

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL

**ON ZERO-DIVISOR GRAPH, IDEAL-BASED ZERO-DIVISOR
GRAPH AND WEAKLY ZERO-DIVISOR GRAPH OF
COMMUTATIVE RINGS**

ÖZLEM KAMBERİ

A THESIS OF MASTER OF SCIENCE
DEPARTMENT OF MATHEMATICS

ADVISOR: ASSOC. PROF. ROGHAYEH HAFEZIEH

OCTOBER 2024

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LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

SIFIR-BÖLEN ÇİZGE, İDEAL-TABANLI SIFIR-BÖLEN
ÇİZGE VE DEĞİŞKEN HALKALARIN ZAYIF-SIFIR
BÖLEN ÇİZGESİ ÜZERİNE

ÖZLEM KAMBERİ

YÜKSEK LİSANS TEZİ
MATEMATİK ANABİLİM DALI

DANIŞMAN: DOÇ. DR. ROGHAYEH HAFEZIEH

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JURY

MEMBER

(THESIS ADVISOR) : Assoc. Prof. Roghayeh HAFEZIEH

MEMBER : Assoc. Prof. Fatma KARAOĞLU CEYHAN

MEMBER : Assoc. Prof. Temha ERKOÇ

APPROVAL

Gebze Technical University Graduate Administration Board/...../.....
date and/..... numbered decision.

SIGNATURE/SEAL

*This thesis is dedicated to my beloved husband,
and to my dear parents.*

ABSTRACT

Given a commutative ring R with 1_R , there are three structural non-directed graphs associated to R , namely the zero-divisor graph denoted by $\Gamma(R)$, the ideal based zero-divisor graph associated to a given ideal I denoted by $\Gamma_I(R)$, and the weakly zero-divisor graph denoted by $W\Gamma(R)$. In this thesis, on one side, we investigate some combinatorial properties of these three graphs and provide comprehensive proofs for the related results; on the other side, we examine the algebraic structure of R by considering some special properties for the associated graphs. As a particular case, we give a brief explanation together with some results of the zero-divisor graph of the idealization ring. Finally, we give some new results related to the weakly zero-divisor graph.

Keywords: Zero-divisor, Combinatorial Properties, Algebraic Structure, Idealization Ring, Commutative ring.

ÖZET

Verilen bir birim elemanı 1_R olan deęişmeli halka R için, R ile ilişkili üç yapısal çizge vardır: sıfır-bölen çizge $\Gamma(R)$, belirli bir ideal I ile ilişkili ideal tabanlı sıfır-bölen çizge $\Gamma_I(R)$ ve zayıf sıfır-bölen çizge $W\Gamma(R)$. Bu tezde, bir yandan bu üç çizgenin bazı kombinatorik özelliklerini araştırıyor ve ilgili sonuçlar için kapsamlı kanıtlar sunuyoruz; dięer yandan, ilişkili çizgeler için bazı özel özellikleri dikkate alarak R 'nin cebirsel yapısını inceliyoruz. Özel bir durum olarak, idealizasyon halkasının sıfır-bölen çizgesi ile ilgili bazı sonuçlarla birlikte kısa bir açıklama veriyoruz. Son olarak, zayıf sıfır-bölen çizge ile ilgili bazı yeni sonuçlar sunuyoruz.

Anahtar Kelimeler: Sıfır Bölen, Kombinatoriyel Özellikler, Cebirsel Yapı, İdealleştirme Halkası, Deęişken Halka.

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1. INTRODUCTION

The interest in the connection between ring and graph theory is growing day by day. One of the tools to describe such a connection is the zero-divisor graph, which was introduced by Beck in 1988, see [1]. Given a commutative ring R with 1_R , we let $Z(R)^* = Z(R) \setminus \{0\}$ to be the set of zero-divisors of R . Then, associated with the commutative ring R , the zero-divisor graph of R , denoted by $\Gamma(R)$, is an undirected graph with the vertex set $Z(R)^*$ where distinct vertices x and y are said to be adjacent if and only if $xy = 0$. Considering this nice connection, many authors have studied the zero-divisor graphs and their properties, as are [2], [3], [4], [5], and [6]. Moreover, authors in [7] define the zero-divisor of idealization of a commutative ring, and they look at the preservation of the diameter and girth number of the zero-divisor graph of a ring when it is extended to idealization of the ring. The idealization of a module over a ring is introduced by Nagata in 1955, see [8]. Let M be an R -module, where R is a commutative ring. Then, the idealization of M in R , denoted by $R(+M)$, is defined to be the commutative ring consisting of tuples (r, m) , for $r \in R$ and $m \in M$, together with componentwise addition and the multiplication is defined as follows: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$, for $r_1, r_2 \in R$ and $m_1, m_2 \in M$. Concerning the zero-divisor graphs, recently numerous types of graphs have been investigated in relation with commutative rings, such as; ideal based zero-divisor graphs and weakly zero-divisor graphs.

Redmond in [9], introduced the notion of the ideal based zero-divisor graphs, and denoted it by $\Gamma_I(R)$. Let R be a commutative ring and I be an ideal of it. Then, the graph $\Gamma_I(R)$ is defined to be the graph with vertex set $\{x \in R \setminus I \mid xy \in I \text{ for some } y \in R \setminus I\}$ and two distinct vertices are adjacent if and only if $xy \in I$. So, in the case when $I = 0$ it appears that $\Gamma_I(R) = \Gamma(R)$, and in the case when I is a prime ideal of R , then the zero-divisor graph $\Gamma(R)$ is empty, see Proposition 4.1. On the other hand, the weakly zero-divisor graph was first introduced by M. J. Nikmehr, A. Azadi, and R. Nikandish in 2021, see [10]. Let R be a commutative ring. The weakly zero-divisor graph, denoted by $W\Gamma(R)$, of a commutative ring R , is defined to be the graph with vertex set $Z(R)^*$,

where two distinct vertices x and y are adjacent if there exist $a \in \text{ann}(x)$ and $b \in \text{ann}(y)$ such that $ab = 0$.

The contents of this thesis are outlined as follows: In Chapter 2, we give notations and definitions from graph theory, ring theory, modulo theory, and field theory, which will be used throughout this thesis. Chapter 3 is dedicated to the study of zero-divisor graphs. In this chapter, we have two parts. In the first part, we give basic properties and certain inductive results that will be used in the following chapters. We start the chapter by showing that $\Gamma(R)$ is a connected graph and the diameter does not exceed 3, see Theorem 3.1. Moreover, in Theorem 3.2 we provide a necessary and sufficient condition for R to ensure that $\Gamma(R)$ is a finite graph. Both properties mentioned above will be of fundamental importance in our thesis. Next, in Theorem 3.7, Theorem 3.8, Theorem 3.9 and Theorem 3.10, thanks to [3], [11] and [12], we answer one of the most important questions in our thesis: If R and S are two commutative rings with $\Gamma(R) \cong \Gamma(S)$, then do we have $R \cong S$? Finally, in the second part of this chapter, in Subsection 3.1, we give the definition and proofs of some preliminary properties of the zero-divisor graph of idealization $\Gamma(R(+)M)$ in relation to $\Gamma(R)$, for some commutative ring R and an R -module, M . We start the subsection by describing the set of vertices of $\Gamma(R(+)M)$ in Proposition 3.1. In the next part of it, we focus on the cases when $M \cong \mathbf{Z}_3$ and $M \cong \mathbf{Z}_2$. In this context, the remaining part of this subsection is dedicated to some results related to the girth number of $R(+)M$ for both of the cases of M mentioned above. It is important to point out that all the results in this subsection are from [7].

In Chapter 4, we investigate ideal based zero-divisor graphs. This chapter is dedicated to the work done in [9] and [13]. We partitioned this chapter into two parts. The first part is devoted to the relationship of $\Gamma_I(R)$ with $\Gamma(R/I)$. Among other things we show that $\Gamma_I(R)$ contains $|I|$ disjoint subgraphs isomorphic to $\Gamma(R/I)$, see Corollary 4.1. We continue this part by showing that for any two commutative rings R and S , and radical ideals I and J of them, respectively, if $|I| = |J|$ and $\Gamma(R/I) \cong \Gamma(S/J)$, then $\Gamma_I(R) \cong \Gamma_J(S)$, see Theorem 4.4. Finally, we end the first part of this chapter by examining the $\Gamma_I(R)$ in the case when I is the intersection of two prime ideals. The second part of this chapter is the Subsection 4.1. Here we look into the girth and clique number of $\Gamma_I(R)$ in the case when I is the intersection of prime ideals and when it is a primary ideal.

Finally, in Chapter 5, we study the weakly zero-divisor graphs of a commutative ring R . Here we give detailed explanation of the work done by authors in [10] and prove two new results. To do this, we split this chapter into three parts. The first part is devoted to the investigation of properties of $W\Gamma(R)$ and the relation with $\Gamma(R)$. One of the most important relations between them is the fact that $\Gamma(R)$ is a connected subgraph of $W\Gamma(R)$, see Lemma 5.1. This will be essential in the proof of our main results given in the last part of this chapter. Moreover, it appears that the graph $W\Gamma(R)$ is also a connected graph with diameter at most two, and if in addition, it contains a cycle, then the girth number is at most four, see Theorem 5.1. In Theorem 5.4 and Theorem 5.5, we give a full description of the case when $W\Gamma(R)$ is a star graph and a complete graph, respectively. The next part of this chapter is the Subsection 5.1. Here we explore the cases when $\Gamma(R)$ and $W\Gamma(R)$ are identical, where it shows up that the ring R being a reduced or a non-reduced ring is essential. In the case when R is a non-reduced ring, we give a full description of $\Gamma(R)$ being identical to $W\Gamma(R)$, see Theorem 5.7 and Corollary 5.1. The last part of this chapter is dedicated to our new results. In Lemma 5.2, we categorize all rings R such that $W\Gamma(R)$ is a path of length at most two. In our last result, we show that there is no commutative ring R such that $W\Gamma(R)$ is the graph in Figure 5.4, see Lemma 5.3.

2. PRELIMINARIES

2.1. Graph Theory Preliminaries

First, we review some fundamental definitions and concepts related to graph theory. Unless specified otherwise, all graphs discussed in this work will be finite undirected graphs.

Let G be a finite undirected graph. The set of vertices of G is denoted by $V(G)$, and the set of edges is denoted by $E(G)$. If $V(G) = \emptyset$, then we say that G is an empty graph. Two vertices in G are considered adjacent if an edge connects them. If for some $x \in V(G)$ we have $e = \{x, y\} \in E(G)$ for some $y \in V(G)$, then we say that the edge e is incident with the vertex x or x is an end vertex of e . The degree of a vertex $x \in V(G)$, denoted by $\delta(x)$, is defined to be the number of edges incident with x . We say that $e, f \in E(G)$ are incident if they share an end vertex.

Definition: Let G be a graph on n vertices. If there exists a vertex x with $\delta(x) = n - 1$ and $\delta(y) = 1$ for all $y \in V(G) \setminus \{x\}$, then G is called a star graph.

Let G and H be two graphs. We say that G is isomorphic to H if there exists a bijection $\phi : G \rightarrow H$ such that two vertices x and y are adjacent in G if and only if $\phi(x)$ and $\phi(y)$ are adjacent in H .

A graph permutation is a reordering of vertices of a graph G that maps the graph onto itself, preserving its structure.

A subgraph H of G is a graph where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An induced subgraph H of G is a subgraph such that for any pair of vertices $x, y \in V(H)$, we have $\{x, y\} \in E(H)$ if and only if $\{x, y\} \in E(G)$, and we write $(V(H)) = H$. We say that two subgraphs H_1 and H_2 of a graph G are disjoint if $V(H_1) \cap V(H_2) = \emptyset$ and $E(H_1) \cap E(H_2) = \emptyset$.

If for any pair of vertices x and y of a graph G there exists $\{x, y\} \in E(G)$, then G is called a complete graph. If in this case $|V(G)| = n$, then we denote G by K_n . A complete subgraph within G is called a clique. The clique number denoted by $\omega(G)$ is defined to be the size of the largest clique within G .

A sequence of vertices $x_1, \dots, x_n$ in a graph G with $x_i \neq x_j$ for $i \neq j$ such that $\{x_i, x_{i+1}\} \in E(G)$ for all $i = 1, \dots, n-1$, is called a path of length $n-1$. The distance between two vertices x and y is the number of edges in the shortest path starting at x and ending at y , and it is denoted by $d(x, y)$. The integer $\text{diam}(G) = \max\{d(x, y) \mid x, y \in V(G)\}$ is called the diameter of the graph G .

Two vertices x and y of a graph G are connected if there exists a path in G starting at x and ending at y . If any pair of vertices of G are connected, then G is said to be a connected graph, otherwise it is called disconnected.

A cycle in a graph G is a sequence of vertices $x_1, \dots, x_m$ such that $\{x_i, x_{i+1}\} \in E(G)$ for all $i = 1, \dots, m-1$ and $\{x_1, x_m\} \in E(G)$. The length of the shortest cycle within a graph G is defined as the girth number of the graph G , denoted by $\text{girth}(G)$.

A graph G is said to be bipartite if we have $V(G) = A \cup B$ such that $A \cap B = \emptyset$ and $\{x, y\} \in E(G)$ only if $x \in A$ and $y \in B$. If $\{x, y\} \in E(G)$ for all $x \in A$ and $y \in B$, then the graph G is called a complete bipartite graph. Moreover, if G is a complete bipartite graph with $V(G) = A \cup B$ where $|A| = n$ and $|B| = m$, then we denote it by $K_{n,m}$.

Let G be a graph. A set of edges $M = \{e_1, \dots, e_m\}$ of G is called a matching of G if $e_i \cap e_j = \emptyset$, for all $i, j \in \{1, \dots, m\}$ and $i \neq j$. Moreover, if every vertex of G is incident with exactly one edge in M , then M is called a perfect matching.

A graph G is called regular if each vertex of G has the same degree. We say that G is r -regular if each vertex of G has degree r . Note that, a 1-regular graph is a graph consisting of only disconnected edges.

2.2. Algebraic Preliminaries

In this section we recall some algebraic preliminaries from ring, modulo and field theory. All definitions and theorems in this section are from [14].

Definition: A set R equipped with two binary operations, called addition, denoted by $+$, and multiplication, denoted by $\cdot$, is called a ring if it satisfies the following properties:

(1) The set R form a group under addition $+$, that is:

(i) For all $a, b \in R$, we have $a + b = b + a$.

- (ii) For all $a, b, c \in R$, we have $a + (b + c) = (a + b) + c$.
- (iii) In R , there is the zero element, written as 0 , satisfying $a + 0 = 0 + a = a$, for all $a \in R$.
- (iv) For all $a \in R$, there is the inverse of a in R , written as $a^{-1} = -a$, satisfying $a + (-a) = (-a) + a = 0$.
- (2) The set R is transitive under multiplication: For all $a, b, c \in R$, we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- (3) Multiplication is distributive with respect to addition in R :
 - (i) For all $a, b, c \in R$, we have $a \cdot (b + c) = a \cdot b + a \cdot c$.
 - (ii) For all $a, b, c \in R$, we have $(a + b) \cdot c = a \cdot c + b \cdot c$.

If in addition there is an element $1_R \in R$ such that $a \cdot 1_R = 1_R \cdot a = a$, for all $a \in R$, then 1_R is called a unit of R and R is called a ring with unity. The set of all units in R forms an abelian group under multiplication. Note that, in the following part the multiplication of two elements a and b in R is written ab , instead of $a \cdot b$.

All rings in this thesis are commutative, that is, for any $a, b \in R$ we have $ab = ba$.

If R is a ring, then we define $R^* = R \setminus \{0\}$. An element $a \in R$ is called a zero-divisor if there exists a non-zero $b \in R$ such that $ab = 0$. The set of all zero-divisors is denoted by $Z(R)$ and we define $Z(R)^* = Z(R) \setminus \{0\}$. An element that is not a zero-divisor is called a regular element. The set of all regular elements is denoted by $reg(R)$. A non-zero ring with unity such that $Z(R) = \{0\}$ is called an integral domain.

Let R be a ring and $a \in R$. We say that an element $b \in R$ annihilates the element a if $ba = 0$. The set of all annihilators of a is denoted by $ann(a)$. It is clear that $ann(x)$ is a subset of $Z(R)$, for all $x \in R$.

Let R be a ring. An element $a \in R$ is called a nilpotent element if there exists a non-zero $n \in \mathbb{N}$ such that $a^n = 0$. A ring which does not contain a non-zero nilpotent element is called a reduced ring.

Remark 2.1: Let $a \in R$ be a nilpotent element. Then, $a^n = 0$ for some $n > 0$. So, $aa^{n-1} = 0$, implying that $a \in Z(R)$. Hence, every nilpotent element is a zero-divisor, but the converse is not true in general.

Definition: Let R and S be two rings with a multiplicative unity 1_R and 1_S , respectively.

A map $f : R \rightarrow S$ is called a homomorphism if the followings hold:

- (i) $f(x + y) = f(x) + f(y)$, for all $x, y \in R$,
- (ii) $f(xy) = f(x)f(y)$, for all $x, y \in R$, and
- (iii) $f(1_R) = 1_S$.

Moreover, if f is an injective and surjective homomorphism, then it is called an isomorphism. In such a case, the ring R is said to be isomorphic to the ring S , and we write $R \cong S$.

The image of an element $a \in R$ under the homomorphism f is the element $f(a) \in S$. The image of f is the set $\text{Img}(f) = \{f(a) \in S \mid a \in R\}$. The Kernel of f is the set of all elements of R which has image zero under f , and we denote it by $\text{Ker}(f)$.

Definition: Let R be a ring and $I \subseteq R$. Then, I is called an ideal of R if it satisfies the following:

- (i) If $a, b \in I$, then $a - b \in I$,
- (ii) If $a \in I$ and $r \in R$, then $ar \in I$.

An ideal I is said to be finitely generated if there exists a finite set $A = \{a_1, \dots, a_n\}$, for $n \in \mathbb{N}$, of R , such that

$$I = (A) = \{r_1a_1 + \dots + r_na_n \mid r_i \in R, a_i \in A \text{ for all } i = 1, \dots, n\}.$$

In this case the set A is called the generating set of the ideal I , or we say that I is generated by $\{a_1, \dots, a_n\}$ and we write $I = (a_1, \dots, a_n)$.

An ideal I of a ring R is called a principle ideal if it is generated by a single element $x \in R$, that is, if $I = (x) = \{rx \mid \text{for all } r \in R\}$. A ring whose all ideals are principle is called a principle ideal domain.

Remark 2.2: [14] The intersection of a family of ideals is an ideal.

Let R be a ring and I and J be two ideals. Then, the addition of I and J is defined as follows:

$$I + J = \{x + y \mid x \in I, y \in J\}.$$

Note that, the addition $I + J$ is the smallest ideal of R containing I and J . In a similar way we can define the addition of a family of ideals of R .

Definition: Let R be a ring and I be an ideal of R . The n th power of I is defined to be:

$$I^n = \left\{ \sum_i a_{i_1} \cdots a_{i_n} \mid a_{i_1}, \dots, a_{i_n} \in I \right\},$$

where $\sum_j$ denotes a finite sum.

Definition: Let R be a ring and I be an ideal of R . Then, the ring $R/I = \{r+I \mid r \in R\}$ together with operations:

$$(i) \quad (r + I) + (s + I) = (r + s) + I$$

$$(ii) \quad (r + I)(s + I) = rs + I,$$

for all $r, s \in R$, is called the quotient ring. The elements of R/I are the cosets of I in R .

Among the fundamental theorems in algebra are the isomorphism theorems. Here we will state the first isomorphism theorem of rings.

Theorem 2.1: [14] Let R and S be two rings, and $f : R \rightarrow S$ be a homomorphism of rings. Then, $R/\text{Ker}(f) \cong \text{Img}(f)$.

Example 2.1: Let $R = \mathbb{Z}_{24}$ and $I = (\bar{8})$ be an ideal of R . Then, by Theorem 2.1 we have $R/I \cong \mathbb{Z}_8$.

Let p be an ideal of the ring R . If for any $ab \in p$ it implies that $a \in p$ or $b \in p$, then p is called a prime ideal. An ideal m is called a maximal ideal if there is no ideal I such that $m \subsetneq I \subsetneq R$. Note that, a maximal ideal is always a prime ideal, but the converse has not to be true. The intersection of any two prime ideals need not be prime. For example, we have $(2) \cap (3) = (6)$ in $\mathbb{Z}$, where (2) and (3) are prime ideals, while (6) is not. Also, we call a ring R decomposable if we can write $R = R_1 \times R_2$, where R_1 and R_2 are two rings. The following two results are well known in ring theory, see [14].

Lemma 2.1: If there exists an idempotent element ($a^2 = a$) in R , then the ring R is decomposable.

Lemma 2.2: If R is a decomposable ring, then $Z(R)$ is not an ideal of R .

Definition: A ring R is said to be a semi-local ring if it has a finite number of maximal ideals and it is called a local ring if it has a unique maximal ideal.

Example 2.2: The ring $\mathbb{Z}_m$ is a semi-local ring for any positive integer m . In particular, if m is a prime power, then R is a local ring.

Proposition 2.1: [14, Proposition 1.7] Let R be a ring. The set of all nilpotent elements

of R is an ideal. Moreover, if I is the set of nilpotent elements, then there is no non-zero nilpotent element in R/I .

Let R be a ring and $\mathcal{N}$ be the set of all nilpotent elements. By Proposition 2.1 the set $\mathcal{N}$ is an ideal and it is called the nilradical of the ring R . Moreover, $\mathcal{N}$ is the intersection of all prime ideals of the ring R , as stated in the next theorem.

Proposition 2.2: [14, Proposition 1.8] The nilradical of a ring R is the intersection of all the prime ideals of R .

Remark 2.3: Let I and J be two ideals of a ring R . Then, the ideal

$$(I : J) = \{x \in R \mid xJ \subseteq I\}$$

is called the colon ideal of I and J . In particular, the colon ideal $(0 : I) = \text{ann}(I)$ is called the annihilator of I .

If I is an ideal of a ring R , then the radical of I is defined as

$$\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\}.$$

Moreover, if $I = \sqrt{I}$, then the ideal I is called a radical ideal.

Let $x, y \in \sqrt{I}$. Then, $x^n \in I$ and $y^m \in I$ for some $n, m \in \mathbb{N}$. Then, by binomial expansion $(x - y)^{n+m} \in I$, which implies that $x - y \in \sqrt{I}$. On the other hand, as I is an ideal, it follows that $r^n x^n, r^m y^m \in I$ for all $r \in R$, implying that $rx, ry \in \sqrt{I}$ for all $r \in R$. Hence, $\sqrt{I}$ is ideal and we call it the radical ideal of I .

Example 2.3: Let $R = \mathbb{Z}_{24}$. Then, the ideal $I = (\bar{8})$ is not a radical ideal, because as $\bar{4}^2 = \bar{16} \in I$, it follows that $\bar{4} \in \sqrt{I}$ and obviously $\bar{4} \notin I$. On the other hand, the ideal $J = (\bar{2})$ is a radical ideal of $\mathbb{Z}_{24}$, because $J = \sqrt{J} = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{10}, \bar{12}, \bar{14}, \bar{16}, \bar{18}, \bar{20}, \bar{22}\}$.

Definition: Let I be an ideal of a ring R . Then, we call I a primary ideal if $I \neq R$ and $xy \in I$ implies that either $x \in I$ or $y^n \in I$ for some $n \in \mathbb{N}$.

Example 2.4: All primary ideals of the ring of integers $\mathbb{Z}$ are; the zero ideals (0) and (p^i) , where p is a prime number. For instance, the ideal $I = (12)$ is not a primary ideal because $3 \cdot 4 = 12 \in (12)$, but $4 \notin (12)$ and $3 \notin \sqrt{(12)} = (6)$, because there does not exist n such that $3^n \in (12)$.

Let $(R_\lambda)_{\lambda \in \Lambda}$ be commutative rings, where Λ is the set of indices. The direct product $\prod_{\lambda \in \Lambda} R_\lambda$ is the set $\{(a_1, a_2, a_3, \dots) \mid a_\mu \in R_\mu, \mu \in \Lambda\}$, which forms a commutative

ring under pointwise addition and multiplication. The direct sum $\bigoplus_{\lambda \in \Lambda} R_\lambda$ is the set $\{(a_1, a_2, a_3, \dots) \mid \text{such that only finitely many } a_i \text{'s are non-zero}\}$, which forms a commutative ring under pointwise addition and multiplication. Note that in the case when Λ is a finite set, then $\prod_{i=1}^n R_i = \bigoplus_{i=1}^n R_i$. In our thesis, we use only the case when Λ is a finite set. In this case, if each of $R_1, \dots, R_n$ is a commutative rings with 1_{R_i} , for all $i = 1, \dots, n$, then $\prod_{i=1}^n R_i = \bigoplus_{i=1}^n R_i$ is a commutative ring with $1_R = (1_{R_1}, \dots, 1_{R_n})$.

Definition: Let R be a ring with 1_R . A non-empty set M is called a module over the ring R if it is an abelian group under addition together with the following map

$$R \times M \rightarrow M, (r, m) \rightarrow rm$$

satisfying the following axioms:

- (i) $1_R \cdot m = m$, for all $m \in M$,
- (ii) $r_1(r_2m) = (r_1r_2)m$, for all $r_1, r_2 \in R$ and $m \in M$,
- (iii) $r(m_1 + m_2) = rm_1 + rm_2$, for all $r \in R$ and $m_1, m_2 \in M$.
- (iv) $(r_1 + r_2)m = r_1m + r_2m$, for all $r_1, r_2 \in R$ and $m \in M$.

If M is a module over R we say that M is an R -module.

Example 2.5: (i) Let R be a ring. Then, any ideal I of R is an R -module, and in particular, R itself is an R -module.

(ii) If $R = \mathbb{Z}$, then every abelian group is an R -module.

(iii) If the positive integer m divides the positive integer n , then $\mathbb{Z}_m$ is a $\mathbb{Z}_n$ -module.

Let R be a ring. An ascending and descending chain of ideals of R is said to be the chain

$$\mathcal{C}_1 : I_1 \subseteq I_2 \subseteq \dots \subseteq I_n \dots$$

$$\mathcal{C}_2 : I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \supseteq \dots,$$

respectively, where I_i is an ideal of R , for all $i = 1, 2, 3, \dots$. We say that $\mathcal{C}_1$ (equivalently $\mathcal{C}_2$) is stationary if there exists $m \in \mathbb{N}$ such that $I_m = I_{m+1} = \dots$.

Definition: A ring R is called a Noetherian Ring if one the following equivalent conditions are satisfied:

- (i) Every non-empty set of ideals in R has a maximal element.

(ii) Every ascending chain of ideals in R is stationary.

(iii) Every ideal in R is finitely generated.

Definition: A ring R is said to be an Artinian ring if every descending chain of ideals in R is stationary. If in addition, the ring R is a local ring, then we call it an Artinian local ring.

In [14, Theorem 8.5] it is shown that any Artinian ring is Noetherian.

Example 2.6: The ring of integers $\mathbb{Z}$ is Noetherian, but not Artinian. Indeed, the descending chain of ideals $(2) \supset (4) \supset (8) \supset \cdots \supset (2^n) \supset (2^{n+1}) \supset \cdots$ in $\mathbb{Z}$ is not stationary. An example of a non-Noetherian ring is a polynomial ring on infinite variables over a field K , denoted by $K[x_1, x_2, x_3, \dots]$. Indeed, the ascending chain of ideals $(x_1) \subset (x_1, x_2) \subset (x_1, x_2, x_3) \subset \cdots \subset (x_1, \dots, x_n) \subset (x_1, \dots, x_{n+1}) \subset \cdots$ of $K[x_1, x_2, x_3, \dots]$ is not stationary.

The following theorem which is called the structure theorem for Artinian rings, plays an important role in our discussion.

Theorem 2.2: [14, Theorem 8.7] An Artinian ring A is uniquely (up to isomorphism) determined by a direct product of a finite number of Artinian local rings.

Proposition 2.3: [14, Proposition 8.1] If R is an Artinian ring, then every prime ideal is maximal.

Proposition 2.4: [14, Proposition 8.3] If R is an Artinian ring, then it has a finite number of maximal ideals.

Let R be a ring and M be an R -module. Then, a prime ideal P of the ring R is said to be associated to M if there is an element $m \in M$ such that $Pm = 0$, or equivalently, if P annihilates m . The set of all such prime ideals of R is denoted by $Ass(M)$. The minimal elements, under inclusion, of the set $Ass(M)$ are called minimal associated primes of M , and the set of all such primes is denoted by $Min(M)$.

A prime ideal P is called a minimal prime ideal over an ideal I if P is the smallest prime ideal containing I (i.e., there are no other prime ideals strictly between I and P). Note that if I is a prime ideal, then I itself is the only minimal prime over it. A prime ideal is referred to as a minimal prime ideal if it is a minimal prime over the zero ideal, and the set of all such prime ideals is denoted by $Min(R)$.

In a Noetherian ring R , a minimal prime ideal over an ideal I is exactly a minimal

associated prime of R/I , where R/I is considered as an R -module.

Recall that a reduced ring is a ring with no non-zero nilpotent elements. Such a ring has the following property.

Theorem 2.3: [15, Corollary 2.4] In a reduced ring R , $Z(R) = \cup P$, where P varies over $\text{Min}(R)$.

Definition: A set $\mathbb{F}$ is called a field if it is a commutative ring with unity such that each element of $\mathbb{F} \setminus \{0\}$ has a multiplicative inverse. If $\mathbb{F}$ is a finite set then we call it a finite field. The characteristic of a field $\mathbb{F}$ is the smallest natural number n such that $n \cdot 1 = 0$, and we write $\text{char}(\mathbb{F}) = n$. If such n does not exist, then the characteristic of $\mathbb{F}$ is said to be zero.

In Field Theory, it is well known that for any prime number p and any positive integer n , there exist fields of order p^n . Moreover, any finite field has order p^n , for some positive prime integer p and positive integer n . So, with $\mathbb{F}_m$ we refer to the finite field of order $m = p^n$, for some positive prime integer p and positive integer n . Note that $\text{char}(\mathbb{F}_{p^n}) = p$.

Example 2.7: Examples of fields with characteristic zero are: $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

Definition: Let $\mathbb{F}_{m_1}, \dots, \mathbb{F}_{m_n}$ be finite fields. Then, the direct product finite field

$$\mathbb{F} = \prod_{i=1}^n \mathbb{F}_{m_i}$$

is defined to be the field consisting of sequences $a = (a_1, \dots, a_n)$ with $a_i \in \mathbb{F}_{m_i}$, and equipped with componentwise addition and multiplication.

Proposition 2.5: [16] The following statements are equivalent:

- (i) R is a finite reduced ring;
- (ii) R is a direct product of fields.

Lemma 2.3: [7, Lemma 2.3] Let $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field. Then, any module operation from R to $\mathbb{Z}_2$ is a canonical extension of a module operation either from $\mathbb{Z}_2$ to $\mathbb{Z}_2$, or from F to $\mathbb{Z}_2$ in the case where F is $\mathbb{Z}_2$.

3. ZERO-DIVISOR GRAPH

Let R be a commutative ring with $1_R \neq 0$. The graph $\Gamma(R)$ with the vertex set $Z(R)^*$ and the edge set $\{\{x, y\} \mid x, y \in Z(R)^* \text{ such that } xy = 0\}$ is called the *zero-divisor graph* of the ring R . We first show in the Theorem 3.1 that $\Gamma(R)$ is always a connected graph.

Theorem 3.1: [2, Theorem 2.3] If R is a commutative ring with 1_R , then $\Gamma(R)$ is connected and $\text{diam}(\Gamma(R)) \leq 3$.

Proof. We know that

$$V(\Gamma(R)) = Z(R)^*.$$

Let $x, y \in V(\Gamma(R))$ such that $x \neq y$. Consider the following cases:

1. If $xy = 0$, then $d(x, y) = 1$.
2. If $x^2 = y^2 = 0$, then $x - xy - y$ is a path of length two.
3. Suppose $x^2 = 0$ and $y^2 \neq 0$. As $y \in Z(R)^*$ and $y^2 \neq 0$ there is an element $b \in Z(R)^*$ with $b \neq y$ such that $by = 0$. If $bx = 0$ then $x - b - y$ is a path and $d(x, y) = 2$.
4. If $y^2 = 0$ and $x^2 \neq 0$ then we have a similar case with case 3.

Thus we may suppose that neither x^2 nor y^2 is zero. Then, there is a non-zero zero-divisor $a, b \in Z(R)^*$ such that $ax = by = 0$. We have the following cases:

1. If $a = b$, then $x - a - y$ is a path of length 2. So from here on we may assume that $a \neq b$.
2. If $ab = 0$, then $x - a - b - y$ is a path of length three and hence $d(x, y) = 3$.
3. If $ab \neq 0$, then $x - ab - y$ is a path of length two.

In all of the above cases there is a path between x and y of length less than or equal to three. Since x and y are arbitrary, it follows that the diameter of $\Gamma(R)$ is less than or equal to three. $\square$

As the ring R may be an infinite ring, then it is obvious that the graph $\Gamma(R)$ may also be an infinite graph. Next, we give a necessary and sufficient condition on R such that $\Gamma(R)$ is a finite graph.

Theorem 3.2: [2, Theorem 2.2] Let R be a commutative ring. Then $\Gamma(R)$ is finite if and only if either R is finite or an integral domain. In particular if $1 \leq |\Gamma(R)| < \infty$, then R is finite and not a field.

Proof. Suppose that $\Gamma(R)$ is finite and non-empty. Then, there exists $0 \neq x, y \in R$ with $xy = 0$. Let $I = \text{ann}(x)$. Then $I \subseteq Z(R) = V(\Gamma(R) \cup \{0\})$ is finite. Moreover, as $yx = 0$, it follows that $y \in I$ and $ry \in I$ for all $r \in R$. Suppose R is infinite. For $i \in I$ define $J_i = \{r \in R : ry = i\}$. Suppose that J_i is finite for each $i \in I$. Then, as R is infinite there is $r' \in R$ such that $r'y \neq i$ for each $i \in I$. Therefore, $r'y \notin I$, which is a contradiction. Hence, there exists an $i \in I$ such that J_i is infinite. As for any $r, s \in J_i$, $(r - s)y = 0$, we conclude that $\text{ann}(y)$ is an infinite subset of $Z(R)$, which is a contradiction. Thus R must be finite.

On the other hand, suppose that R is a finite ring. Then, by the definition of $\Gamma(R)$ we conclude that the graph $\Gamma(R)$ is a graph with finite number of vertices, which implies that it is a finite graph. In the case when R is an integral domain, it is obvious that $V(\Gamma(R))$ is an empty set, implying that $\Gamma(R)$ is a finite graph.

Moreover, if $1 \leq |\Gamma(R)| < \infty$, then by the previous part of the proof we conclude that R is a finite ring. Also, as in this case $|\Gamma(R)| \neq \emptyset$ we obviously have at least one non-zero zero-divisor. Hence, R is not an integral domain, implying that it is not a field. $\square$

In the next theorem, in the case when $\Gamma(R)$ contains a cycle, we give an upper bound for the girth number of $\Gamma(R)$.

Theorem 3.3: [6, Page 3541] If $\Gamma(R)$ contains a cycle, then $\text{girth}(\Gamma(R)) \leq 4$.

One of the fundamental questions in the study of zero-divisor graphs is as follows:

If A and B are two commutative rings where $\Gamma(A) \cong \Gamma(B)$, then do we have $A \cong B$?

To answer this question we continue with the following theorem.

Theorem 3.4: [3, Theorem 3.7] Let $R = R_1 \times \cdots \times R_n$, where $R_1, \dots, R_n$ are some integral domains and $(n \geq 2)$. Then, $w(\Gamma(R)) = n$.

Proof. Let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ where 1 is the i^{th} component of e_i . Then, it is clear that each e_i is connected with each e_j for $i \neq j$. So, there exists a clique with length n , implying that $\omega(\Gamma(R)) \leq n$. To prove equality we use induction on n . Let $\omega(\Gamma(R)) = m$, and firstly consider the case where $n = 2$ which means that $R = R_1 \times R_2$,

where R_1, R_2 are integral domains. If $m = \omega(\Gamma(R)) > 2 = n$, then there exists a clique of size $l \geq 3$, which means that there exists $a_1, a_2, \dots, a_l \in R$ such that $a_i \cdot a_j = 0$ whenever $i \neq j$. For each $i \in \{1, \dots, l\}$, let $a_i = (x_{i1}, x_{i2})$. Consider a_1, a_2 and a_3 . In each one, at least one coordinate is non-zero. Without loss of generality we assume that $a_{11} \neq 0$. Then, $a_{21} = a_{31} = 0$. In this case $a_{22} \neq 0$ and $a_{32} \neq 0$, but as $a_2 a_3 = 0$ it follows that $a_{22} a_{32} = 0$. Hence, we have a contradiction with the fact that both R_1 and R_2 are integral domains.

Now, let $X = \{x_1, \dots, x_m\}$ be a complete subgraph of $\Gamma(R)$ with $m \geq 3$ and each $x_i = (x_{i1}, \dots, x_{in})$. We may assume that $x_{11} \neq 0$, thus $x_{21} = \dots = x_{m1} = 0$. Therefore, we may consider $X \setminus x_1 = \{x_2, \dots, x_m\}$ as a complete subgraph of $\Gamma(R_1 \times \dots \times R_n)$. By induction, we have $m - 1 = n - 1$. Hence, $m = n$. $\square$

It should be noted that there exist some distinct (non-isomorphic) commutative rings A and B such that $\Gamma(A) \cong \Gamma(B)$, see Figure 4.2. However, our next theorem shows that for finite reduced rings that are not fields, the ring R is determined by its zero-divisor graph $\Gamma(R)$.

Theorem 3.5: [2, Theorem 2.5] Let R be a commutative ring. Then, there is a vertex of $\Gamma(R)$ which is adjacent to every vertex if and only if either $R \cong \mathbb{Z}_2 \times A$, where A is an integral domain or $Z(R)$ is an annihilator ideal.

Next, we give a proficiency result for $\Gamma(R)$ to be a complete graph, when R is a commutative ring.

Theorem 3.6: [2, Theorem 2.8] Let R be a commutative ring. Then, $\Gamma(R)$ is complete if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$.

Remark 3.1: [3] Given a commutative ring R , it is clear that, for a vertex x of the zero-divisor graph $\Gamma(R)$, we have

$$\delta(x) = |\text{ann}(x) \setminus \{0, x\}|.$$

In special case, let $R = \prod_{i=1}^n K_i$ where $K_1, \dots, K_n$ are finite fields. If $x = (a_1, \dots, a_n)$, $y = (b_1, \dots, b_n) \in R$, then $xy = (a_1, \dots, a_n)(b_1, \dots, b_n) = (a_1 b_1, \dots, a_n b_n) = (0, \dots, 0)$ implies that $a_i b_i = 0$, for all $i = 1, \dots, n$, and as K_i is a field for each $i = 1, \dots, n$, it follows that either $a_i = 0$ or $b_i = 0$. So, if $x = (a_1, \dots, a_n) \in Z(R)^*$, then at least one of a_i 's is zero, for some $i \in \{1, \dots, n\}$. Moreover, if $y = (b_1, \dots, b_n) \in \text{ann}(x) \subseteq Z(R)^*$, then $b_i = 0$ for all $i \in \{1, \dots, n\}$ such that $a_i \neq 0$. Thus, if we set

$\Lambda \subset \{1, \dots, n\}$ to be the set of all indices such that $a_i = 0$, then $|ann(x)| = |\prod_{i \in \Lambda} K_i|$. Hence, the degree of x in $\Gamma(R)$ is $\delta(x) = |\prod_{i \in \Lambda} K_i| - 1$.

Theorem 3.7: [3, Theorem 4.1] Let A and B be finite reduced commutative rings which are not fields. Then, $\Gamma(A)$ and $\Gamma(B)$ are isomorphic graphs if and only if A and B are isomorphic rings.

Proof. Suppose that $\Gamma(A) \cong \Gamma(B)$. So there exists a graph isomorphism

$$\phi : \Gamma(A) \longrightarrow \Gamma(B).$$

We claim that $A \cong B$. As both A and B are reduced rings, Proposition 2.5 implies that A and B can be written as follows:

$$A = F_1 \times \dots \times F_n, \quad B = K_1 \times \dots \times K_m$$

where F_i and K_j are finite fields for each $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, respectively. Moreover,

$$|F_1| \leq \dots \leq |F_n|, \quad |K_1| \leq \dots \leq |K_m|$$

As A and B are not fields we have $n, m \geq 2$. Also, from Theorem 3.4 we have $w(\Gamma(A)) = n$ and $w(\Gamma(B)) = m$. So, as $\Gamma(A) \cong \Gamma(B)$ we conclude that $n = m$.

By construction, it is clear that the minimum degree of a vertex in $\Gamma(A)$ and $\Gamma(B)$ are $|F_1| - 1$ and $|K_1| - 1$, respectively. Thus, $|F_1| = |K_1|$ and then $F_1 \cong K_1$. On the other hand, the maximum degree of a vertex of $\Gamma(A)$ is $|F_2| \cdots |F_n| - 1$ and for $\Gamma(B)$ is $|K_2| \cdots |K_n| - 1$. Thus, $|F_2| \cdots |F_n| = |K_2| \cdots |K_n|$, and then $|F_1| \cdots |F_n| = |K_1| \cdots |K_n|$. If $n = 2$, then $|F_2| = |K_2|$ and so $F_2 \cong K_2$. Thus $A \cong B$. Hence, we may assume that $n \geq 3$. We progress by induction on n .

As, $\Gamma(A) \cong \Gamma(B)$ then $|V(\Gamma(A))| = |V(\Gamma(B))|$. Since $|F_1| \cdots |F_n| = |K_1| \cdots |K_n|$, we have $(|F_1| - 1) \cdots (|F_n| - 1) = (|K_1| - 1) \cdots (|K_n| - 1)$. If $A' = F_2 \times \dots \times F_n$ and $B' = K_2 \times \dots \times K_n$, then since $|F_1| = |K_1|$, we have $(|F_2| - 1) \cdots (|F_n| - 1) = (|K_2| - 1) \cdots (|K_n| - 1)$. Thus, $|V(\Gamma(A'))| = |V(\Gamma(B'))|$.

If we show that $\Gamma(A') \cong \Gamma(B')$, then by induction on n and applying the inductive hypothesis on A' and B' , we conclude that $A' \cong B'$. As $A = F_1 \times A'$ and $B = K_1 \times B'$, we conclude that $A \cong B$. Hence it remains to prove that $\Gamma(A') \cong \Gamma(B')$.

By degree argument we may assume that $\phi(1, 0, \dots, 0) = (a, 0, \dots, 0)$ for some $0 \neq a \in K_1$, and for $0 \neq b \in F_1$ we have $\phi(b, 0, \dots, 0) = (c, 0, \dots, 0)$ for some $0 \neq c \in K_1$.

Given an arbitrary vertex of $\Gamma(A')$, say $(a_2, \dots, a_n)$, we observe that $(0, a_2, \dots, a_n)$ is a vertex of $\Gamma(A)$. Let $(b_1, b_2, \dots, b_n) = \phi((0, a_2, \dots, a_n))$. Now degree argument implies that $b_1 = 0$. Hence, we can define σ as follows:

$$\sigma : \Gamma(A') \longrightarrow \Gamma(B'), \quad \sigma((a_2, \dots, a_n)) = (b_2, \dots, b_n).$$

σ is Well-defined: Let $(x_2, \dots, x_n) = (y_2, \dots, y_n)$ in $K_2 \times \dots \times K_n$. Then, $x_i = y_i$ for each $i \in \{2, 3, \dots, n\}$. Let $\sigma(x_2, \dots, x_n) = (z_2, \dots, z_n)$. Then by the above discussion we see that $\phi(0, y_2, \dots, y_n) = \phi(0, x_2, \dots, x_n) = (0, z_2, \dots, z_n)$. Hence $\sigma(y_2, \dots, y_n) = (z_2, \dots, z_n) = \sigma(x_2, \dots, x_n)$

σ is injective: Let $\sigma(x_2, \dots, x_n) = \sigma(y_2, \dots, y_n)$. Then, $\phi(0, x_2, \dots, x_n) = \phi(0, y_2, \dots, y_n)$. As ϕ is injective we have $(0, x_2, \dots, x_n) = (0, y_2, \dots, y_n)$, so $(x_2, \dots, x_n) = (y_2, \dots, y_n)$.

σ Preserves adjacency: As ϕ is a bijection and $x, y \in V(\Gamma(A'))$ are adjacent if and only if $\phi(x)$ and $\phi(y)$ are adjacent in $\Gamma(B)$. Now due to the structure of σ , we deduce that $\sigma(x)$ and $\sigma(y)$ are adjacent in B' .

It is clear that $(\text{Im } \sigma \subseteq \Gamma(B'))$.

Since $|\Gamma(A')| = |\Gamma(B')|$ which are finite, σ is a graph isomorphism from $\Gamma(A')$ to $\Gamma(B')$.

□

In order to understand Theorem 3.7 we give the following example, given in [3].

Example 3.1: a) The "finite" is needed in the theorem above: Let $A = \mathbb{Z}_2 \times \mathbb{Z}$ and $B = \mathbb{Z}_2 \times \mathbb{Q}$. Then, the infinite star graph $\Gamma(A)$ and $\Gamma(B)$ are isomorphic, since $|\mathbb{Z}| = |\mathbb{Q}|$ as sets. But, A is not isomorphic to B , because $\mathbb{Q}$ is a field while $\mathbb{Z}$ is not.

b) We may have $|V(\Gamma(A))| = |V(\Gamma(B))|$ and $w(\Gamma(A)) = w(\Gamma(B))$, but $A \not\cong B$:

Let $A = \mathbb{F}_{2^5} \times \mathbb{F}_{5^2}$ and $B = \mathbb{F}_{2^3} \times \mathbb{F}_{7^2}$. Then, $w(\Gamma(A)) = 2$ and $w(\Gamma(B)) = 2$.

Moreover, $|V(\Gamma(\mathbb{F}_{2^5} \times \mathbb{F}_{5^2}))| = |\mathbb{F}_{2^5}| |\mathbb{F}_{5^2}| - (|\mathbb{F}_{2^5}| - 1)(|\mathbb{F}_{5^2}| - 1) - 1 = 32 \cdot 25 - (32 - 1)(25 - 1) - 1 = 55$, and $|V(\Gamma(\mathbb{F}_{2^3} \times \mathbb{F}_{7^2}))| = |\mathbb{F}_{2^3}| |\mathbb{F}_{7^2}| - (|\mathbb{F}_{2^3}| - 1)(|\mathbb{F}_{7^2}| - 1) - 1 = 8 \cdot 49 - 7 \cdot 48 - 1 = 55$. However, A and B have different orders, and so they are not isomorphic as rings.

For a ring R , the graph $\Gamma(R)$ of course may be an infinite graph. But, in the case when R is finite or when R is an integral domain we always have $|\Gamma(R)| < \infty$. The converse of this is also true, as proved in the following theorem.

Theorem 3.8: [11, Theorem 4] If $R_1, \dots, R_n$ and $S_1, \dots, S_m$ are finite local rings, then the following hold:

- (i) For $n \geq 2$, $\Gamma(R_1 \times \dots \times R_n) \cong \Gamma(S_1)$ if and only if $n = 2$ and either $R_1 \times R_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $R_1 \times R_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$.
- (ii) For $n, m \geq 2$, $\Gamma(R_1 \times \dots \times R_n) \cong \Gamma(S_1 \times \dots \times S_m)$ if and only if $n = m$ and there exists a permutation σ over $\{1, \dots, n\}$ such that for any i , $1 \leq i \leq n$, $|R_i| = |S_{\sigma(i)}|$ and $\Gamma(R_i) \cong \Gamma(S_{\sigma(i)})$.

Theorem 3.9: [11, Theorem 5] Let R be a finite reduced ring and S be a ring such that S is not an integral domain. If $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$, unless $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_6$ and S is a local ring.

Proof. Since, R is finite, obviously $\Gamma(R)$ is finite. This implies that $\Gamma(S)$ is finite. As S is not an integral domain, Theorem 3.2 implies that S is a finite ring. As $\Gamma(R)$ is not empty, R is not a field. Since R and S are finite rings, they are both Artinian. So Theorem 2.2 implies that $R \cong F_1 \times \dots \times F_n$ and $S \cong S_1 \times \dots \times S_m$, for $n \geq 2$, and F_i is a finite field for each $i \in \{1, \dots, n\}$ and S_i is a finite local ring for each $i \in \{1, \dots, m\}$. If $m = 1$, Theorem 3.8 implies that $n = 2$ and $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. So, suppose that $n, m \geq 2$. Now by Theorem 3.8 we have $n = m$ and there exists a permutation σ over $\{1, \dots, m\}$ such that $\Gamma(S_i) \cong \Gamma(S_{\sigma(i)})$ and $|S_i| = |S_{\sigma(i)}|$. Since the F_i 's are finite fields, $S_i \cong F_{\sigma(i)}$ for any i , $1 \leq i \leq n$. Thus $R \cong S$. $\square$

Theorem 3.10: [12, Theorem 2.1] If $R = \prod_{i=1}^n F_i$ and $S = \prod_{j=1}^n G_j$, where F_i 's are finite fields and G_j 's are integral domains and $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$.

Now, we categorize commutative rings R with 1_R , such that $\text{diam}(\Gamma(R)) = 3$.

Theorem 3.11: [4, Theorem 2.6(4)] Let R be a commutative ring with 1_R . Then, $\text{diam}(\Gamma(R)) = 3$ if and only if there are distinct zero-divisors $a \neq b$ such that $(0 : (a, b)) = (0)$ and either of the following cases occurs:

- (i) R is reduced with more than two minimal primes, or
- (ii) R is non-reduced.

In the next result, we classify all zero-divisor graphs with girth number four, for any reduced ring R .

Theorem 3.12: [17, Theorem 2.2] The following statements are equivalent for any reduced commutative ring R :

1. $\text{girth}(\Gamma(R)) = 4$.
2. $\Gamma(R) = K_{m,n}$ with $m, n \geq 2$.

Finally, we give a categorization of all zero-divisor graphs $\Gamma(R)$ with girth number infinity, for any reduced ring R .

Theorem 3.13: [17, Theorem 2.4] The following statements are equivalent for any reduced commutative ring R :

1. $\Gamma(R)$ is non-empty with $\text{girth}(\Gamma(R)) = \infty$.
2. $\Gamma(R) = K_{1,n}$ for some $n \geq 1$.

3.1. Zero-Divisor Graphs of Idealizations

In this subsection, R stands for a commutative ring with identity 1_R . By D we will denote an integral domain and by M a unitary R or D module.

Definition: Let M be an R -module. Then, the idealization of M in R is a commutative ring with the following operations:

- (i) $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$, for $r_1, r_2 \in R$ and $m_1, m_2 \in M$;
- (ii) $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$, for $r_1, r_2 \in R$ and $m_1, m_2 \in M$;

which is denoted by $R(+M)$.

Through this section we assume that M and R are non-trivial. Let $a, b \in Z^*(R)$ with $ab = 0$. If $bM = 0$, then $(a, m)(b, 0) = (ab, a \cdot 0 + bm) = (0, 0)$. If $bM \neq 0$, then there is an $n \in M$ such that $bn \neq 0$. So, $(a, m)(0, bn) = (a \cdot 0, abn + m \cdot 0) = (0, 0)$. Hence, if $a \in Z^*(R)$, then $(a, m) \in Z^*(R(+M))$ for all $m \in M$.

Proposition 3.1: [7, Proposition 1.1] Let R be a commutative ring and let M be an R -module. Then, $Z(R(+M))^* = \{(0, m) \mid m \in M^*\} \cup \{(a, n) \mid a \in R^*, n \in M \text{ and for some } m \in M^*, am = 0\} \cup \{(a, n) \mid a \in Z^*(R), n \in M\}$.

Proof. If $m \in M^*$, then $(0, m) \in (R(+M))^*$. Also, $(0, m)(0, m) = (0, 0)$, hence $(0, m) \in Z(R(+M))^*$. Let $(a, n) \in (R(+M))^*$ such that for some $m \in M^*$ we have $am = 0$. Then, $(a, n)(0, m) = (a \cdot 0, am + 0 \cdot m) = (0, 0)$. Moreover, by above explanation we obviously have $\{(a, n) \mid a \in Z(R)^*, n \in M\} \subset Z(R(+M))^*$. Hence, in this case $(a, n) \in Z(R(+M))^*$. $\square$

Now, we will study the girth number of the graph $\Gamma(R(+)M)$. First, we look into the case when $M \cong \mathbb{Z}_3$.

Theorem 3.14: [7, Theorem 2.1] Let R be a commutative ring with identity and $M \cong \mathbb{Z}_3$ an R -module. Then:

- (i) $\text{girth}(\Gamma(R(+)\mathbb{Z}_3)) = 3$ if and only if $\text{ann}(\mathbb{Z}_3) \neq \{0\}$.
- (ii) $\text{girth}(\Gamma(R(+)\mathbb{Z}_3)) = \infty$ if and only if $\text{ann}(\mathbb{Z}_3) = \{0\}$. This occurs precisely when $R \cong \mathbb{Z}_3$.

Proof. Let R be a commutative ring with identity and $M \cong \mathbb{Z}_3$.

- (i) Assume that $\text{ann}(\mathbb{Z}_3) \neq \{0\}$. Then, there exists $0 \neq r \in \text{ann}(\mathbb{Z}_3)$. So, by Proposition 3.1

$$(r, 0) - (0, 1) - (0, 2) - (r, 0)$$

form a cycle of length 3, and we are done.

Assume that $\text{girth}(\Gamma(R(+)\mathbb{Z}_3)) = 3$ and $\text{ann}(\mathbb{Z}_3) = 0$. Then, as $\text{girth}(\Gamma(R(+)\mathbb{Z}_3)) = 3$, there exists a 3-cycle in $\Gamma(R(+)\mathbb{Z}_3)$, say

$$(a, b) - (c, d) - (e, f) - (a, b),$$

for some non-zero $a, c, e \in R$ and $b, d, f \in \mathbb{Z}_3$. By Proposition 3.1 without loss of generality we have $b = c = e = 0$, which gives

$$(a, 0) - (0, d) - (0, f) - (a, 0).$$

So, $ad = af = 0$ and $d \neq f$, which is a contradiction with the fact that $\text{ann}(\mathbb{Z}_3) = 0$. Hence, by a contrapositive proof we are done.

- (ii) Assume that $\text{ann}(\mathbb{Z}_3) = \{0\}$. Then, there is no non-zero $r \in R$ such that $r\mathbb{Z}_3 = \{0\}$. Therefore, $r \cdot 1 \neq 0$ for all $r \in R^*$. So, we get $\text{ann}((0, 1)) = \text{ann}((0, 2)) = \{(0, 0), (0, 1), (0, 2)\}$. Note that $\Gamma(R(+)\mathbb{Z}_3)$ is connected. If there exists $r \in Z(R)^*$, then $(r, 0)(r', 0) = (0, 0)$ for all $r' \in \text{ann}(r)$. Also, for any $r' \in \text{ann}(r)$ we have: $(r', 0)(0, 1) \neq 0$, $(r', 0)(0, 2) \neq 0$, $(r, 0)(0, 1) \neq 0$ and $(r, 0)(0, 2) \neq 0$. So, if $\text{ann}(r) = \{r_1, \dots, r_n\}$, then $(r, 0) - (r_1, 0) - \dots - (r_n, 0)$ is a path in $\Gamma(R(+)\mathbb{Z}_3)$ disconnected from other part of the graph, which contradicts the fact that $\Gamma(R(+)\mathbb{Z}_3)$ is connected. Hence, $Z(R) = \{0\}$ and so R is an

integral domain. Now, suppose that R has more than three elements. Assume that $r\mathbb{Z}_3 \neq \{0\}$ for all $r \in R^*$. Then, there exist distinct $r_1, r_2 \in R^*$ such that $r_1 \cdot 1 = r_2 \cdot 1$, so $(r_1 - r_2) \cdot 1 = 0$ where $r_1 - r_2 \neq 0$, which is a contradiction. Thus, $r\mathbb{Z}_3 = \{0\}$, which is again a contradiction. Hence, R has three elements and as it is a finite integral domain, it follows that R is a field, implying that $R \cong \mathbb{Z}_3$. So, since $Z(R(+)\mathbb{Z}_3)^* = \{(0, 1), (0, 2)\}$ we get $\text{girth}(\Gamma(R(+)\mathbb{Z}_3)) = \infty$.

Assume that $\text{girth}(\Gamma(R(+)\mathbb{Z}_3)) = \infty$ and $\text{ann}(\mathbb{Z}_3) \neq \{0\}$. Then, we get a contradiction with the implication proved in (i). $\square$

Now, we investigate the girth number of $\Gamma(R(+)M)$ in the case when $M \cong \mathbb{Z}_2$.

Theorem 3.15: [7, Theorem 2.2] The girth of $\Gamma(R(+)\mathbb{Z}_2) = 3$ if and only if one of the following hold:

- (i) $\text{girth}(\Gamma(R)) = 3$
- (ii) There exists an $r \in R^*$ such that $r^2 = 0$
- (iii) There exist distinct $a, b \in Z(R)^*$ such that $ab = 0 = a\mathbb{Z}_2 = b\mathbb{Z}_2$.

Proof. Let (i) hold. Then, for some $x, y, z \in \Gamma(R)$ we have $xy = yz = xz = 0$, and $(x, 0) - (y, 0) - (z, 0) - (x, 0)$ form a cycle of length 3 in $\Gamma(R(+)\mathbb{Z}_2)$, hence we are done. If (ii) holds, then $r(r \cdot 1) = r \cdot r = r^2 = 0$. Also, note that as $\Gamma(R(+)\mathbb{Z}_2)$ is connected we have that R is an integral domain. So, as $r \neq 0$ we get $r \cdot 1 = 0$. Then, $(r, 0) - (r, 1) - (0, 1) - (r, 0)$ is a cycle of length three. On the other hand, if (iii) holds then for distinct $a, b \in Z(R)^*$ we have $ab = 0 = a \cdot 1 = b \cdot 1$. So, $(a, 0) - (b, 0) - (0, 1) - (a, 0)$ is a cycle of length three, which finishes the proof of this implication.

Assume that $\text{girth}(\Gamma(R(+)\mathbb{Z}_2)) = 3$. Then, we consider two cases:

Case 1: If $(0, 1)$ is part of a cycle with minimal length, then $(0, 1) - (a, i) - (b, j) - (0, 1)$ is the form of that cycle. Note that if $a \neq b$, then it gives (iii), and if $a = b$ then it gives (ii).

Case 2: If $(0, 1)$ is not part of a cycle with minimal length, then $(a, i) - (b, j) - (c, k) -$

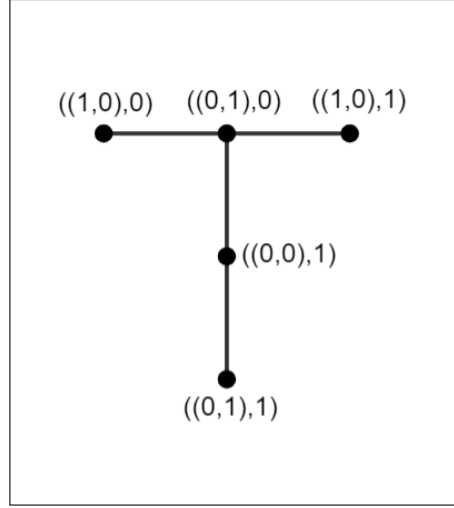


Figure 3.1: Zero-divisor graph of idealization $(\mathbb{Z}_2 \times \mathbb{Z}_2)(+)\mathbb{Z}_2$.

(a, i) is the form of that cycle for non zero $a, b, c \in R$. If all a, b and c are distinct, then $ab = bc = ca = 0$ and $a - b - c - a$ is a cycle of length three in $\Gamma(R)$, hence we get (i). If not all a, b and c are distinct, then we have $a^2 = 0$ or $b^2 = 0$, which gives (ii). $\square$

Using Lemma 2.3, we give the following example to illustrate the above theorems.

Example 3.2: Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and consider $R(+)M$, where $M = \mathbb{Z}_2$. Note that, in this case $R(+)M \cong \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$. By Proposition 3.1, we have $Z(R(+)M)^* = \{((1, 0), 0), ((1, 0), 1), ((0, 1), 0), ((0, 0), 1), ((0, 1), 1)\}$. Moreover, The module operation is, without loss of generality, defined by $(0, 0) \cdot 1 = (0, 1) \cdot 1 = 0$ and $(1, 0) \cdot 1 = (1, 1) \cdot 1 = 1$. Then, by the zero-divisor graph in Figure 3.1, it follows that $\text{girth}(\Gamma(R(+)M)) = \infty$.

Proposition 3.2: [7, Proposition 2.5] Let $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field and $|\mathbb{F}| \geq 3$. Then, $\text{girth}(\Gamma(R(+)\mathbb{Z}_2)) = 4$.

Proof. Since $|\mathbb{F}| \geq 3$, it follows that there exists $1 \neq a \in \mathbb{F}^*$. Also, by Lemma 2.3 we know that the module operation from R to $\mathbb{Z}_2$ is the extension of the module operation from $\mathbb{Z}_2$ to $\mathbb{Z}_2$. So, $((0, 0), 1) - ((0, 1), 1) - ((1, 0), 1) - ((0, a), 0) - ((0, 0), 1)$ is a cycle of length 4. By, Theorem 3.15 in this case $\Gamma(R(+)\mathbb{Z}_2)$ can not have a cycle of length 3. Hence, $\text{girth}(\Gamma(R(+)\mathbb{Z}_2)) = 4$. $\square$

Lemma 3.1: [7, Lemma 2.6] If $\text{diam}(\Gamma(R)) = 3$, then $\text{girth}(\Gamma(R(+)\mathbb{Z}_2))$ is finite.

Proof. Let $a, b, c, d \in R$ be distinct elements and $a - b - c - d$ be a path in $\Gamma(R)$, then $ab = bc = cd = 0$. If $b \cdot 1 = 1$ and $c \cdot 1 = 1$ then $(bc) \cdot 1 = b \cdot 1 \cdot c \cdot 1 = 1 \cdot 1 = 1$,

which is a contradiction as $bc = 0$. So, either $b \cdot 1 = 0$ or $c \cdot 1 = 0$. Assume that $b \cdot 1 = 0$. If in this case $c \cdot 1 = 0$, then $(b, 0) - (c, 0) - (b, 1) - (c, 1) - (b, 0)$ is a cycle of $\Gamma(R(+)Z_2)$. If $c \cdot 1 \neq 0$, then following from the argument above we have $d \cdot 1 = 0$. So, in this case $(b, 0) - (c, 0) - (d, 0) - (c, 1) - (b, 0)$ is a cycle of $\Gamma(R(+)Z_2)$. Hence, $\text{girth}(\Gamma(R(+)Z_2)) \neq \infty$. $\square$

Based on the results above we give the following remark, which is given in [7].

Remark 3.2: By the results above on $R(+)Z_2$ we conclude that $|R/\text{ann}(\mathbb{Z}_2)| = 2$.

Proof. Suppose that $|R/\text{ann}(\mathbb{Z}_2)| > 2$. Then, there exists non-zero distinct $r_1, r_2 \in R$ such that $r_1 + \text{ann}(\mathbb{Z}_2) \neq 0 + \text{ann}(\mathbb{Z}_2)$ and $r_2 + \text{ann}(\mathbb{Z}_2) \neq 0 + \text{ann}(\mathbb{Z}_2)$. Then, $r_1, r_2 \notin \text{ann}(\mathbb{Z}_2)$, implying that $r_1 \cdot 1 = r_2 \cdot 1 = 1$. Then, $(r_1 - r_2) \cdot 1 = 0$ following that $r_1 - r_2 \in \text{ann}(\mathbb{Z}_2)$. Hence, $r_1 + \text{ann}(\mathbb{Z}_2) = r_2 + \text{ann}(\mathbb{Z}_2)$. $\square$

Theorem 3.16: [7, Theorem 2.7] The girth of $\Gamma(R(+)Z_2)$ is infinite if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is an integral domain.

Proof. ($\Leftarrow$) If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then we are done by Example 3.2. If on the other hand R is an integral domain, then $\Gamma(R(+)Z_2)$ is a star graph centered at $(0, 1)$. Hence, in both cases $\text{girth}(\Gamma(R(+)Z_2)) = \infty$.

($\Rightarrow$) Assume that $\text{girth}(\Gamma(R(+)Z_2)) = \infty$, Then, by Lemma 3.1 we have $\text{diam}(\Gamma(R)) \leq 2$. Note that if $Z(R)^* = \emptyset$, then we are done.

Case 1: Suppose that $\text{diam}(\Gamma(R)) = 0$. Then, by [18, Theorem 3.2] we get $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[x]/(x^2)$. Note that as 2 and x are nilpotent elements of $\mathbb{Z}_4$ and $\mathbb{Z}_2[x]/(x^2)$, respectively, then R has a non-zero nilpotent element. Hence, by Theorem 3.15, $\text{girth}(\Gamma(R(+)Z_2)) = 3$, giving a contradiction.

Case 2: Suppose that $\text{diam}(\Gamma(R)) = 1$. Then, $\Gamma(R)$ is a complete graph. So, by [2, Theorem 2.8], if $R \neq \mathbb{Z}_2 \times \mathbb{Z}_2$, then R contains a nilpotent element. Thus, by Theorem 3.15, $\text{girth}(\Gamma(R(+)Z_2)) = 3$, which is a contradiction.

Case 3: Suppose that $\text{diam}(\Gamma(R)) = 2$ and $\Gamma(R)$ is not a star graph. Then, $\text{girth}(\Gamma(R)) < \infty$ and hence $\text{girth}(\Gamma(R(+)Z_2)) < \infty$, which is a contradiction.

So, by [2, Theorem 2.5] we have only the following two possible cases: $R \cong \mathbb{Z}_2 \times D$ where D is an integral domain or $Z(R)$ is an annihilator ideal. If $Z(R)$ is an annihilator ideal, then R has a nilpotent element and by Theorem 3.15 we get a contradiction. So,

let $R \cong \mathbb{Z}_2 \times D$ where D is an integral domain. If D is finite, then it is a field and by Proposition 3.2 we get $\text{girth}(\Gamma(R(+)\mathbb{Z}_2)) = 4$, which is a contradiction. So, suppose that D is an infinite integral domain. If we consider $\mathbb{Z}_2$ as an R -module, then $\text{ann}(\mathbb{Z}_2)$ is an ideal of R . Moreover, it has the following three possible forms: $\mathbb{Z}_2 \times \{0\}$, $\{0\} \times I$ or $\mathbb{Z}_2 \times I$ where I is a non-zero ideal of D . If $\text{ann}(\mathbb{Z}_2) = \mathbb{Z}_2 \times \{0\}$, then $|R/\text{ann}(\mathbb{Z}_2)| = |D| > 2$, which by Remark 3.2 we get a contradiction. Now, suppose that $\text{ann}(\mathbb{Z}_2) = \{0\} \times I$ or $\text{ann}(\mathbb{Z}_2) = \mathbb{Z}_2 \times I$ for a non-zero ideal I of D . Let $(0, a), (0, b) \in \text{ann}(\mathbb{Z}_2)$. Then, $((1, 0), 0) - ((0, a), 0) - ((1, 0), 1) - ((0, b), 0) - ((1, 0), 0)$ is a cycle of length four of $R(+)\mathbb{Z}_2$, which is a contradiction. hence, the proof is complete. $\square$

When analyzing the zero-divisor graph of an idealization, it becomes evident that the diameter of the graph for the ring may not remain constant. Moreover, if the diameter of $\Gamma(R)$ is greater than 1, then the diameter of $\Gamma(R(+)M)$ is also greater than 1. However, there exists a scenario where $\Gamma(R)$ is complete and the diameter of $\Gamma(R(+)M)$ is greater than 1, alongside various other potential combinations between the diameters of $\Gamma(R)$ and $\Gamma(R(+)M)$. The following examples are from [7, Example 3.1, Example 3.2].

Example 3.3: Let $R = \mathbb{Z}_9$. Then, $Z(R)^* = \{3, 6\}$ and hence $\text{diam}(R) = 1$. Now, let $M = \mathbb{Z}_9$ and consider $R(+)M$. Note that, for any non zero distinct $m, n \in M$ we have $(0, m)(0, n) = (0, 0)$. Also, if $r \in \text{reg}(R)$, then $(r, m) \notin Z(R(+)M)^*$. Now, $(0, 3)(r, m) = (0, 0)$ for all $(r, m) \in R(+)M$, but $(3, 1)(3, 3) \neq (0, 0)$. Hence, $\text{diam}(R(+)M) = 2$.

Example 3.4: Let $R = \mathbb{Z}[x]/(x^2)$. Then, $Z(R) = \{ax : a \in \mathbb{Z}^*\}$, hence $\text{diam}(\Gamma(R)) = 1$. Let $M = \mathbb{Z}_6$ and define $(a + bx)m = am$. Note that, $(2, 1)(0, 3) = (3, 1)(0, 2) = (3, 1)(0, 4) = (0, 0)$. Clearly, $\text{ann}((2, 1)) = \{(0, 0), (0, 3)\}$ and $\text{ann}((3, 1)) = \{(0, 0), (0, 2), (0, 3)\}$. Hence, the distance between $(2, 1)$ and $(3, 1)$ is 3, and hence $\text{diam}(R(+)M) = 3$.

4. IDEAL BASED ZERO-DIVISOR GRAPH

In 2003 the author in [9] for the first time introduced the notion of an ideal based zero-divisor graph. In the zero-divisor graph, we considered two vertices x and y to be adjacent if and only if their multiplication is zero, but in the case of an ideal based zero-divisor graph, we let x and y to be adjacent if and only if their multiplication is an element of an ideal I .

Definition: Let I be an ideal of a commutative ring R . We denote by $\Gamma_I(R)$, the graph with vertex set $\{x \in R \setminus I \mid xy \in I \text{ for some } y \in R \setminus I\}$. Moreover, for $x, y \in R \setminus I$, we have $xy \in I$ if and only if, in $\Gamma_I(R)$, x and y are adjacent.

The first question that comes to mind is; when $\Gamma(R)$ and $\Gamma_I(R)$ are the same graphs? The answer of this and more is given in the following proposition.

Proposition 4.1: [9, Proposition 2.2]

- (i) Let R be a commutative ring and I its proper ideal. Then, $\Gamma_I(R)$ is empty if and only if the ideal I is a prime ideal.
- (ii) For $I = (0)$ we have $\Gamma_I(R) = \Gamma(R)$.

Proof.

- (i) First let I be a prime ideal of the ring R . Then, for any $xy \in I$ we have $x \in I$ or $y \in I$. So, from the definition we obviously get $\Gamma_I(R)$ is an empty graph.

On the other hand, let $\Gamma_I(R)$ be an empty graph. Then, if there exist $x, y \in R \setminus I$ such that $xy \in I$, it follows that x and y are adjacent vertices of $\Gamma_I(R)$, which is a contradiction. Hence, I is a prime ideal.

- (ii) Now, if $I = (0)$, then we have $R/I \cong R$. So, by the definition, two non-zero elements x and y of R are adjacent in $\Gamma_I(R)$ if and only if $xy = 0$. Thus, $V(\Gamma_I(R)) = Z(R)^* = V(\Gamma(R))$ and x and y are adjacent in $\Gamma_I(R)$ if and only if they are adjacent in $\Gamma(R)$. Therefore, we conclude that $\Gamma_I(R) = \Gamma(R)$. $\square$

Let R and S be two commutative rings. Let also I be an ideal of R and J be an ideal of S . Then, we want to answer the following question: If $R/I \cong S/J$, then do we have $\Gamma_I(R) \cong \Gamma_J(S)$? To answer this we first prove the following.

Theorem 4.1: [9, Theorem 2.5] Let $x, y \in R \setminus I$ where I is an ideal of a ring R . Then,

- (i) If $x + I$ is adjacent to $y + I$ in $\Gamma(R/I)$, then x is adjacent to y in $\Gamma_I(R)$
- (ii) If x is adjacent to y in $\Gamma_I(R)$ and $x + I \neq y + I$, then $x + I$ is adjacent to $y + I$ in $\Gamma(R/I)$.
- (iii) If x is adjacent to y in $\Gamma_I(R)$ and $x + I = y + I$, then $x^2, y^2 \in I$.

Proof.

- (i) Suppose that $x + I - y + I$ is an edge in $\Gamma(R \setminus I)$. Then, $xy + I = (x + I)(y + I) = 0 + I$. Hence, $x - y$ is an edge in $\Gamma_I(R)$.
- (ii) Suppose that $x - y$ is an edge in $\Gamma_I(R)$ and $x + I \neq y + I$. Then, as $xy \in I$, we have $(x + I)(y + I) = xy + I = 0 + I$. Hence, $x + I - y + I$ is an edge in $\Gamma(R \setminus I)$.
- (iii) Suppose that $x - y$ is an edge in $\Gamma_I(R)$ and $x + I = y + I$. Then, $x^2 + I = (x + I)(x + I) = (x + I)(y + I) = xy + I = 0 + I$ and $y^2 + I = (y + I)(y + I) = (y + I)(x + I) = xy + I = 0 + I$. Hence, $x^2, y^2 \in I$. $\square$

Corollary 4.1: [9, Corollary 2.7] Let I be an ideal of a ring R . Then $\Gamma_I(R)$ contains $|I|$ disjoint subgraphs isomorphic to $\Gamma(R/I)$.

Proof. Consider the quotient ring R/I which contains cosets of I and $\Gamma(R/I)$ with vertex set

$$\{a_\alpha + I \mid \alpha \in \Lambda, a_\alpha \in R\} = Z(R/I) - \{0_{R/I}\}$$

It is clear that, if $\alpha \neq \beta$, then $a_\alpha + I \neq a_\beta + I$. Now, for each $i \in I$, define a graph G_i with vertex set $V(G_i) = \{a_\alpha + i \mid \alpha \in \Lambda\}$, such that two vertices $a_\alpha + i$ and $a_\beta + i$ are adjacent if and only if $a_\alpha + I$ and $a_\beta + I$ are adjacent in $\Gamma(R/I)$. So, there is an edge between $a_\alpha + I$ and $a_\beta + I$ in G_i if and only if $a_\alpha a_\beta \in I$. Then, by Theorem 4.1, G_i is a subgraph of $\Gamma_I(R)$ and $G_i \cong \Gamma(R/I)$ for every $i \in I$. Additionally, if $i \neq j$ then G_i and G_j contain no common vertices. Since $i \in I$, there are $|I|$ many disjoint subgraphs isomorphic to $\Gamma(R/I)$. $\square$

Corollary 4.2: [9, Remark 2.8] Let I be an ideal of a ring R . Then $\Gamma_I(R)$ is a graph on a finite number of vertices if and only if either R is finite or I is a prime ideal. Moreover, if $\Gamma(R/I)$ is a graph on t vertices, then $\Gamma_I(R)$ is a graph on $t|I|$ vertices.

Proof. If I is a prime ideal of R , by Proposition 4.1, it follows that $V(\Gamma_I(R)) = \emptyset$. On

the other hand, if R is a finite ring, then by definition of $\Gamma_I(R)$, $V(\Gamma_I(R))$ is a finite set.

Suppose that $V(\Gamma_I(R))$ is a finite set and I is not a prime ideal. Hence R/I is not an integral domain. Recall,

$$V(\Gamma_I(R)) = \{a_\alpha \in R - I \mid a_\alpha a_\beta \in I \text{ for some } a_\beta \in R - I\},$$

and

$$V(\Gamma(R/I)) = \{a_\alpha + I \in Z(R/I)^* \mid \alpha \in \Lambda, a_\alpha \in R\}.$$

Then obviously $V(\Gamma(R/I))$ is finite. Now, suppose that I is not finite. Let r be an arbitrary non-zero element of I . Then, for any two adjacent vertices a, b in $V(\Gamma_I(R))$, consider the elements of the form $a + r, b + r$. It is easy to observe that these elements are in $R \setminus I$. Moreover, we have $(a + r)(b + r) = ab + ar + br + r^2 \in I$, which means that $a + r$ and $b + r$ are vertices of $\Gamma_I(R)$, for each $r \in I$. As the number of such vertices is infinite, we obtain a contradiction. Hence, I has to be finite. Then, by Theorem 3.2 as $\Gamma(R/I)$ is finite and R/I is not an integral domain, R/I is finite. So, since R/I and I are both finite we conclude that R itself is finite.

As for the “moreover” part; it is clear from Redmond’s three-step construction of $\Gamma_I(R)$, which is given below. $\square$

Remark 4.1: The following steps are called Redmond’s Three Step Construction For Building $\Gamma_I(R)$:

- (i) Let $\{a_\alpha\}_{\alpha \in \Lambda}$ be a set of coset representatives of $V(\Gamma(R/I))$. Then $(V(G_i), E(G_i)) = \{\text{the vertices } \{a_\alpha + i\}_{\alpha \in \Lambda}, \text{ where } a_\alpha + i \text{ is adjacent to } a_\beta + i \text{ if and only if } a_\alpha + I \text{ is adjacent to } a_\beta + I \text{ in } \Gamma(R/I)\}$.
- (ii) For distinct $\alpha, \beta \in \Lambda$ for each $i, j \in I$, $a_\alpha + i$ is adjacent to $a_\beta + j$ if and only if $a_\alpha + I$ is adjacent to $a_\beta + I$ in $\Gamma(R/I)$.
- (iii) For each α and distinct $i, j \in I$, $a_\alpha + i$ is adjacent to $a_\alpha + j$ if and only if $a_\alpha^2 \in I$.
In this step, we are connecting columns as necessary. Such columns are called connected columns.

To illustrate the Remark 4.1 we give the following example, see [9].

Example 4.1: $\Gamma_I(\mathbb{Z}_{24})$ with $I = (8)$. Note that $\mathbb{Z}_{24}/I \cong \mathbb{Z}_8$. Also the columns have height $|I| = 3$. We start by stacking $|I| = 3$ copies of $\Gamma(\mathbb{Z}_8)$ creating edge set 1 of the

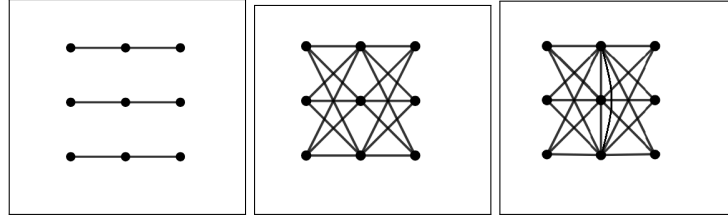


Figure 4.1: Redmond's Three Step Construction For Building $\Gamma_I(R)$.

redmond steps. Then we connect adjacent columns to create edge set 2. Finally we place edges in the connected column $4 + (8)$ to create edge set 3. A graphical representation is given in Figure 4.1.

Theorem 4.2: [9, Corollary 2.6] Let I be a radical ideal of a commutative ring R . Then, $\Gamma_I(R)$ is a complete bipartite graph if and only if $\Gamma(R/I)$ is a complete bipartite graph.

Proof. It is an obvious result of the Theorem 4.1. $\square$

Let G be a simple graph. If H is a subgraph of G such that $V(H) = V(G)$, then H is called a spanning subgraph of G . Recall that G is called r -regular if each vertex has degree r . A 1-regular spanning subgraph of G is called a 1-factor or a perfect matching of G . If we can partition $E(G)$ into 1-factors, then we call the graph G a 1-factorable graph. In [19, page 192], authors show that every r -regular bipartite graph is 1-factorable. Suppose that the edges of G are partitioned into subgraphs $H_1, \dots, H_n$. Then, we write $G \cong H_1 \oplus \dots \oplus H_n$. If in this case $H_i \cong H_j$ for all $1 \leq i, j \leq n$, then we write $G \cong nH$, where $H \cong H_i$ for all $i = 1, \dots, n$.

Theorem 4.3: [13, Theorem 2.1] Let I be a finite ideal of R such that $I = \sqrt{I}$. Then $\Gamma_I(R) \cong |I|^2 \Gamma(R/I)$.

Proof. Let x and y be two adjacent vertices of $\Gamma(R/I)$ and $e = \{x, y\} \in \Gamma(R/I)$. By Remark 4.1 we understand that every $x+r$ is adjacent to every $y+s$ in $\Gamma_I(R)$, for $r, s \in I$. Let us denote by $H^{(e)}$ the subgraph of $\Gamma_I(R)$ with vertex set $\{x+r, y+r \mid \text{for all } r \in I\}$ and edge set $\{\{x+r, y+s\} \mid \text{for all } r, s \in I\}$. It is obvious that $H^{(e)}$ is isomorphic to $K_{|I|, |I|}$. Thus, the graph $H^{(e)}$ is $|I|$ -regular bipartite graph, and it follows from [19, page 192] that $H^{(e)} \cong K_{|I|, |I|} \cong M_1^{(e)} \oplus \dots \oplus M_{|I|}^{(e)}$, where $M_i^{(e)}$ is a perfect matching, for all $i = 1, \dots, |I|$. Let $H_i := \bigoplus_{e \in E(\Gamma(R/I))} M_i^{(e)}$ and note that H_i is a subgraph of $\Gamma_I(R)$, for all $i = 1, \dots, |I|$. As $I = \sqrt{I}$, it follows that $\Gamma_I(R) \cong H_1 \oplus \dots \oplus H_{|I|}$. Also, by understanding of Remark 4.1 we have $M_i^{(e)} \cong \Gamma(R/I)$, for all $i = 1, \dots, |I|$. So,

for each $i = 1, \dots, |I|$, the subgraph H_i is partitioned into $|I|$ edge-disjoint subgraph isomorphic to $\Gamma(R/I)$. Hence, by the explanation above this theorem, we conclude that $\Gamma_I(R) \cong |I|^2 \Gamma(R/I)$. $\square$

Now we prove the following theorem, which shows that under some conditions if $\Gamma(R/I) \cong \Gamma(S/J)$, then $\Gamma_I(R) \cong \Gamma_J(S)$.

Theorem 4.4: [13, Theorem 2.2] Let I be a finite ideal of R and let J be a finite ideal of S such that $I = \sqrt{I}$ and $J = \sqrt{J}$. If $|I| = |J|$ and $\Gamma(R/I) \cong \Gamma(S/J)$, then $\Gamma_I(R) \cong \Gamma_J(S)$, as desired.

Proof. Assume that $|I| = |J|$ and $\Gamma(R/I) \cong \Gamma(S/J)$. Then, from Theorem 4.3 we have the following: $\Gamma_I(R) \cong |I|^2 \Gamma(R/I) \cong |J|^2 \Gamma(S/J) \cong \Gamma_J(S)$. $\square$

The following example shows that in Theorem 4.4, the conditions $I = \sqrt{I}$ and $J = \sqrt{J}$ and $|I| = |J|$ are necessary, see [13] and [9].

Example 4.2: (i) Let $R = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and $S = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

Also, let $I = 0 \oplus 0 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and $J = 0 \oplus 0 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ be ideals of R and S , respectively. Note that $R/I \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \cong S/J$ and both I and J are radical ideals. On the other hand, the number of edges of $\Gamma_I(R)$ is not equal to the number of edges of $\Gamma_J(S)$, and hence $\Gamma_I(R) \not\cong \Gamma_J(S)$.

(ii) Let $R = \mathbb{Z}_6 \times \mathbb{Z}_3$ and $S = \mathbb{Z}_{24}$. Also, let $I = 0 \times \mathbb{Z}_3$ and $J = (8)$ be ideals of R and S , respectively. It is obvious that $R/I \cong \mathbb{Z}_6$ and $S/J \cong \mathbb{Z}_8$. From Figure 4.2 we have $\Gamma(R/I) \cong \Gamma(S/J)$, and from Figure 4.3 we have $\Gamma_I(R) \not\cong \Gamma_J(S)$. Note that in this example neither I nor J are radical ideals.

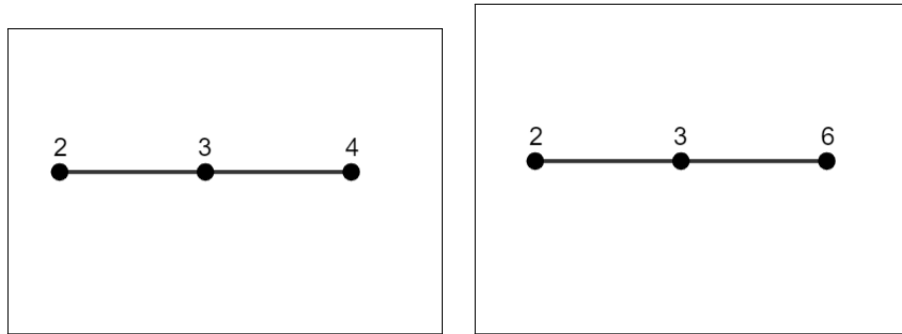


Figure 4.2: Zero-divisor graph of $\mathbb{Z}_6$ and $\mathbb{Z}_8$, respectively.

Theorem 4.5: [13, Theorem 2.4] Let R be a commutative ring and I be a nonzero

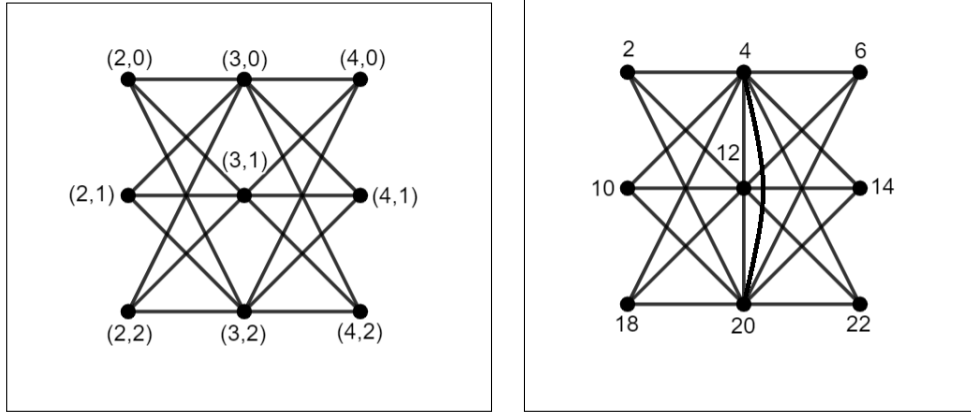


Figure 4.3: Ideal based zero-divisor graph of $\mathbb{Z}_6 \times \mathbb{Z}_3$ and $\mathbb{Z}_{24}$, respectively.

ideal of R . If $a \in \Gamma_I(R)$ is adjacent to every vertex of $\Gamma_I(R)$, then $(I : a)$ is a maximal element (with respect to inclusion) of the set $\{(I : x) \mid x \in R\}$. Moreover, $(I : a)$ is a prime ideal.

Proof. Let $V = V(\Gamma_I(R))$ and $0 \neq x \in R$. One can observe that as a is adjacent to every element of V , it follows that $(I : a) = \{r \in R \mid ar \in I\} = V \cup I$. Since x is non-zero, it is clear that a and $a + x$ are two different vertices in $\Gamma_I(R)$. Moreover, by the hypothesis these vertices are adjacent, which means that $a(a + x) \in I$. As $x \in I$, it follows that $ax \in I$. So, $a^2 + ax - ax = a^2 \in I$. Therefore, for any $x \in R$ we have $(I : x) \subseteq V \cup I = (I : a)$.

Let x and y be two elements of the ring R such that $xy \in (I : a)$. Then, $xya \in I$, and in particular, $x \in (I : ya)$. Note that, the previous part implies that $(I : ya) \subseteq (I : a)$. Suppose that $x, y \notin (I : a)$. Then, $ya \notin I$. Let $z \in (I : a)$. As $za \in I$ and $z(ya) \in I$, we have $(I : a) \subseteq (I : ya)$. Therefore, $(I : a) = (I : ya)$. So, $x \in (I : ya) = (I : a)$, which is a contradiction. Hence, $(I : a)$ is a prime ideal. $\square$

Theorem 4.6: [13, Theorem 2.5] Let R be a commutative ring and I be an ideal of R . If S is a clique of maximal size in $\Gamma_I(R)$ such that $x^2 = 0$ for all $x \in S$, then $S \cup I$ is an ideal of R .

Proof. Obviously, $S \cup I$ is non-empty. Let $x, y \in S \cup I$ be two arbitrary elements. If both x and y are elements of I , then $x - y \in I$ and so $x - y \in S \cup I$.

Let $x, y \in S$ and $x - y \notin I$. Then, for any $z \in S$, since it is adjacent to both x and y in $\Gamma_I(R)$, it follows that $zx, zy \in I$. So, $z(x - y) = zx - zy \in I$ and $S \cup \{x - y\}$ is also

a clique. Hence, as S is a clique of maximal size, $x - y \in S$ and so $x - y \in S \cup I$.

Let $x \in I$ and $y \in S$. Since y is a vertex of $\Gamma_I(R)$, it follows that $y \in R \setminus I$ and $x - y \notin I$. If z is an arbitrary element of S , then it is adjacent to y . Also, $x \in I$ implies that $zx \in I$. So, $z(x - y) = zx - zy \in I$. Hence, $x - y \in S$ and so $x - y \in S \cup I$.

Now, let $x \in S \cup I$ and $r \in R$. If $x \in I$ or $r \in I$, then $rx \in I$. So, suppose that $x, r \notin I$ and $x \in S$. If $rx \in I$, then we are done. If $rx \notin I$, then any element z of S is adjacent to x in $\Gamma_I(R)$. So, $xz \in I$ and so $rxz \in I$. Hence, rx is adjacent to z in $\Gamma_I(R)$ and so $rx \in S$. $\square$

Theorem 4.7: [13, Theorem 2.6] Let R be a commutative ring and I be an ideal of R . If $S = \sqrt{I} \setminus I$ is a nonempty set, then the subgraph spanned by S , denoted by (S) is connected.

Proof. Let $x, y \in S$. If $xy \in I$, then x and y are adjacent in $\Gamma_I(R)$ and we are done. So, suppose that $xy \notin I$. Since $x, y \in \sqrt{I}$, it follows that $x^n, y^m \in I$ for some $n, m \geq 2$ and $x^{n-1} = y^{m-1} \notin I$. Hence, the path

$$x - x^{n-1} - xy - y^{m-1} - y$$

is a path connecting x and y . As x and y was arbitrary elements of S , the assertion follows. $\square$

Corollary 4.3: [13, Corollary 2.7] Let R be a commutative ring and N be the nilradical of R or a nilpotent ideal of R . If N is nontrivial, then the graph spanned by $(N \setminus \{0\})$ is a connected subgraph of $\Gamma_I(R)$.

Proof. Let N be the nilradical of R or a nilpotent ideal of R . Then, for $x, y \in N \setminus \{0\}$ there are some integers $n, m \geq 1$ such that $x^n = x^m = 0$. So, by Theorem 4.7 the assertion follows. $\square$

In the case when I is a prime ideal of a commutative ring R , we showed that $\Gamma_I(R) = \emptyset$. Now, we investigate $\Gamma_I(R)$, where $I = p_1 \cap p_2$ for two prime ideals p_1 and p_2 of R .

Theorem 4.8: [13, Theorem 3.1] Let R be a commutative ring. Then,

- (i) If p_1 and p_2 are prime ideals of R and $I = p_1 \cap p_2 \neq 0$, then $\Gamma_I(R)$ is a complete bipartite graph.
- (ii) If I is a nonzero ideal of R such that $I = \sqrt{I}$, then $\Gamma_I(R)$ is a complete bipartite graph if and only if there exist prime ideals p_1 and p_2 of R such that $I = p_1 \cap p_2$.

Proof.

(i) Suppose $0 \neq I = p_1 \cap p_2$ where p_1 and p_2 are prime ideals. First we prove that $\Gamma_I(R)$ is a bipartite graph. To prove this, let $x, y \in R \setminus I$ be two adjacent vertices in $\Gamma_I(R)$. This means that $xy \in I$, so $xy \in p_1$ and $xy \in p_2$. As both p_1 and p_2 are prime ideals, we have $x \in p_1$ or $y \in p_1$ and $x \in p_2$ or $y \in p_2$. If x (respectively y) is an element of both p_1 and p_2 , then x (respectively y) is an element of I , which is a contradiction. If $x \in p_1 \setminus p_2$ and $y \in p_1 \setminus p_2$, then $xy \in p_1 \setminus p_2$ or equivalently $xy \notin I$, which is a contradiction. So, we suppose without loss of generality that $x \in p_1 \setminus p_2$ and $y \in p_2 \setminus p_1$. Hence, $\Gamma_I(R)$ is bipartite graph with $V_1 = p_1 \setminus p_2$ and $V_2 = p_2 \setminus p_1$.

Now, we claim that $\Gamma_I(R)$ is a complete bipartite graph. Let $x \in p_1 \setminus p_2$. As p_1 is an ideal, it is closed under multiplication by all elements of R . In particular, for each $y \in p_2 \setminus p_1$, we have $xy \in p_1$. If for some $y_0 \in p_2 \setminus p_1$, $xy_0 \notin I$, then $xy_0 \notin p_2$, which is not possible as p_2 is an ideal.

(ii) The reverse direction is proved in part (i). Now, suppose that $I = \sqrt{I} \neq 0$ and $\Gamma_I(R)$ is a complete bipartite graph with parts V_1 and V_2 . Let $p_1 = V_1 \cup I$ and $p_2 = V_2 \cup I$. Then, since $V_1 \cap V_2 = \emptyset$ we obviously have $p_1 \cap p_2 = I$. So, it is left to prove that both p_1 and p_2 are prime ideals. we firstly prove that p_1 is an ideal of R .

Let $x, y \in p_1$. If both $x, y \in I$, then $x - y \in I$ and so $x - y \in p_1$. If $x, y \in V_1$, then there is a vertex $z \in V_2$ adjacent to both x and y in $\Gamma_I(R)$. Then, $zx, zy \in I$. So, $z(x - y) = zx - zy \in I$. Thus, as $z \in V_2$ either $x - y \in I$ or $x - y \in V_1$. Both cases imply that $x - y \in p_1$.

Now, if $x \in V_1$ and $y \in I$, then there exists $z \in V_2$ such that $zx \in I$. As $y \in I$ and I is an ideal, we have $(x - y)z \in I$. As $z \in V_2$, we have $x - y \in I$, and so $x - y \in p_1$.

Let $x \in p_1$ and $r \in R$. If $x \in I$, then $rx \in I$. If $x \in V_1$, then there exists a vertex $z \in V_2$ adjacent to x in $\Gamma_I(R)$, or equivalently $zx \in I$. Consequently, we have $zrx \in I$. If $rx \in I$, then $rx \in p_1$. On the other hand, if $rx \notin I$ then $rx \in V_1$, so it is in p_1 . From both cases we conclude that $rx \in p_1$. Hence, p_1 is an ideal of R .

Similarly, we can observe that p_2 is an ideal of R .

We claim that p_1 is a prime ideal; Let $x, y \in R \setminus p_1$ such that $xy \in p_1$. Then, either $xy \in I$ or $xy \in V_1$. In both cases there exists $z \in V_2$ such that $zxy \in I$. If $zy \in I$, then since $y \notin I$, by definition of $\Gamma_I(R)$ it follows that z and y are adjacent in $\Gamma_I(R)$. So, we have $y \in V_1$, which is contradiction. Thus, $zy \notin I$ and so $zy \in V_1$. Moreover, we have $z^2y \in I$ and since $I = \sqrt{I}$, $z^2 \notin I$. So, again by definition we have $z^2 \in V_1$. As z is an element of V_2 and the graph is complete bipartite, we have $z^3 \in I$. Now the equality $I = \sqrt{I}$ implies that $z \in I$, which is a contradiction. Hence, p_1 is a prime ideal of R . $\square$

The converse of Theorem 4.8 part (i) is not true. As a counter example we consider $R = \mathbb{Z}_8$ and $I = (4)$, where $\Gamma_I(R)$ is bipartite, but we can not write I as the intersection of two prime ideals.

4.1. Girth and Clique Number of $\Gamma_I(R)$

Now we investigate the girth and clique number of $\Gamma_I(R)$, where I is equal to the intersection of prime ideals. First, we look into the case when $I = p_1 \cap p_2$, for p_1 and p_2 as prime ideals of the ring R .

Theorem 4.9: [13, Theorem 4.1] Let p_1 and p_2 be distinct prime ideals of R and $I = p_1 \cap p_2$. Then, either $\text{gr}(\Gamma_I(R)) = 4$ or $R/I = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Proof. We claim that either $|p_2 \setminus p_1| \geq 2$ and $|p_1 \setminus p_2| \geq 2$; or both have one element.

First consider the case that $|p_1 \setminus p_2| = 1$ and $|p_2 \setminus p_1| \geq 2$. Assume that $p_1 \setminus p_2 = \{x\}$, then $xy \in p_1 \cap p_2 = I$ for all $y \in p_2 \setminus p_1$. So, $\Gamma_I(R)$ is a star graph. In particular, x is a cut vertex which is a contradiction due to [9, Theorem 3.2]. A similar explanation follows in the case when $|p_2 \setminus p_1| = 1$ and $|p_1 \setminus p_2| \geq 2$.

Hence $|p_1 \setminus p_2| \geq 2$ and $|p_2 \setminus p_1| \geq 2$. Now Theorem 4.8 implies that $\Gamma_I(R)$ is a complete bipartite graph, so $\text{gr}(\Gamma_I(R)) = 4$.

Now, suppose that $|p_1 \setminus p_2| = 1$ and $|p_2 \setminus p_1| = 1$. Then, $p_1 \setminus p_2 = \{x\}$ for some

$x \in R$. As $I = p_1 \cap p_2$, it follows that $p_1 = \{x\} \cup I$. If $r \in R \setminus p_2$, then $rx \in p_1$. If also $rx \in I$, then as $x, r \in R \setminus I$ it follows that $x - r$ is a vertex in $\Gamma_I(R)$. Then, $r \in p_2$ which is a contradiction. So, $rx \in p_1 \setminus I$ for all $r \in R \setminus p_2$, and then $rx = x$. Therefore, $x - rx = x(1 - r) = 0 \in p_2$ and as p_2 is a prime ideal we have $1 - r \in p_2$, for all $r \in R \setminus p_2$. Then, as $1 - r \in p_2$, it follows that $1 + p_2 = r + p_2$ in R/p_2 , for all $r \in R \setminus p_2$. Hence we have two different cases for the elements of this quotient ring:

- For $r \in p_2$ and $r + p_2 = p_2 = 0 + p_2$;
- $r \in R \setminus p_2$ and $r + p_2 = 1 + p_2$.

So, $R/p_2 = \{0 + p_2, 1 + p_2\} \cong \mathbb{Z}_2$. On the other hand, using the fact that $|p_2 \setminus p_1| = 1$ and following the same argument as above we get $R/p_1 \cong \mathbb{Z}_2$. As $\mathbb{Z}_2$ is a field, it follows that p_1 and p_2 are maximal ideals of R . Obviously $x \in p_1 \setminus p_2$ implies that $x \in p_1 \subsetneq p_1 + p_2$. Since p_1 is a maximal ideal of R , we get $p_1 + p_2 = R$. Hence, $R/I \cong R/(p_1 \cap p_2) \cong R/p_1 \times R/p_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. $\square$

In the case when I is the intersection of finite prime ideals of R , the clique number of $\Gamma_I(R)$ is given in the next theorem.

Theorem 4.10: [13, Theorem 4.2] Let I be an ideal of R such that $I = \bigcap_{1 \leq i \leq n} p_i$ and for each $1 \leq j \leq n$, $I \neq \bigcap_{1 \leq i \leq n, i \neq j} p_i$ where p_i 's are prime ideals of R . Then, $w(\Gamma_I(R)) = n$.

Proof. Let $x_j \in \bigcap_{1 \leq i \leq n, i \neq j} p_i \setminus p_j$. Then, $x \notin I$ implying that $x \in R \setminus I$. Note that $x_j \in (p_1 \setminus p_j) \cap \dots \cap (p_{j-1} \setminus p_j) \cap (p_{j+1} \setminus p_j) \cap \dots \cap (p_n \setminus p_j)$. So, as all p_i 's are prime ideals we have $x_j x_k \in p_i$ for all $1 \leq j, k \leq n$ such that $j \neq k$ and all $1 \leq i \leq n$. Thus, $x_j x_k \in I = \bigcap_{1 \leq i \leq n} p_i$ for all $1 \leq j, k \leq n$ such that $j \neq k$. Hence, $\{x_1, \dots, x_n\}$ is a clique in $\Gamma_I(R)$, following that $w(\Gamma_I(R)) \geq n$. So, we claim that $w(\Gamma_I(R)) \leq n$. To prove the claim we use induction on n . Let $n = 2$. By Theorem 4.8, $\Gamma_I(R)$ is a bipartite graph and so $w(\Gamma_I(R)) = 2$.

Now, suppose that $n > 2$ and the result follows for all integers less than n . Let $\{x_1, \dots, x_m\}$ be a clique in $\Gamma_I(R)$. Then, $x_1 x_j \in \bigcap_{1 \leq i \leq n} p_i = I$ for all $2 \leq j \leq m$. Suppose without loss of generality that $x_1 \notin p_1$. Then, $x_2, \dots, x_m \in p_1$ and $x_2, \dots, x_m \notin \bigcap_{2 \leq i \leq n} p_i$. If we let $J = \bigcap_{2 \leq i \leq n} p_i$, then $\{x_2, \dots, x_m\}$ is a clique in $\Gamma_J(R)$. Therefore, by induction step we have $m - 1 = n - 1$, implying that $m = n$. $\square$

Next, we collect some conclusions from the above theorems.

Corollary 4.4: [13, Corollary 4.3] The following hold:

- (a) If $I = \bigcap_{1 \leq i \leq n} p_i \neq 0$ and $J = \bigcap_{1 \leq j \leq m} q_j$ where p_i 's and q_j 's are prime ideals such that $\Gamma_I(R) = \Gamma_J(R)$, then $n = m$.
- (b) If for any $p \in \text{Min}(R)$, p is a finitely generated ideal, then $w(\Gamma_{\text{nil}(R)}(R)) = |\text{Min}(R)|$ (which is finite by [20]).
- (c) If R is a semi-local ring and not local, then $w(\Gamma_{J(R)}(R)) = |\text{Max}(R)|$.
- (d) If n is a square-free integer, then $w(\Gamma_{n\mathbb{Z}}(\mathbb{Z})) = k$, where k is the number of primes in the decomposition of n into primes.

Lastly, we show that in the case when I is a primary ideal that is not prime with at least three elements or when $\text{ass}(R/I)$ contains at least three elements, the girth number of $\Gamma_I(R)$ is always three.

Theorem 4.11: [13, Theorem 4.4] Let I be an ideal of R . Suppose either I is a primary ideal of R that is not prime and $|I| \geq 3$, or $|\text{ass}(R/I)| \geq 3$. Then, $\text{gr}(\Gamma_I(R)) = 3$.

Proof. Firstly suppose that I is a primary ideal which is not prime and $|I| \geq 3$. Let $a, b \in R/I$ such that $ab \in I$. Then, $b^n \in I$ and $b^{n-1} \notin I$ for some $n \in \mathbb{N}$. So, $a, b^{n-1} \notin I$. Hence, $a - b - b^{n-1} - a$ is a cycle in $\Gamma_I(R)$. Therefore, $\text{gr}(\Gamma_I(R)) = 3$.

Now, suppose that $|\text{Ass}(R/I)| \geq 3$. Then, by [21, Corollary 2.2] it follows that $\text{gr}(\Gamma(R/I)) = 3$. Hence, $\text{gr}(\Gamma_I(R)) = 3$. $\square$

Note that one of two conditions in the Theorem 4.11 is necessary. For example, if $R = \mathbb{Z}_8$ and $I = (4)$, then $|\text{Ass}(R/I)| = 1$ and $\text{gr}(\Gamma(R/I)) = \infty$.

5. WEAKLY ZERO-DIVISOR GRAPHS

The concept of weakly zero-divisor graphs, denoted by $W\Gamma(R)$, is firstly given in 2021 in [10], where authors investigate the properties of $W\Gamma(R)$ and relations with $\Gamma(R)$.

Definition: Let R be a commutative ring. Then, the weakly zero-divisor graph of R , denoted by $W\Gamma(R)$, is the graph with vertex set $Z(R)^*$, where two vertices x and y are adjacent if and only if there exist $r \in \text{ann}(x)$ and $s \in \text{ann}(y)$ such that $rs = 0$.

An example of a weakly zero divisor graph is as follows.

Example 5.1: Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. Then, the vertex set of $W\Gamma(R)$ is $Z(R)^* = \{(1, 0), (0, 1), (2, 0), (0, 2)\}$. So, the graph $W\Gamma(R)$ is given in Figure 5.1. Note that in this case $\Gamma(R) = W\Gamma(R)$, but in general this is not the case.

One of the fundamental properties of $W\Gamma(R)$ in relation to $\Gamma(R)$ is that $\Gamma(R)$ is a connected subgraph of $W\Gamma(R)$. This property plays a crucial role throughout this thesis.

Lemma 5.1: [10, Lemma 2.1] If R is a ring then the following hold:

- (i) Let $x, y \in Z(R)^*$. If $x - y$ is an edge of $\Gamma(R)$, then $x - y$ is an edge of $W\Gamma(R)$.
- (ii) If $x \in \text{Nil}(R)^*$, then x is adjacent to all the other vertices in $W\Gamma(R)$.
- (iii) The vertices in $\text{Nil}(R)^*$ generates a clique in $W\Gamma(R)$.

Proof.

(i) Let $x, y \in Z(R)^*$ such that $x - y$ is an edge of $\Gamma(R)$. It means that $xy = 0$, so

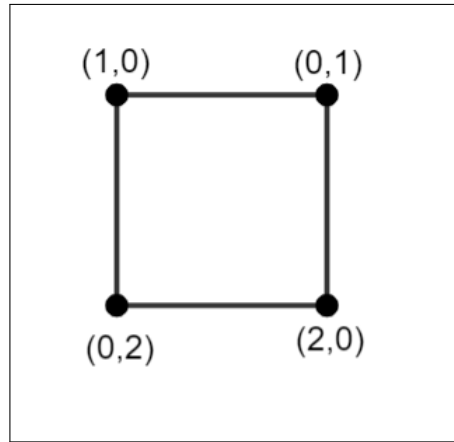


Figure 5.1: Weakly zero-divisor graph of $\mathbb{Z}_3 \times \mathbb{Z}_3$.

$x \in \text{ann}(y)$ and $y \in \text{ann}(x)$. Hence, $x - y$ is an edge of $W\Gamma(R)$.

(ii) Let $x \in \text{Nil}(R)^*$, $y \in Z(R)^*$ and $r \in \text{ann}(y)$. So, $\exists n \in \mathbb{N}$ such that $x^n = 0$ and $x^i \neq 0$, for $1 \leq i \leq n - 1$. Therefore, $x^n = x^{n-1}x = 0$ implies $x^{n-1} \in \text{ann}(x)$.

Now we have two cases:

1. If $x^{n-1}r = 0$, then by the definition of $W\Gamma(R)$, $x - y$ is an edge of $W\Gamma(R)$.
2. If $x^{n-1}r \neq 0$, then by $xx^{n-1}r = 0$ we get $x^{n-1}r \in \text{ann}(x)$. Also, we know that $yr = 0$, and then $yrx^{n-1} = 0$. So, $x^{n-1}r \in \text{ann}(y)$. Therefore, $x^{n-1}r \in \text{ann}(x) \cap \text{ann}(y)$. So, we have $x^{n-1}rx^{n-1}r = x^{2n-2}r^2 = x^n r^2 x^{n-2} = 0$. Hence, $x - y$ is an edge of $W\Gamma(R)$.

As y is an arbitrary vertex, we are done.

(iii) From (ii) every element of $\text{Nil}(R)^*$ is adjacent to all the other vertices, and in particular to all the other elements of $\text{Nil}(R)^*$. Hence, as $\text{Nil}(R)^* \subseteq Z(R)^*$, it is a complete subgraph of $W\Gamma(R)$. $\square$

Now, we give the following example to illustrate the graph structure of $\Gamma(R)$ as a connected subgraph of $W\Gamma(R)$.

Example 5.2: If $R = \mathbb{Z}_{12}$, then $Z(R)^* = \{2, 3, 4, 6, 8, 9, 10\}$. Then, the zero-divisor and weakly zero-divisor graphs of R are given in Figure 5.2. It is obvious that $\Gamma(R)$ is a connected subgraph of $W\Gamma(R)$. Moreover, as 6 is a nilpotent element of $\mathbb{Z}_{12}$ then it is adjacent to all other vertices of $W\Gamma(R)$. Note that, the converse of the second part of the Lemma 5.1 is obviously not true, because 2, 4, 8 and 10 are adjacent to all other vertices of $W\Gamma(R)$, but none of them is nilpotent.

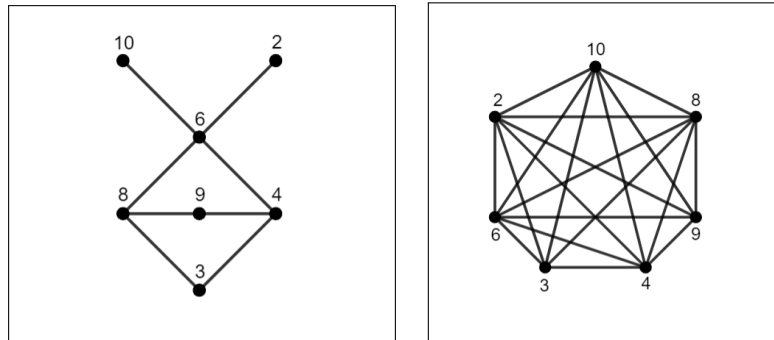


Figure 5.2: Zero-divisor and weakly zero-divisor graphs of $\mathbb{Z}_{12}$, respectively.

Next, we show that, for any commutative ring R , the diameter of $W\Gamma(R)$ does not exceed 2, and in the case when it contains a cycle the girth number is at most 4. This will be another fundamental property of the graph $W\Gamma(R)$, which is very useful in our thesis.

Theorem 5.1: [10, Theorem 2.2] If R is a commutative ring, then $W\Gamma(R)$ is connected and $\text{diam}(W\Gamma(R)) \leq 2$. Moreover, if $W\Gamma(R)$ contains a cycle, then $\text{girth}(W\Gamma(R)) \leq 4$.

Proof. Lemma 5.1 implies that every edge of $\Gamma(R)$ is an edge of $W\Gamma(R)$. Hence, from Theorem 3.1 it follows that $W\Gamma(R)$ is connected. Suppose that $W\Gamma(R)$ contains a cycle, then by Theorem 3.3 $\text{girth}(W\Gamma(R)) \leq 4$. Now we need to prove $\text{diam}(W\Gamma(R)) \leq 2$. Let x, y be two distinct vertices of $W\Gamma(R)$. Suppose that $x - y$ is not an edge of $W\Gamma(R)$. Which means that $rs \neq 0$, for every $r \in \text{ann}(x)$ and $s \in \text{ann}(y)$. Let $r \in \text{ann}(x)$ and $s \in \text{ann}(y)$. Then, $rx = 0$ and $sy = 0$. Since $rxs = rys = 0$, we have the following path in $W\Gamma(R)$;

$$x - rs - y.$$

Hence $\text{diam}(W\Gamma(R)) \leq 2$. $\square$

As an example for Theorem 5.1, one can consider the graph in Figure 5.2. To have the girth number equal to four it is necessary and sufficient for R to be a reduced ring with exactly two minimal prime ideals.

Theorem 5.2: [10, Theorem 2.3] Let R be a ring and $W\Gamma(R)$ contain a cycle. Then $\text{girth}(W\Gamma(R)) = 4$ if and only if R is reduced with $|\text{Min}(R)| = 2$.

Proof. Suppose that $\text{girth}(W\Gamma(R)) = 4$. If $\text{Nil}(R) \neq 0$, then by Lemma 5.1 $\text{girth}(W\Gamma(R)) = 3$, which is a contradiction. So, $\text{Nil}(R) = 0$ and hence R is reduced. We claim that $W\Gamma(R) = \Gamma(R)$;

If not, then there are distinct elements $x, y \in Z(R)^*$ such that $x - y$ is an edge of $W\Gamma(R)$ and it is not edge of $\Gamma(R)$, that is $xy \neq 0$. So, there are $r \in \text{ann}(x)$ and $s \in \text{ann}(y)$ such that $rs = 0$. As $\text{Nil}(R) = (0)$ and $xy \neq 0$, we have $r \neq x \neq y$ and $s \neq y \neq x$. We now consider the following two cases:

Case 1: If there exists some $0 \neq b \in \text{ann}(x) \cap \text{ann}(y)$, then $b - x - y - b$ is a cycle in $W\Gamma(R)$ of length three, which contradicts the assumption.

Case 2: Suppose $\text{ann}(x) \cap \text{ann}(y) = 0$, then we have $x \neq y$. As $\text{ann}(x) \subseteq \text{ann}(xy)$,

and $\text{ann}(x) \cap \text{ann}(y) = 0$ we conclude that $xy \neq x, y$. So, x, y, xy are pairwise distinct. Since $r \in \text{ann}(x) \subseteq \text{ann}(xy)$ and $rs = 0$, we deduce that $xy - y$ is an edge of $W\Gamma(R)$. In the same way we can see that $xy - x$ is also an edge of $W\Gamma(R)$. Hence, $xy - x - y - xy$ is a cycle in $W\Gamma(R)$ of length three, which is a contradiction. Hence $W\Gamma(R) = \Gamma(R)$.

So $\text{girth}(W\Gamma(R)) = \text{girth}(\Gamma(R)) = 4$. Now Theorem 3.12 implies that $W\Gamma(R) = \Gamma(R) = K_{m,n}$. Hence, there are two prime ideals P_1 and P_2 of R such that $|P_1^*| = m$ and $|P_2^*| = n$, and $\{P_1, P_2\} = \text{Min}(R)$.

Suppose that R is reduced and $\text{Min}(R) = \{P_1, P_2\}$. Since R is reduced we have $Z(R) = P_1 \cup P_2$ by Corollary 2.3 and $P_1 \cap P_2 = (0)$, if not then $P_1 \cap P_2 \notin \text{Min}(R)$, which is not true. Now, we claim that P_1 and P_2 are independent sets of $W\Gamma(R)$:

If $ma = 0$ then, $m(a + b) = 0$ implies $mb = 0$. Hence, $(a + b) - a$ is an edge of $W\Gamma(R)$. Let $x, y \in P_1$ and $0 \neq r \in \text{ann}(x)$, $0 \neq s \in \text{ann}(y)$. Obviously $r, s \in Z(R) = P_1 \cup P_2$ where $P_1 \cap P_2 = (0)$. So, $sy = 0 \in P_2$ implies $s \in P_2$ or $y \in P_2$, but as $y \in P_1$ and $P_1 \cap P_2 = 0$ we get $y \notin P_2$ and so $s \in P_2$. Similarly, $rx = 0 \in P_2$ implies $r \in P_2$ or $x \in P_2$, but as $x \in P_1$ and $P_1 \cap P_2 = 0$ we get $x \notin P_2$ and $r \in P_2$. Moreover, if $rs = 0 \in P_1$ then $r \in P_1$ or $s \in P_1$, which contradicts $P_1 \cap P_2 = 0$. Hence, $r, s \in P_2$ and $rs \neq 0$. So, similarly for P_2 and we showed that P_1 and P_2 are independent. Then, we get $W\Gamma(R) = K_{|P_1^*|, |P_2^*|}$. By hypothesis $W\Gamma(R)$ contains a cycle and so $\text{girth}(W\Gamma(R)) = 4$. $\square$

An example for the Theorem 5.2 is the graph of $W\Gamma(R)$ in Figure 5.1, where the girth number is 4 and $R = \mathbb{Z}_3 \times \mathbb{Z}_3$ is a reduced ring with only two prime ideals $P_1 = 0 \times \mathbb{Z}_3$ and $P_2 = \mathbb{Z}_3 \times 0$.

Theorem 5.3: [10, Theorem 2.4] If R is a reduced ring where $Z(R)^*$ is an ideal of R , then $W\Gamma(R) \neq \Gamma(R)$ and $\text{girth}(W\Gamma(R)) = 3$.

Proof. Let $a \in Z(R)^*$ and $b \in \text{ann}(a) \setminus \{0\}$. Then, also $b \in Z(R)^*$ as $ab = 0$. So, by the fact that $Z(R)^*$ is an ideal of R we have $a + b \in Z(R)^*$ and $a + b \neq 0$ because $a \neq 0$. Also, note that $a(b + a) = ab + a^2 = a^2 \neq 0$ because R is a reduced ring. Hence, $a - (a + b)$ is not an edge of $\Gamma(R)$. If for some $c \in R$ we have $(a + b)c = 0$, then $a(a + b)c = 0$, which is $\text{ann}(a + b) \subseteq \text{ann}(a(a + b)) = \text{ann}(a^2)$. Hence,

$\text{ann}(a+b) \subseteq \text{ann}(a^2)$. Let $m \in \text{ann}(a+b)$, then $m \in \text{ann}(a^2)$. We have two cases:

Case 1: If $ma \neq 0$ then, $m(a+b) = 0$ implies that $ma(a+b) = 0$, implying $ma \in \text{ann}(a+b)$, and $ma^2 = 0$ implies that $(ma)a = 0$, implying $ma \in \text{ann}(a)$. So, $(ma)(ma) = m^2a^2 = m(ma^2) = 0$, which is impossible as R is supposed to be reduced. Hence, $(a+b) - a$ is an edge of $W\Gamma(R)$.

Case 2: If $ma = 0$ then, $m(a+b) = 0$ implies $mb = 0$. Since b is in $\text{ann}(a)$, m is in $\text{ann}(a+b)$ and $bm = 0$ we conclude that $(a+b) - a$ is an edge of $W\Gamma(R)$.

Hence, $W\Gamma(R) \neq \Gamma(R)$.

Now, we claim that $\text{girth}(W\Gamma(R)) = 3$: Firstly, note that none of $a+b$, $(a+b)a$, a is zero. So, we claim that $a+b \neq (a+b)a \neq a$:

Suppose that $(a+b)a = a$. Then, we have $a^2 = a$, and so a is an idempotent. By Lemma 2.1 and Lemma 2.2 we have $Z(R)$ is not an ideal of R , which contradicts the hypothesis.

Suppose that $a+b = (a+b)a$. Then, $a^2 = a+b$ and hence $a^2 \neq a$, because $b \neq 0$. This implies that; $a^2 = a^2 + ab = a(a+b) = a \cdot a^2 = a^2$. Again we get contradiction.

Now, due to the previous parts we know that $a+b-a$ is an edge of $W\Gamma(R)$. From the fact that $\text{ann}(a+b) \subseteq \text{ann}((a+b)a)$ we get that $(a+b)a - a$ is an edge of $W\Gamma(R)$. Also, as $\text{ann}(a) \subseteq \text{ann}(a^2) \subseteq \text{ann}(a(a+b))$ we have $a+b-a(a+b)$ as an edge of $W\Gamma(R)$. Thus, $(a+b) - a - a(a+b) - (a+b)$ is a 3-cycle of $W\Gamma(R)$, and we are done. $\square$

Theorem 5.4: [10, Theorem 2.5] Let R be a ring. Then $W\Gamma(R)$ is a star graph if and only if one of the following holds:

1. $R \cong \mathbb{Z}_2 \times D$, where D is an integral domain.
2. $|\text{Nil}(R)| = |Z(R)| = 3$.

Proof. ($\Leftarrow$) If $|\text{Nil}(R)| = |Z(R)| = 3$, then we have only two non-zero zero-divisors. Let $Z(R) = \{0, a, b\}$ and assume that ab is non-zero.

As $a \in Z(R)^*$ there exists $0 \neq r \in R$ such that $ar = 0$, so $r \in Z(R)^* = \{a, b\}$. If

$ab \neq 0$, then $r \neq b$ and so $a^2 = 0$. Hence, $a \in \text{Nil}(R)$. Similarly, as $b \in Z(R)^*$ there exists $0 \neq s \in R$ such that $bs = 0$ and so $s \in Z(R)^*$, Since $ab \neq 0$ we have $s = b$ and so $b^2 = 0$. Hence, $b \in \text{Nil}(R)$.

So, $(ab)^2 = a^2b^2 = 0$ implies $ab \in \text{Nil}(R)$. Now we have two cases:

1. If $ab = a$ we have $a(b - 1) = 0$ which implies $b - 1 \in Z(R)$. But as $Z(R) = \text{Nil}(R)$ is an ideal of R , we get $b - (b - 1) = 1 \in Z(R)$, which is a contradiction.
2. If $ab = b$, in the same way we get a contradiction.

So, $ab = 0$ and we have only one edge in $\text{WT}(R)$, hence $\text{WT}(R)$ is a star.

If $R \cong \mathbb{Z}_2 \times D$, where D is an integral domain, then the vertex $(1, 0)$ is adjacent to all other vertices of the form $(0, a)$ as a changes in F , indeed, $(1, 0)$ is the center of the star graph $\text{WT}(R)$.

($\Rightarrow$) Suppose that $\text{WT}(R)$ is a star graph. By Lemma 5.1 it follows that $\text{Nil}(R)^*$ is a complete subgraph of $\text{WT}(R)$. As a star graph is acyclic, we deduce that $\text{Nil}(R)$ has at most three distinct elements. So, we have the following three cases:

Case 1: If $|\text{Nil}(R)| = 1$, then $\text{Nil}(R) = \{0\}$, and so R is reduced ring. Suppose that $a \in V(\text{WT}(R))$ is adjacent to all other vertices. Then we claim that a is an idempotent. Assume that $a \neq a^2$. Then, as $\text{ann}(a) \subseteq \text{ann}(a^2)$, let $x \in \text{ann}(a^2)$, then we have $xa^2 = 0$, but as R is reduced we have $a^2 \neq 0$ which gives $xa \in \text{ann}(a)$. If $xa \neq 0$, then $(xa)^2 = x^2a^2 = xxa^2 = 0$, and so $0 \neq xa \in \text{Nil}(R)$, which gives a contradiction because $|\text{Nil}(R)| = 1$. So, $xa = 0$ and $x \in \text{ann}(a)$. Hence, $\text{ann}(a) = \text{ann}(a^2)$, and so a and a^2 are adjacent to all other vertices. So $Z(R) = \{0, a, a^2\}$, for if not, then we have a triangle in $\text{WT}(R)$. As R is reduced then any power of a cannot be zero. So, either $a^3 = a$ or $a^3 = a^2$.

If $a^3 = a$ then $a(a^2 - 1) = 0$, and $(a^2 - 1) \in Z(R)$. Therefore, we have the following subcases:

1. $a^2 - 1 = 0$ which is not possible as $a^2 = 1 \notin Z(R)$.

2. $a^2 - 1 = a^2$ which is not possible as $-1 \neq 0$.

3. $a^2 - 1 = a$ which gives $a^2 = 0$ and it is not possible as R is reduced.

If $a^3 = a^2$ then $a^2(a - 1) = 0$, and $(a - 1) \in Z(R)$. Therefore, we have the following subcases:

1. $a - 1 = 0$ which is impossible as $a = 1 \notin Z(R)$.

2. $a - 1 = a$ which is impossible as $-1 \neq 0$.

3. $a - 1 = a^2$ which gives $a^4 = 0$ and it is not possible as R is reduced.

So a is an idempotent element.

Now, as R has an idempotent element it can be written as $R \cong R_1 \times R_2$ where R_1 and R_2 are two rings. So, we claim that $R_1 \cong \mathbb{Z}_2$ and $R_2 \cong D$, where D is an integral domain.

If $R_1 \cong \mathbb{Z}_2$ and $R_2 \cong \mathbb{Z}_2$, then we are done as $\mathbb{Z}_2$ is an integral domain.

Suppose that $|R_2^*| \geq 2$. Then, there is $1 \neq b \in R_2^*$. If $1 \neq r \in R_1$, then $(r, 0)$ is a zero divisor in R . So, $(r, 0) - (0, 1) - (1, 0) - (0, b) - (r, 0)$ is a cycle in $WT(R)$, which is a contradiction with the fact that $WT(R)$ is a star graph. Hence, $r = 0$ and $R_1 \cong \mathbb{Z}_2$. Now, let $x \in Z(R_2^*)$ and $a \in \text{ann}(x)$. then, $(1, 0) - (0, x) - (0, a) - (1, 0)$ is a triangle in $WT(R)$, which is a contradiction, which gives $x = 0$. So, $Z(R_2) = (0)$, and hence $R_2 \cong D$ where D is an integral domain.

Suppose that $|Nil(R)| = 2$. As $WT(R)$ is a star graph, it has a central vertex say x . Now Lemma 5.1(1), verifies that $0 \neq x \in Nil(R)$, and so $Nil(R) = \{0, x\}$. Moreover, as $\Gamma(R)$ is a connected subgraph of $WT(R)$ we deduce that $\Gamma(R)$ is a star graph with center x . We claim that $WT(R)$ is complete which means $\text{ann}(x) = Z(R)$.

Let $z, y \in V(WT(R))$ such that $x \neq y \neq z$. We know that $xy = 0$ and $xz = 0$. As x is a nilpotent element, there exists some positive integer as small as possible, say n , such that $x^n = 0$. Let l and t be those integers for which we have $n = l + t$. Then obviously $(x^l y = 0$ and $x^t z = 0$ and $x^l x^t = x^n = 0$. Thus $y - z$ is an edge of $WT(R)$. As z and y are arbitrary, we conclude that $WT(R)$ is complete.

So, we have $WT(R) = K_2$ and then $Z(R) = \{0, x, b\}$. We know that $|Nil(R)| = 2$,

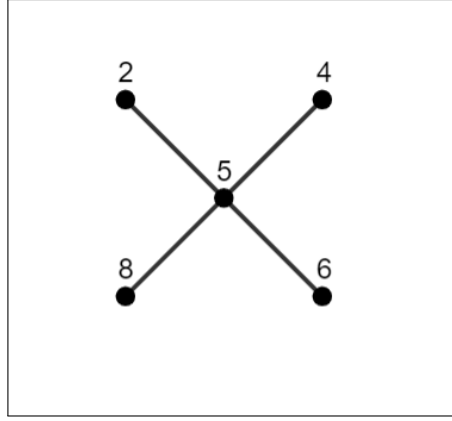


Figure 5.3: Weakly zero-divisor graph of $\mathbb{Z}_{10}$.

so $\text{Nil}(R) = \{0, x\}$ where $x^2 = 0$. Since $Z(R)$ is an ideal of R we get $x + b \in Z(R)$, since $x \neq 0$ we get $x + b \neq b$ and since $b \neq 0$ we get $x + b \neq x$. So, $Z(R) = \{0, x, -x\}$ which implies $x + b = 0 \Rightarrow b = -x$. Therefore, $b^2 = (-x)^2 = 0$ and so $b \in \text{Nil}(R)$, which implies $b = x$ and it is a contradiction.

Case 2: Suppose $|\text{Nil}(R)| = 3$. So we have only to prove that $|\text{Nil}(R)| = |Z(R)| = 3$. By Lemma 5.1 we know that if $x \in \text{Nil}(R)^*$, then x is adjacent to all other vertices in $W\Gamma(R)$. Let x be not the center of the star $W\Gamma(R)$. Then, by Lemma 5.1 we have $W\Gamma(R) \cong K_2$. So, we have only two non-zero zero-divisors and we are done. $\square$

Example 5.3: Consider $R = \mathbb{Z}_{10}$. Then, $R \cong \mathbb{Z}_2 \times D$, where $D = \mathbb{Z}_5$ is an integral domain. Thus, the weakly zero-divisor graphs of R is a star graph given in Figure 5.3.

Finally, we give a necessary and sufficient condition for $W\Gamma(R)$ to be a complete graph, when R is an Artinian ring.

Theorem 5.5: [10, Theorem 2.6] Let R be an Artinian ring. Then $W\Gamma(R)$ is a complete graph if and only if one of the following hold:

1. $R \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$
2. $R \cong R_1 \times \cdots \times R_m$, where R_i is a non-reduced Artinian local ring, for every $1 \leq i \leq m$.

Proof. ($\Rightarrow$) Suppose that $W\Gamma(R)$ is complete. Then, due to Theorem 2.2 we know that $R \cong R_1 \times \cdots \times R_m$, where R_i 's are Artinian local ring for every $1 \leq i \leq m$. If all R_i 's are non-reduced then part 2 holds and we are done. So, suppose that at least one of R_i 's is reduced. Without loss of generality, assume that R_1 is a reduced Artinian

ring. Hence, R_1 is a field. Now, we consider the following two cases.

Case 1: Suppose that $R_i \cong \mathbb{Z}_2$, for every $i \neq 1$. Now we claim that $R \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. By the contrary suppose that the claim is not correct and assume that $R_1 \not\cong \mathbb{Z}_2$. Let $1 \neq u \in R_1^*$. Consider $x = (u, 1, \dots, 1, 0)$ and $y = (1, 1, \dots, 1, 0)$ where both are vertices of $W\Gamma(R)$. If $r = (r_1, r_2, \dots, r_m) \in \text{ann}(x)$ then $(r_1 u, r_2, r_3, \dots, r_{m-1}, 0) = (0, \dots, 0)$. So, we get $r_2 = r_3 = \cdots = r_{m-1} = 0$ and $r_1 u = 0$. As R_1 is a field $u^{-1} \in R_1$ we have $r_1 u u^{-1} = 0$ which gives $r_1 = 0$. Therefore, $r = (0, \dots, 0, r_m)$, where $r_m \in R_m \cong \mathbb{Z}_2$. Hence, $r = (0, \dots, 0)$ or $r = (0, \dots, 0, 1)$ and so $\text{ann}(x) = \{(0, \dots, 0), (0, \dots, 0, 1)\}$. Similarly we can see that $\text{ann}(y) = \{(0, \dots, 0), (0, \dots, 0, 1)\}$. Since $(0, \dots, 0, 1)$ is the only non-zero element of $\text{ann}(x)$ and $\text{ann}(y)$, and $(0, \dots, 0, 1)(0, \dots, 0, 1) \neq (0, \dots, 0)$ we conclude that x and y are not adjacent in $W\Gamma(R)$, which is a contradiction.

Case 2: Suppose that $R_i \not\cong \mathbb{Z}_2$, for some $i \neq 1$. Now we claim that this is impossible. Without loss of generality suppose that $R_m \not\cong \mathbb{Z}_2$. Let $1 \neq u \in R_m - Z(R)$. Consider $x = (0, 1, \dots, 1, u)$ and $y = (0, 1, \dots, 1, 1)$ both in $V(W\Gamma(R))$. Let $r = (r_1, r_2, \dots, r_m) \in \text{ann}(x)$. Then $rx = (r_1, \dots, r_m)(0, 1, \dots, 1, u) = (0, r_2, \dots, r_{m-1}, r_m u) = (0, \dots, 0)$. So, $r_2 = \cdots = r_{m-1} = 0$ and $r_m u = 0$. As R_m is a field we get $r_m u u^{-1} = 0$, which means $r_m = 0$. Hence, as $r_1 \in R_1$ we get $\text{ann}(x) = \{(0, \dots, 0), (1, 0, \dots, 0)\}$. Similarly we have $\text{ann}(y) = \{(0, \dots, 0), (1, 0, \dots, 0)\}$. As $(1, 0, \dots, 0)$ is the only non-zero element of $\text{ann}(x)$ and $\text{ann}(y)$, and $(1, 0, \dots, 0)(1, 0, \dots, 0) \neq (0, \dots, 0)$ we conclude that x and y are not adjacent in $W\Gamma(R)$, which is a contradiction.

($\Leftarrow$) Now firstly suppose that $R \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. If $x = (1, \dots, 1)$ then for any non-zero $r = (r_1, \dots, r_m) \in \text{ann}(x)$ we have $xr = (1, \dots, 1)(r_1, \dots, r_m) = (r_1, \dots, r_m) \neq (0, \dots, 0)$. Hence, $V(W\Gamma(R)) = \{(x_1, \dots, x_m) : x_i = 0 \text{ for some } i\}$. Let $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ be two arbitrary different vertices of $W\Gamma(R)$. Define $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ for all $1 \leq i \leq m$ where i stands for the i^{th} non-zero entry of the vector e_i . Note that there has to be $i \neq j$ such that $x_i = 0$ and $y_j = 0$ because $x \neq y$. Then, $x \cdot e_i = 0$, $y \cdot e_j = 0$ and $e_i \cdot e_j = 0$. Hence, x and y are adjacent in $W\Gamma(R)$. As x and y was arbitrary we conclude that $W\Gamma(R)$ is complete.

Now suppose that $R \cong R_1 \times \cdots \times R_m$, where R_i is a non-reduced Artinian local ring for every $1 \leq i \leq m$. Let

$$A = \{(x_1, \dots, x_m) \in R : x_i \in \text{Nil}(R_i)^* \text{ for some } 1 \leq i \leq m\}$$

and

$$B = \{(x_1, \dots, x_m) \in R : x_i \notin \text{Nil}(R_i)^*, \text{ for all } 1 \leq i \leq m \text{ and } x_i = 0 \text{ for some } 1 \leq i \leq m\}.$$

By construction of A and B obviously $A \cap B = \emptyset$. We also claim that $V(W\Gamma(R)) = A \cup B$.

Firstly, let $x = (x_1, \dots, x_m) \in V(W\Gamma(R))$, then there is $y = (y_1, \dots, y_m) \in R^*$ such that $xy = 0$. So, $xy = (x_i y_i, \dots, x_m y_m) = (0, \dots, 0)$, which gives that $x_i y_i = 0$ for all i . Look at the coordinates of x ; if there exists $r \in \{1, \dots, m\}$ such that $x_r \in \text{Nil}(R_r)^*$, then by the definition $x \in A$, and so $x \in A \cup B$. So, suppose that there is no $r \in \{1, \dots, m\}$ such that $x_r \in \text{Nil}(R_r)^*$. Then, for each $i \in \{1, \dots, m\}$, $x_i \notin \text{Nil}(R)^*$, so $x \in B$ and hence $x \in A \cup B$. Conversely, if $x \in A$ or $x \in B$, then obviously x is a zero-divisor and x is a vertex of $W\Gamma(R)$. So, now we separate this part of the proof into three cases.

Case 1: Here we claim that any two elements of A are adjacent in $W\Gamma(R)$. Let $x = (x_1, \dots, x_m), y = (y_1, \dots, y_m) \in A$. Then, $x_i \in \text{Nil}(R_i)^*$ and $y_j \in \text{Nil}(R_i)^*$ for some $1 \leq j \leq m$. Let n and k be the least positive integers for which $x_i^n = 0$ and $y_j^k = 0$. Now we have two subcases:

Subcase a: If $i \neq j$. Then, $x_i \cdot x_i^{n-1} \cdot e_i = 0$ and $y_j \cdot y_j^{k-1} \cdot e_j = 0$, which gives $x_i^{n-1} \cdot e_i \cdot y_j^{k-1} \cdot e_j = 0$. Hence, $x - y$ is adjacent in $W\Gamma(R)$.

Subcase b: If $i = j$. Then, if $x_i^{n-1} \cdot y_i^{k-1} = 0$ we have $x_i^{n-1} \cdot e_i \cdot y_i^{k-1} \cdot e_i = 0$ and so $x - y$ are adjacent. If $x_i^{n-1} \cdot y_i^{k-1} \neq 0$, we consider $r = x_i^{n-1} \cdot y_i^{k-1} \cdot e_i$, then $r \in \text{ann}(x)$ and $r \in \text{ann}(y)$ and also $r^2 = 0$. Hence, $x - y$ is adjacent.

Case 2: Here we claim that any two elements of B are adjacent in $W\Gamma(R)$. Let $x, y \in B$. Similarly, suppose that $x_i = 0$ and $y_j = 0$, and then we have two subcases:

Subcase a: If $i \neq j$. Then, $x e_i = 0, y e_j = 0$ and $e_i e_j = 0$. Hence, $x - y$ are adjacent.

Subcase b: If $i = j$. As R_i is non-reduced for all $1 \leq i \leq m$, then there exists a non-zero

nilpotent element r_i in R_i . Let, n_i be the smallest natural number such that $r_i^{n_i} = 0$, for all $1 \leq i \leq m$. Then, if $r = (r_1, \dots, r_m)$ we have $y \cdot r \cdot e_i = 0$, $x \cdot r^{n-1} \cdot e_i = 0$, and $r^{n-1} \cdot e_i \cdot r \cdot e_j = 0$. Hence, $x - y$ are adjacent.

Case 3: Here we claim that if $x \in A$ and $y \in B$, then $x - y$ is adjacent in $W\Gamma(R)$. Suppose that $x_i = 0$ and $y_j = 0$. Again we have two subcases:

Subcase a: If $i \neq j$ then we have $x_i^{n-1} \cdot e_i \cdot e_j = 0$, $x_i \cdot x_i^{n-1} \cdot e_i = 0$, and $y \cdot e_j = 0$. Hence, $x - y$ are adjacent.

Subcase b: If $i = j$, then set $r = x_i^{n-1} \cdot e_i$. Then, $xr = 0$, $yx_ie_i = 0$, and $rx_ie_i = 0$. Hence, $x - y$ are adjacent.

Hence, $W\Gamma(R)$ is complete. $\square$

5.1. The Relationship Between the $\Gamma(R)$ and the $W\Gamma(R)$

As we already know that $\Gamma(R)$ is a connected subgraph of $W\Gamma(R)$, a natural question would be as follows: When the graph $W\Gamma(R)$ and the graph $\Gamma(R)$ are identical? We first answer this question in the case when R is a reduced ring and not an integral domain.

Theorem 5.6: [10, Theorem 3.1] If R is a reduced ring that is not an integral domain, then $W\Gamma(R) = \Gamma(R)$ if and only if $|Min(R)| = 2$.

Proof. ($\Rightarrow$) If $W\Gamma(R) = \Gamma(R)$ and R is reduced, we claim that if there are distinct a and b in $Z(R)^*$, then $ann((a, b)) = (0)$. Let $0 \neq r \in ann((a, b))$, then $ra = 0$ and $rb = 0$, but $r \neq a$ and $r \neq b$. So in $\Gamma(R)$ we have the star $a - r - b$ with center r , but in $W\Gamma(R)$, a and b are adjacent, which implies that $W\Gamma(R) \neq \Gamma(R)$, and hence a contradiction. Therefore, we can apply Theorem 3.11. If $|Min(R)| \geq 3$, then by Theorem 3.11 we get $diam(\Gamma(R)) = diam(W\Gamma(R)) = 3$, which contradicts Theorem 5.1. We now claim that $|Min(R)| \neq 1$. If not, then $Min(R) = \{(0)\}$, where (0) states for zero ideal. Then, $R/(0) = R$ is an integral domain, which contradicts the assumption.

($\Leftarrow$) Let R be a reduced ring which is not an integral domain. Assume that $|Min(R)| = 2$.

Case (1): If $W\Gamma(R)$ contains a cycle, then as we discussed in the proof of Theorem 5.2, we can observe that $W\Gamma(R) = \Gamma(R)$.

Case (2): Assume that $girth(W\Gamma(R)) = \infty$. As $\Gamma(R)$ is a subgraph of $W\Gamma(R)$, we conclude that $girth(\Gamma(R)) = \infty$. As R is a reduced ring which is not an integral domain and $girth(\Gamma(R)) = \infty$ Theorem 3.13 implies that $\Gamma(R) = K_{1,n}$, for some $n \geq 1$. On the other hand, $dim(W\Gamma(R)) \leq 2$ which implies that $W\Gamma(R) = \Gamma(R) = K_{1,n}$. $\square$

Note that an example when $\Gamma(R)$ and $W\Gamma(R)$ are identical is the Example 5.3, where one can check that the zero-divisor and weakly zero-divisor graphs of $R = \mathbb{Z}_{10}$ are both star graphs on 5 vertices, as in Figure 5.3. Also, note that $\mathbb{Z}_{10}$ is a reduced ring because 10 is a squarefree integer. Moreover, the only minimal prime ideals of $\mathbb{Z}_{10}$ are $I = \{\bar{0}, \bar{5}\}$ and $J = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}\}$. Finally, we answer our main question in the case when R is not a reduced ring.

Theorem 5.7: [10, Theorem 3.2] Let R be a non-reduced ring. Then the followings are equivalent:

1. $W\Gamma(R) = \Gamma(R)$.
2. $Z(R)^2 = 0$.
3. $\Gamma(R)$ is a complete graph.

Proof. (1) $\Rightarrow$ (2) As R is non-reduced, $Nil(R)$ is non zero, so there exists $x \in Nil(R)^*$. As $W\Gamma(R) = \Gamma(R)$, we deduce that x is adjacent to all other vertices by Lemma 5.1. We have one of the following cases: R is isomorphic with $\mathbb{Z}_2 \times A$, where A is an integral domain, or $Z(R)$ is an annihilator ideal. Note that the first case does not occure as a ring with such structure is a reduced ring. Hence, $Z(R)$ is an annihilator ideal. Now, we claim that $Z(R) = ann(x)$. It is clear that $ann(x) \subseteq Z(R)$. If there exists $y \in Z(R) \setminus ann(x)$, then y is a vertex of $\Gamma(R)$ (so $W\Gamma(R)$) and $xy = 0$. Therefore, $y \in ann(x)$, which is a contradiction. Hence, $ann(x) = Z(R)$.

Firstly suppose that $|Z(R)^*| = 2$. Then, $Z(R)^* = \{x, r\}$, and we have $\Gamma(R) = K_2$. Now, let $|Z(R)^*| \geq 3$. Then, there exists non-equal $y, z \in Z(R)^* \setminus \{x\}$. Also, we have $ann(y) \subseteq Z(R) = ann(x)$. So, for each $\alpha \in ann(y)$ we have $\alpha x = 0$. If $\alpha = x$, then $x^2 = 0$. So, suppose $\alpha \neq x$, then we have the following cases:

- (i) if $\alpha = y$, then $y \in Nil(R)^*$. So, y is a complete vertex and $yz = 0$.
- (ii) if $\alpha \neq y$ and $\alpha = z$, then $zy = 0$.
- (iii) Let $\alpha \neq y$ and $\alpha \neq z$. Then, from $x \in ann(z)$, $\alpha \in ann(y)$, and $x\alpha = 0$ we get y is adjacent with z .

Hence, $y - z$ is an edge in $W\Gamma(R)$. As y and z are arbitrary vertices of $W\Gamma(R)$ we conclude that $W\Gamma(R)$ is complete. Then, also $\Gamma(R)$ is complete as subgraph of $\Gamma(R)$. Now as $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a reduced ring Theorem 3.6. verifies that $Z(R)^2 = \{xy : x, y \in Z(R)\} = 0$.

(2) $\Rightarrow$ (3) Let $Z(R)^2 = 0$. Then, all elements of $Z(R)^*$ are adjacent in $\Gamma(R)$, concluding that $\Gamma(R)$ is complete.

(3) $\Rightarrow$ (1) Assume that $\Gamma(R)$ is complete graph. We know that $V(\Gamma(R)) = V(W\Gamma(R)) = Z(R)^*$. Hence, $W\Gamma(R)$ is complete and so $\Gamma(R) = W\Gamma(R)$. $\square$

Corollary 5.1: [10, Corollary 3.3] If R is a non-reduced ring, then the following conditions are equivalent:

1. $W\Gamma(R)$ is a star graph.
2. $girth(W\Gamma(R)) = \infty$.
3. $W\Gamma(R) = \Gamma(R)$ and $girth(\Gamma(R)) = \infty$.
4. $|Z(R)^*| = |Nil(R)^*| = 2$.
5. $W\Gamma(R) = \Gamma(R) = K_{1,1}$.

Proof. (1) $\Rightarrow$ (2) It is clear that a star graph has no cycle.

(2) $\Rightarrow$ (3) Let $girth(W\Gamma(R)) = \infty$. Then, there is no cycle in $W\Gamma(R)$. Let $a \in Nil(R)^*$. By Lemma 5.1 it follows that a is adjacent to all other vertices of $W\Gamma(R)$. Since $W\Gamma(R)$ has no cycle and $\Gamma(R)$ is a connected subgraph of $W\Gamma(R)$ we get $W\Gamma(R) = \Gamma(R)$.

(3) $\Rightarrow$ (4) By Theorem 5.7, the fact that $W\Gamma(R) = \Gamma(R)$ implies that they are complete. Since $girth(\Gamma(R)) = \infty$ and R is non-reduced we conclude that the graph $\Gamma(R)$ is just

an edge. So we are done.

(4) $\Rightarrow$ (5) We suppose that R is non-reduced and $|Z(R)^*| = |Nil(R)^*| = 2$. Let $Z(R)^* = \{x, y\}$, we know that $\Gamma(R)$ is connected. So, in $\Gamma(R)$ there is an edge between x and y . There exists a non-zero nilpotent element say r such that $r^k = 0$ for some $k \geq 1$. This implies that r is a non-zero zero-divisor, hence it is either x or y . Say, without loss of generality x is a non-zero zero-divisor. Then, for some positive integer n , $x^n = 0$ and x is adjacent to all the other vertices of $W\Gamma(R)$ by Lemma 1.1.. Hence, we conclude that $W\Gamma(R) = \Gamma(R) = K_{1,1}$.

(5) $\Rightarrow$ (1) It is obvious. $\square$

5.2. New Results

Our first result classifies all rings R such that $W\Gamma(R)$ is a path of length at most 2, up to isomorphism.

Lemma 5.2: Suppose R is a commutative ring with $1 \neq 0$. Suppose also that $W\Gamma(R)$ is a path of length at most 2. Then, one of the following cases holds:

1. $W\Gamma(R)$ is a singleton if and only if $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$,
2. $W\Gamma(R)$ is $P_1 = K_2$ if and only if $|Nil(R)| = |Z(R)| = 3$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$,
3. $W\Gamma(R)$ is P_2 if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$.

Proof.

1. ($\Rightarrow$) Suppose $W\Gamma(R)$ is a singletone. Hence, both graphs $\Gamma(R)$ and $W\Gamma(R)$ have $Z(R)$ as the vertex set, so we conclude that $\Gamma(R)$ is a singletone. Let $0 \neq x \in Z(R)^*$. Then, there exists $y \in R$ such that $xy = 0$, but if $y \neq x$, then $0 \neq y \in Z(R)^*$, which is a contradiction because $Z(R)$ has a unique non-zero element. Hence, $x^2 = 0$. Claim that $(x) = \{0, x\}$. Let $z \in (x)$ such that $z \neq 0$ and $z \neq x$. Then, there exists $0 \neq y \in R$ such that $xy = z$, and so $0 = x^2y = xz$. So, $z \in Z(R)^*$, which is a contradiction. Hence, $(x) = \{0, x\}$.

Consider the following sequence:

$$0 \longrightarrow \text{ann}(x) \longrightarrow R \longrightarrow (x) \longrightarrow 0,$$

defined by the maps $\phi : \text{ann}(x) \rightarrow R$; $\phi(a) = a$ for each $a \in \text{ann}(x)$, and $\psi : R \rightarrow (x)$; $\psi(a) = ax$ for each $a \in R$. We claim that this sequence is exact. By the definition we have:

$$\text{Ker}(\phi) = \{a \in \text{ann}(x) : a = 0 \text{ in } R\} = \{0\},$$

and

$$\text{Im}(\phi) = \{b \in R : \phi(a) = b, \text{ for each } a \in \text{ann}(x)\} = \{0, x\}.$$

Also, we have

$$\text{Ker}(\psi) = \{a \in R : \psi(a) = 0 \text{ in } (x)\} = \{0, x\},$$

and

$$\text{Im}(\psi) = \{b \in (x) : \psi(a) = b \text{ for each } a \in R\}.$$

So, $\text{Im}(\phi) = \text{Ker}(\psi)$. Moreover, obviously ϕ is injective and ψ is surjective. Hence, the sequence is exact. Then, by the properties of exact sequences we get

$$R = \text{ann}(x) \oplus (x) = \{(0, 0), (0, x), (x, 0), (x, x)\}.$$

So, as $|R| = 4$ and we know that $\text{char}(R) \mid |R|$, we conclude that $\text{char}(R) \in \{2, 4\}$. If $\text{char}(R) = 4$ we have $R \cong \mathbb{Z}_4 \cong \{0, 1, 2, 3\}$, and if $\text{char}(R) = 2$ we have $R \cong \mathbb{Z}_2[x]/(x^2) = \{0, 1, x, x + 1\}$.

($\Leftarrow$) Now suppose that $R \cong \mathbb{Z}_4 = \{0, 1, 2, 3\}$, then $Z(R)^* = \{2\}$ and $W\Gamma(R)$ is a singleton. On the other hand, if $R \cong \mathbb{Z}_2[x]/(x^2) = \{0, 1, x, x + 1\}$ we have:

- $1 \cdot x \neq 0$
- $1(1 + x) = 1 + x \neq 0$
- $x(1 + x) = x \neq 0$
- $(1 + x)(1 + x) = 1 \neq 0$.
- $x \cdot x = x^2 = 0$

So, $Z(R)^* = \{x\}$. Hence, $W\Gamma(R)$ is a singleton.

2. ($\Rightarrow$) Suppose $W\Gamma(R)$ is $P_1 = K_2 = K_{1,1}$. Then, as $W\Gamma(R)$ is a connected graph and $Z(R)^*$ is its vertex set, we have $|Z(R)| = 3$. As $W\Gamma(R)$ is a star graph, by Theorem 5.4 we have either $|Nil(R)| = |Z(R)| = 3$ or $R \cong \mathbb{Z}_2 \times D$, where D is an integral domain. If the first one holds, then we are done. Suppose that the latter one holds. Furthermore, suppose that $|D| > 2$. Then, there are at least two non-zero elements of D , say a and b . So, we have at least three non-zero zero-divisors in R , namely $(0, a)$, $(0, b)$ and $(1, 0)$. This contradicts the fact that $|Z(R)| = 3$. Hence, $|D| = 2$ and since it is an integral domain, it is a field, implying that $D \cong \mathbb{Z}_2$.

($\Leftarrow$) Suppose that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then, obviously $W\Gamma(R) = P_1$.

3. ($\Leftarrow$) This part is clear.

($\Rightarrow$) Let $W\Gamma(R) = P_2$. Then, as $P_2 = K_{1,2}$ is a star graph, we can use Theorem 5.4. Moreover, as $W\Gamma(R)$ is a connected graph with vertex set $Z(R)^*$, we have $|Z(R)| = 4$. So, by Theorem 5.4 it follows that $R \cong \mathbb{Z}_2 \times D$, where D is an integral domain. By Part 2 we know that $D \not\cong \mathbb{Z}_2$. Suppose that $|D| > 3$. Then, the ring D contains at least three non-zero zero-divisors. By following a similar argument to Part 2, we get a contradiction. Hence, we have $D \cong \mathbb{Z}_3$, as desired.

□

Now, we finally prove the last result of this thesis.

Lemma 5.3: There is no commutative ring R with $1 \neq 0$ such that $W\Gamma(R)$ is the graphs in Figure 5.4.

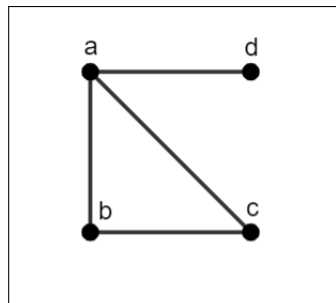


Figure 5.4: $W\Gamma(R)$ in Lemma 5.3.

Proof. Firstly note that by Lemma 5.1 we know that $\Gamma(R)$ is a connected subgraph

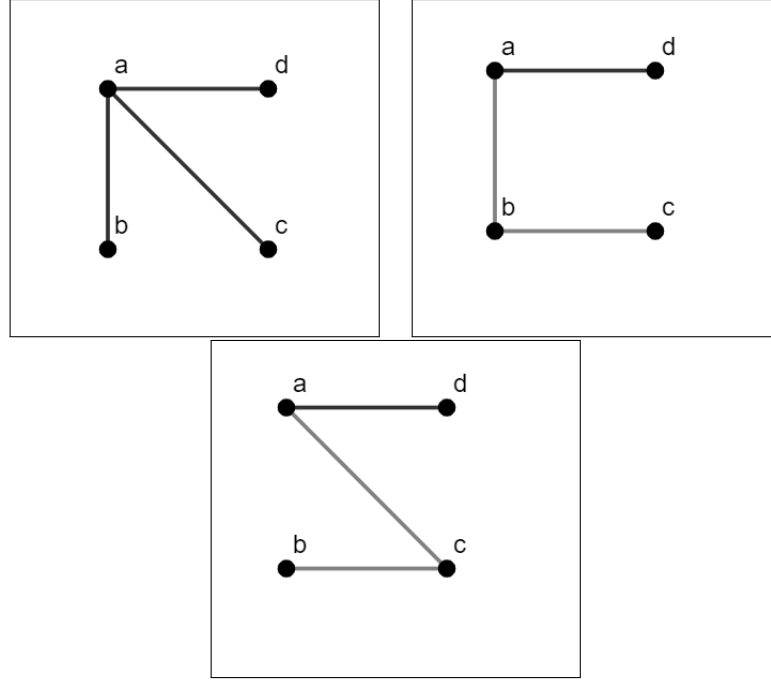


Figure 5.5: All possible connected subgraphs of from Figure 5.4.

of $W\Gamma(R)$. Now, we assume that $W\Gamma(R) = G$ and get a contradiction in each of the following steps:

Step (1): Suppose that $\Gamma(R)$ is a star graph. Then $W\Gamma(R) \neq \Gamma(R)$ and we note that $V(W\Gamma(R)) = V(\Gamma(R)) = Z(R)^* = \{a, b, c, d\}$. Hence, the graph $\Gamma(R)$ is a connected subgraph of $W\Gamma(R)$ with vertices $\{a, b, c, d\}$, and it is supposed to be a star graph. So, the graph $\Gamma(R)$ is the graph G_1 in Figure 5.5. Therefore, we have $ab = ac = ad = 0$. As $Z(R)^* = \{a, b, c, d\}$, $\text{ann}(b) = \text{ann}(c) = \{a\}$ and $b - c$ is an edge of $W\Gamma(R)$, it follows that $a^2 = 0$. So, the fact that $ab = ad = a^2 = 0$ implies that $b - d$ is an edge in $W\Gamma(R)$, which is a contradiction.

Step (2): As $\Gamma(R)$ is not a star graph, then $a - d$ and $b - c$ are edges of $\Gamma(R)$, otherwise it is not a connected graph, which is a contradiction.

Step (3): By the previous two steps we conclude that $\Gamma(R)$ is the graph G or one of G_2 or G_3 in Figure 5.5.

Step (4): Suppose that $\Gamma(R) = G_2$. Then $ad = ab = bc = 0$, and so $d - c$ is an edge in $W\Gamma(R)$, which is a contradiction.

Step (5): Suppose that $\Gamma(R) = G_3$. Then, we have $ad = ac = bc = 0$, and so $d - b$ is an edge in $W\Gamma(R)$, which is a contradiction.

Step (6): Suppose that $\Gamma(R) = G$. If R is a non-reduced ring, then by Theorem 5.7, it

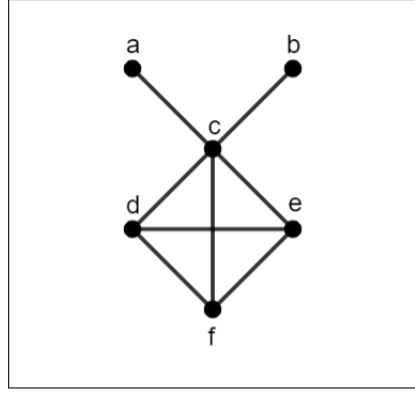
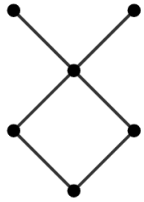
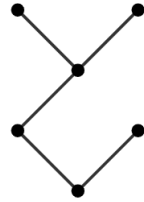
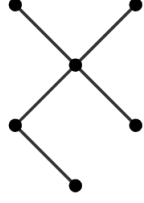
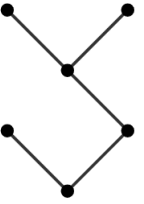
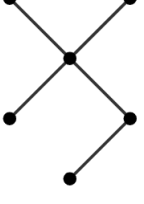


Figure 5.6: The graph $H = WT(R)$ in Remark 5.1.

follows that $\Gamma(R)$ is a complete graph, which is a contradiction. On the other hand, if R is a reduced ring, then by Theorem 5.6, we have $\text{girth}(WT(R)) = 4$, which is a contradiction. Hence, the assertion follows. $\square$

Remark 5.1: Note that there is no commutative ring R with $1 \neq 0$ such that $WT(R) = H$ in Figure 5.6 and $\Gamma(R)$ is one of the graphs in Table 5.1. To show this we follow a similar way as in Lemma 5.3. Assume that it is not true. Note that, by Lemma 5.1, if the ring R is a non-reduced ring, then there is a vertex adjacent to all other vertices in $WT(R)$, which is a contradiction. So, we assume that the ring R is a reduced ring. If $\Gamma(R)$ is one of H_2, H_3, H_4 , or H_5 in Table 5.1, then the girth number of $\Gamma(R)$ is ∞ . So, by Theorem 3.13 it follows that $\Gamma(R) = K_{1,n}$, which is a contradiction. On the other hand, if $\Gamma(R) = H_3$ in Table 5.1, then the girth number of $\Gamma(R)$ is 4. So, by Theorem 3.12 we have $\Gamma(R) = K_{n,m}$, hence a contradiction.

Table 5.1: $\Gamma(R)$, as a subgraph of H .

	$diam(H_i)$	$girth(H_i)$	
H_1	3	4	
H_2	4	∞	
H_3	3	∞	
H_4	4	∞	
H_5	3	∞	

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BIOGRAPHY

Özlem KAMBERI graduated with a Bachelor's degree in Mathematics from Gebze Technical University in 2019. Demonstrating her commitment to furthering her education, she was accepted into the graduate program at Gebze Technical University in 2022.