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A THESIS SUBMITTED FOR THE DEGREE OF
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# ÇİZGE KURAMINDA BASKINLIK VE BAĞIMSIZLIĞIN YENİ YAPISAL YÖNLERİ

**HADI ALIZADEH**
**DOKTORA TEZİ**
**BİLGİSAYAR MÜHENDİSLİĞİ ANABİLİM DALI**

DANIŞMANI
DOÇ. DR. DİDEM GÖZÜPEK

**GEBZE**
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# SUMMARY

Independence and domination are two related graph theory concepts, which have many applications in computer science. In this thesis, we are mainly interested in independence and domination related problems. In particular, this thesis covers two topics: almost well-dominated graphs and paired domination. We introduce almost well-dominated graphs as graphs with only two different sizes of minimal dominating sets, where the difference between these two sizes is one. We obtain a complete structural characterization for almost well-dominated graphs without induced cycles of sizes 3, 4, 5, and 7. Next, we proceed with almost well-dominated bipartite graphs. We initially establish an upper bound for the order of bipartite graphs with domination gap $k$, where $k \geq 1$, and minimum degree at least two. We then give a complete structural characterization of almost well-dominated bipartite graphs with minimum degree at least two.

Paired domination is another topic of interest in this thesis. We particularly focus on two graph parameters: upper domination number, which is denoted by $\Gamma(G)$, and upper paired domination number, which is denoted by $\Gamma_{pr}(G)$. We specify the relationship between these two parameters by proving that $\Gamma_{pr}(G) \leq 2\Gamma(G)$ for any graph $G$. As a first step, we determine the structure of bipartite and unicyclic graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ by characterizing such graphs. Next, we obtain two other characterization results: one for graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ and girth at least 6 and the other for triangle-free cactus graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$.

**Key Words: Independence, Domination, Almost well-dominated graphs, Paired domination, Upper domination number, Upper paired domination number.**

# ÖZET

Baskınlık ve bağımsızlık, çizge kuramının bilgisayar bilimlerinde birçok uygulaması bulunan iki ilişkili kavramıdır. Bu tezde genel olarak baskınlık ve bağımsızlıkla ilgili problemleri ele alıyoruz. Bu bağlamda, neredeyse iyi baskınlanmış çizgeler ve eşli baskınlık konularını özel olarak ele aldık. Neredeyse iyi-baskınlanmış çizgeleri minimal baskın kümelerinin büyüklüklerinin sadece iki farklı değer alabildiği ve en büyük ile en küçük arasındaki farkın bir olduğu çizgeler olarak tanımlıyoruz. Bu tezde 3, 4, 5, ve 7 boyunda endüklenmiş döngüler içermeyen neredeyse iyi-baskınlanmış çizgeler için yapısal bir karakterizasyon sunuyoruz. Daha sonra iki parçalı neredeyse iyi-baskınlanmış çizgelerle devam edip ilk önce $k \geq 1$ olmak üzere baskınlık farkı $k$ ve minimum derecesi en az iki olan iki parçalı neredeyse iyi-baskınlanmış çizgelerin düğüm sayısı için bir üst sınır buluyoruz. Ayrıca, minimum derecesi iki olan iki-parçalı neredyse iyi-baskınlanmış çizgeler için bir yapısal karakterizasyon elde ediyoruz.

Bu tez kapsamında ilgilendiğimiz bir diğer konu ise eşli baskınlıktır. Bu bağlamda özellikle $\Gamma(G)$ ile gösterilen üst baskınlık sayısı ve $\Gamma_{pr}(G)$ ile simgelenen üst eşli baskınlık sayısı olarak ifade edilen iki çizge parametresini ele alıyoruz. Sözü geçen iki çizge parametresi arasında her $G$ çizgesi için $\Gamma_{pr}(G) \leq 2\Gamma(G)$ şeklinde bir ilişki olduğunu ispatlıyoruz. Başlangıçta, $\Gamma_{pr}(G) = 2\Gamma(G)$ özelliğine sahip iki parçalı ve tek döngülü çizgeleri karakterize ederek bu tür çizgelerin yapılarını belirliyoruz. Ek olarak, $\Gamma_{pr}(G) = 2\Gamma(G)$ özelliğine sahip çizgelerle ilgili başka karakterizasyon sonuçları sunuyoruz. Bu sonuçlar $\Gamma_{pr}(G) = 2\Gamma(G)$ özelliğine sahip beli en az 6 olan çizgeler ve $\Gamma_{pr}(G) = 2\Gamma(G)$ özelliğine sahip üçgensiz kaktüs çizgelerinin karakterizasyonlarını içermektedir.

**Anahtar Kelimeler: Baskınlık, Bağımsızlık, Neredeyse iyi baskınlanmış çizgeler, Eşli baskınlık, Üst baskınlık sayısı, Üst eşli baskınlık sayısı.**

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# TABLE of CONTENTS

# LIST OF ABBREVIATIONS AND ACRONYMS

| **Abbreviations and Acronyms** | | **Explanations** |
|---|---|---|
| $\gamma(G)$ | : | Domination number of $G$ |
| $\Gamma(G)$ | : | Upper domination number of $G$ |
| $\mu_d(G)$ | : | Domination gap of $G$ |
| $\alpha(G)$ | : | Independence number of $G$ |
| $i(G)$ | : | Independent domination number of $G$ |
| $\gamma_{pr}(G)$ | : | Paired domination number of $G$ |
| $\Gamma_{pr}(G)$ | : | Upper paired nomination number of $G$ |
| $\delta(G)$ | : | Minimum degree of $G$ |
| $\Delta(G)$ | : | Maximum degree of $G$ |
| $K_n$ | : | A complete graph on $n$ vertices |
| $C_n$ | : | A cycle on $n$ vertices |
| $P_n$ | : | A path on $n$ vertices |
| AWC | : | Almost well-covered |
| AWD | : | Almost well-dominated |
| NP | : | Nondeterministic polynomial-time |
| PDS | : | Paired dominating set |

# LIST OF FIGURES

# LIST OF TABLES

# 1. INTRODUCTION

Graph theory is a branch of mathematics that studies graphs, which are structures used for modeling the relationship between objects. In this regard, a graph is composed of vertices (objects) and edges (relationships) that connect some pairs of vertices. In computer science, graphs are used to represent communication networks, computer hardware design, resource allocation in operating systems, and social media-related concepts.

The concept of independence is used for a group of objects no two of which have a relationship with each other. An independent set in a graph is a set of pair-wise non-adjacent vertices. The problem of finding an independent set of maximum size, a.k.a. the maximum independent set problem, is one of the oldest problems in graph theory. This problem arises in a variety of fields such as scheduling, time allotment, and coding theory. The maximum independent set problem can be further explained with an example regarding scheduling as follows: given a set of jobs that has to be executed on a computer, the objective is to find a maximum set of jobs that can be executed without interference between any two jobs. The jobs can be represented by vertices and the interference between them by edges in a graph. Then the objective corresponds to finding the largest set of non-adjacent vertices (non-conflicting jobs) in the graph.

An independent set which is not properly contained in another one is called a maximal independent set. A maximal independent set of the largest size in a graph $G$ is called a maximum independent set, and its cardinality is called the independence number of $G$. While a maximal independent set can be easily found by a greedy algorithm in a general graph, the problem of finding maximum independent set has been proved to be NP-complete [1]. However, if all maximal independent sets of a graph have the same size, that is, if every maximal independent set is a maximum independent set, then the maximum independent set can be found in polynomial time. Graphs having such property were first termed as well-covered graphs by Plummer [2].

Many results have been found regarding well-coveredness property since the concept was first introduced. Particularly, the survey paper by Plummer [3] has provided a comprehensive investigation of the class of well-covered graphs. In addition, in terms of computational complexity, Chavatal et al. [4] have shown that it is NP-complete to determine whether a given graph is not well-covered; hence, the problem of determining whether a given graph is well-covered is co-NP-complete.

Another concept of interest in this thesis is domination in graphs. A dominating set in a graph $G$ is a set $S$ of vertices of $G$ such that every vertex of $G$ is either in $S$ or is adjacent to a vertex in $S$. From an application perspective, domination concept can be redefined in the following way. Suppose that there exist a number of objects (customers) each with a certain location. Domination is about selecting minimum number of locations to put facilities to serve all customers.

A dominating set is minimal if it does not contain another dominating set. A minimal dominating set of the minimum size is a minimum dominating set. The cardinality of a minimum dominating set is referred to as domination number and denoted by $\gamma(G)$, whereas the maximum cardinality of a minimal dominating set is called the upper domination number and denoted by $\Gamma(G)$. Furthermore, the domination gap of a graph $G$, which is denoted by $\mu_d(G)$, is the difference between $\Gamma(G)$ and $\gamma(G)$.

Domination is in close connection with independence concept since every maximal independent set in a graph is a minimal dominating set. It is NP-complete to determine the domination number of a general graph [1]. Finbow et al. [5] introduced the concept of well-dominated graphs whose minimal dominating sets have the same cardinality. It is easy to see that the domination number can be found polynomially for well-dominated graphs. In addition, Finbow et al. [5] have shown that every well-dominated graph is a well-covered graph, which implies that well-dominated graphs are a subclass of well-covered graphs. Well-dominated graphs were further detailed in a book by Hynes et al. [6]. From a computational point of view, Ananchuen et al. [7] showed that the complexity of determining whether a given graph is well-dominated is not known.

There exist several variants of domination concept in the literature. Paired domination is one of these variants, which we inspect in this thesis. Note that a matching $M$ in a graph $G$ is a set of pairwise non-adjacent edges. Furthermore, if a matching $M$ matches all vertices of a graph $G$, we call $M$ a perfect matching. A paired dominating set (PDS) of a graph G is a dominating set $D$ of $G$ with the additional property that the subgraph induced by $D$ has a perfect matching. In the context of the customers-facilities example, paired domination can simply be explained as a type of domination with an additional constraint that each facility serving the customers has a unique backup. The paired domination number of a graph $G$, denoted by $\gamma_{pr}(G)$, is the minimum cardinality of a PDS in $G$. In addition, the upper paired domination number of a graph G, denoted by $\Gamma_{pr}(G)$, is the maximum cardinality of a minimal PDS in $G$.

In this thesis, we will mainly deal with independence and domination-related topics with focus on well-covered graphs, well-dominated graphs, and paired domination. The remainder of this thesis is organized as follows:

In Chapter 1, after explaining graph-theoretic definitions and notations, we provide a comprehensive review of the known results in the literature regarding well-covered graphs, well-dominated graphs, and paired domination.

In Chapter 2, we study the graphs whose domination gap is one. Inspired by the work of Ekim et al. [8], which call the graphs with independence gap one almost well-covered graphs, we term the graphs with domination gap one almost well-dominated graphs. The most related work in the literature to our study in this chapter is due to Finbow et al. [5], whose results have implications for almost well-dominated graphs with girth at least 8. In this chapter we extend the results of Finbow et al. [5] to almost well-dominated graphs without induced cycles of sizes 3,4,5, and 7 by giving a complete structural characterization for such graphs.

In Chapter 3, we investigate almost well-dominated bipartite graphs. The first result of our study in this chapter provides an upper bound for the cardinality of bipartite graphs with domination gap $k$, where $k \geq 1$, and minimum degree at least two, that is, we prove that $|V(G)| \leq 10k$. This result implies that an almost well-dominated bipartite graph ($k = 1$) with minimum degree at least two has at most 10 vertices. Another main result of this chapter is a complete structural characterization of almost well-dominated bipartite graphs with minimum degree at least two. While a 4-cycle is the only well-dominated bipartite graph with minimum degree at least two due to Finbow et al. in [5], our results in this thesis show that there exist exactly 31 almost well-dominated bipartite graphs with minimum degree at least two.

Chapter 4 covers our study regarding paired domination with focus on two graph parameters: upper paired domination number and upper domination number. Our first result determines the relationship between these two parameters, that is, we show that $\Gamma_{pr}(G) \leq 2\Gamma(G)$ for any graph G. In addition, this chapter includes an analysis of the graphs which hold the property $\Gamma_{pr}(G) = 2\Gamma(G)$. In particular, by using the results of [9], we characterize two special graph classes: bipartite and unicyclic graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$. Next, we address the graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ from a restricted girth perspective. Our major result here includes two characterizations: graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ and girth at least 6, and $C_3$-free cactus graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$. We conclude our study in this chapter by leaving the characterization of the general case of $C_3$-free graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ as an open question.

Chapter 5 concludes the thesis by providing a summary of the main results obtained throughout this thesis together with a discussion for possible future work.

# 2. LITERATURE RESEARCH

In this chapter, we first give definitions of some basic graph-theoretical terms which we frequently use in this thesis. In addition, further definitions will be provided wherever required throughout the thesis and the reader is referred to [10] for the terms whose definitions are not included in this thesis. We then review the available research works in the literature and compare the most related ones with our work.

## 2.1. Preliminaries

A graph G is an ordered pair $\left(V(G), E(G)\right)$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges each connecting a pair of vertices. An undirected graph is a graph whose edges have no orientation. An edge with identical endpoint vertices is called loop and edges having same endpoint vertices are named multiple edges. Throughout this thesis, $G$ is a simple graph, that is, a finite, undirected, and loopless graph without multiple edges.

We abbreviate an edge $\{u,v\}$ between two vertices $u$ and $v$ as $uv$. An edge uv connecting two vertices $u$ and $v$ is said to be incident to $u$ and $v$. If there exists an edge $uv$ between the vertices $u$ and $v$, we say that $u$ and $v$ are adjacent. The neighborhood of a vertex $v$, denoted by $N(v)$, is the set of all vertices adjacent to the vertex $v$. In addition, the closed neighborhood of a vertex $v$, denoted by $N[v]$, is the set $N(v) \cup \{v\}$.

The degree of a vertex $v$ is the number of vertices adjacent to $v$, that is, $|N(v)|$. By $\delta(G)$ (resp. $\Delta(G)$), we denote the minimum (resp. maximum) degree of $G$, that is, the degree of the vertex with the smallest (resp. greatest) degree in $G$. A vertex of degree zero in $G$ is referred as an isolated vertex of $G$ whereas a vertex of degree one in $G$ is called a pendant vertex (or a leaf) of $G$.

If all vertices of a graph are pairwise adjacent, it is called a complete graph. A complete graph on $n$ vertices is denoted by $K_n$. A path $P$ in a graph is a sequence of edges which connect a sequence of distinct vertices. A cycle is a sequence of consecutively adjacent vertices starting and ending at the same vertex with no repetitions of vertices and edges. We denote a path on $n$ vertices by $P_n$ and we use $C_n$

for a cycle on $n$ vertices. The girth of a graph $G$ is the length of a shortest cycle in $G$. A graph $G$ is said to be connected if any pair of vertices $u$ and $v$ in $G$ can be joined by a path in $G$; otherwise, $G$ is said to be disconnected. If a graph does not contain any cycles, then it is said to be acyclic. A tree is defined as a connected acyclic graph.

A subgraph of a graph $G = \big(V(G), E(G)\big)$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Furthermore, a subgraph of $G$ induced by a set $S \subseteq V(G)$, denoted by $G[S]$, is a graph $H$ such that $V(H) = S$ and $E(H) = \{uv \in E(G): u, v \in S\}$.

A set $I$ of vertices in a graph $G$ is said to be an independent set if no two vertices in $I$ are adjacent. An independent set which is not properly contained in another one is called a maximal independent set. A maximal independent set of the largest size in a graph $G$ is called a maximum independent set whose size is the independence number of $G$ denoted by $\alpha(G)$. On the other hand, by $i(G)$, we denote the cardinality of a minimum maximal independent set. The difference between $\alpha(G)$ and $i(G)$ is referred as the independence gap of $G$.

A dominating set in a graph $G$ is a set $S$ of vertices of $G$ such that every vertex of $G$ is either in $S$ or has a neighbor in $S$. A dominating set is minimal if it does not contain another dominating set. A minimal dominating set of the minimum size is called a minimum dominating set. While the cardinality of the minimum dominating set is referred to as domination number denoted by $\gamma(G)$, the cardinality of a maximum minimal dominating set is called upper domination number denoted by $\mathit{\Gamma}(G)$. The domination gap of a graph $G$ is the difference between $\mathit{\Gamma}(G)$ and $\gamma(G)$.

Since every maximal independent set is a minimal dominating set, mentioned graph parameters are related by the following inequalities (domination chain). For any graph $G$ we have:

$$\gamma(G) \leq i(G) \leq \alpha(G) \leq \mathit{\Gamma}(G) \tag{2.1}$$

A graph is said to be well-covered if all its maximal independent sets have the same cardinality, i.e., $i(G) = \alpha(G)$. Similarly, a graph is defined to be well-dominated if all its minimal dominating sets have the same size, i.e., $\gamma(G) = \mathit{\Gamma}(G)$.

Notice that $\gamma(G) = \Gamma(G)$ yields $i(G) = \alpha(G)$, which in turn implies that a well-dominated graph is a well-covered one, as well.

## 2.2. Related work

In this section, we give an overview of related work in the literature with an emphasis on the research works about well-covered graphs, well-dominated graphs, and paired domination.

### 2.2.1. Well-covered and well-dominated graphs

The concept of well-coveredness was first introduced by Plummer [2] from vertex cover perspective. From this perspective, a graph is said to be well-covered if every minimal vertex cover is a minimum vertex cover. The fact that a vertex cover is the complement of an independent set implies that in a well-covered graph, all maximal independent sets have the same cardinality. Throughout this thesis, we consider the definition of well-covered graphs from independent set viewpoint.

While finding the independence number of a well-covered graph is easy, the determination of the independence number of a general graph is in the class of NP-complete problems [1]. Then it would be natural to ask the following question: Is it NP-complete to determine whether a given graph is well-covered? Although there exist many results regarding well-coveredness property in the literature, this question remains unanswered. What is known so far is that Chvatal et al. [4] and Sankaranarayana et al. [11] have independently shown that determining whether a general graph G is not well-covered is an NP-complete problem, hence the problem of determining well-coveredness for a graph falls into the class of co-NP-complete problems.

It seems that giving a complete characterization of well-covered graphs is a difficult problem since no structural characterization has been given for well-covered graphs so far in the literature. This fact has led the researchers to investigate certain subclasses of well-covered graphs. Campbell [12] characterized all cubic well-covered graphs with connectivity at most two. Plummer [13] showed that 3-connected cubic planar well-covered graphs contain only four graphs. Later, Royle and Ellingham [14] characterized all cubic well-covered graphs. Prisner et al. [15]

characterized well-covered simplicial, chordal, and circular arc graphs. Randerath and Volkmann [16] characterized well-covered block-cactus graphs.

Ravindra in [17] characterized bipartite well-covered graphs. Ravindra's characterization implies a polynomial-time recognition algorithm for bipartite well-covered graphs. Moreover, there exist other examples in the literature where checking well-coveredness property can be done in polynomial time for certain graph classes. When restricted to line graphs, Lesk et al. [18] showed that there exists a polynomial-time algorithm for checking well-coveredness using the fact that the line-graph of a graph $G$ is well-covered if and only if $G$ is equimatchable. Later, Tarsi and Tankus [19] came up with a stronger result showing that there is a polynomial-time algorithm for checking well-coveredness property for claw-free graphs which is a larger graph family including line graphs.

In the literature, it is observed that a number of special families of well-covered graphs have been defined and studied by researchers since the concept of well-coveredness was first introduced. It is known that the independence number is bounded by half the graph order in a graph with no isolated vertices. A well-covered graph whose independence number is exactly half the graph order is referred to as very well-covered graph. Staples [20] and Ravindra [17] independently characterized bipartite very well-covered graphs. Later, Favaron in [21] gave a characterization for very well-covered graphs and showed some of their properties. Sankarayana et al. [11] studied very well-covered graphs from an algorithmic viewpoint by focusing on a number of problems which are NP-complete for general graphs. They showed that a number of problems which are NP-complete for well-covered graphs have polynomial time solutions in very well-covered graphs. Table 2.1 shows some of their results.

On the other hand, there exist other subclasses of well-covered graphs defined based on vertex/edge removal. Staples [20] defined $W_n$ well-covered graphs as follows:

*Definition 2.1: Let* $n$ *be a positive integer. A graph* $G$ *belongs to class* $W_n$ *if* $|V(G)| \geq n$ *and every* $n$ *disjoint independent sets in* $G$ *are contained in* $n$ *disjoint maximum independent sets.*

Table 2.1: Complexity results for well-covered and very well-covered graphs.

| Problem | well-covered | very well-covered |
|---|---|---|
| Independent set | P | P |
| Vertex cover | P | P |
| Dominating set | NP-c | P |
| Independent dominating set | P | P |
| Dominating cycle | NP-c | NP-c |
| Hamiltonian cycle | NP-c | P |
| Hamiltonian path | NP-c | NP-c |
| Clique | NP-c | NP-c |
| Clique partition | NP-c | P |
| Chromatic number | NP-c | NP-c |

By this definition, $W_1$ is the class of well-covered graphs and the $W_n$ classes form the following relation:

$$W_1 \supseteq W_2 \supseteq W_3 \supseteq \ldots \supseteq W_n \tag{2.2}$$

Staples investigated the properties of $W_n$ graphs and gave a characterization for these graphs stated in Theorem 2.1:

*Theorem 2.1: [20] Let* $n \geq 2$. *Then* $G \in W_n$ *if and only if* $\alpha(G - v) = \alpha(G)$ *and* $G - v \in W_{n-1}$ *for all* $v \in V(G)$.

Taking into account the above theorem, the subclass $W_2$, known also as 1-well-covered graphs, contains well-covered graphs $G$ such that for every $v$ in $G$, $G - v$ is also well-covered and has the same independence number. 1-well-covered graphs were later studied in detail by Pinter [22]. Pinter determined 1-well-covered planar regular graphs in [23] and 1-well-covered planar with girth four in [24], and later extended his work to 1-well-covered graphs of girth four in [25]. Hartnel [26] addressed 1-well-covered graphs without 4-cycles and provided a characterization for these graphs. Let $F$ be a graph on $3t$ vertices such that $F$ has no 4-cycles and such that the vertices can be partitioned into $t$ subsets each of which includes a 3-cycle in $F$ (see Figure 2.1). Moreover, at least two of the three vertices on these 3-

cycles are of degree 2 in $F$. Let $\mathcal{F}$ represent the family of all such graphs. Then the main result of Hartnell's work is:

*Theorem 2.2: [26] Let $G$ be a connected graph without 4-cycles. Then $G$ is 1-well-covered if and only if it is isomorphic to $K_2$, $C_5$ or a member of the family $\mathcal{F}$.*

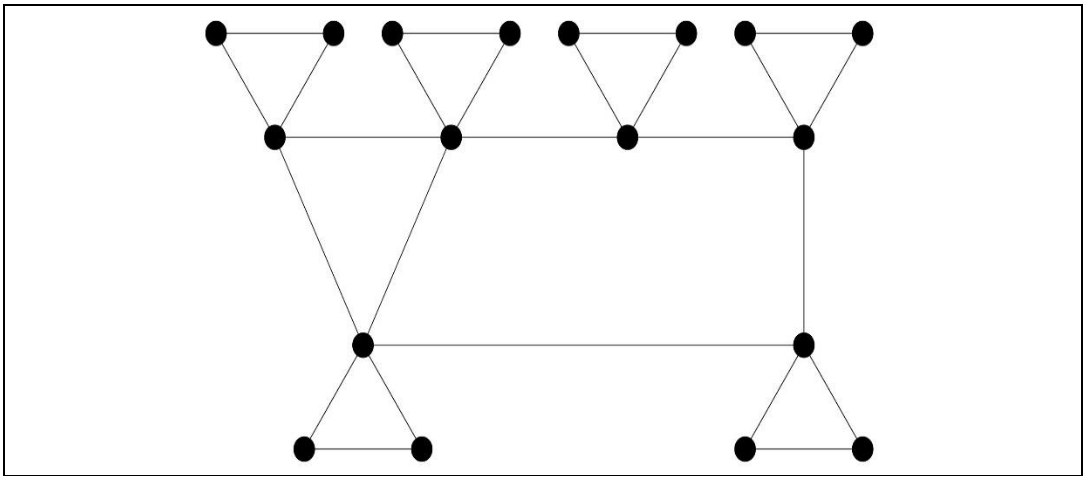

Figure 2.1: A graph in the family $\mathcal{F}$.

If the removal of any edge from a well-covered graph leaves a graph which is also well-covered, the graph is called strongly well-covered. For example, the only complete graphs which are strongly well-covered are $K_1$ and $K_2$, the only strongly well-covered complete bipartite graphs are $K_{1,1}$ and $K_{2,2}$, and $C_4$ is the only strongly well-covered cycle. Pinter [27] characterized strongly well-covered graphs with independence number two and showed that a strongly well-covered graph on at least four vertices is 3-connected and has minimum degree at least four. Furthermore, Pinter constructed infinite families of strongly well-covered graphs with arbitrarily large (even) independence number in [22].

The literature about well-covered graphs includes some structural results for well-covered graphs under restricted girth conditions. Finbow and Hartnell [28] characterized well-covered graphs with girth at least 8. The main result of their work is stated in Theorem 2.3.

*Theorem 2.3: [28] If $G$ is a graph in which all cycles have size 8 or greater, then $G$ is well-covered if and only if every interior vertex of $G$ has exactly one leaf.*

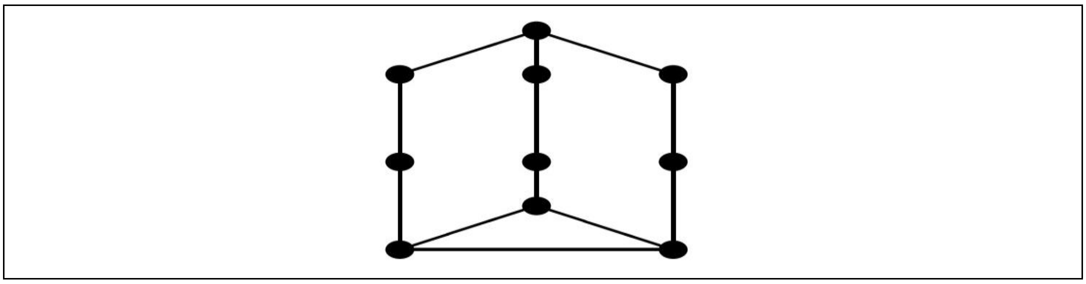

Figure 2.2: The graph $T_{10}$.

Finbow et al. in [29] characterized well-covered graphs with girth at least five. Later, Finbow et al. came up with a stronger result in [30] providing a characterization for well-covered graphs containing neither 4-nor 5-cycles. Before stating their main result, we need to mention the following definition. We say that a graph $G$ is in the family $\mathcal{F}$ if there exists $\{x_1, x_2, \dots, x_k\} \subseteq V(G)$ where for each $i$, $x_i$ is simplicial, $|N[x_i]| \leq 3$ and $\{N[x_i] | i = 1, 2, \dots, k\}$ is a partition $V(G)$. Furthermore, a vertex $v$ of a graph $G$ is called a simplicial vertex if any two vertices of $N(v)$ are adjacent in $G$. The main result of Finbow et al. in [30] is stated in Theorem 2.4.

*Theorem 2.4: [30] A graph G containing neither 4-cycles nor 5-cycles is well-covered if and only if $G$ is isomorphic to $C_7$ (a 7-cycle), $T_{10}$ (shown in Figure 2.2) or $G \in \mathcal{F}$.*

Note that checking membership of a graph in the family $\mathcal{F}$ can be done in polynomial time [30]; therefore, the characterization in Theorem 2.4 provides a polynomial-time recognition algorithm as well. For more details on subclasses and properties of well-covered graphs, the reader is referred to a comprehensive survey paper by Plummer [3].

Well-dominated graphs are closely connected to well-covered graphs since a maximal independent set in a graph is a minimal dominating set. In fact, well-dominated graphs are a subclass of well-covered graphs. Thus, the works related to well-dominated graphs often contain results about well-coveredness property as well.

Finding a minimum dominating set in a general graph is an NP-complete problem [1]. However, if the input is restricted to well-dominated graphs, then the problem has a polynomial-time solution. Finbow et al. [5] introduced for the first time the concept of well-dominated graphs. A graph is said to be well-dominated if all its minimal dominating sets have the same cardinality, i.e., $\gamma(G) = \Gamma(G)$. The computational complexity of determining whether a given graph is well-dominated is not known to be in class NP [7]. Like well-covered property, no complete structural characterization has been given for well-dominated graphs so far in the literature. The literature about well-dominated graphs mainly contains results for special subclasses of well-dominated graphs and well-dominated graphs with restricted girth.

Finbow et al. in [31] provide two characterization results: one for well-dominated graphs of girth at least five and the other for well-dominated bipartite graphs. Their characterization of well-dominated graphs of girth at least five is mainly based on the result of their previous research work on well-covered graphs of girth at least five in [29], whose main result is stated in Theorem 2.5.

*Theorem 2.5: [29] Let G be a connected well-covered graph of girth* $\geq 5$. *Then G belongs to the family* $\mathcal{PC}$ *or G is isomorphic to* $K_1$, $C_7$, $P_{10}$, $P_{13}$, $Q_{13}$, *or* $P_{14}$.

Before explaining the family $\mathcal{PC}$, we need the definition for a basic 5-cycle ($C_5$). A 5-cycle in a graph $G$ is called basic if it does not contain two adjacent vertices of degree three or more in $G$. A graph $G$ is in the family $\mathcal{PC}$ if its vertices can be partitioned into two subsets: $\mathcal{P}$ which contains the vertices incident with the pendant edges, where in addition the pendant edges form a perfect matching of $\mathcal{P}$; $\mathcal{C}$ which contains the vertices of the basic 5-cycles and the basic 5-cycles form a partition of $\mathcal{C}$. Moreover, $K_1$ is a complete graph on a single vertex, $C_7$ is a 7-cycle and $P_{10}$, $P_{13}$, $Q_{13}$, and $P_{14}$ are shown in Figure 2.3. Using the result in Theorem 2.5, Finbow et al. [31] prove that the graphs in the family $\mathcal{PC}$ are well-dominated and among the other well-covered graphs stated in Theorem 2.5, $K_1$, $C_7$, $P_{10}$, and $P_{14}$ are well-dominated. The main result of [31] regarding well-dominated graphs of girth at least 5 is stated in Theorem 2.6.

*Theorem 2.6: [31] Let G be a connected well-dominated graph of girth* $\geq 5$. *Then G is in* $\mathcal{PC}$ *or G is isomorphic to* $K_1$, $C_7$, $P_{10}$, *or* $P_{14}$.

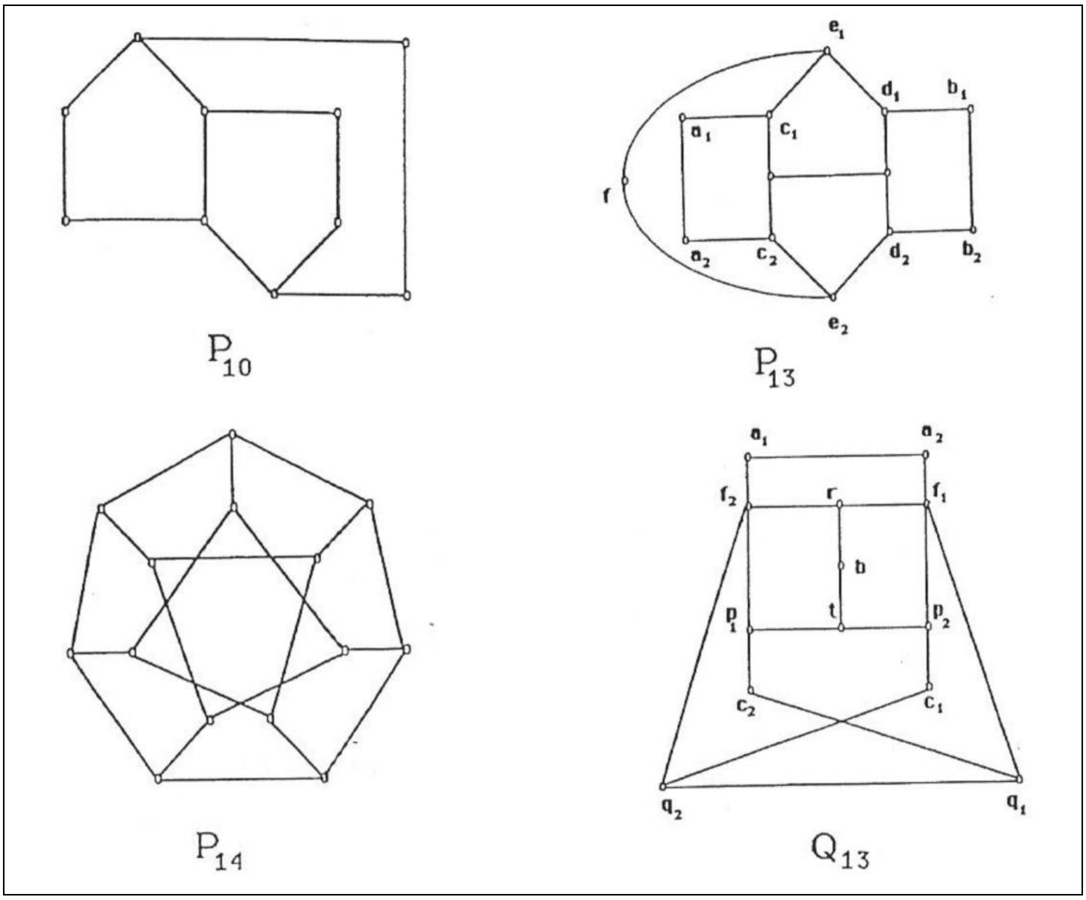

Figure 2.3: The graphs $P_{10}$, $P_{13}$, $Q_{13}$, and $P_{14}$.

Furthermore, Finbow et al. in [31] showed that a bipartite well-dominated graph is either in the family $\mathcal{P}$ or is a 4-cycle. This characterization implies a polynomial-time recognition algorithm for bipartite well-dominated graphs as well. If the input is restricted to the graphs of girth at least 6, the following Corollary is obtained from Theorem 2.5 which states that under girth at least 6 condition, the classes of well-covered and well-dominated graphs coincide.

*Corollary 2.1: Let G be a graph of girth at least 6. Then G is well-dominated if and only if G is well-covered.*

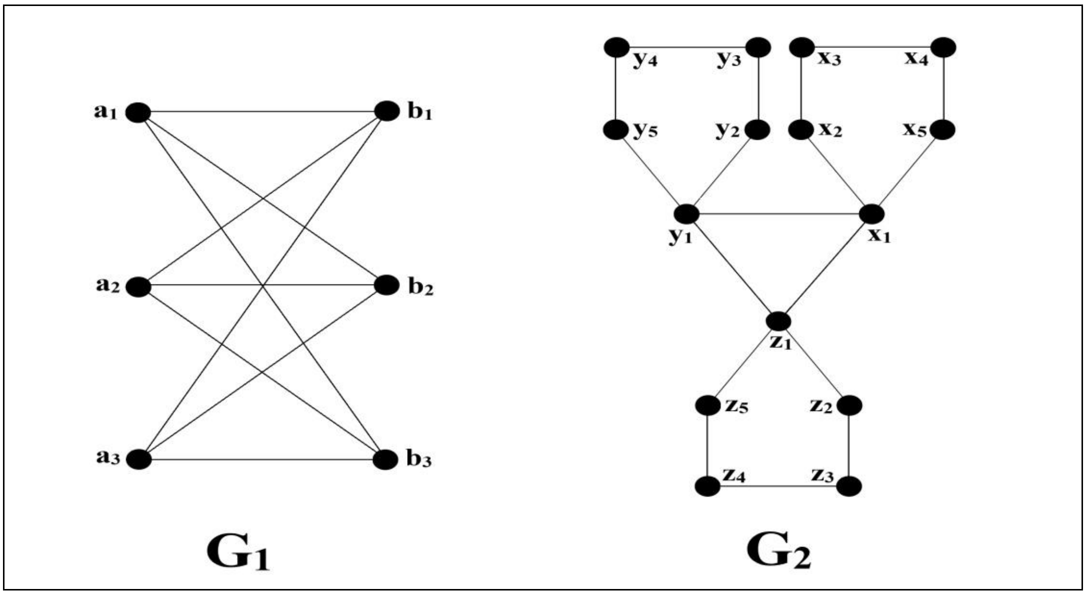

Figure 2.4: The graphs $G_1$ and $G_2$.

Topp and Volkmann [32] provided characterizations for well-dominated and well-covered block graphs and unicyclic graphs. Zverovich et al. [33] gave characterizations for locally well-dominated graphs and locally independent well-dominated graphs. In addition, they showed that the domination number and the independent domination number can be computed in polynomial time for a number of graph classes such as locally well-dominated graphs. King [34] characterized well-dominated graphs that are 3-connected, planar, and claw-free, and then extended the results to a characterization of all well-covered graphs with mentioned properties. Gionet et al. [35] characterized 4-connected, 4-regular, and claw-free well-dominated graphs.

Building upon the result of Finbow et al. in [30], Levit and Tankus [36] show that a well-dominated graph containing neither 4-cycles nor 5-cycles can be recognized in polynomial time. Levit and Tankus gave a stronger result than that in Corollary 2.1 in the following theorem:

*Theorem 2.7: [36] Let G be a graph containing neither $C_4$ nor $C_5$. Then G is well-dominated if and only if G is well-covered.*

In terms of coincidence between well-covered and well-dominated graph classes, the result stated in Theorem 2.7 is the strongest possible result since the example graphs $G_1$ and $G_2$ (Figure 2.4) show that $G_1$, which contains $C_4$ but not $C_5$, is a well-covered graph whose independent sets are all of size three; however, it is not well-dominated since it has minimal dominating sets of sizes two and three. On the other hand, the graph $G_2$, which contains $C_5$ but not $C_4$, is well-covered and all of its independent sets have size 6; however, it is not well-dominated since it has minimal dominating sets of sizes 6 and 7.

Thus far, we have reviewed results about well-covered and well-dominated graphs. These graphs have independence and domination gap equal to zero, respectively. While well-covered and well-dominated graphs are well-studied topics, there exist few research works in the literature about the graphs whose independence and domination gaps are not equal to zero.

Finbow et al. in [37] use the notation $M_n$ to denote the graphs having exactly $n$ distinct sizes of maximal independent sets. With this notation, $M_2$ is the class of graphs with two distinct sizes of maximal independent sets. Finbow et al. [37] gave a characterization for the graphs in $M_2$ with girth at least 8. Hartnell and Rall [38] studied the graphs with $\delta \geq 2$ and maximal independent sets with exactly $t$ different sizes for a positive integer $t$. The authors provided a characterization for graphs with $t$ cardinalities of maximal independent sets, minimum degree at least two, and girth at least $6t - 6$. In another research work, Barbosa et al. [39] showed that there exists a finite number of connected graphs with $\delta \geq 2$, $\Delta \leq D$, and girth at least 7 that have maximal independent sets with $r$ distinct sizes for $r \geq 2$ and $D \geq 3$.

Ekim et al. [8] investigated a subclass of graphs in $M_2$ where the difference between the two distinct sizes of maximal independent sets is one. They used the term almost well-covered to name these graphs whose independence gap is one. Ekim et al. [8] characterized a subclass of almost well-covered graphs with girth at least 6 and further gave an algorithm which recognizes $\{C_3, C_4, C_5, C_7\}$-free almost well-covered graphs in polynomial time. They left the characterization of the general case of almost well-covered graphs with girth at least 6 as an open question. The same authors in another study [40] dealt with the independence gap from algorithmic and structural perspective with an emphasis on perfect graphs.

In domination side, Dunbar et al. [41] used the notation $D_n$ to denote graphs with exactly $n$ distinct sizes of minimal dominating sets. With this notation, $D_2$ is the class of graphs having minimal dominating sets with exactly two different sizes. Dunbar et al. in [41] characterized a subclass of bipartite graphs in $D_2$ having a vertex adjacent to more than one leaf. In this thesis, we study a subclass of graphs in $D_2$ where the difference between the two sizes of minimal domination sets is one. We call such graphs almost well-dominated graphs.

When restricted to girth at least 8, almost well-dominated graphs are a subclass of $M_2$ graphs defined by Finbow et al. [37]. Thus, the results in [37] have implications for almost well-dominated graphs with girth at least 8. Chapter 2 of thesis is devoted for our study on almost well-dominated graphs with restricted girth. In this study, we improve the results of Finbow et al. [37] by providing a complete structural characterization for $(C_3, C_4, C_5, C_7)$-free almost well-dominated graphs. Besides, almost well-dominated graphs are a subclass of almost well-covered graphs when restricted to girth at least 6. Hence by characterization of $(C_3, C_4, C_5, C_7)$-free almost well-dominated graphs, we respond in part to the open question posed by Ekim et al. in [8].

There exists a result due to Finbow et al. [5] in the literature that shows that when restricted to domination gap zero and minimum degree at least two, there exists a single bipartite graph, that is, a 4-cycle. In our study on almost well-dominated bipartite graphs in Chapter 3, we have extended the result of Finbow et al. [5] to include the case of domination gap one by showing that there exist precisely 31 almost well-dominated bipartite graphs with minimum degree at least two.

## 2.3. Paired domination

Paired domination is a well-studied subject in the literature. A number of research works in the literature study paired domination number in special graph classes such as trees [42], claw-free cubic graphs [43], and generalized claw-free graphs [44]. Another group of research works provides upper bounds for paired domination number such as Hynes an Slater [45] whose result is stated in Theorem 2.8.

*Theorem 2.8: [45] Let G be a connected graph of order* $n \geq 3$*, then* $\gamma_{pr}(G) \leq n - 1$.

In terms of characterization results, Hynes and Slater in [45] provide characterizations for graphs with $\gamma_{pr}(G) = n$ and $\gamma_{pr}(G) = n - 1$ where n is the graph order. Ulatowski [46] extends these results by giving a characterization for graphs with $\gamma_{pr}(G) = n - 2$. Furthermore, there are several research works in the literature that deal with the relationship between paired domination number and different domination types such as total domination number e.g., [47][48].

In this thesis, we pay much more attention to the concept of upper paired domination which is a gap in the literature on paired domination. Hening et al. in [49] approach the concept of upper paired domination from an algorithmic point of view. Note that the authors define Upper-PDS as the problem of finding a $\Gamma_{\mathrm{pr}}$-set in a graph G. One result in [49] shows that the decision version of Upper-PDS problem is NP-complete for general graphs. The authors further show that Upper-PDS has a polynomial-time solution for some special graph classes such as chain graphs, threshold graphs, and proper interval graphs.

There are few research works in the literature putting forward structural results about upper paired domination. Dorbec et al. in [50] investigated the relationship between the upper paired domination number and the upper total domination number of a graph. Their result is stated in Theorem 2.9.

*Theorem 2.9: [50] Let $G$ be a graph with no isolated vertex. Then $\Gamma_t \geq 1/2(\Gamma_{pr} + 2)$.*

In the same research work, the authors provide a characterization for the trees satisfying the equality in Theorem 2.9. Dorbec et al. in [51] come up with some upper bounds for $\Gamma_{\mathrm{pr}}(\mathrm{G})$ in terms of graph parameters such as minimum degree and graph order. Theorem 2.10 expresses these upper bounds for $\Gamma_{\mathrm{pr}}(\mathrm{G})$.

*Theorem 2.10: [51] Let $G$ be a connected claw-free graph with order $n$ and minimum degree $\delta$. Then*

$$\Gamma_{pr}(G) \leq \begin{cases} 4n/5 & if\, \delta{=}1 \text{ and } n \geq 3 \\ 3n/4 & if\, \delta{=}2 \text{ and } n \geq 6 \\ 2n/3 & if\, \delta \geq 3 \end{cases} \tag{2.3}$$

The most interesting work in the literature is due to Ulatowski [9]. Ulatowski gives two characterization results: one for graphs with $\Gamma_{pr} = n$ and the other for graphs with $\Gamma_{pr} = n - 1$. We mention these results in Theorems 2.11 and 2.12.

*Theorem 2.11: [9] For a graph $G$ of order $n$, $\Gamma_{pr} = n$ if and only if $G$ is isomorphic to $mK_2$.*

Note that $mK_2$ is a graph composed of $m \geq 1$ disjoint $K_2$. The result in Theorem 2.11 implies that $K_2$ is the only connected graph whose upper paired domination number is equal to its order. Before mentioning the next reult of Ulatowski in Theorem 2.12, we need to define two notations. The graph $K^{*}_{1,t}$, which is called a subdivided star, is obtained from a star $K_{1,t}$ by subdividing every edge once. If we attach $\Delta$ triangles to the central vertex of a $K^{*}_{1,t}$, then we obtain the graph family $K^{*\Delta}_{1,t}$ for $\Delta \geq 0$.

*Theorem 2.12: [9] Let $G$ be a connected graph of order $n \geq 3$. Then $\Gamma_{pr} = n - 1$ if and only if $G \in \{C_3, C_5, K^{*\Delta}_{1,t}\}$.*

In our study on paired domination, which is included in Chapter 4, we use the result in Theorem 2.11 for characterization of bipartite graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ and that in Theorem 2.12 for characterization of unicyclic graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$.

# 3. ALMOST WELL-DOMINATED GRAPHS

In this chapter, we investigate almost well-dominated graphs. We start with providing some results for the general case of almost well-dominated graphs and then proceed with results for almost well-dominated graphs with restricted girth in Sections 3.1 and 3.2.

Recall that the domination gap of a graph $G$, denoted by $\mu_d(G)$, is defined as the difference $\Gamma(G) - \gamma(G)$. While well-dominated graphs are those graphs having domination gap zero, almost well-dominated graphs are the ones having domination gap one. The result in Lemma 3.1 prompts us to focus on connected graphs for the remainder of this chapter.

*Lemma 3.1. A graph is almost well-dominated if and only if all its components are well-dominated, except one, which is almost well-dominated.*

*Proof.* Let $G$ be an almost well-dominated graph and let $H_1$, $H_2$,..., $H_k$ be the components of $G$. By the definition of $\mu_d(G)$, we have $\mu_d(G) = \sum_{n=1}^{k} \mu_d\,(H_n)$. Since $G$ is almost well-dominated, then $\mu_d(G) = 1$. Thus, the domination gap is one for only one of the components and it is zero for all the other components. The converse is easy to verify. ☐

For a given integer $k \geq 0$, a vertex $x$ in graph $G$ is said to be of type-k if it is adjacent to $k$ leaves. Lemma 3.2 determines the types of vertices that can exist in a graph with domination gap $k$.

*Lemma 3.2. If $\mu_d(G) = k$ for any $k \geq 0$, then every internal vertex of $G$ is adjacent to at most $k+1$ leaves.*

*Proof.* Suppose to the contrary that there exists an internal vertex x with $p \geq k+2$ leaves $l_1, l_2, \ldots, l_p$. Since the leaves of x are private neighbors of it, there exists a minimal dominating set $D$ including $x$. Consider the set $D' = D - \{x\} \cup \{l_1, l_2, \ldots, l_p\}$. Then, there exists a minimal dominating set $D''$ in $G$ with $|D''| \geq |D'| = |D| + p - 1$. This implies that $\mu_d(G) \geq k+1$, contradicting the assumption $\mu_d(G) = k$. ☐

Corollary 3.1 states an implication of Lemma 3.2 for almost well-dominated graphs.

*Corollary 3.1. Let $G$ be an almost well-dominated graph. Then every internal vertex of $G$ is adjacent to at most 2 leaves.*

By Corollary 3.1, the internal vertices of an almost well-dominated graph are of type-0, type-1, or type-2. In addition, we use the following lemma frequently in our arguments.

*Lemma 3.3. For every independent set $I$ in G, $\mu_d(G - N[I]) \leq \mu_d(G)$.*

*Proof.* Let $H = G - N[I]$. Suppose to the contrary that $\mu_d(H) > \mu_d(G)$. Then there exist two minimal dominating sets $D_1$ and $D_2$ in $H$ such that $|D_1| - |D_2| = \mu_d(H)$. Clearly, adding $I$ to $D_1$ and $D_2$ results in two minimal dominating sets $D_1'$ and $D_2'$ in $G$ such that $|D_1'| - |D_2'| > \mu_d(G)$, which is a contradiction. ☐

An immediate result of Lemma 3.3 for almost well-dominated graphs is stated in the following corollary.

*Corollary 3.2. Let $G$ be an almost well-dominated graph. Then for every independent set $I$ in G, the graph $G - N[I]$ is either an almost well-dominated or a well-dominated graph.*

Our first result on almost well-dominated graphs is stated in the following lemma, which provides a basis for our characterization by restricting the number of vertices of type-2 existing in an almost well-dominated graph.

*Lemma 3.4. Let $G$ be an almost well-dominated graph. Then G has at most one vertex of type-2.*

*Proof.* Suppose to the contrary that G has at least two vertices of type-2, say $x$ and $y$ with leaves $\{l_1, l_2\}$ and $\{l_3, l_4\}$, respectively. Since both of $x$ and $y$ have leaves (private neighbors), then there exists a minimal dominating set $D_1$ containing $x$ and $y$. Consider the set $D = D_1 - \{x, y\} \cup \{l_1, l_2, l_3, l_4\}$. Then $G$ has another minimal dominating set $D_2$ with $|D_2| \geq |D| = |D_1| + 2$, which implies that $\mu_d(G) \geq 2$, a contradiction. ☐

Based on the result of Lemma 3.4, we continue our characterization in the following cases:

- almost well-dominated graphs containing a single vertex of type-2
- almost well-dominated graphs containing no vertex of type-2

## 3.1. Almost well-dominated graphs containing a single vertex of type-2

Our result in this section on almost well-dominated graphs of girth at least 6 with a single vertex of type-2 is stated in Lemma 3.7, which follows from the results in the following two lemmas.

*Lemma 3.5. [41] If* $G \in \mathcal{D}_2$ *and* $G$ *has a vertex* $x$ *adjacent to a set of leaves* $L'$*, where* $|L'| \geq 2$*, then* $G - (\{x\} \cup L')$ *must be in* $\mathcal{D}_1$*.*

*Lemma 3.6. [29] Let G be a connected well-dominated graph of girth at least 6. Then G belongs to the family* $\mathcal{P}$ *or G is isomorphic to* $K_1$ *or* $C_7$*.*

However, before stating the main lemma, we need to define the following graph family $\mathcal{G}_1$.

*Definition 3.1. A graph* $G$ *with girth at least 6 is in the family* $\mathcal{G}_1$ *if it has a single vertex of type-2 and the rest of the internal vertices, if any, are of type-1.*

*Lemma 3.7. Let G be a connected graph of girth at least 6 with a single vertex of type-2. Then* $G$ *is almost well-dominated if and only if* $G \in \mathcal{G}_1$*.*

*Proof.* Let $x$ be a vertex of type-2 in $G$ with two leaves, say $\{\ell_1, \ell_2\}$. We first prove that if $G$ is almost well-dominated, then $G \in \mathcal{G}_1$. Let $G' = G - \{x, \ell_1, \ell_2\}$ and note that $G'$ might have more than one component. By Lemma 3.5, we have $G' \in \mathcal{D}_1$. This means that every component of $G'$ is well-dominated. In addition, by Lemma 3.6, the graphs $K_1$, $C_7$, and the family $\mathcal{P}$ are the only possible candidates for the components of $G'$. If there exists a component of $G'$ isomorphic to $K_1$, then denote the single vertex of $K_1$ by $y$. Then the vertex $x$ is a vertex of type-3 in $G$, a contradiction by Lemma 3.2. On the other hand, if there exists a component of $G'$ isomorphic to a $C_7 = (abcdefg)$, then due to girth at least 6, $x$ is adjacent to exactly one vertex, say

$c$, on $C_7$. Consider the independent set $I = \{a, e\}$. Then the vertex $x$ is of type-3 in $G - N[I]$, a contradiction by Lemma 3.2. Now we turn our attention to the case where a component of $G'$ belongs to the family $\mathcal{P}$. We show that $x$ is adjacent to the components of $G'$ through the stems of these components. Suppose to the contrary that $x$ is adjacent to a leaf $\ell$ in a component $H \in \mathcal{P}$. Let $s$ be the stem of $\ell$. The stem $s$ has at least one neighbor, say $u$, different from $\ell$ since otherwise it would not be a stem. The vertex $u$ is not adjacent to $x$ since otherwise $\{x, l, s, u\}$ forms a 4-cycle. Consider the independent set $I = \{u\}$. The vertex $x$ is of type-3 in $G - N[I]$, a contradiction by Lemma 3.2. Hence, $x$ is adjacent to the components of $G'$ through the stems of these components. Thus, $G \in \mathcal{G}_1$.

In order to prove the converse, assume that $G \in \mathcal{G}_1$ and $x$ is the only vertex of type-2. Note that from each internal vertex of type-1 and its respective leaf, only one vertex is included in any minimal dominating set $D$ in $G$. Further, $D$ includes either $x$ and hence has cardinality $|L_G| - 1$ or $D$ includes the leaves of $x$ and hence has cardinality $|L_G|$. Thus, $\mu_d(G) = 1$. ☐

## 3.2. Almost well-dominated graphs containing no vertex of type-2

In this section we focus on almost well-dominated graphs whose internal vertices are of type-0 or type-1. Our starting point is the following proposition.

*Proposition 3.1. Let $G$ be an almost well-dominated graph. If $G$ does not contain a vertex of type-2, then it contains a vertex of type-0.*

*Proof.* Suppose to the contrary that there exists no vertex of type-0 in $G$. Then all internal vertices in $G$ are of type-1, thus $G \in \mathcal{P}$ *a*nd hence $G$ is well-dominated, a contradiction. ☐

Our next result restricts the number of type-0 neighbors of a type-0 vertex in $(C_3, C_4, C_5, C_7)$-free almost well-dominated graphs.

*Lemma 3.7. Let $x$ be a vertex of type-0 in a $(C_3, C_4, C_5, C_7)$-free almost well-dominated graph $G$. Then $x$ has at most two neighbors of type-0.*

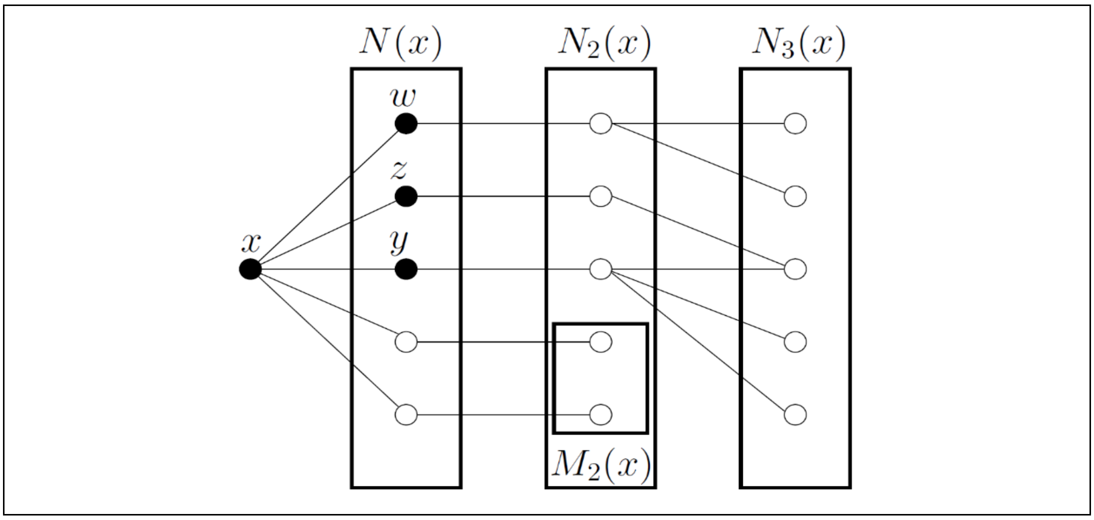

Figure 3.1: Type-0 vertex $x$ with three type-0 neighbors.

*Proof.* Suppose to the contrary that x has at least three neighbors of type-0, say $y$, $z$, and $w$ in $G$ (see Figure 3.1). Note that x may also have neighbors of type-1 as shown in Figure 3.1. Let $N_2(x)$ and $N_3(x)$ denote the vertices at distance 2 and 3 from $x$, respectively. Since $G$ is a $(C_3, C_5, C_7)$-free graph, both $N_2(x)$ and $N_3(x)$ are independent sets. Let $M_2(x)$ be the leaves of type-1 neighbors of $x$. Note that $I = N_3(x) \cup M_2(x)$ is an independent set in $G$. Let $H = G - N[I]$. The graph $H$ has a vertex $x$ with 3 leaves and hence $\mu_d(H) \geq 2$, a contradiction by Corollary 3.2. □

In the rest of the chapter, a component of the subgraph induced by the vertices of type-0 is called type-0 component. Lemma 3.7 provides a tool to determine the structure of type-0 components in a $(C_3, C_4, C_5, C_7)$-free almost well-dominated graph.

*Corollary 3.3. Let $G$ be a $(C_3, C_4, C_5, C_7)$-free almost well-dominated graph with no vertex of type-2. Then the graph induced by the vertices of type-0 is composed of components isomorphic to a path $P_i \in \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_{10}\}$ or a cycle $C_j \in \{C_6, C_8, C_9, C_{10}, C_{11}, C_{13}\}$.*

*Proof.* Note that the graph induced by vertices of type-0 corresponds to $G - N[L_G]$ and by Lemma 3.7, the vertices of $G - N[L_G]$ are of degrees 0, 1 or 2. The only graph classes satisfying this degree restriction are the paths and the cycles. It follows from Lemma 3.3 that every component of $G - N[L_G]$ has domination gap at most 1. Note that $\gamma(P_n) = \lceil n/3 \rceil$ and $\gamma(C_n) = \lceil n/3 \rceil$, whereas $\Gamma(P_n) = \lceil n/2 \rceil$ and $\Gamma(C_n) =$

$\lfloor n/2 \rfloor$. Thus, $P_2, P_3, P_4, P_5, P_6, P_7, P_8$, and $P_{10}$ are the only paths having domination gap at most 1. Similarly, $C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}, C_{11}$, and $C_{13}$ are the only cycles having domination gap at most 1. ☐

The following lemma shows that a $(C_3, C_4, C_5, C_7)$-free almost well-dominated graph with no vertex of type-2 contains exactly one type-0 component.

*Lemma 3.8. Let $G$ be a $(C_3, C_4, C_5, C_7)$-free almost well-dominated graph with no vertex of type-2. Then $G$ has exactly one type-0 component.*

*Proof.* Suppose to the contrary that $G$ has at least two type-0 components and let $H_1, H_2, \ldots, H_k$ represent the set of all type-0 components where $k \geq 2$. If $k \geq 3$, choose a minimum dominating set $S_i$ of $H_i$ for $3 \leq i \leq k$ and let $S = \cup_{i=3}^{k} S_i$. By Corollary 3.3, a type-0 component in a $(C_3, C_4, C_5, C_7)$-free almost well-dominated graph is either a path $P_i$, where $i \in \{1,2,3,4,5,6,7,8,10\}$ or a cycle $C_j$, where $j \in \{6,8,9,10,11,13\}$.

First suppose that both $H_1$ and $H_2$ are cycles, say $H_1 \cong C_{m_1}$ and $H_2 \cong C_{m_2}$. Recall that a cycle $C_n$ has a minimal dominating set of size $\lfloor n/2 \rfloor$. Let $D_{H_1}$ and $D_{H_2}$ be two minimal dominating sets of sizes $\lfloor m_1/2 \rfloor$ and $\lfloor \mathrm{m}_2/2 \rfloor$ in $H_1$ and $H_2$, respectively. Observe that there exists a minimal dominating set $D_1$ in G such that $D_1 = L_G \cup D_{H_1} \cup D_{H_2} \cup S$. Then we have $|D_1| = |L_G| + \lfloor m_1/2 \rfloor + \lfloor m_2/2 \rfloor + |S|$. Note that the number of vertices of type-1 is equal to the number of leaves in $G$ and further note that type-0 components have at least one neighbor of type-1. Let $L'$ be a set which includes one of the vertices of type-1 adjacent to each of $H_1$ and $H_2$, and the leaves of other vertices of type-1. It is obvious that $|L'| = |L_G|$. Hence, by taking the set $L'$, at least one vertex from each of $H_1$ and $H_2$ is dominated. Furthermore, the remaining vertices of $H_1$ and $H_2$, which induce two paths $P_{m_1-1}$ and $P_{m_2-1}$, have minimal dominating sets of sizes $\lceil (m_1-1)/3 \rceil$ and $\lceil (m_2-1)/3 \rceil$, respectively. Then, there exists a minimal dominating set $D_2$ such that $|D_2| \leq |L_G| + \lceil (m_1-1)/3 \rceil + \lceil (m_2-1)/3 \rceil + |S|$. However, 3.1 is a contradiction.

$$|D_1| - |D_2| \geq \lfloor m_1/2 \rfloor - \lceil (m_1-1)/3 \rceil + \lfloor m_2/2 \rfloor - \lceil (m_2-1)/3 \rceil \geq 2 \qquad (3.1)$$

Next assume that both $H_1$ and $H_2$ are paths, say $H_1 \cong P_{m_1}$ and $H_2 \cong P_{m_2}$. Note that a path $P_n$ has minimal dominating sets of sizes $\lceil n/2 \rceil$ and $\lceil n/3 \rceil$. Let $D_{H_1}$ and

$D_{H_2}$ be two minimal dominating sets of sizes $\lceil m_1/2 \rceil$ and $\lceil m_2/2 \rceil$ in $H_1$ and $H_2$, respectively. Observe that the set $D_1 = L_G \cup D_{H_1} \cup D_{H_2} \cup S$ is a minimal dominating set of $G$. Thus, we have $|D_1| = |L_G| + \lceil m_1/2 \rceil + \lceil m_2/2 \rceil + |S|$. Note that the end vertices of a type-0 path have at least one neighbor of type-1 in $G$. Let $L'$ be a set including the vertices of type-1 adjacent to the end vertices of $H_1$ and $H_2$ and the leaves of other vertices of type-1. Hence, by taking the set $L'$, at least the end vertices of each of $H_1$ and $H_2$ are dominated. Moreover, the remaining vertices of $H_1$ and $H_2$, which induce two paths $P_{m_1-2}$ and $P_{m_2-2}$, have minimal dominating sets of sizes $\lceil (m_1 - 2)/2 \rceil$ and $\lceil (m_2 - 2)/2 \rceil$, respectively. Thus, there exists a minimal dominating set $D_2$ such that:

$$|\mathrm{D}_2| \leq |\mathrm{L}_\mathrm{G}| + \left\lceil \frac{(\mathrm{m}_1 - 2)}{2} \right\rceil + \left\lceil \frac{(\mathrm{m}_2 - 2)}{2} \right\rceil + |\mathrm{S}| \tag{3.2}$$

It follows from (3.2) that $|\mathrm{D}_2| \leq |\mathrm{L}_\mathrm{G}| + \left\lceil \frac{\mathrm{m}_1}{2} \right\rceil + \left\lceil \frac{\mathrm{m}_2}{2} \right\rceil - 2 + |\mathrm{S}|$. This in turn implies that $|D_1| - |D_2| \geq 2$, which is a contradiction.

In the last case, we suppose that one of the components, say $H_1$, is a cycle $C_{m_1}$, and the other, namely $\mathrm{H}_2$, is a path $\mathrm{P}_{\mathrm{m}_2}$. Let $D_{H_1}$ and $D_{H_2}$ be two minimal dominating sets of sizes $\lfloor \mathrm{m}_1/2 \rfloor$ and $\lceil \mathrm{m}_2/2 \rceil$ in $\mathrm{H}_1$ and $\mathrm{H}_2$, respectively. Similarly, the set $D_1 = L_G \cup D_{H_1} \cup D_{H_2} \cup S$ is a minimal dominating set of G. Thus, we have $|D_1| = |L_G| + \lfloor m_1/2 \rfloor + \lceil m_2/2 \rceil + |S|$. Notice that $H_1$ has at least one neighbor of type-1 and the end vertices of $\mathrm{H}_2$ both have neighbors of type-1. Let $L'$ be a set including the vertices of type-1 adjacent to the type-0 components and the leaves of other vertices of type-1. Hence, by taking the set $\mathrm{L}'$, at least one vertex from $\mathrm{H}_1$ and two end vertices of $\mathrm{H}_2$ are dominated. Therefore, the remaining vertices of $\mathrm{H}_1$, which induce a path $\mathrm{P}_{\mathrm{m}_1-1}$ and the remaining vertices of $\mathrm{H}_2$, which induce a path $\mathrm{P}_{\mathrm{m}_2-2}$ have minimal dominating sets of sizes $\lceil (m_1 - 1)/3 \rceil$ and $\lceil (m_2 - 2)/2 \rceil$, respectively. Hence, there exists a minimal dominating set $\mathrm{D}_2$ such that:

$$|\mathrm{D}_2| \leq |\mathrm{L}_\mathrm{G}| + \left\lceil \frac{(\mathrm{m}_1 - 1)}{3} \right\rceil + \left\lceil \frac{(\mathrm{m}_2 - 2)}{2} \right\rceil + |\mathrm{S}| \tag{3.3}$$

It follows from (3.3) that $|\mathrm{D}_2| \leq |\mathrm{L}_\mathrm{G}| + \left\lceil\frac{(\mathrm{m}_1-1)}{3}\right\rceil + \lceil \mathrm{m}_2/2 \rceil - 1 + |\mathrm{S}|$. This implies that $|D_1| - |D_2| = \lfloor m_1/2 \rfloor - \lceil (m_1-1)/3 \rceil + 1 \geq 2$, a contradiction. □

From here onwards, we denote the type-0 component of $G$ by $G_0$. Recall that $L_G$ denotes the set of leaves in a graph $G$. We will use the following proposition frequently in our proofs.

*Proposition 3.2. Let $G$ be a graph with no vertex of type-k for $k \geq 2$. Then, $\Gamma(G) = |L_G| + \Gamma(G_0)$.*

*Proof.* Let $G$ be a graph with no vertex of type-k for $k \geq 2$. Note that the set of leaves of G together with a maximum minimal dominating set of $\mathrm{G}_0$ is a minimal dominating set of size $|L_G| + \Gamma(G_0)$ in $G$. Furthermore, we show that there is no minimal dominating set of size at least $|L_G| + \Gamma(G_0) + 1$ in G. First notice that any minimal dominating set of $G$ contains exactly one vertex from each stem-leaf pair since otherwise it is not minimal. Now consider a dominating set $D$ of size at least $|L_G| + \Gamma(G_0) + 1$ in $G$. Then $D$ contains either at least $\Gamma(G_0) + 1$ vertices from $\mathrm{G}_0$ or at least $|L_G| + 1$ vertices from the stem-leaf pairs. Both cases imply that $D$ is not minimal. Thus, $\Gamma(G) = |L_G| + \Gamma(G_0)$. □

In what follows, we focus on the cases where $G_0$ is isomorphic to one of the paths or cycles mentioned in Corollary 3.3. Using the previous results and lemmas, we show that some of these cases yield families of $(C_3, C_4, C_5, C_7)$-free almost well-dominated graphs.

### 3.2.1. Type-0 component is a path

In this section, we analyze almost well-dominated graphs with a type-0 component isomorphic to a path $P_n$. Recall that a path $P_n$ has $\gamma(P_n) = \lceil n/3 \rceil$ and $\Gamma(P_n) = \lceil n/2 \rceil$. First let $G_0 \cong P_1$. We define the graph family $\mathcal{G}_2$ and then state the result for this case in Lemma 3.9.

*Definition 3.1. A $(C_3, C_4, C_5, C_7)$-free graph $G$ is in the family $\mathcal{G}_2$, if it has a single vertex of type-0 with at least two neighbors of type-1 and the rest of the internal vertices, if any exist, are of type-1.*

*Lemma 3.9. Let $G$ be a ($C_3, C_4, C_5, C_7$)-free graph without a vertex of type-2. Then $G$ is almost well-dominated with $G_0 \cong P_1$ if and only if $G \in \mathcal{G}_2$.*

*Proof.* Let $G$ be a ($C_3, C_4, C_5, C_7$)-free graph without a vertex of type-2. If $G$ is almost well-dominated with $G_0 \cong P_1$, then $G \in \mathcal{G}_2$ by definition of $\mathcal{G}_2$.

To prove the converse, we assume that $G \in \mathcal{G}_2$ and let $v$ be the vertex of type-0 in $G$. By Proposition 3.2, we have $\Gamma(G) = |L_G| + 1$. Note further that every minimal dominating set $D$ includes exactly one vertex from each stem-leaf pair; thus, $|D| \geq |L_G|$. If any stem adjacent to $v$ is included in a minimal dominating set $D_1$, then $v \notin D_1$ and thus $|D_1| = |L_G|$. On the other hand, if none of the stems adjacent to $v$ are included in a minimal dominating set $D_2$, then $v \in D_2$, and thus $|D_2| = |L_G| + 1$. Hence, $\mu_d(G) = 1$. □

Next suppose that $G_0 \cong P_2$. In this case we obtain a graph family $\mathcal{G}_3$ which is defined in Definition 3.2.

*Definition 3.2. A ($C_3, C_4, C_5, C_7$)-free graph $G$ is in the family $\mathcal{G}_3$, if it has one type-0 component $H \cong P_2$ where the end vertices of $H$ have at least one neighbor of type-1 in $G$ and the rest of the internal vertices, if any, are of type-1.*

*Lemma 3.10. Let $G$ be a ($C_3, C_4, C_5, C_7$)-free graph without a vertex of type-2. Then $G$ is almost well-dominated with $G_0 \cong P_2$ if and only if $G \in \mathcal{G}_3$.*

*Proof.* Let $G$ be a ($C_3, C_4, C_5, C_7$)-free graph without a vertex of type-2. If $G$ is almost well-dominated with $G_0 \cong P_2$, then each end vertex of $P_2$ has at least one neighbor of type-1 in $G$ and the rest of the internal vertices (if any) are of type-1. Hence, $G \in \mathcal{G}_3$.

To prove the converse, let $G \in \mathcal{G}_3$. Note that every minimal dominating set includes exactly one vertex from each stem-leaf pair; thus, each minimal dominating set is of size at least $|L_G|$. Furthermore, by Proposition 3.2, we have $\Gamma(G) = |L_G| + 1$. Therefore, it remains to show that $G$ has two minimal dominating sets of sizes $|L_G|$ and $|L_G| + 1$. If both stems adjacent to the end vertices of $P_2$ are included in a minimal dominating set, then no vertex from $P_2$ can be added to this minimal dominating set; hence, such a minimal dominating has size $|L_G|$. However, if none of the stems adjacent to the $P_2$ are included in a minimal dominating set, one vertex from $P_2$ can be added to this minimal dominating set, which has size $|L_G| + 1$. Thus,

$G$ is an almost well-dominated graph since all minimal dominating sets are of size either $|L_G|$ or $|L_G| + 1$. □

In the case of $G_0 \cong P_3$, we define the graph family $\mathcal{G}_4$ in Definition 3.3 and state the result for this case in Lemma 3.11.

*Definition 3.3. A $(C_3, C_4, C_5, C_7)$-free graph $G$ is in the family $\mathcal{G}_4$ if it has one type-0 component $H \cong P_3$, where the end vertices of $H$ have at least one neighbor of type-1 in $G$, the middle vertex in $H$ has no neighbors of type-1 in $G$, and the rest of the internal vertices, if any, are of type-1.*

*Lemma 3.11. Let $G$ be a $(C_3, C_4, C_5, C_7)$-free graph without a vertex of type-2. Then $G$ is almost well-dominated with $G_0 \cong P_3$ if and only if $G \in \mathcal{G}_4$.*

*Proof.* Let $G$ be a $(C_3, C_4, C_5, C_7)$-free graph without a vertex of type-2. Suppose that $G$ is almost well-dominated with $G_0 \cong P_3$ and let $P_3 = [abc]$. Since $a$ and $c$ are of type-0, they have at least one neighbor of type-1. Further, we claim that the middle vertex of $P_3$, namely $b$, does not have a neighbor of type-1. Suppose to the contrary that $b$ has at least one neighbor of type-1. Then, the set of leaves $L_G$ together with $\{a, c\}$ form a minimal dominating set $D_1$ of size $|L_G| + 2$. On the other hand, consider a minimal dominating set $D_2$ which includes the type-1 neighbors of $a$, $b$ and $c$. Such a minimal dominating set includes no vertices from $\{a, b, c\}$ and is of size $|L_G|$. Hence $\mu_d(G) \geq 2$, a contradiction. Thus, $c$ has no neighbor of type-1 and hence $G \in \mathcal{G}_4$.

To prove the converse, suppose that $G \in \mathcal{G}_4$. By Proposition 3.2, we have $\Gamma(G) = |L_G| + 2$. Moreover, note that every minimal dominating set in a graph $G \in \mathcal{G}_4$ includes $|L_G|$ vertices from stem-leaf pairs and either one (the vertex $b$) or two vertices ($a$ and $c$) from $P_3$. Thus, all minimal dominating sets are of size either $|L_G| + 1$ or $|L_G| + 2$ and hence $G$ is an almost well-dominated graph. □

We proceed with the case $G_0 \cong P_4$. This case yields another family of almost well-dominated graphs $\mathcal{G}_5$ defined in Definition 3.4.

*Definition 3.4. A $(C_3, C_4, C_5, C_7)$-free graph $G$ is in the family $\mathcal{G}_5$ if it has one type-0 component $H \cong P_4 = [abcd]$ where the end vertices of $H$, namely $a$ and $d$ have at least one neighbor of type-1 in $G$, at least one of the middle vertices of $H$, say $b$ has no neighbors of type-1 in $G$, and the rest of the internal vertices, if any, are of type-1.*

*Lemma 3.12. Let $G$ be a ($C_3, C_4, C_5, C_7$)-free graph without a vertex of type-2. Then $G$ is almost well-dominated with $G_0 \cong P_4$ if and only if $G \in \mathcal{G}_5$.*

*Proof.* Let $G$ be a $(C_3, C_4, C_5, C_7)$-free graph without a vertex of type-2. Suppose that $G$ is almost well-dominated with $G_0 \cong P_4$ and let $P_4 = [abcd]$. Since the end vertices $a$ and $d$ are of type-0, they have at least one neighbor of type-1 in $G$. Furthermore, we show that since $G$ is almost well-dominated, at least one of the middle vertices, namely $b$ or $c$, does not have a neighbor of type-1 in $G$. Suppose to the contrary that both of $b$ and $c$ have neighbors of type-1 in $G$. Then the set of leaves $L_G$ together with two vertices from $P_4$, say $\{a, c\}$, form a minimal dominating set $D_1$ of size $|L_G| + 2$ in $G$. On the other hand, consider a minimal dominating set $D_2$ which includes the type-1 neighbors of $a, b, c$ and $d$. While such a minimal dominating set includes no vertices from $P_4$, it includes exactly one vertex from each stem-leaf pair and has size $|L_G|$. Thus, $\mu_d(G) \geq 2$, a contradiction. Therefore, at least one of the middle vertices of $P_4$ has no type-1 neighbors in $G$. Hence, $G \in \mathcal{G}_5$.

For the converse, assume that $G \in \mathcal{G}_5$. By Proposition 3.2, we have $\Gamma(G) = |L_G| + 2$. Moreover, notice that every minimal dominating set in a graph $G \in \mathcal{G}_5$ includes $|L_G|$ vertices from stem-leaf pairs and either one or two vertices from $P_4$. Thus, all minimal dominating sets are of size either $|L_G| + 1$ or $|L_G| + 2$ and hence $G$ is almost well-dominated. ◻

The last case which we analyze in this section is the case of $G_0 \cong P_6$. However, we first deal with the cases where $G_0 \in \{P_5, P_7, P_8, P_{10}\}$ and show that in these cases $G$ is not almost well-dominated.

*Lemma 3.13. Let $G$ be a ($C_3, C_4, C_5, C_7$)-free graph without a vertex of type-2 and suppose that the graph induced by the vertices of type-0 in $G$ is isomorphic to a $P_m$ for $m \in \{5,7,8,10\}$. Then $G$ is not almost well-dominated.*

*Proof.* Let $G$ be a $(C_3, C_4, C_5, C_7)$-free graph without a vertex of type-2 and let $H$ be the graph induced by the vertices of type-0 in $G$. Suppose that $H \cong P_m$ where $m \in \{5,7,8,10\}$. Note that a path on $m$ vertices has two minimal dominating sets of sizes $\lceil m/2 \rceil$ and $\lceil m/3 \rceil$. Consider a minimal dominating set of cardinality $\lceil m/2 \rceil$ in $H$, say $D_H$. Then, the set of leaves $L_G$ together with $D_H$ form a minimal dominating set $D_1$ of size $|L_G| + \lceil m/2 \rceil$ in $G$. On the other hand, let $S$ be the set of stems in $G$. Note that

$|S| = |L_G|$ and $S$ definitely dominates the end vertices of $H$. Let further $H'$ be the graph induced by the internal vertices of $H$. Since $H' \cong P_{m-2}$, a minimal dominating set $D_{H'}$ in $H'$ together with $S$ form a minimal dominating set $D_2$ of size at most $|L_G| + \lceil (m-2)/3 \rceil$ in $G$. For $m \in \{5,7,8,10\}$, we get $|D_1| - |D_2| \geq 2$, thus $G$ is not almost well-dominated. □

The case of $G_0 \cong P_6$ leads to a family of almost well-dominated graphs $\mathcal{G}_6$ defined in Definition 3.5.

*Definition 3.5. A $(C_3, C_4, C_5, C_7)$-free graph $G$ is in the family $\mathcal{G}_6$ if it has one type-0 component $H \cong P_6 = [abcdef]$ where the end vertices of $H$ have at least one neighbor of type-1 in $G$, the vertices adjacent to the end vertices of $H$, namely $\{b, e\}$, have no neighbors of type-1 in $G$, and the rest of the internal vertices, if any, are of type-1.*

*Lemma 3.14. Let $G$ be a $(C_3, C_4, C_5, C_7)$-free graph without a vertex of type-2. Then $G$ is almost well-dominated with $G_0 \cong P_6$ if and only if $G \in \mathcal{G}_6$.*

*Proof.* Let $G$ be a $(C_3, C_4, C_5, C_7)$-free graph without a vertex of type-2. Suppose that $G$ is almost well-dominated with $G_0 \cong P_6$ and let $P_6 = [abcdef]$. As the end vertices $a$ and $f$ are of type-0, they have at least one neighbor of type-1 in $G$. Furthermore, it is easy to see that $G$ has two minimal dominating sets $D_1 = L_G \cup \{a, c, e\}$ and $D_2 = L_G \cup \{b, e\}$ of sizes $|L_G| + 3$ and $|L_G| + 2$, respectively. Thus, in order for $G$ to be almost well-dominated, the minimal dominating sets of size smaller than $|L_G| + 2$ must be avoided. We show that the vertices $b$ and $e$ have no neighbors of type-1 in $G$. Suppose for a contradiction that at least one of $b$ and $e$, say $b$, has neighbors of type-1 in $G$. Then, consider a minimal dominating set $D_3$ which includes the vertex $d$ from $P_6$ and the type-1 neighbors of $a$, $b$, and $f$. Such a minimal dominating set includes exactly one vertex from each stem-leaf pair and the vertex $d$. Hence, $|D_3| = |L_G| + 1$. Thus, $\mu_d(G) \geq 2$, a contradiction. Therefore, none of $b$ and $e$ have neighbors of type-1 in $G$. Thus, $G \in \mathcal{G}_6$.

To prove the converse, suppose that $G \in \mathcal{G}_6$. By Proposition 3.2, we have $\Gamma(G) = |L_G| + 3$. Furthermore, note that every minimal dominating set in a graph $G \in \mathcal{G}_6$ includes $|L_G|$ vertices from stem-leaf pairs and either two or three vertices from $P_6$. Thus, all minimal dominating sets of $G$ have size either $|L_G| + 2$ or $|L_G| + 3$. Hence, $G$ is almost well-dominated. □

### 3.2.2. Type-0 component is a cycle

In this section we investigate the cases where the type-0 component is isomorphic to a cycle $C_n$, where $n \in \{6,8,9,10,11,13\}$. Recall that a cycle $C_n$ has $\gamma(C_n) = \lceil n/3 \rceil$ and $\Gamma(C_n) = \lfloor n/2 \rfloor$. Let us first assume that $G_0 \cong C_6$. We will define the following graph family in order to state our result in Lemma 3.15.

*Definition 3.6. A ($C_3, C_4, C_5, C_7$)-free graph $G$ is in the family $\mathcal{G}_7$ if it has one type-0 component $H \cong C_6$ where no three consecutive vertices on $H$ have neighbors of type-1 in $G$ and the rest of the internal vertices, if any, are of type-1.*

*Lemma 3.15 Let $G$ be a ($C_3, C_4, C_5, C_7$)-free graph without a vertex of type-2. Then $G$ is almost well-dominated with $G_0 \cong C_6$ if and only if $G \in \mathcal{G}_7$.*

*Proof.* Let $G$ be a ($C_3, C_4, C_5, C_7$)-free graph without a vertex of type-2. Suppose that $G$ is almost well-dominated with $G_0 \cong C_6$. Note that $C_6$ has two minimal dominating sets of sizes 2 and 3; thus, together with the set $L_G$, the graph $G$ has minimal dominating sets of sizes $|L_G| + 2$ and $|L_G| + 3$. By Proposition 3.2, we have that $\Gamma(G) = |L_G| + 3$. Now it remains to ensure that the cases which lead to minimal dominating sets with size at most $|L_G| + 1$ are avoided. These cases are as follows:

- If all vertices of $C_6$ are adjacent to stems, then the stems constitute a minimal dominating set of size $|L_G|$ in $G$.
- If the vertices of $C_6$ which are not adjacent to stems induce a path $P_i$ where i $\in \{1,2,3\}$, then one vertex from $P_i$ together with the stems form a minimal dominating set of size $|L_G| + 1$ in $G$.

In order to avoid the above cases, no three consecutive vertices on $C_6$ must have neighbors of type-1; therefore $G \in \mathcal{G}_7$.

To prove the converse suppose that $G \in \mathcal{G}_7$. By Proposition 3.2, we have $\Gamma(G) = |L_G| + 3$. Furthermore, by the definition of $\mathcal{G}_7$, no three consecutive vertices on $C_6$ have type-1 neighbors, which implies that all the minimal dominating sets of G include at least two vertices from $C_6$ and hence of size at least $|L_G| + 2$. Hence, $G$ is almost well-dominated. □

Next we assume $G_0 \cong C_8$. In Definition 3.7, we define an almost well-dominated graph family $\mathcal{G}_8$ which has a type-0 component isomorphic to $C_8$.

*Definition 3.7. A ($C_3, C_4, C_5, C_7$)-free graph $G$ is in the family $\mathcal{G}_8$ if it has one type-0 component $H \cong C_8 = (abcdefgh)$ where neither two consecutive vertices nor two vertices at distance 4 (say for example $a$ and $e$) on $H$ have type-1 neighbors in $G$, and the rest of the vertices, if any, are of type-1.*

*Lemma 3.16. Let $G$ be a ($C_3, C_4, C_5, C_7$)-free graph without a vertex of type-2. Then $G$ is almost well-dominated with $G_0 \cong C_8$ if and only if $G \in \mathcal{G}_8$.*

Proof. Let $G$ be a ($C_3, C_4, C_5, C_7$)-free graph without a vertex of type-2. Suppose that $G$ is almost well-dominated with $G_0 \cong C_8$ and let $C_8 = (abcdefgh)$. Note that $C_8$ has two minimal dominating sets of sizes 3 and 4; thus, together with the set $L_G$, the graph $G$ has minimal dominating sets of sizes $|L_G| + 3$ and $|L_G| + 4$. By Proposition 3.2, we have $\Gamma(G) = |L_G| + 4$. Furthermore, since $G$ is almost well-dominated, the cases leading to minimal dominating sets of size at most $|L_G| + 2$ must be avoided. The cases which require that at most two vertices from $C_8$ being included in a minimal dominating set are as follows:

- If the vertices of $C_8$ which are not adjacent to type-1 neighbors induce a single path $P_m$ where $m \leq 6$, then $\lceil m/3 \rceil$ vertices from $P_m$ together with the stems constitute a minimal dominating set $D$ of size $|L_G| + \lceil m/3 \rceil$ in $G$. For $m \leq 6$, we have that $\lceil m/3 \rceil \leq 2$. Hence, $|D| \leq |L_G| + 2$.
- If two vertices of $C_8$ with distance 4, say $a$ and $e$, have neighbors of type-1, then the stems together with two vertices from $C_8$, namely $c$ and $g$, form a dominating set of size $|L_G| + 2$, which in turn includes a minimal dominating set of size at most $|L_G| + 2$ in $G$.

In order to avoid the above cases, neither two consecutive vertices nor two vertices at distance 4 on $C_8$ have type-1 neighbors in $G$. Therefore, $G \in \mathcal{G}_8$.

To prove the converse suppose that $G \in \mathcal{G}_8$. By Proposition 3.2, $\Gamma(G) = |L_G| + 4$. Then, by definition of $\mathcal{G}_8$, all the minimal dominating sets of G include at least three vertices from $C_8$ and thus, have size at least $|L_G| + 3$. Therefore, $G$ is almost well-dominated. □

We proceed with the case where $G_0 \cong C_9$.

*Definition 3.8. A ($C_3, C_4, C_5, C_7$)-free graph $G$ is in the family $\mathcal{G}_9$, if it has one type-0 component $H \cong C_9 = (abcdefghi)$ with the following properties:*

- *No three consecutive vertices on $H$ have type-1 neighbors in $G$.*
- *No two consecutive vertices on $H$, say $\{a, b\}$, together with a vertex at distance 4 from both a and b on $H$, say $f$, have type-1 neighbors in $G$.*

*Lemma 3.17. Let $G$ be a ($C_3, C_4, C_5, C_7$)-free graph without a vertex of type-2. Then $G$ is almost well-dominated with $G_0 \cong C_9$ if and only if $G \in \mathcal{G}_9$.*

*Proof*. Let $G$ be a ($C_3, C_4, C_5, C_7$)-free graph without a vertex of type-2. Suppose that $G$ is almost well-dominated with $G_0 \cong C_9$. Note that $C_9$ has two minimal dominating sets of sizes 3 and 4; thus, together with the set $L_G$, the graph $G$ has minimal dominating sets of sizes $|L_G| + 3$ and $|L_G| + 4$. By Proposition 3.2, we have that $\Gamma(G) = |L_G| + 4$. Then, it suffices to guarantee that the cases which lead to minimal dominating sets of size at most $|L_G| + 2$ are prevented; since otherwise, the domination gap becomes at least two. The cases which require that at most two vertices from $C_9$ are included in a minimal dominating set are as follows:

- If the vertices of $C_9$ which do not have neighbors of type-1 in $G$ induce a single path $P_m$ for $m \leq 6$, then the stems together with $\lceil m/3 \rceil$ vertices from $P_m$ form a minimal dominating set $D$ of size $|L_G| + \lceil m/3 \rceil$. Since $m \leq 6$, we have that $\lceil m/3 \rceil \leq 2$. Then $|D| \leq |L_G| + 2$.

- If the vertices of $C_9$ which are not adjacent to neighbors of type-1 in $G$ induce two disjoint paths $P_i$ and $P_j$ on $C_9$ for $i \leq 3$ and $j \leq 3$, then the stems together with $\lceil i/3 \rceil$ vertices from $P_i$ and $\lceil j/3 \rceil$ vertices from $P_j$ constitute a minimal dominating set $D$ of size $|L_G| + \lceil i/3 \rceil + \lceil j/3 \rceil$ in $G$. However, $|D| \leq |L_G| + 2$ since we have that $\lceil i/3 \rceil \leq 1$ and $\lceil j/3 \rceil \leq 1$ for $i \leq 3$ and $j \leq 3$.

In order to avoid the first case, no three consecutive vertices on $C_9$ must have type-1 neighbors in $G$. Furthermore, to prevent the second case, no two consecutive vertices together with a vertex at distance 4 from these consecutive vertices on $C_9$ must have type-1 neighbors in $G$. Hence, $G \in \mathcal{G}_9$.

To prove the converse, suppose that $G \in \mathcal{G}_9$. It follows from Proposition 3.2 that $\Gamma(G) = |L_G| + 4$. Furthermore, by definition of $\mathcal{G}_9$, any minimal dominating set of G includes at least three vertices from $C_9$ and hence has size at least $|L_G| + 3$. Therefore, $G$ is almost well-dominated. □

In the case of $G_0 \cong C_m$ where $m \in \{10,13\}$, we show that there exists a unique almost well-dominated graph for each value of m.

*Lemma 3.18. Let G be a $(C_3, C_4, C_5, C_7)$-free graph without a vertex of type-2. Then G is almost well-dominated with $G_0 \cong C_m$ for $m \in \{10,13\}$ if and only if $G \cong C_m$ for $m \in \{10,13\}$.*

*Proof.* Let $G$ be a $(C_3, C_4, C_5, C_7)$-free graph without a vertex of type-2. Suppose that $G$ is almost well-dominated with $G_0 \cong C_m$ for $m \in \{10,13\}$. Suppose to the contrary that $C_m$ has at least one neighbor of type-1 in $G$, say $u$. Let $l$ be the leaf neighbor of $u$ in $G$. Note that $C_m$ has two minimal dominating sets of sizes $\lfloor m/2 \rfloor$ and $\lceil m/3 \rceil$; thus, together with the set $L_G$, $G$ has two minimal dominating sets $D_1$ of size $|L_G| + \lfloor m/2 \rfloor$ and $D_2$ of size $|L_G| + \lceil m/3 \rceil$. Note that the vertex u has at least one neighbor, say $v$, on $C_m$. Observe that the set $L_G - \{l\} \cup \{u\}$, which is of size $|L_G|$, dominates at least the vertex $v$ from $C_m$. Hence, the vertices of $C_m$ different from $v$, which induce a path $P_{m-1}$, has a minimal dominating set of size $\lceil (m-1)/3 \rceil$. Thus, the set $L_G - \{l\} \cup \{u\}$ together with $\lceil (m-1)/3 \rceil$ vertices from $P_{m-1}$ form a dominating set $D$ of size $|L_G| + \lceil (m-1)/3 \rceil$, which implies a minimal dominating set $D_3$ of size at most $|L_G| + \lceil (m-1)/3 \rceil$. However, $|D_1| - |D_3| \geq 2$ for $m \in \{10,13\}$, a contradiction.

The proof for the converse is straightforward since it is easy to verify that $C_{10}$ and $C_{13}$ are almost well-dominated graphs. □

The last case we settle in this section is the case where $G_0 \cong C_{11}$. We obtain a family of almost well-dominated graphs $\mathcal{G}_{10}$ defined in Definition 3.9.

*Definition 3.9. A $(C_3, C_4, C_5, C_7)$-free graph $G$ is in the family $\mathcal{G}_{10}$ if it has a type-0 component $H \cong C_{11} = (abcdefghijk)$ with the following properties:*

- *No two consecutive vertices on $H$ have type-1 neighbors in $G$*
- *No two vertices at distance 4 on $H$, say $a$ and $e$, have type-1 neighbors in $G$*

*Lemma 3.19. Let $G$ be a ($C_3, C_4, C_5, C_7$)-free graph without a vertex of type-2. Then $G$ is almost well-dominated with $G_0 \cong C_{11}$ if and only if $G \in \mathcal{G}_{10}$.*

$Proof$. Let $G$ be a ($C_3, C_4, C_5, C_7$)-free graph without a vertex of type-2. Suppose that $G$ is almost well-dominated with $G_0 \cong C_{11} = (abcdefghijk)$. Note that $C_{11}$ has two minimal dominating sets of sizes 4 and 5; thus, together with the set of leaves $L_G$, the graph $G$ has minimal dominating sets of sizes $|L_G| + 4$ and $|L_G| + 5$. Notice that $\Gamma(G) = |L_G| + 5$ by Proposition 3.2. Therefore, the cases leading to a minimal dominating set of size at most $|L_G| + 3$ must be prevented since otherwise, the domination gap becomes at least two. The cases which require that at most three vertices from $C_{11}$ be included in a minimal dominating set are as follows:

- If the vertices of $C_{11}$ which do not have neighbors of type-1 in $G$ induce a single path $P_m$ for $m \leq 9$, then the stems together with $\lceil m/3 \rceil$ vertices from $P_m$ form a minimal dominating set $D$ of size $|L_G| + \lceil m/3 \rceil$. Since $m \leq 9$, we have that $\lceil m/3 \rceil \leq 3$. Thus, $|D| \leq |L_G| + 3$.
- If the vertices of $C_{11}$ which do not have neighbors of type-1 in $G$ induce two disjoint paths $P_3$ and $P_6$, say $[abc]$ and $[efghij]$, respectively, then the set $L_G$ together with one vertex from $P_3$, namely $b$, and two vertices from $P_6$, namely $f$ and $i$, form a dominating set $D$ of size $|L_G| + 3$, which implies a minimal dominating set of size at most $|L_G| + 3$ in $G$.

In order to avoid the first case, no two consecutive vertices on $C_{11}$ must have type-1 neighbors in $G$. Furthermore, to prevent the second case, no two vertices at distance 4 on $C_{11}$ must have type-1 neighbors in $G$. Hence, $G \in \mathcal{G}_{10}$.

To prove the converse suppose that $G \in \mathcal{G}_{10}$. It follows from Proposition 3.2 that $\Gamma(G) = |L_G| + 5$. Moreover, by the definition of $\mathcal{G}_{10}$, all minimal dominating sets in $G$ include at least four vertices from $C_{11}$ and thus, have size at least $|L_G| + 4$. Hence, $G$ is almost well-dominated. ☐

Our main result for $(C_3, C_4, C_5, C_7)$-free almost well-dominated graphs is stated in the following theorem.

*Theorem 3.1. Let $G$ be a $(C_3, C_4, C_5, C_7)$-free graph. Then, $G$ is an almost well-dominated graph if and only if one of the following holds:*

- *$G$ has a single vertex of type-2 and $G \in \mathcal{G}_1$.*
- *$G$ has no vertex of type-2 and $G \in \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4 \cup \mathcal{G}_5 \cup \mathcal{G}_6 \cup \mathcal{G}_7 \cup \mathcal{G}_8 \cup \mathcal{G}_9 \cup \mathcal{G}_{10} \cup \{C_{10}, C_{13}\}$.*

*Proof.* It is first followed by Lemma 3.4 that a $(C_3, C_4, C_5, C_7)$-free graph $G$ has at most one vertex of type-2. Then we proceed the proof in two cases: $G$ has a single vertex of type-2 and $G$ has no vertex of type-2. While the first case follows from Lemma 3.7, the latter follows from Lemmas 3.9, 3.10, 3.11, 3.12, 3.13, 3.14, 3.15, 3.16, 3.17, 3.18, and 3.19. ☐

# 4. ALMOST WELL-DOMINATED BIPARTITE GRAPHS

In this chapter, we focus on a special subclass of almost well-dominated graphs, that is, bipartite graphs with domination gap one. We first provide an upper bound for the order of bipartite graphs with domination gap $k$ and then characterize almost well-dominated bipartite graphs with minimum degree at least two.

Throughout this chapter, we denote by $d_v$ the degree of a vertex $v$ in graph $G$. We frequently use the following lemmas, which show that the domination number of graphs with minimum degree at least 2 and 3 is restricted by an upper bound. Note that in Lemma 4.1, $\mathcal{A}$ is the family of graphs shown in Figure 4.1.

*Lemma 4.1: [52] Let* $G$ *be a connected graph with* $\delta(G) \geq 2$ *and* $G \notin \mathcal{A}$. *Then* $\gamma(G) \leq \lfloor 2n/5 \rfloor$.

Among the graphs in the family $\mathcal{A}$, the only bipartite graphs are $A_5$ and $A_6$. While $A_5$ is a well-dominated graph, $A_6$ is an almost well-dominated bipartite graph where the minimal dominating sets have size either three or four. The result [53] in Lemma 4.2 establishes an upper bound for the domination number of connected graphs with minimum degree at least three.

*Lemma 4.2: [53] Let* $G$ *be a connected graph with* $\delta(G) \geq 3$. *Then* $\gamma(G) \leq \lfloor 3n/8 \rfloor$.

In Lemma 4.3, we state our result on bipartite graphs with $\mu_d(G) = k$ and $\delta(G) \geq 2$.

*Lemma 4.3: Let* $G$ *be a bipartite graph with* $\delta(G) \geq 2$ *and* $\mu_d(G) = k$, *where* $k \geq 1$. *Then* $|V(G)| \leq 10k$.

Proof. Let $G$ be a bipartite graph with $\delta(G) \geq 2$ and $\mu_d(G) = k$ and let $A$ and $B$ be the partite sets of $G$, where $|A| \geq |B|$ and $|V(G)| = n$. By Lemma 4.1, $G$ has a minimal dominating set $D_1$ of size at most $\lfloor 2n/5 \rfloor$. Note that in a connected bipartite graph every partite set is a minimal dominating set. Hence, since $|A| + |B| = n$ and $|A| \geq |B|$, we have that $|A| \geq n/2$, implying that $G$ has another minimal dominating set $D_2$ of size at least $n/2$. Since the domination gap of $G$ is $k$, $|D_2| - |D_1| \leq k$. On the other hand, $|D_2| - |D_1| \geq n/2 - \lfloor 2n/5 \rfloor$. Thus, we have that $n/2 - \lfloor 2n/5 \rfloor \leq k$, which yields $n \leq 10k$. It remains to check the exceptional graphs in the family $\mathcal{A}$, where the only bipartite graph with $\mu_d \geq 1$ is $A_6$. It is easy to see that $\mu_d(A_6)$=1 and its order is 7. Hence, $|V(A_6)| \leq 10$. □

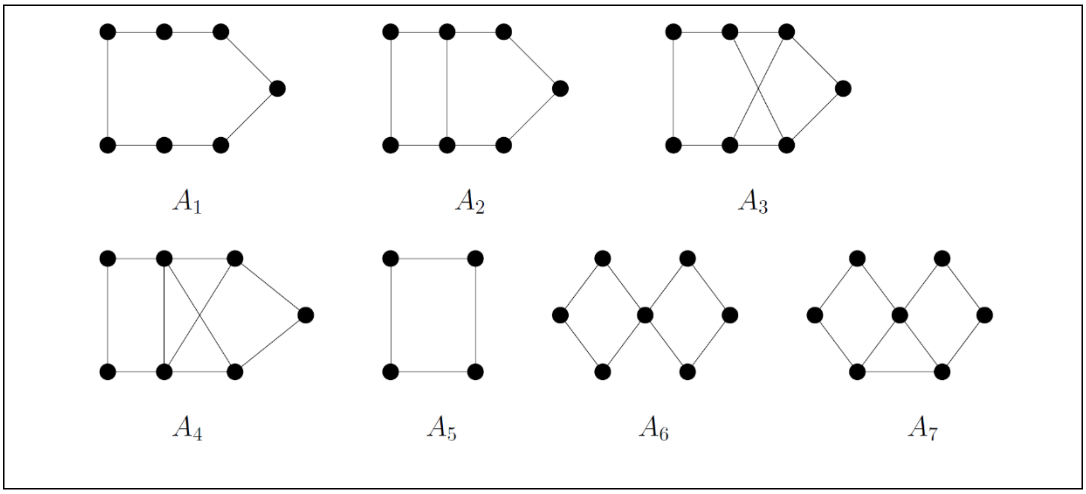

Figure 4.1: The graphs in family $\mathcal{A}$.

An immediate implication of Lemma 4.3 for almost well-dominated graphs is stated in the following corollary.

*Corollary 4.1: Let $G$ be an almost well-dominated bipartite graph with $\delta(G) \geq 2$. Then $|V(G)| \leq 10$.*

While the result in Corollary 4.1 restricts the order of almost well-dominated bipartite graphs with $\delta(G) \geq 2$ by 10, the following lemma establishes an upper bound for the maximum degree of each vertex in almost well-dominated bipartite graphs with minimum degree 2.

*Lemma 4.4: Let $G$ be an almost well-dominated bipartite graph with $\delta(G) \geq 2$. Then $\Delta(G) \leq 4$.*

*Proof.* Let $G$ be an almost well-dominated bipartite graph with $\delta(G) \geq 2$. Assume to the contrary that there exists a vertex $x$ with degree at least 5 in $G$, i.e., $|N(x)| \geq 5$. Let $N_i(x)$ be the set of vertices at distance i from x. We first show that $N_{i\geq 3}(x) = \emptyset$. Suppose to the contrary that $N_3(x) \neq \emptyset$. Then $N(x) \cup N_3(x)$ is an independent set of size at least 6, which implies a maximal independent (minimal dominating) set $D_1$ of size at least 6. However, by Corollary 4.1, $G$ has at most 10 vertices and thus, by

Lemma 4.1, $\gamma(G) \leq 4$. This contradicts with $G$ being almost well-dominated since $|D_1| - \gamma(G) \geq 2$ and thus $N_{i\geq 3}(x) = \emptyset$.

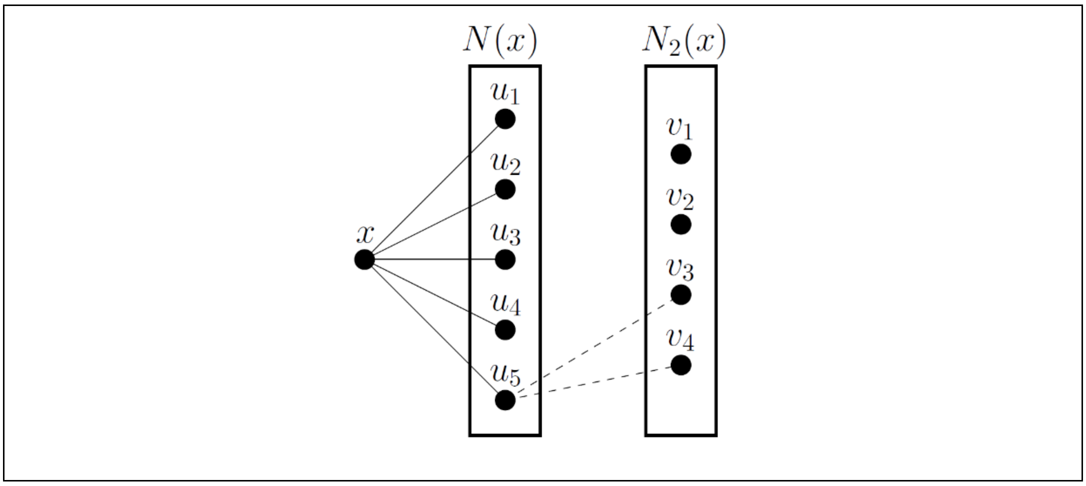

Figure 4.2: The graph $G$ with a vertex $x$ of degree 5.

Note that $x \cup N_2(x)$ and $N(x)$ are two minimal dominating sets in $G$. If $|N(x)| \geq 6$, then $N(x)$ is a minimal dominating set of size at least 6, contradiction with $G$ being almost well-dominated. Therefore, $N(x)$ is of size exactly 5 (see Figure 4.2). Since $G$ is an almost well-dominated graph and $|V(G)| \leq 10$, the possible size for $x \cup N_2(x)$ is either 5 or 4, i.e., the possible sizes of $N_2(x)$ are 3 and 4. If $|N_2(x)| = 3$, then $|V(G)| = 9$, which implies that $\gamma(G) \leq 3$ and $|N(x)| - \gamma(G) \geq 2$, contradiction.

Now consider the case where $|N_2(x)| = 4$. Suppose for a contradiction that a vertex $u_1$ in $N(x)$ has three neighbors $\{v_1, v_2, v_3\}$ in $N_2(x)$. Then the set $\{x, u_1, v_4\}$ forms a dominating set of size three; however, the domination gap becomes 2 since $N(x)$ is a minimal dominating set of size 5, contradiction. Therefore, in this case any vertex in $N(x)$ has at most two neighbors in $N_2(x)$. Furthermore, since each vertex in $\mathrm{N}_2(\mathrm{x})$ has degree at least two, there are at least 8 edges between $\mathrm{N}(\mathrm{x})$ and $\mathrm{N}_2(\mathrm{x})$. Since $|\mathrm{N}(\mathrm{x})| = 5$, by pigeonhole principle, there exists a vertex in $\mathrm{N}(\mathrm{x})$, say $\mathrm{u}_5$, with two neighbors in $\mathrm{N}_2(\mathrm{x})$. Without loss of generality, let $\{\mathrm{v}_3, \mathrm{v}_4\}$ be the two neighbors of $\mathrm{u}_5$ in $\mathrm{N}_2(\mathrm{x})$. Each of the vertices $\mathrm{v}_1$ and $\mathrm{v}_2$ have at least two neighbors in $\mathrm{N}(\mathrm{x})$. The vertex $\mathrm{u}_5$ is not among the possible neighbors for these vertices since it has its own two neighbors $\{\mathrm{v}_3, \mathrm{v}_4\}$ in $\mathrm{N}_2(\mathrm{x})$. Here, two cases arise: neighborhoods of $\mathrm{v}_1$ and $\mathrm{v}_2$ are either disjoint or not. In the first case, let $\{\mathrm{u}_1, \mathrm{u}_2\}$ and $\{\mathrm{u}_3, \mathrm{u}_4\}$ be the disjoint neighborhoods of $v_1$ and $v_2$ in N(x), respectively. In this case, the set $\{u_5, v_1, v_2\}$ forms a dominating set of size 3, contradiction. In the latter case, let $u_2$ be a common neighbor for $v_1$ and $v_2$ in $N(x)$. Then the set $\{x, u_5, u_2\}$ forms a dominating set of size 3, contradiction. ☐

The following result determines the upper domination number of an almost well-dominated bipartite graph.

*Lemma 4.5: Let $G$ be a connected almost well-dominated bipartite graph. Then $G$ has two partite sets $A$ and $B$ with $|A| \geq |B|$ such that $|A| - |B| \leq 1$ and $\Gamma(G) = |A|$.*

*Proof.* Let $G$ be a connected bipartite graph with $A$ and $B$ as its partite sets. Observe that each partite is a minimal dominating set. Hence $|A| - |B| \leq 1$. It remains to prove that $\Gamma(G) = |A|$. Suppose to the contrary that there exists a minimal dominating set $D$ of size at least $|A| + 1$. Let $D' = V(G) - D$. Every vertex in $D$ has at least one neighbor in $D'$ since $D$ is a minimal dominating set; therefore, $D'$ is a dominating set including a minimal dominating set $D''$ of size at most $|D''| \leq |D'| = n - |D| \leq n - |A| - 1 \leq |A| - 1$. Thus, $|D| - |D''| \geq 2$, a contradiction. ☐

## 4.1. Almost well-dominated bipartite graphs with $\delta(G) \geq 2$

By the aid of the Corollary 4.1 and Lemma 4.4, we give a complete structural characterization of almost well-dominated bipartite graphs with $\delta(G) \geq 2$. We first focus on the case of almost well-dominated bipartite graphs with $\delta(G) \geq 3$ and then proceed with the case $\delta(G) = 2$.

### 4.1.1. Almost well-dominated bipartite graphs with $\delta(G) \geq 3$

In the case of $\delta(G) \geq 3$, Lemma 4.2 provides an upper bound for the domination number of graphs with $\delta(G) \geq 3$. Using this upper bound, we state the result on almost well-dominated bipartite graphs with $\delta(G) \geq 3$ in Lemma 4.6.

*Lemma 4.6: Let $G$ be a connected bipartite almost well-dominated graph with $\delta(G) \geq 3$. Then $G$ is isomorphic to $G_{6-1}$ in Figure 4.4.*

*Proof.* Notice that, by Lemma 4.5, an almost well-dominated bipartite graph $G$ of even order $n$ has two partite sets, each of size $n/2$, and that of odd order $n$ has two partite sets with size $(n+1)/2$ and $(n-1)/2$. Furthermore, by Lemma 4.2, $G$ has a minimal dominating set of size at most $\lfloor 3n/8 \rfloor$. In the case of odd $n$, $(n+1)/2 - \lfloor 3n/8 \rfloor \leq 1$ yields $n \leq 3$, implying that there is no almost well-dominated bipartite graph with $\delta(G) \geq 3$ and odd order. On the other hand, for even $n$, $n/2 - \lfloor 3n/8 \rfloor \leq 1$ yields $n \leq 8$. Hence, we focus on the cases with $|V(G)| = 6$ and $|V(G)| = 8$.

Let $|V(G)| = 6$. Then $G$ has two partites each of size three. Let $A = \{a, b, c\}$ and $B = \{d, e, f\}$ be the two partites of $G$. Since $\delta(G) \geq 3$, each vertex has at least three neighbors in the other partite. Since the size of each partite is three, each vertex is adjacent to exactly three vertices in the other partite. The graph obtained in this case is shown as $G_{6-1}$ in Figure 4.4.

Next assume that $|V(G)| = 8$. Then $G$ has two partites each of size four. Let $A = \{a, b, c, d\}$ and $B = \{e, f, g, h\}$ be the two partites of $G$. Both $A$ and $B$ are minimal dominating sets of size 4. We show that $G$ has a minimal dominating set of size two and hence $G$ is not almost well-dominated. Note that, by Lemma 4.4, the vertices in $G$ have degrees either 3 or 4. Suppose that a vertex of $G$, say $a$, is of degree 4. Then the total degree of the vertices in partite $A$ is at least 13, implying that there exists at least one vertex of degree 4 in the partite $B$, say $e$. Then $\{a, e\}$ forms a minimal dominating set of size two, contradiction. On the other hand, if $a$ is of degree three, it has three neighbors, say $\{e, f, g\}$, in partite $B$ and since $h$ has degree at least 3, it is adjacent to $\{b, c, d\}$, implying that $\{a, h\}$ is a minimal dominating set of size two, contradiction. ☐

### 4.1.2. Almost well-dominated bipartite graphs with $\delta(G) = 2$

Recall that an almost well-dominated bipartite graph $G$ of even order n has two partite sets, each of size $n/2$, and the one with odd order $n$ has two partite sets with sizes $(n+1)/2$ and $(n-1)/2$. Furthermore, by Lemma 4.1, $G$ has a minimal dominating set of size at most $\lfloor 2n/5 \rfloor$. Note that in the case of odd $n$, we may assume that $G$ has two minimal dominating sets $D_1$ of size $(n+1)/2$ and $D_2$ of size at most $\lfloor 2n/5 \rfloor$, respectively. Since we are interested in graphs with $\mu_d = 1$, we have that $|D_1| - |D_2| \leq 1$. In other words, $(n+1)/2 - \lfloor 2n/5 \rfloor \leq 1$, implying that $n \leq 5$.

Notice further that while for odd $n \geq 7$, $|D_1| - |D_2| \geq 2$, a bipartite graph with $\delta = 2$ has at least 4 vertices. Thus, for the case of almost well-dominated bipartite graphs with $\delta = 2$ and odd order, we have to investigate graphs with order 5 together with the only exceptional bipartite graph with odd order in the family $\mathcal{A}$, namely $A_6$ of order 7 (see Figure 4.1) for which Lemma 2.1 does not hold. On the other hand, for even n, we may assume that G has two minimal dominating sets $D_1$ of size $n/2$ and $D_2$ of size at most $\lfloor 2n/5 \rfloor$, respectively. Since $\mu_d = 1$, it follows that $|D_1| - |D_2| \leq 1$, which in turn implies that $n \leq 10$. For $|V(G)| = 4$, there is a single bipartite graph with $\delta(G) = 2$, that is $C_4$, which is not almost well-dominated. Therefore, for the case of almost well-dominated bipartite graphs with $\delta(G) = 2$ and even order, we must check orders 6, 8, and 10. Hence, from now onwards, we will focus on orders 5, 6, 7, 8, and 10 separately and prove that there are precisely 30 almost well-dominated bipartite graphs with $\delta(G) = 2$ (see Figure 4.4).

The case $|V(G)| = 5$ leads to a single almost well-dominated graph. This result is stated in Lemma 4.7.

*Lemma 4.7: Let $G$ be a connected bipartite almost well-dominated graph with $\delta(G) = 2$ and $|V(G)| = 5$. Then $G$ is isomorphic to $G_5$ in Figure 4.4.*

*Proof.* Let $G$ be a connected bipartite almost well-dominated graph with $\delta(G) = 2$ and $|V(G)| = 5$. Then $G$ has two partites $A$ and $B$ of sizes 3 and 2, respectively. Since $\delta(G) = 2$, every vertex in $A$ is adjacent to both vertices in $B$. This case results in the graph $G_5$ shown in Figure 4.4. ☐

We next assume that $|V(G)| = 6$. This case yields 3 almost well-dominated bipartite graphs. Lemma 4.8 states the result of this case.

*Lemma 4.8: Let $G$ be a connected almost well-dominated bipartite graph with $\delta(G) = 2$ and $|V(G)| = 6$. Then $G \in \{G_{6-2}, G_{6-3}, G_{6-4}\}$ in Figure 4.4.*

*Proof.* Here $G$ has two partites each of size three. Let $A = \{a, b, c\}$ and $B = \{d, e, f\}$ be the two partites of $G$. In this case, a vertex can be adjacent to at most three vertices in the other partite; hence, $\Delta(G) \leq 3$. Suppose first that $\Delta(G) = 2$. Then all vertices of $G$ are of degree 2. Hence $G$ is isomorphic to $C_6$, which is shown as $G_{6-2}$ in Figure 4.4. Now suppose that $\Delta(G) = 3$. Note that there is at least one vertex of degree two in $G$. Let a in the partite $A$ be such a vertex. Then partite $A$ has at least a vertex of degree 3 since otherwise all vertices of $A$ would be of degree 2 implying that all vertices in $B$ are of degree two. Let $d_c = 3$. Then b has degree either 2 or 3. We consider these two possible degrees for b separately in the sequel.

Let $d_b = 2$, i.e., $(d_a, d_b, d_c) = (2,2,3)$. This degree sequence for partite $A$ implies the same degree sequence for partite $B$, i.e., $\left(d_d, d_e, d_f\right) = (2,2,3)$. Hence, vertices $c$ and $f$ are adjacent to all vertices in $B$ and $A$, respectively. The vertex $a$ is adjacent to one of $\{d, e\}$, say d. Hence b is adjacent to e. The graph obtained here is an almost well-dominated graph shown as $G_{6-3}$ in Figure 4.4.

Next suppose that $d_b = 3$, i.e., $(d_a, d_b, d_c) = (2,3,3)$. This degree sequence for partite A implies that $\left(d_d, d_e, d_f\right) = (2,3,3)$ for partite $B$. In this case, the vertices $b$ and $c$ are adjacent to all vertices in partite $B$ and the vertices $e$ and $f$ are adjacent to all vertices in partite $A$. The graph obtained here is an almost well-dominated graph shown as $G_{6-4}$ in Figure 4.4. □

In the case $|V(G)| = 7$, we obtain a single almost well-dominated bipartite graph. Lemma 4.9 states our result for $|V(G)| = 7$.

*Lemma 4.9: Let G be a connected almost well-dominated bipartite graph with* $\delta(G) = 2$ *and* $|V(G)| = 7$. *Then* $G$ *is isomorphic to* $G_7$, *shown in Figure 4.4.*

*Proof.* Let $G$ be a bipartite graph of order 7 such that $G \notin \mathcal{A}$ (shown in Figure 4.1). Then by Lemma 4.1, $\gamma(G) \leq 2$ for $n = 7$. Furthermore, $G$ has two partites of sizes 4 and 3. Hence $\mu_d(G) \geq 2$, a contradiction. Next we assume that $G \in \mathcal{A}$. Among the graphs in the family $\mathcal{A}$, the only bipartite graph on 7 vertices is $A_6$, which is an almost well-dominated graph shown as $G_7$ in Figure 4.4. □

Lemma 4.10 and Lemma 4.11 state the results for the case $|V(G)| = 8$. Propositions 4.1 and 4.2 provide us tools to characterize almost well-dominated bipartite graphs with $\delta(G) = 2$ and order 8.

*Proposition 4.1: Let G be a connected almost well-dominated bipartite graph with* $\delta(G) = 2$ *and* $|V(G)| = 8$. *Then any two vertices of degree 3 in different partites of* $G$ *are adjacent.*

*Proof.* Note that $G$ has two partites each of size 4. Let $\mathrm{A} = \{a, b, c, d\}$ and $B = \{e, f, g, h\}$ be the two partites of $G$. Suppose to the contrary that $a$ and e are two nonadjacent vertices of degree three in different partites. Then $a$ is adjacent to the vertices in $\{f, g, h\}$ and $e$ is adjacent to the vertices in $\{b, c, d\}$. Then $\{a, e\}$ is a minimal dominating set of size two. However, each partite set of $G$ is a minimal dominating set of size 4 and hence $\mu_d(G) \geq 2$, contradiction to $G$ being almost well-dominated. ☐

*Proposition 4.2: Let $G$ be a connected almost well-dominated bipartite graph with $\delta(G) = 2$ and $|V(G)| = 8$. Then*

*i) There exists no two vertices of degree 4 in different partites.*
*ii) There exist at most two vertices of degree 4 in one partite.*

*Proof*. Note that $G$ has two partites each of size 4. For (*i*), it is easy to see that two vertices of degree 4 in different partites form a minimal dominating set of size two, a contradiction with $G$ being almost well-dominated. For (*ii*), suppose to the contrary that there exist at least three vertices of degree 4 in one partite, say partite $A$. Then the sum of the degrees of the vertices in partite $A$ is at least 14 which enforces the existence of at least two vertices of degree 4 in the partite $B$, a contradiction to (*i*). ☐

From here onwards, we focus on the cases with $\Delta(G) \leq 3$ and $\Delta(G) = 4$, separately. For $\Delta(G) \leq 3$, Lemma 4.10 and for $\Delta(G) = 4$, Lemma 4.11state the main results for almost well-dominated bipartite graphs with $\delta(G) = 2$ and $|V(G)| = 8$.

*Lemma 4.10: Let $G$ be a connected almost well-dominated bipartite graph with $\delta(G) = 2$, $\Delta(G) \leq 3$, and $|V(G)| = 8$. Then $G \in \{G_{8-5}, G_{8-6}, G_{8-7}, G_{8-8}, G_{8-9}\}$ shown in Figure 4.4.*

*Proof*. Note that each vertex of $G$ has degree 2 or 3. If all vertices have degree two, $G \cong C_8$, which is an almost well-dominated graph shown as $G_{8-5}$ in Figure 4.4. Hence, we proceed with $\Delta(G) = 3$. Note that G has two partites each of size 4. Let $A = \{a, b, c, d\}$ and $B = \{e, f, g, h\}$ be the two partites of $G$. Observe that if all vertices in one partite have degree 3 (or 2) then all vertices in the other partite have degree 3 (or 2) as well, a contradiction with $\delta(G) = 2$ and $\Delta(G) = 3$. Hence, each partite $A$ and $B$ has at least one vertex with degree 3 and one with degree 2. Let $d_a = 3$ and $d_b = 2$ in $A$. Then the degrees of the remaining two vertices, namely $(d_c, d_d)$ is either (2,2), (2,3), or (3,3). We proceed with the proof by considering the possible sequences for $(d_c, d_d)$ separately in the sequel.

Assume first that $(d_c, d_d) = (2,2)$, i.e., $(d_a, d_b, d_c, d_d) = (3,2,2,2)$. This degree sequence for partite $A$ implies that $\left(d_e, d_f, d_g, d_h\right) = (3,2,2,2)$. By Proposition 4.1, the vertices of degree 3 in different partites, namely $a$ and $e$ are adjacent. Furthermore, $a$ has two other neighbors in $\{f, g, h\}$, say $\{f, g\}$. Similarly, $e$ is adjacent to two other vertices in $\{b, c, d\}$, say $\{b, c\}$. We first assume that $dh \in E(G)$. Then $d$ is adjacent to one of $\{f, g\}$, say $g$. Similarly, $h$ is adjacent to one of $\{b, c\}$, say $c$. Then $f$ is adjacent to $b$. The graph obtained in this case is shown as $G_{8-6}$ in Figure 4.4. Next suppose that $dh \notin E(G)$. Then $d$ is adjacent to both $f$ and $g$, and $h$ is adjacent to both $b$ and $c$. The graph obtained here is shown as $G_{8-9}$ in Figure 4.4.

Assume next that $(d_c, d_d) = (3,2)$, i.e., $(d_a, d_b, d_c, d_d) = (3,2,3,2)$. This degree sequence for partite $A$ implies that $\left(d_e, d_f, d_g, d_h\right) = (3,2,3,2)$. By Proposition 4.1, vertices of degree three in partite $A$, namely $a$ and $c$ are adjacent to the vertices of degree 3 in partite $B$, namely $e$ and $g$. We proceed with the proof based on the possible connections between vertices of degree two in the sequel.

Suppose first that there exists at least one vertex of degree two with two neighbors of degree two. Let $b$ in partite $A$ be such a vertex adjacent to the vertices in $\{f, h\}$. Then the vertices $e$ and $g$ are adjacent to $d$ as their third neighbor. Further, $a$ is adjacent to one of $\{f, h\}$, say $f$. Hence, $c$ is adjacent to $h$. This case results in $G_{8-8}$ depicted in Figure 4.4.

Suppose next that a vertex of degree two is adjacent to at most one vertex of degree two. In this case, we first show that there exists at least one pair of adjacent vertices of degree two. Suppose to the contrary that there exists no two adjacent vertices of degree two. Then each vertex of degree 2 has two neighbors of degree 3. Then the resulting graph is disconnected, contradiction. Thus, there is at least one pair of adjacent vertices of degree two. Let $b$ and $f$ be such a pair. Since $b$ is not adjacent to another vertex of degree two, it is adjacent to one of the vertices of degree three $\{e, g\}$, say $e$. Similarly $d$ is not adjacent to $f$ and hence it is adjacent to both $g$ and $h$. Further, $f$ is adjacent to one of $\{a, c\}$, say $a$, and finally $c$ is adjacent to $h$. This case yields $G_{8-7}$ depicted in Figure 4.4.

For the last case, we assume that $(d_c, d_d) = (3,3)$, i.e., $(d_a, d_b, d_c, d_d) = (3,2,3,3)$. This degree sequence for partite $A$ implies that $\left(d_e, d_f, d_g, d_h\right) = (3,2,3,3)$. By Proposition 4.1, each of the vertices $a$, $c$, and $d$ are adjacent to each of the vertices $e$, $g$, and $h$. Hence, $b$ and $f$ remain disconnected. Thus, this case does not lead to any connected almost well-dominated bipartite graph. □

*Lemma 4.11: Let $G$ be a connected almost well-dominated bipartite graph with $\delta(G) = 2$, $\Delta(G) = 4$, and $|V(G)| = 8$. Then $G \in \{G_{8-1}, G_{8-2}, G_{8-3}, G_{8-4}\}$ shown in Figure 4.4.*

*Proof*. Let $G$ be an almost well-dominated bipartite graph with $\delta(G) = 2$, $\Delta(G) = 4$, and $|V(G)| = 8$. Let further $A = \{a, b, c, d\}$ and $B = \{e, f, g, h\}$ be the two partites of $G$. By Proposition 4.2, there are at most two vertices of degree 4 in one partite and no two vertices of degree 4 in different partites.

First suppose that $G$ has two vertices of degree 4, say $a$ and $b$. Then, by Proposition 4.2, the vertices in partite $B$ are of degree either 2 or 3. Note that the sum of the degrees of the vertices in partite $A$ is at least 12, while the corresponding sum for partite $B$ is at most 12. Thus, the only possible degrees for $c$ and $d$ is 2, i.e., $(d_a, d_b, d_c, d_d) = (4,4,2,2)$. This degree sequence for partite $A$ implies that $\left(d_e, d_f, d_g, d_h\right) = (3,3,3,3)$ for partite $B$. Note that the vertices of degree 4, namely $a$ and $b$, are both adjacent to all vertices in partite $B$. The vertex $c$ is adjacent to any two vertices in partite $B$, say $\{e, f\}$. Finally, $d$ is adjacent to $g$ and $h$. The graph obtained here is shown as $G_{8-3}$ in Figure 4.4.

Next suppose that $G$ has a single vertex of degree 4, say $a$ in the partite $A$. Note that the vertices in partite $B$ are of degree either 2 or 3 by Proposition 4.2. Since the sum of the degrees of the vertices in partite $B$ is at most 12, the possible degree sequences for partite $A$ are $(4,2,2,2)$, $(4,3,2,2)$, and $(4,3,3,2)$. We address these cases separately in the sequel.

Suppose first that $(d_a, d_b, d_c, d_d) = (4,2,2,2)$. With this degree sequence for partite $A$, the only possible degree sequence for the partite $B$ is $(2,2,3,3)$. Without loss of generality suppose that $\left(d_e, d_f, d_g, d_h\right) = (2,2,3,3)$. Naturally, $a$ is adjacent to all vertices in partite $B$. Then $h$ is adjacent to any two vertices from $\{b, c, d\}$, say $\{c, d\}$. Now, $b$ is adjacent to at least one of $\{e, f\}$, say $e$. Now consider the second neighbor of $b$ in partite $B$. Suppose first that $bf \in E(G)$. Then $g$ is adjacent to $c$ and $d$. Hence, $G \cong G_{8-4}$ shown in Figure 4.4. Now assume that $bg \in E(G)$. Then $g$ is adjacent to one of $c$ or $d$, say $d$. Then $fc \in E(G)$ and $G \cong G_{8-1}$ shown in Figure 4.4.

In the case of $(d_a, d_b, d_c, d_d) = (4,3,2,2)$, partite $B$ has the degree sequence $(d_e, d_f, d_g, d_h) = (2,3,3,3)$. Naturally, $a$ is adjacent to all vertices in partite $B$. Then $b$ is is adjacent to all vertices of degree 3 in $B$, namely $\{f, g, h\}$, by Proposition 4.1. Then $e$ is adjacent to one of $\{c, d\}$, say $c$. Then $c$ is adjacent to one vertex in $\{f, g, h\}$, say $f$. Then $d$ is adjacent to $g$ and $h$. This case yields the almost well-dominated graph $G_{8-2}$ in Figure 4.4.

Assume next that $(d_a, d_b, d_c, d_d) = (4,3,3,2)$. In this case, partite $B$ has the degree sequence $(d_e, d_f, d_g, d_h) = (3,3,3,3)$. By Proposition 4.1, a vertex of degree 3 in one partite $A$, say $b$, has to be adjacent to all vertices of degree 3 in partite $B$; however, there exist 4 vertices of degree 3 in $B$, contradiction. This case does not lead to any almost well-dominated graph. □

Let us now focus on the case where $|V(G)| = 10$. In this case, $G$ has two partites each of size 5. By Lemma 4.4, $\Delta(G) \leq 4$. We focus on the cases with $\Delta(G) \leq 3$ and $\Delta(G) = 4$ separately in Lemma 4.12 and Lemma 4.13, respectively.

*Lemma 4.12: Let $G$ be a connected almost well-dominated bipartite graph with $\delta(G) = 2$, $\Delta(G) \leq 3$, and $|V(G)| = 10$. Then $G \in \{ G_{10-1}, G_{10-2}, G_{10-3}, G_{10-7}, G_{10-10}, G_{10-12}, G_{10-13}, G_{10-14}, G_{10-15}, G_{10-16} \}$.*

*Proof.* Let $A = \{a, b, c, d, e\}$ and $B = \{f, g, h, i, j\}$ be the two partites of $G$. Notice that each partite $A$ or $B$ is a minimal dominating set of size 5. Thus, a minimal dominating set of size at most 3 contradicts with $G$ being almost well-dominated. Moreover, the degrees of the vertices of $G$ are either 2 or 3. If all vertices have degree two, then $G$ is isomorphic to a $C_{10}$, which is an almost well-dominated bipartite graph shown as $G_{10-1}$ in Figure 4.4. Thus, we assume that there exists at least one vertex with degree three in $G$. Note that if all vertices in one partite have degree 2 (or 3), then the vertices in the other partite all have degree 2 (or 3). Since $\delta(G) = 2$, we may suppose that there are two vertices of degree 3 and 2 in one partite, say $a$ and $b$, respectively, in partite $A$. Now, the degrees of the remaining three vertices, namely $(d_c, d_d, d_e)$, are either (2,2,2), (2,2,3), (2,3,3), or (3,3,3). In what follows, we consider these cases separately.

Claim 1. Let $G$ be a connected almost well-dominated bipartite graph with $\delta(G) = 2$, $\Delta(G) = 3$, and $|V(G)| = 10$. If $(d_a, d_b, d_c, d_d, d_e) = (3,2,2,2,2)$, then $G \in \{G_{10-2}, G_{10-14}\}$.

Proof of Claim 1. Let $(d_a, d_b, d_c, d_d, d_e) = (3,2,2,2,2)$. This degree sequence for partite $A$ implies the same degree sequence for partite $B$. Without loss of generality, assume that $\left(d_f, d_g, d_h, d_i, d_j\right) = (3,2,2,2,2)$.

First assume that $af \notin E(G)$. Then $a$ has three neighbors of degree 2 in the partite $B$, say $\{g, h, i\}$. Similarly, $f$ has three neighbors in $\{b, c, d, e\}$, say $\{b, c, d\}$. Note that $e$ is not adjacent to $j$ since otherwise $\{a, f, e\}$ forms a minimal dominating set of size 3, a contradiction. Thus, $e$ has two neighbors from $\{g, h, i\}$, say $\{h, i\}$. Furthermore, $j$ has two neighbors from $\{b, c, d\}$, say $\{c, d\}$. Finally, $b$ is adjacent to $g$. This case results in $G_{10-14}$ in Figure 4.4.

Now suppose that $af \in E(G)$. Then $a$ has two other neighbors from $\{g, h, i, j\}$, say $\{g, h\}$. Similarly, $f$ has two other neighbors from $\{b, c, d, e\}$, say $\{b, c\}$. Since $f$ is adjacent to all vertices $\{a, b, c\}$, we show that $d$ and $e$ do not dominate all vertices $\{g, h, i, j\}$. Suppose to the contrary that $\{d, e\}$ dominates $\{g, h, i, j\}$. Then $\{f, d, e\}$ forms a minimal dominating set of size 3, contradiction. Thus, $d$ and $e$ have at least one common neighbor from $\{g, h, i, j\}$ which is either $i$ or $j$. Let $i$ be the common neighbor of $d$ and $e$. Similarly, since $a$ is adjacent to the vertices $\{f, g, h\}$, the vertices $i$ and $j$ must not dominate $\{b, c, d, e\}$ in order to avoid a minimal dominating set $\{a, i, j\}$ of size 3. This implies that $i$ and $j$ have at least one common neighbor in $\{b, c, d, e\}$ which can be either $d$ or $e$. Let $e$ be the common neighbor of $i$ and $j$. Note that $j$ is not adjacent to $d$ since otherwise $G$ is disconnected. Therefore, $j$ is adjacent to one of $\{b, c\}$, say $c$. Then $d$ is adjacent to one of $\{g, h\}$, say $h$. Finally, $b$ is adjacent to $g$. This case yields $G_{10-2}$ depicted in Figure 4.4.

Claim 2. Let $G$ be a connected almost well-dominated bipartite graph with $\delta(G) = 2$, $\Delta(G) = 3$, and $|V(G)| = 10$. If $(d_a, d_b, d_c, d_d, d_e) = (3,2,2,2,3)$ then $G \in \{G_{10-3}, G_{10-7}, G_{10-12}, G_{10-13}\}$.

Proof of Claim 2. Let $(d_a, d_b, d_c, d_d, d_e) = (3,2,2,2,3)$. This degree sequence for partite $A$ implies that $\left(d_f, d_g, d_h, d_i, d_j\right) = (3,2,2,2,3)$. Let further $H_3$ be the subgraph induced by the vertices of degree 3, namely $\{a, e, f, j\}$. Figure 4.3 shows the six possible cases for $H_3$. We proceed the proof by analyzing each of these six cases separately in the sequel.

If $H_3 \cong J_1$, the vertices of degree 3 are adjacent to all three vertices of degree 2 in the other partite. The resulting graph is disconnected, contradiction. Next assume that $H_3 \cong J_2$. In this case, each of $a$ and $f$ is adjacent to all three vertices of degree 2 in the other partite. Then, $\{a, f, e\}$ is minimal dominating set of size 3, contradiction.

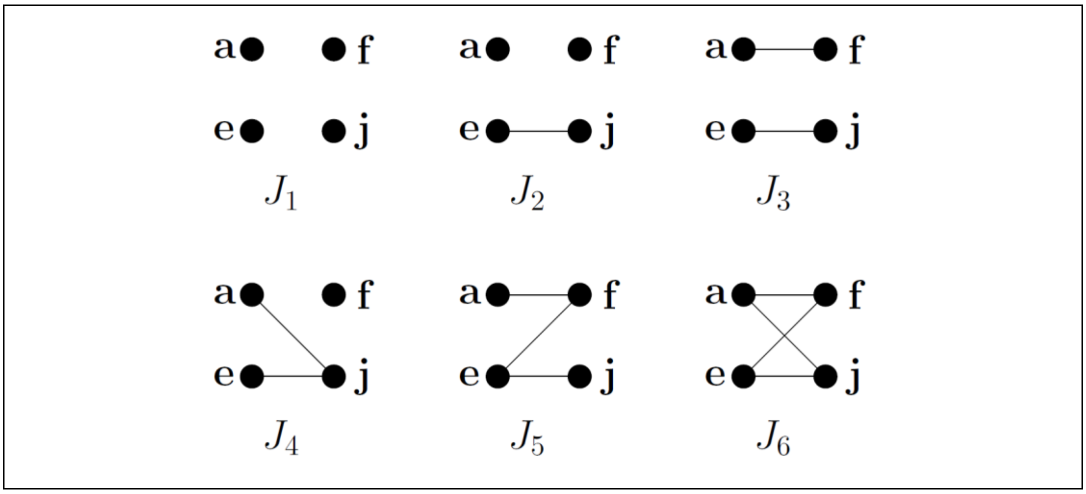

Figure 4.3: The graph $H_3$ induced by the vertices of degree three.

We then suppose that $H_3 \cong J_3$. In this case, a has two neighbors from $\{g, h, i\}$, say $\{g, h\}$. Similarly, $f$ has two neighbors from $\{b, c, d\}$, say $\{b, c\}$. Now we claim that $i$ and $j$ do not dominate $\{b, c, d, e\}$. Suppose to the contrary that $i$ and $j$ dominate $\{b, c, d, e\}$. Then $\{a, i, j\}$ forms a minimal dominating set of size three, contradiction. Note that the neighbors of $i$ and $j$ are from the set $\{b, c, d, e\}$ which has size 4. Since $j$ has degree 3, the vertex $i$ has two common neighbors with $j$, since otherwise $i$ and $j$ dominate $\{b, c, d, e\}$. The common neighbors of $i$ and j are $d$ and $e$. Now, as its third neighbor, $e$ is adjacent to one of $\{g, h\}$, say $h$. Similarly, $j$ is adjacent to one of $\{c, b\}$, say $c$. Finally, $g$ is adjacent to b and the graph obtained in this case is $G_{10-3}$ shown in Figure 4.4.

In the case of $H_3 \cong J_4$, the vertex $f$ is adjacent to $\{b, c, d\}$. We will now show that $a$ and $e$ do not dominate $\{g, h, i, j\}$. Suppose to the contrary that $a$ and $e$ dominate $\{g, h, i, j\}$. Then $\{f, a, e\}$ forms a minimal dominating set of size 3, a contradiction. Observe that the candidate neighbors of $a$ and $e$ are from the set $\{g, h, i, j\}$. Since $a$ and $e$ are both of degree 3, they must have three common neighbors, since otherwise they dominate the set $\{g, h, i, j\}$. Moreover, note that one of these common neighbors is $j$ and the other two must be selected from $\{g, h, i\}$, say

$\{h, i\}$. Then $g$ has two neighbors from $\{b, c, d\}$, say $\{b, c\}$. Finally, $d$ is adjacent to $j$. The graph obtained in this case is $G_{10-13}$ depicted in Figure 4.4.

Next suppose that $H_3 \cong J_5$. First, $a$ is adjacent to two vertices from $\{g, h, i\}$, say $\{g, h\}$. Similarly, $j$ is adjacent to two vertices from $\{b, c, d\}$, say $\{c, d\}$. The vertex $b$ is not adjacent to $i$ since otherwise $\{a, j, b\}$ forms a minimal dominating set of size 3, contradiction. Then $b$ has at least one neighbor from $\{g, h\}$, say $g$. Similarly, $i$ has at least one neighbor from $\{c, d\}$, say $d$. At this point, as its third neighbor, $f$ has two candidates, namely $\{b, c\}$. If $fc \in E(G)$, then $b$ is adjacent to $h$ and $\{f, b, d\}$ is a minimal dominating set of size 3, contradiction. Hence, $f$ is adjacent to $b$ as its third neighbor. In addition, $e$ is not adjacent to $h$ since otherwise $i$ is adjacent to $c$ and $\{e, i, g\}$ is a minimal dominating set of size 3, contradiction. Thus, $e$ is adjacent to $i$ as its third neighbor. Finally, $c$ is adjacent to $h$. This case leads to almost well-dominated graph $G_{10-7}$ in Figure 4.4.

Finally, in the case of $H_3 \cong J_6$, the vertex $f$ is adjacent to one of $\{b, c, d\}$, say $b$, as its third neighbor. Similarly, $e$ is adjacent to one of $\{g, h, i\}$, say $i$, as its third neighbor. We now show that the vertices $c$ and $d$ do not dominate $\{g, h, i, j\}$. Suppose to the contrary that $c$ and $d$ do not dominate $\{g, h, i, j\}$. Then $\{f, c, d\}$ is a minimal dominating set of size 3, contradiction. Thus, $c$ and $d$ do not dominate $\{g, h, i, j\}$. As $c$ and $d$ are both of degree two, they have at least one common neighbor in order not to dominate $\{g, h, i, j\}$, where the only candidates are $g$ and $h$. Let $h$ be the common neighbor of $c$ and $d$. At this point, notice that $g$ is not adjacent to both $a$ and $b$, since otherwise $\{g, h, e\}$ forms a minimal dominating set of size 3, contradiction. Therefore, $g$ has at least one neighbor from $\{c, d\}$. Let $c$ be the neighbor of $g$. The vertices $d$ and $g$ are not adjacent since otherwise the graph is disconnected. Observe that if $bg \in E(G)$ and i$d \in E(G)$, then $j$ and $a$ remain without a third neighbor, contradiction. Hence, at least one of $bg \notin E(G)$ and $id \notin E(G)$ must hold. By symmetry, suppose that $bg \notin E(G)$. Recall that $g$ is not adjacent to $d$. Therefore, $g$ is adjacent to $a$. Now we show that $i$ is not adjacent to $b$, because otherwise $\{a, h, i\}$ is a minimal dominating set of size 3, contradiction. Thus, $i$ is adjacent to $d$. Finally, $j$ is adjacent to $b$. The graph obtained in this case is $G_{10-12}$ in Figure 4.4.

Claim 3. Let $G$ be a connected almost well-dominated bipartite graph with $\delta(G) = 2$, $\Delta(G) = 3$, and $|V(G)| = 10$. If $(d_a, d_b, d_c, d_d, d_e) = (3,2,2,3,3)$ then $G \in \{G_{10-10}, G_{10-15}\}$.

Proof of Claim 3. Let $(d_a, d_b, d_c, d_d, d_e) = (3,2,2,3,3)$. This degree sequence for partite $A$ implies that $\left(d_f, d_g, d_h, d_i, d_j\right) = (3,2,2,3,3)$. In this case, we first claim that a vertex of degree two is adjacent to at most one vertex of degree two in the other partite. Suppose to the contrary that $b$ is adjacent to two vertices of degree two in partite $B$, namely $g$ and $h$. We now show that $c$ is adjacent to at least one of $g$ or $h$. Suppose to the contrary that $c$ is adjacent to two vertices of degree 3, say $\{i, j\}$. Then $f$ is adjacent to all vertices $\{a, d, e\}$ and $\{f, b, c\}$ is a minimal dominating set of size 3, contradiction. Therefore, $c$ is adjacent to at least one of $g$ or $h$, say $g$. At this point, we show that any two vertices of degree 3 in different partites are adjacent. Suppose to the contrary that two vertices of degree 3 in different partites, say $a$ and $f$, are nonadjacent. Then $a$ is adjacent to each of $\{h, i, j\}$ and $f$ is adjacent to each of $\{c, d, e\}$ and hence $\{a, f, b\}$ is a minimal dominating set of size 3, contradiction. Therefore, each vertex of degree 3 in partite $A$, namely $\{a, d, e\}$, is adjacent to each vertex of degree 3 in partite $B$, namely $\{f, i, j\}$. Then, $c$ is adjacent to $h$ and the graph is disconnected, contradiction. Therefore, a vertex of degree two is adjacent to at most one vertex of degree two in the other partite. We proceed with the proof of this case based on the connection between vertices of degree two in the sequel.

Suppose first that no two vertices of degree two are adjacent. In this case $c$ has two neighbors from vertices of degree 3 $\{f, i, j\}$, say $\{i, j\}$. Similarly, $h$ has two neighbors from $\{a, d, e\}$, say $\{d, e\}$. In addition, $g$ has at least one neighbor from $\{d, e\}$, say $d$. Similarly, $b$ has at least one neighbor from $\{i, j\}$, say $i$. The vertex $d$ is adjacent to one of $\{i, j, f\}$ as its third neighbor. If $di \in E(G)$, then $f$ is adjacent to the vertices $\{a, b, e\}$ and hence $\{d, f, j\}$ is a minimal dominating set of size 3, contradiction. Thus, $di \notin E(G)$. If $dj \in E(G)$, then $f$ is adjacent to the vertices $\{a, b, e\}$ and hence $\{c, d, f\}$ is a minimal dominating set of size 3, contradiction. Thus, $dj \notin E(G)$ as well. Therefore, $d$ is adjacent to $f$. If the vertices $i$ and $j$ dominate $\{a, b, c, e\}$, then $\{d, i, j\}$ is a minimal dominating set of size 3, contradiction. Thus, $i$ and j must have 3 common neighbors (same neighborhood) in order not to dominate $\{a, b, c, e\}$. Since $i$ is adjacent to $b$, then $j$ is also adjacent to $b$. On the other hand, vertex $a$ has 3 neighbors in $\{f, g, i, j\}$, where at least one of these neighbors is either $i$ or $j$. However, since $i$ and $j$ have the same neighborhood, both $i$ and $j$ are adjacent to $a$. Furthermore, $e$ is adjacent to both $f$ and $g$ and hence $a$ is adjacent to $f$. This case results in $G_{10-15}$ shown in Figure 4.4.

Next assume that at least a pair of vertices of degree two, say $b$ and $g$, are adjacent. Then $b$ and $g$ have their second neighbor from vertices of degree 3, say $bf \in E(G)$ and $ga \in E(G)$. Note that if two vertices of degree 3, one from $\{d, e\}$ and the other from $\{i, j\}$, say $d$ and $i$, are not adjacent, then $d$ is adjacent to the vertices $\{f, h, j\}$ and $i$ is adjacent to the vertices $\{a, c, e\}$. Then $\{d, i, b\}$ is a minimal dominating set of size 3, contradiction. Thus, both vertices $d$ and $e$ are adjacent to both vertices $i$ and $j$. At this point, suppose that the vertices $\{d, e\}$ are both adjacent to $h$ and the vertices $\{i, j\}$ are both adjacent to $c$. Then the graph is disconnected, contradiction. Therefore, we may suppose that one of the conditions does not hold; in other words, at least one of $d$ or $e$ is not adjacent to $h$, say $dh \notin E(G)$. Hence $d$ is adjacent to $f$. Notice further that since d dominates $\{f, i, j\}$, the vertices $g$ and $h$ must not dominate $\{a, b, c, e\}$ since otherwise $\{d, g, h\}$ is a minimal dominating set of size 3, contradiction. Therefore, $g$ and $h$ have at least one common neighbor, where the only candidate is $a$ and hence $ha \in E(G)$. The vertices $\{a, c\}$ are the candidates for the third neighbors of $i$ and $j$. Since $a$ can be adjacent to only one of $i$ and $j$, then at least one of $i$ and $j$ is adjacent to $c$, say $jc \in E(G)$. Now, if $ai \in E(G)$, then $\{a, j, b\}$ is a minimal dominating set of size 3, contradiction. Hence $ai \notin E(G)$ and $a$ is adjacent to $f$ as its third neighbor. Finally, $i$ is adjacent to $c$ and further $h$ is adjacent to $e$. The graph obtained in this case is $G_{10-10}$ shown in Figure 4.4.

Claim 4. Let $G$ be a connected almost well-dominated bipartite graph with $\delta(G) = 2$, $\Delta(G) = 3$, and $|V(G)| = 10$. If $(d_a, d_b, d_c, d_d, d_e) = (3,2,3,3,3)$, then $G$ is isomorphic to $G_{10-16}$.

Proof of Claim 4. Let $(d_a, d_b, d_c, d_d, d_e) = (3,2,3,3,3)$. This degree sequence for partite $A$ implies that $\left(d_f, d_g, d_h, d_i, d_j\right) = (3,2,3,3,3)$. In this case, we first show that the vertices of degree two, namely $b$ and $g$, are not adjacent. Suppose to the contrary, that $bg \in E(G)$. The vertices $b$ and $g$ have a neighbor from vertices of degree three. Let $b$ be adjacent to $f$ and $g$ be adjacent to $a$. Notice that if two vertices of degree 3, one from $\{c, d, e\}$ and the other from $\{h, i, j\}$, say $c$ and $h$, are not adjacent, then $c$ is adjacent to $\{f, i, j\}$ and $h$ is adjacent to $\{a, d, e\}$ and hence $\{c, h, b\}$ is a minimal dominating set of size 3, contradiction. Then each vertex in $\{c, d, e\}$ is adjacent to each vertex in $\{h, i, j\}$ and the graph is disconnected, contradiction. Thus, the vertices $b$ and $g$ are nonadjacent. Therefore, we may suppose that $b$ is adjacent to two vertices of degree 3, say $\{f, h\}$, and $g$ is adjacent to two vertices in $\{a, c, d, e\}$, say $\{a, c\}$. The vertex $e$ has at least one neighbor in $\{i, j\}$, say $j$ and at least one neighbor in $\{h, f\}$, say $h$. In addition, $j$ has at least one neighbor in $\{c, a\}$, say $c$. We now show that $c$ is not adjacent to $h$. Suppose for a contradiction that $c$ is adjacent to $h$. Then $d$ is adjacent to all vertices $\{f, i, j\}$ and $\{d, h, a\}$ is a minimal dominating set of size 3, contradiction. Thus, $ch \notin E(G)$. Suppose further that $cf \in E(G)$ and $ha \in E(G)$. Then $d$ is adjacent to $i$ and $\{c, h, d\}$ is a minimal dominating set of size 3, contradiction. Thus, at least one of $cf \in E(G)$ and $ha \in E(G)$ does not hold. By symmetry, we may assume that $ha \notin E(G)$ and hence $hd \in E(G)$. Now if $a$ is adjacent to both $f$ and i, then $\{h, a, c\}$ is a minimal dominating set of size 3, contradiction. Hence, $a$ has at least one neighbor different from $f$ and $i$, namely $aj \in E(G)$. Furthermore, $d$ is adjacent to both $f$ and $i$. At this point, we show that $a$ is not adjacent to $f$. Suppose to the contrary that $a$ is adjacent to $f$. Then $c$ is adjacent to $i$ and then $\{c, h, a\}$ is a minimal dominating set of size 3, contradiction. Thus $af \notin E(G)$ and in turn $a$ is adjacent to $i$ as its third neighbor. If $f$ is adjacent to $c$, then $\{a, h, f\}$ is a minimal dominating set of size 3, contradiction. Hence $fc \notin E(G)$ and $f$ is adjacent to $e$. Finally, $c$ is adjacent to $i$. This case results in $G_{10-16}$ shown in Figure 4.4.

In summary, if $\Delta(G) = 2$, then $G$ is isomorphic to $G_{10-1}$ and in the case of $\Delta(G) = 3$, the proof of Lemma 4.12 follows from Claims 1, 2, 3, and 4. ☐

*Lemma 4.13: Let $G$ be a connected almost well-dominated bipartite graph with $\delta(G) = 2$, $\Delta(G) = 4$, and $|V(G)| = 10$. Then $G \in \{G_{10-4}, G_{10-5}, G_{10-6}, G_{10-8}, G_{10-9}, G_{10-11}\}$.*

*Proof.* Let $A = \{a, b, c, d, e\}$ and $B = \{f, g, h, i, j\}$ be the two partites of $G$. Since $\Delta(G) = 4$, there exists at least one vertex of degree 4, say $e$, in $G$. Let further the vertices $\{g, h, i, j\}$ be the neighbors of $e$ in partite $B$. Note that if $d_f = 4$, then $f$ is adjacent to all vertices in $\{a, b, c, d\}$ and $\{e, f\}$ is a minimal dominating set of size 2, contradiction. Furthermore, if $d_f = 3$, then $f$ is adjacent to three vertices in $\{a, b, c, d\}$, say $\{b, c, d\}$. Then, $\{e, f, a\}$ is a minimal dominating set of size 3, contradiction. Thus, $d_f = 2$. Let $a$ and $b$ be the neighbors of $f$ in the partite $A$. Note that since $e$ dominates the vertices $\{g, h, i, j\}$ and $f$ dominates the vertices $\{a, b\}$, if $c$

and $d$ have a common neighbor, say $h$, then $\{e, f, h\}$ is a minimal dominating set of size 3, contradiction. Thus, $c$ and $d$ have disjoint neighborhoods in $\{g, h, i, j\}$, which implies that $c$ and $d$ are both of degree 2. Thus far, it is known that $(d_c, d_d, d_e, d_f) = (2,2,4,2)$, the vertex $e$ is adjacent to the vertices in $\{g, h, i, j\}$, and $f$ is adjacent to the vertices in $\{a, b\}$. We proceed with the proof by focusing on the possible degrees for the remaining vertices $a$ and $b$ in partite $A$ in the sequel.

Suppose first that there exist more than one vertex of degree 4 in one partite. Then at least one of $a$ or $b$, say $a$, is of degree 4. Since $c$ and $d$ have disjoint neighborhoods, we suppose that $c$ is adjacent to two vertices in $\{g, h, i, j\}$, say $g$ and $h$. Then $d$ is adjacent to $i$ and $j$. Note that $a$ is adjacent to $f$ and further has 3 neighbors in $\{g, h, i, j\}$, say $\{h, i, j\}$. Now observe that $a$ dominates $\{f, h, i, j\}$ and $g$ has two neighbors $c$ and $e$ in $A$. If $g$ is further adjacent to $b$, then $\{a, g, d\}$ is a minimal dominating set of size 3, contradiction. Therefore, $g$ is not adjacent to $b$. Hence, $d_g = 2$. If the vertex $b$ has a neighbor in $\{i, j\}$, say $i$, then $\{a, g, i\}$ is a minimal dominating set of size 3, contradiction. Therefore, $b$ has no neighbor in $\{i, j\}$ and hence is adjacent to $h$. Therefore, $d_b = 2$ and $d_h = 4$. The graph obtained in this case, where $(d_a, d_b, d_c, d_d, d_e) = (4,2,2,2,4)$ and $(d_f, d_g, d_h, d_i, d_j) = (2,2,4,3,3)$, is shown as $G_{10-8}$ in Figure 4.4.

Next suppose that there is only one vertex of degree 4 in one partite, i.e., $d_a \neq 4$ and $d_b \neq 4$. In this case, the possible options for $(d_a, d_b)$ are $(2,2)$, $(3,2)$, and $(3,3)$. We proceed with analyzing each possible degree sequence separately in the following.

Suppose first that $(d_a, d_b) = (2,2)$, i.e., $(d_a, d_b, d_c, d_d, d_e) = (2,2,2,2,4)$, which implies two possible degree sequences $(2,2,2,2,4)$ and $(2,2,2,3,3)$ for the partite $B$. Let first $(d_f, d_g, d_h, d_i, d_j) = (2,2,2,2,4)$. Notice that at least one of $c$ or $d$, say $d$ is adjacent to $j$. Then $d$ is adjacent to one vertex in $\{g, h, i\}$, say $i$. In addition, $c$ is adjacent to $g$ and $h$ since it has a disjoint neighborhood from $d$. Finally, $j$ is adjacent to $a$ and $b$ as its third and fourth neighbors. The graph obtained here is $G_{10-6}$ shown in Figure 4.4.

In the case of $(d_f, d_g, d_h, d_i, d_j) = (2,2,2,3,3)$, since $c$ and $d$ have no common neighbor, $j$ has exactly one neighbor from $\{c, d\}$, say $d$. Then, the third neighbor of $j$ is from $\{a, b\}$, say $b$. Similarly, $i$ has exactly one neighbor from $\{c, d\}$ and its third neighbor is different from $c$ and $d$, i.e., it is adjacent to $a$. Now, if $i$ is adjacent to $d$, then $c$ is adjacent to the vertices in $\{g, h\}$ and $\{i, b, c\}$ is a minimal dominating set of size 3, contradiction. Therefore, $id \notin E(G)$ and $i$ is adjacent to $c$ as its third neighbor. Then $c$ is adjacent to one of $g$ or $h$, say $g$. Then $d$ is adjacent to $h$. The graph obtained here is $G_{10-4}$ depicted in Figure 4.4.

Next let $(d_a, d_b) = (3,2)$, in other words, $(d_a, d_b, d_c, d_d, d_e) = (3,2,2,2,4)$. With this degree sequence for partite $A$, there are two possible degree sequences $(2,2,2,3,4)$ and $(2,2,3,3,3)$ for the partite $B$.

First suppose that $(d_f, d_g, d_h, d_i, d_j) = (2,2,2,3,4)$. Since $c$ and $d$ have no common neighbor, $j$ is adjacent to only one of $c$ or $d$, say $d$. Furthermore, $j$ is adjacent to $a$ and $b$ as its third and fourth neighbors. Observe that if $ic \notin E(G)$, then $i$ is adjacent to the vertices $\{d, a\}$ and $c$ is adjacent to the vertices $\{g, h\}$. Then, $\{i, c, b\}$ is a minimal dominating set of size 3, contradiction. Therefore, $ic \in E(G)$ and further $i$ is adjacent to $a$ as its third neighbor. Now $c$ is adjacent to one of $g$ or $h$, say $g$, as its second neighbor and finally, $d$ is adjacent to $h$. The graph obtained here is $G_{10-5}$ shown in Figure 4.4.

We then suppose that $(d_f, d_g, d_h, d_i, d_j) = (2,2,3,3,3)$. One of $c$ or $d$, say $c$ is adjacent to $g$. The vertex $a$ is adjacent to two vertices in $\{h, i, j\}$, say $h$ and $i$. Notice that since $j$ is adjacent to only one of $c$ or $d$, the other neighbor of $j$ is $b$. Note further that if $j$ is adjacent to $d$, then $\{a, c, j\}$ is a minimal dominating set of size 3, contradiction. Therefore, $jd \notin E(G)$ and hence $j$ is adjacent to $c$. Finally, $d$ is adjacent to both $h$ and $i$. The graph obtained in this case is $G_{10-11}$ shown in Figure 4.4.

The last option for $(d_a, d_b)$ is $(3,3)$. When $(d_a, d_b, d_c, d_d, d_e) = (3,3,2,2,4)$, there are two possible degree sequences $(2,2,3,3,4)$ and $(2,3,3,3,3)$ for the partite $B$. Note that the degree sequence $(2,2,2,4,4)$ for the partite $B$ is covered in the case where there exist more than one vertex of degree 4 in one partite, where the resulting graph is $G_{10-8}$ in Figure 4.4.

In the case of $(d_f, d_g, d_h, d_i, d_j) = (2,2,3,3,4)$, the vertex $j$ is adjacent to exactly one of $c$ or $d$, say $d$. In addition, $j$ is adjacent to $a$ and $b$. The vertex $h$ is adjacent to at most one of $c$ and $d$, thus its other neighbor is in $\{a, b\}$, say $a$. Similarly, $i$ has exactly one neighbor in $\{c, d\}$; thus, $i$ is also adjacent to $b$. Notice that if $i$ is adjacent to $d$, then $c$ is adjacent to $g$ and $h$ and $\{a, i, c\}$ is a minimal dominating set of size 3, contradiction. Therefore, $id \notin E(G)$ and $i$ is adjacent to $c$. Note that if $d$ is adjacent to $g$, then $\{a, i, d\}$ is a minimal dominating set of size 3, contradiction. Thus, $d$ is adjacent to $h$ and $g$ is adjacent to $c$. However, $\{b, h, c\}$ is a minimal dominating set of size 3, contradiction. This case does not lead to any almost well-dominated graph.

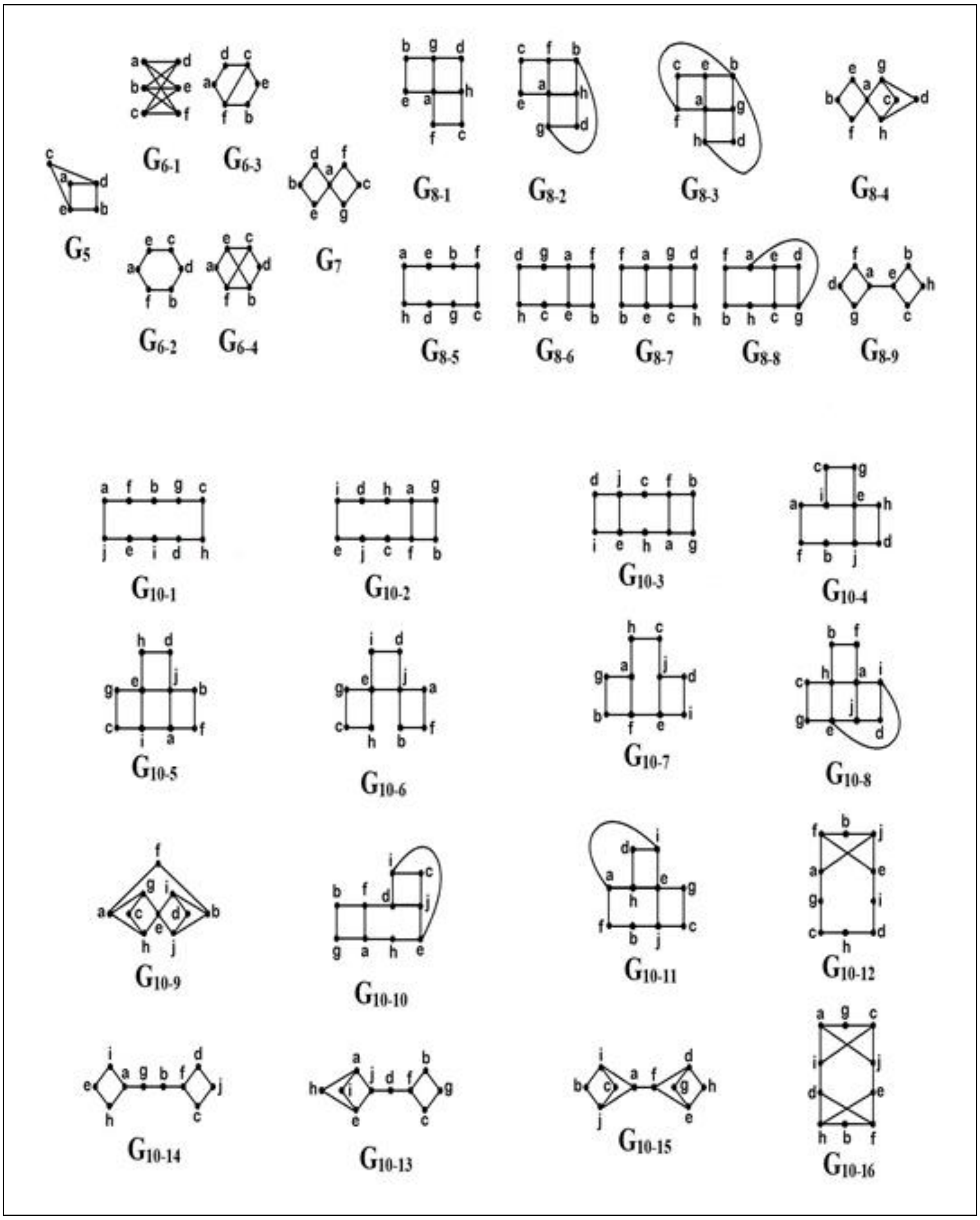

Figure 4.4: Almost well-dominated bipartite graphs with $\delta(G) \geq 2$.

We then suppose that $(d_f, d_g, d_h, d_i, d_j) = (2,3,3,3,3)$. In this case, $d$ is adjacent to any two vertices in $\{g, h, i, j\}$, say $i$ and $j$. Then $c$ is adjacent to $g$ and $h$. The vertex $j$ is adjacent to one of $a$ or $b$, say $b$. Moreover, $a$ has at least one neighbor in $\{g, h\}$, say $g$. Now observe that $a$ is not adjacent to $i$, since otherwise $\{a, c, j\}$ is a minimal dominating set of size 3, contradiction. Hence, $a$ is adjacent to $h$ and $b$ is adjacent to $i$. The graph obtained here is shown as $G_{10-9}$ in Figure 4.4. ☐

Our main result in this section for almost well-dominated bipartite graphs with $\delta(G) \geq 2$ is stated in the following theorem.

*Theorem 4.1. Let $G$ be a bipartite graph with $\delta(G) \geq 2$. Then $G$ is almost well-dominated if and only if $G$ is isomorphic to one of the following:*

- *If $|V(G)| = 5$, then $G \cong G_5$.*
- *If $|V(G)| = 6$, then $G \in \{G_{6-1}, G_{6-2}, G_{6-3}, G_{6-4}\}$.*
- *If $|V(G)| = 7$, then $G \cong G_7$.*
- *If $|V(G)| = 8$, then $G \in \{G_{8-1}, G_{8-2}, \dots, G_{8-9}\}$.*
- *If $|V(G)| = 10$, then $G \in \{G_{10-1}, G_{10-2}, \dots, G_{10-16}\}$.*

*Proof*. Let $G$ be a bipartite graph with $\delta(G) \geq 2$. We proceed the proof in two cases: $\delta(G) \geq 3$ and $\delta(G) = 2$. The first case follows from Lemma 4.6 where $G \cong G_{6-1}$. For the second case, first note that the smallest order of an almost well-dominated bipartite graph with $\delta(G) = 2$ is 5 and further it is shown by Corollary 4.1 that $|V(G)| \leq 10$. Then the result follows for each possible value of $|V(G)|$, separately, from Lemmas 4.7, 4.8, 4.9, 4.10, 4.11, 4.12, and 4.13.

For the converse direction (⇐), the property of being almost well-dominated can be easily verified for the 31 graphs in Theorem 4.1. ☐

# 5. UPPER PAIRED DOMINATION

In this chapter we study a different variant of domination concept, that is, paired domination. We particularly pay attention to upper paired domination, which is a relatively uncovered topic in the literature on paired domination.

Recall that upper paired domination number of a graph $G$, denoted by $\Gamma_{pr}(G)$, is the maximum cardinality of a minimal paired dominating set in $G$. Recall further that upper domination number of a graph $G$, denoted by $\Gamma(G)$, is the largest cardinality of a minimal dominating set in $G$. We begin this chapter by determining the relationship between these two graph parameters. In what follows, we provide some lemmas and known results from the literature which assist us in achieving this goal.

One of the arguments which we will frequently use in our forthcoming proofs is that any independent set $S$ in a graph $G$ can be extended to a maximal independent set $I$ in $G$. The other argument, which is stated in Lemma 5.1, is that every maximal independent set is a minimal dominating set.

*Lemma 5.1: Every maximal independent set is a minimal dominating set.*

*Proof*. Let $I$ be any maximal independent set in $G$. Due to the maximality of $I$, any vertex in $V\backslash I$ has a neighbor in $I$. Hence, $I$ is a dominating set. Furthermore, if we remove any vertex $v \in I$, the set $I - \{v\}$ does not dominate $v$. Hence, $I$ is a minimal dominating set. ☐

The most related results in the literature which provide useful tools for our work in this chapter is due to Ulatowski [9]. We state the first result of Ulatowski in Lemma 5.2, which describes the graphs with upper domination number equal to their order.

*Lemma 5.2: [9] For a graph $G$ of order $n$, $\Gamma_{pr}(G) = n$ if and only if $G$ is isomorphic to $mK_2$.*

Here, $mK_2$ denotes a graph with $m \geq 1$ disjoint $K_2$s. The result in Lemma 5.2 implies that $K_2$ is the only connected graph satisfying $\Gamma_{pr}(G) = n$. The next result establishes an upper bound for the upper domination number of a connected graph.

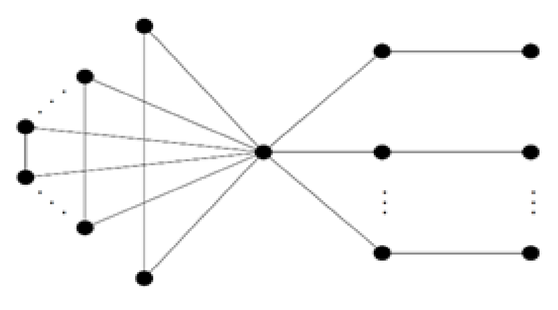

Figure 5.1: A graph in the family $K_{1,t}^{*\Delta}$.

*Lemma 5.3: [9] If $G$ is a connected graph of order $n \geq 3$, then $\Gamma_{pr}(G) \leq n-1$.*

In the same work, Ulatowski characterized the graphs satisfying the equality in the bound of Lemma 5.3. However, before stating this result, we recall some definitions and notations. The subdivided star $K_{1,t}^{*}$ is the graph obtained from a star $K_{1,t}$ by subdividing every edge once. Let $K_{1,t}^{*\Delta}$ for $\Delta \geq 0$ be a family of graphs obtained by attaching $\Delta$ triangles to the central vertex of a $K_{1,t}^{*}$ (see Figure 5.1).

Lemma 5.4 states the second result of Ulatowski regarding the graphs with upper domination number equal to one less than their order.

*Lemma 5.4: [9] Let $G$ be a connected graph of order $n \geq 3$. Then $\Gamma_{pr}(G) = n-1$ if and only if $G \in \{C_3, C_5, K_{1,t}^{*\Delta}\}$.*

The following two lemmas give necessary conditions for the minimality of a paired dominating set. Notice that the notation $epn(u,v;S)$ which is used by [51] is defined as follows:

$$\forall u,v \in S, epn(u,v;S) = \{w \in N(u) \cup N(v) \backslash S \mid N(w) \cap S\{u,v\}\} \qquad (5.1)$$

In other words, for a vertex $w \in epn(u,v;S)$ it holds that either $w \in epn(u,S)$, or $w \in epn(v,S)$, or $w$ is adjacent to both $u$ and $v$ and no other vertex in $S \backslash \{u,v\}$.

*Lemma 5.5: [51] Let S be a minimal PDS in a connected graph $G$ of order at least 3 and let $\{u,v\} \subset S$ and $S' = S \backslash \{u,v\}$. If $S'$ dominates both $u$ and $v$ and $G[S']$ contains a perfect matching, then $|epn(u,v;S)| \geq 1$.*

*Lemma 5.6: [51] Let $S$ be a minimal PDS in a connected graph $G$ of order at least 3 and let $M$ be a perfect matching in $G[S]$. If $uv \in M$ and both $u$ and $v$ have degree at least 2 in $G[S]$, then $|epn(u, v; S)| \geq 1$.*

Our first observation which is stated in Lemma 5.7 is useful in determining the relationship between the upper domination number and the upper paired domination number.

*Lemma 5.7: Every minimal paired dominating set $P$ in $G$ contains a minimal dominating set $S$ such that $|S| \geq |P|/2$.*

*Proof*. Let $P$ be a minimal paired dominating set of $G$. The set $P$ has a perfect matching $M$ which includes pairs of matched vertices $(u_i, v_i)$ for $1 \leq i \leq |P|/2$. Notice that $P$ is not necessarily a minimal dominating set. Thus, we may assume that $S = P \backslash T$ is a minimal dominating set, where $T$ is the set of vertices that are needed to be removed from $P$ to obtain a minimal dominating set. We claim that from each pair of matched vertices $(u_i, v_i)$ in $P$ at most one vertex can be removed to attain a minimal dominating set. We prove our claim by contradiction. Let $(u_1, v_1)$ be a pair of matched vertices in $P$. Suppose to the contrary that $S = P \backslash \{u_1, v_1\}$ is a minimal dominating set in $G$. Notice that $S$ has a perfect matching $M'$ including pairs of matched vertices $(u_i, v_i)$ for $2 \leq i \leq |P|/2$. Hence, $S$ is a paired dominating set in $G$, a contradiction to the minimality of $P$ which completes the proof of our claim. Therefore, from each pair of matched vertices $(u_i, v_i)$ for $1 \leq i \leq |P|/2$ in $P$, at most one vertex can be removed to obtain a minimal dominating set $S$; that is, $T \leq |P|/2$. This in turn implies that $|S| \geq |P|/2$. ☐

Here we state the relationship between the upper domination number and the upper paired domination number in Corollary 5.1, which is an immediate result of Lemma 5.7.

*Corollary 5.1: For any graph $G$, $\Gamma_{pr}(G) \leq 2\Gamma(G)$.*

*Proof*. Let $P$ be a $\Gamma_{pr}$-set of $G$. By Lemma 5.7, $P$ contains a minimal dominating set $S$ with $|S| \geq |P|/2 = \Gamma_{pr}(G)/2$. In addition, we know that $S$ is a minimal dominating set and therefore $|S| \leq \Gamma(G)$. Hence, $\Gamma(G) \geq |S| \geq \Gamma_{pr}(G)/2$. Thus, we have $\Gamma_{pr}(G) \leq 2\Gamma(G)$. ☐

In the remainder of this section, we will investigate the properties of the graphs satisfying $\Gamma_{pr}(G) = 2\Gamma(G)$.

## 5.1. Graphs with $\boldsymbol{\Gamma_{pr}(G) = 2\Gamma(G)}$

The following result for the graphs with the property $\Gamma_{pr}(G) = 2\Gamma(G)$ is obtained from Lemma 5.7.

*Lemma 5.8: Let $G$ be a graph with the property $\Gamma_{pr}(G) = 2\Gamma(G)$. Then every $\Gamma_{pr}$-set of $G$ contains an independent $\Gamma$-set (see Figure 5.2).*

*Proof*. Let $P$ be a $\Gamma_{pr}$-set of $G$ with a perfect matching $M$. By Lemma 5.7, $P$ contains a minimal dominating set $D$ such that $|D| \geq \Gamma_{pr}/2$. Since $\Gamma_{pr}(G) = 2\Gamma(G)$, it follows that $|D| \geq \Gamma(G)$. Thus, $D$ is a $\Gamma$-set. Now it suffices to prove that $D$ is an independent set. Suppose to the contrary that $D$ has two adjacent vertices $v_1$ and $v_2$. Suppose further that $u_1v_1, u_2v_2 \in M$. Let $M' = M\setminus\{u_1v_1, u_2v_2\} + v_1v_2$ and $P' = P\setminus\{u_1, u_2\}$. Notice that $P'$ is a dominating set in $G$ since it contains $D$. Furthermore, it has a perfect matching $M'$; therefore, $P'$ is a paired dominating set, a contradiction to the minimality of P. Thus, $D$ is an independent set. □

In the following, we first state our results for two special graph classes with the property $\Gamma_{pr}(G) = 2\Gamma(G)$, namely bipartite and unicyclic graphs. We then investigate special graph classes with $\Gamma_{pr}(G) = 2\Gamma(G)$ and restricted girth.

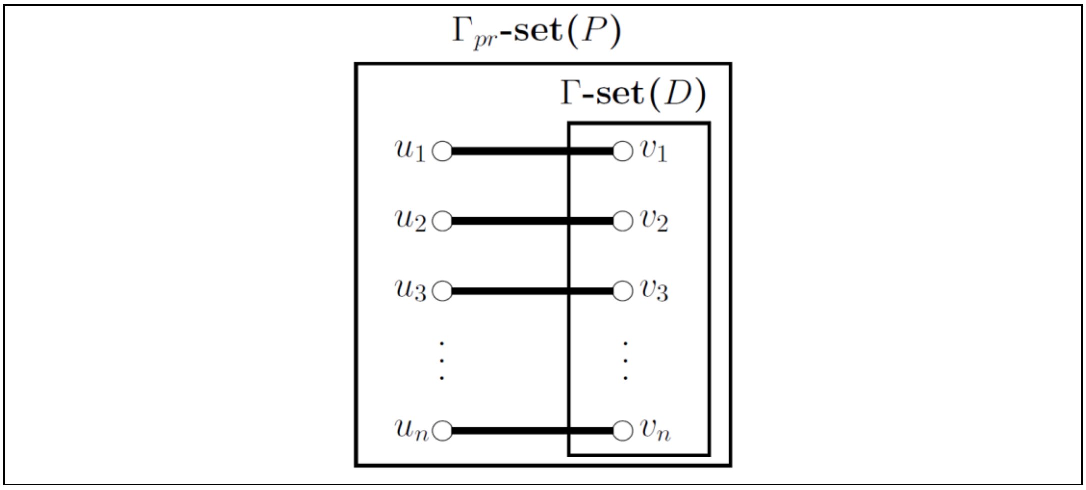

Figure 5.2: A $\Gamma_{pr}$-set $P$ in a graph with $\Gamma_{pr}(G) = 2\Gamma(G)$

### 5.1.1. Bipartite graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$

In this section, we characterize bipartite graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$. We state our obtained result for bipartite graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ in Theorem 5.1.

*Theorem 5.1: Let $G$ be a connected bipartite graph. Then $\Gamma_{pr}(G) = 2\Gamma(G)$ if and only if $G$ is isomorphic to $K_2$.*

*Proof.* Let $G$ be a connected bipartite graph with $\Gamma_{pr}(G) = 2\Gamma(G)$ and order $n$. Note that $G$ has at least one partite of size at least $n/2$, which implies a minimal dominating set of size at least $n/2$. Hence $\Gamma(G) \geq n/2$. Then we have $\Gamma_{pr}(G) \geq n$ which implies that $\Gamma_{pr}(G) = n$. By Lemma 5.2, $G$ is isomorphic to $mK_2$. Since $G$ is connected, it is isomorphic to $K_2$.

For the converse direction, it is easy to see that $\Gamma(K_2) = 1$ and $\Gamma_{pr}(K_2) = 2$ and hence $\Gamma_{pr}(K_2) = 2\Gamma(K_2)$. □

### 5.1.2. Unicyclic graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$

The aim of this section is to describe unicyclic graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$. Before stating our result on unicyclic graphs with the property $\Gamma_{pr}(G) = 2\Gamma(G)$ in Theorem 5.2, we mention the following lemma which establishes lower bounds for upper domination number in unicyclic graphs:

*Lemma 5.9: Let $G$ be a connected unicyclic graph of order $n$. Then the following hold:*

- *For even $n$, $\Gamma(G) \geq n/2$*
- *For odd $n$, $\Gamma(G) \geq (n-1)/2$*

*Proof.* Let $G$ be a connected unicyclic graph of order $n$. Note that $G$ has a single cycle, say $C$. Let $x$ and $y$ be two adjacent vertices of $G$ on $C$. Let further $G'$ be a graph obtained by removing the edge between $x$ and $y$; that is, $V(G') = V(G)$ and $E(G') = E(G) - xy$. Since $G'$ has no cycles, it is a tree and consequently a bipartite graph. If $n$ is even, then $G'$ has either two partites of size $n/2$ or at least one partite, say $A'$ of size at least $n/2+1$. In the former case, at least one of the partites of size $n/2$ in $G'$ is also an independent set in $G$ and hence $\Gamma(G) \geq n/2$. In the latter case, one possibility is that $x$ and $y$ reside in different partites, in which case $A'$ is also an independent set in $G$ of size at least $n/2$+1. However, the other possibility is that $x$ and $y$ reside in the same partite $A'$, in which case $A' - x$ is an independent set of size at least $n/2$ in $G$. Both possibilities imply that $\Gamma(G) \geq n/2$ for even $n$.

On the other hand, if $n$ is odd, then $G'$ has a partite, say $A'$, of size at least $(n+1)/2$. Here two possibilities exist. One is that $x$ and $y$ are in different partites, in which case $A'$ is also an independent set in $G$ and thus, $\Gamma(G) \geq (n+1)/2$. The other possibility is that $x$ and $y$ reside in $A'$, in which case $A' - x$ is an independent set of size at least $(n+1)/2 - 1 = (n-1)/2$ in $G$ and hence, $\Gamma(G) \geq (n-1)/2$. Both possibilities yield $\Gamma(G) \geq (n-1)/2$ for odd $n$. ☐

Now we are ready to state the main result of this section in Theorem 5.2.

*Theorem 5.2: Let $G$ be a connected unicylic graph. Then $\Gamma_{pr}(G) = 2\Gamma(G)$ if and only if $G \in \{C_3, C_5, K_{1,t}^{*1}\}$.*

*Proof.* Let $G$ be a connected unicylic graph with $\Gamma_{pr}(G) = 2\Gamma(G)$ and order $n$. In the case of even $n$, by Lemma 5.9, we have $\Gamma(G) \geq n/2$, which yields $\Gamma_{pr}(G) = n$. Then, by Lemma 5.2, $G$ is isomorphic to $mK_2$ which is not unicyclic, contradiction. Thus, the case of even $n$ does not lead to a unicyclic graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. However, for odd $n$, it follows from Lemma 5.9 that $\Gamma(G) \geq (n-1)/2$. This, in turn, leads to $\Gamma_{pr}(G) \geq (n-1)$. Since n is odd we have $\Gamma_{pr}(G) = (n-1)$. Then, it follows from Lemma 5.4 that $G$ is isomorphic to $\{C_3, C_5, K_{1,t}^{*\Delta}\}$. Obviously, $C_3$ and $C_5$ are unicyclic graphs and $K_{1,t}^{*1}$ (for $\Delta = 1$) is the only unicyclic graph in the family $K_{1,t}^{*\Delta}$. Therefore, $G \in \{C_3, C_5, K_{1,t}^{*1}\}$.

For the converse direction, we show that if $G \in \{C_3, C_5, K_{1,t}^{*1}\}$, then $\Gamma_{pr}(G) = 2\Gamma(G)$. For the case of $C_3$ and $C_5$, it is easy to verify that $\Gamma(C_3) = 1$, $\Gamma_{pr}(C_3) = 2$ and $\Gamma(C_5) = 2$, $\Gamma_{pr}(C_5) = 4$. Furthermore, $\Gamma\left(K_{1,t}^{*1}\right) = t + 1$ and $\Gamma_{pr}\left(K_{1,t}^{*1}\right) = 2(t+1)$. ☐

### 5.1.3. Graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ and restricted girth

In this section, we address the problem of describing graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ from a girth point of view. We begin with the case of graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ and girth at least 6.

We first give some definitions and notations that we frequently use in the forthcoming proofs. Let $P$ be any $\Gamma_{pr}$-set of a graph $G$ with $\Gamma_{pr}(G) = 2\Gamma(G)$. By $G[P]$, we refer to the subgraph induced by the set $P$. Furthermore, if a vertex in $P$ is only adjacent to a single vertex in $P$, we name it a leaf in $G[P]$.

*Theorem 5.3: Let $G$ be a graph of girth at least 6. Then $\Gamma_{pr}(G) = 2\Gamma(G)$ if and only if $G$ is isomorphic to $mK_2$ (*for $m \geq 1$).

*Proof*. Let $G$ be a graph of girth at least 6 with $\Gamma_{pr}(G) = 2\Gamma(G)$. If $\Gamma_{pr}(G) = n$, by Lemma 5.2, $G$ is isomorphic to $mK_2$ (for $m \geq 1$) and we are done.

We will now show that the case $\Gamma_{pr}(G) \leq (n-1)$ does not lead to a graph with $\Gamma_{pr}(G) = 2\Gamma(G)$ and complete the proof. Suppose that $\Gamma_{pr}(G) \leq (n-1)$. By Lemma 5.8, $G$ has a $\Gamma_{pr}$-set $P$, which has an independent $\Gamma$-set $B$ inside it. We further define set $A = P \setminus B$ as the set of partners of the vertices in $B$. Let $A = \{a_i\}$ and $B = \{b_i\}$ for $1 \leq i \leq \Gamma(G)$. Note that $P$ has a perfect matching including pairs of matched vertices $(a_i, b_i)$ for $a_i \in A$, $b_i \in B$, and $1 \leq i \leq \Gamma(G)$. Since $\Gamma_{pr}(G) \leq (n-1)$, it implies that there exists at least one vertex $x$ in $V(G) \setminus P$.

We first show that $G[P] \neq mK_2$ where $m = \Gamma(G)$. Suppose to the contrary that $G[P] = \Gamma(G)K_2$. Note that the vertex $x$ is adjacent to at most one vertex of each pair of matched vertices $(a_i, b_i)$ since the girth is at least 6. Let $Z$ be a set including one vertex from each pair of vertices $(a_i, b_i)$ in $P$ which is not adjacent to $x$. Since $G[P] = \Gamma(G)K_2$, the set $Z$ is an independent set. Thus, $Z \cup x$ forms an independent set of size $\Gamma(G) + 1$, a contradiction to $B$ being a $\Gamma$-set. Therefore, $G[P] \neq mK_2$ and without loss of generality, we assume that there exist at least two pairs of matched vertices, say $(a_1, b_1)$ and $(a_2, b_2)$, which have two adjacent endpoints, say $a_1a_2 \in E(G)$. Now by Lemma 5.5, it holds that $|epn(b_1, b_2; P)| \geq 1$. Let $y$ be a vertex in $epn(b_1, b_2; P)$. Due to the girth restriction, $y$ is not adjacent to both of $b_1$ and $b_2$.

Thus, $y$ is adjacent to exactly one of $b_1$ and $b_2$, say $b_1$. Notice that for each pair of matched vertices $(a_i, b_i)$ for $2 \leq i \leq \Gamma(G)$, one of the following three cases holds:

- Case 1: $b_i a_1 \notin E(G)$
- Case 2: $b_i a_1 \in E(G)$ and $a_i$ is a leaf in $G[P]$
- Case 3: $b_i a_1 \in E(G)$ and $a_i$ is not a leaf in $G[P]$

Note that in Case 3 a vertex $b_i$ for $2 \leq i \leq \Gamma(G)$ has a neighbor $a_1$ which is different from its partner $a_i$. Hence $b_i$ has degree at least two in $G[P]$. Besides, the partner of $b_i$, namely $a_i$, has degree at least two in $G[P]$ since it is not a leaf in $G[P]$. Therefore, it follows by Lemma 5.6 that $|epn(a_i, b_i; P)| \geq 1$. This in turn implies that there exists at least one vertex $c_i$ in $V(G)\setminus P$ which is a private neighbor of $a_i$ and $b_i$. Due to girth at least 6 restriction, $c_i$ is adjacent to exactly one of $a_i$ and $b_i$. Since $b_i$ is a vertex in the $\Gamma$-set $B$, $c_i$ is only adjacent to $b_i$ (see Figure 5.3).

Now let us define the three sets $A'$, $B'$, and $C$ as follows: for each pair of matched vertices $(a_i, b_i)$ for $2 \leq i \leq \Gamma(G)$ if Case 1 holds, then put $b_i$ in $B'$; if Case 2 holds, then put $a_i$ in $A'$, and if Case 3 holds, then put $c_i$ in $C$. It is easy to see that $B'$ is an independent set since $B' \subseteq B$. Furthermore, $I = A' \cup C \cup \{y\}$ is an independent set since $I \subset N_2(a_1)$ and the girth of $G$ is at least 6. The vertices in $A'$ are leaves in $G[P]$; that is, they are only adjacent to their partners $b_i$ in $B\setminus B'$. Thus, no vertex in $A'$ is adjacent to a vertex in $B'$. Besides, the vertices in $C$ are private neighbors which are only adjacent to a vertex $b_i$ in $B\setminus B'$. Hence, no vertex in $C$ is adjacent to a vertex in $B'$. By definition no vertex in $B'$ is adjacent to $a_1$ and the vertex $y$ is adjacent only to $b_1$. Hence we have that $\{y, a_1\} \cup A' \cup B' \cup C$ is an independent set. Note that the sets $A'$, $B'$, and $C$ are mutually disjoint sets since for each pair of matched vertices $(a_i, b_i)$ for $2 \leq i \leq \Gamma(G)$, exactly one of the three aforementioned cases holds. Thus, $|A' \cup B' \cup C| = \Gamma(G) - 1$. Hence, the set $\{y, a_1\} \cup A' \cup B' \cup C$ is an independent set of size $\Gamma(G) + 1$, a contradiction to $B$ being a $\Gamma$-set. Therefore, there exists no graph $G$ with $\Gamma_{pr}(G) \leq (n-1)$, $\Gamma_{pr}(G) = 2\Gamma(G)$, and girth at least 6. Hence, $G$ is isomorphic to $mK_2$ (for $m \geq 1$) and we are done.

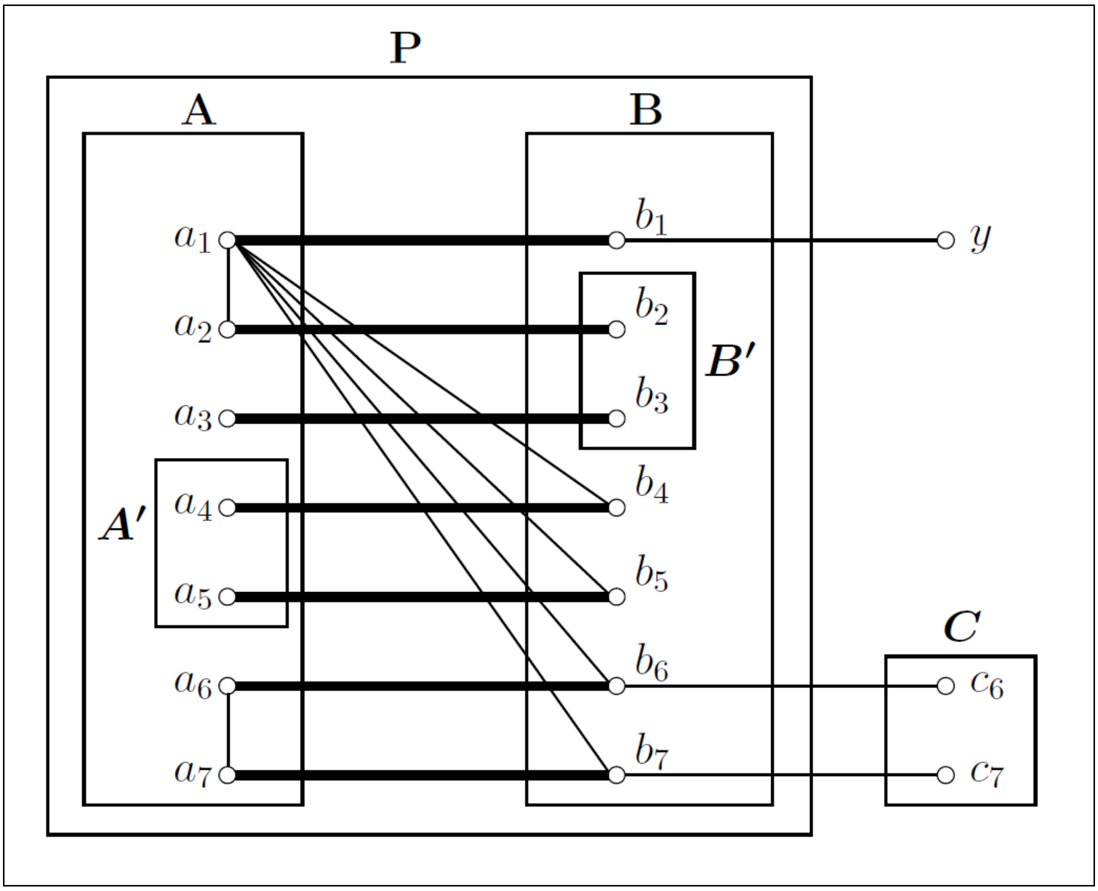

Figure 5.3: The sets $A', B'$, and $C$.

For the converse direction, it can easily be verified that $\Gamma(mK_2)= m$ and $\Gamma_{pr}(mK_2)=2m$. Therefore, $\Gamma_{pr}(mK_2)=\Gamma(mK_2)$. ☐

In what follows, we proceed to the graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ and girth smaller than 6. We focus on graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ and girth at least 4 and provide a characterization for a special family of graphs with the mentioned properties, that is, $C_3$-free cactus graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$.

From here onward, we assume that $G$ is a $C_3$-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Furthermore, recall that by Lemma 5.8, $G$ has a $\Gamma_{pr}$-set $P$ with an independent $\Gamma$-set inside it. Let further $P = A \cup B$, where $B$ is an independent $\Gamma$-set and $A$ is the set of partners of the vertices in $B$. Let $A = \{a_i\}$ and $B = \{b_i\}$ for $1 \leq i \leq \Gamma(G)$. Note that $P$ has a perfect matching including pairs of matched vertices $(a_i, b_i)$ for $a_i \in A$, $b_i \in B$, and $1 \leq i \leq \Gamma(G)$. We now continue with presenting a number of lemmas which provide essential tools for the characterization of $C_3$-free cactus graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$.

*Lemma 5.10: Let $G$ be $C_3$-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Let further $P$ be any $\Gamma_{pr}$-set of $G$. Then, for $1 \leq i \leq \Gamma(G)$, at least one vertex of each pair of matched vertices $(a_i, b_i)$ is a leaf in $G[P]$.*

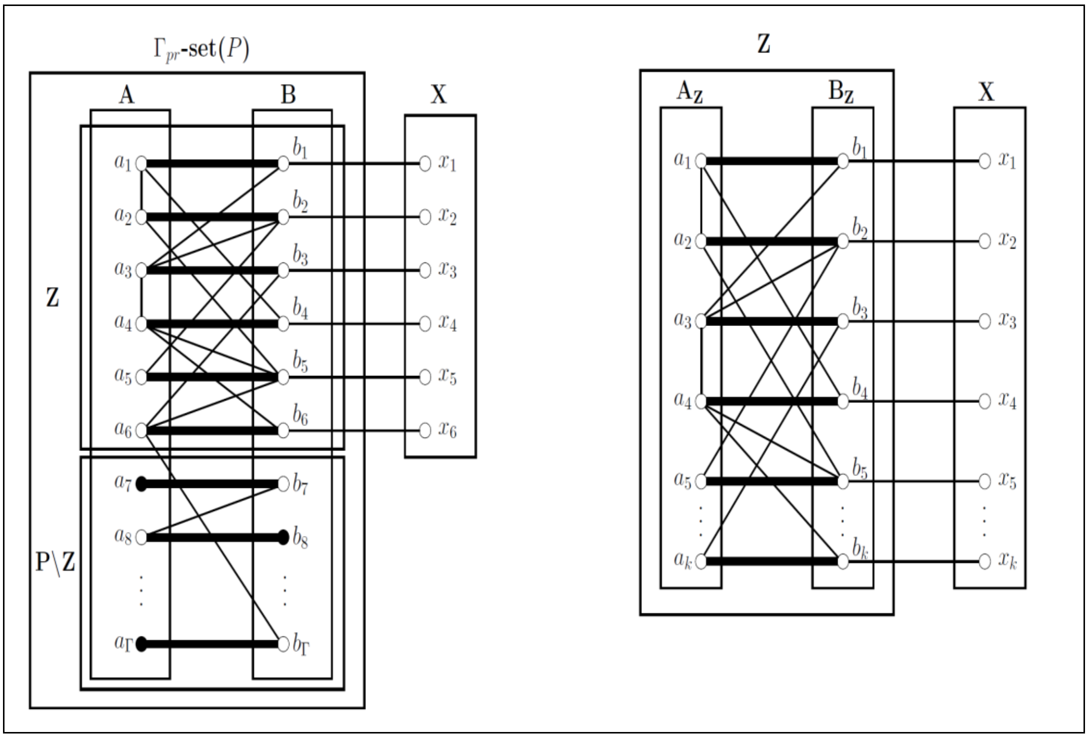

Figure 5.4: The sets $P, Z, P\backslash Z$ and $X$ in $G$ and the subgraph $G[Z \cup X]$.

*Proof*. Suppose to the contrary that there exist $k$ pairs of matched vertices $(a_i, b_i)$ in $P$ such that both $a_i$ and $b_i$ have degree at least 2 in $G[P]$ for $1 \leq k \leq \Gamma(G)$. We first look at the case $k = 1$, where there exists a single pair of matched vertices, say $(a_1, b_1)$, in $P$ such that both $a_1$ and $b_1$ have degree at least 2 in $G[P]$. By Lemma 5.6, $|epn(a_1, b_1; P)| \geq 1$, which implies that $a_1$ and $b_1$ have a private neighbor $x_1$ in $V(G)\backslash P$. Since $G$ is a $C_3$-free graph, $x_1$ is not adjacent to both $a_1$ and $b_1$. Thus, $x_1$ is adjacent to exactly one of $a_1$ and $b_1$. Since $B$ is a $\Gamma$-set, $x_1$ is adjacent to $b_1$. We define $I_L$ as a set containing one leaf in $G[P]$ from each pair of matched vertices $(a_i, b_i)$ for $2 \leq i \leq \Gamma(G)$ in. It is clear that $I_L$ is an independent set. Thus, $\{x_1, a_1\} \cup I_L$ is an independent set of size $\Gamma(G) + 1$, which implies a minimal dominating set of size at least $\Gamma(G) + 1$, a contradiction to $B$ being a $\Gamma$-set of $G$. Hence, we are done with the case $k = 1$.

Then we proceed to the case with $k$ pairs of matched vertices $(a_i, b_i)$ in $P$ such that both $a_i$ and $b_i$ have degree at least 2 in $G[P]$ for $2 \leq k \leq \Gamma(G)$. Let $Z$ be the set containing pairs of matched vertices $(a_i, b_i)$ in $P$ such that both $a_i$ and $b_i$ have degree at least 2 in $G[P]$. We further assume that $Z = A_z \cup B_z$ where $A_z \subseteq A$ and $B_z \subseteq B$ (see Figure 5.4). By Lemma 5.6, for each pair of $(a_i, b_i)$ in $Z$, we have that $|epn(a_i, b_i; P)| \geq 1$ for $1 \leq i \leq k$. This implies that each pair of vertices $a_i$ and $b_i$ in $Z$ have a private neighbor $x_i$ in $V(G)\backslash P$. The vertex $x_i$ is not adjacent to both $a_i$ and $b_i$ since $G$ is a $C_3$-free graph. Thus, each $x_i$ is adjacent to exactly one of $a_i$ and $b_i$. Since $B$ is a $\Gamma$-set, $x_i$ is adjacent only to $b_i$. We define $X$ as a set containing $x_i$ for $1 \leq i \leq k$ (see Figure 5.4). Notice that from each pair of matched vertices $(a_i, b_i)$ in $P\backslash Z$ at least one vertex is a leaf in $G[P]$. The leaves in $G[P]$ are shown with filled circles in Figure 5.4. Let $I_L$ be a set containing one vertex from each pair of matched vertices $(a_i, b_i)$ in $P\backslash Z$ which is a leaf in $G[P]$. Therefore, $|I_L| = \Gamma(G) - |Z|$. We continue with the following claims.

Claim 1: Each $a_i \in A_z$ has at least one neighbor in $B_z$ different from its partner $b_i$.

Proof of Claim 1: Suppose to the contrary that a vertex $a_i \in A_z$, say $a_1$, is adjacent only to its partner $b_1$ in $G[Z]$. Then $\{a_1, x_1\} \cup (B_z\backslash\{b_1\}) \cup I_L$ is an independent set of size $2 + \Gamma(G) - 1 = \Gamma(G) + 1$, a contradiction to $B$ being a $\Gamma$-set. ☐

Claim 2: For any $a_i \in A_Z$, it holds that at least two vertices in $N_2(a_i) \cap X$ are adjacent.

Proof of Claim 2: Suppose to the contrary that there exists a vertex in $A_Z$, say $a_1$, such that $(N_2(a_1) \cap X)$ is an independent set (see Figure 5.5). Then, $a_1 \cup (N_2(a_1) \cap X) \cup \big(B_Z\backslash N(a_1)\big) \cup I_L$ is an independent set of size $\Gamma(G) + 1$, a contradiction to $B$ being a $\Gamma$-set. Thus, at least two vertices in $N_2(a_i) \cap X$ are adjacent. ☐

The argument in Claim 2 implies that for each vertex $a_i \in A_Z$, there is at least one pair of adjacent vertices $(x_k, x_l)$ in $X$. Since $|A_Z| = k$, there must exist at least $k$ pairs of adjacent vertices in $X$. However, since $|X| = k$, there exist at most $k/2$ pairs with disjoint vertices in $X$. Therefore, there exist at least two vertices in $A_z$, say $a_1$ and $a_2$, whose corresponding pairs of adjacent vertices in $X$ are not disjoint; that is, these pairs have either one or two vertices in common. Recall that each vertex $x_i \in X$ is a private neighbor of a vertex $b_i \in B_z$; that is, each $x_i$ is adjacent to a single vertex $b_i$ in $B_z$. Now let $x_1$ and $x_2$ be the corresponding pair of adjacent vertices for $a_1$ in $X$.

This implies that $b_2$ is also adjacent to $a_1$ and we have a 5-cycle $C_1 = (a_1, b_1, x_1, x_2, b_2)$. Note that if $x_1$ and $x_2$ are also the corresponding pair of adjacent vertices for $a_2$, then $b_1$ is also adjacent to $a_2$ and we have a 5-cycle $C_2 = (a_2, b_1, x_1, x_2, b_2)$. However, $C_1$ and $C_2$ are two cycles with a common edge $x_1x_2$, a contradiction to $G$ being a cactus graph. In the other case, if the corresponding pair of adjacent vertices for $a_2$ has only one vertex, say $x_2$, in common with that of $a_1$, then $x_2$ is adjacent to another vertex in $X$, say $x_3$. This in turn implies that $b_3$ is also adjacent to $a_2$ and we have a 5-cycle $C_3 = (a_2, b_2, x_2, x_3, b_3)$. In this case, we have two cycles $C_1$ and $C_3$ with a common edge $b_2x_2$, a contradiction to $G$ being a cactus graph. Therefore, there are no pairs of matched vertices $(a_i, b_i)$ in $P$ such that both $a_i$ and $b_i$ have degree at least 2 in $G[P]$ for $1 \leq k \leq \Gamma(G)$. □

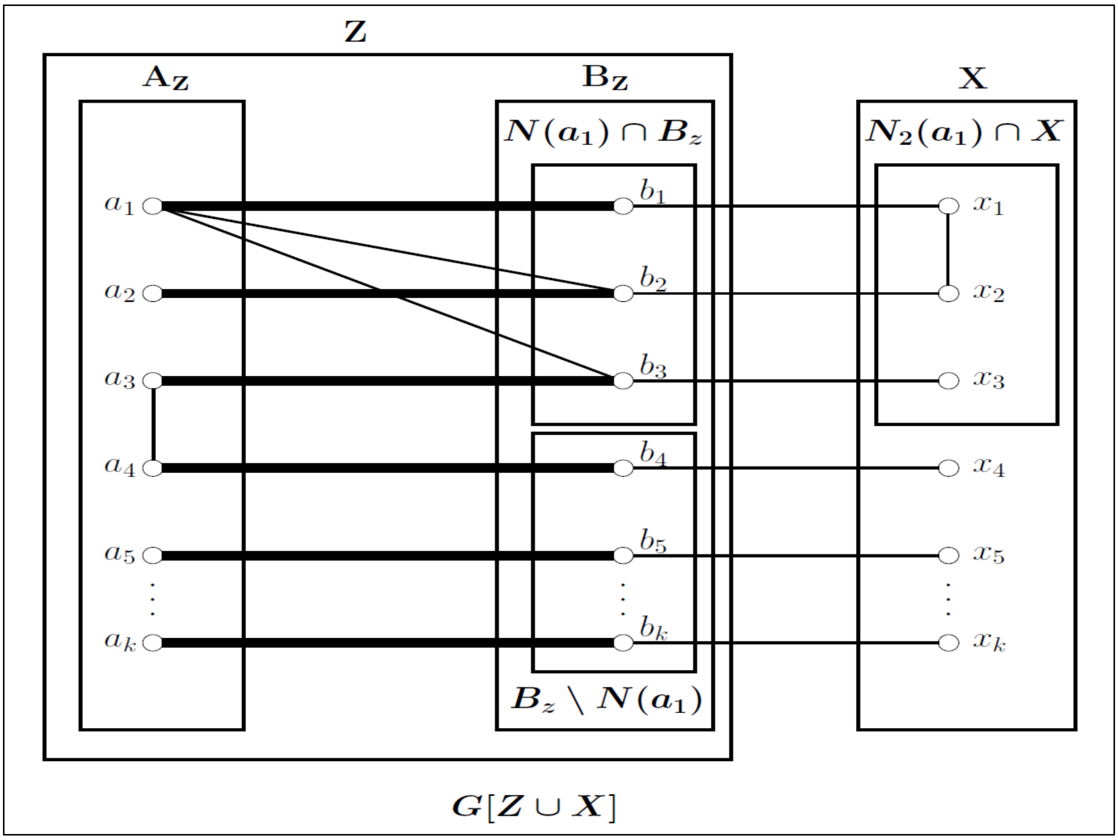

Figure 5.5: The sets $N(a_1) \cap B_z$ and $N_2(a_1) \cap X$ in $G[Z \cup X]$.

Lemma 5.10 implies that at least one vertex of each pair of matched vertices $(a_i, b_i)$ in $P$ is a leaf in $G[P]$. We define the set $L_p$ as a set containing one leaf from each pair of matched vertices $(a_i, b_i)$ in $G[P]$ for $1 \leq i \leq \Gamma(G)$. It is clear that $L_p$ is an independent set in $G[P]$ and $|L_p| = \Gamma(G)$. In the following lemmas, we obtain some other properties of $C_3$-free cactus graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$.

*Lemma 5.11: Let $G$ be a $C_3$-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Let further $P$ be any $\Gamma_{pr}$-set of $G$. If there exists a vertex $x$ in $V(G)\backslash P$, it has exactly two neighbors in $P$.*

*Proof*. We first prove that if there exists a vertex $x$ in $V(G)\backslash P$, it has at least two neighbors in $P$.

Claim 1. Every vertex $x$ in $V(G)\backslash P$ has at least two neighbors in $P$.

Proof of Claim 1: Suppose to the contrary that there exists a vertex $x$ in $V(G)\backslash P$ which has exactly one neighbor in $P$, say $b_1$. By Lemma 5.10, one vertex from each pair of matched vertices $(a_i, b_i)$ in $P$ is a leaf in $G[P]$. Let further $L'_p$ be a set containing one leaf in $G[P]$ from each pair of matched vertices $(a_i, b_i)$ for $2 \leq i \leq \Gamma(G)$. Thus, $|L'_p| = \Gamma(G) - 1$. Then, $\{a_1, x\} \cup L'_p$ is a minimal dominating set of size $2 + \Gamma(G) - 1 = \Gamma(G) + 1$, a contradiction to $B$ being a $\Gamma$-set of $G$. Therefore, each vertex $x$ in $V(G)\backslash P$ has at least two neighbors in $P$.

Now we proceed by showing that the case of a vertex $x$ in $V(G)\backslash P$ with at least three neighbors in $P$ leads to a contradiction and complete the proof of Lemma 5.11. Suppose to the contrary that $x$ is a vertex in $V(G)\backslash P$ with at least three neighbors in $P$. We define a set $Z$ as follows: for each pair of matched vertices $(a_i, b_i)$ in $P$, if $a_i \in N(x)$, put $b_i$ in $Z$; otherwise, if $b_i \in N(x)$, put $a_i$ in $Z$.

We first show that $Z$ is an independent set. Suppose to the contrary that two vertices in $Z$, say $a_1$ and $a_2$ are adjacent. By definition of $Z$, the partners of these vertices, namely $b_1$ and $b_2$ are neighbors of $x$. Moreover, since $a_1$ and $a_2$ are adjacent, by Lemma 5.5, the vertices $b_1$ and $b_2$ have a private neighbor, say $y$, in $V(G)\backslash P$. Definitely, the vertex $y$ is different from $x$ since $x$ has at least three neighbors in $P$ and cannot be a private neighbor for $b_1$ and $b_2$. However, $y$ is adjacent to exactly one of $b_1$ or $b_2$ since otherwise we have two cycles $(yb_1a_1a_2b_2)$ and $(xb_1a_1a_2b_2)$ with a common edge $a_1a_2$, a contradiction to $G$ being a cactus graph. Thus, $y$ is adjacent to one of $b_1$ or $b_2$, say $b_1$. Then, $y$ is a vertex in $V(G)\backslash P$ with exactly one neighbor $b_1$ in $P$, a contradiction to Claim 1. Therefore, $Z$ is an independent set.

Let $L'_p$ be a set containing one leaf in $G[P]$ from each pair of matched vertices $(a_i, b_i)$ in $P$ such that neither $a_i$ nor $b_i$ is adjacent to $x$. It is obvious that $|L'_p| = \Gamma(G) - |Z|$. Then, $\{x\} \cup Z \cup L'_p$ is an independent set of size $\Gamma(G) + 1$, which implies a minimal dominating set of size at least $\Gamma(G) + 1$, a contradiction to $B$ being a $\Gamma$-set. Therefore, any vertex $x$ in $V(G) \backslash P$ has exactly two neighbors inside $P$. ☐

*Lemma 5.12: Let $G$ be $C_3$-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Let further $P$ be any $\Gamma_{pr}$-set of $G$. If there exists a vertex $x$ in $V(G) \backslash P$, then the partners of the two neighbors of $x$ in $P$ are adjacent.*

Proof. Let $x$ be a vertex in $V(G) \backslash P$. By Lemma 5.11, the vertex x has exactly two neighbors in $P$, say $b_1$ and $b_2$. Suppose to the contrary that the partners of $b_1$ and $b_2$, namely $a_1$ and $a_2$, are non-adjacent. By Lemma 5.10, we know that at least one vertex from each pair of matched vertices $(a_i, b_i)$ in $P$ is a leaf in $G[P]$. Let $L'_p$ be the set containing one leaf in $G[P]$ from each pair of matched vertices $(a_i, b_i)$ for $3 \leq i \leq \Gamma(G)$. Note that $|L'_p| = \Gamma(G) - 2$. Thus, $\{x, a_1, a_2\} \cup L'_p$ is a minimal dominating set of size $\Gamma(G) + 1$, a contradiction to $B$ being a $\Gamma$-set. Therefore, the partners of the neighbors of $x$ in $P$, namely $a_1$ and $a_2$, are adjacent. ☐

*Lemma 5.13: Let $G$ be a $C_3$-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Let further $P$ be any $\Gamma_{pr}$-set of $G$. If there exist two vertices $x_1$ and $x_2$ in $V(G) \backslash P$, then they have no common neighbor in $P$.*

*Proof.* Suppose to the contrary that $x_1$ and $x_2$ are two vertices in $V(G) \backslash P$, which have common neighbors in $P$. By Lemma 5.11, each of $x_1$ and $x_2$ has exactly two neighbors in $P$. Let $a_1$ and $a_2$ be the two neighbors of $x_1$ in $P$. By Lemma 5.12, the partners of $a_1$ and $a_2$, namely $b_1$ and $b_2$, are adjacent. If $x_1$ and $x_2$ have two common neighbors in $P$, that is, if $x_2$ is also adjacent to $a_1$ and $a_2$, then we have two cycles $(x_1 a_1 b_1 b_2 a_2)$ and $(x_2 a_1 b_1 b_2 a_2)$ with a common edge $b_1 b_2$, a contradiction to $G$ being a cactus graph. On the other hand, if $x_1$ and $x_2$ have only one common neighbor, say $a_2$, then $x_2$ has another neighbor in $P$, say $a_3$. By Lemma 5.12, the partners of $a_2$ and $a_3$, namely $b_2$ and $b_3$, are adjacent. Then we have two cycles

$(x_1a_1b_1b_2a_2)$ and $(x_2a_2b_2b_3a_3)$ with a common edge $a_2b_2$, a contradiction to $G$ being a cactus graph. Hence, $x_1$ and $x_2$ have no common neighbor in $P$. ☐

*Lemma 5.14: Let $G$ be a $C_3$-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Let further $P$ be any $\Gamma_{pr}$-set of $G$. Then, $\Delta(G[P]) \leq 2$.*

*Proof.* Let $G$ be a $C_3$-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Let further $P$ be any $\Gamma_{pr}$-set of $G$ which includes pairs of matched vertices $(a_i, b_i)$ for $1 \leq i \leq \Gamma(G)$. Suppose to the contrary that a vertex in $P$, say $a_1$, has at least three neighbors in $P$. One of these three neighbors is the partner of $a_1$, namely $b_1$. Without loss of generality, let $b_2$ and $b_3$ be the other two neighbors of $a_1$ in $P$. Since $a_1$ is adjacent to $b_2$, by Lemma 5.5, we have $|epn(a_2, b_1; P)| \geq 1$, which implies that $a_2$ and $b_1$ have a private neighbor $x$ in $V(G)\backslash P$. By Lemma 5.11, $x$ is adjacent to both $a_2$ and $b_1$. In addition, since $a_1$ is adjacent to $b_3$, by Lemma 5.5, we have $|epn(a_3, b_1; P)| \geq 1$. This implies that $a_3$ and $b_1$ have a private neighbor $y$ in $V(G)\backslash P$. By Lemma 5.11, $y$ is adjacent to both $a_3$ and $b_1$. However, $x$ and $y$ are two vertices in $V(G)\backslash P$ with a common neighbor $b_1$ in $P$, a contradiction to Lemma 5.13. Thus, a vertex in $P$ has at most two neighbors in $P$; that is, $\Delta(G[P]) \leq 2$. ☐

*Lemma 5.15: Let $G$ be a $C_3$-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Let further $P$ be any $\Gamma_{pr}$-set of $G$. At most one vertex from each pair of matched vertices $(a_i, b_i)$ in $P$ has a neighbor in $V(G)\backslash P$.*

*Proof.* Suppose to the contrary that there exists a pair of matched vertices in $P$, say $(a_1, b_1)$, such that both $a_1$ and $b_1$ have neighbors in $V(G)\backslash P$. Let further $x_1$ be the neighbor of $b_1$ and $x_2$ be the neighbor of $a_1$ in $V(G)\backslash P$. It is clear that $x_1 \neq x_2$ since $G$ is a $C_3$-free graph. By Lemma 5.11, $x_1$ has two neighbors in $P$. Hence, we may assume that $x_1$ is adjacent to another vertex in $P$, say $b_2$. Similarly, $x_2$ has two neighbors in $P$; however, by Lemma 5.13, $x_2$ has no common neighbor with $x_1$ in $P$. Thus, we may assume that $x_2$ is adjacent to $b_3$. By Lemma 5.12, $a_1$ is adjacent to $a_2$ and $b_1$ is adjacent to $a_3$. Then, $(a_1, b_1)$ is a pair of matched vertices both of which have degree at least two in $G[P]$, a contradiction to Lemma 5.10. Thus, at most one vertex from each pair of matched vertices $(a_i, b_i)$ in $P$ has a neighbor in $V(G)\backslash P$. ☐

*Lemma 5.16: Let $G$ be a $C_3$-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Let further $P$ be any $\Gamma_{pr}$-set of $G$. Then any two vertices $x_1$ and $x_2$ in $V(G)\backslash P$ are non-adjacent.*

*Proof.* Let $G$ be a $C_3$-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Let further $P$ be any $\Gamma_{pr}$-set of $G$ which includes pairs of matched vertices $(a_i, b_i)$ for $1 \leq i \leq \Gamma(G)$. Suppose to the contrary that $x_1$ and $x_2$ are two adjacent vertices in $V(G)\backslash P$. We know that by Lemma 5.11 and Lemma 5.13, $x_1$ is adjacent to exactly two vertices in $P$, say $\{b_1, b_2\}$, and $x_2$ is adjacent to two different vertices, say $\{b_3, b_4\}$. By Lemma 5.12, the partners of $b_1$ and $b_2$, namely $a_1$ and $a_2$, and the partners of $b_3$ and $b_4$, namely $a_3$ and $a_4$, are adjacent. By Lemma 5.14, $a_1$, $a_2$, $a_3$, and $a_4$ have no other neighbors in $G[P]$. Then there exists an independent set $I = B\backslash\{b_1, b_2, b_3, b_4\}$ in $G[P]$ such that $|I| = \Gamma - 4$. Then $\{x_1, b_1, b_2, a_3, a_4\} \cup I$ is a minimal dominating set of size $5 + |I| = 5 + \Gamma(G) - 4 = \Gamma(G) + 1$, a contradiction to $B$ being a $\Gamma$-set. Therefore, any two vertices $x_1$ and $x_2$ in $V(G)\backslash P$ are non-adjacent. ☐

Now we are ready to give our main result in this section in Theorem 5.4, which describes the structure of $C_3$-free cactus graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$. Notice that the graph $m_1C_5 + m_2K_2$, which is stated in Theorem 5.4, is a graph composed of $m_1$ copies of disjoint $C_5$s and $m_2$ copies of disjoint $K_2$s.

*Theorem 5.4: Let $G$ be a $C_3$-free cactus graph. Then $\Gamma_{pr}(G) = 2\Gamma(G)$ if and only if $G$ is isomorphic to $m_1K_2 + m_2C_5$ for $m_1 + m_2 \geq 1$.*

*Proof.* Let $G$ be a $C_3$-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. By Lemma 5.8, $G$ has a $\Gamma_{pr}$-set $P$ with an independent $\Gamma$-set $B$ inside it. Let further $A$ be the set of partners of the vertices in $B$. Hence, $P = A \cup B$. Moreover, $P$ has a perfect matching including pairs of matched vertices $(a_i, b_i)$ for $1 \leq i \leq \Gamma(G)$. We start with the case where there exist no vertices in $V(G)\backslash P$, that is, $\Gamma_{pr}(G) = n$. By Lemma 5.2, $G$ is isomorphic to $m_1K_2$ for $m_1 \geq 1$, which is a cactus graph and we are done with this case.

Next, we proceed with the case where there exists at least one vertex $x_1$ in $V(G)\backslash P$, that is, $\Gamma_{pr}(G) \leq n - 1$. By Lemma 5.11, $x_1$ has two neighbors in $P$, say $b_1$ and $b_2$. By Lemma 5.12, the partners of $b_1$ and $b_2$, namely $a_1$ and $a_2$, are adjacent. Since $a_1$ and $a_2$ each has two neighbors in $P$, by Lemma 5.14, they have no other neighbors in $P$. By Lemma 5.15, $a_1$ and $a_2$ have no neighbors in $V(G)\backslash P$ since their partners, namely $b_1$ and $b_2$, have a neighbor $x_1$ in $V(G)\backslash P$. As $a_1$ and $a_2$ each has two neighbors in $G[P]$, by Lemma 5.10, their partners, namely $b_1$ and $b_2$, are only adjacent to their partners and have no other neighbors in $G[P]$. Moreover, $b_1$ and $b_2$ have no neighbors in $V(G)\backslash P$ other than $x_1$ by Lemma 5.13. The vertex $x_1$ has two neighbors $b_1$ and $b_2$ in $P$ and has no other neighbors in $V(G)\backslash P$ by Lemma 5.16. Hence, the vertices $\{x_1, b_1, a_1, a_2, b_2\}$ form a disjoint 5-cycle in $G$. We can make the previous arguments for any vertex in $V(G)\backslash P$; that is, any vertex in $V(G)\backslash P$ together with four vertices from $P$ form a disjoint 5-cycle in $G$. Therefore, $G$ is composed of components which are either $K_2$ or $C_5$.

For the converse direction, it can easily be verified that if $G$ is isomorphic to $m_1K_2 + m_2C_5$ for $m_1 + m_2 \geq 1$, then we have that $\Gamma(m_1K_2 + m_2C_5) = m_1 + 2m_2$, and $\Gamma_{pr}(m_1K_2 + m_2C_5) = 2m_1 + 4m_2$ and hence $\Gamma_{pr}(m_1K_2 + m_2C_5) = 2\Gamma(m_1K_2 + m_2C_5)$. ☐

An immediate result of Theorem 5.4 for connected graphs is stated in Corollary 5.2.

*Corollary 5.2: Let $G$ be a connected $C_3$-free cactus graph. Then $\Gamma_{pr}(G) = 2\Gamma(G)$ if and only if $G$ is either $C_5$ or $K_2$.*

Note that some of the arguments used in Lemmas 5.11- 5.16 are not restricted to cactus graphs and can be used for the general case of $C_3$-free graphs. Then the question that arises here is whether all lemmas mentioned above can be extended for the general case of $C_3$-free graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$. Hence, we pose the following as an open question:

Question: Does Theorem 5.4 hold for $C_3$-free graphs?

# 6. CONCLUSION

In this thesis we deal with two interrelated graph theory concepts: independence and domination. We particularly pay attention to three topics: well-covered graphs, well-dominated graphs, and paired domination. We first give a detailed review of the known results in the literature with an emphasis on the aforementioned topics. The literature on this area is mostly dominated by the research works which focus on the scenarios where the independence or domination gap is zero. In Chapters 2 and 3, we turn our attention to the graphs whose domination gap is one. We name such graphs almost well-dominated graphs.

In Chapter 2, we determine the structure of $(C_3, C_4, C_5, C_7)$-free almost well-dominated graphs by presenting a complete structural characterization of such graphs. This result is an extension of the previous work by Finbow et al., which has implications for almost well-dominated graphs with girth at least 8. In this direction, studying the general case of almost well-dominated graphs with girth at least 6 or special classes of almost well-dominated graphs with girth at least 6 are possible options for future study.

In Chapter 3, we investigate almost well-dominated bipartite graphs. The first result of our study in this chapter provides an upper bound for the cardinality of bipartite graphs with domination gap $k$, where $k \geq 1$, and minimum degree at least two, that is, we prove that $|V(G)| \leq 10k$. The main result of this chapter is a complete structural characterization of almost well-dominated bipartite graphs with minimum degree at least two. While a 4-cycle is the only well-dominated bipartite graph with minimum degree at least two due to Finbow et al. in [5], our results in this thesis show that there exist exactly 31 almost well-dominated bipartite graphs with minimum degree at least two. In line with this study, investigating the structure of almost well-dominated bipartite graphs with domination gap at least two is a possible direction for future research.

In Chapter 4 we focus on paired domination by concentrating on two parameters: upper paired domination number and upper domination number. As a first step, we specify the relationship between these two parameters by showing that $\Gamma_{pr}(G) \leq 2\Gamma(G)$ for any graph G. We then focus on the graphs achieving equality in the mentioned relationship, that is, graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$. Our study in

Chapter 4 has two directions. In one direction, by using the results of Ulatowski [9], we determine the structure of bipartite and unicyclic graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$. In the other direction, we approach the problem of graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ by restricting the graph girth, where we obtain two results: one is the characterization of graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ and girth at least 6 and the other is the characterization of $C_3$-free cactus graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$. As a future research, we conclude Chapter 4 by leaving the characterization of the general case of $C_3$-free graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ as an open question.

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# BIOGRAPHY

Hadi Alizadeh received his B.Sc. in Computer Engineering from K.N. Toosi University of Technology in 2011 and M.s degree in Computer Engineering from Istanbul Technical University in 2015, respectively. He is currently a PhD student in Computer Engineering at Gebze Technical University, Graduate School of Natural and Applied Sciences. His research interests are graph theory and network optimization problems.

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# APPENDICES

## Appendix A: Publications Based on this Thesis

Alizadeh H., Gözüpek D., (2021), “Almost well-dominated bipartite graphs with minimum degree two”, RAIRO Operations Research, 55, 1633-1646.

Alizadeh H., Gözüpek D., Ekinci, G. B., (2020), “(C3,C4,C5,C7)-free almost well-dominated graph”, Discussiones Mathematicae Graph Theory, 1–19.

Alizadeh H., Gözüpek D., (2021), “Upper paired domination versus upper domination”, Submitted to Discrete Mathematics and Theoretical Computer Science.