made Under Thesis

- [1] Yilmaz A. F., Ülkü H. A., (2018), “On the conforming combination of the electric and magnetic field integral equations”, 2018 International Applied Computational Electromagnetics Society Symposium (ACES), 6–7, Denver, CO, USA.
- [2] Yilmaz A. F., Ülkü H. A., (2018), “Uyumlu ayrıklaştırılmış elektrik ve manyetik alan integral denklemlerinin hedef uzayı uyumlu birleşimi”, Gebze Teknik Üniversitesi Fen Bilimleri Enstitüsü, Lisansüstü Araştırmalar Sempozyumu, 14-16 Mayıs.

Appendix B: Rao-Wilton-Glisson Functions

An RWG function [27] is defined on a pair of triangles on the initial mesh as

$$\mathbf{f}_n(\mathbf{r}) = \begin{cases} \frac{\pm l_n}{2A_n^\pm}(\mathbf{r} - \mathbf{r}_n^\pm); \mathbf{r} \in S_n^\pm \\ 0 & ; \mathbf{r} \notin S_n^\pm \end{cases} \quad (\text{B.1})$$

where, $S_n^\pm$ denotes the triangle pair that share a common edge, l_n is the length of the common edge, $A_n^\pm$ is the area of $S_n^\pm$, $\mathbf{r}_n^\pm$ is the uncommon vertex of $S_n^\pm$, as illustrated in Figure B.1. The divergence of an RWG function can be found as

$$\nabla \cdot \mathbf{f}_n(\mathbf{r}) = \begin{cases} \frac{\pm l_n}{A_n^\pm} & ; \mathbf{r} \in S_n^\pm \\ 0 & ; \mathbf{r} \notin S_n^\pm \end{cases} \quad (\text{B.2})$$

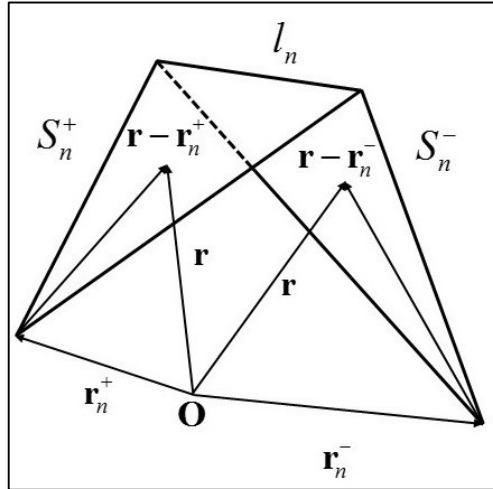


Figure B.1: Illustration of an RWG function.

Barycentrically refined mesh is obtained by adding three median to the each triangle defined on the initial mesh as shown in Figure B.2. Transformation matrix $\mathbf{R}$ maps the RWG functions defined on the barycentrically refined mesh (bRWG) to the RWG functions defined on the initial mesh [43]. Coefficients of the transformation matrix $\mathbf{R}$ for an RWG function are calculated as follows,

$$\begin{aligned}
c_1 = 0, c_2 = -\frac{l}{6l_2}, c_3 = \frac{l}{6l_3}, c_4 = -\frac{l}{3l_4}, c_5 = 0, c_6 = \frac{l}{3l_6}, c_7 = 1, \\
c_8 = 1, c_9 = -\frac{l}{3l_9}, c_{10} = 0, c_{11} = \frac{l}{3l_{11}}, c_{12} = -\frac{l}{6l_{12}}, c_{13} = 0, c_{14} = \frac{l}{6l_{14}}
\end{aligned} \tag{B.3}$$

where l is the length of the reference edge and l_i is the length of the associated barycentric edge. Edges of RWG functions defined on the barycentrically refined mesh are illustrated in Figure B.2.

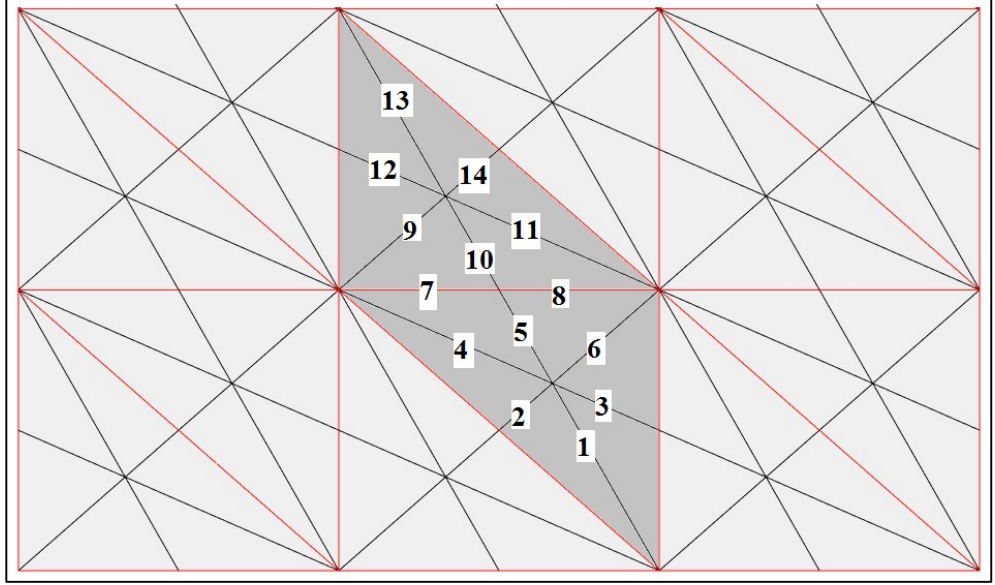


Figure B.2: Barycentric edges used to define RWG functions.

Modelled current flow considering bRWG functions is illustrated in the Figure B.3 on the barycentrically refined mesh.

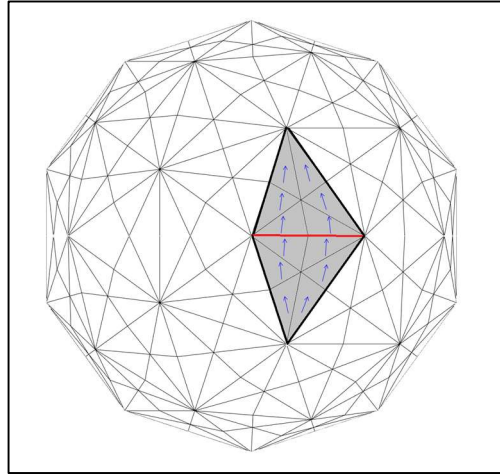


Figure B.3: Illustration of an RWG function on the barycentrically refined mesh.

Appendix C: Buffa-Christiansen-Chen-Wilton Functions

Computation of the BCCW functions can be implemented with little tweaking of the former implementation of the bRWG using transformation matrix $\mathbf{P}$. Transformation matrix $\mathbf{P}$ maps the RWG functions defined on the barycentrically refined mesh to the BCCW functions [33]. Coefficients of the transformation matrix $\mathbf{P}$ is calculated as follows for the bRWGs located at the left part of the reference edge,

$$c_i = \frac{N_c - i}{2l_i N_c}, i = 1, \dots, 2N_c - 1 \quad (\text{C.1})$$

and for the bRWGs located at the right part,

$$c_{\tilde{i}} = \frac{\tilde{N}_c - \tilde{i}}{2l_{\tilde{i}} \tilde{N}_c}, \tilde{i} = 1, \dots, 2\tilde{N}_c - 1 \quad (\text{C.2})$$

bRWGs located around the reference edge, named 0 and $\tilde{0}$ have coefficients $1/2l_0$ and $-1/2l_{\tilde{0}}$, respectively. Here, N_c and $\tilde{N}_c$ are the number of cells located at the right and left of the reference edge. Edges of BBCW functions defined on the barycentrically refined mesh is illustrated in Figure C.1. Detailed information about transformation matrix $\mathbf{P}$ can be found at [33], [43].

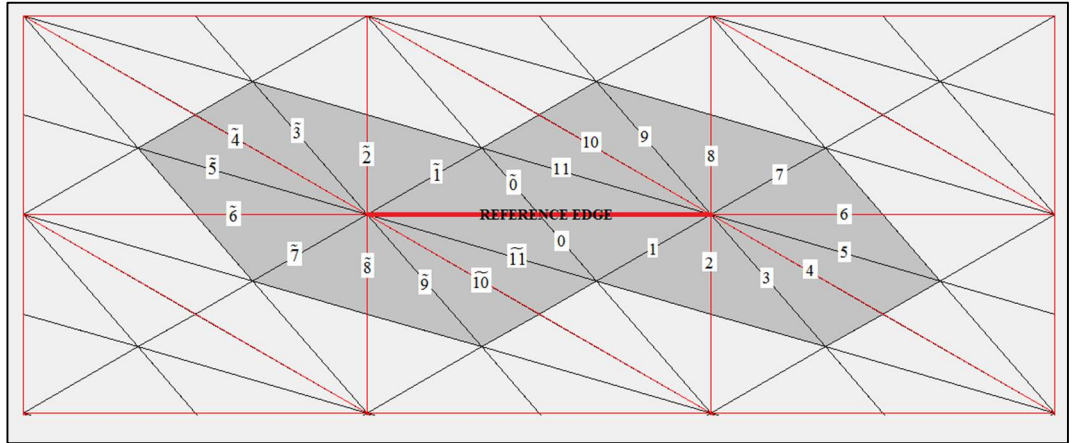


Figure C.1: Barycentric edges used to define BCCW functions.

Modelled current flow considering BCCW functions is illustrated in Figure C.2.

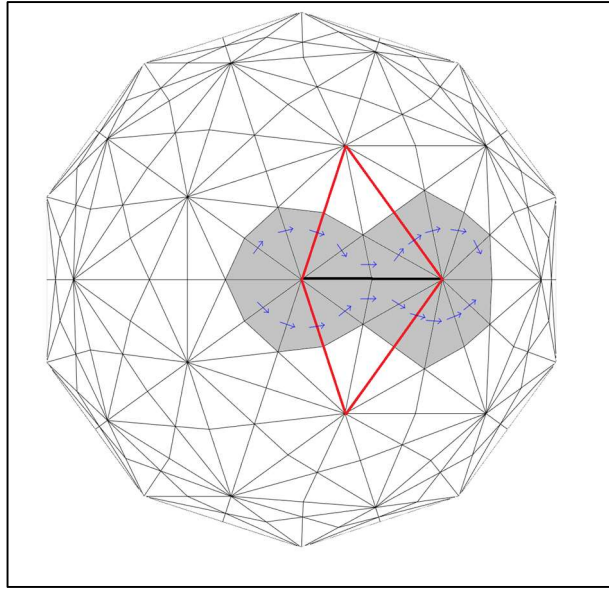


Figure C.2: Illustration of a BCCW function on the barycentrically refined mesh.